Lecture Notes in Functional Analysis

Lectures by Dr. Md Ramiz Reza *

Contents

*Notes by Ashish Kujur, Last Updated: February 20, 2023

References

The following textbooks will be used for this course:

1. John B. Conway – A Course in Functional Analysis
2. Walter Rudin – Real and Complex Analysis
3. Bhatia – Notes on Functional Analysis
4. Erwin Kreyzsig – Introductory functional analysis with applications

1 Lecture 1 — *Introduction to Hilbert Spaces and some examples* — 9th January, 2023

1.1 Inner Product Spaces

Definition 1.1.1 (Inner Product)**.** Let V be a vector space over a field $\mathbb{F}$ (where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$). A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ is called an *inner product* if it satisfies the following properties

1. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
2. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
3. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
4. $\langle x, x \rangle \geq 0$
5. $\langle x, x \rangle = 0$ only if $x = 0$.

for all $x, y, z \in V$ and $\alpha \in \mathbb{F}$. A vector space V with an inner product is called an *inner product space.*

Example 1.1.2 (Examples of inner product spaces)**.** Here are some examples of inner product spaces:

1. The obvious first example is that of $\mathbb{C}^n$ with the standard 2-inner product given by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$$

2. One can then consider the space $\ell^2 (\mathbb{N})$ which is the vector space of all square summable sequences on $\mathbb{C}$. That is,

$$\ell^2 (\mathbb{N}) = \left\{ (x_n) \in \mathbb{C}^{\mathbb{N}} \mid \sum_{i \in \mathbb{N}} |x_i|^2 < \infty \right\}$$

We define an inner product on this vector space by

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$$

One can show using Holder's inequality that the sum turns out to be finite and the "inner product" is indeed an inner product.

3. Next, we consider the vector space of all polynomials over $\mathbb{C}$ which we denote by $\mathbb{C}[x]$. If $p, q \in \mathbb{C}[x]$, we define an inner product on $\mathbb{C}[x]$ by

$$\langle p, q \rangle = \int_0^1 p \overline{q} \, \mathrm{dx}$$

4. One can define inner products on $C[0,1]$ and $L^2(X, \mathscr{A}, \mu)$ in an similar fashion as in item 3. Note that $(X, \mathscr{A}, \mu)$ is a measure space.

Definition 1.1.3. Let V be an inner product space. We can define a function $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ by

$$\|x\| = \sqrt{\langle x, x \rangle}$$

We call this function *norm induced by the inner product.* (This norm is indeed a norm as one can check!)

The proof of the following theorems are skipped:

Theorem 1.1.4 (Cauchy Schwarz inequality)**.** *Let V be an inner product space, $x, y \in V$. Then we have that*

$$|\langle x, y \rangle \leq \|x\| \, \|y\|$$

Theorem 1.1.5 (Triangle Inequality)**.** *Let V be an inner product space, $x, y \in V$. Then we have that*

$$\|x + y\| \leq \|x\| + \|y\|$$

1.2 Hilbert Spaces

Definition 1.2.1. Let V be an inner product space. One can consider V as a metric space by defining the following metric d:

$$d(v, w) = \|v - w\|$$

for all $v, w \in V$. Then (V, d) is a metric space (Check!). We say that V is a *Hilbert Space* if (V, d) is a complete metric space.

Example 1.2.2. We consider some examples and not-so-example of Hilbert Space:

1. $\mathbb{R}^n$ and $\mathbb{C}^n$ with the standard inner product are complete!

2. $\ell^2(\mathbb{N})$ is complete.

3. $L^2(X)$ is complete where $(X, \mathscr{A}, \mu)$ is a measure space.

4. $\mathcal{C}[0,1]$ is not complete w.r.t $L^2[0,1]$ inner product.

5. Consider $c_{00} = \left\{ (x_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N}) : (x_n)_{n \in \mathbb{N}} \text{ is eventually zero} \right\}$. c_{00} has the induced inner product. We show that c_{00} with this induced product is not complete! One consider the sequence of sequences given by

$$f_n = \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, 0, \ldots\right)$$

One can then easily show that (f_n) is Cauchy sequence in c_{00} but it does not converge in c_{00}.

2 Lecture 2 — *Hilbert Spaces!* — 11th January, 2023

The important goal of this lecture is to show that if H is a Hilbert Space then we show that under certain conditions an element can be projected onto a set. But before that, we prove the following theorem:

Theorem 2.0.1 (Norm is uniformly continuous)**.** *Let H be a Hilbert space. The norm function on H, that is, $\|\cdot\| : H \to \mathbb{R}$ given by $\|x\| = \sqrt{\langle x, x\rangle}$, $x \in H$, is continuous.*

Proof. Let $x, y \in H$. Then by the triangle inequality, we have the following:

$$\|x\| = \|(x-y)+y\| \leq \|x-y\| + \|y\|$$

and hence

$$\|x\| - \|y\| \leq \|x-y\|$$

Interchanging the role of x and y in the previous inequality, we have htat

$$\|y\| - \|x\| \leq \|x-y\|$$

and thus, we have proved that

$$|\|x\| - \|y\|| \leq \|x-y\|$$

which says that $\|\cdot\|$ is uniformly continuous. □

Note that theorem 2.0.1 holds for any normed linear space, that is, there is no use of completeness there.

2.1 Closed and Convex!

Theorem 2.1.1. *Let S be a closed convex set in a Hilbert space H. Let $x \in H$. The distance of x from S, denoted as $d(x, S)$, is given by*

$$d(x, S) = \inf\{\|x-y\| : y \in S\}.$$

It follows that there exist a unique $s_0 \in S$ such that $d(x, S) = \|x - s_0\|$.

Proof. First of all, recall the parallelogram identity which holds for any innter product spaces, and hence in particular for Hilbert spaces,

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

The parallelogram law plays a crucial role in the proof of this theorem. Now, let's get busy to prove the theorem. First of all, by definition of infimum, we can find a sequence (s_n) in S such that $d(s_n, x) \to d\,(x, S)$. To be economical, let us denote $\delta := d\,(x, S)$. We show that (s_n) is Cauchy sequence in H. To do so, let $\varepsilon > 0$ be given.

Observe that for any $n, m \in \mathbb{N}$,

$$\left\|\frac{x-s_n}{2} - \frac{x-s_m}{2}\right\|^2 + \left\|\frac{x-s_n}{2} + \frac{x-s_m}{2}\right\|^2 = \frac{1}{2}\left(\|x-s_n\|^2 + \|x-s_m\|^2\right)$$

and hence

$$\frac{1}{4}\|s_m - s_n\|^2 = \frac{1}{2}\left(\|x - s_n\|^2 + \|x - s_m\|^2\right) - \frac{1}{4}\left\|x - \frac{s_n + s_m}{2}\right\|^2 \tag{2.1.1}$$

Now since $d(s_n, x)$ converges to δ, we must have that $d(s_n, x)^2$ converges to δ^2 and hence there is some $K \in \mathbb{N}$ such that for all $i \geq K$,

$$\|x - s_i\|^2 < \delta^2 + \frac{\varepsilon^2}{4}$$

Now for all $n, m \geq K$ and from equation 2.1.1, we have that

$$\begin{aligned}\|s_m - s_n\|^2 &= 2\left(\|x - s_n\|^2 + \|x - s_m\|^2\right) - \left\|x - \frac{s_n + s_m}{2}\right\|^2 \\ &\overset{(!)}{<} 2 \cdot 2\left(\delta^2 + \frac{\varepsilon^2}{4}\right) - 4\delta^2 \\ &= \varepsilon^2\end{aligned}$$

Note that in inequality (!), we made use of the convexity of S to conlude that $\frac{s_n+s_m}{2} \in S$. This shows that (s_n) is Cauchy. Now, since H is a Hilbert space, (s_n) must converge to some $s_0 \in H$. Closedness of S allows us to conclude that s_0 must be in S.

Hence, $x - s_n$ converges to $x - s_0$. By Theorem 2.0.1, we conclude that $\|x - s_n\|$ converges to $\|x - s_0\|$. Since $\|x - s_n\|$ also converges to δ, we have by uniqueness of limits that $\delta = \|x - s_0\|$.

It remains to prove the uniqueness of such a vector. Let us suppose that s_0 and t_0 be two vectors such that $\|x - s_0\| = \|x - t_0\| = \delta$.

Applying parallelogram identity on the vectors s_0 and t_0 as in Equation 2.1.1, we get

$$\begin{aligned}\frac{1}{4}\|s_0 - t_0\|^2 &= \frac{1}{2}\left(\|x - s_0\|^2 + \|x - t_0\|^2\right) - \frac{1}{4}\left\|x - \frac{s_0 + t_0}{2}\right\|^2 \\ &\leq \delta^2 - \left\|x - \frac{s_0 + t_0}{2}\right\|^2 \\ &\leq 0\end{aligned}$$

Hence, $s_0 = t_0$ and this completes the proof of the theorem. $\square$

Example 2.1.2 (distance is achieved but the vector may not be unique)**.** Consider the normed linear space $(\mathbb{R}^2, \|\cdot\|_1)$. Now consider the subset S of $\mathbb{R}^2$ given by

$$S = \{(x_1, x_2) : x_1 + x_2 = 1\}.$$

Note that $d((0,0), S) = 1 = d(0,0), (1,0)) = d((0,0), (0,1))$. Hence, the uniqueness is not guaranteed.

Exercise 2.1.3. *Consider the space* $(C[0,1])$ *with the supremum norm* $\|\cdot\|_\infty$*, that is,* $\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}$*. Let* S *be the set*

$$S = \left\{ f \in C[0,1] \,:\, \int_0^{1/2} f(x)\,dx - \int_{1/2}^1 f(x)\,dx = 1 \right\}.$$

Show that the set S *is closed and convex but the distance* $d(0,S) = 1$*, is never achieved at any point in* S*. That is, it is not the case that there is some* $f \in S$ *such that* $d(0,f) = \|f\|_\infty = 1$*.*

Solution. We begin by showing that S is convex. Let $f, g \in S$ and $t \in [0,1]$. Then we have that

$$\begin{aligned}\int_0^{1/2} (tf(x) + (1-t)g(x))\,dx - \int_{1/2}^1 (tf(x) + (1-t)g(x)\,dx) &= t + (1-t) \\ &= 1\end{aligned}$$

Note that the second equality follows by the virtue of $f, g \in S$.

