Solutions to Euclidean Harmonic Analysis Assignment 2

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1 Question 1

Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces, and let f be a product measurable function on $X \times Y$. If $f \geq 0$ and $1 \leq p < \infty$, then

$$\left[\int_X \left(\int_Y f(x,y)d\nu(y)\right)^p d\mu(x)\right]^{\frac{1}{p}} \leq \int_Y \left[\int_X f(x,y)^p d\mu(x)\right]^{\frac{1}{p}} d\nu(y) \tag{1.1}$$

If $1 \leq p \leq \infty$, $f(\cdot, y) \in L^p(\mu)$ for a.e. y, and the function $y \to ||f(\cdot, y)||_p$ is in $L^1(\nu)$, then $f(x, \cdot) \in L^1(\nu)$ for a.e. x, the function $x \to \int f(x,y)d\nu(y)$ is in $L^p(\mu)$, and

$$\left|\left|\int f(\cdot, y)d\nu(y)\right|\right|_p \leq \int ||f(\cdot, y)||_p d\nu(y).$$

To prove the above you can do it by the following steps.

(a) Let $1 \leq p < \infty$ and $g \in L_p(X, \mathcal{M}, \mu)$. Then $\|g\|_p = \{\int_X gh : \|h\|_q = 1\}$ where $\frac{1}{p} + \frac{1}{q} = 1$. (Hint: Use Hölder's inequality and then consider the function $h(x) = |g(x)|^{q-1} \frac{\text{sgn } g(x)}{\|g\|_q^{q-1}}$ where $\text{sgn } g(x) := \frac{g(x)}{|g(x)|}$ when $g(x) \neq 0$ and $\text{sgn } g(x) = 0$ otherwise.)

(b) For $1 < p < \infty$, use (a) and Fubini's theorem to prove 1.

Solution. To prove this question, we use the following result due to Tonelli:

Proposition 1.1 (Tonelli)**.** *Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be σ-finite measure spaces, and let $f : X \times Y \to [0, +\infty]$ be $\mathscr{A} \times \mathscr{B}$-measurable. Then*

(a) the function $x \mapsto \int_Y f(x,y)\, d\nu(y)$ is $\mathscr{A}$-measurable and the function $y \mapsto \int_X f(x,y)\, d\mu(x)$ is $\mathscr{B}$-measurable, and

(b) f satisfies

$$\int_{X \times Y} f d(\mu \times \nu) = \int_X \left(\int_Y f(x,y)\, d\nu(y)\right) d\mu(x) = \int_Y \left(\int_X f(x,y)\, d\mu(x)\right) d\nu(y)$$

The statement and proof of the above proposition can be found in [2] in Chapter 5, Section 2.

Also, we appeal to the following result which can be found in Theorem 6.16. in [3] and also in the proof of duality of L^p in Proposition 3.5.5. in [2].

Theorem 1.2. *Suppose $1 \leq p < \infty$, μ is a σ-finite positive measure on X and Φ is a bounded linear functional on $L^p(\mu)$. Then there is a unique $g \in L^q(\mu)$ where q is the conjugate exponent of p such that*

$$\Phi(f) = \int_X fg\, d\mu \ \ (f \in L^p(\mu)).$$

Moreover, if Φ and g are related as in the previous equaiton then we have

$$\|\Phi\| = \|g\|_q .$$

We can start the proof now. Let f be a measurable function on $X \times Y$. We first do it for the case where $f \geq 0$. For $p = 1$, we are done because then it is Proposition 1.1 in disguise. So, suppose that $1 < p < \infty$. Also, we may assume $\int_X \left(\int_Y f(x,y)^p d\mu(x)\right)^{1/p} d\nu(y) < \infty$ for otherwise the inequality 1.1 is always true.

Now, define $F(x) = \int_Y f(x,y) d\nu(y)$. By Proposition 1.1, F is $\mathscr{M}$-measurable.

Now, we define a linear functional $\Phi : L^q(\mu) \to \mathbb{C}$ by the following way:

$$\Phi(g) = \int_X gF \, d\mu \, (g \in L^q \ (\mu))$$

We now show that Φ is bounded linear functional. To this end, let $g \in L^q(\mu)$ and consider the following:

$$\begin{aligned}
\left| \int_X gF d\mu \right| &\leq \int_Y |g(x)| \, |F(x)| \, d\mu(x) \\
&\leq \int_X |g(x)| \left(\int_Y f(x,y) d\nu(y) \right) d\mu(x) \\
&\leq \int_X \left(\int_Y f(x,y) \, |g(x)| \, d\nu(y) \right) d\mu(x) \\
&\leq \int_X \left(\int_Y f(x,y) \, |g(x)| \, d\mu(x) \right) d\nu(y) && \text{(Proposition 1.1)} \\
&\leq \int_X \left(\int_Y f(x,y)^p d\mu(x) \right)^{1/p} \|g\|_{L^q(\mu)} \, d\nu(y) && \text{(Holder's inquality)} \\
&\leq \|g\|_{L^q(\mu)} \underbrace{\int_X \left(\int_Y f(x,y)^p d\mu(x) \right)^{1/p} d\nu(y)}_{<\infty \text{ by assumption}} .
\end{aligned}$$

