Reproducing kernel Hilbert spaces & (Complete) Pick property

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Notations

- The *open unit disc* $\{z \in \mathbb{C} : |z| < 1\}$ of the complex plane will be denoted by $\mathbb{D}$.
- The boundary of $\mathbb{D}$ namely *the unit circle of the complex plane* will be denoted by $\mathbb{T}$. That is $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.
- For any open subset $\Omega \subset \mathbb{C}$, $\mathcal{O}(\Omega)$ will denote the set of all holomorphic functions on Ω.
- The set of all *bounded* holomorphic functions on the open unit disc $\mathbb{D}$ will be denoted by $H^\infty(\mathbb{D})$.
- If V, W is a normed linear spaces then the set of all bounded operators is denoted by $\mathcal{B}(V, W)$.

Pick Nevanlinna Problem

Question (Motivation?)
Given initial data of n distinct points $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{D}$ and target data of n points $w_1, w_2, \ldots, w_n \in \mathbb{C}$, is there a holomorphic function $\varphi : \mathbb{D} \to \mathbb{C}$ such that $\varphi(\lambda_i) = w_i$ for each $i = 1, \ldots, n$?

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Question (Pick (1916), Nevanlinna (1919))
Given initial data of n *points* $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{D}$ *and target data of* n *points* $w_1, w_2, \ldots, w_n \in \overline{\mathbb{D}}$*, is there a holomorphic function* $\varphi : \mathbb{D} \to \mathbb{C}$ *such that* $\varphi(\lambda_i) = w_i$ *for each* $i = 1, \ldots, n$ *and furthermore* $|\varphi(\lambda)| \leqslant 1$ *for each* $\lambda \in \mathbb{D}$*?*

Answer to Pick Nevanlinna Problem

Theorem (Pick-Nevanlinna Theorem)
Let $(\lambda_i)_{1 \leqslant i \leqslant n}$ *be* n *distinct points in* $\mathbb{D}$ *and let* $(w_i)_{1 \leqslant i \leqslant n} \subset \mathbb{C}$*.*
Then the following are equivalent:

1. *There is a holomorphic function such that*

$$f\left(\lambda_i\right) = w_i \qquad \left(1 \leqslant i \leqslant n\right),$$

   and, moreover, $\|f\|_\infty \leqslant 1$*.*

2. *The matrix* $P = \left(P_{j,k}\right)_{1 \leqslant j,k \leqslant n}$ *is positive semidefinite where*

$$P_{j,k} = \frac{1 - w_j \overline{w_k}}{1 - \lambda_j \overline{\lambda_k}} \qquad (j, k = 1, \ldots, n)$$

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Definition (Pick matrix)
The matrix P associated with the initial data $(\lambda_i)_{1\leqslant i\leqslant n} \subset \mathbb{D}$ and $(w_i)_{1\leqslant i\leqslant n} \subset \mathbb{C}$ is called the (associated) *Pick matrix*.

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Let Ω be a set and let $\mathcal{F}(\Omega, \mathbb{C})$ denote the set of the functions from Ω to $\mathbb{C}$. A Hilbert space $\mathcal{H} \subset \mathcal{F}(\Omega, \mathbb{C})$ is called a *reproducing kernel Hilbert space* (rkHs, in short) if

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1. For each $z \in \Omega$, the evaluation functional

$$\begin{aligned} \mathcal{E}_z : \mathcal{H} &\to \mathbb{C} \\ f &\mapsto f(z) \end{aligned}$$

is bounded.

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1. For each $z \in \Omega$, the evaluation functional
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is bounded.
2. For each $z \in \Omega$, there is some function $f_z \in \mathcal{H}$ such that $f_z(z) \neq 0$.

The kernel function associated to a rkHs

By the definition of rkHs, the evaluation functional $\mathcal{E}_w$ is bounded for each $w \in \Omega$. Therefore, by the *Riesz representation theorem*, there is a unique function $K(\cdot, w) \in \mathcal{H}$ such that

$$f(w) = \langle f, K(\cdot, w) \rangle \qquad (f \in \mathcal{H}) \,. \tag{1}$$

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Definition (the kernel function)
Let $\mathcal{H}$ be rkHs on Ω. The *kernel function associated to* $\mathcal{H}$ is the function

$$K : \Omega \times \Omega \to \mathbb{C}$$

satisfying Equation 1. For each $w \in \Omega$, the function $K(\cdot, w) \in \mathcal{H}$ is called *the reproducing kernel at the point* w.

