Problems & Solutions in Functional Analysis

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§1 Question 1

Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space and x, y be two non zero vector in V. Show that $\|x+y\| = \|x\| + \|y\|$ holds if and only if $x = cy$ for some scalar $c > 0$.

Solution. Let x, y be two nonzero vectors in an inner product space V. Consider the following equivalences:

$$\begin{aligned} \|x+y\| = \|x\| + \|y\| &\iff \|x+y\|^2 = (\|x\| + \|y\|)^2 \\ &\iff \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \\ &\iff \Re\langle x, y\rangle = \|x\|\|y\| \end{aligned}$$

Suppose that $\|x+y\| = \|x\| + \|y\|$ holds. Then we have from the above equivalence that $\Re\langle x, y\rangle = \|x\|\|y\|$. Since $\|x\|\|y\| \leq \Re\langle x, y\rangle \leq |\langle x, y\rangle| \leq \|x\|\|y\|$, we have that $\langle x, y\rangle = \|x\|\|y\|$. Since the equality in Cauchy Schwarz inequaliy holds iff x and y are linearly dependent, we must have that $x = cy$ for some $c \in \mathbb{C}$. Thus, we must have that

$$\begin{aligned} \Re\langle cy, y\rangle = \|cy\|\|y\| &\iff \Re c\langle y, y\rangle = |c|\|y\|\|y\| \\ &\iff \langle y, y\rangle \Re c = |c|\|y\|^2 \\ &\iff \Re c = |c| \\ &\iff c > 0 \end{aligned}$$

The argument is reversible and the proof is complete! $\square$

§2 Question 3

Fix a $n \times n$ strictly positive definite matrix $A = \big(a_{i,j}\big)$. Consider $(\mathbb{C}^n, \langle \cdot, \cdot \rangle)$ where

$$\langle x, y\rangle = \sum_{i,j=1}^{n} a_{i,j} x_j \bar{y}_i = \langle Ax, y\rangle_2,\ \ x, y \in \mathbb{C}^n.$$

Prove that $(\mathbb{C}^n, \langle \cdot, \cdot \rangle)$ is indeed an inner product space. Conversely show that if $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{C}^n$, then there exist a $n \times n$ strictly positive definite matrix $A = \big(a_{i,j}\big)$ such that

$$\langle x, y\rangle = \sum_{i,j=1}^{n} a_{i,j} x_j \bar{y}_i = \langle Ax, y\rangle_2,\ \ x, y \in \mathbb{C}^n.$$

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§3 Question 4

Let M be a subspace of an inner product space $(V, \langle \cdot, \cdot \rangle)$. Show that $\overline{M}$, the closure of M, in V is also a subspace. Moreover show that $M^{\perp} = \overline{M}^{\perp}$.

Solution. We make the following claims:

Claim §3.0.1 (orthogonal complement of a set and the orthogonal completement of its closure are same!)**.** *Let M be a subset of a inner product space H. Then $M^{\perp} = \left(\overline{M}\right)^{\perp}$*

Proof. It follows by definition that $M \subset \overline{M}$ and hence $\left(\overline{M}\right)^{\perp} \subset M^{\perp}$. Now for reverse the inclusion, let $v \in M^{\perp}$ and let $y \in \overline{M}$. We need to show that $\langle v, y \rangle = 0$. Since $y \in \overline{M}$ there is a sequence (y_n) in M such that $y_n \to y$. Since $v \in M^{\perp}$, we have that $\langle v, y_n \rangle = 0$ for all $n \in \mathbb{N}$. Since $\langle v, y_n \rangle \to \langle v, y \rangle$, we have by uniqueness of limits that $\langle v, y \rangle = 0$. This completes the proof. □

Claim §3.0.2 (orthogonal complement of orthogonal complement)**.** *Let M be a closed subspace of the Hilbert space H. Then*

$$M = \left(M^{\perp}\right)^{\perp}$$

Proof of Claim. Let us first show that $M \subset \left(M^{\perp}\right)^{\perp}$ (which in fact holds for any set M). Let $v \in M$ and $w \in M^{\perp}$. It is clear by definition of $M^{\perp}$ that $\langle v, w \rangle = 0$. Hence, $v \in \left(M^{\perp}\right)^{\perp}$.