Now, we proceed to show that the S is closed. Let (f_n) be a sequence of functions in S converging to $f \in C[0,1]$. We need to prove that $f \in S$. Now convergence in supremum norm is the same as the uniform convergence, so, we have that following:

$$\lim_{n\to\infty} \left(\int_0^{1/2} f_n(x)\,dx - \int_{1/2}^1 f_n(x)\,dx \right) = 1$$

implies

$$\int_0^{1/2} f(x)\,dx - \int_{1/2}^1 f(x)\,dx = 1$$

and thus $f \in S$. Consider the zero function and the set S, we show that that there is no $f \in S$ such that $d(0,S) = d(f,0) = \|f\|_\infty$. Incomplete! □

2.2 Projections

Theorem 2.2.1. *Let* H *be a Hilbert space. For any fixed* $y \in H$*, consider the map* $L_y : H \to \mathbb{C}$ *defined by* $L_y(x) = \langle x, y\rangle,\ x \in H$*. Then* L_y *is a continuous linear functional on* H*.*

Proof. Let $y \in H$ be fixed. Consider the function $L_y : H \to \mathbb{C}$ given by $L_y(x) = \langle x, y\rangle$ for each $x \in H$. We show that L_y is Lipschitz continuous. Let $x_0 \in H$. If $x \in H$, we have that

$$\begin{aligned}|L_y(x) - L_y x_0| &= |\langle x, y\rangle - \langle x_0, y\rangle| \\ &= |\langle x - x_0, y\rangle| \\ &\le \|x - x_0\|\,\|y\|\end{aligned}$$

Note the inequality follows from Cauchy Schwarz and this completes the proof. □

Definition 2.2.2. Let H be a Hilbert space. For any $y \in H$, the symbol $y^{\perp}$ denote the subspace defined by

$$y^{\perp} := \{x \in H : \langle x, y \rangle = 0\}$$

Observe that $y^{\perp}$ is a closed subspace of H. This is because $y^{\perp}$ is the kernel of the continuous map L_y as given by Theorem 2.2.1.

Definition 2.2.3. Let H be a Hilbert space. Let M be any subspace of H. Let the symbol $M^{\perp}$ denote the subspace given by

$$M^{\perp} = \{x \in H : \langle x, y \rangle = 0 \, \text{for all} \, y \in M\} = \bigcap_{y \in M} y^{\perp}.$$

Observe that $M^{\perp}$ is always closed since it is intersection of closed subspaces of H.

Theorem 2.2.4 (Existence of an Orthogonal Projection onto a closed subspaces)**.** *Let M be a closed subspace of a Hilbert space H. Then*

(a) Every $x \in H$ has a unique decomposition

$$x = Px + Qx$$

into a sum of $Px \in M$ and $Qx \in M^{\perp}$. Thus $H = M \oplus M^{\perp}$.

(b) Px and Qx are the nearest points to x in M and in $M^{\perp}$ respectively.

(c) The mappings $P : H \to M$ and $Q : H \to M^{\perp}$ are linear and satisfies $P^2 = P$ and $Q^2 = Q$. The map P and Q are called the ***orthogonal projection onto** M **and** $M^{\perp}$* *respectively.*

(d) $\|x\|^2 = \|Px\|^2 + \|Qx\|^2$ for every $x \in H$.

Proof. Since subspaces are convex, we can appeal to Theorem 2.1.1 as we please. We now start to prove each of the statements of theorem:

(a) Let $x \in H$ be arbitrary. Then by the Theorem 2.1.1 there is a unique vector $Px \in M$ such that

$$d(x, M) = \|x - Px\|$$

Define $Qx \in M$ by $Qx = x - Px$. We need to show that $Qx \in M^{\perp}$. Let $y \in M$. We want to show that $\langle x - Px, y \rangle = 0$. To do so, observe that

$$\left\langle Qx - \langle Qx, y \rangle \frac{y}{\|y\|^2}, y \right\rangle = \langle Qx, y \rangle - \left\langle \langle Qx, y \rangle \frac{y}{\|y\|^2}, y \right\rangle = 0 \tag{2.2.1}$$

Now,

$$Qx = \underbrace{\left(Qx - \langle Qx, y \rangle \frac{y}{\|y\|^2} \right)}_{=:v_1} + \underbrace{\langle Qx, y \rangle \frac{y}{\|y\|^2}}_{=:v_2}$$

Note that by equation 2.2.1, v_1 and v_2 are orthogonal and hence by Pythagoras theorem[1] for inner product spaces, we may write

$$\begin{aligned}
\delta^2 = \|Qx\|^2 &= \left\| Qx - \langle Qx, y\rangle \frac{y}{\|y\|^2} \right\|^2 + \left\| \langle Qx, y\rangle \frac{y}{\|y\|^2} \right\|^2 \\
&= \left\| Qx - \langle Qx, y\rangle \frac{y}{\|y\|^2} \right\|^2 + \frac{|\langle Qx, y\rangle|}{\|y\|^2} \\
&= \left\| x - Px - \langle Qx, y\rangle \frac{y}{\|y\|^2} \right\|^2 + \frac{|\langle Qx, y\rangle|}{\|y\|^2} \qquad \geq \delta^2 + \frac{|\langle Qx, y\rangle|}{\|y\|^2}
\end{aligned}$$

and thus, we have that $|\langle Qx, y\rangle| = 0$. This completes the proof of (a).

(b) By uniqueness of part (a), it follows that Px is the nearest point to x in M. It remains to prove that Qx is the nearest point to x in $M^\perp$. Note that $x - Qx = Px \in M$. Now for any $y \in M^\perp$ we have $Qx - y \in M^\perp$. Thus we get

$$\|x - y\|^2 = \|(x - Qx) + (Qx - y)\|^2 = \|x - Qx\|^2 + \|Qx - y\|^2 \geqslant \|x - Qx\|^2.$$

This shows that Qx is the nearest point to x in $M^\perp$.

(c) Let $x_1, x_2 \in M$. By part (a), we have that

$$\begin{aligned}
x_1 &= Px_1 + Qx_1 \\
x_2 &= Px_2 + Qx_2 \\
x_1 + x_2 &= P(x_1 + x_2) + Q(x_1 + x_2)
\end{aligned}$$

Now taking sums and rearranging, we have that

$$\underbrace{Px_1 + Px_2 - P(x_1 + x_2)}_{\in M} = \underbrace{Q(x_1 + x_2) - Qx_1 - Qx_2}_{\in M^\perp}$$

Since $M \cap M^\perp = \{0\}$, the linearity of P and Q follows.

Now, let $x \in P$. We need to prove that $P^2 x = Px$. Now note that $Px \in M$. Thus by part (a) we have

$$Px = P^2 x + QPx$$

By uniqueness of part (a), we must have that $Px = P^2 x$. This completes the proof. $Q^2 = Q$ can be proved similarly.

(d) This follows immediately from Pythagoras theorem.

□

Corollary 2.2.5. *Let M be a closed subspace of a Hilbert space H. Then $(M^\perp)^\perp = M$. In case M is a subspace then $(M^\perp)^\perp = \overline{M}$, the closure of M in H.*

[1] In inner product space, if $\langle v_1, v_2\rangle = 0$ then $\|v_1 + v_2\|^2 = \|v_1\|^2 + \|v_2\|^2$

3 Lecture 3 — *Riesz Representation Theorem for Hilbert Spaces* — 13th January, 2023

3.1 Lecture 2 continued ...

Before we move onto prove the Riesz Representation Theorem, we finish the proof of Corollary 2.2.5. It follows immediately from the following results:

Proposition 3.1.1 (orthogonal complement of a set and the orthogonal completement of its closure are same!)**.** *Let M be a subset of a inner product space H. Then* $M^{\perp} = \left(\overline{M}\right)^{\perp}$

Proof. It follows by definition that $M \subset \overline{M}$ and hence $\left(\overline{M}\right)^{\perp} \subset M^{\perp}$. Now for reverse the inclusion, let $v \in M^{\perp}$ and let $y \in \overline{M}$. We need to show that $\langle v, y \rangle = 0$. Since $y \in \overline{M}$ there is a sequence (y_n) in M such that $y_n \to y$. Since $v \in M^{\perp}$, we have that $\langle v, y_n \rangle = 0$ for all $n \in \mathbb{N}$. Since $\langle v, y_n \rangle \to \langle v, y \rangle$, we have by uniqueness of limits that $\langle v, y \rangle = 0$. This completes the proof. ☐

Proposition 3.1.2 (orthogonal complement of orthogonal complement)**.** *Let M be a closed subspace of the Hilbert space H. Then*

$$M = \left(M^{\perp}\right)^{\perp}$$

Proof. Let us first show that $M \subset \left(M^{\perp}\right)^{\perp}$ (which in fact holds for any set M). Let $v \in M$ and $w \in M^{\perp}$. It is cleary by definition of $M^{\perp}$ that $\langle v, w \rangle = 0$. Hence, $v \in \left(M^{\perp}\right)^{\perp}$.

Let us proceed to show the inclusion in the other direction. Let $v \in \left(M^{\perp}\right)^{\perp}$. Since M is closed, by Theorem 2.2.4, we have that $v = Pv + Qv$ where $Pv \in M$ and $Qv \in M^{\perp}$. By the previous paragraph, we have that $M \subset \left(M^{\perp}\right)^{\perp}$ and hence $Pv \in \left(M^{\perp}\right)^{\perp}$. Hence, we have that $Qv \in \left(M^{\perp}\right)^{\perp}$. Now, $Qv \in M^{\perp} \cap \left(M^{\perp}\right)^{\perp}$. Hence, $Qx = 0$ and thus, $v = Pv \in M$. ☐

Note that Proposition 3.1.1 does not depend on H being a Hilbert Space while Proposition 3.1.2 does!

Now, proof of Corollary 2.2.5 follows immediately:

Proof of Corollary 2.2.5. The first part of Corollary 2.2.5 is basically Proposition 3.1.2. Now to prove the second part, observe that

$$\begin{aligned} \left(M^{\perp}\right)^{\perp} &= \left(\left(\overline{M}\right)^{\perp}\right)^{\perp} && \text{by Proposition 3.1.1} \\ &= \overline{\overline{M}} && \text{by Proposition3.1.2} \\ &= \overline{M} \end{aligned}$$

☐

3.2 Existence of closed subspaces of Hilbert Spaces

Let H be a Hilbert space of dimension at least 1. Does there always exist a closed subspace of H? The answer is *Yes*!

Let us proceed to prove this: Let H be any Hilbert space of dimension at least one. So, there is at least one nonzero vector v. Let M be the subspace spanned by v. We show that M is closed. Let (y_n) be a sequence in M converging to some $x \in H$. By definition of M, we have that for every $n \in \mathbb{N}$, $y_n = c_n v$ for some $c_n \in \mathbb{F}$. We claim that c_n is a Cauchy sequence in $\mathbb{F}$.