By Theorem 1.1, we have that there is some unique $h \in L^p(\mu)$ such that $\Phi(g) = \int_X hg d\mu$ for each $g \in L^q(\mu)$.

By uniqueness, we also have that $F = h$ μ-almost everywhere. Hence F is in $L^p(\mu)$. It follows that $\|F\|_{L^p(\mu)} = \left[\int_X \left(\int_Y f(x,y) d\nu(y)\right)^p d\mu(x)\right]^{\frac{1}{p}} \leq \int_X \left(\int_Y f(x,y)^p d\mu(x)\right)^{1/p} d\nu(y)$.

Now, we proceed to complete the second part. Let f be a measurable function on $X \times Y$ such that $f(\cdot, y) \in L^p(\mu)$ for ν-almost every y and the function $y \mapsto \|f(\cdot, y)\|_p$ is in $L^1(\nu)$. First, consider the case where $1 \leq p < \infty$.

Since the function $y \mapsto \|f(\cdot, y)\|_p$ is in $L^1(\nu)$, we have that

$$\int_X \left(\int_Y |f(x,y)|^p \, d\mu(x) \right)^{1/p} d\nu(y) < \infty.$$

So, we repeat the above proof taking f as $|f|$ (as all the hypothesis are met) and we have that the function $x \to \int f(x,y)d\nu(y)$ is in $L^p(\mu)$ and

$$\left[\int_X \left(\int_Y |f(x,y)|\, d\nu(y)\right)^p d\mu(x)\right]^{\frac{1}{p}} \leq \int_X \left(\int_Y |f(x,y)|^p\, d\mu(x)\right)^{1/p} d\nu(y).$$

which is the same as

$$\left|\left|\int f(\cdot,y)d\nu(y)\right|\right|_p \leq \int ||f(\cdot,y)||_p d\nu(y).$$

To finish the proof for $p=\infty$. Fix $y \in Y$. Hence for μ-almost every x,

$$f(x,y) \leq \|f(\cdot,y)\|_\infty .$$

It follows that for each $x \in X$,

$$\int_Y f(x,y)\, d\nu(y) \leq \int_Y \|f(\cdot,y)\|\, d\nu(y)$$

Taking (essential)-supremum over $x \in X$, we have

$$\left\|\int_Y f(x,y)\, d\nu(y)\right\|_\infty \leq \int_Y \|f(\cdot,y)\|_\infty\, d\nu(y).$$

$\square$

2 Question 2

Let $1 \leq p < \infty$ and $f \in L_p(\mathbb{T})$. For any $t \in \mathbb{R}$ define $\tau_t f(x) := f(x-t)$. Prove that $\|\tau_t f - f\|_p \to 0$ as $t \to 0$. Show that the conclusion fails for $p = \infty$.

Solution. Let $1 \leq p < \infty$ and $f \in L^p(\mathbb{T})$. To show $\lim_{t\to 0^+} \|\tau_t f - f\|_p = 0$, let $\varepsilon > 0$ be given. Since $C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$, there is some $g \in C(\mathbb{T})$ such that $\|f - g\|_p < \frac{\varepsilon}{3}$.

Since $g \in C(\mathbb{T})$, g is uniformly continuous on $\mathbb{T}$. Hence, there is some $\delta > 0$ such that

$$\|g(\cdot - t) - g(t)\|_\infty < \frac{\varepsilon}{3}$$

whenever $|t| < \delta$.

Hence, we have that $\|\tau_t g - g\|_\infty < \frac{\varepsilon}{3}$ whenever $|t| < \delta$. Consequently, we have that

$$\|g(\cdot - t) - g(t)\|_p^p = \int_{\mathbb{T}} |g(x-t) - g(x)|^p \, dx \leq \|g(\cdot - t) - g(t)\|_\infty^p = \|\tau_t g - g\|_\infty^p$$

and hence $\|\tau_t g - g\| < \frac{\varepsilon}{3}$ whenever $|t| < \delta$.

Also, note that $\|\tau_t(f-g)\|_p = \|f - g\|_p$ because Lebesgue measure is translation invariant.

