Solutions to Functional Analysis Assignment 1

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Note

A checkmark ✓ indicates the question has been done.

Contents

1 Question 1 ✓

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space and x, y be two non zero vector in V. Show that $\|x+y\| = \|x\| + \|y\|$ holds if and only if $x = cy$ for some scalar $c > 0$.

Solution. Let x, y be two nonzero vectors in an inner product space V. Consider the following equivalences:

$$\begin{aligned}
\|x+y\| = \|x\| + \|y\| &\iff \|x+y\|^2 = (\|x\| + \|y\|)^2 \\
&\iff \langle x+y, x+y \rangle = \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \\
&\iff \Re \langle x, y \rangle = \|x\|\|y\|
\end{aligned}$$

Suppose that $\|x+y\| = \|x\| + \|y\|$ holds. Then we have from the above equivalence that $\Re \langle x, y \rangle = \|x\|\|y\|$. Since $\|x\|\|y\| \leq \Re \langle x, y \rangle \leq |\langle x, y \rangle| \leq \|x\|\|y\|$, we have that $\langle x, y \rangle = \|x\|\|y\|$. Since the equality in Cauchy Schwarz inequaliy holds iff x and y are linearly dependent, we must have that $x = cy$ for some $c \in \mathbb{C}$. Thus, we must have that

$$\begin{aligned}
\Re \langle cy, y \rangle = \|cy\|\|y\| &\iff \Re c \langle y, y \rangle = |c|\|y\|\|y\| \\
&\iff \langle y, y \rangle \Re c = |c|\|y\|^2 \\
&\iff \Re c = |c| \\
&\iff c > 0
\end{aligned}$$

The argument is reversible and the proof is complete! □

2 Question 2 ✓

Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space. Let $\overline{B(0,1)}$ denotes the closed unit ball in V, that is,

$$\overline{B(0,1)} = \{x \in V : \|x\| \leqslant 1\}.$$

Show that $\overline{B(0,1)}$ is strictly convex, that is, for any two distinct vector $x, y \in H$, if $\|x\| = 1, \|y\| = 1$, then $\|tx + (1-t)y\| < 1$ for each $t \in (0,1)$.

Solution. Let $x, y \in V$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A simple computation shows that:

$$\begin{aligned}
\|tx + (1-t)y\|^2 - 1 &= t^2 \|x\|^2 + 2t(1-t)\Re(\langle x, y\rangle) + (1-t)^2 \|y\|^2 - 1 \\
&= t^2 + 2t\,(t-1)\,\Re\,\langle x, y\rangle + 1 + t^2 - 2t - 1 \\
&= 2t^2 - 2t + 2t\,(t-1)\,\Re\,(\langle x, y\rangle) \\
&= 2t(1-t)\,(\Re(\langle x, y\rangle) - 1)
\end{aligned}$$

Since $t \neq 0$ and $t \neq 1$, we will be done if we show that $\Re(\langle x, y\rangle) < 1$. If it happened that $\Re\,(\langle x, y\rangle) \geq 1$ then we would have

$$1 \leq \Re\,(\langle x, y\rangle) \leq |\langle x, y\rangle| \leq \|x\|\,\|y\| \leq 1$$

This shows that $|\langle x, y\rangle| = \|x\|\,\|y\|$. Equality in CS inequality means that x and y are linearly dependent. Thus, we may assume that $x = \lambda y$ for some $\lambda \in \mathbb{C}$. But then

$$x = \lambda y \Rightarrow \|x\| = |\lambda|\,\|y\| \Rightarrow |\lambda| = 1.$$

Hence, we may write $x = e^{i\theta} y$ for some $\theta \in [0, 2\pi)$. Thus, we have

$$\begin{aligned}
\Re\langle x, y\rangle &= \Re\left\langle x, e^{i\theta} x\right\rangle \\
&= \Re(e^{-i\theta}) \\
&= \cos\theta
\end{aligned}$$

By assumption that $\Re\,(\langle x, y\rangle) \geq 1$, we have that $\cos\theta \geq 1$. This implies $\cos\theta = 1$. Thus, $\theta = 0$ by our assumption that $\theta \in [0, 2\pi)$. This implies $x = y$ which contradicts our assumption $x \neq y$. $\square$

3 Question 3 ✓

Fix a $n \times n$ strictly positive definite matrix $A = \big(a_{i,j}\big)$. Consider $(\mathbb{C}^n, \langle \cdot, \cdot \rangle)$ where

$$\langle x, y \rangle = \sum_{i,j=1}^{n} a_{i,j} x_j \bar{y}_i = \langle Ax, y \rangle_2, \ \ x, y \in \mathbb{C}^n.$$

Prove that $(\mathbb{C}^n, \langle \cdot, \cdot \rangle)$ is indeed an inner product space. Conversely show that if $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{C}^n$, then there exist a $n \times n$ strictly positive definite matrix $A = \big(a_{i,j}\big)$ such that

$$\langle x, y \rangle = \sum_{i,j=1}^{n} a_{i,j} x_j \bar{y}_i = \langle Ax, y \rangle_2, \ \ x, y \in \mathbb{C}^n.$$

Solution. ($\Longrightarrow$) This direction is easy to check and follows immediately by the fact that A is *strictly* positive definite matrix.

