Chapter 3 — Convergence in Measure

Last Updated: July 30, 2022

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§1 Modes of Convergence

For simplicity, we will deal with $\mathbb{R}$-valued functions only. For further remarks, see the footnote in the book.

Definition §1.0.1. Let $\left(X, \mathscr{A}, \mu\right)$ be a measure space. Let $f, f_1, f_2, \ldots : X \to \mathbb{R}$ be a sequence of $\mathscr{A}$-measurable functions. We say that f_n converges to f *in measure* if for every $\varepsilon > 0$, we have that

$$\lim_n \mu\left(\left\{x \in X : \left|f_n(x) - f(x)\right| > \varepsilon\right\}\right) = 0$$

Remark §1.0.2. ⚠ General convergence in measure is neither implied nor implies convergence almost everywhere! As the following examples show:

Example §1.0.3.

1. Consider $(X, \mathscr{B}(\mathbb{R}), \lambda)$. Consider the sequence of functions $\{\chi_{[n,\infty)}\}$. Then $\chi_{[n,\infty)} \to 0$ function everywhere (hence, almost everywhere).But it does not converge in measure! To see this take $\varepsilon = \frac{1}{2}$.

2. Consider the Borel subsets of $[0,1)$ with the Lebesgue measure. Consider the sequence whose first term is the characteristic function of $[0,1)$. The next two terms are the characteristic functions of $[0, \frac{1}{2}$ and $[\frac{1}{2}, 1)$. The next four terms are the characteristic functions of $[0,1/4)$, $[1/4,1/2)$, $[1/2,3/4)$ and $[3/4,1)$. Well this goes on...

   Now, let $\varepsilon > 0$ be given. Clearly this sequence converges to zero in measure but for each $x \in [0,1)$, $\left\{f_n(x)\right\}$ has infinitely many zeroes and ones, so, it does not converge!

Proposition §1.0.4 ($\mu < \infty$ and $\left\{f_n\right\} \to f$ a.e. implies $\left\{f_n\right\} \to f$ in measure)**.** *Let $(X, \mathscr{A}, \mu)$ be a measure space. Let $f, f_1, f_2 \ldots$ be a sequence of $\mathscr{A}$-measurable real valued functions on X. If μ is finite and $\left\{f_n\right\}$ converges almost everywhere to f then $\left\{f_n\right\}$ converges in measure to f.*

Proof. We need to prove that

$$\lim_n \mu\left(\left\{x \in X : |f_n(x) - f(x)| > \varepsilon\right\}\right) = 0$$

for every $\varepsilon > 0$. Let $\varepsilon > 0$ be given. Define $A_n = \{x \in X : |f_n(x) - f(x)| > \varepsilon\}$ and $B_n = \cup_{k \geq n} A_k$.

It is easy to see that $\{B_n\}$ is a decreasing sequence of subsets of X. We claim that $\cap_n B_n$ is contained in the set $\{x \in X : \{f_n(x)\}$ does not converge to $f(x)\}$. To see this, let $x \in \cap B_n$, then $x \in A_n$ for infinitely many n. If $\{f_n(x)\} \to f(x)$ then there must be some $N \in \mathbb{N}$ such that $|f_n(x) - f(x) < \varepsilon$ for every $n \geq N$. Since $x \in A_n$ for infinitely many n, we can select a $k \geq N$ such that $x \in A_k$. Then we have that $\varepsilon < |f_k(x) - f(x)| < \varepsilon$ which is absurd! Note that the first inequality is due to $x \in A_k$ and the second inequality holds due to $k \geq N$. Thus, this completes the proof of our claim.

Notice that the set $\{x \in X : \{f_n(x)\}$ does not converge to $f(x)\}$ is μ- negligible set. Thus $\mu(\cap_n B_n) = 0$.

Since $\{B_n\}$ is a decreasing sequence of sets and $\mu < \infty$, Proposition 1.2.5 in the book implies that $\lim_n \mu(B_n) = \mu(\cap_n B_n) = 0$.

