Lecture Notes in Functional Analysis

Lectures by Dr. Md Ramiz Reza *

Contents

References

The following textbooks will be used for this course:

*Notes by Ashish Kujur, Last Updated: January 22, 2023

1. John B. Conway – A Course in Functional Analysis
2. Walter Rudin – Real and Complex Analysis
3. Bhatia – Notes on Functional Analysis
4. Erwin Kreyzsig – Introductory functional analysis with applications

§1 Lecture 1 — *Introduction to Hilbert Spaces and some examples* — 9th January, 2023

§1.1 Inner Product Spaces

Definition §1.1.1 (Inner Product)**.** Let V be a vector space over a field $\mathbb{F}$ (where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$). A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ is called an *inner product* if it satisfies the following properties

1. $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
2. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
3. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
4. $\langle x, x \rangle \geq 0$
5. $\langle x, x \rangle = 0$ only if $x = 0$.

for all $x, y, z \in V$ and $\alpha \in \mathbb{F}$. A vector space V with an inner product is called an *inner product space.*

Example §1.1.2 (Examples of inner product spaces)**.** Here are some examples of inner product spaces:

1. The obvious first example is that of $\mathbb{C}^n$ with the standard 2-inner product given by
$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$$

2. One can then consider the space $\ell^2(\mathbb{N})$ which is the vector space of all square summable sequences on $\mathbb{C}$. That is,

$$\ell^2(\mathbb{N}) = \left\{ (x_n) \in \mathbb{C}^{\mathbb{N}} \mid \sum_{i \in \mathbb{N}} |x_i|^2 < \infty \right\}$$

We define an inner product on this vector space by

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$$

One can show using Holder's inequality that the sum turns out to be finite and the "inner product" is indeed an inner product.

3. Next, we consider the vector space of all polynomials over $\mathbb{C}$ which we denote by $\mathbb{C}[x]$. If $p, q \in \mathbb{C}[x]$, we define an inner product on $\mathbb{C}[x]$ by

$$\langle p, q \rangle = \int_0^1 p\overline{q}\,\mathrm{dx}$$

4. One can define inner products on $C[0,1]$ and $L^2(X, \mathscr{A}, \mu)$ in an similar fashion as in item 3. Note that $(X, \mathscr{A}, \mu)$ is a measure space.

Definition §1.1.3. Let V be an inner product space. We can define a function $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ by

$$\|x\| = \sqrt{\langle x, x \rangle}$$

We call this function *norm induced by the inner product.* (This norm is indeed a norm as one can check!)

The proof of the following theorems are skipped:

Theorem §1.1.4 (Cauchy Schwarz inequality)**.** *Let V be an inner product space, $x, y \in V$. Then we have that*

$$|\langle x, y \rangle \leq \|x\| \, \|y\|$$

Theorem §1.1.5 (Triangle Inequality)**.** *Let V be an inner product space, $x, y \in V$. Then we have that*

$$\|x + y\| \leq \|x\| + \|y\|$$

§1.2 Hilbert Spaces

Definition §1.2.1. Let V be an inner product space. One can consider V as a metric space by defining the following metric d:

$$d(v, w) = \|v - w\|$$

for all $v, w \in V$. Then (V, d) is a metric space (Check!). We say that V is a *Hilbert Space* if (V, d) is a complete metric space.

Example §1.2.2. We consider some examples and not-so-example of Hilbert Space:

1. $\mathbb{R}^n$ and $\mathbb{C}^n$ with the standard inner product are complete!
2. $\ell^2(\mathbb{N})$ is complete.
3. $L^2(X)$ is complete where $(X, \mathscr{A}, \mu)$ is a measure space.
4. $\mathscr{C}[0,1]$ is not complete w.r.t $L^2[0,1]$ inner product.
5. Consider $c_{00} = \left\{(x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) : (x_n)_{n\in\mathbb{N}} \text{ is eventually zero}\right\}$. c_{00} has the induced inner product. We show that c_{00} with this induced product is not complete! One consider the sequence of sequences given by

   $$f_n = \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, 0, \ldots\right)$$

   One can then easily show that $\left(f_n\right)$ is Cauchy sequence in c_{00} but it does not converge in c_{00}.

§2 Lecture 2 — *Hilbert Spaces!* — 11th January, 2023

The important goal of this lecture is to show that if H is a Hilbert Space then we show that under certain conditions an element can be projected onto a set. But before that, we prove the following theorem:

Theorem §2.0.1 (Norm is uniformly continuous)**.** *Let H be a Hilbert space. The norm function on H, that is, $\|\cdot\| : H \to \mathbb{R}$ given by $\|x\| = \sqrt{\langle x, x\rangle}$, $x \in H$, is continuous.*

Proof. Let $x, y \in H$. Then by the triangle inequality, we have the following:

$$\|x\| = \|(x-y)+y\| \le \|x-y\| + \|y\|$$

and hence

$$\|x\| - \|y\| \le \|x-y\|$$

Interchanging the role of x and y in the previous inequality, we have htat

$$\|y\| - \|x\| \le \|x-y\|$$

and thus, we have proved that

$$\big|\|x\| - \|y\|\big| \le \|x-y\|$$

which says that $\|\cdot\|$ is uniformly continuous. □

Note that theorem §2.0.1 holds for any normed linear space, that is, there is no use of completeness there.