($\Longleftarrow$) Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathbb{C}^n$. To complete the proof of this direction, we consider the Gramian matrix of the standard orthonormal basis of $\{e_1, e_2, \ldots, e_n\}$ with standard inner product. Let us define the $n \times n$ matrix A whose entries are given by

$$A_{ij} = \langle e_j, e_i \rangle \ \text{ for each } 1 \le i, j \le n.$$

It suffices to show that holds for the standard orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ for linearity of the inner product does the rest of the job. Observe that

$$\langle e_j, e_i \rangle = \langle Ae_j, e_i \rangle_2$$

for all $i, j \in \{1, \ldots, n\}$.

This completes the proof. □

4 Question 4 ✓

Let M be a subspace of an inner product space $(V, \langle \cdot, \cdot \rangle)$. Show that $\overline{M}$, the closure of M, in V is also a subspace. Moreover show that $M^{\perp} = \overline{M}^{\perp}$.

Solution. We make the following claims:

Claim 4.0.1 (orthogonal complement of a set and the orthogonal completement of its closure are same!)**.** *Let M be a subset of a inner product space H. Then $M^{\perp} = \left(\overline{M}\right)^{\perp}$*

Proof. It follows by definition that $M \subset \overline{M}$ and hence $\left(\overline{M}\right)^{\perp} \subset M^{\perp}$. Now for reverse the inclusion, let $v \in M^{\perp}$ and let $y \in \overline{M}$. We need to show that $\langle v, y \rangle = 0$. Since $y \in \overline{M}$ there is a sequence (y_n) in M such that $y_n \to y$. Since $v \in M^{\perp}$, we have that $\langle v, y_n \rangle = 0$ for all $n \in \mathbb{N}$. Since $\langle v, y_n \rangle \to \langle v, y \rangle$, we have by uniqueness of limits that $\langle v, y \rangle = 0$. This completes the proof. ☐

Claim 4.0.2 (orthogonal complement of orthogonal complement)**.** *Let M be a closed subspace of the Hilbert space H. Then*

$$M = \left(M^{\perp}\right)^{\perp}$$

Proof of Claim. Let us first show that $M \subset \left(M^{\perp}\right)^{\perp}$ (which in fact holds for any set M). Let $v \in M$ and $w \in M^{\perp}$. It is clear by definition of $M^{\perp}$ that $\langle v, w \rangle = 0$. Hence, $v \in \left(M^{\perp}\right)^{\perp}$.

Let us proceed to show the inclusion in the other direction. Let $v \in \left(M^{\perp}\right)^{\perp}$. Since M is closed, by Projection Theorem, we have that $v = Pv + Qv$ where $Pv \in M$ and $Qv \in M^{\perp}$. By the previous paragraph, we have that $M \subset \left(M^{\perp}\right)^{\perp}$ and hence $Pv \in \left(M^{\perp}\right)^{\perp}$. Hence, we have that $Qv \in \left(M^{\perp}\right)^{\perp}$. Now, $Qv \in M^{\perp} \cap \left(M^{\perp}\right)^{\perp}$. Hence, $Qx = 0$ and thus, $v = Pv \in M$. ☐

Now, we start the proof. Let M be subspace of V. Consider the following:

$$\begin{aligned}
\left(M^{\perp}\right)^{\perp} &= \left(\left(\overline{M}\right)^{\perp}\right)^{\perp} && \text{by Claim 1} \\
&= \overline{\overline{M}} && \text{by Claim 2} \\
&= \overline{M}
\end{aligned}$$

☐

5 Question 5 ✓

Let M be a subspace of a Hilbert space H. Show that $(M^\perp)^\perp = \overline{M}$.

Proof. See Claim 4.0.2 of Question 4. $\square$

6 Question 6 ✓

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Let $\overline{B(0,1)}$ denotes the closed unit ball in V, that is,

$$\overline{B(0,1)} = \{x \in V : \|x\| \leqslant 1\}.$$

Show that $\overline{B(0,1)}$ is compact if and only if dimension of V is finite.
(Hint : if $\mathcal{B} = \{u_\alpha : \alpha \in I\}$ is a collection of orthonormal vectors in V, then $\|u_\alpha - u_\beta\| = \sqrt{2}$ for every $\alpha, \beta \in I$ and $\alpha \neq \beta$.)

