Problems & Solutions in Functional Analysis

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§1 Question 15

Consider the map $\varphi : V \to \mathbb{C}^n$ given by

$$\varphi(v) = \begin{bmatrix} \langle v, b_1 \rangle \\ \vdots \\ \langle v, b_n \rangle \end{bmatrix}$$

for all $v \in V$. We show that this map φ is injective. Our proof will be then complete by the rank nullity theorem.

So, let $v \in V$ and suppose that $\varphi(v) = 0$. Then $\langle v, b_i \rangle = 0$ for all $i = 1, 2, \ldots, n$. Since $b_1, \ldots, b_n$ is a basis for V, there exists $\alpha_1, \ldots, \alpha_n$ such that

$$v = \alpha_1 b_1 + \ldots + \alpha_n b_n$$

Hence, we have that

$$\begin{aligned} \langle v, v \rangle &= \langle v, \alpha_1 b_1 + \ldots + \alpha_n b_n \rangle \\ &= \sum_{i=1}^{n} \overline{\alpha_i} \langle v, b_i \rangle \\ &= 0 \end{aligned}$$

Hence $v = 0$. This completes the proof!