§2.1 Closed and Convex!

Theorem §2.1.1. *Let S be a closed convex set in a Hilbert space H. Let $x \in H$. The distance of x from S, denoted as $d(x,S)$, is given by*

$$d(x,S) = \inf\{\|x-y\| : y \in S\}.$$

It follows that there exist a unique $s_0 \in S$ such that $d(x,S) = \|x - s_0\|$.

Proof. First of all, recall the parallelogram identity which holds for any innter product spaces, and hence in particular for Hilbert spaces,

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

The parallelogram law plays a crucial role in the proof of this theorem. Now, let's get busy to prove the theorem. First of all, by definition of infimum, we can find a sequence (s_n) in S such that $d(s_n, x) \to d(x,S)$. To be economical, let us denote $\delta := d(x,S)$. We show that (s_n) is Cauchy sequence in H. To do so, let $\varepsilon > 0$ be given.

Observe that for any $n, m \in \mathbb{N}$,

$$\left\|\frac{x-s_n}{2} - \frac{x-s_m}{2}\right\|^2 + \left\|\frac{x-s_n}{2} + \frac{x-s_m}{2}\right\|^2 = \frac{1}{2}\left(\|x-s_n\|^2 + \|x-s_m\|^2\right)$$

and hence

$$\frac{1}{4}\left\|s_m - s_n\right\|^2 = \frac{1}{2}\left(\left\|x - s_n\right\|^2 + \left\|x - s_m\right\|^2\right) - \frac{1}{4}\left\|x - \frac{s_n + s_m}{2}\right\|^2 \tag{§2.1.1}$$

Now since $d(s_n, x)$ converges to δ, we must have that $d(s_n, x)^2$ converges to δ^2 and hence there is some $K \in \mathbb{N}$ such that for all $i \geq K$,

$$\left\|x - s_i\right\|^2 < \delta^2 + \frac{\varepsilon^2}{4}$$

Now for all $n, m \geq K$ and from equation §2.1.1, we have that

$$\begin{aligned}\left\|s_m - s_n\right\|^2 &= 2\left(\left\|x - s_n\right\|^2 + \left\|x - s_m\right\|^2\right) - \left\|x - \frac{s_n + s_m}{2}\right\|^2 \\ &\overset{(!)}{<} 2 \cdot 2\left(\delta^2 + \frac{\varepsilon^2}{4}\right) - 4\delta^2 \\ &= \varepsilon^2\end{aligned}$$

Note that in inequality (!), we made use of the convexity of S to conlude that $\frac{s_n + s_m}{2} \in S$. This shows that (s_n) is Cauchy. Now, since H is a Hilbert space, (s_n) must converge to some $s_0 \in H$. Closedness of S allows us to conclude that s_0 must be in S.

Hence, $x - s_n$ converges to $x - s_0$. By Theorem §2.0.1, we conclude that $\|x - s_n\|$ converges to $\|x - s_0\|$. Since $\|x - s_n\|$ also converges to δ, we have by uniqueness of limits that $\delta = \|x - s_0\|$.

It remains to prove the uniqueness of such a vector. Let us suppose that s_0 and t_0 be two vectors such that $\|x - s_0\| = \|x - t_0\| = \delta$.

Applying parallelogram identity on the vectors s_0 and t_0 as in Equation §2.1.1, we get

$$\begin{aligned}\frac{1}{4}\left\|s_0 - t_0\right\|^2 &= \frac{1}{2}\left(\left\|x - s_0\right\|^2 + \left\|x - t_0\right\|^2\right) - \frac{1}{4}\left\|x - \frac{s_0 + t_0}{2}\right\|^2 \\ &\leq \delta^2 - \left\|x - \frac{s_0 + t_0}{2}\right\|^2 \\ &\leq 0\end{aligned}$$

Hence, $s_0 = t_0$ and this completes the proof of the theorem. $\square$

Example §2.1.2 (distance is achieved but the vector may not be unique)**.** Consider the normed linear space $\left(\mathbb{R}^2, \|\cdot\|_1\right)$. Now consider the subset S of $\mathbb{R}^2$ given by

$$S = \{(x_1, x_2) : x_1 + x_2 = 1\}.$$

Note that $d((0,0),S) = 1 = d(0,0),(1,0)) = d((0,0),(0,1))$. Hence, the uniqueness is not guaranteed.

