Reproducing kernel Hilbert spaces & (Complete) Pick property

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Notations

- The *open unit disc* $\{z \in \mathbb{C} : |z| < 1\}$ of the complex plane will be denoted by $\mathbb{D}$.
- The boundary of $\mathbb{D}$ namely *the unit circle of the complex plane* will be denoted by $\mathbb{T}$. That is $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.
- For any open subset $\Omega \subset \mathbb{C}$, $\mathcal{O}(\Omega)$ will denote the set of all holomorphic functions on Ω.
- The set of all *bounded* holomorphic functions on the open unit disc $\mathbb{D}$ will be denoted by $H^{\infty}(\mathbb{D})$.
- If V, W is a normed linear spaces then the set of all bounded operators is denoted by $\mathcal{B}(V, W)$.

Pick Nevanlinna Problem

Question (Motivation?)

Given initial data of n distinct points $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{D}$ and target data of n points $w_1, w_2, \dots, w_n \in \mathbb{C}$, is there a holomorphic function $\varphi : \mathbb{D} \to \mathbb{C}$ such that $\varphi(\lambda_i) = w_i$ for each $i = 1, \dots, n$?

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Unsuprisingly, existence of such a holomorphic function is given by Lagrange interpolating polynomial.

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Question (Pick (1916), Nevanlinna (1919))

Given initial data of n points $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{D}$ and target data of n points $w_1, w_2, \ldots, w_n \in \overline{\mathbb{D}}$, is there a holomorphic function $\varphi : \mathbb{D} \to \mathbb{C}$ such that $\varphi(\lambda_i) = w_i$ for each $i = 1, \ldots, n$ and furthermore $|\varphi(\lambda)| \leq 1$ for each $\lambda \in \mathbb{D}$?

Answer to Pick Nevanlinna Problem

Theorem (Pick-Nevanlinna Theorem)

Let $(\lambda_i)_{1 \le i \le n}$ *be* n *distinct points in* $\mathbb{D}$ *and let* $(w_i)_{1 \le i \le n} \subset \mathbb{C}$.
Then the following are equivalent:

1. *There is a holomorphic function such that*

$$f(\lambda_i) = w_i \qquad (1 \le i \le n),$$

   and, moreover, $\|f\|_\infty \le 1$.

2. *The matrix* $P = (P_{j,k})_{1 \le j,k \le n}$ *is positive semidefinite where*

$$P_{j,k} = \frac{1 - w_j \overline{w_k}}{1 - \lambda_j \overline{\lambda_k}} \qquad (j, k = 1, \ldots, n)$$

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$$P_{j,k}=\frac{1-w_j\overline{w_k}}{1-\lambda_j\overline{\lambda_k}} \qquad (j,k=1,\ldots,n)$$

Definition (Pick matrix)

The matrix P associated with the initial data $(\lambda_i)_{1\leq i\leq n}\subset\mathbb{D}$ and $(w_i)_{1\leq i\leq n}\subset\mathbb{C}$ is called the (associated) *Pick matrix*.

A general setup: Reproducing kernel Hilbert space

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Let Ω be a set and let $\mathcal{F}(\Omega, \mathbb{C})$ denote the set of the functions from Ω to $\mathbb{C}$. A Hilbert space $\mathcal{H} \subset \mathcal{F}(\Omega, \mathbb{C})$ is called a *reproducing kernel Hilbert space* (rkHs, in short) if

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1. For each $z \in \Omega$, the evaluation functional

$$\begin{aligned} \mathcal{E}_z : \mathcal{H} &\to \mathbb{C} \\ f &\mapsto f(z) \end{aligned}$$

is bounded.

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is bounded.
2. For each $z \in \Omega$, there is some function $f_z \in \mathcal{H}$ such that $f_z(z) \neq 0$.

