Shift Invariant Subspaces

...more specifically 1-unilateral shift.

by Ashish Kujur

on September 06, 2023

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Given any operator $T : V \to V$, the following are always invariant under T:

1. $\{0\}$,
2. V,
3. $\ker T$,
4. $\operatorname{im} T$.

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1. One may ask whether every linear operator on the finite dimensional vector space V over $\mathbb{R}$ or $\mathbb{C}$ admits a invariant subspace different from $\{0\}$ and the whole space V. Answer: Yes! It always admits a one or two dimensional invariant subspace (See [Axl15]).

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1. One may ask whether every linear operator on the finite dimensional vector space V over $\mathbb{R}$ or $\mathbb{C}$ admits a invariant subspace different from $\{0\}$ and the whole space V. Answer: Yes! It always admits a one or two dimensional invariant subspace (See [Axl15]).
2. The same question also has an affirmative response in infinite dimensions (...but this is an algebraic question.)

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Question

Let H be a separable complex Hilbert space of dimension > 1. Does every bounded linear operator on $T : H \to H$ admit a nontrivial closed T-invariant subspace, that is, a closed linear subspace W of H, which is different from $\{0\}$ and H such that $T(W) \subset W$?

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This problem is still open. For the Banach space case, counterexamples were provided by Per Enflo (1976, 1987) and Charles Read on ℓ^1 (1984, 1985).

Unilateral Shifts

» Right Shift on $\ell^2(\mathbb{N}_0)$

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
$\ell^2(\mathbb{N}_0) := \left\{(a_0, a_1, a_2, \ldots) \mid \forall i \in \mathbb{N}_0, a_i \in \mathbb{C}; \sum_{i=0}^{\infty} |a_i|^2 < \infty\right\}$.
$\ell^2(\mathbb{N}_0)$ is an Hilbert space with the following inner product:

$$\left\langle (a_n)_{n \in \mathbb{N}_0}, (b_n)_{n \in \mathbb{N}_0} \right\rangle = \sum_{n=0}^{\infty} a_n \bar{b}_n.$$

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Consider the map $\mathcal{S} : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$ given by

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For each $n \in \mathbb{N}_0$,
$\mathcal{M}_n := \left\{(a_n)_{n\in\mathbb{N}_0} \in \ell^2(N_0) \mid a_0 = a_1 = \ldots = a_{n-1} = 0\right\}$. Then

$$\mathcal{S}\mathcal{M}_n = \mathcal{M}_{n+1} \subset \mathcal{M}_n.$$

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Are these all the invariant subspaces on $\ell^2(\mathbb{N}_0)$?

» Hardy Space

Definition (The Hardy Hilbert Space)

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. We define

$$H^2(\mathbb{D}) = \left\{ f : \mathbb{D} \to \mathbb{C} \mid f(z) = \sum_{n=0}^{\infty} a_n z^n, z \in \mathbb{D}; (a_n)_{n \in \mathbb{N}_0} \in \ell^2(\mathbb{N}_0) \right\}$$

For $f = \sum_{n=0}^{\infty} a_n z^n$ and $g = \sum_{n=0}^{\infty} b_n z^n$ both in $H^2(\mathbb{D})$, we define

$$\langle f, g \rangle = \left\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \right\rangle := \sum_{n=0}^{\infty} a_n \bar{b}_n.$$

$H^2(\mathbb{D})$ is an Hilbert space with the given inner product.

» Shift on the Hardy Space

Define $S : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ given by

$$S(a_0 + a_1 z + \ldots + a_n z^n + \ldots) = a_0 z + a_1 z^2 + \ldots + a_n z^{n+1} + \ldots.$$

This is the same as

$$[Sf](z) = zf(z) \text{ for each } f \in H^2 \text{ and each } z \in \mathbb{D}.$$

We will call it the shift on the Hardy space.