Exercise §2.1.3. *Consider the space* $(C[0,1])$ *with the supremum norm* $\|\cdot\|_\infty$*, that is,* $\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}$. *Let S be the set*

$$S = \left\{ f \in C[0,1] : \int_0^{1/2} f(x)\,dx - \int_{1/2}^1 f(x)\,dx = 1 \right\}.$$

Show that the set S is closed and convex but the distance $d(0,S) = 1$*, is never achieved at any point in S. That is, it is not the case that there is some* $f \in S$ *such that* $d(0,f) = \|f\|_\infty = 1$.

Solution. We begin by showing that S is convex. Let $f, g \in S$ and $t \in [0,1]$. Then we have that

$$\begin{aligned}\int_0^{1/2} \left(tf(x) + (1-t)g(x)\right)dx - \int_{1/2}^1 \left(tf(x) + (1-t)g(x)\,dx\right) &= t + (1-t) \\ &= 1\end{aligned}$$

Note that the second equality follows by the virtue of $f, g \in S$.

Now, we proceed to show that the S is closed. Let (f_n) be a sequence of functions in S converging to $f \in C[0,1]$. We need to prove that $f \in S$. Now convergence in supremum norm is the same as the uniform convergence, so, we have that following:

$$\lim_{n\to\infty}\left(\int_0^{1/2} f_n(x)\,dx - \int_{1/2}^1 f_n(x)\,dx\right) = 1$$

implies

$$\int_0^{1/2} f(x)\,dx - \int_{1/2}^1 f(x)\,dx = 1$$

and thus $f \in S$. Consider the zero function and the set S, we show that that there is no $f \in S$ such that $d(0,S) = d(f,0) = \|f\|_\infty$. Incomplete! □

§2.2 Projections

Theorem §2.2.1. *Let H be a Hilbert space. For any fixed* $y \in H$*, consider the map* $L_y : H \to \mathbb{C}$ *defined by* $L_y(x) = \langle x, y\rangle$, $x \in H$. *Then* L_y *is a continuous linear functional on H.*

Proof. Let $y \in H$ be fixed. Consider the function $L_y : H \to \mathbb{C}$ given by $L_y(x) = \langle x, y \rangle$ for each $x \in H$. We show that L_y is Lipschitz continuous. Let $x_0 \in H$. If $x \in H$, we have that

$$\begin{aligned} \left|L_y(x) - L_y x_0\right| &= \left|\langle x, y\rangle - \langle x_0, y\rangle\right| \\ &= \left|\langle x - x_0, y\rangle\right| \\ &\leq \|x - x_0\| \, \|y\| \end{aligned}$$

Note the inequality follows from Cauchy Schwarz and this completes the proof.

□

Definition §2.2.2. Let H be a Hilbert space. For any $y \in H$, the symbol $y^\perp$ denote the subspace defined by

$$y^\perp := \{x \in H : \langle x, y\rangle = 0\}$$

Observe that $y^\perp$ is a closed subspace of H. This is because $y^\perp$ is the kernel of the continuous map L_y as given by Theorem §2.2.1.

Definition §2.2.3. Let H be a Hilbert space. Let M be any subspace of H. Let the symbol $M^\perp$ denote the subspace given by

$$M^\perp = \{x \in H : \langle x, y\rangle = 0 \text{ for all } y \in M\} = \bigcap_{y \in M} y^\perp.$$

Observe that $M^\perp$ is always closed since it is intersection of closed subspaces of H.

Theorem §2.2.4 (Existence of an Orthogonal Projection onto a closed subspaces)**.** *Let M be a closed subspace of a Hilbert space H. Then*

(a) *Every $x \in H$ has a unique decomposition*

$$x = Px + Qx$$

into a sum of $Px \in M$ and $Qx \in M^\perp$. Thus $H = M \oplus M^\perp$.

(b) *Px and Qx are the nearest points to x in M and in $M^\perp$ respectively.*

(c) *The mappings $P : H \to M$ and $Q : H \to M^\perp$ are linear and satisfies $P^2 = P$ and $Q^2 = Q$. The map P and Q are called the* ***orthogonal projection onto*** *M* ***and*** *$M^\perp$ respectively.*

(d) $\|x\|^2 = \|Px\|^2 + \|Qx\|^2$ *for every* $x \in H$.

