Lecture Notes in Differential Geoemtry

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Last Updated: January 8, 2023

Introduction

This is a set of lecture notes which I took for reviewing stuff that I typed after taking class from *Dr. Saikat Chatterjee*. All the typos and errors are of mine.

Contents

§1 Lecture 1 — 3rd January — Baby Stuff

§1.1 References for the Course

- De Carmo – Curves and Surfaces
- Tu – Introduction to Smooth Manifolds
- Lee – Introduction to Smooth Manifolds

§1.2 Geometry of curves in three dimensions

Definition §1.2.1 (smooth)**.** A real function of real variable is *smooth* if it has, at all points, derivatives of all orders.

Definition §1.2.2 (parameterized curve)**.** Let $I \subset \mathbb{R}$ be an open interval. A *parameterized curve* in $\mathbb{R}^n$ is a smooth map $\gamma : I \to \mathbb{R}^n$.

Example §1.2.3. Some of examples of curves are:

1. $\gamma : \mathbb{R} \to \mathbb{R}^2$, $t \overset{\gamma}{\mapsto} (\cos t, \sin t)$
2. $\gamma : \mathbb{R} \to \mathbb{R}^2$, $t \overset{\gamma}{\mapsto} (t, mt)$ for any real number m.

§1.3 Brief Review of Inverse Function Theorem & Reparameterization of Curves

Let us recall what the Inverse Function Theorem[1] says:

Theorem §1.3.1. *Suppose f is $1-1$ and continuous on an open interval I. If f is differentiable at a point $x_0 \in I$ and $f'(x_0) \neq 0$, then f^{-1} is differentiable at $f(x_0)$, and*

$$\left(f^{-1}\right)\left(f(x_0)\right) = \frac{1}{f'(x_0)}$$

It follows from this theorem §1.3.1 that:

Theorem §1.3.2 (Inverse Function Theorem in 1-D)**.** *Let U be an open subset of $\mathbb{R}$, $\varphi : U \to \mathbb{R}$ be a smooth map and u be some point in U such that $\varphi'(u) \neq 0$ for some $u \in U$. Then there is an open nbhd $V \subset U$ containing u such that $\varphi|_V : V \to \varphi(V)$ is a diffeomorphism.*

Proof (sketch). Let u be some point in U such that $\varphi'(u) \neq 0$. Since φ is smooth, we have that φ' is continuous. Hence by the continuity of φ', we have that there is an open interval $V \subset U$ such that $\varphi'(y) \neq 0$ for all $y \in V$. Applying the meean value theorem on $\varphi|_V$, we have that $\varphi|_V$ is $1-1$ and the theorem follows from the previous theorem. □

[1] See Denlinger, Elements of Real Analysis: Theorem 6.2.4

Example §1.3.3 (Inverse function theorem does not imply that the function is a diffeomorphism!)**.** Let $U = \mathbb{R} \setminus \{0\}$. Consider the map $\varphi : U \to \mathbb{R}$, $x \overset{\varphi}{\mapsto} x^2$. Observe that $\varphi'(u) \neq 0$ for all $u \in U$ but φ is not a diffeomorphism since φ is not injective.

Observation §1.3.4. *If $\varphi : I \to \mathbb{R}$ is a $\mathscr{C}^1$ function such that $\varphi'(u) \neq 0$ for any $u \in I$ then $\varphi : I \to \varphi(I)$ is a diffeomorphism and $\varphi'(u) \neq 0$ for any $u \in I$. Consequently, if φ is smooth then $\varphi^{(k)}$ is nonzero on I for any $k \in \mathbb{Z}_{\geq 0}$.*

Proof. It follows immediately from mean value theorem that φ is injective and it follows from Theorem §1.3.1 that it is differentiable and cannot be zero anywhere. □

Remark §1.3.5. Consider I to be an open interval and $\varphi : I \to \mathbb{R}$ be a smooth map such that $\varphi'(u) \neq 0$ for all $u \in I$. Hence, it follows that φ' is injective. Thus, $\varphi : I \to \varphi(I)$ is a diffeomorphism from Theorem §1.3.1.

Definition §1.3.6 (Reparametrization)**.** Let $\gamma : I \to \mathbb{R}^n$ be a smooth curve and $\varphi : J \to I$ be a diffeomorphism where J is an open interval. Define $\beta = \gamma \circ \varphi : J \to \mathbb{R}^n$. Then β is a smooth curve (by the Chain Rule) and β is called the *reparametrization* of γ. Note that since $\gamma = \beta \circ \varphi^{-1}$, we call β and γ *reparameterizations of each other.*

The proof of the following proposition is so easy that it is skipped:

Proposition §1.3.7. *If β and γ are reparameterization of each other then $im\left(\beta\right) = im\left(\gamma\right)$.*

Definition §1.3.8 (regular curve)**.** Let $\gamma : I \to \mathbb{R}^n$ be a smooth curve. Then $\gamma'(t_0)$ is called the *tangent* of γ at $t_0 \in I$. If $\gamma'(t) \neq 0$ for every $t \in I$, we say that γ is *regular.*

Now, suppose that $\gamma : I \to \mathbb{R}^n$ be a regular curve. We want to a find $\beta : J \to \mathbb{R}^n$, reparameterization of γ such that $\|\beta'(t)\| = 1$ for all $t \in I$. To achieve this, we define…

§1.4 Arc Length Parameterization

Definition §1.4.1. Let $\gamma : I \to \mathbb{R}^n$ be a regular curve, the *arc length* between t_1 and t_2 in I is

$$L_\gamma(t_1, t_2) = \int_{t_1}^{t_2} \left\|\gamma'(t)\right\| \mathrm{dt}$$

Let us fix $t_0 \in I$. Define $L_\gamma : I \to \mathbb{R}$ by $L_\gamma(t) = \int_{t_0}^{t} \left\|\gamma'(x)\right\| \mathrm{dx}$ for every $t \in I$.