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We will call it the shift on the Hardy space. Consider the map $U : H^2(\mathbb{D}) \to \ell^2(\mathbb{N}_0)$ given by

$$z_i \overset{U}{\mapsto} e_i = \left(0, 0, \ldots, \underbrace{1}_{\text{ith entry}}, 0, \ldots\right)$$

for each $i \in \mathbb{N}_0$. Thus, U is a unitary transformation and $U^* \mathcal{S} U = S$. Consequently, the invariant subspaces of S and the invariant subspaces of $\mathcal{S}$ are in one-to-one correspondence.

Some Examples in $H^2(\mathbb{D})$

» Some Invariant Subspaces of $H^2(\mathbb{D})$

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \mathbb{D}$. Consider the subspace

$$\mathcal{M} = \left\{ f \in H^2(\mathbb{D}) \mid f(\alpha_i) = 0 \text{ for all } 1 \leq i \leq n \right\}.$$

This forms a closed invariant subspace of shift S on the Hardy Space, $1 \notin \mathcal{M}$ and $p(z) = \prod_{i=1}^{n} (\alpha_i - z) \in \mathcal{M}$ and $p \not\equiv 0$.

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This forms a closed invariant subspace of shift S on the Hardy Space, $1 \notin \mathcal{M}$ and $p(z) = \prod_{i=1}^{n} (\alpha_i - z) \in \mathcal{M}$ and $p \not\equiv 0$.
Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\} \subset \mathbb{D}$ be an infinite subset. Like the previous case, it is easily seen that

$$\mathcal{M} = \left\{ f \in H^2(\mathbb{D}) \mid f(\alpha_i) = 0 \text{ for all } i \in \mathbb{N} \right\}$$

is an closed invariant subspace of $H^2(\mathbb{D})$. But is it the case that

$$\{0\} \subsetneq \mathcal{M} \subsetneq H^2(\mathbb{D})?$$

Naive Guess: $p(z) = \prod_{i=1}^{\infty} (\alpha_i - z)$. But is $p \in H^2(\mathbb{D})$?

» Inner functions

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Definition (Inner Function)

Let $f : \mathbb{D} \to \mathbb{C}$ and let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Then f is said to be an *inner function* if f is a bounded holomorphic function and the function $\tilde{f}$ defined almost everywhere on $\mathbb{T}$ by

$$\tilde{f}\left(e^{i\theta}\right) = \lim_{r \to 1} f\left(re^{i\theta}\right)$$

has unit modulus almost everywhere.

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Finite Blaschke Product

Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{D}$ and $\beta \in \mathbb{R}$. Then the finite Blaschke product is the function given by

$$B(z) = e^{i\beta} \prod_{k=1}^{n} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z}.$$

» Infinite Blaschke Product

Theorem (Blaschke, F. Riesz)

Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}\setminus\{0\}$. Then the infinite product

$$B(z) = \prod_{n=1}^{\infty} \frac{|\alpha_i|}{\alpha_i} \frac{\alpha_i - z}{1 - \bar{\alpha}_i z}$$

converges uniformly on compact subsets of $\mathbb{D}$

$$\iff \sum_{i=1}^{\infty} (1 - |\alpha_i|) < \infty.$$

In either cases, B defines an inner function.

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In either cases, B defines an inner function.

For a proof of this, take a look at [Rud74], [Hof07] or [Mas09].

Beurling's Theorem

» $L^2(\mathbb{T})$

Definition (Lebesgue Measure on $\mathbb{T}$)

A subset $A \subset \mathbb{T}$ is called measurable subset of $\mathbb{T}$ if $\{t \in (-\pi, \pi] : e^{it} \in A\}$ is Lebesgue measurable subset of $(-\pi, \pi]$. The Lebesgue measure $m_{\mathbb{T}}$ on the measurable subset of $\mathbb{T}$ is given by

$$\lambda_{\mathbb{T}}(A) = \frac{\lambda\left(\{t \in (-\pi, \pi] : e^{it} \in A\}\right)}{2\pi}$$

where λ is the Lebesgue measure on $(-\pi, \pi]$.