The kernel function associated to a rkHs

By the definition of rkHs, the evaluation functional $\mathcal{E}_w$ is bounded for each $w \in \Omega$. Therefore, by the *Riesz representation theorem*, there is a unique function $K(\cdot, w) \in \mathcal{H}$ such that

$$f(w) = \langle f, K(\cdot, w) \rangle \qquad (f \in \mathcal{H}) . \tag{1}$$

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Definition (the kernel function)

Let $\mathcal{H}$ be rkHs on Ω. The *kernel function associated to* $\mathcal{H}$ is the function

$$K : \Omega \times \Omega \to \mathbb{C}$$

satisfying Equation 1. For each $w \in \Omega$, the function $K(\cdot, w) \in \mathcal{H}$ is called *the reproducing kernel at the point* w.

Primary Examples of rkHs: Analytic Function Spaces

Let $\Omega = \mathbb{D}$ the open unit disc. For each $s \in \mathbb{R}$, consider the set of holomorphic functions $\mathcal{H}_s$ on $\mathbb{D}$

$$\mathcal{H}_s := \left\{ f = \sum_{n \geq 0} \hat{f}(n) z^n \in \mathcal{O}(\mathbb{D}) : \sum_{n \geq 0} (n+1)^{-s} \left| \hat{f}(n) \right|^2 < \infty \right\}. \tag{2}$$

Each $\mathcal{H}_s$ is a rkHs where $\|f\|_{\mathcal{H}_s}^2 := \sum_{n \geq 0} (n+1)^{-s} \left| \hat{f}(n) \right|^2$.

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s	Name	Kernel Function
-1	Bergman space $A^2(\mathbb{D})$	$K(z,w) = \frac{1}{(1-\bar{w}z)^2}$
0	Hardy space $H^2(\mathbb{D})$	$K(z,w) = \frac{1}{(1-\bar{w}z)}$
1	Dirichlet space $\mathcal{D}$	$K(z,w) = \begin{cases} \frac{1}{z\bar{w}} \log\left(\frac{1}{1-z\bar{w}}\right) & w \neq 0 \\ 1 & w = 0 \end{cases}$

The Hardy Space

Hardy space can be equivalently defined as the set of all functions $f \in \mathcal{O}(\mathbb{D})$ such that

$$\sup_{0 \le r < 1} \int_0^{2\pi} \left| f\left(re^{it}\right) \right|^2 \, dt < \infty.$$

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It can be easily shown that

$$\|f\|_{H^2(\mathbb{D})}^2 := \sum_{n \geq 0} |a_n|^2 = \sup_{0 \leq r < 1} \int_0^{2\pi} \left| f\left(re^{it}\right) \right|^2 dt$$

for each $f = \sum_{n \geq 0} a_n z^n \in H^2(\mathbb{D})$.

The Multiplier Algebra associated to a rkHs

Definition (Multiplier Algebra)

Let $\mathcal{H}$ be a rkHs on Ω. We define the *multiplier algebra* $\mathcal{M}(\mathcal{H})$ *associated with the rkHs* $\mathcal{H}$ to be

$$\mathcal{M}(\mathcal{H}) := \{\varphi : \Omega \to \mathbb{C} \mid \varphi f \in \mathcal{H} \text{ for each } f \in \mathcal{H}\}.$$

For each $\varphi \in \mathcal{M}(\mathcal{H})$, we define the multiplier norm of φ to be

$$\|\varphi\|_{\mathcal{M}(\mathcal{H})} := \|M_\varphi\|$$

where $M_\varphi : \mathcal{H} \to \mathcal{H}$ is the multiplication by φ operator.

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Some Properties of the Multiplier Algebra

Proposition

Let $\mathcal{H}$ be a rkHs on Ω. If $\varphi \in \mathcal{M}(\mathcal{H})$ then for each $w \in \Omega$,

$$M_\varphi^* k(\cdot, w) = \overline{\varphi(w)} k(\cdot, w).$$

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Corollary

With setup in the previous proposition, if $\varphi \in \mathcal{M}(\mathcal{H})$ then we have

$$\sup_{w \in \Omega} |\varphi(w)| \leq \|\varphi\|_{\mathcal{M}(\mathcal{H})}.$$

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As a consequence of the previous corollary, we have that $\mathcal{M}(\mathcal{H}) \subset \mathcal{H} \cap \mathcal{B}(\Omega)$ where $\mathcal{B}(\Omega)$ is the set of bounded functions on Ω.