Proof. Since subspaces are convex, we can appeal to Theorem §2.1.1 as we please. We now start to prove each of the statements of theorem:

(a) Let $x \in H$ be arbitrary. Then by the Theorem §2.1.1 there is a unique vector $Px \in M$ such that

$$d(x, M) = \|x - Px\|$$

Define $Qx \in M$ by $Qx = x - Px$. We need to show that $Qx \in M^{\perp}$. Let $y \in M$. We want to show that $\langle x - Px, y \rangle = 0$. To do so, observe that

$$\left\langle Qx - \langle Qx, y \rangle \frac{y}{\|y\|^2}, y \right\rangle = \langle Qx, y \rangle - \left\langle \langle Qx, y \rangle \frac{y}{\|y\|^2}, y \right\rangle = 0 \qquad (§2.2.1)$$

Now,

$$Qx = \underbrace{\left(Qx - \langle Qx, y \rangle \frac{y}{\|y\|^2} \right)}_{=:v_1} + \underbrace{\langle Qx, y \rangle \frac{y}{\|y\|^2}}_{=:v_2}$$

Note that by equation §2.2.1, v_1 and v_2 are orthogonal and hence by Pythagoras theorem for inner product spaces, we may write

$$\begin{aligned}
\delta^2 = \|Qx\|^2 &= \left\| Qx - \langle Qx, y \rangle \frac{y}{\|y\|^2} \right\|^2 + \left\| \langle Qx, y \rangle \frac{y}{\|y\|^2} \right\|^2 \\
&= \left\| Qx - \langle Qx, y \rangle \frac{y}{\|y\|^2} \right\|^2 + \frac{|\langle Qx, y \rangle|}{\|y\|^2} \\
&= \left\| x - Px - \langle Qx, y \rangle \frac{y}{\|y\|^2} \right\|^2 + \frac{|\langle Qx, y \rangle|}{\|y\|^2} \qquad \geq \delta^2 + \frac{|\langle Qx, y \rangle|}{\|y\|^2}
\end{aligned}$$

and thus, we have that $|\langle Qx, y \rangle| = 0$. This completes the proof of (a).

(b) By uniqueness of part (a), it follows that Px is the nearest point to x in M. It remains to prove that Qx is the nearest point to x in $M^{\perp}$. Note that $x - Qx = Px \in M$. Now for any $y \in M^{\perp}$ we have $Qx - y \in M^{\perp}$. Thus we get

$$\|x - y\|^2 = \|(x - Qx) + (Qx - y)\|^2 = \|x - Qx\|^2 + \|Qx - y\|^2 \geqslant \|x - Qx\|^2.$$

This shows that Qx is the nearest point to x in $M^{\perp}$.

(c) Let $x_1, x_2 \in M$. By part (a), we have that

$$\begin{aligned} x_1 &= Px_1 + Qx_1 \\ x_2 &= Px_2 + Qx_2 \\ x_1 + x_2 &= P(x_1 + x_2) + Q(x_1 + x_2) \end{aligned}$$

Now taking sums and rearranging, we have that

$$\underbrace{Px_1 + Px_2 - P(x_1 + x_2)}_{\in M} = \underbrace{Q(x_1 + x_2) - Qx_1 - Qx_2}_{\in M^\perp}$$

Since $M \cap M^\perp = \{0\}$, the linearity of P and Q follows.

Now, let $x \in P$. We need to prove that $P^2 x = Px$. Now note that $Px \in M$. Thus by part (a) we have

$$Px = P^2 x + QPx$$

By uniqueness of part (a), we must have that $Px = P^2 x$. This completes the proof. $Q^2 = Q$ can be proved similarly.

(d) This follows immediately from Pythagoras theorem.

□

Corollary §2.2.5. *Let M be a closed subspace of a Hilbert space H. Then $(M^\perp)^\perp = M$. In case M is a subspace then $(M^\perp)^\perp = \overline{M}$, the closure of M in H.*

§3 Lecture 3 — *Riesz Representation Theorem for Hilbert Spaces* — 13th January, 2023

§3.1 Lecture 2 continued ...

Before we move onto prove the Riesz Representation Theorem, we finish the proof of Corollary §2.2.5. It follows immediately from the following results:

Proposition §3.1.1 (orthogonal complement of a set and the orthogonal completement of its closure are same!)**.** *Let M be a subset of a inner product space H. Then $M^\perp = \left(\overline{M}\right)^\perp$*

Proof. It follows by definition that $M \subset \overline{M}$ and hence $\left(\overline{M}\right)^{\perp} \subset M^{\perp}$. Now for reverse the inclusion, let $v \in M^{\perp}$ and let $y \in \overline{M}$. We need to show that $\langle v, y \rangle = 0$. Since $y \in \overline{M}$ there is a sequence (y_n) in M such that $y_n \to y$. Since $v \in M^{\perp}$, we have that $\langle v, y_n \rangle = 0$ for all $n \in \mathbb{N}$. Since $\langle v, y_n \rangle \to \langle v, y \rangle$, we have by uniqueness of limits that $\langle v, y \rangle = 0$. This completes the proof. □

Proposition §3.1.2 (orthogonal complement of orthogonal complement)**.** *Let M be a closed subspace of the Hilbert space H. Then*

$$M = \left(M^{\perp}\right)^{\perp}$$

Proof. Let us first show that $M \subset \left(M^{\perp}\right)^{\perp}$ (which in fact holds for any set M). Let $v \in M$ and $w \in M^{\perp}$. It is cleary by definition of $M^{\perp}$ that $\langle v, w \rangle = 0$. Hence, $v \in \left(M^{\perp}\right)^{\perp}$.