Let us proceed to show the inclusion in the other direction. Let $v \in \left(M^{\perp}\right)^{\perp}$. Since M is closed, by Projection Theorem, we have that $v = Pv + Qv$ where $Pv \in M$ and $Qv \in M^{\perp}$. By the previous paragraph, we have that $M \subset \left(M^{\perp}\right)^{\perp}$ and hence $Pv \in \left(M^{\perp}\right)^{\perp}$. Hence, we have that $Qv \in \left(M^{\perp}\right)^{\perp}$. Now, $Qv \in M^{\perp} \cap \left(M^{\perp}\right)^{\perp}$. Hence, $Qx = 0$ and thus, $v = Pv \in M$. □

Now, we start the proof. Let M be subspace of V. Consider the following:

$$\begin{aligned}
\left(M^{\perp}\right)^{\perp} &= \left(\left(\overline{M}\right)^{\perp}\right)^{\perp} && \text{by Claim 1} \\
&= \overline{\overline{M}} && \text{by Claim 2} \\
&= \overline{M}
\end{aligned}$$

□

§4 Question 6

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Let $\overline{B(0,1)}$ denotes the closed unit ball in V, that is,

$$\overline{B(0,1)} = \{x \in V : \|x\| \leqslant 1\}.$$

Show that $\overline{B(0,1)}$ is compact if and only if dimension of V is finite.
(Hint : if $\mathcal{B} = \{u_\alpha : \alpha \in I\}$ is a collection of orthonormal vectors in V, then $\|u_\alpha - u_\beta\| = \sqrt{2}$ for every $\alpha, \beta \in I$ and $\alpha \neq \beta$.)

Solution. ($\Longleftarrow$) Suppose that $\dim V$ is finite. In Lecture 5, we showed that V is isometrically isomorphic to $\mathbb{C}^n$ with Euclidean norm, that is, there exists a linear map $T : V \to \mathbb{C}^n$ which is an isometry. Since every isometry is an homeomorphism[1], we have that $T^{-1}\left(\overline{B_{\mathbb{C}^n}(0,1)}\right) = \overline{B_V(0,1)}$. Since a continuous image of a compact set is compact, we have that $\overline{B_V(0,1)}$ is compact!

($\Longrightarrow$) (Contrapositive proof) Suppose that $\dim V$ is infinite. Then by Zorn's Lemma, it has a maximal orthonormal set $\{u_\alpha : \alpha \in I\}$. Since V is an inner product space, we have that $\|u_\alpha - u_\beta\| = \sqrt{2}$ for every $\alpha, \beta \in I$ with $\alpha \neq \beta$. Let $\{e_i : i \in \mathbb{N}\}$ be any countable subset of $\{u_\alpha : \alpha \in I\}$. Now, $\{e_i\}_{i \in \mathbb{N}}$ is a sequence in $\overline{B(0,1)}$ but it cannot possibly have a convergent subsequence, hence, $\overline{B(0,1)}$ is not compact. □

[1] proof here!

§5 Question 15

Consider the map $\varphi : V \to \mathbb{C}^n$ given by

$$\varphi(v) = \begin{bmatrix} \langle v, b_1 \rangle \\ \vdots \\ \langle v, b_n \rangle \end{bmatrix}$$

for all $v \in V$. We show that this map φ is injective. Our proof will be then complete by the rank nullity theorem.

So, let $v \in V$ and suppose that $\varphi(v) = 0$. Then $\langle v, b_i \rangle = 0$ for all $i = 1, 2, \ldots, n$. Since $b_1, \ldots, b_n$ is a basis for V, there exists $\alpha_1, \ldots, \alpha_n$ such that

$$v = \alpha_1 b_1 + \ldots + \alpha_n b_n$$

Hence, we have that

$$\begin{aligned} \langle v, v \rangle &= \langle v, \alpha_1 b_1 + \ldots + \alpha_n b_n \rangle \\ &= \sum_{i=1}^{n} \overline{\alpha_i} \langle v, b_i \rangle \\ &= 0 \end{aligned}$$

Hence $v = 0$. This completes the proof!

