Solutions to Functional Analysis Assignment 2

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Note

A checkmark ✓ indicates the question has been done.

Contents

1 Question 1 ✓

Let $a, b \in \mathbb{R}$ and $\mathscr{C}[a, b]$ denotes the space of real valued continuous function on $[a, b]$. Let $\|f\|_p$ and $\|f\|_\infty$ denotes the norm on $\mathscr{C}[a, b]$ given by $\|f\|_p = (\int |f(x)|^p dx)^{\frac{1}{p}}$, for $p \geqslant 1$ and $\|f\|_\infty = \sup\{|f(x)| : x \in [a, b]\}$. Establish the following inequalities :

(i) $|\displaystyle\int f(x)g(x)dx| \leqslant \|f\|_p \|g\|_q, \;\; f, g \in \mathscr{C}[a, b], p > 1$ and $1/p + 1/q = 1$.

(ii) $|\displaystyle\int f(x)g(x)dx| \leqslant \|f\|_1 \|g\|_\infty, \;\; f, g \in \mathscr{C}[a, b]$.

(iii) $\|f + g\|_p \leqslant \|f\|_p + \|g\|_p, \;\; f, g \in \mathscr{C}[a, b]$, for every $p \geqslant 1$ and $p = \infty$.

Proof. Check any measure theory text. ✓ □

2 Question 2 ✓

Let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence of function in $\mathscr{C}[0,1]$ given by $f_n(x) = x^n$, $x \in [0,1]$, $n \in \mathbb{N}$. Let d_1 and d_∞ be the metric induced by $\|\cdot\|_1$ and $\|\cdot\|_\infty$ as discussed in the previous question, that is, $d_1(f,g) = \|f-g\|_p$ and $d_\infty(f,g) = \|f-g\|_\infty$, $f,g \in \mathscr{C}[0,1]$. Show that the sequence $d_1(f_n, 0) \to 0$ as $n \to \infty$ but $d_\infty(f_n, 0)$ does not tends to 0 as $n \to \infty$. Show that even $\{f_n\}_{n\in\mathbb{N}}$ has no convergent subsequence in the metric space $(\mathscr{C}[0,1], d_\infty)$. Can you conclude from the above that $(\mathscr{C}[0,1], d_\infty)$ is not equivalent to $(\mathscr{C}[0,1], d_1)$? Show that $(\mathscr{C}[0,1], d_2)$ is not equivalent to $(\mathscr{C}[0,1], d_1)$?

Solution. We first show that $f_n := x^n$ converge to 0 in the 1-norm. This is easy to see:

$$\begin{aligned}\|f_n - 0\|_1 &= \int_0^1 |f_n|\, dx \\ &= \int_0^1 x^n dx \\ &= \frac{1}{n+1} \to 0 \text{ as } n \to \infty\end{aligned}$$

We now show that f_n has no convergent subsequence. For the sake of contradiction, let $\{f_{n_k}\}$ be sequence which converges uniformly (same as convergence in sup norm) to some $f \in C[0,1]$. To obtain contradiction, we make use of the following fact:

If $f_n : X \to \mathbb{C}$ is a sequence of function which converge to $f : X \to \mathbb{C}$ uniformly and $\{x_n\}$ is a sequence in X which converges to $x \in X$ then $\{f_n(x_n)\}$ converges to $f(x)$.

Now, since uniform convergence is stronger than pointwise convergence, using the above fact, we have that

$$f(1) = \lim_{n\to\infty} f_{n_k}(1) = 1$$

Also, since $(1 - 1/n)$ converges to 1, we have that

$$f(1) = \lim_{k\to\infty} f_{n_k}(1 - 1/n_k) = \left(1 - \frac{1}{n_k}\right)^{n_k} = \frac{1}{e}$$

It is obvious to see now that that the two metrics are not equivalent. □

3 Question 3 ✓

Let $(X, \|\cdot\|)$ be a normed linear space (in short NLS) and (X, d) be the associated metric, that is, $d(a,b) = \|a-b\|,\ a, b \in X$. Show that a ball $B_d(a,r)$ is always a convex subset of X.