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Then $L^2(\mathbb{T})$ is a Hilbert space and if we define $\zeta : \mathbb{T} \to \mathbb{T}$ to be function given by $\zeta(e^{it}) = e^{it}$ for each $e^{it} \in \mathbb{T}$ then $\{\zeta^n : n \in \mathbb{Z}\}$ forms an orthonormal basis for $L^2(\mathbb{T})$. (Proof: Refer [Axl20] or [Rud74]).

» Hardy Spaces on $\mathbb{T}$

For $f \in L^2(\mathbb{T})$, define $\hat{f}(n) = \langle f, \zeta^n \rangle_{L^2(\mathbb{T})} = \int_{\mathbb{T}} f\zeta^{-n} d\lambda_{\mathbb{T}}$.

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Definition (Hardy Spaces on $\mathbb{T}$)

We define the Hardy space $H^2(\mathbb{T})$ to be given by

$$H^2(\mathbb{T}) := \left\{ f \in L^2(\mathbb{T}) : \hat{f}(n) = 0 \text{ for each } n \leq -1 \right\}.$$

Easy to see: $H^2(\mathbb{T})$ is a closed subspace of $L^2(\mathbb{T})$.

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Easy to see: $H^2(\mathbb{T})$ is a closed subspace of $L^2(\mathbb{T})$. Furthermore, if $f \in H^2(\mathbb{D})$ then the function $\tilde{f} : \mathbb{T} \to \mathbb{C}$

$$\tilde{f}(e^{it}) = lim_{r \to 1^-} f(re^{it})$$

for each $e^{it} \in \mathbb{T}$ is $H^2(\mathbb{T})$ function, and, furthermore, we have that

$$\langle f, g \rangle_{H^2(\mathbb{D})} = \left\langle \tilde{f}, \tilde{g} \right\rangle_{H^1(\mathbb{T})}$$

for each $f, g \in H^2(D)$ and $\tilde{f}, \tilde{g} \in H^2(\mathbb{T})$ are their corresponding boundary functions.

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Consider the operator $M_\theta : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ given by $[M_\theta(f)](z) = \theta(z)f(z)$ for each $f \in H^2(\mathbb{D})$ and $z \in D$.

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To show it is invariant, consider the following:

$$S(\theta H^2(\mathbb{D})) = z\left(\theta H^2(\mathbb{D})\right) = \theta\left(zH^2(\mathbb{D})\right) \subset \theta H^2(\mathbb{D})$$

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Theorem (Beurling's Theorem)

Every nonzero (closed) invariant subspace of $H^2(\mathbb{D})$ is of the form $\theta H^2(\mathbb{D})$ for some inner function θ.

» Sketch of Proof of Beurling's Theorem

Theorem (von Neumann-Wold decomposition)

Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$ and $\mathcal{W} = \mathcal{H} \ominus T\mathcal{H}$.

1. $T^m\mathcal{H} \perp T^n\mathcal{H}$ *for every* $m, n \geq 0$ *and* $m \neq 0$.
2. $\mathcal{W}_\infty = \cup_{n\geq 0} T^n\mathcal{H}$ *then* $T\mathcal{W}_\infty$ *and* $T|_{\mathcal{W}_\infty}$ *is unitary.*
3. $\mathcal{W}_0 = \bigoplus_{n\geq 0} T^n\mathcal{W}$ *then* $T\mathcal{W}_0 \subset \mathcal{W}_0$ *and there is no nonzero invariant subspace of* $\mathcal{W}_0$ *upon which the restriction of* T *is unitary.*
4. $\mathcal{H} = \mathcal{W}_0 \oplus \mathcal{W}_\infty$.

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4. $\mathcal{H} = \mathcal{W}_0 \oplus \mathcal{W}_\infty$.

For a proof, see [Nag+10].

» Sketch Part 1: $\mathcal{SM} \subsetneq \mathcal{M}$

» Sketch Part 2: $(\mathcal{M} \ominus S\mathcal{M})$ has an inner function θ

» Sketch Part 3: Use Wold Decomposition

» Final Remarks: Singular Inner Function & Factorization

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Thank You! Questions?