Primary Examples of rkHs: Analytic Function Spaces

Let $\Omega = \mathbb{D}$ the open unit disc. For each $s \in \mathbb{R}$, consider the set of holomorphic functions $\mathcal{H}_s$ on $\mathbb{D}$

$$\mathcal{H}_s := \left\{ f = \sum_{n \geqslant 0} \hat{f}\,(n)\, z^n \in \mathcal{O}\,(\mathbb{D}) : \sum_{n \geqslant 0} (n+1)^{-s} \left| \hat{f}(n) \right|^2 < \infty \right\}.$$

Each $\mathcal{H}_s$ is a rkHs where $\|f\|^2_{\mathcal{H}_s} := \sum_{n \geqslant 0} (n+1)^{-s} \left| \hat{f}(n) \right|^2$.

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Each $\mathcal{H}_s$ is a rkHs where $\|f\|_{\mathcal{H}_s}^2 := \sum_{n \geqslant 0} (n+1)^{-s} \left| \hat{f}(n) \right|^2$.

s	Name	Kernel Function
-1	Bergman space $A^2(\mathbb{D})$	$K(z,w) = \frac{1}{(1-\bar{w}z)^2}$
0	Hardy space $H^2(\mathbb{D})$	$K(z,w) = \frac{1}{(1-\bar{w}z)}$
1	Dirichlet space $\mathcal{D}$	$K(z,w) = \begin{cases} \frac{1}{z\bar{w}} \log\left(\frac{1}{1-z\bar{w}}\right) & w \neq 0 \\ 1 & w = 0 \end{cases}$

Kernel functions are PSD

If K is a kernel function associated to a reproducing kernel Hilbert space $\mathcal{H}$ on Ω then K is positive semidefinite, that is, for each distinct elements $\lambda_1, \ldots, \lambda_n$ of Ω, the matrix $\left[K\left(\lambda_j, \lambda_l\right)\right]_{1 \leqslant j,l \leqslant n}$ is positive semidefinite.

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Theorem (Aronszajn[1])
If K is a positive semidefinite function on a set Ω, there is a unique reproducing kernel Hilbert space $\mathcal{H}$ on Ω whose reproducing kernel is K.

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The Hardy Space

Hardy space can be equivalently defined as the set of all functions $f \in \mathcal{O}(\mathbb{D})$ such that

$$\sup_{0 \leqslant r < 1} \int_0^{2\pi} \left| f\left(re^{it}\right) \right|^2 \, dt < \infty.$$

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It can be easily shown that

$$\|f\|_{H^2(\mathbb{D})}^2 := \sum_{n \geqslant 0} |a_n|^2 = \sup_{0 \leqslant r < 1} \int_0^{2\pi} \left| f\left(re^{it}\right) \right|^2 \frac{dt}{2\pi}$$

for each $f = \sum_{n \geqslant 0} a_n z^n \in H^2(\mathbb{D})$.

The Multiplier Algebra associated to a rkHs

Definition (Multiplier Algebra)
Let $\mathcal{H}$ be a rkHs on Ω. We define the *multiplier algebra* $\mathcal{M}(\mathcal{H})$ *associated with the rkHs* $\mathcal{H}$ to be

$$\mathcal{M}(\mathcal{H}) := \{\varphi : \Omega \to \mathbb{C} \mid \varphi f \in \mathcal{H} \text{ for each } f \in \mathcal{H}\}.$$

For each $\varphi \in \mathcal{M}(\mathcal{H})$, we define the multiplier norm of φ to be

$$\|\varphi\|_{\mathcal{M}(\mathcal{H})} := \|M_\varphi\|$$

where $M_\varphi : \mathcal{H} \to \mathcal{H}$ is the multiplication by φ operator.