Solution. It suffices to show that $B(0,1)$ is convex because every other ball is just this (modulo translations and dilations). We proceed to show that $B(0,1)$ is convex. Let $x, y \in B(0,1)$. Then we have that for $t \in [0,1]$,

$$\begin{aligned} \|tx + (1-t)y\| &= t\|x\| + (1-t)\|y\| \\ &< t + (1-t) \\ &= 1 \end{aligned}$$

And we're done. $\square$

4 Question 4 ✓

Let $(X, \|\cdot\|)$ be a normed linear space and Y be a proper subspace of X. Show that Y°, the set of interior point of Y, is empty.

Solution. Let Y be proper subspace of a normed linear space X. Suppose that the interior of Y was nonempty. Then there exists $y \in Y$ and $r > 0$ such that $B(y, r) \subset Y$. Since Y is a subspace, we have that $B(0, r) \subset Y$. We now show that $X \subset Y$ which would contradict that Y is a proper subspace of X.

Let $x \in X \setminus \{0\}$. Then note that $\frac{r}{2}\frac{x}{\|x\|} \in B(0, r)$ and hence $\frac{r}{2}\frac{x}{\|x\|} \in Y$. But then we have that $x \in Y$ because any Y is a subspace and hence

$$x = \frac{2\|x\|}{r}\left(\frac{r}{2}\frac{x}{\|x\|}\right) \in Y$$

And we're done. $\square$

5 Question 5 ✓

Let X be a finite-dimensional normed linear space and Y be any normed linear space. If $T : X \to Y$ is a linear transformation then show that T is continuous.

Solution. Let $\mathcal{I} : (X, \|\cdot\|) \to (\mathbb{C}^n, \|\|_2)$ be an isometric isomorphism (such a thing exists provided X is finite dimensional!). Let $T : (\mathbb{C}^n, \|\|_2) \to (Y, \|\|_Y)$ be any linear transformation. Consider the following:

$$\begin{aligned} \|Tx\|_Y &= \left\| T\left(\sum_{i=1}^{n} x_i e_i\right) \right\| \\ &= \left\| \sum_{i=1}^{n} x_i T e_i \right\| \\ &\leq \sqrt{\sum_{i=1}^{n} \|Te_i\|^2}\, \|x\|_2 \end{aligned}$$

This shows that T is continuous. Hence $T \circ \mathcal{I} : (X, \|\|_X) \to (Y, \|\|_Y)$ is continuous. □

6 Question 6 ✓

Let X be a finite-dimensional normed linear space and $E \subset X$. Show that E is compact if and only if E is closed and bounded subset of X.

Solution. Let X be n dimensional normed linear space with the norm $\|\|_1$. We can also give an inner product structure X in the following way: Fix a basis $\{v_i : i \in \{1, 2, \ldots, n\}\}$ of X and we define an inner product:

$$\left\langle \sum_{i=1}^{n} x_i v_i, \sum_{i=1}^{n} y_i v_i \right\rangle = \sum_{i=1}^{n} x_i \overline{y_i} \tag{6.0.1}$$

Since every finite dimensional inner product space is isometrically isomorphic to $\mathbb{C}^n$ with the 2-norm, we have that X with the norm induced by the inner product defined in Equation 6.0.1 is isometrically isomorphic with $\mathbb{C}^n$ with the 2-norm. Let's call this norm induced by the inner product by $\|\|_2$. Let $T : (X, \|\|_2) \to (\mathbb{C}^n, \|\|_2)$ be an isometric isomorphism.

We claim that in $(X, \|\|_2)$, E is compact iff E is closed and bounded. Since in every metric space, we have that compact subsets are closed and bounded, we need to check only one direction. To do so, let E be closed and bounded subset of X. Then $T(E)$ is closed and bounded because T is an isometric isomorphism (and hence it is an homeomorphism and preserves length). Since $T(E)$ is a closed and bounded subset of $\mathbb{C}^n$, we have that $T(E)$ is compact. Since T is an homeomorphism, $E = T^{-1}(T(E))$ is compact. This completes the proof of the claim.