Solution. ($\Longleftarrow$) Suppose that $\dim V$ is finite. In Lecture 5, we showed that V is isometrically isomorphic to $\mathbb{C}^n$ with Euclidean norm, that is, there exists a linear map $T : V \to \mathbb{C}^n$ which is an isometry. Since every isometry is an homeomorphism[1], we have that $T^{-1}\left(\overline{B_{\mathbb{C}^n}(0,1)}\right) = \overline{B_V(0,1)}$. Since a continuous image of a compact set is compact, we have that $\overline{B_V(0,1)}$ is compact!

($\Longrightarrow$) (Contrapositive proof) Suppose that $\dim V$ is infinite. Then by Zorn's Lemma, it has a maximal orthonormal set $\{u_\alpha : \alpha \in I\}$. Since V is an inner product space, we have that $\|u_\alpha - u_\beta\| = \sqrt{2}$ for every $\alpha, \beta \in I$ with $\alpha \neq \beta$. Let $\{e_i : i \in \mathbb{N}\}$ be any countable subset of $\{u_\alpha : \alpha \in I\}$. Now, $\{e_i\}_{i \in \mathbb{N}}$ is a sequence in $\overline{B(0,1)}$ but it cannot possibly have a convergent subsequence, hence, $\overline{B(0,1)}$ is not compact. □

[1] proof here!

7 Question 7 ✓

(Direct sum of two Hilbert spaces) : Let H_1 and H_2 be two Hilbert spaces. Now consider the vector space $H_1 \times H_2$. For two vector $h = (h_1, h_2)$ and $g = (g_1, g_2)$ in $H_1 \times H_2$, define

$$\langle h, g \rangle = \langle h_1, g_1 \rangle_{H_1} + \langle h_2, g_2 \rangle_{H_2}.$$

Show that $(H_1 \times H_2, \langle \cdot, \cdot \rangle)$ is a Hilbert space. This Hilbert space is called as direct sum of H_1 and H_2 and denoted as $H_1 \oplus H_2$.

Solution. I pass the burden to Question 8. □

8 Question 8 ✓

(Direct sum of family of Hilbert spaces) : Let $\{H_k\}_{k\in\mathbb{N}}$ be a sequence of Hilbert space. Consider the vector space H defined by

$$H = \left\{(h_k)_{k\in\mathbb{N}} : h_k \in H_k \,\text{for all}\, n \in \mathbb{N}, \text{ and } \sum_{k=1}^{\infty} \|h_k\|^2 < \infty\right\}.$$

For $h = (h_k)_{k\in\mathbb{N}} \in H$ and $g = (g_k)_{k\in\mathbb{N}} \in H$, define

$$\langle h, g\rangle = \sum_{k=1}^{\infty} \langle h_k, g_k\rangle_{H_k}.$$

Show that $H, \langle\cdot,\cdot\rangle$ is a Hilbert space. This H is called the direct sum of the family of Hilbert spaces $\{H_k\}_{k\in\mathbb{N}}$ and denoted as $\bigoplus\limits_{k\in\mathbb{N}} H_k$.

Solution. First, we need to show that H is a vector space. To do so, it suffices to show that it is closed under sum and scalar multiplication (as this is a subspace of the vector space of functions from $\mathbb{N}$ to $\cup_{i=1}^{n} H_i$).

Let $(h_k)_{k\in\mathbb{N}}$, $(g_k)_{k\in\mathbb{N}}$ be two elements of H. Then for any $k \in \mathbb{N}$, we have that

$$\begin{aligned}
\|h_k + g_k\| &\leq \|h_k\| + \|g_k\| && \text{triangle inequality for } H_k \\
\rightsquigarrow (\|h_k + g_k\|)^2 &\leq (\|h_k\| + \|g_k\|)^2 && \text{squaring both sides} \\
\rightsquigarrow \sum_{k=1}^{\infty} (\|h_k + g_k\|)^2 &\leq \sum_{k=1}^{\infty} (\|h_k\| + \|g_k\|)^2 && \text{taking sum both sides} \\
&\leq 2\sum_{k=1}^{\infty} \left(\|h_k\|^2 + \|g_k\|^2\right) && \text{CS for } \mathbb{R}^2 \\
&\leq 2\sum_{k=1}^{\infty} \|h_k\|^2 + 2\sum_{k=1}^{\infty} \|g_k\|^2 \\
&< \infty
\end{aligned}$$

If $\alpha \in \mathbb{C}$ then we have that

$$\sum_{k=1}^{\infty} \|\alpha h_k\|^2 \leq \sum_{k=1}^{\infty} \alpha \|h_k\|^2 < \infty$$

This shows that H is a vector space. Now, we proceed to show that the prescribed inner product is indeed an inner product.