Let us proceed to show the inclusion in the other direction. Let $v \in \left(M^{\perp}\right)^{\perp}$. Since M is closed, by Theorem §2.2.4, we have that $v = Pv + Qv$ where $Pv \in M$ and $Qv \in M^{\perp}$. By the previous paragraph, we have that $M \subset \left(M^{\perp}\right)^{\perp}$ and hence $Pv \in \left(M^{\perp}\right)^{\perp}$. Hence, we have that $Qv \in \left(M^{\perp}\right)^{\perp}$. Now, $Qv \in M^{\perp} \cap \left(M^{\perp}\right)^{\perp}$. Hence, $Qx = 0$ and thus, $v = Pv \in M$. □

Note that Proposition §3.1.1 does not depend on H being a Hilbert Space while Proposition §3.1.2 does!

Now, proof of Corollary §2.2.5 follows immediately:

Proof of Corollary §2.2.5. The first part of Corollary §2.2.5 is basically Proposition §3.1.2. Now to prove the second part, observe that

$$\begin{aligned}
\left(M^{\perp}\right)^{\perp} &= \left(\left(\overline{M}\right)^{\perp}\right)^{\perp} && \text{by Proposition §3.1.1} \\
&= \overline{\overline{M}} && \text{by Proposition§3.1.2} \\
&= \overline{M}
\end{aligned}$$

□

§3.2 Existence of closed subspaces of Hilbert Spaces

Let H be a Hilbert space of dimension at least 1. Does there always exist a closed subspace of H? The answer is *Yes*!

Let us proceed to prove this: Let H be any Hilbert space of dimension at least one. So, there is at least one nonzero vector v. Let M be the subspace spanned by v. We show that M is closed. Let $\left(y_n\right)$ be a sequence in M converging to some $x \in H$. By definition of M, we have that for every $n \in \mathbb{N}$, $y_n = c_n v$ for some $c_n \in \mathbb{F}$. We claim that c_n is a Cauchy sequence in $\mathbb{F}$.

To show that (c_n) is Cauchy in $\mathbb{F}$, let $\varepsilon > 0$ be given. Since $(c_n v)$ is convergent, it is Cauchy. So there is some $N \in \mathbb{N}$ such that for $n, m \geq N$, we have $\|c_n v - c_m v\| < \|v\| \varepsilon$. Which is turn implies that for $n, m \geq N$, $|c_n - c_m| < \varepsilon$.

Now, since (c_n) is Cauchy in $\mathbb{F}$, it must converge to some $c \in F$. Now, the sequence $(c_n v)$ converges to cv in M and by the uniqueness of limits, we have that $y = cv$ and hence $y \in M$.

This argument generalises, *mutatis mutandis*, and the following result holds:

Theorem §3.2.1. *Every finite dimensional subspace of a Hilbert space is closed.*

Proof. We do a proof by induction on the dimension of finite dimensional subspace. The base case is clear by the argument given before the statement of this theorem.

Suppose the theorem is true for all subspaces of dimension n.

Let H be an inner product space and U be a finite dimensional subspace of H of dimension $n+1$.

Let v be a nonzero vector of U. Then let

$$v_U^{\perp} = v^{\perp} \cap U$$

It is easy to see that $U = \operatorname{span} v \oplus v_U^{\perp}$. Since $v_U^{\perp}$ is a subspace of dimension n, it is closed by the induction hypothesis.

We now proceed to show that U is closed in H. Let (u_n) be a sequence in U converging to $x \in H$. Then for each $n \in \mathbb{N}$ we have that $u_n = c_n v + v_n$ for some $c_n \in F$ and some $v_n \in v_U^{\perp}$. Since (u_n) is convergent, we have that (u_n) is Cauchy. Thus,

$$|c_n - c_m|^2 \|v\|^2 + \|v_n - v_m\|^2 \to 0 \text{ as } m, n \to \infty$$

and note that this is due to Pythagoras theorem.

Then (c_n) converges to c and (v_n) is Cauchy and hence converges to some $y \in H$. Since $v_U^{\perp}$ is closed, we have that $y \in v_U^{\perp}$. Thus, u_n converges to $cv + y$ and by uniqueness of limits, we have that $x = cv + y \in U$. □

Example §3.2.2. Consider the subspace c_{00} in $\ell^2(\mathbb{N})$. We showed that c_{00} is not complete with inner product on $\ell^2(\mathbb{N})$, so, it cannot be closed (because closed subspaces of a complete metric space are closed!). So, we may ask what is the closure of c_{00} in $\ell^2(\mathbb{N})$?