Now, every norm on finite dimensional vector space is equivalent, so, the topology generated by 1-norm is the same as the topology generated by the 2-norm. So, we have that following:

$$\begin{aligned}
E \text{ is compact in } (X, \|\|_1) &\Leftrightarrow E \text{ is compact in } (X, \|\|_2) && \text{topologies are the same!} \\
&\Leftrightarrow E \text{ is closed and bounded in } (X, \|\|_2) && \text{by the above claim} \\
&\overset{(\star)}{\Leftrightarrow} E \text{ is closed and bounded in } (X, \|\|_1) && \text{topologies are the same!}
\end{aligned}$$

Observe that in $(\star)$, closed is a topological property but not boundedness. But boundness is due to definition of equivalence of norms. This completes the proof. □

7 Question 7 ✓

Let X be a vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on X. Let τ_1 and τ_2 be the topology (that is the collection of all open set) associated to the normed spaces $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ respectively. Show that $\|\cdot\|_1$ and $\|\cdot\|_2$ are two equivalent norms on X if and only if $\tau_1 = \tau_2$.

Solution. Let $\|\|_1$ and $\|\|_2$ be two norms on X and let τ_1 and τ_2 be the respective topologies generated by $\|\|_1$ and $\|\|_2$.

($\Longrightarrow$) Suppose that $\|\|_1$ and $\|\|_2$ be equivalent. To show that τ_1 and τ_2 are equivalent, it suffices to show that the basis of the topologies are equivalent, that every basis element of one topology contains a basis element of the other. But this follows immediately from the definition of the equivalent norms.

($\Longleftarrow$) Lazy to write this down :($\square$

8 Question 8 ✓

Let X be a normed linear space and $F : X \to \mathbb{C}$ be a non zero linear functional. Suppose $F(x_0) \neq 0$ for some $x_0 \in X$. Show that $X = \ker F \oplus \operatorname{span}\{x_0\}$, that is,

(i) $\ker F \cap \operatorname{span}\{x_0\} = \{0\}$.

(ii) $X = \ker F + \operatorname{span}\{x_0\}$.

Show that F is continuous if and only if $\ker F$ is a closed subspace in X. (Hint : Use the continuity of the projection map $\pi : X \to X/\ker F$ defined by $\pi(x) = [x]$, $x \in X$.)

Proof. Let X be a normed linear space and $F : X \to \mathbb{C}$ be nonzero linear functional. Since F is nonzero, there must be some $x_0 \in X$ such that $F(x_0) \neq 0$. We now proceed to show that $X = \ker F \oplus \operatorname{span}\{x_0\}$.

We first show that $X = \ker F + \operatorname{span}\{x_0\}$. Let $x \in X$. Then $F(x) \in \mathbb{C}$. Since $F(x_0) \neq 0$. There must be some $\lambda \in \mathbb{C}$ such that $F(x) = \lambda F(x_0)$. Thus, we have that $F(x - \lambda x_0) = 0$. Thus, $x - \lambda x_0 \in \ker F$. Hence, $x = \lambda x_0 + y$ for some $y \in \ker F$. This shows that $X = \ker F + \operatorname{span}\{x_0\}$.

Now, we proceed to show that $\ker F \cap \operatorname{span}\{x_0\} = \{0\}$. To do so, let $y \in \ker F \cap \operatorname{span}\{x_0\}$. Then we have thath $y = \lambda x_0$ for some $\lambda \in \mathbb{C}$. Hence, we have that $F(y) = \lambda F(x_0) = 0$. Since $F(x_0) \neq 0$, we have that $\lambda = 0$ and thus, $y = 0$. This completes the proof of the claim.

The above two paragraphs show that $X = \ker F \oplus \operatorname{span}\{x_0\}$.

Now, we proceed to show that F is continuous iff $\ker F$ is a closed subspace of X. Let's begin the proof in the ($\Rightarrow$) direction. Suppose that F is continuous. Then we have that $\ker F = F^{-1}(\{0\})$ and hence it must be closed.