Now observe that $L_\gamma(t) = \left\|\gamma^{'}(t)\right\|$ for every $t \in I$. Since γ is regular, we have that L'_γ is nonzero in I (by Observation §1.3.4). Hence L_γ is smooth (why?).

Hence, $L_\gamma : I \to L_\gamma(I)$ is a diffeomorphism. Hence, $L_\gamma^{-1} : J \to I$ is smooth where $J := L_\gamma(I)$. Now, $\beta = \gamma \circ L_\gamma^{-1} : J \to \mathbb{R}^n$ is a reparametrization of γ. Let $S_\gamma = L_\gamma^{-1}$. Thus for all $s \in J$,

$$\beta^{'}(s) = \gamma'\left(S_\gamma(s)\right) \cdot S'_\gamma(s)$$

Hence if $s \in S$ then $L_\gamma(t) = s$ for some $t \in I$ and hence,

$$\begin{aligned}
S'_\gamma(s) &= S'_\gamma\left(L_\gamma(t)\right) \\
&= \frac{1}{L'_\gamma(t)} && \text{(by Theorem §1.3.1)} \\
&= \frac{1}{\left\|\gamma'(t)\right\|} \\
&= \frac{1}{\gamma'\left(S_\gamma(s)\right)}
\end{aligned}$$

Hence, we have that

$$\beta'(s) = \frac{\gamma'\left(S_\gamma(s)\right)}{\left\|\gamma'\left(S_\gamma(s)\right)\right\|}$$

and

$$\left\|\beta'(s)\right\| = 1$$

for all $s \in J$.

This proves the following theorem:

Theorem §1.4.2. *Let γ be a regular curve then there is a parameterization $S_\gamma : J \to I$ scuch that*

$$\left\|\beta'(s)\right\| = 1$$

for all $s \in J$ where $\beta = \gamma \circ S_\gamma$.

Definition §1.4.3. The parameterization in Theorem §1.4.2 is called *arc length parameterization.*

Now, with the aforemention definition and theorem, we can assume that all regular curves are *unit speed parametrization.*

Let $\gamma : I \to \mathbb{R}^3$ be a regular curve, $\gamma'(t) \neq 0$ for each $t \in I$. Sinceγ is a unit speed paramterization, we have that $\gamma'(t) \cdot \gamma'(t) = 0$ for each $t \in I$. By differentiating we have that $\gamma'(t) \cdot \gamma''(t) = 0$ for each $t \in I$.

Hence, $\gamma''(t)$ is perpendicular to $\gamma'(t)$ for each $t \in I$. This begs us to make the following definition:

Definition §1.4.4 (Normal Vector at a point t)**.** Let γ be as in the previous paragraph. The unit vector $\hat{\eta}(t)$ be the unit vector in the direction of $\gamma''(t)$. We call $\hat{\eta}(t)$ is called the *normal vector* at t.

Proposition §1.4.5. *The norm function on* $\mathbb{R}^n$ *is smooth.*

Proof. We first show that $f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ given by $f(x) = \langle x, x \rangle$ is smooth. This is easy to see: the projection functions are smooth. Now, note that the result follows immediately from Theorem 1.3.1 of Differential Geometry of Manifolds by Lovett. Now the square root function $\sqrt{\cdot} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is smooth. Since composition of smooth functions is smooth, we have that f is smooth. $\square$

Now, let γ be as in previous paragraph. Then there exist a map $K_\gamma : I \to \mathbb{R}$ such that $\gamma''(t) = K_\gamma(t)\hat{\eta}(t)$ for each $t \in I$. We call this function K_γ as the curvature function of γ. Observe that $|K_\gamma(t)| = \left\| \gamma''(t) \right\|$ is smooth by Proposition §1.4.5.

Observe that if K_γ is the zero function, then $\gamma''(t) = 0$ for all $t \in I$ and hence by the Mean value theorem, γ must be straight line.

Now, let $\gamma'(t) = \hat{t}(t)$. The plane defined by $\left(\hat{t}(t), \hat{\eta}(t)\right)$ is called the *oscillating plane* of t.

Let $\hat{b}(t) = \hat{t}(t) \times \hat{\eta}(t)$. Then

$$\begin{aligned}
\hat{b}'(t) &= \hat{t}'(t) \times \hat{\eta}(t) + \hat{t}(t) \times \hat{\eta}'(t) \\
&= \gamma''(t) \times \hat{\eta}(t) + \hat{t}(t) \times \hat{\eta}'(t) \\
&= 0 + \hat{t}(t) \times \hat{\eta}'(t)
\end{aligned}$$

Thus, $\hat{b}'(t)$ is perpendicular to $\hat{t}(t)$. Hence, $\hat{b}'(t)$ is perpendicular to $\hat{b}(t)$ since $\hat{b}(t) \cdot \hat{b}(t) = 1$. Thus, $\hat{b}'(t) = \tau(t)\hat{\eta}(t)$ for some $\tau(t) \in \mathbb{R}$. Thus, we have a smooth map $\tau : I \to \mathbb{R}$.