To show that (c_n) is Cauchy in $\mathbb{F}$, let $\varepsilon > 0$ be given. Since $(c_n v)$ is convergent, it is Cauchy. So there is some $N \in \mathbb{N}$ such that for $n, m \geq N$, we have $\|c_n v - c_m v\| < \|v\| \varepsilon$. Which is turn implies that for $n, m \geq N$, $|c_n - c_m| < \varepsilon$.

Now, since (c_n) is Cauchy in $\mathbb{F}$, it must converge to some $c \in F$. Now, the sequence $(c_n v)$ converges to cv in M and by the uniqueness of limits, we have that $y = cv$ and hence $y \in M$.

This argument generalises, *mutatis mutandis*, and the following result holds:

Theorem 3.2.1. *Every finite dimensional subspace of a Hilbert space is closed.*

Proof. We do a proof by induction on the dimension of finite dimensional subspace. The base case is clear by the argument given before the statement of this theorem.

Suppose the theorem is true for all subspaces of dimension n.

Let H be an inner product space and U be a finite dimensional subspace of H of dimension $n + 1$.

Let v be a nonzero vector of U. Then let

$$v_U^\perp = v^\perp \cap U$$

It is easy to see that $U = \text{span } v \oplus v_U^\perp$. Since $v_U^\perp$ is a subspace of dimension n, it is closed by the induction hypothesis.

We now proceed to show that U is closed in H. Let (u_n) be a sequence in U converging to $x \in H$. Then for each $n \in \mathbb{N}$ we have that $u_n = c_n v + v_n$ for some $c_n \in F$ and some $v_n \in v_U^\perp$. Since (u_n) is convergent, we have that (u_n) is Cauchy. Thus,

$$|c_n - c_m|^2 \|v\|^2 + \|v_n - v_m\|^2 \to 0 \text{ as } m, n \to \infty$$

and note that this is due to Pythagoras theorem.

Then (c_n) converges to c and (v_n) is Cauchy and hence converges to some $y \in H$. Since $v_U^\perp$ is closed, we have that $y \in v_U^\perp$. Thus, u_n converges to $cv + y$ and by uniqueness of limits, we have that $x = cv + y \in U$. □

Example 3.2.2. Consider the subspace c_{00} in $\ell^2(\mathbb{N})$. We showed that c_{00} is not complete with inner product on $\ell^2(\mathbb{N})$, so, it cannot be closed (because closed subspaces of a complete metric space are closed!). So, we may ask what is the closure of c_{00} in $\ell^2(\mathbb{N})$?

It is precisely $\ell^2(\mathbb{N})$. One can show this as follows: if $f \in \ell^2(\mathbb{N})$ then we may consider the sequence (g_n) in c_{00} given by

$$g_n = (f_1, f_2, \ldots, f_n, 0, 0, 0, \ldots)$$

It is easily seen that g_n converges to f.

Exercise 3.2.3. *If M is a subspace of a Hilbert Space H then so is $\overline{M}$.*

3.3 Statement and Proof of Riesz Representation Theorem

In Theorem 2.2.1, we saw that for any $y \in H$ where H is a Hilbert space, the linear functional $L_y : H \to \mathbb{C}$ given by $x \mapsto \langle x, y \rangle$ is continuous. But does it happen that given a continuous linear functional $L : H \to \mathbb{C}$ there is a $y_0 \in H$ such that $L = L_{y_0}$? Theanswer to this question is:

Theorem 3.3.1 (Riesz Representation Theorem for Hilbert Spaces)**.** *Let $L : H \to \mathbb{C}$ be continuous linear functional. Then there exists a $y_0 \in H$ such that $L = L_{y_0}$.*

But before proving Riesz Representation Theorem, let us prove that following theorem:

Proposition 3.3.2 (necessary and sufficient condition for a subspace to be dense)**.** *Suppose M is a subspace of a Hilbert space V. Then $\overline{M} = V$ iff $M^\perp = 0$.*

Proof of Proposition 3.3.2. First suppose $\overline{M} = V$. We need to show that $M^\perp = 0$. By 3.1.1, we have that $V^\perp = \left(\overline{M}\right)^\perp = M^\perp$. But since $V^\perp = \{0\}$, we have that $M^\perp = \{0\}$.

To prove the converse, suppose that $M^\perp = 0$. Taking $\perp$ both sides, we have that $\overline{M} = 0$ by using 2.2.5. $\square$

Proof of 3.3.1. Let $L : H \to \mathbb{C}$ be continuous linear functional. Since L is continuous, $\ker L$ is closed. Let $M := \ker L$.

Now, there are two possible cases: either $M^\perp = 0$ or $M^\perp \supsetneq \{0\}$.

If $M^\perp = \{0\}$, then $M = \overline{M} = (M^\perp)^\perp = \{0\}^\perp = H$. Hence, we have that $L = L_0$ and this case is done.

Let us consider the other case where $M^\perp \supsetneq \{0\}$. We claim that $\dim M^\perp = 1$. Since $M^\perp \supsetneq \{0\}$, we may select a nonzero vector $w \in M^\perp$. We show that $M^\perp = \text{span } \{w\}$. To do so, let $\tilde{w} \in M^\perp$. Consider the vector $x := L(\tilde{w})\, w - L(w)\tilde{w}$. Clearly, $L(x) = 0$. Hence $x \in M$. Since $w, \tilde{w} \in M^\perp$, we also have that $x \in M^\perp$. But then $x \in M \cap M^\perp = \{0\}$and hence $x = 0$. Thus, $L(\tilde{w})\, w = L(w)\tilde{w}$. Observe that $L(w) \neq 0$ for otherwise $w \in M$ and $w \in M \cap M^\perp$ forcing $w = 0$. Hence $\tilde{w} = (L(w))^{-1} L(\tilde{w})w$. Hence $\tilde{w} \in \text{span } \{w\}$ which completes the proof of the claim.

Let w be as in previous paragraph. We have that $H = M \oplus M^\perp = \ker L \oplus \text{span } w$. Hence, $(\text{span } \{w\})^\perp = (M^\perp)^\perp = \overline{M} = M$. Since $(\text{span } \{w\}) = w^\perp$, we have that $w^\perp = M$.

Now, let $x \in H$. Now,

$$x = \underbrace{x - \frac{\langle x, w\rangle\, w}{\|w\|^2}}_{w^\perp = M} + \underbrace{\frac{\langle x, w\rangle\, w}{\|w\|^2}}_{\in M^\perp}$$

Hence, we have that

$$L(x) = \left\langle x, \frac{\overline{L(w)} w}{\|w\|^2} \right\rangle$$

This completes the proof! $\square$

4 Lecture 4 — *Projection for subspaces having countable orthonormal basis* — 17th January, 2022

4.1 Projections and Orthonormal Sets in finite dimensions...

Suppose M is a finite dimensional subspace of an Hilbert space H and dimension of M is n. Let $\mathcal{B} = \{u_i : 1 \leqslant i \leqslant n\}$ be an orthonormal basis of M (existence of such basis follows from the Gram-Schmidt orthogonalization process to any basis of M). For any $x \in H$, consider the vector $\sum\limits_{j=1}^{n} \langle x, u_j \rangle u_j$ in M. Thus we obtain

$$x = \Big(x - \sum_{j=1}^{n} \langle x, u_j \rangle u_j \Big) + \Big(\sum_{j=1}^{n} \langle x, u_j \rangle u_j \Big)$$

We verify that $x - \sum_{j=1}^{n} \langle x, u_j \rangle u_j \in M^{\perp}$. Observe that for all $1 \leq i \leq n$,

$$\begin{aligned} \left\langle x - \sum_{j=1}^{n} \langle x, u_j \rangle u_j, u_i \right\rangle &= \langle x, u_i \rangle - \sum_{j=1}^{n} \langle x, u_j \rangle \langle u_j, u_i \rangle \\ &= \langle x, u_i \rangle - \langle x, u_i \rangle = 0 \end{aligned}$$

Now by the uniqueness of the decomposition (see Theorem 2.2.4) $H = M \oplus M^{\perp}$, it follows that

$$P_{_M} x = \sum_{j=1}^{n} \langle x, u_j \rangle u_j,$$

$$P_{_{M^{\perp}}} x = x - \sum_{j=1}^{n} \langle x, u_j \rangle u_j,$$

where $P_{_M}$ and $P_{_{M^{\perp}}}$ denotes the orthogonal projection onto M and $M^{\perp}$ respectively. Furthermore, it follows that

$$\|P_{_M} x\|^2 = \sum_{j=1}^{n} |\langle x, u_j \rangle|^2 = \|x\|^2 - \|P_{_{M^{\perp}}} x\|^2 \leqslant \|x\|^2, \ x \in H. \tag{4.1.1}$$

4.2 Generalising projections to infinite dimensions...

Proposition 4.2.1. *Suppose $\mathcal{B} = \{u_i : i \in \mathbb{N}\}$ is an orthonormal set in a Hilbert space H. Then for any $x \in H$, we have*

$$\sum_{j=1}^{\infty} |\langle x, u_j \rangle|^2 \leqslant \|x\|^2.$$

This is known as the Bessel's inequality. Let $M = \overline{span\ \{u_i : i \in \mathbb{N}\}}$ be the smallest closed subspace spanned by the orthonormal set $\mathcal{B}$. It follows that $S_n(x) = \sum_{j=1}^{n} \langle x, u_j \rangle u_j$ is a Cauchy *sequence in M. Hence the limit $\lim_{n\to\infty} S_n(x) = \sum_{j=1}^{\infty}\langle x, u_j\rangle u_j$ exist in the Hilbert space M. Moreover, $P_{_M}(x)$, the orthogonal projection of x onto M, is given by*

$$P_{_M}(x) = \sum_{j=1}^{\infty}\langle x, u_j\rangle u_j = \lim_{n\to\infty} S_n(x).$$

Furthermore equality occurs in the Bessel's inequality if and only if $x \in M$.

