Solutions to Euclidean Harmonic Analysis Assigment 1

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21.01.2024

Contents

1 Lecture Note Question 1

What is the measure of $\left\{z \in \mathbb{T} : z = e^{2\pi i\theta}, \frac{1}{2} \leq \theta < 1\right\}$?

Solution. Let $e : [0, 1) \to \mathbb{T}$ be the measurable map given by

$$e(t) = e^{2\pi i t}$$

for each $t \in [0, 1)$. Let λ be the Lebesgue measure on $[0, 1)$. Denote the pushforward measure on $\mathbb{T}$ by λe^{-1}. Then

$$\begin{aligned}
\lambda e^{-1}\left(\left\{z \in \mathbb{T} : z = e^{2\pi i\theta}, \frac{1}{2} \leq \theta < 1\right\}\right) &= \lambda\left(e^{-1}\left(\left\{z \in \mathbb{T} : z = e^{2\pi i\theta}, \frac{1}{2} \leq \theta < 1\right\}\right)\right) \\
&= \lambda\left([1/2, 1)\right) \\
&= \frac{1}{2}.
\end{aligned}$$

$\square$

2 Lecture Note Question 2

Show that if $f : \mathbb{R} \to \mathbb{C}$ is Lebesgue measurable and 1-periodic with $\int_0^1 |f(t)|\, dt < \infty$ we have

$$\int_a^{a+1} f(t)dt = \int_0^1 f(t)dt$$

for each $a \in \mathbb{R}$.

Use this to show that the arc length measure is rotation invariant, that is, for each $f \in L^1(\mathbb{T})$ and $\zeta \in T$,

$$\int_{\mathbb{T}} f_\zeta = \int_{\mathbb{T}} f$$

where $f_\zeta(z) := f(\zeta z)$ for each $z \in \mathbb{T}$.

Solution. Let $f : \mathbb{R} \to \mathbb{C}$ is Lebesgue measurable and 1-periodic with $\int_0^1 |f(t)|\, dt < \infty$. Let $a \in \mathbb{R}$ and k be the unique integer such that $a \leq k < a+1$. Consider the following:

$$\begin{aligned}
\int_a^{a+1} f(t)\, dt &= \int_a^k f(t)\, dt + \int_k^{a+1} f(t)\, dt \\
&= \int_{\mathbb{R}} f(t)\chi_{[a,k]}(t)\, dt + \int_{\mathbb{R}} f(t)\chi_{[k,a+1]}(t)\, dt \\
&= \int_{\mathbb{R}} f(t-(k-1))\, \chi_{[a,k]}(t-(k-1))\, dt + \int_{\mathbb{R}} f(t-k)\chi_{[k,a+1]}(t-k)\, dt \qquad (\star) \\
&= \int_{\mathbb{R}} f(t-(k-1))\, \chi_{[a-(k-1),1]}(t)\, dt + \int_{\mathbb{R}} f(t-k)\chi_{[0,a-(k-1)]}(t)\, dt \\
&= \int_{a-(k-1)}^1 f(t-(k-1))\, dt + \int_0^{a-(k-1)} f(t-k)\, dt \\
&= \int_{a-(k-1)}^1 f(t)\, dt + \int_0^{a-(k-1)} f(t)\, dt \qquad (\star\star) \\
&= \int_0^1 f(t)dt.
\end{aligned}$$

Note that $(\star)$ is true because Lebesgue measure is translation invariant and $(\star\star)$ is true because f is 1-periodic.

To prove the second part of the question, let $f \in L^1(\mathbb{T})$. Then $f \circ e : \mathbb{R} \to \mathbb{C}$ meets the hypothesis of the previous statement that we proved. Now let $\zeta \in \mathbb{T}$ then $\zeta = e(\theta)$ for some $\theta \in \mathbb{R}$. Hence we have that

$$\begin{aligned}
f_\zeta\left(e^{2\pi i t}\right) &= f\left(\zeta e^{2\pi i t}\right) \\
&= f\left(e^{2\pi i\theta} e^{2\pi i t}\right) \\
&= f\left(e^{2\pi i(\theta+t)}\right) \\
&= (f \circ e)(\theta + t).
\end{aligned}$$

for each $t \in \mathbb{R}$ and hence we have that

$$\begin{aligned}
\int_{\mathbb{T}} f_\zeta &= \int_0^1 (f \circ e)(\theta + t)\ dt \\
&= \int (f \circ e)(\theta + t)\, \chi_{[0,1]}\, dt \\
&= \int (f \circ e)(t)\, \chi_{[-\theta, 1-\theta]}\, dt \\
&= \int_{-\theta}^{-\theta+1} (f \circ e)(t)\, dt \\
&= \int_0^1 (f \circ e)(t) dt \\
&= \int_{\mathbb{T}} f.
\end{aligned}$$

$\square$

3 Lecture Note Question 3

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1. Compute $\hat{f}(k)$ when

$$f(e^{2\pi it}) = \sum_{k=-N}^{N} a_k e^{2\pi ikt}$$

2. Compute $\hat{f}(k)$ when

$$f(e^{2\pi it}) = \begin{cases} 1 & \text{for } a < t < b \\ 0 & \text{otherwise} \end{cases}$$

where $[a, b] \subset [0, 1)$.