To show the reverse direction, namely ($\Leftarrow$), we first show that the projection map is continuous. First, we observe that for any $x \in X$, we have that

$$\begin{aligned} \|[x]\| &= \inf_{y \in \ker F} \|x - y\| && \text{by definition} \\ &\leq \|x\| && 0 \in \ker F \end{aligned}$$

Now, this shows that the projection map $\pi : X \to X/\ker F$ is bounded and since it is a linear map, it is continuous.

Now, consider the map $\tilde{T} : X/\ker F \to \mathbb{C}$ given by

$$[x] \overset{\tilde{T}}{\mapsto} F(x)$$

We showed that $X = \ker F \oplus \operatorname{span}\{x_0\}$. By the first isomorphism theorem for vector spaces, we have that $X/\ker F \cong \operatorname{span}\{x_0\}$. This shows that $X/\ker F$ is finite dimensional. Since $\tilde{T}$ is linear and $X/\ker F$ is finite dimensional, we have thath $\tilde{T}$ is continuous.

Observe that $T = \tilde{T} \circ \pi : X \to \mathbb{C}$ is continuous linear functional by virtue of being composition of two continuous linear maps. This completes the proof. $\square$

9 Question 9 ✓

Let X be a normed linear space and $S = \{x \in X : \|x\| = 1\}$ be the unit sphere in X. Show that X is complete if and only if S is complete.

Solution. ($\Rightarrow$) Suppose that (V, d) is complete. Then S is complete because S is closed in V.

($\Leftarrow$) Suppose that (S, d) is complete. Let $\{x_n\}$ be a Cauchy sequence in V.

We consider two different cases, namely.

(i) $x_n = 0$ for infinitely many $n \in \mathbb{N}$ and

(ii) $x_n = 0$ for at most finitely many $n \in \mathbb{N}$.

We conisder the first case. Suppose that $x_n = 0$ for infinitely many $n \in \mathbb{N}$. We, therefore, can select a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} = 0$ for every $k \in \mathbb{N}$. We now show that $\{x_n\}$ converges to 0.

Let $\varepsilon > 0$ be given. Then there is some $N \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \varepsilon \text{ whenever } n, m \geq N.$$

Select $k \in \mathbb{N}$ such that $n_k \geq N$. Now, we have that

$$\begin{aligned} \|x_n\| &= \|x_n - x_{n_k}\| \\ &< \varepsilon \end{aligned}$$

This shows that $\{x_n\}$ converges to 0.

We now consider the second case. Suppose that $x_n = 0$ for at most finitely many $n \in \mathbb{N}$. Therefore, there is some $N \in \mathbb{N}$ such that $x_n \neq 0$ for $n \geq N$. Convergence of sequence depends only on its tail, so, we assume without loss of generality, that $x_n \neq 0$ for every $n \in \mathbb{N}$.

Now, consider the sequence $\{y_n\}$ in S given by

$$y_n = \frac{x_n}{\|x_n\|}$$

for each $n \in \mathbb{N}$.

We now claim that $\|x_n\|$ converges to some $\lambda \geq 0$. Since $\{x_n\}$ is Cauchy, so, is $\{\|x_n\|\}$. But Cauchy in $\mathbb{R}$ implies convergence, and, hence $\{\|x_n\|\}$ converges to some $\lambda \geq 0$. If $\|x_n\|$ converges to 0 then we have that $\{x_n\}$ converges to 0. And we would be done. So, suppose that $\lambda > 0$. So, by the definition of convergence, there must be some $K \in \mathbb{N}$ such that $\|x_n\| > \frac{\lambda}{2}$ for every $n \geq K$.

Consider the following for $n, m \in \mathbb{N}$,

$$\begin{aligned}
\|y_m - y_n\| &= \left\| \frac{x_m}{\|x_m\|} - \frac{x_n}{\|x_m\|} \right\| \\
&= \left\| \frac{\|x_n\| x_m - \|x_m\| x_n}{\|x_m\| \|x_n\|} \right\| \\
&= \left\| \frac{\|x_n\| x_m - x_m \|x_m\| + \|x_m\| x_m - \|x_m\| x_n}{x_n \|x_m \|x_n\|\|} \right\| \\
&= \left\| \frac{x_m (\|x_n\| - \|x_m\|) + \|x_m\| (x_m - x_n)}{\|x_m\| \|x_n\|} \right\| \\
&\leq \frac{1}{\|x_n\| \|x_m\|} (\|x_m\| |\|x_n\| - \|x_m\|| + \|x_m\| \|x_m - x_n\|) \\
&\leq \frac{2}{\|x_n\|} \|x_n - x_m\|
\end{aligned}$$