Proof. Fix a natural number k and consider $M_k = \operatorname{span}\{u_j : 1 \leqslant j \leqslant k\}$. By Equation 4.1.1 we find that

$$\|P_{_{M_k}} x\|^2 = \sum_{j=1}^{k} |\langle x, u_j\rangle|^2 \leqslant \|x\|^2,\ x \in H,$$

where where $P_{_{M_k}}$ denotes the orthogonal projection onto M. Since this holds true for every $k \in \mathbb{N}$, it follows that

$$\sum_{j=1}^{\infty} |\langle x, u_j\rangle|^2 \leqslant \|x\|^2.$$

For $S_n(x) = \sum_{j=1}^{n}\langle x, u_j\rangle u_j$, note that

$$\|S_m(x) - S_k(x)\|^2 = \sum_{j=k+1}^{m} |\langle x, u_j\rangle|^2, \qquad k \geqslant m.$$

Since $\sum_{j=1}^{\infty} |\langle x, u_j\rangle|^2 < \infty$, it follows that $\{S_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in M and converges to some vector in M. Let $z = \lim_{n\to\infty} S_n(x)$. It is straightforward to verify that $\langle x - S_n(x), u_j\rangle = 0$ for every $n \geqslant j$. Since $x - z = \lim_{n\to\infty}(x - S_n(x))$, it follows that $\langle x - z, u_j\rangle = 0$. This holds true for any $j \in \mathbb{N}$. Hence we get that $x - z \in M^{\perp}$. Thus we have

$$x = z + (x - z),\ \ x - z \in M^{\perp},\ z \in M.$$

By the uniqueness of the decomposition $H = M \oplus M^{\perp}$, we obtain that

$$P_{_M}(x) = z = \sum_{j=1}^{\infty}\langle x, u_j\rangle u_j = \lim_{n\to\infty} S_n(x),\ \ P_{_{M^\perp}}(x) = x - z.$$

Now, we proceed to show that the Bessel inequality holds iff $x \in M$. Suppose that $x \in M$. Then we have that $P_M(x) = x$ and hence $x = z + (x - z) = x + (x - z)$ and hence $x = z$. But note that $\|z\|^2 = \left\|\sum_{j=1}^{\infty} \langle x, u_j\rangle u_j\right\| = \sum_{j=1}^{\infty} |\langle x, u_j\rangle|$ but this follows from

$$\left\|\sum_{j=1}^{\infty} \langle x, u_j\rangle u_j\right\| = \sum_{j=1}^{\infty} |\langle x, u_j\rangle|$$

and then taking $n \to \infty$, we get

$$\left\|\sum_{j=1}^{n} \langle x, u_j\rangle u_j\right\| = \sum_{j=1}^{n} |\langle x, u_j\rangle| \tag{4.2.1}$$

Conversely, suppose that equality in Bessel's inequality holds then from equalion 4.2.1 and the fact that $\|x\|^2 = \|z\|^2 \|x - z\|^2$ that $\|x - z\|^2 = 0$ and hence $x \in M$. □

Corollary 4.2.2. *Suppose* $\mathcal{B} = \{u_i : i \in \mathbb{N}\}$ *is a maximal orthonormal set in a Hilbert space* H*. Let* $M = \overline{span\{u_i : i \in \mathbb{N}\}}$ *be the smallest closed subspace spanned by the orthonormal set* $\mathcal{B}$*. Then it follows that* $M = H$ *and for any* $x \in H$ *we have*

$$x = \sum_{j=1}^{\infty} \langle x, u_j \rangle u_j = \lim_{n \to \infty} S_n(x), \quad \|x\|^2 = \sum_{j=1}^{\infty} |\langle x, u_j \rangle|^2.$$

Proof. If $M^{\perp} \neq 0$, then consider a non zero unit vector u in $M^{\perp}$. Then $\mathcal{B} \cup \{u\}$ is another family of orthonormal set containing $\mathcal{B}$. This contradicts the maximality of $\mathcal{B}$. Hence by the maximality of $\mathcal{B}$, it follows that $M^{\perp} = \{0\}$. Now the corollary follows from the proposition. □

4.3 Existence of a maximal orthonormal set in a inner product space

Let $(X, \leqslant)$ be the collection of all orthonormal set in V equipped with the partial ordering of set inclusion, that is, for $A, B \in X$, we have $A \leqslant B$ if $A \subseteq B$. It is straightforward to verify that if $\mathscr{C}$ is a chain (totally ordered set) in the partially ordered set $(X, \leqslant)$, then the chain $\mathscr{C}$ has an upper bound in X namely the union of the members of $\mathscr{C}$. Hence by Zorn's Lemma it follows that X has a maximal element, that is, V has a maximal orthonormal set. Now, we make a definition:

Definition 4.3.1 (orthonormal basis)**.** A maximal orthonormal set in a Hilbert space is called an *orthonormal basis of the Hilbert space.*

4.4 Separability of Hilbert Spaces

Proposition 4.4.1. *Let* H *be a Hilbert space. Then* H *is separable (that is it has a countable dense set) if and only if* H *admits an at most countable orthonormal basis.*

Proof. Suppose $\mathcal{B} = \{u_\alpha : \alpha \in I\}$ be an collection of orthonormal set in H. It is straightforward to verify that $\|u_\alpha - u_\beta\| = \sqrt{2}$ for every $\alpha, \beta \in I$ with $\alpha \neq \beta$. Thus the collection of balls $\{B(u_\alpha, \frac{\sqrt{2}}{2}) : \alpha \in I\}$ are pairwise disjoint. If I is uncountable then we have uncountable such balls which are pairwise disjoint. This contradicts any existence of countable dense set in H. Thus if H is separable then any orthonormal collection in H has to be at most countable (finite or countably infinite). Hence any maximal orthonormal set in H must be at most countable. This proves that for a separable Hilbert space H we have an at most countable orthonormal basis.

For the converse direction assume that H admits a countable orthonormal basis, say $\mathcal{B} = \{u_i : i \in \mathbb{N}\}$. Let $D = \cup_{n \in \mathbb{N}} D_n$, where D_n is given by

$$D_n = \Big\{ \sum_{j=1}^{n} c_j u_j : c_j \in \mathbb{Q} + i\mathbb{Q} \Big\}$$

Note that each D_n is countable and hence D is countable. It is straightforward to see that $\overline{D_n} = \text{span}\ \{u_j : 1 \leqslant j \leqslant n\}$ for each $n \in \mathbb{N}$. It follows that span $\{u_j : j \in \mathbb{N}\} \subseteq \overline{D}$. This gives us that $\overline{\operatorname{span}\{u_j : j \in \mathbb{N}\}} \subseteq \overline{D}$. In view of the Corollary 4.2.2 we obtain that D is dense in H and hence H is separable. $\square$

5 Lecture 5 — *Examples Galore!* — 18th January, 2023

5.1 Properties of Finite Dimensional Hilbert Spaces

Proposition 5.1.1. *Suppose H is a finite dimensional Hilbert space. Then H is isometrically isomorphic to the Euclidean space $(\mathbb{C}^n, \langle\cdot,\cdot\rangle_2)$, where $\langle\cdot,\cdot\rangle_2$ is the standard Euclide*

Proposition 5.1.2. *Suppose H is a separable infinite dimensional Hilbert space. Then H is isometrically isomorphic to $\ell^2(\mathbb{N})$.*

5.2 Revisiting the examples of Hilbert spaces

(a) Consider $\ell^2(\mathbb{N})$. Let e_n be the sequence in $\ell^2(\mathbb{N})$ defined by $e_n(i) = \delta_{i,n}$ for every $i, n \in \mathbb{N}$. Then the set $\{e_n : n \in \mathbb{N}\}$ forms an orthonormal basis for $\ell^2(\mathbb{N})$.

(b) Consider c_{00} as a subspace of $\ell^2(\mathbb{N})$. It is an inner product space but not complete w.r.t the metric induced by the associated inner product. In fact c_{00} is dense in $\ell^2(\mathbb{N})$, that is, the closure of c_{00} in $\ell^2(\mathbb{N})$ is the space $\ell^2(\mathbb{N})$ itself.

(c) Consider $(L^2(\mathbb{T}), d\sigma)$, where σ is the Lebesgue measure (the normalised arc length measure) on the unit circle $\mathbb{T}$. Consider the function $z^n : \mathbb{T} \to \mathbb{C}$ given by

$$z^n(e^{it}) = e^{int},\ t \in [0, 2\pi).$$

Note that

$$\langle z^n, z^m\rangle_{L^2(\mathbb{T})} = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)t}dt,\ \ n, m \in \mathbb{Z}.$$

Thus $\{z^n : n \in \mathbb{Z}\}$ forms an orthogonal set. In view of Stone-Weirstrass Theorem, we have span $\{z^n : n \in \mathbb{Z}\}$ is dense in $\mathscr{C}(\mathbb{T})$ in uniform norm $\|\cdot\|_\infty$. Since $\|f\|_2 \leqslant \|f\|_\infty$ for every $f \in \mathscr{C}(\mathbb{T})$, it follows that $span\{z^n : n \in \mathbb{Z}\}$ is dense in $\mathscr{C}(\mathbb{T})$ in L^2 norm $\|\cdot\|_2$.

6 Lecture 6 — *Nonseparable Hilbert Spaces & Generalising Sums...* — 23rd January, 2023

6.1 Non separable Hilbert Spaces

In previous lectures, we saw a couple of examples of separable Hilbert spaces but did not even a see a example to a nonseparable Hilbert space. To remedy this situation, we define a Hilbert space $\ell^2(\mathbb{R})$.

First of all, note how we define $\ell^2(\mathbb{N})$

$$\ell^2(\mathbb{N}) = \left\{ f : \mathbb{N} \to \mathbb{C} \mid \sum_{i=1}^{\infty} |f(i)| < \infty \right\}$$

where the inner product is given by the standard 2-inner product. Can we do the same for $\ell^2(\mathbb{R})$? To answer that question, we make a definition:

Definition 6.1.1. Let $g : \mathbb{R} \to \mathbb{R}_{\geq 0}$. For any finite $F \subset \mathbb{R}$, define $S_F = \sum_{x \in F} g(x)$. We say that $\sum_{x \in R} g(x) < +\infty$ if $\sup_F S_F < \infty$ where F varies over all finite subsets of $\mathbb{R}$.

With the above definition, we have an easy proposition:

Proposition 6.1.2. *Let $g : \mathbb{R} \to \mathbb{R}_{\geq 0}$. Suppose that $\sum_{x \in \mathbb{R}} g(x) < \infty$. Then $A := \{x \in \mathbb{R} : g(x) \neq 0\}$ is countable!*

Proof. For each $n \in \mathbb{N}$, define $A_n = \{x \in \mathbb{R} : |g(x)| > 1/n\}$. It is easy to see that $A = \cup_{n=1}^{\infty} A_n$ (the infamous Archimedean property) and $\{A_i\}_{i \in \mathbb{N}}$ is an increasing sequence of sets.

We claim that for all $n \in \mathbb{N}$, A_n is finite. If our claim is false then there must be some $n_0 \in \mathbb{N}$ such that A_{n_0} is infinite. Let $\{x_i : i \in \mathbb{N}\}$ be a countable subset of A_{n_0}. Then note that $\sup_F S_F \geq \sum_{i=1}^{n} g(x_i)$ for all $n \in \mathbb{N}$. But then $\sup_F S_F \geq \frac{n}{n_0}$ for each $n \in \mathbb{N}$. Hence $\sup_F S_F = +\infty$ which contradicts our assumption. This proves our claim.