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where $M_\varphi : \mathcal{H} \to \mathcal{H}$ is the multiplication by φ operator. With the above setup, $\mathcal{M}(\mathcal{H})$ becomes a commutative Banach algebra with the multiplication being pointwise product. In practice, rkHs contain the analytic polynomials, hence, in particular, the constant function 1. Consequently, $\mathcal{M}(\mathcal{H}) \subset \mathcal{H}$.

Some Properties of the Multiplier Algebra

Proposition
Let $\mathcal{H}$ be a rkHs on Ω. If $\varphi \in \mathcal{M}(\mathcal{H})$ then for each $w \in \Omega$,

$$M_\varphi^* k(\cdot, w) = \overline{\varphi(w)} k(\cdot, w).$$

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Corollary
With setup in the previous proposition, if $\varphi \in \mathcal{M}(\mathcal{H})$ then we have

$$\sup_{w \in \Omega} |\varphi(w)| \leqslant \|\varphi\|_{\mathcal{M}(\mathcal{H})}.$$

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As a consequence of the previous corollary, we have that $\mathcal{M}(\mathcal{H}) \subset \mathcal{H} \cap \mathcal{B}(\Omega)$ where $\mathcal{B}(\Omega)$ is the set of bounded functions on Ω.

Example: Multiplier of the Hardy Space

Consider the Hardy space $H^2(\mathbb{D})$. From the previous slide, we conclude that

$$\mathcal{M}\left(H^2\right) \subset H^2 \cap \mathcal{B}(\mathbb{D}) = H^\infty(\mathbb{D}).$$

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In fact, if $\varphi \in H^\infty(\mathbb{D})$ and $f \in H^2$, we have that

$$\begin{aligned} \|\varphi f\|_{H^2}^2 &= \sup_{0 \leqslant r < 1} \int_0^{2\pi} \left|\varphi\left(re^{it}\right) f\left(re^{it}\right)\right|^2 \frac{dt}{2\pi} \\ &\leqslant \|\varphi\|_{H^\infty}^2 \sup_{0 \leqslant r < 1} \int_0^{2\pi} \left|f\left(re^{it}\right)\right|^2 \frac{dt}{2\pi} \\ &= \|\varphi\|_{H^\infty}^2 \|f\|_{H^2}^2 . \end{aligned}$$

This shows that $\mathcal{M}(H^2) = H^\infty(\mathbb{D})$.

The Pick Nevanlinna Problem in the language of rkHs

Sarason[2] realised that the Pick Nevanlinna Problem can be recasted for rkHs. In the following theorem, let $\mathcal{H} = H^2(\mathbb{D})$.

Theorem
Let $(\lambda_j)_{1 \leqslant j \leqslant n}$ be n distinct points in $\mathbb{D}$ and let $(w_j)_{1 \leqslant j \leqslant n} \subset \mathbb{C}$. Then the following are equivalent:

1. *There is $\varphi \in \mathcal{M}(\mathcal{H}) = H^\infty(\mathbb{D})$ with $\|\varphi\|_{\mathcal{M}(\mathcal{H})} = \|M_\varphi\| = \|\varphi\|_{H^\infty} \leqslant 1$ and*

$$f(\lambda_j) = w_j \qquad (1 \leqslant j \leqslant n),$$

2. *The matrix $P = (P_{j,k})_{1 \leqslant j,k \leqslant n}$ is positive semidefinite where*

$$P_{j,k} = (1 - w_j \overline{w_k})\, k(\lambda_i, \lambda_j) = \frac{1 - w_j \overline{w_k}}{1 - \lambda_j \overline{\lambda_k}} \qquad (j,k = 1, \ldots, n)$$

[2] Donald Sarason. "Generalized Interpolation in H^∞". In: *Transactions of the American Mathematical Society* (1967).

Existence of an interpolating contractive multiplier implies positivity of Pick matrix

Let $\mathcal{H}$ be a rkHs in Ω. Let $(\lambda_i)_{1 \leqslant i \leqslant n}$ be n distinct points in Ω and let $(w_i)_{1 \leqslant i \leqslant n} \subset \mathbb{C}$.

Suppose that there exists a $\varphi \in \mathcal{M}(\mathcal{H})$ such that $\|\varphi\|_{\mathcal{M}(\mathcal{H})} \leqslant 1$ and $\varphi(\lambda_i) = w_i$ for each $i = 1, 2, \ldots, n$. Let P be the corresponding Pick matrix.