It is precisely $\ell^2(\mathbb{N})$. One can show this as follows: if $f \in \ell^2(\mathbb{N})$ then we may consider the sequence (g_n) in c_{00} given by

$$g_n = (f_1, f_2, \ldots, f_n, 0, 0, 0, \ldots)$$

It is easily seen that g_n converges to f.

Exercise §3.2.3. *If M is a subspace of a Hilbert Space H then so is $\overline{M}$.*

§3.3 Statement and Proof of Riesz Representation Theorem

In Theorem §2.2.1, we saw that for any $y \in H$ where H is a Hilbert space, the linear functional $L_y : H \to \mathbb{C}$ given by $x \mapsto \langle x, y \rangle$ is continuous. But does it happen that given a continuous linear functional $L : H \to \mathbb{C}$ there is a $y_0 \in H$ such that $L = L_{y_0}$? Theanswer to this question is:

Theorem §3.3.1 (Riesz Representation Theorem for Hilbert Spaces)**.** *Let $L : H \to \mathbb{C}$ be continuous linear functional. Then there exists a $y_0 \in H$ such that $L = L_{y_0}$.*

But before proving Riesz Representation Theorem, let us prove that following theorem:

Proposition §3.3.2 (necessary and sufficient condition for a subspace to be dense)**.** *Suppose M is a subspace of a Hilbert space V. Then $\overline{M} = V$ iff $M^\perp = 0$.*

Proof of Proposition §3.3.2. First suppose $\overline{M} = V$. We need to show that $M^\perp = 0$. By §3.1.1, we have that $V^\perp = \left(\overline{M}\right)^\perp = M^\perp$. But since $V^\perp = \{0\}$, we have that $M^\perp = \{0\}$.

To prove the converse, suppose that $M^\perp = 0$. Taking $\perp$ both sides, we have that $\overline{M} = 0$ by using §2.2.5. □

Proof of §3.3.1. Let $L : H \to \mathbb{C}$ be continuous linear functional. Since L is continuous, $\ker L$ is closed. Let $M := \ker L$.

Now, there are two possible cases: either $M^\perp = 0$ or $M^\perp \supsetneq \{0\}$.

If $M^\perp = \{0\}$, then $M = \overline{M} = (M^\perp)^\perp = \{0\}^\perp = H$. Hence, we have that $L = L_0$ and this case is done.

Let us consider the other case where $M^{\perp} \supsetneq \{0\}$. We claim that $\dim M^{\perp} = 1$. Since $M^{\perp} \supsetneq \{0\}$, we may select a nonzero vector $w \in M^{\perp}$. We show that $M^{\perp} = \operatorname{span}\{w\}$. To do so, let $\tilde{w} \in M^{\perp}$. Consider the vector $x := L(\tilde{w})\, w - L(w)\,\tilde{w}$. Clearly, $L(x) = 0$. Hence $x \in M$. Since $w, \tilde{w} \in M^{\perp}$, we also have that $x \in M^{\perp}$. But then $x \in M \cap M^{\perp} = \{0\}$and hence $x = 0$. Thus, $L(\tilde{w})\, w = L(w)\,\tilde{w}$. Observe that $L(w) \neq 0$ for otherwise $w \in M$ and $w \in M \cap M^{\perp}$ forcing $w = 0$. Hence $\tilde{w} = (L(w))^{-1} L(\tilde{w}) w$. Hence $\tilde{w} \in \operatorname{span}\{w\}$ which completes the proof of the claim.

Let w be as in previous paragraph. We have that $H = M \oplus M^{\perp} = \ker L \oplus \operatorname{span} w$. Hence, $\left(\operatorname{span}\{w\}\right)^{\perp} = (M^{\perp})^{\perp} = \overline{M} = M$. Since $\left(\operatorname{span}\{w\}\right) = w^{\perp}$, we have that $w^{\perp} = M$.

Now, let $x \in H$. Now,

$$x = \underbrace{x - \frac{\langle x, w\rangle\, w}{\|w\|^2}}_{w^{\perp}=M} + \underbrace{\frac{\langle x, w\rangle\, w}{\|w\|^2}}_{\in M^{\perp}}$$

Hence, we have that

$$L(x) = \left\langle x, \frac{\overline{L(w)}\, w}{\|w\|^2} \right\rangle$$

This completes the proof! $\square$

§4 Lecture 4 — *Projection for subspaces having countable orthonormal basis* — 17th January, 2022

§4.1 Projections and Orthonormal Sets in finite dimensions…

Suppose M is a finite dimensional subspace of an Hilbert space H and dimension of M is n. Let $\mathscr{B} = \{u_i : 1 \leqslant i \leqslant n\}$ be an orthonormal basis of M (existence of such basis follows from the Gram-Schmidt orthogonalization process to any basis of M). For any $x \in H$, consider the vector $\sum_{j=1}^{n} \langle x, u_j\rangle u_j$ in M. Thus we obtain

$$x = \left(x - \sum_{j=1}^{n} \langle x, u_j\rangle u_j\right) + \left(\sum_{j=1}^{n} \langle x, u_j\rangle u_j\right)$$