3. Show that $\lim_{n\to\infty} \left|\hat{f}(n)\right| = 0$ in the above example.

Solution. 1. A quick computation shows that

$$\hat{f}(n) = \begin{cases} a_k & \text{for } -N \leq k \leq N \\ 0 & \text{otherwise.} \end{cases}$$

2. Again a simple computation shows that

$$\hat{f}(n) = \frac{i}{2\pi n}\left(e^{-2\pi inb} - e^{-2\pi ina}\right)$$

3. For the trignometric polynomial, the Fourier coefficients is eventually zero, hence, $\lim_{n\to\infty} \left|\hat{f}(n)\right| = 0$. While for the second one,

$$\begin{aligned} \left|\hat{f}(n)\right| &= \left|\frac{i}{2\pi n}\left(e^{-2\pi inb} - e^{-2\pi ina}\right)\right| \\ &\leq \frac{2}{2\pi n} = \frac{1}{\pi n} \to 0 \text{ as } n \to \infty. \end{aligned}$$

□

4 Lecture Note Question 4

Calculate $S_N f$ when $f = \sum_{k=-M}^{M} a_k e^{2\pi i k t}$ (such an f is called a trignometric polynomial). Show that

$$\lim_{N\to\infty} S_N f = f$$

pointwise and in L^p-norm for all $1 \leq p \leq \infty$.

Solution. Note that it follows by part 1 of the previous question that $S_N f = f$ for all $N \geq M$. Therefore we have pointwise and convergence in L^p norm for each $1 \leq p \leq \infty$. $\square$

5 Lecture Note Question 5

We define the Dirichlet kernel D_N for each $N \in \mathbb{N}$ in the following way:

$$D_N(t) = \sum_{k=-N}^{N} e^{2\pi i k t}$$

Show the following:

1. $D_N(t) = \frac{\sin(\pi(2N+1)t)}{\sin(\pi t)} \qquad (t \neq 0).$
2. $\int_0^1 D_N(t) dt = 1$
3. $|D_N(t)| \leq \frac{1}{\sin(\pi\delta)}$ for $0 < \delta \leq |t| < \frac{1}{2}$.

Solution. 1. Note that $D_N(t)$ is real number for each $t \neq 0$ because $\sum_{k=-N}^{N} e^{2\pi i k t} = 1 + \sum_{k=1}^{N} \left(e^{2\pi i k t} + \overline{e^{2\pi i k t}} \right) = \Re\left(1 + 2\sum_{k=1}^{N} e^{2\pi i k t}\right)$.

Hence, we have that

$$\begin{aligned}
\sum_{k=-N}^{N} e^{2\pi i k t} &= \left(\frac{e^{-2\pi i N t}\left(1 - e^{2\pi i (2N+1)t}\right)}{1 - e^{2\pi i t}} \right) && \text{(using the geometric formula)} \\
&= \Re\left(\frac{e^{-2\pi i N t}\left(1 - e^{2\pi i (2N+1)t}\right)}{1 - e^{2\pi i t}} \right) && \text{(as justified)} \\
&= \Re\left(\frac{e^{-2\pi i N t} e^{\pi i (2N+1)t}\left(e^{-\pi i (2N+1)t} - e^{\pi i (2N+1)t}\right)}{e^{\pi i t}(e^{-\pi i t} - e^{\pi i t})} \right) \\
&= \Re\left(\frac{e^{-\pi i (2N+1)t} - e^{\pi i (2N+1)t}}{e^{-\pi i t} - e^{\pi i t}} \right) \\
&= \frac{\sin\left(\pi\left(2N+1\right)t\right)}{\sin\left(\pi t\right)}
\end{aligned}$$

2. Note

$$\begin{aligned}
\int_0^1 \sum_{k=-N}^{N} e^{2\pi i k t} dt &= \int_0^1 1 + \sum_{k=1}^{N} \left(e^{2\pi i k t} + \overline{e^{2\pi i k t}} \right) dt \\
&= \int_0^1 \Re\left(1 + 2\sum_{k=1}^{N} e^{2\pi i k t} \right) dt \\
&= \Re\left(1 + 2\sum_{k=1}^{N} \int_0^1 e^{2\pi i k t} dt \right) \\
&= \Re\left(1 + \frac{2}{2\pi i} \sum_{k=1}^{N} \int_{C_k} z^{k-1} dz \right) && (C_k \text{ is circle of radius } 2\pi k) \\
&= 1
\end{aligned}$$