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Suppose that there exists a $\varphi \in \mathcal{M}(\mathcal{H})$ such that $\|\varphi\|_{\mathcal{M}(\mathcal{H})} \leqslant 1$ and $\varphi(\lambda_i) = w_i$ for each $i = 1, 2, \ldots, n$. Let P be the corresponding Pick matrix. For each $a_1, a_2, \ldots, a_n \in \mathbb{C}$, we have

$$\left\langle \left(I - M_\varphi M_\varphi^*\right) \left(\sum_{j=1}^n a_j k(\cdot, \lambda_j)\right), \left(\sum_{i=1}^n a_i k(\cdot, \lambda_i)\right) \right\rangle_{\mathcal{H}}$$

$$= \sum_{i,j=1}^n a_j \bar{a_i} \left(1 - \bar{w}_j w_i\right) k(\lambda_i, \lambda_j) = \sum_{i,j=1}^n a_j \bar{a}_i P_{ij}$$

$$= \left\langle P\left(\sum_{j=1}^n a_j e_j\right), \sum_{i=1}^n a_i e_i \right\rangle_{\mathbb{C}^n}.$$

Pick Spaces

We showed that in any rkHs, existence of contractive multiplier implies the positivity of the Pick matrix. The converse need to be true (for instance, this does not hold in the Bergman space).

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Definition (Pick space)
Let $\mathcal{H}$ be an rkHs on Ω. We say that $\mathcal{H}$ is a *Pick space* if for every distinct n points $\lambda_1, \lambda_2, \ldots, \lambda_n \in \Omega$ and every n points $w_1, w_2, \ldots, w_n \in \mathbb{C}$, the positivity of the corresponding Pick matrix implies that there is some $\varphi \in \mathcal{M}(\mathcal{H})$ such that $\|\varphi\|_{\mathcal{M}(\mathcal{H})} \leqslant 1$ and $\varphi(\lambda_i) = w_i$ for each $i = 1, \ldots, n$.

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Example
Hardy and Dirichlet space are Pick spaces but Bergman space is not.

Interpolating Operators?

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Question

When can we find a multiplier $\Phi \in \mathcal{M}_{\mathcal{H}}(\mathcal{E}, \mathcal{E}')$ (what does this mean?) such that $\|M_\Phi\| \leqslant 1$ and $\Phi(\lambda_i) = W_i$ for each $i = 1, \ldots, n$?

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To this end, we define the notion of ...

Vector Valued Reproducing Kernel Hilbert Spaces

Definition (Vector Valued rkHs)
Let $\mathcal{E}$ be a Hilbert space. Let $\mathcal{H}$ be a reproducing kernel Hilbert space on Ω with kernel K. Consider the tensor product $\mathcal{H} \otimes \mathcal{E}$.

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We identify each element $f \in \mathcal{H} \otimes \mathcal{E}$ as a function from Ω to $\mathcal{E}$ which satisfy the following property:

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The aforementioned product $\mathcal{H} \otimes \mathcal{E}$ is said to be $\mathcal{E}$*-valued reproducing kernel Hilbert space*.

An example of vector valued rkHs

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Let $\mathcal{H}$ be a rkHs on Ω and let $\mathbb{C}^n$ be the (usual) inner product space. If $f \in \mathcal{H}$ and e_j is an element of the standard orthonormal basis of $\mathbb{C}^n$ then by definition, we have that for $w \in \Omega$ and $e_i \in \mathbb{C}^n$,

$$\begin{aligned}\langle f \otimes e_j, K(\cdot, w) \otimes e_i \rangle_{\mathcal{H} \otimes \mathcal{E}} &= \langle f, K(\cdot, w) \rangle_{\mathcal{H}} \langle e_j, e_i \rangle_{\mathbb{C}^n} \\ &= f(w) \delta_{ij}\end{aligned}$$

and hence $\langle (f \otimes e_j)(w), e_j \rangle_{\mathbb{C}^n} = f(w) \delta_{ij}$.

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and hence $\langle (f \otimes e_j)(w), e_j \rangle_{\mathbb{C}^n} = f(w)\delta_{ij}$.