Example: Multiplier of the Hardy Space

Consider the Hardy space $H^2(\mathbb{D})$. From the previous slide, we conclude that

$$\mathcal{M}\left(H^2\right) \subset H^2 \cap \mathcal{B}(\mathbb{D}) = H^\infty(\mathbb{D}).$$

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In fact, if $\varphi \in H^\infty(\mathbb{D})$ and $f \in H^2$, we have that

$$\begin{aligned}
\|\varphi f\|_{H^2}^2 &= \sup_{0 \le r < 1} \int_0^{2\pi} \left|\varphi\left(re^{it}\right) f\left(re^{it}\right)\right|^2 \, dt \\
&\le \|\varphi\|_{H^\infty}^2 \sup_{0 \le r < 1} \int_0^{2\pi} \left|f\left(re^{it}\right)\right|^2 \, dt \\
&= \|\varphi\|_{H^\infty}^2 \|f\|_{H^2}^2 .
\end{aligned}$$

This shows that $\mathcal{M}(H^2) = H^\infty(\mathbb{D})$.

Rephrasing the Pick Nevanlinna Problem in the language of rkHs

Sarason [Sar67] realised that the Pick Nevanlinna Problem can be recasted for rkHs. In the following theorem, let $\mathcal{H} = H^2(\mathbb{D})$.

Theorem

Let $(\lambda_i)_{1 \le i \le n}$ be n distinct points in $\mathbb{D}$ and let $(w_i)_{1 \le i \le n} \subset \mathbb{C}$. Then the following are equivalent:

1. *There is $\varphi \in \mathcal{M}(\mathcal{H}) = H^\infty(\mathbb{D})$ with $\|\varphi\|_{\mathcal{M}(\mathcal{H})} = \|M_\varphi\| = \|\varphi\|_{H^\infty} \le 1$ and*

$$f(\lambda_i) = w_i \qquad (1 \le i \le n),$$

2. *The matrix $P = (P_{j,k})_{1 \le j,k \le n}$ is positive semidefinite where*

$$P_{j,k} = (1 - w_j \overline{w_k})\, k(\lambda_i, \lambda_j) = \frac{1 - w_j \overline{w_k}}{1 - \lambda_j \overline{\lambda_k}} \qquad (j, k = 1, \dots, n)$$

Existence of an interpolating contractive multiplier implies positivity of Pick matrix

Let $\mathcal{H}$ be a rkHs in Ω. Let $(\lambda_i)_{1 \leq i \leq n}$ be n distinct points in Ω and let $(w_i)_{1 \leq i \leq n} \subset \mathbb{C}$.
Suppose that there exists a $\varphi \in \mathcal{M}(\mathcal{H})$ such that $\|\varphi\|_{\mathcal{M}(\mathcal{H})} \leq 1$ and $\varphi(\lambda_i) = w_i$ for each $i = 1, 2, \ldots, n$. Let P be the corresponding Pick matrix.

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$$\begin{aligned}
&\left\langle \left(I-M_\varphi M_\varphi^*\right)\left(\sum_{j=1}^{n} a_j k(\cdot,\lambda_j)\right), \left(\sum_{i=1}^{n} a_i k(\cdot,\lambda_i)\right)\right\rangle_{\mathcal{H}}\\
&=\sum_{i,j=1}^{n} a_j\bar{a}_i\left(1-\bar{w}_j w_i\right)k(\lambda_i,\lambda_j)=\sum_{i,j=1}^{n} a_j\bar{a}_i P_{ij}\\
&=\left\langle P\left(\sum_{j=1}^{n} a_j e_j\right), \sum_{i=1}^{n} a_i e_i\right\rangle_{\mathbb{C}^n}.
\end{aligned}$$

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Definition (Pick space)

Let $\mathcal{H}$ be an rkHs on Ω. We say that $\mathcal{H}$ is a *Pick space* if for every distinct n points $\lambda_1, \lambda_2, \ldots, \lambda_n \in \Omega$ and every n points $w_1, w_2, \ldots, w_n \in \mathbb{C}$, the positivity of the corresponding Pick matrix implies that there is some $\varphi \in \mathcal{M}(\mathcal{H})$ such that $\|\varphi\|_{\mathcal{M}(\mathcal{H})} \leq 1$ and $\varphi(\lambda_i) = w_i$ for each $i = 1, \ldots, n$.