First, we show that if $h, g \in H$ then $\langle h, k\rangle < \infty$. To do so, let $h, k \in H$ then

$$\begin{aligned}
\left|\sum_{k=1}^{\infty} \langle h_k, g_k\rangle\right| &\leq \sum_{k=1}^{\infty} |\langle h_k, g_k\rangle| && \text{continuity of absolute modulus} \\
&\leq \sum_{k=1}^{\infty} \|h_k\| \, \|g_k\| && \text{CS for each of IPS} \\
&\leq \left(\sum_{k=1}^{\infty} \|h_k\|^2\right) \left(\sum_{k=1}^{\infty} \|g_k\|^2\right) && \text{Holder's inequality} \\
&< \infty
\end{aligned}$$

I'm feeling a little lazy at the moment but trust me, the rest are easy to check. ✓

□

9 Question 9 ✓

Consider the normed linear space $(\mathbb{R}^2, \|\cdot\|_1)$, where the the distance d is given by

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|,\ x = (x_1, x_2) \in \mathbb{R}^2,\ y = (y_1, y_2) \in \mathbb{R}^2.$$

Now consider the set $S = \{(x_1, x_2) : x_1 + x_2 = 1\}$. Show that the distance of the zero vector from S, that is, $d(0, S)$ is achieved at infinitely many points in S.

Solution. Observe that

$$\begin{aligned} d\left((0,0), S\right) &= \inf\left\{d\left((0,0), (x,y) : (x,y) \in S\right)\right\} \\ &= \inf\left\{|x| + |1-x| : x + y = 1\right\} \\ &= \inf\left\{|x| + |1-x| : x \in \mathbb{R}\right\} \\ &= \inf[1, \infty) = 1 \end{aligned}$$

Also, note that $d\left((0,0), S\right) = 1 = d\left((0,0), (x, 1-x)\right)$ for every $x \in [0,1]$. This completes the solution. □

10 Question 10

Consider $C[0,1]$, the space of all complex valued continuous function on the interval $[0,1]$, equipped with the supremum norm, $\|\cdot\|_\infty$, that is, $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$. Let S be the subset

$$S = \left\{ f \in C[0,1] \,:\, \int_0^{1/2} f(x)\, dx - \int_{1/2}^1 f(x)\, dx = 1 \right\}$$

Show that the set S is closed and convex but the distance is never achieved. That is, there is no $f \in S$ such that $\|f\|_\infty = d(0,S)$.

Solution. We begin by showing that S is convex. Let $f, g \in S$ and $t \in [0,1]$. Then we have that

$$\begin{aligned} \int_0^{1/2} (tf(x) + (1-t)\, g(x))\, dx - \int_{1/2}^1 (tf(x) + (1-t)\, g(x)\, dx) &= t + (1-t) \\ &= 1 \end{aligned}$$

Note that the second equality follows by the virtue of $f, g \in S$.

Now, we proceed to show that the S is closed. Let (f_n) be a sequence of functions in S converging to $f \in C[0,1]$. We need to prove that $f \in S$. Now convergence in supremum norm is the same as the uniform convergence, so, we have that following:

$$\lim_{n\to\infty} \left(\int_0^{1/2} f_n(x)\, dx - \int_{1/2}^1 f_n(x)\, dx \right) = 1$$

implies

$$\int_0^{1/2} f(x)\, dx - \int_{1/2}^1 f(x)\, dx = 1$$

and thus $f \in S$. Consider the zero function and the set S, we show that that there is no $f \in S$ such that $d(0,S) = d(f,0) = \|f\|_\infty$. We show this in gentle steps as it follows.

Now, we proceed to show that $d(0,S) = 1$. To do so, observe that we need to show that $\inf\{\|f\|_\infty : f \in S\} = 1$. First of all, if $f \in S$ then we have that

$$\begin{aligned} \int_0^{1/2} f - \int_{1/2}^1 f = 1 &\rightsquigarrow \left| \int_0^{1/2} f - \int_{1/2}^1 f \right| = 1 \\ &\rightsquigarrow \left| \int_0^{1/2} f \right| + \left| \int_{1/2}^1 f \right| \geq 1 \\ &\rightsquigarrow \|f\|_\infty \geq 1 \end{aligned}$$

Hence, we have that 1 is a lowerbound for the set S. Now, let $\varepsilon > 0$ be given. We show that there is some function $f \in S$ such that $1+\varepsilon > \|f\|_\infty$. This will establish that $d(0,S) = 1$. Select a $1/n < \varepsilon$. Consider the function

$$f(x) = \begin{cases} \frac{1+\varepsilon}{2/(n+1)} & 0 \leq x \leq 2/(n+1) \\ 1+\varepsilon & 2/(n+1) \leq 0 \leq 1 \end{cases}$$

It can be shown that this function f has sup norm equals $1+\varepsilon$ and is a member of S. Now, we proceed to show that there is no function $f \in S$ such that $\|f\|_\infty = 1$.