3. Let $0 < \delta \leq |t| < \frac{1}{2}$. Since sin is an increasing function on $[0, \frac{\pi}{2}]$, we have that

$$\frac{1}{\sin(\pi|t|)} \leq \frac{1}{\sin(\pi\delta)}$$

and consequently, we have that

$$|D_N(t)| \leq \frac{1}{|\sin(\pi t)|} = \frac{1}{\sin(\pi|t|)} \leq \frac{1}{\sin(\pi\delta)}.$$

□

6 Lecture Note Question 6

Show that for any $0 < \delta < \frac{1}{2}$ there exists $c, d > 0$ such that for any $|t| < \delta$ we have

$$d\,|\pi t| \leq |\sin(\pi t)| \leq c\,|\pi t|\,.$$

Solution. Consider the function $f(t) = \sin(\pi t)$. Let $0 < \delta < \frac{1}{2}$ and let $t \in \mathbb{R}$ with $|t| < \delta$. Suppose that $t \geq 0$. Then there is some $t_0 \in [0, t]$ such that $\sin(\pi t) = \cos(\pi t_0)\,\pi t$.

Since cos is decreasing on the interval $[0, \pi/2]$, we have that

$$\sin(\pi t) \geq \cos(\pi\delta)\,\pi t \rightsquigarrow |\sin(\pi t)| \geq |\cos(\pi\delta)|\,|\pi t|\,.$$

If $t < 0$, we use what we have proved immediately for $-t$ and the lower bound is established with $d = |\cos(\pi\delta)|$.

The upper bound is established immediately with $c = 1$. $\square$

7 Assignment Question 2

Let D_N be the Dirichlet kernel. Prove that

$$\frac{4}{\pi^2}\sum_{k=1}^{N}\frac{1}{k} \le \|D_N\|_{L^1(\mathbb{T})} \le 2+\frac{\pi}{4}+\frac{4}{\pi^2}\sum_{k=1}^{N}\frac{1}{k}.$$

Solution. First, we prove the lower bound estimate for $\|D_N\|_{L^1(\mathbb{T})}$. We have already shown that

$$D_N(t)=\frac{\sin((2N+1)\pi t)}{\sin(\pi t)}.$$

To estimate the lower bound, consider the following:

$$\begin{aligned}
\|D_N\|_{L^1(\mathbb{T})} &= \int_0^1\left|\frac{\sin((2N+1)\pi t)}{\sin(\pi t)}\right| dt \\
&= 2\int_0^{\frac{1}{2}}\left|\frac{\sin((2N+1)\pi t)}{\sin(\pi t)}\right| dt && \text{(symmetric about } x=1/2) \\
&= \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left|\frac{\sin((2N+1)t)}{\sin t}\right| dt && \text{(substitute } u=\pi t) \\
&\ge \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left|\frac{\sin((2N+1)t)}{t}\right| dt \\
&= \frac{2(2N+1)}{\pi}\int_0^{\frac{(2N+1)\pi}{2}}\left|\frac{\sin(t)}{t}\right| dt && \text{(substitute } u=(2N+1)t) \\
&\ge \frac{2}{\pi}\int_0^{\frac{(2N+1)\pi}{2}}\left|\frac{\sin(t)}{t}\right| dt \\
&= \frac{2}{\pi}\left(\sum_{k=0}^{n-1}\int_{k\pi}^{(k+1)\pi}\frac{|\sin t|}{t}\,dt+\int_{(k+1)\pi}^{\frac{(2N+1)}{2}}\frac{|\sin t|}{t}\,dt\right) \\
&\ge \frac{2}{\pi}\sum_{k=0}^{n-1}\int_{k\pi}^{(k+1)\pi}\frac{|\sin t|}{t}\,dt \\
&\ge \frac{2}{\pi}\sum_{k=0}^{n-1}\int_0^{\pi}\frac{\sin t}{t+k\pi}\,dt && \text{(substitute } u=t+k\pi) \\
&\ge \frac{2}{\pi}\sum_{k=0}^{n-1}\int_0^{\pi}\frac{\sin t}{\pi+k\pi}\,dt \\
&= \frac{2}{\pi^2}\sum_{k=0}^{n-1}\frac{1}{k+1}\int_0^{\pi}\sin t\,dt \\
&= \frac{4}{\pi^2}\sum_{k=1}^{n}\frac{1}{k}.
\end{aligned}$$

To estimate the upper bound, consider the following: □

8 Assignment Question 3

Let $f \in L^1(\mathbb{T})$ and p be a trignometric polynomial. Define

$$f * p(x) = \int_0^1 f(x-t)\, p(t) dt$$

for each $x \in [0,1)$. Show that $f * p \in C(\mathbb{T})$. Prove that $(f * p)(x) = \sum \hat{p}(m) \hat{f}(m) e^{2\pi i m x}$.

