# Lecture Notes in Functional Analysis

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## Introduction

This is a set of lecture notes which I took for reviewing stuff that I typed after taking class from *Dr. Md. Ramiz Reza*. All the typos and errors are of mine.

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## §1 Lecture 1 — Introduction to Hilbert Spaces and some examples — 9th January, 2023

**Definition §1.0.1** (Inner Product)**.** Let $V$ be a vector space over a field $\mathbb{F}$ (where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$). A function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{F}$ is called an *inner product* if it satisfies the following properties

1. $\langle x, x\rangle \geq 0$
2. $\left\langle x+y, z\right\rangle = \langle x, z\rangle + \left\langle y, z\right\rangle$
3. $\left\langle \alpha x, y\right\rangle = \alpha \left\langle x, y\right\rangle$
4. $\left\langle x, y\right\rangle = \overline{\left\langle y, x\right\rangle}$

for all $x, y, z \in V$ and $\alpha \in \mathbb{F}$. A vector space $V$ with an inner product is called an *inner product space*.

**Example §1.0.2** (Examples of inner product spaces)**.** Here are some examples of inner product spaces:

1. The obvious first example is that of $\mathbb{R}^n$ with the standard 2-inner product given by
$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y_i}$$
2. One can then consider the space $\ell^2(\mathbb{N})$ which is the vector space of all square summable sequences on $\mathbb{C}$. That is,
$$\ell^\infty = \left\{ (x_n) \in \mathbb{C}^{\mathbb{N}} \mid \sum_{i\in\mathbb{N}} |a_i|^2 < \infty \right\}$$
We define an inner product on this vector space by
$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$$
One can show using Holder's inequality that the sum turns out to be finite and the "inner product" is indeed an inner product.
3. Next, we consider the vector space of all polynomials over $\mathbb{C}$ which we denote by $\mathbb{C}[x]$. If $p, q \in \mathbb{C}[x]$, we define an inner product on $\mathbb{C}[x]$ by
$$\langle p, q\rangle = \int_0^1 p\overline{q}\,\mathrm{dx}$$
4. One can define inner products on $C[0,1]$ and $L^2(X, \mathscr{A}, \mu)$ in an similar fashion as in item 3. Note that $(X, \mathscr{A}, \mu)$ is a measure space.

**Definition §1.0.3.** Let $V$ be an inner product space. We can define a function $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ by
$$\|x\| = \sqrt{\langle x, x\rangle}$$
We call this function *norm induced by the inner product.* This norm is indeed a norm as one can check!