Now, let $\varepsilon > 0$ be given. Since $\{x_n\}$ is Cauchy, we have that there is some $M \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \frac{\lambda}{4}\varepsilon \text{ whenever } n, m \geq M.$$

For $m, n \geq \max\{M, K\}$ we have that

$$\begin{aligned}
\|y_m - y_n\| &\leq \frac{2}{\|x_n\|} \|x_n - x_m\| \\
&< \frac{2}{\lambda/2} \frac{\lambda}{4}\varepsilon = \varepsilon
\end{aligned}$$

This shows that $\{y_n\}$ is Cauchy and hence convergent. Since $x_n = y_n \|x_n\|$ for all $n \in \mathbb{N}$ and product of two convergent sequences is convergent. We have that $\{x_n\}$ is convergenet. $\square$

10 Question 10 ✓

Let X be a normed linear space and $F : X \to \mathbb{C}$ be a continuous, non zero linear functional.

(a) Show that $N(F) = \sup\left\{\frac{|F(x)|}{\|x\|} : x \in X, x \neq 0\right\} < \infty$.

(b) Suppose $M = \ker F$ and $x_0 \notin M$. Show that $d(x_0, M) = \frac{|F(x_0)|}{N(F)}$.

Solution. Since F is continuous, we have that there must be some $M > 0$ such that

$$|F(x)| \leq M \|x\| \text{ for all } x \in X.$$

Thus, we have that

$$\frac{|F(x)|}{\|x\|} \leq M \text{ for all } x \in X.$$

This shows that $N(F) < \infty$. [1]

Now, let $M = \ker F$. Observe that by definition of quotient norm, we have that

$$\|[x]\|_{X/M} = \inf_{m \in M} \|x - m\|_M = d(x, M)$$

for every $x \in X$. Therefore, we need to show that $|F(x_0)| = N(F)\,\|[x_0]\|_{X/M}$. Let $x \in M$ be arbitrary. Then by Question 8, we have that $x = y + \lambda x_0$ for some $y \in \ker F$ and some $\lambda \in \mathbb{C}$. In case, $\lambda = 0$, we have that

$$\begin{aligned}\frac{|F(x)|}{\|x\|} &= \frac{F(y)}{\|y\|} \\ &= 0 \leq \frac{|F(x_0)|}{\|[x_0]\|_{X/M}}\end{aligned}$$

Else if $\lambda \neq 0$, we have that

$$\begin{aligned}\frac{|F(x)|}{\|x\|} &= \frac{|F(y + \lambda x_0)|}{\|\lambda x_0 + y\|} \\ &= \frac{|F(x_0)|}{\left\|x_0 + \frac{1}{\lambda} y\right\|} \leq \frac{|F(x_0)|}{\|[x_0]\|_{X/M}}\end{aligned}$$

Since $x \in X$ was arbitrary, taking supremum, we have

$$N(F) \leq \frac{|F(x_0)|}{\|x_0\|_{X/M}} \rightsquigarrow N(F)\,\|x_0\|_{X/M} \leq |F(x_0)|\,. \tag{10.0.1}$$

[1] One can easily check that $N(F)$ is $\|F\|_{X^*}$ in disguise!

To prove the reverse inequality, observe that if $y \in M$ then

$$\frac{|F(x_0)|}{\|x_0 - m\|} \leq N(F) \rightsquigarrow \frac{|F(x_0)|}{N(F)} \leq \|x_0 - m\|.$$

Since m is arbitrary, we have that

$$\frac{|F(x_0)|}{N(F)} \leq \|[x_0]\|_{X/M} \rightsquigarrow |F(x_0)| \leq N(F) \|[x_0]\|_{X/M}. \tag{10.0.2}$$

Combining 10.0.1 and 10.0.2, we have what we wanted. $\square$

11 Question 11 ✓

Let X be a finite dimensional normed linear space and M be a proper closed subspace of X. Show that the unit sphere $S := \{x : \|x\| = 1\}$ on X is compact. Use this to show that there exist a unit vector x such that $\operatorname{dist}(x, M) = 1$. This need not to be true if X is infinite dimensional. Show that the choice

$$X = \{f \in C[0,1] : f(0) = 0\}$$
$$M = \{f \in X : \int_0^1 f = 0\}$$

provides a counter example. (This also shows that in F. Riesz's Lemma the constant t can not be taken to be equal to 1 in general.)