□

11 Question 11 ✓

Let H be a Hilbert space. If $x_0 \in H$ and M is a closed subspace of H, prove that

$$\min\{\|x - x_0\| : x \in M\} = \max\{|\langle x_0, y\rangle| : y \in M^\perp, \|y\| = 1\}.$$

Proof. Let $\alpha = \sup\left\{|\langle x_0, y\rangle| : y \in M^\perp, \|y\| = 1\right\}$ and $\beta = \inf\{\|x - x_0\| : x \in M\}$. Let $\varepsilon > 0$ be arbitrary. Then there exists $y_0 \in M^\perp$ with $\|y_0\| = 1$ such that $\alpha - \varepsilon < |\langle x_0, y_0\rangle|$. Then we have that

$$\begin{aligned}
\alpha - \varepsilon &< |\langle x_0, y_0\rangle| \\
&= |\langle P(x_0) + Q(x_0), y_0\rangle| && \text{projection decomposition for Hilbert spaces} \\
&= |\langle Q(x_0), y_0\rangle| && Px_0 \perp y_0 \\
&\leq \|Q(x_0)\| \, \|y_0\| && \text{CS inequality} \\
&= \|Q(x_0)\| = \beta && \text{again by projection decomposition thm for Hilbert spaces}
\end{aligned}$$

Since ε is arbitrary, we have that $\alpha \leq \beta$. Now we proceed to show the reverse inequality.

Now let $y = \frac{Qx_0}{\|Qx_0\|}$. Note that $y \in M^\perp$ and $\|y\| = 1$. We have that

$$\begin{aligned}
\alpha \geq \left|\left\langle x_0, \frac{Qx_0}{\|Qx_0\|}\right\rangle\right| &= \frac{1}{\|Qx_0\|}\langle x_0, Qx_0\rangle \\
&= \frac{1}{\|Qx_0\|}\langle Px_0 + Qx_0, Qx_0\rangle \\
&= \|Qx_0\| = \beta.
\end{aligned}$$

This shows that $\alpha = \beta$ and completes the proof. □

12 Question 12

Compute

$$\min_{a,b,c\in\mathbb{R}} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2 dx$$

and find

$$\max_{g\in S} \int_{-1}^{1} x^3 g(x) dx,$$

where $S = \left\{ g \in L^2[-1,1] : \int_{-1}^{1} g(x)dx = \int_{-1}^{1} xg(x)dx = \int_{-1}^{1} x^2 g(x)dx = 0, \int_{-1}^{1} |g(x)|^2 dx = 1 \right\}$.

13 Question 13

Compute

$$\min_{a,b,c\in\mathbb{R}} \int_0^\infty |x^3 - a - bx - cx^2|^2 e^{-x} dx$$

14 Question 14 ✓

Fix a positive integer N, put $\omega = e^{2\pi i/N}$. prove the following orthogonality relations

$$\frac{1}{N}\sum_{n=1}^{N}\omega^{nk} = \begin{cases} 1, & k = 0, \\ 0, & 1 \leqslant k \leqslant N-1. \end{cases} \tag{14.0.1}$$

Using this identity show that

$$\langle x, y\rangle = \frac{1}{N}\sum_{n=1}^{N}\|x + \omega^n y\|^2\omega^n \tag{14.0.2}$$

holds true in every inner product space provided $N \geqslant 3$.

Solution. We first prove 14.0.1. Observe that if $k = 0$ then $\omega^{n0} = 1$ for every $n \in \{1, 2, \ldots, N\}$ and thus we have that

$$\frac{1}{N}\sum_{n=1}^{N}\omega^{n0} = 1$$

Now suppose that $1 \leq k \leq N-1$. Then we have that

$$\omega^k = e^{i2\pi k/N} \neq 1$$

and thus we can apply the geometric sum formula. Observe that we have

$$\begin{aligned}
\omega^k + \omega^{2k} + \ldots + \omega^{Nk} &= \omega^k\sum_{n=0}^{N-1}\omega^{nk} \\
&= \omega^k\frac{1-\omega^{Nk}}{1-\omega^k} \\
&= \omega^k\frac{1-\left(e^{2\pi i/N}\right)^{Nk}}{1-\omega^k} \\
&= \omega^k\frac{1-e^{2\pi ik}}{1-\omega^k} \\
&= 0.
\end{aligned}$$

Now, we proceed to prove 14.0.2. Let $N \geq 3$. Then we have that

$$\begin{aligned}
\frac{1}{N}\sum_{n=1}^{N}\|x+\omega^n y\|^2\omega^n &= \frac{1}{N}\sum_{n=1}^{N}\langle x+\omega^n y, x+\omega^n y\rangle\,\omega^n \\
&= \cancelto{0}{\frac{\|x\|^2}{N}\sum_{i=1}^{N}\omega^n} + \frac{\langle x, y\rangle}{N}\sum_{i=1}^{N}1 + \cancelto{0}{\frac{\langle y, x\rangle}{N}\sum_{i=1}^{N}\omega^{2n}} + \cancelto{0}{\frac{\|y\|^2}{N}\sum_{i=1}^{N}\omega^n} \quad \text{using 14.0.1} \\
&= \langle x, y\rangle
\end{aligned}$$