Therefore, $f \otimes e_j$ is the following $\mathbb{C}^n$ valued function (when each element in $\mathbb{C}^n$ is viewed as row vector):

$$(f \otimes e_j)(w) = \left(0, \ldots, \underbrace{f(w)}_{j\text{th entry}}, \ldots, 0 \right) \qquad (w \in \Omega).$$

Multipliers of Vector Valued rkHs

Let $\mathcal{E}, \mathcal{E}'$ be two Hilbert space and $\mathcal{H}$ be a reproducing kernel Hilbert space on Ω. Consider $\mathcal{H} \otimes \mathcal{E}$ and $\mathcal{H} \otimes \mathcal{E}'$ as vector valued reproducing kernel Hilbert spaces on Ω.

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A *multiplier* Φ is a $B\left(\mathcal{E}, \mathcal{E}'\right)$-valued function on Ω such that $\Phi f \in \mathcal{H} \otimes \mathcal{E}'$ for each $f \in \mathcal{H} \otimes \mathcal{E}$. Note that Φf is viewed as a $\mathcal{E}'$-valued function in the following sense:

$$(\Phi f)(w) = \Phi(w) f(w) \qquad (w \in \Omega).$$

The set of all multipliers from $\mathcal{H} \otimes \mathcal{E}$ to $\mathcal{H} \otimes \mathcal{E}'$ is denoted by $\mathcal{M}_{\mathcal{H}}\left(\mathcal{E}, \mathcal{E}'\right)$.

Multipliers of Vector Valued rkHs

Let $\mathcal{E}, \mathcal{E}'$ be two Hilbert space and $\mathcal{H}$ be a reproducing kernel Hilbert space on Ω. Consider $\mathcal{H} \otimes \mathcal{E}$ and $\mathcal{H} \otimes \mathcal{E}'$ as vector valued reproducing kernel Hilbert spaces on Ω.

A *multiplier* Φ is a $B(\mathcal{E}, \mathcal{E}')$-valued function on Ω such that $\Phi f \in \mathcal{H} \otimes \mathcal{E}'$ for each $f \in \mathcal{H} \otimes \mathcal{E}$. Note that Φf is viewed as a $\mathcal{E}'$-valued function in the following sense:

$$(\Phi f)(w) = \Phi(w) f(w) \qquad (w \in \Omega).$$

The set of all multipliers from $\mathcal{H} \otimes \mathcal{E}$ to $\mathcal{H} \otimes \mathcal{E}'$ is denoted by $\mathcal{M}_{\mathcal{H}}(\mathcal{E}, \mathcal{E}')$.

Lemma
If $\Phi \in \mathcal{M}_{\mathcal{H}}(\mathcal{E}, \mathcal{E}')$ then $M_\Phi^* (K(\cdot, w) \otimes e') = K(\cdot, w) \otimes \Phi(w)^* e'$ for each $w \in \Omega$ and $e' \in \mathcal{E}'$.

Operator Pick Interpolation

Let $\mathcal{H}$ be a rkHs on Ω and $\mathcal{E}, \mathcal{E}'$ be two Hilbert spaces.

Operator Pick Interpolation

Let $\mathcal{H}$ be a rkHs on Ω and $\mathcal{E}, \mathcal{E}'$ be two Hilbert spaces. Suppose that $\lambda_1, \ldots, \lambda_n \in \Omega$ be distinct and $W_1, \ldots, W_n \in \mathcal{B}(\mathcal{E}, \mathcal{E}')$. A couple of slides ago, we asked the following question:

Question
When can we find $\Phi \in \mathcal{M}_{\mathcal{H}}(\mathcal{E}, \mathcal{E}')$ *such that* $\|M_\Phi\| \leqslant 1$ *and* $\Phi(\lambda_i) = W_i$ *for each* $i = 1, \ldots, n$*?*

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A necessary condition is that the matrix of operators

$$P_n = \left(\left(1 - W_i W_j^*\right) K(\lambda_i, \lambda_j)\right)_{i,j=1}^{n}$$

is a *positive matrix of operators*. But it is not *sufficient*!