Since A is countable union of finite sets, we have that A is countable. $\square$

What the above proposition says is that every nonnegative function whose domain is the reals is zero almost everywhere![2]

But then we want to define this notion for $f : \mathbb{R} \to \mathbb{C}$ as is the case in $\ell^2(\mathbb{N})$ functions. How do we do that?

Definition 6.1.3. Let $g : X \to \mathbb{C}$ be a function. Let $S_F = \sum_{i \in X} g(i)$ where F is any finite subset of $\mathbb{R}$. We will say that S_F converges to some $\lambda \in \mathbb{C}$ if for every $\varepsilon > 0$, there is a finite subset F_0 of X such that for every finite subset $F \supset F_0$, we have that $|S_F - \lambda| < \varepsilon$.

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[2]this is both in the measure theoretic sense and literal sense!

$\ell^2(\mathbb{N}) = \{f : \mathbb{N} \to \mathbb{C} \mid \sum_{i=1}^{\infty} |f(i)| < \infty\}$ consider $\ell^2(\mathbb{R})$

for define $g : \mathbb{R} \to \mathbb{R}_{\geq 0}$

we trying to define $\sum_{x\in\mathbb{R}} g(x)$

for $g : \mathbb{N} \to \mathbb{R}_{\geq 0}$, we define $S_n = g(1) + \ldots + g(n)$. note $\sum_{i=1}^{\infty} g(i)$ iff S_n is bounded

back to $\sum_{x\in\mathbb{R}}$ case, we have

for any $F \subset \mathbb{R}$ finite, define $S_F = \sum_{x\in F} g(x)$. we say that $\sum_{x\in F} g(x) < +\infty$ if $\sup_F S_F < \infty$ where F varies over all finite subsets of $\mathbb{R}$.

remark: this is same as integral with the counting measure!

Proposition 6.1.4. *Suppose that $\sum_{x\in\mathbb{R}} g(x) < \infty$. Then $A := \{x : g(x) \neq 0\}$ is countable!*

Proof. For each $n \in \mathbb{N}$, define $A_n = \{x : |g(x)| > 1/n\}$. $A = \cup_{n=1}^{\infty} A_n$ and observe that A_i is an increasing class.

Observe that for all $n \in \mathbb{N}$, A_n is finite. (prove this!)

Since A is countable union of finite sets, we have that A is countable. $\square$

back to $\ell^2(\mathbb{N})$

define $\langle f, g\rangle = \sum_{x\in F} f(x)\overline{g(x)}$

we need to make a sense for function taking values in $\mathbb{C}$

Definition 6.1.5. $\{S_F\}$ is a net $\lim_F S_F$ if there is a $\lambda \in \mathbb{C}$ such that for any $\varepsilon > 0$, there is a finite set F_0 such that $|S_F - \lambda| \leq \varepsilon$ for any set $F \supset F_0$.

one can check that the notions when $g : \mathbb{N} \to \mathbb{R}$ then $\sum_{i=1}^{\infty} |g(i)| < +\infty$ iff S_F is convergent where $S_F = \sum_{i\in F} g(i)$ where F is a finite set convergnt.

back to definijng the inner product

define $\langle f, g\rangle = \lim_F \sum_{x\in F} f(x)\overline{g(x)}$

we now show that that for any f, g the sum s indeed finite consider $\left|\sum_{i=1}^{n} f(x_i)\overline{g(x_i)}\right| \leq \left(\sum_{i=1}^{n} |f(x_i)|\right)^{1/2} \left(\sum_{i=1}^{n} |g(x_i)|\right)^{1/2}$

then $\sup_F |S_F| \leq \|f\| \|g\|$

define $e_x : \mathbb{R} \to \mathbb{C}$ in the natural wasy

$\underline{span\{e_x : x \in \mathbb{R}\}}^{\perp} = \{0\}$

$\overline{span\{e_x : x \in \mathbb{R}\}} = \ell^2(\mathbb{R})$

hence $\langle f, e_x\rangle = f(x)$

hence $f \in span\{e_x : x \in \mathbb{R}\}^{\perp}$ hence $f(x) = 0$ for every $x \in \mathbb{R}$

now suppose M has onb $\{u_i : i \in I\}$ then one can show that $P_M(x) = \lim_F \sum_{i\in F} \langle x, u_i\rangle u_i$.

now we intend to show the bessel inequality, this immediately follows from the finite case though:

$$\sum_{i\in I} \left|\langle x, u_i\rangle^2\right| \leq \|x\|^2$$

then one can show that $x - \sum_{i\in I} \langle x, u_i\rangle u_i$ is orthogonal to u_i for each $i \in I$

hence, it follows that $P_M(x) = \sum_{i\in I} \langle x, u_i\rangle u_i$

---

one can possibly show that $\{u_i : i \in I\}$ is an orthonormal set where I is any indexing set then

$$H \cong \ell^2 (I)$$

We saw $\mathbb{C}^n, \langle\rangle_2$, $\ell^2 (\mathbb{N})$, $L^2[0,1]$, $L^2 (X, \mu)$

$H^2 (\mathbb{D})$ set of all analytic functions on the disc such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $\sum_{i=0}^{\infty} |a_i|^2 < \infty$ and we have that $a_n = \frac{f^{(n)}(0)}{n!}$

given $f, g \in H^2 (\mathbb{D})$, define

$\langle f, g\rangle = \sum_{n=1}^{\infty} a_n \overline{b_n}$

one can establish a one-to-one correspondence between the $H^2 (\mathbb{D})$ and $\ell^2 (\mathbb{C})$

$$f \mapsto (a_0, a_1, a_2, \ldots)$$

where a_i is defined as above. this is a separable hilbert space which can be easily by the isometry as one can check!

7 Lecture 7 — *Banach Spaces and some examples* — 25th January, 2023

7.1 Possibly, final

7.2 Class Sketch

7.2.1 Definition

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A norm $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ is a funtion (we call this the length) if the following properties are satisfied:

1. $\|v\| \geq 0$ for all $v \in V$
2. $\|v\| = 0$ iff $v = 0$
3. $\|cv\| = |c| \, \|v\|$ for all $c \in F$
4. $\|v + w\| \leq \|v\| + \|w\|$

One can define a function on a normed linear space $d(v, w) = \|v - w\|$. It is not hard to see that (V, d) is a metric space. $(V, \|\|)$ is called a Banach space if the metric space (V, d) is complete where d is the metric induced by the norm.

7.2.2 Examples

Every hilbert space is a banach space since inner product induces a norm.

$L^p(X, \mu)$ is Banach space where (X, μ) is any measure space where $p \in [1, \infty)$.

$X = \{1, \ldots, n\}$ and μ is the counting measure then $L^p(X, \mu)$ is the $\mathbb{C}^n$ with the $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ for $p \in [1, \infty)$ and $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$

One can also consider case such that $X = [0, 1]$ or $X = \mathbb{R}$ with $\mu = \lambda$ the Lebesgue measure.

Also, $X = C[0, 1]$ with $\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$.

Consider $X = \mathbb{N}$ and μ be the counting measure, we get what we call $\ell^p(\mathbb{N})$ whenever $p \in [1, \infty)$. If $p = \infty$, we have $\ell^\infty(\mathbb{N})$, we have the set of all bounded sequences. explicitly try to write out the norms!

Consider $C[0, 1]$ as a subset of $L^p[0, 1]$, $1 \leq p < \infty$. For any p, then $C(X)$ is a dense subset of $L^p(X)$ where X is a locally compact Hausdorff space.

See tutorial problem for an example of a Banach space such that the distance of point and a closed convex set is not achieved!

Let B be a Banach space. Let M be a subspace of B. Does there exist subspace N such that $B = M \oplus N$?

Consider $\mathbb{R}^2$ with 1-norm. Consider the subspace M spanned by e_1. Then any subspace spanned by a single vector linearly dependent does the work.

If H is a Hilbert space then $\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$.

Suppose $(V, \|\|)$ is a normed space. Suppose that for all $x, y \in V$, we have that $\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$. We claim that $\|\cdot\|$ is induced by a unique inner product.

One can define

$$\langle v, w \rangle = \frac{1}{4} \sum_{k=1}^{4} \left\| v + i^k w \right\|^2$$

It can be shown that the above defines an inner product indeed!

Let V be a real inner product space. Then we have that

$$\|v+w\|^2 - \|v-w\|^2 = 4 \langle v, w \rangle$$

Suppose $V, \|\|$ satisfies the parallelogram law then one can define the inner product as above and one can check that it is indeed an inner product.

A set $\{u_i : i \in \mathbb{N}\}$ is called a Schauder basis of a normed linear space $(V, \|\|)$ if there exists a *unique* sequence in $\mathbb{F}$, $\{c_i : i \in \mathbb{N}\}$, $x = \lim_n \sum_{i=1}^n c_i u_i = \sum_{i=1}^\infty c_i u_i$ for every $x \in V$.

Any basis of a finite dimensional vector space is a Schauder basis.

Consider $\ell^1(\mathbb{N})$. As expected, $\{e_i : i \in \mathbb{N}\}$ is a Schauder basis for $\ell^1(\mathbb{N})$. Let $x = (x_i) \in \ell^1(\mathbb{N})$. Then the sequence terms give the necessary sequence, that is,

$$S_n = \sum_{i=1}^{n} x_i e_i$$

We claim that $S_n \to x$ in $\ell^1(\mathbb{N})$. that is, $\|S_n - x\|_1 \to 0$. Note that

$$\|S_n - x\|_1 = \sum_{i=n+1}^{\infty} |x_i| \to 0 \text{ as } n \to \infty$$

It remains to show uniqueness. Let $x = (x_i)$. Suppose $x = \sum c_i e_i = \sum d_i e_i$. But this is the same as $\sum c_i e_i = 0$. Consider $p_n = \sum_{i=1}^n c_i e_i$ and hence $p_n \to 0$. Hence $\|p_n\| \to 0$. Thus, $\sum_{i=1}^n |c_i| \to 0$ and hence $c_i = 0$ for all $i \in \mathbb{N}$ and this completes the proof of the claim.

Orthonormal basis are always an example of a separable Hilbert space (Verify!)

Proposition 7.2.1. *Let B be a Banach space which admits a Schauder basis $\{u_i : i \in \mathbb{N}\}$. Then B is separable that is, it admits a countable dense set.*

We consider an example first. In the case of $\ell^2(\mathbb{N})$, consider the set

$$\cup_{n=1}^{\infty}\left\{\sum c_i e_i \mid \mathbb{Q}+i\mathbb{Q}\right\}$$

This does the job!