□

15 Question 15 ✓

Let V be a finite dimensional inner product space with inner product $\langle \cdot, \cdot \rangle$. Suppose $\{b_1, b_2, \ldots, b_n\}$ is a basis for V and $c_1, c_2, \ldots, c_n$ are n scalar numbers. Show that there exist a unique vector x in V such that $\langle x, b_j \rangle = c_j$ for each $j = 1, 2, \ldots, n$.

Solution. Consider the map $\varphi : V \to \mathbb{C}^n$ given by

$$\varphi(v) = \begin{bmatrix} \langle v, b_1 \rangle \\ \vdots \\ \langle v, b_n \rangle \end{bmatrix}$$

for all $v \in V$. We show that this map φ is injective. Our proof will be then complete by the rank nullity theorem.

So, let $v \in V$ and suppose that $\varphi(v) = 0$. Then $\langle v, b_i \rangle = 0$ for all $i = 1, 2, \ldots, n$. Since $b_1, \ldots, b_n$ is a basis for V, there exists $\alpha_1, \ldots, \alpha_n$ such that

$$v = \alpha_1 b_1 + \ldots + \alpha_n b_n$$

Hence, we have that

$$\begin{aligned} \langle v, v \rangle &= \langle v, \alpha_1 b_1 + \ldots + \alpha_n b_n \rangle \\ &= \sum_{i=1}^{n} \overline{\alpha_i} \langle v, b_i \rangle \\ &= 0 \end{aligned}$$

Hence $v = 0$. This completes the proof! □

16 Question 16 ✓

Let V be a finite dimensional inner product space with inner product $\langle\cdot,\cdot\rangle$. Suppose $\beta_b = (b_1, b_2, \ldots, b_n)$ and $\beta_e = (e_1, e_2, \ldots, e_n)$ are two ordered basis for V which are related by the following relation: $e_j = \sum_{k=1}^{n} P_{k,j} b_k$, for $j = 1, 2, \ldots, n$. In short (in matrix multiplication notation) they are related by the following :

$$(e_1, e_2, \ldots, e_n) = (b_1, b_2, \ldots, b_n) \begin{pmatrix} P_{1,1} & P_{1,2} & \ldots & P_{1,n} \\ P_{2,1} & P_{2,2} & \ldots & P_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n,1} & P_{n,2} & \ldots & P_{n,n} \end{pmatrix}, \text{ that is, } \beta_e = \beta_b P.$$

Let G_e and G_b be the Grammian matrix given by

$$G_e = \begin{pmatrix} \langle e_1, e_1\rangle & \langle e_2, e_1\rangle & \ldots & \langle e_n, e_1\rangle \\ \langle e_1, e_2\rangle & \langle e_2, e_2\rangle & \ldots & \langle e_n, e_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle e_1, e_n\rangle & \langle e_2, e_n\rangle & \ldots & \langle e_n, e_n\rangle \end{pmatrix}, \ G_b = \begin{pmatrix} \langle b_1, b_1\rangle & \langle b_2, b_1\rangle & \ldots & \langle b_n, b_1\rangle \\ \langle b_1, b_2\rangle & \langle b_2, b_2\rangle & \ldots & \langle b_n, b_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle b_1, b_n\rangle & \langle b_2, b_n\rangle & \ldots & \langle b_n, b_n\rangle \end{pmatrix}$$

(a) Show that $G_e = \bar{P}^t G_b P$, where P is the matrix $(\!(P_{i,j})\!)$.
(b) Show that the matrix G_b is positive definite, that is,

(i) $\bar{G_b}^t = G_b$, that is, G_b is self adjoint,

(ii) $\langle G_b x, x\rangle_2 > 0$ for every non zero $x \in \mathbb{C}^n$. Here $\langle\cdot,\cdot\rangle_2$ denotes the standard Eucledian inner product on $\mathbb{C}^n$.