Hence, we have for $|t| < \delta$, we have that

$$\begin{aligned}
\|\tau_t f - f\|_p &= \|\tau_t f - \tau_t g + \tau_t g - g + g - f\|_p \\
&\leq \|\tau_t(f-g)\|_p + \|\tau_t g - g\|_p + \|g - f\|_p \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

This completes the proof. $\square$

3 Question 3

Let $f \in L_1(\mathbb{R})$. Define $\hat{f}(\zeta) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\zeta}dx$. Show that if $\int_{\mathbb{R}} |x||f(x)|dx < \infty$, then we must have that $\hat{f}$ is continuously differentiable. Find a condition on f for which $\hat{f}$ will be a smooth function.

Solution. We need to show that the function $\hat{f}$ is continuously differentiable. Fix a point $\zeta_0 \in \mathbb{R}$. Consider the sequence of function $F_n : \mathbb{R} \to \mathbb{C}$ given by

$$F_n(x) = f(x)\, e^{-2\pi i x\zeta_0} \left(\frac{e^{\left(-2\pi i \frac{x}{n}\right)} - 1}{1/n} \right)$$

for each $x \in \mathbb{R}$. It is easy to see that $F_n(x)$ converges pointwise everywhere to $-2\pi i x f(x)e^{-2\pi i x\zeta_0}$ because $\lim_{n\to\infty} \frac{e^{\left(-2\pi i \frac{x}{n}\right)}-1}{1/n} = 0$ for each $x \in \mathbb{R}$. Also, we claim that $|F_n(x)| \leq 2\pi\,|xf(x)|$. To show this, we make use of the fact that $|e^{ix} - 1| \leq |x|$ for each $x \in \mathbb{R}$. [1] To this end, let $x \in \mathbb{R}$ and consider the following:

$$\begin{aligned} |F_n(x)| &= \left| f(x)\, e^{-2\pi i x\zeta_0} \left(\frac{e^{\left(-2\pi i \frac{x}{n}\right)} - 1}{1/n} \right) \right| \\ &\leq |f(x)|\, \frac{\left|-2\pi i \frac{x}{n}\right|}{1/n} \\ &= 2\pi\, |xf(x)| \end{aligned}$$

We went through the trouble of defining the sequence of function $(F_n)_{n\in\mathbb{N}}$ for the reason

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[1] One can show this by using that fact that $|\sin x| \leq |x|$ for each $x \in \mathbb{R}$ as follows:

$$\begin{aligned} \left|e^{ix} - 1\right| &= |\cos x + i\sin x - 1| \\ &= |1 - \cos x + i\sin x| \\ &= \left|2\sin^2\left(\frac{x}{2}\right) + 2i\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\right| \\ &= \left|2i\sin\left(\frac{x}{2}\right)e^{\frac{ix}{2}}\right| \\ &\leq 2\frac{|x|}{2} = |x|\,. \end{aligned}$$

that:

$$\begin{aligned}
\frac{d\hat{f}}{d\zeta}(\zeta_0) &= \lim_{n\to\infty} \frac{\hat{f}\left(\zeta_0 + \frac{1}{n}\right) - \hat{f}(\zeta_0)}{\frac{1}{n}} \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} F_n(x)\, dx \\
&= \int_{\mathbb{R}} \lim_{n\to\infty} F_n(x)\, dx && \text{(by DCT)} \\
&= \int_{\mathbb{R}} -2\pi i x f(x) e^{-2\pi i x \zeta_0}\, dx \\
&= -2\pi i \widehat{(xf)}(\zeta_0).
\end{aligned}$$

This shows that $\hat{f}$ is differentiable and $\frac{d\hat{f}}{d\zeta}(\zeta) = -2\pi i \widehat{(xf)}(\zeta)$. To see that this is continuous, let $\zeta_0 \in R$ and let (h_n) be any sequence converging to 0 and we again apply DCT in the following way:

$$\begin{aligned}
\lim_{n\to\infty} \widehat{xf}(\zeta_0 + h_n) &= \lim_{n\to\infty} \int_{-\infty}^{\infty} xf(x)\, e^{-2\pi i(\zeta_0 + h_n)} \\
&\overset{\text{(DCT)}}{=} \int_{-\infty}^{\infty} \lim_{n\to\infty} xf(x)\, e^{-2\pi i(\zeta_0 + h_n)} \\
&= \int_{-\infty}^{\infty} xf(x)\, e^{-2\pi i \zeta_0} \\
&= \widehat{xf}(\zeta_0)
\end{aligned}$$

This would complete the proof provided we justify the step at DCT step. That is easily justified by the fact that $x \mapsto xf(x)$ is in $L^1(\mathbb{R})$.