Now, observe that $A_n \subseteq B_n$ for every $n \in \mathbb{N}$. This observation implies that $\mu(A_n) \leq \mu(B_n)$ and hence $\mu(A_n) = 0$ as $n \to \infty$.

This completes the proof. □

Proposition §1.0.5. *Let $(X, \mathscr{A}, \mu)$ be a measure space. Let f and $f_1, f_2, \ldots$ be a sequence of $\mathscr{A}$ measurable real valued functions on X. If $\{f_n\}$ converges to f in measure then then there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which converges almost everywhere to f.*

Let X be a set. A sequence of real valued functions $\{f_n\}$ on X is said to *converge uniformly* to a real valued function f on X if for every $\varepsilon > 0$ there is a $N \in \mathbb{N}$ such that for every $n \geq N$ and every $x \in X$, we have that $|f_n(x) - f(x)| < \varepsilon$.

Proposition §1.0.6 (Egoroff's Theoren)**.** *Let $(X, \mathscr{A}, \mu)$ be a measure space. Let f and $f_1, f_2, \ldots$ be a sequence of $\mathscr{A}$-measurable real valued functions on X. If μ is finite and $\{f_n\}$ converges to f almost everywhere then for each $\varepsilon > 0$ there is some $B \in \mathscr{A}$ satisfying $\mu(B^c) < \varepsilon$ and $\{f_n\}$ converges to f uniformly on B.*

Outline of Proof. Let $\varepsilon > 0$. For each n, let $g_n = \sup_{k \geq n} |f_n - f|$.

Let $F = \{x \in X : \{f_n(x)\}$ does not converge to $f(x)\}$. Then by assumption F is a μ-negligible set.

We claim that g_n is finite almost everywhere for every n. Let $G_n = \{x \in X : g_n(x) = +\infty\}$. We show that $G_n \subseteq F$. If we show this then $\mu(G_n) = 0$ and we will done with the proof of our claim.

It is rather easy to show that $F^c \subseteq G_n^c$. Let $x \in F$. Then there is some $N \in \mathbb{N}$ such that $|f_k(x) - f(x)| < 1$ for every $k \geq N$.

If $n \geq N$ then we have that $g_n(x) = \sup_{k \geq n} |f_k(x) - f(x)| \leq 1$.

If $n < N$ then $g_n(x) \leq \max\{|f_n(x) - f(x)|, \ldots, |f_{N-1} - f(x)|, 1\}$.

Either ways, $g_n(x) < +\infty$. Thus this shows that $x \in G_n^c$.

Now, let $G = \{x \in X : \{g_n(x)\}$ does not converge to $0\}$. Complete the proof by showing that $G \subseteq F$ and following up the textbook!

□

Egoroff's Theorem provides a motivation for the following definition:

Definition §1.0.7. Let $(X, \mathscr{A}, \mu)$ be a measure space. Let f and $f_1, f_2, \ldots$ be a sequence of $\mathscr{A}$-measurable real valued functions on X. Then $\{f_n\}$ converges to f *almost uniformly* if for each $\varepsilon > 0$ there is some $B \in \mathscr{A}$ such that $\mu(B^c) < \varepsilon$ and $\{f_n\}$ converges to f uniformly on B.

We remark that if $\{f_n\}$ converges to f almost uniformly then $\{f_n\}$ converges to f almost everywhere. To see this, for $n \in \mathbb{N}$, select $B_n \in \mathscr{A}$ such that $\mu\left(B_n^c\right) < \frac{1}{n}$ and $\{f_n\}$ converges to f uniformly on B_n. Let $B = \bigcap_n B_n$. Then $\mu(B^c) \leq \mu(B_n^c) < \frac{1}{n}$ for every $n \in \mathbb{N}$. Thus $\mu(B^c) = 0$. We claim that $\lim_n f_n = f$ everywhere on B. This is easy to see: if $x \in B$ then $x \in B_n$ for some $n \in \mathbb{N}$ and since $\{f_n\}$ converges to f uniformly on B_n, we have that $\lim_n f_n(x) = f(x)$.

It follows from Egoroff's theorem that on a finite measure space, almost everywhere convergence is equivalent to almost uniform convergence.