(c) Show that $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of V if and only if $P\bar{P}^t = {G_b}^{-1}$.
(d) Let T be a linear map from V into itself. Suppose the matrix representation of the linear map T w.r.t the basis β_b and β_e is given by $[T]_{\beta_b}$ and $[T]_{\beta_e}$ respectively. Show that

$$[T]_{\beta_e} = [T]_{\beta_b P} = P^{-1}[T]_{\beta_b} P.$$

Solution. (a) Let G_e and G_b be the matrix whose entries are given by

$$(G_e)_{ij} = \langle e_j, e_i\rangle$$

and

$$(G_b)_{ij} = \langle b_j, b_i\rangle$$

for each $i, j \in \{1, 2, \ldots, n\}$. Consider the following:

$$\begin{aligned}
(G_e)_{ij} &= \langle e_j, e_i \rangle \\
&= \left\langle \sum_{k=1}^{n} P_{kj} b_k, \sum_{l=1}^{n} P_{li} b_l \right\rangle \\
&= \sum_{k=1}^{n} \sum_{l=1}^{n} P_{kj} \langle b_k, b_l \rangle \overline{P_{li}} \\
&= \sum_{k=1}^{n} \sum_{l=1}^{n} (P^*)_{il} (G_b)_{lk} P_{kj} \\
&= (P^* G_b P)_{ij}
\end{aligned}$$

(b) Clearly, we have that

$$\begin{aligned}
(G_e^*)_{ij} &= \left(\overline{G_e}\right)_{ji} \\
&= \overline{\langle e_i, e_j \rangle} \\
&= \langle e_j, e_i \rangle \\
&= (G_e)_{ij}
\end{aligned}$$

for all i, j. Hence G_e is self-adjoint. Observe that

$$\langle G_b e_j, e_i \rangle_2 = \langle b_j, b_i \rangle$$

for all i, j. Now let $x \in V$. Then there exists unique $x_1, x_2, \ldots, x_n \in \mathbb{C}$ such that

$$x = \sum_{i=1}^{n} x_i e_i$$

Then we have that

$$\begin{aligned}
\langle G_b x, x \rangle_2 &= \left\langle G_b \left(\sum_{j=1}^{n} x_j e_j \right), \sum_{i=1}^{n} x_i e_i \right\rangle \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} x_j \overline{x_i} \langle G_b e_j, e_i \rangle \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} x_j \overline{x_i} \langle b_j, b_i \rangle \\
&= \left\langle \sum_{j=1}^{n} x_j b_j, \sum_{i=1}^{n} x_i b_i \right\rangle \\
&\geq 0
\end{aligned}$$

(c) By definition we have that $G_e = I$ iff $\{e_1, \ldots, e_n\}$ is orthonormal. ALso from item (a) we have that

$$\begin{aligned}
\{e_1, e_2, \ldots, e_n\} \text{ is orthonormal } &\Leftrightarrow I = P^* G_b P \\
&\Leftrightarrow \left(P^*\right)^{-1} P^{-1} = G_b \\
&\Leftrightarrow \left(P P^*\right)^{-1} G_b \\
&\Leftrightarrow P P^* = G_b^{-1}
\end{aligned}$$

(d) Clearly, $P = (I)_{\beta_e}^{\beta^b}$. Then

$$\begin{aligned}
P\,(T)_{\beta_e} &= (I)_{\beta^e}^{\beta_e} \\
&= (I \circ T)_{\beta_e}^{\beta^b} \\
&= (T \circ I)_{\beta_e}^{\beta^b} \\
&= (T)_{\beta_b}^{\beta_b} (I)_{\beta_e}^{\beta_b} \\
&= (T)_{\beta_b} P
\end{aligned}$$

Hence, we have that

$$(T)_{\beta_e} = P^{-1} (T)_{\beta_b} P$$

This completes the proof. $\square$

17 Question 17 ✓

Let $(V, \|\cdot\|)$ be a normed linear space where the norm $\|\cdot\|$ on V satisfies the parallelogram law, that is,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad x, y \in V.$$

Show that the norm $\|\cdot\|$ is induced by an inner product on V, that is, $\|x\|^2 = \langle x, x\rangle$ for some inner product $\langle\cdot,\cdot\rangle$ on V.

Solution (as it was done in class). Assume $(V, \|\|)$ is a real normed linear space which satisfies the parallelogram identity, that is, for all $a, b \in V$,

$$\|a+b\|^2 + \|a-b\|^2 = 2\|a\|^2 + 2\|b\|^2$$

We intend to define the inner product on V by

$$\langle v, w\rangle = \frac{\|x+y\|^2 - \|x-y\|^2}{4}$$

We show that $\langle\cdot,\cdot\rangle$ is an inner product.
The symmetric property is evident.
We proceed to show linearity in the first variable, that is, we need to show that

$$\|x_1+x_2+y\|^2 - \|x_1+x_2-y\|^2 = \|x_1+y\|^2 - \|x_1-y\|^2 + \|x_2+y\|^2 - \|x_2-y\|^2$$