We verify that $x - \sum_{j=1}^{n} \langle x, u_j \rangle u_j \in M^{\perp}$. Observe that for all $1 \le i \le n$,

$$\begin{aligned}\left\langle x - \sum_{j=1}^{n} \langle x, u_j \rangle u_j, u_i \right\rangle &= \langle x, u_i \rangle - \sum_{j=1}^{n} \langle x, u_j \rangle \langle u_j, u_i \rangle \\ &= \langle x, u_i \rangle - \langle x, u_i \rangle = 0\end{aligned}$$

Now by the uniqueness of the decomposition (see Theorem §2.2.4) $H = M \oplus M^{\perp}$, it follows that

$$P_M x = \sum_{j=1}^{n} \langle x, u_j \rangle u_j,$$

$$P_{M^{\perp}} x = x - \sum_{j=1}^{n} \langle x, u_j \rangle u_j,$$

where P_M and $P_{M^{\perp}}$ denotes the orthogonal projection onto M and $M^{\perp}$ respectively. Furthermore, it follows that

$$\|P_M x\|^2 = \sum_{j=1}^{n} |\langle x, u_j \rangle|^2 = \|x\|^2 - \|P_{M^{\perp}} x\|^2 \leqslant \|x\|^2, \; x \in H. \tag{§4.1.1}$$

§4.2 Generalising projections to infinite dimensions...

Proposition §4.2.1. *Suppose $\mathscr{B} = \{u_i : i \in \mathbb{N}\}$ is an orthonormal set in a Hilbert space H. Then for any $x \in H$, we have*

$$\sum_{j=1}^{\infty} |\langle x, u_j \rangle|^2 \leqslant \|x\|^2.$$

This is known as the Bessel's inequality. Let $M = \overline{span\{u_i : i \in \mathbb{N}\}}$ be the smallest closed subspace spanned by the orthonormal set $\mathscr{B}$. It follows that $S_n(x) = \sum_{j=1}^{n} \langle x, u_j \rangle u_j$ is a Cauchy sequence in M. Hence the limit $\lim_{n\to\infty} S_n(x) = \sum_{j=1}^{\infty} \langle x, u_j \rangle u_j$ exist in the Hilbert space M. Moreover, $P_M(x)$, the orthogonal projection of x onto M, is given by

$$P_M(x) = \sum_{j=1}^{\infty} \langle x, u_j \rangle u_j = \lim_{n\to\infty} S_n(x).$$

Furthermore equality occurs in the Bessel's inequality if and only if $x \in M$.

Proof. Fix a natural number k and consider $M_k = \operatorname{span}\{u_j : 1 \leqslant j \leqslant k\}$. By Equation §4.1.1 we find that

$$\|P_{M_k} x\|^2 = \sum_{j=1}^{k} |\langle x, u_j\rangle|^2 \leqslant \|x\|^2,\ x \in H,$$

where where P_{M_k} denotes the orthogonal projection onto M. Since this holds true for every $k \in \mathbb{N}$, it follows that

$$\sum_{j=1}^{\infty} |\langle x, u_j\rangle|^2 \leqslant \|x\|^2.$$

For $S_n(x) = \sum_{j=1}^{n} \langle x, u_j\rangle u_j$, note that

$$\|S_m(x) - S_k(x)\|^2 = \sum_{j=k+1}^{m} |\langle x, u_j\rangle|^2, \qquad k \geqslant m.$$

Since $\sum_{j=1}^{\infty} |\langle x, u_j\rangle|^2 < \infty$, it follows that $\{S_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in M and converges to some vector in M. Let $z = \lim_{n\to\infty} S_n(x)$. It is straightforward to verify that $\langle x - S_n(x), u_j\rangle = 0$ for every $n \geqslant j$. Since $x - z = \lim_{n\to\infty}(x - S_n(x))$, it follows that $\langle x - z, u_j\rangle = 0$. This holds true for any $j \in \mathbb{N}$. Hence we get that $x - z \in M^{\perp}$. Thus we have

$$x = z + (x - z),\ \ x - z \in M^{\perp}, z \in M.$$

By the uniqueness of the decomposition $H = M \oplus M^{\perp}$, we obtain that

$$P_{M}(x) = z = \sum_{j=1}^{\infty} \langle x, u_j\rangle u_j = \lim_{n\to\infty} S_n(x),\ P_{M^{\perp}}(x) = x - z.$$