Finally, we showed in class that if $f \in \mathcal{S}(\mathbb{R})$, that is, the Schwartz class of $\mathbb{R}$ then $\hat{f}$ must be smooth. $\square$

4 Question 4

Let $f : \mathbb{R} \to \mathbb{C}$ be such that $\int_{\mathbb{R}} |f(x)|dx < \infty$ and $g \in C_c^\infty(\mathbb{R})$. Prove that the function $f * g$ defined by $f * g(x) := \int_{\mathbb{R}} f(y)g(x - y)dy$ is a well-defined smooth function.

(a) Is $f * g$ will also be compactly supported? What if $f \in C_c^\infty(\mathbb{R})$?

(b) Prove that $\|f * g\|_p \leq \|f\|_1 \|g\|_p$ for all $1 \leq p \leq \infty$.

Solution. (a) If $f \in L^1(\mathbb{R})$ and $g \in C_c^\infty(\mathbb{R})$ then it can be shown that $f * g$ may not be compactly supported.

On the other hand, we show that if both f and g are both compactly supported then $f * g$ must be compactly supported. In fact, we show that if $\operatorname{supp}(f * g) \subset \overline{\operatorname{supp} f + \operatorname{supp} g}$. [2].

First, we claim that for a fixed $x \in R$, $f(x - y)\, g(y) \neq 0$ implies that $y \in (x - \operatorname{supp} f) \cap \operatorname{supp} g$. To see this, let $y \in \mathbb{R}$ be such that $f(x - y)\, g(y) \neq 0$. Consequently, $f(x - y) \neq 0$ and $g(y) \neq 0$. If $g(y) \neq 0$ then $y \in \operatorname{supp} g$ and if $f(x - y) \neq 0$ then $x - y \in \operatorname{supp} f$, that is, $y \in x - \operatorname{supp} f$. Let $C_x = (x - \operatorname{supp} f) \cap \operatorname{supp} g$ for each $x \in \mathbb{R}$.

Now, note that for a fixed $x \in \mathbb{R}$, we have that

$$
\begin{aligned}
(f * g)(x) &= \int_{\mathbb{R}} f(x - y)\, g(y)\, dy \\
&= \int_{C_x} f(x - y)\, g(y)\, dy + \cancelto{0}{\int_{C_x^c} f(x - y)\, g(y)\, dy} \quad \text{(by the previous paragraph)} \\
&= \int_{(x - \operatorname{supp} f) \cap \operatorname{supp} g} f(x - y)\, g(y)\, dy.
\end{aligned}
$$

We claim that if $x \notin \operatorname{supp} f + \operatorname{supp} g$ then $(x - \operatorname{supp} f) \cap \operatorname{supp} g = \emptyset$. This is to see for if $y \in \operatorname{supp}(x - \operatorname{supp} f) \cap \operatorname{supp} g$ then $x - y \in \operatorname{supp} f$ and $y \in \operatorname{supp} g$ which implies $x \in \operatorname{supp} f + \operatorname{supp} g$. Consequently, if $x \notin \operatorname{supp} f + \operatorname{supp} g$ then we have that $(f * g)(x) = 0$. Hence, we have that $(f * g) = 0$ a.e. on $(\operatorname{supp} f + \operatorname{supp} g)^c$. Hence, $f * g = 0$ a.e. in particular on the interior of $(\operatorname{supp} f + \operatorname{supp} g)^c$ which equals $\overline{\operatorname{supp} f + \operatorname{supp} g}$.

Since sum of two compact sets is compact, we are done.[3].

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[2] The proof of the claim of $\operatorname{supp}(f * g) \subset \overline{\operatorname{supp} f + \operatorname{supp} g}$ of the solution was borrowed from [1].

[3] because sum is jointly continuous and product of compact sets is compact.

(b) One can prove this with the weaker hypothesis that $g \in L^p(\mathbb{R})$ in the following way:

$$\begin{aligned}
\|f * g\|_p &= \left\| \int_{\mathbb{R}} f(y) g(\cdot - y)\, dy \right\|_p \\
&\leq \int_{\mathbb{R}} \|f(y) g(\cdot - y)\|_p \, dy && \text{(See question 1)} \\
&= \int_{\mathbb{R}} |f(y)| \, \|g(\cdot - y)\|_p \, dy \\
&= \int_{\mathbb{R}} |f(y)| \, \|g\|_p \, dy && \text{(translation invariance)} \\
&= \|f\|_1 \|g\|_p .
\end{aligned}$$

□

References

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