§6 Question 17

Assume $(V, \|\|)$ is a real normed linear space which satisfies the parallelogram identity, that is, for all $a, b \in V$,

$$\|a+b\|^2 + \|a-b\|^2 = 2\|a\|^2 + 2\|b\|^2$$

We intend to define the inner product on V by

$$\langle v, w\rangle = \frac{\|x+y\|^2 - \|x-y\|^2}{4}$$

We show that $\langle \cdot, \cdot \rangle$ is an inner product.
The symmetric property is evident.
We proceed to show linearity in the first variable, that is, we need to show that

$$\|x_1 + x_2 + y\|^2 - \|x_1 + x_2 - y\|^2 = \|x_1 + y\|^2 - \|x_1 - y\|^2 + \|x_2 + y\|^2 - \|x_2 - y\|^2$$

Setting $a = x_1$ and $b = x_2 + y$ in the parallelogram identity, we get

$$\|x_1 + y + x_2\|^2 + \|x_1 - y - x_2\|^2 = 2\|x_1\|^2 + 2\|x_2 + y\|^2$$

Doing the same for $a = x_2 - y$ and $b = x_2$, we have

$$\|x_1 - y + x_2\|^2 + \|x_1 - y - x_2\|^2 = 2\|x_1 - y\|^2 + 2\|x_2\|^2$$

Subtracting the above two equations, we get

$$\|x_1 + x_2 + y\|^2 - \|x_1 - y + x_2\|^2 = 2\|x_1\|^2 + 2\|x_2 + y\|^2 - 2\|x_1 - y\|^2 - 2\|x_2\|^2$$

Switching the roles of x_2 and x_1, we get

$$\|x_2 + x_1 + y\|^2 - \|x_2 - y + x_1\|^2 = 2\|x_2\|^2 + 2\|x_1 + y\|^2 - 2\|x_2 - y\|^2 - 2\|x_1\|^2$$

Adding the above two equations, we get

$$2\|x_1 + x_2 + y\|^2 - 2\|x_1 + x_2 - y\|^2 = 2\|x_2 + y\|^2 - 2\|x_1 - y\|^2 + 2\|x_1 + y\|^2 - 2\|x_2 - y\|^2$$

Rearranging the above equation, we observe that we have established what we wanted to prove!

Linearity is the other variable follows by symmetry and the linearity in the first variable!
Now, finally we proceed to show that for any $\lambda \in \mathbb{R}$, we have that

$$\langle \lambda x, y\rangle = \lambda \langle x, y\rangle$$

Note that by linearity in the first variable, we have that for $n \in \mathbb{Z}$,

$$\langle nx, y\rangle = n\langle x, y\rangle$$

In a similar fashion, it can be shown that for $r \in \mathbb{Q}$,

$$\langle rx, y\rangle = r\langle x, y\rangle$$

Let us assume Cauchy-Schwarz! at the moment. Let $r \in \mathbb{R}$. Let r_n be a sequence of rationals converging to $r \in \mathbb{R}$.

Observe that fixing $y \in V$, it is easily seen that

$$x \mapsto \frac{\|x+y\|^2 - \|x-y\|^2}{4}$$

is continuous by virtue of translation, norm and square of a function being continuous! Then the result follows!

Irregardless, we prove Cauchy Schwarz! It can be seen by minimizing r is the function $r \mapsto \|rx+y\|^2$. One needs to see that for $r \in \mathbb{Q}$

$$\|rx+y\|^2 = \langle rx+y, rx+y\rangle = r^2\|x\|^2 + 2r\langle x, y\rangle + \|y\|^2 \geq 0$$

Hence the above holds for any $r \in \mathbb{R}$ by taking limits. Minimizing the function, we get the Cauchy Schwarz inequality.

We now proceed to the complex case!

Let $V, \|\|$ be a complex normed linear space. By the polarization identity, we have that for $x, y \in V$,

$$\begin{aligned}
\langle x, y\rangle &= \frac{1}{4}\sum_{k=1}^{4}\left\|x+i^k y\right\|^2 i^k \\
&= \frac{\|x+y\|^2 - \|x-y\|^2}{4} - \frac{\|x+iy\|^2 - \|x-iy\|^2}{4} \\
&= q(x, y) + iq(x, iy)
\end{aligned}$$

where $q(x, y) = \frac{\|x+y\|^2 - \|x-y\|^2}{4}$. We have already shown that q is an inner product over $\mathbb{R}$.

Now one can use the properties of inner product for q to show that $\langle\cdot,\cdot\rangle$ is an inner product over $\mathbb{C}$.