Solution. First, we make use of a fact that convolution is commutative, that is, if $f, g \in L^1(\mathbb{T})$ then $f * g = g * f$.[1].

Let $f \in L^1(\mathbb{T})$ and p be a trignometric polynomial. Thus, we have that $f * p = p * f$. Also, for each $n \in \mathbb{Z}$, we denote $e^n : [0,1) \to \mathbb{C}$ given by

$$e^n(t) = \exp(2\pi i n t)$$

for each $t \in [0,1)$.

Let p be a trignometric polynomial. Now, we show that $f * p$ is continuous on $\mathbb{T}$. It is enough to show that $p * f$ is continuous in the view that $f * p = p * f$. To this end, let (x_n) be any sequence in $[0,1)$ to $x \in [0,1)$. Then note that:

$$\begin{aligned}
\lim_{n\to\infty} (p * f)(x_n) &= \lim_{n\to\infty} \int_0^1 p(x_n - t) f(t)\, dt \\
&\overset{(\star)}{=} \int_0^1 \lim_{n\to\infty} p(x_n - t) f(t)\, dt \\
&= \int_0^1 p(x-t) f(t) && \text{(trignometric polynomials are continuous)} \\
&= (p * f)(x).
\end{aligned}$$

The aforementioned series of equality will show that $p * f$ is continuous at x provided we justify the equality at the step $(\star)$. To justify the equality $(\star)$, we appeal to Dominated Convergence Theorem. For each $n \in \mathbb{N}$, define

$$F_n(t) = p(x_n - t) f(t)$$

for each $t \in [0,1)$. Note since p is continuous, we have that $\lim_{n\to\infty} F_n(t) = p(x-t) f(t)$ at each $t \in [0,1)$. Also note that

$$|F_n(t)| = |p(x-t) f(t)| \le \|p\|_\infty |f(t)|$$

for each $t \in [0,1)$. Since $\|p\|_\infty f \in L^1(\mathbb{T})$, the equality at $(\star)$ makes sense via Dominated Convergence Theorem.

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[1]A proof of this can be found in Katznelson, An Introduction To Harmonic Analysis, Page 5

We now show that $(f * p)(x) = \sum \hat{p}(m)\, \hat{f}(m)\, e^{2\pi imx}$. To this end, consider the following:

$$
\begin{aligned}
(f * p)(x) &= \int_0^1 f(x-t)\, p(t)\, dt \\
&= \int_0^1 f(x-t) \left(\sum_m \hat{p}(m)\, e^{2\pi imt} \right) dt \quad \text{(as p is trig. polynomial and sum is a finite sum)} \\
&= \sum_m \hat{p}(m) \left(\int_0^1 f(x-t)\, e^{2\pi imt} dt \right) \\
&= \sum_m \hat{p}(m)\, (f * e^m)(x) \\
&= \sum_m \hat{p}(m)\, (e^m * f)(x) \\
&= \sum_m \hat{p}(m) \int_0^1 e^{2\pi im(x-t)} f(x)\, dt \\
&= \sum_m \hat{p}(m)\, \hat{f}(m)\, e^{2\pi imx}.
\end{aligned}
$$

□

9 Assignment Question 4

Suppose that $f(t) := \sum_{k=-N}^{N} a_k e^{2\pi ikt}$. Show that $\sum_{k=-N}^{N} \left|\hat{f}(k)\right|^2 = \|f\|_{L^2(\mathbb{T})}^2$.

Solution. Let us denote $F_N = \{0, \pm 1, \ldots, \pm N\}$. Let $f(t) = \sum_{k=-N}^{N} a_k e^{2\pi ikt}$. We have already seen that $\hat{f}(k) = a_k$ for each $k \in F_N$. Thus, $f(t) = \sum_{k=-N}^{N} \hat{f}(k) e^{2\pi ikt}$. Now,

$$
\begin{aligned}
\|f\|_{L^2}^2 (\mathbb{T}) &= \langle f, f \rangle \\
&= \left\langle \sum_{k \in F_N} \hat{f}(k) e^{2\pi ikt}, \sum_{j \in F_N} \hat{f}(j) e^{2\pi ijt} \right\rangle \\
&= \sum_{k,j \in F_N} \hat{f}(k) \overline{\hat{f}(j)} \left\langle e^{2\pi ikt}, e^{2\pi ijt} \right\rangle \\
&= \sum_{k,j \in F_N} \hat{f}(k) \overline{\hat{f}(j)} \delta_{kj} \\
&= \sum_{k \in F_N} \left|\hat{f}(k)\right|^2
\end{aligned}
$$

which completes the proof. $\square$