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Example

Hardy and Dirichlet space are Pick spaces but Bergman space is not.

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Question

When can we find a multiplier $\Phi \in \mathcal{M}_{\mathcal{H}}(\mathcal{E}, \mathcal{E}')$ (what does this mean?) such that $\|M_\Phi\| \leq 1$ and $\Phi(\lambda_i) = W_i$ for each $i = 1, \ldots, n$?

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To this end, we define the notion of ...

Vector Valued Reproducing Kernel Hilbert Spaces

Definition (Vector Valued rkHs)

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$$\langle f(w), e \rangle_{\mathcal{E}} = \langle f, K(\cdot, w) \otimes e \rangle_{\mathcal{H} \otimes \mathcal{E}}$$

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$$\langle f(w), e \rangle_{\mathcal{E}} = \langle f, K(\cdot, w) \otimes e \rangle_{\mathcal{H} \otimes \mathcal{E}}$$

The aforementioned product $\mathcal{H} \otimes \mathcal{E}$ is said to be *$\mathcal{E}$-valued reproducing kernel Hilbert space*.

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$$\begin{aligned} \langle f \otimes e_j, K(\cdot, w) \otimes e_i \rangle_{\mathcal{H} \otimes \mathcal{E}} &= \langle f, K(\cdot, w) \rangle_{\mathcal{H}} \langle e_j, e_i \rangle_{\mathbb{C}^n} \\ &= f(w)\delta_{ij} \end{aligned}$$

and hence $\langle (f \otimes e_j)(w), e_j \rangle_{\mathbb{C}^n} = f(w)\delta_{ij}$.

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and hence $\langle (f \otimes e_j)(w), e_j\rangle_{\mathbb{C}^n} = f(w)\delta_{ij}$. Therefore, $f \otimes e_j$ is the following $\mathbb{C}^n$ valued function (when each element in $\mathbb{C}^n$ is viewed as column vector):

$$(f \otimes e_j)(w) = \begin{bmatrix} 0 \\ \vdots \\ \underbrace{f(w)}_{\text{jth entry}} \\ \vdots \\ 0 \end{bmatrix} \qquad (w \in \Omega)$$

Multipliers of Vector Valued rkHs

Let $\mathcal{E}, \mathcal{E}'$ be two Hilbert space and $\mathcal{H}$ be a reproducing kernel Hilbert space on Ω. Consider $\mathcal{H} \otimes \mathcal{E}$ and $\mathcal{H} \otimes \mathcal{E}'$ as vector valued reproducing kernel Hilbert spaces on Ω.

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A *multiplier* Φ is a $B(\mathcal{E}, \mathcal{E}')$-valued function on Ω such that $\Phi f \in \mathcal{H} \otimes \mathcal{E}'$ for each $f \in \mathcal{H} \otimes \mathcal{E}$. Note that Φf is viewed as a $\mathcal{E}'$-valued function in the following sense:

$$(\Phi f)(w) = \Phi(w) f(w) \qquad (w \in \Omega).$$

The set of all multipliers from $\mathcal{H} \otimes \mathcal{E}$ to $\mathcal{H} \otimes \mathcal{E}'$ is denoted by $\mathcal{M}_{\mathcal{H}}(\mathcal{E}, \mathcal{E}')$.

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Lemma

If $\Phi \in \mathcal{M}_{\mathcal{H}}(\mathcal{E}, \mathcal{E}')$ then $M_\Phi^*(K(\cdot, w) \otimes e') = K(\cdot, w) \otimes \Phi(w)^* e'$ for each $w \in \Omega$ and $e' \in \mathcal{E}'$.