We claim the set

$$\cup_{n=1}^{\infty}\left\{\sum c_i u_i \mid \mathbb{Q}+i\mathbb{Q}\right\}$$

Consider $M_n = \text{span}\{u_1,\ldots,u_n\}$ and D_n be the set of vectors which are span of $u_1,\ldots,u_n$ with rational coefficients. Then closure of D_n is M_n. Then

$$\bigcup_{n\in\mathbb{N}} \overline{D_n} \subset M_n \subset \text{span }\{u_i : i \in \mathbb{N}\}$$

Since $\{u_i\}$ is a Schauder basis, we can write $B = \overline{\text{span }\{u_i : i \in \mathbb{N}\}}$. Observe that $\cup_{n=1}^{\infty}\overline{D_n} \subset \overline{D}$. But then $\overline{D_n} = M_n$. Hence, we have that span $\{u_i : i \in \mathbb{N}\} = \cup_{n=1}^{\infty} M_n \subset \overline{D}$. Taking closure again, we have that complete this!

One can show that $\{e_i : i \in \mathbb{N}\}$ is not a Schauder basis for $\ell^{\infty}(\mathbb{N})$. In fact, it does not admit a Schauder basis at all!

8 Lecture 8 — *Equivalence of norms* — 30th January, 2023

8.1 Equivalence of Norms

Definition 8.1.1. Let V be a vector space. Let $\|\|_1$ and $\|\|_2$ be two norms on V. We say that $\|\|_1$ and $\|\|_2$ are equivalent if there exists positive constants C_1, C_2 such that

$$C_2 \|v\|_1 \leq \|v\|_2 \leq C_1 \|v\|_1$$

for all $v \in V$.

Let $B_i(0,1)$ be the unit ball of i norm. Then by definition, we have that

$$B_1(0,1) \subset B_2(0,c_2)$$

and

$$B_2(0,1) \subset B_1\left(0, \frac{1}{c_2}\right)$$

Example 8.1.2. Consider $\mathbb{R}^2$ with 1-norm and 2-norm. It can be shown that

$$\|x\|_2 \leq \|x\|_1$$

It follows from Cauchy Schwarz that

$$|x_1| + |x_2| \leq \sqrt{2}\sqrt{|x_1|^2 + |x_2|^2}$$

and hence we have that

$$\|x\|_1 \leq \sqrt{2}\,\|x\|_2$$

Hence, the norms are equivalent.

Proposition 8.1.3. *Let V be a vector space. Let $\|\|_1$ and $\|\|_2$ be two equivalent norms on V. A sequence $\{x_n\}$ converges to x in one norm iff it converges in the other.*

Corollary 8.1.4. *Let V be a vector space. Let $\|\|_1$ and $\|\|_2$ be two norms on V. Let A be subset of V. Then closure of A in 1-norm is the same as the closure of A in 2-norm.*

Corollary 8.1.5. *Let V be a vector space. Let $\|\|_1$ and $\|\|_2$ be two norms on V. Let A be subset of V. A is closed (corr. open) in 1-norm iff A is closed (corr. open) in 2-norm.*

Example 8.1.6. It follows by Holder's inequality that in $\mathbb{R}^n$, any p-norm is equivalent to 1-norm.

Proof of Example 8.1.6. Recall Holder's inequality, which states that if $a, b \in \mathbb{C}^n$ and p, q are conjugate exponents then

$$\sum_{i=1}^{n} |a_i b_i| \leq \|a\|_p \|b\|_q$$

Now if $x \in \mathbb{C}^n$ and taking $y = (1, \ldots, 1)$ we have that

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \leq 2^{1/q} \|x\|_p$$

and also note that

$$\|x\|_\infty \leq \|x\|_1$$

Hence it follows that 1-norm and p-norm are equivalent for any $p \geq 1$. $\square$

Example 8.1.7 (Looking for norms which are not equivalent)**.** Consider c_{00} which is the span of e_i. It is evident that

$$\|x\|_\infty \leq \|x\|_1$$

for every $x \in c_{00}$. But consider the sequence $A_n = \left(\underbrace{\frac{1}{n}, \ldots, \frac{1}{n}}_{n \text{ terms}}, \ldots\right)$. Note that this sequence converges to 0 in ∞ norm, however, this does not converge in the 1 norm for the very simple reason that

$$\|A_n\|_1 = 1$$

for every $n \in \mathbb{N}$.

Proposition 8.1.8. *Suppose V be a normed linear space. Let $Y = span \ \{v_{1,\ldots,v_n}\} \subset V$ where v_i is a basis. Then there exists positive constants such that*

$$C \|x\|_2 \leq \left\| \sum_{i=1}^{n} x_i v_i \right\| \leq M \|x\|_2$$

Proof. Observe that

$$\left\| \sum_{i=1}^{n} x_i v_i \right\| \leq \sum_{i=1}^{n} |x_i| \|v_i\| \overset{\text{CS inequality}}{\leq} M \|x\|_2$$

where $M = \left(\left|\|v_1\|^2 + \ldots + \|v_n\|^2\right|\right)^{1/2}$.

Consider the function f on the unit sphere S given by $f(x) = \|\sum_{i=1}^{n} x_i v_i\|$ for every $x \in S$.

Now we claim that f is continuous. Consider

$$|f(x) - f(y)| = \left| \left\| \sum_i x_i v_i \right\| - \left\| \sum_i y_i v_i \right\| \right| \leq M \left\| x - y \right\|_2$$

Since $f(x) > 0$ for every $x \in S$ and since f is continuous, we have hthat there is some $x_0 \in S$ such that $\delta = f(x_0) = \inf_{x \in S} f(x)$.

Thus, we have that $\| \sum_{i=1}^n x_i v_i \| \geq \delta$ for every $x \in S$.

Now, let $x \in V$. Then $\left(\frac{x_1}{\|x\|}, \ldots, \frac{x_n}{\|x\|} \right) \in S$ and the result follows. □

Corollary 8.1.9. *Let $(C^n, \|\|)$ be a normed linear space. Then this normed linear space is equivalent with the usual Euclidean norm on $\mathbb{C}^n$.*

Proof. Take v_i to be standard basis. □

Corollary 8.1.10. *Let Y be a finite dimensional subspsace of a normed linear space V. Then Y is complete (and hence closed).*

Proof. Let Y be a finite dimensional closed subspace. Let $v_1, \ldots, v_k$ be a basis for Y. Let $\{x_n\}$ be a Cauchy sequence in Y. Suppose that for each $n \in \mathbb{N}$, we have that

$$x_n = a_{n1} v_1 + \ldots + a_{nk} v_k \ .$$

Let $\{a_n\}$ be the sequence in $\mathbb{C}^n$ given

$$a_n = (a_{n1}, \ldots, a_{nk})$$

for each $n \in \mathbb{N}$. Then from Proposition 8.1.8, we have for $n, m \in \mathbb{N}$ and $i \in \{1, 2, \ldots, k\}$, we have that

$$C|a_{ni} - a_{mi}| \leq C \left\| a_n - a_m \right\|_2 \leq \left\| x_n - x_m \right\| \leq M \left\| a_n - a_m \right\|_2$$

Since $\{x_n\}$ is Cauchy, we have that each a_{ni} is Cauchy and hence converges at some a_i.

Now, we show that $\{x_n\}$ converges to the vector

$$x := a_1 v_1 + \ldots + a_k v_k$$

This is easy to see by Cauchy Schwarz inequality that

$$\left\| x_n - x \right\| \leq M \left\| a_n - a \right\|_2$$

This completes the proof. □

Exercise 8.1.11. *Consider $C[0, 1]$. Show that that sup norm and any p norm are not equivalent.*

Solution. Does this work? □

Exercise 8.1.12. *c_{00} and the set of all polynomials are not complete.*
Hint: Use BCT!

Proof. A space generated by an infinite but countably many number of linearly independent vectors $v_1, v_2, \ldots$ cannot be complete! Consider the subspaces $M_n = \operatorname{span}[v_1, v_2, \ldots, v_n]$. Each of which are finite dimensional and hence closed and do not have an interior,so, by Baire's Theorem, we have that they cannot be complete! □

9 Lecture 9 — *Baire Category Theorem, Riesz Lemma, Quotient Spaces* — 1st February, 2023

9.1 Consequence of Baire's Theorem

It follows from Baire's theorem that

Proposition 9.1.1. *If X is a Banach space then any Hamel basis of X is uncountable.*

Some interesting other consequences are that: $\mathbb{R}^2$ is not a union of countably many straight lines.

9.2 Riesz says NO to compact unit balls in infinite dimensions

We saw that in a Hilbert Space, the unit ball in compact iff the space is of finite dimension.

Now, consider the unit ball of ℓ^1

$$B_1(0) = \{(x) \mid \|x\| \leq 1\}$$

Note that $\{e_i\}$ is a sequence in the unit ball. Note that for $n, m \in \mathbb{N}$,

$$\|e_n - e_m\|_1 = 2$$

This says that ℓ^1 is not complete.

Now consider ℓ^∞. The same argument works but the distance between any two distinct sequences e_n and e_m is 1.

The same argument works for ℓ^p.

This proves the following:

Proposition 9.2.1. *Unit ball of ℓ^p is not compact.*

Proposition 9.2.2 (Riesz Lemma)**.** *Let X be a NLS, M is a closed subspace of X. Fix $t \in (0,1)$. Then there is $x_0 \in X$ such that $\|x_0\| = 1$ and $d(x_0, M) \geq t$.*

We see an application before the proof: Suppose $\dim X = \infty$. So, let M_1 be the span of some nonzero vector $v \in X$ whose norm is 1. This M_1 is closed. By the lemma, there is v_2 in the unit circle such that $d(v_2, M_1) \geq 1/2$.

Now consider M_2 be the span of v_1 and v_2. Again by the lemma, there is v_3 in the unit circle such that $d(v_3, M_2) \geq 1/2$.

Suppose that we have obtained a sequence $v_1, v_2, \ldots, v_n$ such that M_n is a span of $v_1, \ldots, v_n$ and then repeating the argument, we can obtained v_{n+1} in the unit circle such that $d(v_{n+1}, M_n) \geq 1/2$.

Now, observe that $\|v_j - v_k\| \geq 1/2$. Hence the unit ball cannot be closed.

Proof of Riesz Lemma. Consider $y \notin M$. Then $\delta := d(y, M) > 0$ because M is closed.Consider $\delta/t > \delta$. Since δ is the infimum of distances between y and the points of M. We can find $m_0 \in M$ such that $\|y - m_0\| < \frac{\delta}{t}$.