Setting $a = x_1$ and $b = x_2 + y$ in the parallelogram identity, we get

$$\|x_1+y+x_2\|^2 + \|x_1-y-x_2\|^2 = 2\|x_1\|^2 + 2\|x_2+y\|^2$$

Doing the same for $a = x_2 - y$ and $b = x_2$, we have

$$\|x_1-y+x_2\|^2 + \|x_1-y-x_2\|^2 = 2\|x_1-y\|^2 + 2\|x_2\|^2$$

Subtracting the above two equations, we get

$$\|x_1+x_2+y\|^2 - \|x_1-y+x_2\|^2 = 2\|x_1\|^2 + 2\|x_2+y\|^2 - 2\|x_1-y\|^2 - 2\|x_2\|^2$$

Switching the roles of x_2 and x_1, we get

$$\|x_2+x_1+y\|^2 - \|x_2-y+x_1\|^2 = 2\|x_2\|^2 + 2\|x_1+y\|^2 - 2\|x_2-y\|^2 - 2\|x_1\|^2$$

Adding the above two equations, we get

$$2\|x_1+x_2+y\|^2 - 2\|x_1+x_2-y\|^2 = 2\|x_2+y\|^2 - 2\|x_1-y\|^2 + 2\|x_1+y\|^2 - 2\|x_2-y\|^2$$

Rearranging the above equation, we observe that we have established what we wanted to prove!

Linearity is the other variable follows by symmetry and the linearity in the first variable!

Now, finally we proceed to show that for any $\lambda \in \mathbb{R}$, we have that

$$\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$$

Note that by linearity in the first variable, we have that for $n \in \mathbb{Z}$,

$$\langle nx, y \rangle = n \langle x, y \rangle$$

In a similar fashion, it can be shown that for $r \in \mathbb{Q}$,

$$\langle rx, y \rangle = r \langle x, y \rangle$$

Let us assume Cauchy-Schwarz! at the moment. Let $r \in \mathbb{R}$. Let r_n be a sequence of rationals converging to $r \in \mathbb{R}$.

Observe that fixing $y \in V$, it is easily seen that

$$x \mapsto \frac{\|x+y\|^2 - \|x-y\|^2}{4}$$

is continuous by virtue of translation, norm and square of a function being continuous! Then the result follows!

Irregardless, we prove Cauchy Schwarz! It can be seen by minimizing r is the function $r \mapsto \|rx+y\|^2$. One needs to see that for $r \in \mathbb{Q}$

$$\|rx+y\|^2 = \langle rx+y, rx+y \rangle = r^2 \|x\|^2 + 2r \langle x, y \rangle + \|y\|^2 \geq 0$$

Hence the above holds for any $r \in \mathbb{R}$ by taking limits. Minimizing the function, we get the Cauchy Schwarz inequality.

We now proceed to the complex case!

Let $V, \|\|$ be a complex normed linear space. By the polarization identity, we have that for $x, y \in V$,

$$\begin{aligned}
\langle x, y \rangle &= \frac{1}{4} \sum_{k=1}^{4} \left\|x + i^k y\right\|^2 i^k \\
&= \frac{\|x+y\|^2 - \|x-y\|^2}{4} - \frac{\|x+iy\|^2 - \|x-iy\|^2}{4} \\
&= q(x, y) + iq(x, iy)
\end{aligned}$$

where $q(x, y) = \frac{\|x+y\|^2 - \|x-y\|^2}{4}$. We have already shown that q is an inner product over $\mathbb{R}$.

Now one can use the properties of inner product for q to show that $\langle \cdot, \cdot \rangle$ is an inner product over $\mathbb{C}$.

$\square$

18 Question 18

Let V be a finite dimensional inner product space with inner product $\langle \cdot, \cdot \rangle$. Suppose W is a subspace of V and $\beta_k = (v_1, v_2, \ldots, v_k)$ is a ordered basis for W. Let $y = v_{k+1}$ is a vector outside W.

A distance formula to keep in mind (Proof not needed)

Then the distance of v_{k+1} from W is given by $d(v_{k+1}, W) = \frac{\sqrt{\det G_{\beta_{k+1}}}}{\sqrt{\det G_{\beta_k}}}$, where G_{β_k} and $G_{\beta_{k+1}}$ are the Grammian matrix associated to the vectors $\beta_k = (v_1, v_2, \ldots, v_k)$ and $\beta_{k+1} = (v_1, v_2, \ldots, v_k, v_{k+1})$.

Sketch of the proof : Volume of the k dimensional parallelepiped formed by the vectors in $\beta_k \times d(v_{k+1}, W) =$ Volume of the (k+1) dimensional parallelepiped formed by the vectors in β_{k+1}.

Problem Let $\mathcal{P}_3$ be the vector space of all polynomials over $\mathbb{R}$ of degree less than or equal to 3, with the inner product

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt, \;\; f, g \in \mathcal{P}_3.$$

Let $\mathcal{P}_2$ be the subspace of $\mathcal{P}_3$ given by the set of all polynomials over $\mathbb{R}$ of degree less than or equal to 2. Find the distance of x^3 from $\mathcal{P}_2$.