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Question

When can we find $\Phi \in \mathcal{M}_{\mathcal{H}}(\mathcal{E}, \mathcal{E}')$ *such that* $\|M_\Phi\| \leq 1$ *and* $\Phi(\lambda_i) = W_i$ *for each* $i = 1, \ldots, n$*?*

Operator Pick Interpolation

Let $\mathcal{H}$ be a rkHs on Ω and $\mathcal{E}, \mathcal{E}'$ be two Hilbert spaces. Suppose that $\lambda_1, \ldots, \lambda_n \in \Omega$ be distinct and $W_1, \ldots, W_n \in \mathcal{B}(\mathcal{E}, \mathcal{E}')$. A couple of slides ago, we asked the following question:

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A necessary condition is that the matrix of operators

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is a *positive matrix of operators*. But it is not *sufficient*!

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Definition

A matrix $T = (T_{ij})$ consisting of operators in $\mathcal{H}$ is said to be *positive* if $\langle Tv, v \rangle_{\mathcal{H}^n} \geq 0$ for each $v \in \mathcal{H}^n$ or equivalently, for each $v_1, \ldots, v_n \in \mathcal{H}$,

$$\sum_{i,j=1}^{n} \langle T_{i,j} v_j, v_i \rangle_{\mathcal{H}} \geq 0.$$

Complete Pick Property

Definition (Complete Pick Property)

A rkHs $\mathcal{H}$ on Ω with kernel K is said to have the *complete pick property* if for each $s, t \in \mathbb{N}$, we take any set of distinct points $\lambda_1, \ldots, \lambda_m \in \Omega$ and $W_1, \ldots, W_m \in \mathcal{B}(\mathbb{C}^t, \mathbb{C}^s)$, positivity of the Pick matrix, namely,

$$P_m = \left(\left(1 - W_i W_j^*\right) K(\lambda_i, \lambda_j)\right)_{i,j=1}^{m}$$

implies the existence of $\Phi \in \mathcal{M}_{\mathcal{H}}(\mathbb{C}^t, \mathbb{C}^s)$ such that $\Phi(\lambda_i) = W_i$ and $\|M_\Phi\| \leq 1$.

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Remark: Complete Pick Property implies the Pick property!

McCullough-Quiggin Theorem

Quiggin [Qui93] showed that for a rkHs with (irreducible) kernel K with the Complete Pick Property, it is necessary that for each $\lambda_1, \lambda_2, \ldots, \lambda_{n+1} \in \Omega$, the matrix

$$Q_n = \left(1 - \frac{K(\lambda_i, \lambda_{n+1}) K(\lambda_{n+1}, \lambda_j)}{K(\lambda_i, \lambda_j) K(\lambda_{n+1}, \lambda_{n+1})}\right)_{i,j=1}^{n}$$

is positive semidefinite. McCullough [McC94] showed that it is necessary. (Remark: The current formulation of this theorem is due to Agler and McCarthy [AM00].)

If a rkHs $\mathcal{H}$ has a normalised kernel, that is there is some point $w \in \Omega$ such that $K(z, w) = 1$ for each $z \in \Omega$. Then the matrix Q_n can be made pretty, that is, one just needs to check that the matrix

$$Q_n = \left(1 - \frac{1}{k(\lambda_i, \lambda_j)}\right)_{i,j=1}^{n} \geq 0$$

Final Words and some Open Questions

McCullough-Quiggin Theorem allows us to show that a large class of rkHs are Complete Pick Spaces, for instance, the Hardy Space, the Dirichlet space, the $\mathcal{H}_s$-spaces for $s \leq 0$.

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Is there a "nice" characterisation of Pick Spaces, that is, the rkHs with the Pick property like we do for Complete Pick spaces?

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Serra [Ser05] showed there is a class of rkHs which are Pick but not Complete Pick but none of them were analytic function spaces. Are there any analytic rkHs on $\mathbb{D}$ which are Pick but not Complete Pick?

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Thank you for your attention!