Take $x_0 = (y - m_0)/\|y - m_0\|$. Note that norm of x_0 is 1.

Then we have that

$$\begin{aligned}
\|x_0 - m\| &= \left\| \frac{y - m_0}{\|y - m_0\|} - m \right\| \\
&= \frac{1}{\|y - m_0\|} \left\| y - \underbrace{m_0 - m \|y - m_0\|}_{\in M} \right\| \\
&\geq \frac{\delta}{\|y - m_0\|} > t
\end{aligned}$$

This shows that $d(x_0, M) \geq t$. $\square$

Theorem 9.2.3. *In an infinite dimensional normed linear space, neither the unit circle nor the unit disc is compact.*

Proof. Let X be an infifnite dimensional normed linear space. Select a vector x_1 in the unit circle, that is, x_1 has unit norm. Now span of x_1 is a finite dimensional subspace of X hence must be closed, in view, of previous lecture. Hence, we can apply Riesz lemma, to obtain a vector x_2 of unit norm such that

$$\|x_2 - \alpha x_1\| > \frac{1}{2} \text{ for every } \alpha \in \mathbb{F}.$$

Now, we can do the same with span of the vector x_2 and x_1 to get a third vector x_3 of unit norm such that

$$\|x_3 - \alpha x_1 - \beta x_2\| > \frac{1}{2} \text{ for every } \alpha, \beta \in \mathbb{F}.$$

Continuing this way, we obtain a sequence of vectors in the unit circle, satisfying, $\|x_n - x_m\| > \frac{1}{2}$. Hence, we have obtained a sequence in the unit circle which has no hope of having a converging subsequence, hence, the unit circle is not compact.

Note that the same argument shows that the closed unit disc is not compact as well. $\square$

Let X, Y be two normed linear space. One can construct another NLS by $X \oplus Y$ in the following way:

$$\|(x_1, x_2)\|_1 = \|x_1\|_X + \|x_2\|_Y$$

and in general in the p norm style.

9.3 Quotient Space

Let X be a normed linear space, M be a closed subspace. Then

$$X/M = \{[x] : x \in X\}$$

One can define $\|[x]\| = \inf \{\|x - m\| : m \in M\}$. One can show that with this norm, X/M becomes a NLS indeed!

But is X/M a Banach space? Yes:

Proposition 9.3.1. *If X is complete then X/M is also complete.*

example: $M = \{(x) : x_1 = 0\}$ of c_{00}.

Lemma 9.3.2. *Let X be a Banach space. If $\{v_j\}$ is absolutely summable, that is, $\sum_{j=1}^{\infty} \|v_j\| < \infty$. Then $S_n = \sum_{j=1}^{n} v_j$ is convergent.*

Proof. For $n \geq m$, consider the following:

$$\begin{aligned} \|S_n - S_m\| &= \left\| \sum_{j=m+1}^{n} v_j \right\| \\ &\leq \sum_{j=m+1}^{n} \|v_j\| \end{aligned}$$

Since $\sum_{j=1}^{\infty} \|v_j\| < \infty$, we have that its sequence of partial sums is Cauchy and the former inequality is also Cauchy. Since X is a Banach space, S_n converges. This completes the proof. □

Theorem 9.3.3. *Let X be a NLS. The following are equivalent:*

1. *X is complete*
2. *Every absolutely summable sequence is summable.*

10 Lecture 10 — *Bounded Linear Maps* — 6th February, 2023

10.1 Continuing from where we left from...

Theorem 10.1.1. *Let X be a NLS. The following are equivalent:*

1. *X is complete.*
2. *Every absolutely summable sequence is summable.*

Proof. ($\Longrightarrow$) This is just Lemma 9.3.2.

($\Longleftarrow$) Let $\{y_n\}$ be Cauchy in Z. It suffices to find a subsequence of $\{y_n\}$ which converges. Since $\{y_n\}$ is Cauchy, for every $k \in \mathbb{N}$, there is some $N_k \in \mathbb{N}$ such that

$$\|y_i - y_j\| < \frac{1}{2^k} \text{ for every } i, j \geq N_k.$$

We may select N_k's in a way that $N_1 \leq N_2 \leq \ldots$. Now defne the sequence

$$t_j = y_{N_j}$$

for every $j \in \mathbb{N}$.

Hence, we have that for every $j \in \mathbb{N}$,

$$\|t_j - t_{j+1}\| \leq \left\|y_{N_j} - y_{N_{j+1}}\right\| \leq \frac{1}{2^j}$$

Now note that the sequence $\{t_j - t_{j+1}\}_j$ is absolutely summable. By hypothesis, we have that $\{t_j - t_{j+1}\}_j$is summable. But that is the same as saying $\sum_j t_j$ is convergent. Hence, we have that a subsequence of $\{y_n\}$, namely, $\{t_j\}_j$ is convergent and hence we are done. $\square$

We proceed to prove Proposition 9.3.1:

Proof. Let $[x_n]$ be an absolutely summable in X/M, that is,

$$\sum_{n=1}^{\infty} \|[x_n]\| < \infty$$

We want to show that

$$\sum_{k=1}^{n} [x_k] \to [x]$$

for some $x \in X$ as $n \to \infty$.

By the definition of the norm in the quotient space, for each $k \in \mathbb{N}$, we can find $m_k \in M$ such that

$$\|x_k + m_k\| < \|[x_k]\| + \frac{1}{2^k}$$

From the above, we conclude that $\sum_{k=1}^{\infty}(x_k + m_k)$ is summable. Hence, $\sum_{k=1}^{\infty}(x_k + m_k)$ converges to some $x \in M$. By the continuity of the projection, we are done (see below). $\square$

Proposition 10.1.2. *Projection map is continuous.*

10.2 Continuous Linear Maps on Normed Linear Spaces

Example 10.2.1. Consider the linear map $L : c_{00} \to \mathbb{R}$ given by

$$(x_j) \mapsto x_1 + 2x_2 + \ldots + nx_n + \ldots$$

This map is not continuous because we can consider the sequence $\{e_j/j\}$. Note that $e_j/j \to 0$ but $T(e_j)/j = 1$ for every $j \in \mathbb{N}$.

Proposition 10.2.2. *Let $T : X \to Y$ be a linear map. The following are equivalent:*

1. *T is continuous map.*
2. *T is continuous at $x = 0$.*
3. *$T\left(\overline{B_X(0,1)}\right)$ is bounded, that is, where $\overline{B_X(0,1)}$ denotes the closed unit ball in X with centre at the origin. Equivalently speaking*

$$\sup\{\|Tx\|_Y : \|x\| \leqslant 1\} < \infty.$$

4. *$\|Tx\| \leq M\|x\|$ for every $x \in X$.*

Proof. It is easy to see that item 1 implies item 2.

We proceed to show that item 2 implies item 3. Suppose tha T is continuous at $x = 0$. We need to show that the set $T\left(\overline{B_X(0,1)}\right)$ is bounded. Since T is continuous at 0, there is some $\delta > 0$ such that

$$\|x\|_X < \delta \rightsquigarrow \|Tx\|_Y < 1$$

Now, let $x \in X$ with $\|x\|_X \leq 1$. Then we have that $\left\|\frac{\delta}{2}x\right\|_X < \delta$ and hence we have that

$$\left\|T\left(\frac{\delta}{2}x\right)\right\|_Y < 1 \rightsquigarrow \|Tx\|_Y < \frac{2}{\delta}$$

This shows that $T\left(\overline{B_X(0,1)}\right)$ is bounded. Now, we proceed to show that item 3 implies item 4. Suppose that $T\left(\overline{B_X(0,1)}\right)$ is bounded. That is, there is some $M > 0$ such that

$$\|x\|_X \leq 1 \rightsquigarrow \|Tx\|_Y \leq M$$

Now, let $y \in X$ be arbitrary. If $y = 0$ then we have that $\|Ty\| = 0 \leq M\|y\|$. Suppose that $y \neq 0$. Then we have that the vector $y/\|y\|$ has norm at most 1 and therefore we have that

$$\|T(y/\|y\|_Y)\| \leq M \rightsquigarrow \|T(y)\| \leq M\|y\|$$

Now, we proceed to prove that item 4 implies item 1. Because item 4 with linearity implies Lipschitz continuity. $\square$

Proposition 10.2.3. *Every linear map on a finite dimensional normed linear space is continuous.*

Example 10.2.4. $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$

$$(x_1, x_2, \ldots) \mapsto (\lambda_1 x_1, \lambda_2 x_2, \ldots)$$

We show that T is continuous iff $\{\lambda_i\}$ is bounded. ($\Rightarrow$) Suppose that T is continuous. By Proposition 10.2.2, we have that there is some $M > 0$ such that

$$\|Tx\| \leq M \|x\|$$

for each $x \in \ell^2(\mathbb{N})$. Therefore for any $i \in \mathbb{N}$, we have that

$$|\lambda_i| = \|Te_1\|_2 \leq M \|e_i\|_2 = M.$$

This shows that $\{\lambda_i\}$ is a bounded sequence. Conversely suppose that $\{\lambda_i\}$ is a bounded sequence. Therefore, there is some $M > 0$ such that $|\lambda_i| \leq M$ for all $i \in \mathbb{N}$. Now for any $x \in \ell^2(\mathbb{N})$, we have that

$$\begin{aligned}
\|T(x_1, x_2, \ldots)\| &= \|(\lambda_1 x_1, \lambda_2 x_2, \ldots)\|_2 \\
&= \left(\sum_{i=1}^{\infty} |\lambda_i x_i|^2\right)^{\frac{1}{2}} \\
&\leq M \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}} \\
&= M \|(x_1, x_2, \ldots)\|_2 .
\end{aligned}$$

Example 10.2.5 (A discontinuous linear functional)**.** Consider the normed linear space $X = (c_{00}, \|\cdot\|_1)$. Consider the linear map $T : X \to \mathbb{C}$ defined by

$$T\Big((x_j)_{j\in\mathbb{N}}\Big) = \sum_{k=1}^{\infty} k x_k, \quad ((x_j)_{j\in\mathbb{N}} \in c_{00}.$$

Note that $\frac{e_n}{n} \to 0$ as $n \to \infty$ in $(c_{00}, \|\cdot\|_1)$ but $T(\frac{e_n}{n}) = 1$ for all $n \in \mathbb{N}$. Hence T is not continuous. (Here e_n denote the coordinate sequence whose n-th term is 1 and all other terms are 0.)