Proof. Note that S is compact in view of Question 6. Consider the map $f : S \to \mathbb{C}$ given by

$$x \overset{f}{\mapsto} d\left(x, M\right)$$

Since we have $|d\left(x, M\right) - d\left(y, M\right)| \leq d\left(x, y\right)$, f is continuous. We now show that $\sup f\left(S\right) = 1$. Let $x \in S$. Then we have that

$$\begin{aligned} d(x, M) &= \inf_{y \in M} \|x - y\| \\ &\leq \|x\| = 1. \end{aligned}$$

Thus, $\sup f\left(S\right) \leq 1$. Now, let $\varepsilon > 0$ be arbitrary. Then by F. Riesz lemma, there exists a vector $x_0 \in M$ such that

$$\begin{aligned} 1 - \varepsilon < \|y - x_0\| \text{ for all } y \in M \rightsquigarrow 1 - \varepsilon &\leq \inf_{y \in M} \|y - x_0\| = d(x_0, M) \\ \rightsquigarrow 1 &\leq \sup f(S) + \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have that $\sup f\left(S\right) \geq 1$. This shows that $\sup f\left(S\right) = 1$.

Now, continuous functions on compact sets achieve their supremum, therefore, there must be some vector $x \in S$ such that $f(x) = d(x, M) = 1$.

Now, we move on to the next part of the question. Consider the following sets:

$$X = \{f \in C[0,1] : f(0) = 0\}$$
$$M = \left\{f \in X : \int_0^1 f = 0\right\}$$

We need to show that there is no vector $f \in X$ whose $\|f\|_\infty = 1$ but $d(f, M) = 1$. Assume the contrary and suppose that such an f exists.

Consider the linear map $F : X \to \mathbb{R}$ given by

$$T(g) = \int_0^1 g$$

It is evident that $\ker F = M$. Hence, by Question 10, we have that

$$|T(g)| = d\,(g, M)\,\|T\|$$

holds for every $g \in X$. (In case, $g \in M$ then both sides are zero!)

Hence, we have, in particular, that

$$|T(f)| = \|T\|$$

Now, we claim that $\|T\| = 1$. To prove this, let $f \in X$ with $\|f\|_\infty \leq 1$. Then we have that

$$\begin{aligned} |T\,(f)| &= \left| \int_0^1 f \right| \\ &\leq \int_0^1 |f| \\ &\leq \|f\|_\infty \leq 1. \end{aligned}$$

This shows that $\|T\| \leq 1$. To show the reverse inequality, let $\varepsilon > 0$. It is not hard to draw a "trapezoid like" function whose area is larger than $1 - \varepsilon$. This shows that $\|T\| = 1$.

Thus, we have that $|T(f)| = \left| \int_0^1 f \right| = 1 = \|f\|_\infty$. Consider the following:

$$1 = \left| \int_0^1 f \right| \leq \int_0^1 |f| \leq \|f\|_\infty = 1.$$

This shows that $\int_0^1 |f| = 1$ and $\|f\|_\infty = 1$. Now, we show that $|f| \equiv 1$ on $[0, 1]$. If we do so, we will be done because then f cannot be in X as $|f\,(0)| = 1 \neq 0$.

To show that $|f| \equiv 1$ on [0,1], we define a function $g : [0, 1]$ given by $g = \|f\|_\infty - |f| = 1 - f$. Observe that g is nonnegative and $\int g = 0$ and hence $g \equiv 0 \rightsquigarrow |f| \equiv 1$ on $[0, 1]$.

This completes the proof. $\square$