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Definition
A matrix $T = (T_{ij})$ consisting of operators in $\mathcal{H}$ is said to be *positive* if $\langle Tv, v\rangle_{\mathcal{H}^n} \geqslant 0$ for each $v \in \mathcal{H}^n$.

Complete Pick Property

Definition (Complete Pick Property)
A rkHs $\mathcal{H}$ on Ω with kernel K is said to have the *complete pick property* if for each $s, t \in \mathbb{N}$, we take any set of distinct points $\lambda_1, \ldots, \lambda_m \in \Omega$ and $W_1, \ldots, W_m \in \mathcal{B}\left(\mathbb{C}^t, \mathbb{C}^s\right)$, positivity of the Pick matrix, namely,

$$P_m = \left(\left(1 - W_i W_j^*\right) K\left(\lambda_i, \lambda_j\right)\right)_{i,j=1}^{m}$$

implies the existence of $\Phi \in \mathcal{M}_{\mathcal{H}}\left(\mathbb{C}^t, \mathbb{C}^s\right)$ such that $\Phi\left(\lambda_i\right) = W_i$ and $\|M_\Phi\| \leqslant 1$.

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Remark: Complete Pick Property implies the Pick property!

McCullough-Quiggin Theorem

Quiggin[3]: for a rkHs with (irreducible) kernel K with the Complete Pick Property, it is necessary that for each $\lambda_1, \lambda_2, \ldots, \lambda_{n+1} \in \Omega$, the matrix

$$Q_n = \left(1 - \frac{K(\lambda_i, \lambda_{n+1}) K(\lambda_{n+1}, \lambda_j)}{K(\lambda_i, \lambda_j) K(\lambda_{n+1}, \lambda_{n+1})}\right)_{i,j=1}^{n}$$

is positive semidefinite. McCullough[4] showed that it is necessary. (The current formulation is due to Agler and McCarthy[5]).

[3] Peter Quiggin. "For which reproducing kernel Hilbert spaces is Pick's theorem true!" In: *Integral Equations Oper. Theory* (1993).

[4] Scott McCullough. "The local de Branges-Rovnyak construction and complete Nevanlinna-Pick kernels". In: *Algebraic methods in operator theory*. Birkhäuser Boston, Boston, MA, 1994. ISBN: 0-8176-3745-1.

[5] Jim Agler and John E. McCarthy. "Complete Nevanlinna-Pick kernels". In: *J. Funct. Anal.* (2000).

McCullough-Quiggin Theorem (pretty version)

If a rkHs $\mathcal{H}$ has a normalised kernel, that is there is some point $w \in \Omega$ such that $K(z, w) = 1$ for each $z \in \Omega$. Then the matrix Q_n can be made pretty, that is, one just needs to check that the matrix

$$Q_n = \left(1 - \frac{1}{K(\lambda_i, \lambda_j)}\right)_{i,j=1}^{n} \geqslant 0.$$

for each $\lambda_1, \lambda_2, \ldots, \lambda_n \in \Omega$.

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Example (Hardy Space has the Complete Pick Property)
The kernel for the Hardy Space is given by $K(z, w) = \frac{1}{1-\overline{w}z}$. Note that $1 - \frac{1}{K(z,w)} = z\overline{w}$. Hence, by McCullough-Quiggin Theorem, Hardy space has the Complete Pick Property.

Final Words and some Open Questions

McCullough-Quiggin Theorem allows us to show that a large class of rkHs are Complete Pick Spaces, for instance, the Hardy Space, the Dirichlet space, the $\mathcal{H}_s$-spaces for $s \leqslant 0$.

[6] António Serra. "New examples of non-complete Pick kernels". In: *Integral Equations Operator Theory* 53.4 (2005), pp. 553–572. ISSN: 0378-620X,1420-8989. DOI: 10.1007/s00020-005-1363-7. URL: https://doi.org/10.1007/s00020-005-1363-7.

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Question
Is there a "nice" characterisation of Pick Spaces, that is, the rkHs with the Pick property like we do for Complete Pick spaces?

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Serra[6] showed there is a class of rkHs which are Pick but not Complete Pick but none of them were analytic function spaces. Are there any analytic rkHs on $\mathbb{D}$ which are Pick but not Complete Pick?

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Thank you for your attention!