11 Lecture 11 — *More Bounded Linear Maps, I guess!* — 08th February, 2023

11.1 Bounded Linear Maps form a Banach space!

Let X and Y be two normed linear spaces. We define

$$L(X,Y) = \{T : X \to Y \ : \ T \text{ linear and continuous } \}$$

This is a vector space with the following operations:

$$\begin{aligned}(T_1 + T_2)(x) &= T_1(x) + T_2(x)\\ (\alpha T)(x) &= \alpha T(x)\end{aligned}$$

for all $x, y \in X$.

We can also give it a norm by:

$$\|T\| = \sup\left\{\|Tx\|_Y : \|x\|_X \leq 1\right\}$$

Note that this indeed defines a norm because if T is continuous then there is some $M > 0$ such that $\|Tx\| \leq M\|x\|$ for each $x \in V$. So $\|T\| < \infty$. (check the other properties like triangle inequality!)

Observation 11.1.1. *If T is continuous then $\|Tx\| \leq \|T\|\,\|x\|$ for each $x \in V$. This is easy to see if $x = 0$. If $x \neq 0$ then*

$$\begin{aligned}\left\|T\left(\frac{x}{\|x\|}\right)\right\| &\leq \|T\|\\ \rightsquigarrow \|Tx\|_Y &\leq \|T\|\,\|x\|\end{aligned}$$

Lemma 11.1.2. *Let X, Y be two normed linear spaces. If Y is a Banach space then so is $B(X,Y)$ with the operator norm.*

Proof. Let $\{T_n\}$ be a Cauchy sequence in $B(X,Y)$. We claim that for each $x \in X$, we have that $\{T_n x\}$ is Cauchy in Y. This is easy to see by the following:

$$\|T_n x - T_m x\| \leq \|T_n - T_m\|\,\|x\|$$

Thus, $\{T_n x\}$ is Cauchy in Y. Hence, we define the following function $T : X \to Y$ we define

$$\lim_{n\to\infty} T_n x =: T(x)$$

It is easy to see that T is linear. Now, we need to check that T is continuous. Since $\{T_n\}$ is Cauchy, we have that $\{T_n\}$ are bounded. So there is some $M > 0$ such that $\|T_n\| \leq M$. Then we have that

$$\begin{aligned}\|T_n x\| &\leq \|T_n\|\,\|x\| \leq M\|x\| &\\ \rightsquigarrow \|Tx\| &\leq M\|x\| & \text{taking limits}\end{aligned}$$

Thus, T is continuous. Now, we proceed to show that T_n actually converges to the linear operator T.

Let $\varepsilon > 0$ be given. Then there is some $K \in \mathbb{N}$ such that for every $n, m \geq K$, we have that

$$\|T_n - T_m\| < \varepsilon$$

$\square$

Corollary 11.1.3. *Dual space of any normed linear space is complete!*

12 Lecture 12 — Hahn Banach Theorem & Its Consequences — 15th February, 2023

12.1 Class Work

12.1.1 Extensions

$T : X \to Y$ continuous/bounded, Y is Banach X is dense in $\hat{X}$ then there is $\hat{T} : \hat{X} \to Y$ continuous, $\hat{T}$ is unique such that

$$\hat{T}x = Tx \text{ for all } x \in X$$

and $\|T\|_X = \left\|\hat{T}\right\|_{\hat{X}}$

Note that $\|Tx\| \leq \|T\| \|x\| \rightsquigarrow \|Tx - Ty\| \leq \|T\| \|x - y\|$ for every $x, y \in X$

If $\{x_n\}$ is Cauchy as

$$0 \leq \|Tx - Ty\| \leq M \|x - y\|$$

Now, if $x_n \to \pi$ then Tx_n is Cauchy and define

$$\hat{T}(p) = \lim_n Tx_n$$

Let's try to show welldefinedness. Suppose $x_n \to p$ and $y_n \to p$. Then $x_n - y_n \to 0$ then $T(x_n - y_n) \to 0$. This shows uniqueness of limits.

We proceed to show linearity. Let $x_n \to p$ and $y_n \to q$. Then $x_n + y_n \to p + q$ then

$$\begin{aligned}
\hat{T}(p+q) &= \lim T(x_n + z_n) \\
&= \lim (Tx_n + Tz_n) \\
&= \hat{T}(p) + \hat{T}(q)
\end{aligned}$$

Show that $\|T\| = \left\|\hat{T}\right\|$.

12.1.2 Finite Rank Operator

Let H be a Hilbert space. Let $x, y \in H$. Define

$$x \otimes y : H \to H$$

$$f \mapsto \langle f, y \rangle x$$

for every $f \in H$.

$$\begin{aligned}
\|T_{x,y}(f)\| &= \|\langle f, y \rangle x\| \\
&= |\langle f, y \rangle| \|x\| \\
&\leq \|f\| \|y\| \|x\|
\end{aligned}$$

This is a operator of rank 1. The converse is also true, any rank 1 operator looks something like the above.

12.1.3 Completing the converse

Let X and Y be NLS. Fix a vector $y \in Y$ and fix $f \in X^*$. Define $T_{y,f} : X \to Y$ by $T_{y,f}x = f(x)y$. It is easy to see that $T_{y,f}$ is continuous, in fact, it can be shown, by Hahn Banach, that $\|T_{y,f}\| = \|f\|_{X^*} \|y\|$.

Let $\{y_n\}$ be Cauchy. Consider the sequence of operators $T_{y_n,f}$ is again Cauchy, hence, we have that $T_{y_n,f}$ converges to some T. Therefore, $T_{y_n,f}(x) \to Tx$ for some $f(x) \neq 0$. Hence $y_n \to \frac{Tx}{f(x)}$.

12.1.4 Hahn Banach Theorem

Let X be a NLS. A map $p : X \to \mathbb{R}$ is called sublinear if

1. $p(x+y) \leq p(x) + p(y)$
2. $p(\alpha x) = \alpha p(x)$

If besides the above, the condition

$$p(\alpha x) = |\alpha| \, p(x)$$

for all $\alpha \in F$. Then p is called a seminorm.

Theorem 12.1.1 (Hahn Banach Theorem for real vector space)**.** *Suppose X is a real NLS, M be a subspace of X, $p : X \to \mathbb{R}$ is a sublinear map, let $T : M \to \mathbb{R}$ be a linear functionals satisfying*

$$Tx \leq p(x)$$

Then there exists a linear map $\hat{T} : X \to \mathbb{R}$ so that

1. $\hat{T}_M = T$
2. $\hat{T}x \leq p(x)$

13 Lecture 13 — *insert catchy title here...* — 20th February, 2023

13.1 Completing the proof of (Real) Hahn Banach Theorem ...

Theorem 13.1.1 (Hahn Banach theorem)**.** *Let X be a vector space over $\mathbb{R}$, p be a sublinear functional on X, M be a subspace of X and M is proper. Let $T : M \to \mathbb{R}$ be a linear functional such that $Tm \leq p(m)$ then there exists a extension $\hat{T} : X \to \mathbb{R}$ so that $\hat{T}$ is linear, $\hat{T}(m) = T(m)$ for all $m \in M$ and $\hat{T}(x) \leq p(x)$ for all $x \in X$.*

Collect the set $\{(W, T_W)\}$ where $M \subset W$, $T_W \mid_M = T$ and $T_W(x) \leq p(x)$ for all $x \in W$. We give an ordering on the set $(W, T_{W_1}) \leq (W_2, T_{W_2})$ iff $W_1 \subset W_2$ and $T_{W_2} \mid W_1 = T_{W_1}$. It is easy to check that $\leq$ is a partial order on this set.

Let $\mathcal{C} = \{(W_i, T_{W_i})\}_{i \in I}$ is a chain. Let $W = \bigcup_{i \in I} W_i$. So W is a subspace and let T be the union of the maps T_i, $i \in I$. It is also easily seen that $T_W(w) \leq p(w)$ for each $w \in W$. Note that (W, T) is an upperbound for this chain $\mathcal{C}$.

By Zorn's Lemma, let (V, T) be a maximal element. By maximality, we have that $V = X$.

Definition 13.1.2. Let X be a NLS. A seminorm $p : X \to \mathbb{F}$ is map satisfying

1. $p(x) \geq 0$
2. $p(x + y) \leq p(x) + p(y)$
3. $p(tx) = |t| \, p(x)$ for every $t \in F$.

Example 13.1.3 (Seminorm)**.** Let X be a NLS. Consider the map $p(x) = c \|x\|$ for every $x \in X$ where $c > 0$.

Theorem 13.1.4. *Suppose X is a NLS over $\mathbb{R}$ or $\mathbb{C}$. Let $M \subset X$ be a proper subspace, p is a seminorm on X. Let $T : M \to \mathbb{F}$ linear, $|Tx| \leq p(x)$ for every $m \in M$. Then there exists $\hat{T} : X \to \mathbb{F}$ linear, $\left|\hat{T}x\right| \leq p(x)$ for each $x \in X$ and $\hat{T}m = Tm$ for every $m \in M$.*

Corollary 13.1.5. *Every linear functional on a subspace of a NLS can extended so that the norm is preserved.*

Proof. Take $p(x) = \|T\| \, \|x\|$. (fill the gap!) □

Lemma 13.1.6. *Let X be a vector space over $\mathbb{C}$, $f : X \to \mathbb{R}$ be a $\mathbb{R}$-linear map.*

1. *Define $\hat{f} : X \to \mathbb{C}$ defined by $\hat{f}(x) = f(x) - if(ix)$ for every $x \in X$. Then $\hat{f}$ is $\mathbb{C}$-linear and $\Re \hat{f} = f$.* [3]
2. *Let $g : X \to \mathbb{C}$ be $\mathbb{C}$ -linear, then $g(x) = f(x) + iv(x)$. Then f is $\mathbb{R}$-linear and $g = \hat{f}$, that is, $g(x) = f(x) - if(ix)$.*

[3] Let $g : X \to \mathbb{C}$ be a linear functional. Let $g = u + iv$. Then $u = \Re g$ and $\Im g$. Then u and v are $\mathbb{R}$-linear.

3. *Suppose $g : X \to C$ is $\mathbb{C}$-linear, $g(x) = f(x) - if(ix)$ where $f = \Re g$. If p is a seminorm on X, $|f(x)| \leq p(x) \iff |g(x)| \leq p(x)$ for every $x \in X$. This tells us that $\|f\| = \|g\|$.*

Proof. Linearity of $\hat{f}$ is clear. Note that

$$\begin{aligned}\hat{f}(x_1 + x_2) &= f(x_1 + x_2) - if(ix_1 + ix_2) \\ &= f(x_1) f(x_2) - if(ix_1) - if(ix_2)\end{aligned}$$

$$\hat{f}((a + ib)x) =$$

□

13.2 Application of Hahn-Banach Theorem