Now, we proceed to show that the Bessel inequality holds iff $x \in M$. Suppose that $x \in M$. Then we have that $P_M(x) = x$ and hence $x = z + (x - z) = x + (x - z)$ and hence $x = z$. But note that $\|z\|^2 = \left\|\sum_{j=1}^{\infty} \left\langle x, u_j \right\rangle u_j\right\| = \sum_{j=1}^{\infty} \left|\left\langle x, u_j\right\rangle\right|$ but this follows from

$$\left\|\sum_{j=1}^{\infty} \left\langle x, u_j \right\rangle u_j\right\| = \sum_{j=1}^{\infty} \left|\left\langle x, u_j \right\rangle\right|$$

and then taking $n \to \infty$, we get

$$\left\|\sum_{j=1}^{n} \left\langle x, u_j \right\rangle u_j\right\| = \sum_{j=1}^{n} \left|\left\langle x, u_j \right\rangle\right| \tag{§4.2.1}$$

Conversely, suppose that equality in Bessel's inequality holds then from equalion §4.2.1 and the fact that $\|x\|^2 = \|z\|^2 \|x-z\|^2$ that $\|x-z\|^2 = 0$ and hence $x \in M$. $\square$

Corollary §4.2.2. *Suppose $\mathscr{B} = \{u_i : i \in \mathbb{N}\}$ is a maximal orthonormal set in a Hilbert space H. Let $M = \overline{span\{u_i : i \in \mathbb{N}\}}$ be the smallest closed subspace spanned by the orthonormal set $\mathscr{B}$. Then it follows that $M = H$ and for any $x \in H$ we have*

$$x = \sum_{j=1}^{\infty} \langle x, u_j\rangle u_j = \lim_{n\to\infty} S_n(x), \ \|x\|^2 = \sum_{j=1}^{\infty} |\langle x, u_j\rangle|^2.$$

Proof. If $M^{\perp} \neq 0$, then consider a non zero unit vector u in $M^{\perp}$. Then $\mathscr{B} \cup \{u\}$ is another family of orthonormal set containing $\mathscr{B}$. This contradicts the maximality of $\mathscr{B}$. Hence by the maximality of $\mathscr{B}$, it follows that $M^{\perp} = \{0\}$. Now the corollary follows from the proposition. $\square$

§4.3 Existence of a maximal orthonormal set in a inner product space

Let $(X, \leqslant)$ be the collection of all orthonormal set in V equipped with the partial ordering of set inclusion, that is, for $A, B \in X$, we have $A \leqslant B$ if $A \subseteq B$. It is straightforward to verify that if $\mathscr{C}$ is a chain (totally ordered set) in the partially ordered set $(X, \leqslant)$, then the chain $\mathscr{C}$ has an upper bound in X namely the union of the members of $\mathscr{C}$. Hence by Zorn's Lemma it follows that X has a maximal element, that is, V has a maximal orthonormal set. Now, we make a definition:

Definition §4.3.1 (orthonormal basis)**.** A maximal orthonormal set in a Hilbert space is called an *orthonormal basis of the Hilbert space.*

§4.4 Separability of Hilbert Spaces

Proposition §4.4.1. *Let H be a Hilbert space. Then H is separable (that is it has a countable dense set) if and only if H admits an at most countable orthonormal basis.*

Proof. Suppose $\mathscr{B} = \{u_\alpha : \alpha \in I\}$ be an collection of orthonormal set in H. It is straightforward to verify that $\|u_\alpha - u_\beta\| = \sqrt{2}$ for every $\alpha, \beta \in I$ with $\alpha \neq \beta$. Thus the collection of balls $\{B(u_\alpha, \frac{\sqrt{2}}{2}) : \alpha \in I\}$ are pairwise disjoint. If I is uncountable then we have uncountable such balls which are pairwise disjoint. This contradicts any existence of countable dense set in H. Thus if H is separable then any orthonormal collection in H has to be at most countable (finite or countably infinite). Hence any maximal orthonormal set in H must be at most countable. This proves that for a separable Hilbert space H we have an at most countable orthonormal basis.

For the converse direction assume that H admits a countable orthonormal basis, say $\mathscr{B} = \{u_i : i \in \mathbb{N}\}$. Let $D = \cup_{n\in\mathbb{N}} D_n$, where D_n is given by

$$D_n = \left\{ \sum_{j=1}^{n} c_j u_j : c_j \in \mathbb{Q} + i\mathbb{Q} \right\}$$

Note that each D_n is countable and hence D is countable. It is straightforward to see that $\overline{D_n} = \text{span}\,\{u_j : 1 \leqslant j \leqslant n\}$ for each $n \in \mathbb{N}$. It follows that span $\{u_j : j \in \mathbb{N}\} \subseteq \overline{D}$. This gives us that $\overline{\text{span}\,\{u_j : j \in \mathbb{N}\}} \subseteq \overline{D}$. In view of the Corollary §4.2.2 we obtain that D is dense in H and hence H is separable. $\square$