A REVIEW OF INTERPOLATING SEQUENCES IN SPACES WITH THE COMPLETE PICK PROPERTY

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In this paper under review titled "Interpolating sequences in spaces with the complete Pick property" (see [Ale+19]), Aleman, Hartz, McCarthy, and Richter generalise a result due to Carleson for the Hardy space [Car58b] and of Marshall and Sundberg for the Dirichlet space [MS94].

Carleson showed in [Car58b] that for a complex sequence (λ_i) satisfying $|\lambda_i| < 1$ for each $i \in \mathbb{N}$, the following are equivalent:

(1) For any bounded complex sequence (w_i), there is $f \in H^\infty(\mathbb{D})$ (which is the *multiplier algebra* of the Hardy-Hilbert space $H^2(\mathbb{D})$) such that $f(z_i) = w_i$ for each $i \in \mathbb{N}$.

(2) For each $i \in \mathbb{N}$, there is some $c > 0$ such that

$$\prod_{j \neq i} \left| \frac{\lambda_i - \lambda_j}{1 - \bar{\lambda}_i \lambda_j} \right| \geq c > 0. \tag{0.1}$$

In modern terminology, a sequence $(\lambda_i) \subset \mathbb{D}$, the open unit disk, is called *an interpolating sequence for* $H^\infty(\mathbb{D})$ if it satisfies at least one of the condition (hence, both) of Carleson's result. Intuitively, it means a sequence is interpolating for $H^\infty(\mathbb{D})$ iff the points are sufficiently spread out in the hyperbolic metric of the open unit disc. This is an conclusion that one can make by observing equation 0.1. An analogous result can be formulated in the context of *reproducing kernel Hilbert spaces* (or rkHs, inshort).

Aleman, Hartz, McCarthy, and Richter in [Ale+19] show that for a complete Pick space $\mathcal{H}$, a sequence is interpolating for its multiplier algebra $\mathcal{M}(\mathcal{H})$ iff it is separated and generates a Carleson measure. More specifically, if $\mathcal{H}$ is complete Pick space (a special rkHs where positivity of Pick matrix with matricial entries implies interpolation by multipliers, see [AM02] for a precise definition) then the following are equivalent:

(IM): $(\Lambda) = (\lambda_i) \subset X$ is *interpolating for* $\mathcal{M}(\mathcal{H})$, that is, whenever $(w_i)_{i \in \mathbb{N}} \in \ell^\infty$, there is a multiplier $\varphi \in \mathcal{M}(\mathcal{H})$ such that $\varphi(\lambda_i) = w_i$ for each i, and,

(S+C): the following two hold:

(S): Λ is *separated with the pseudometric* d on X given by

$$d(z, w) = \sqrt{1 - \frac{|k(z, w)|^2}{k(z, z)\, k(w, w)}} \qquad (z, w \in X),$$

that is, there is some $c > 0$ such that $d(\lambda_i, \lambda_j) \geq c > 0$ for each $i \neq j$ and

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(C): the atomic measure μ given by $\mu = \sum_i \frac{\delta_{\lambda_i}}{k(\lambda_i, \lambda_i)}$ is a *Carleson measure for $\mathcal{H}$*, in other words, there is some $c > 0$ such that

$$\sum_i \frac{|f(\lambda_i)|^2}{k(\lambda_i, \lambda_i)} \leq c \|f\|^2 \qquad (f \in \mathcal{H}).$$

This settled a 20 year problem posed by Agler and McCarthy in Chapter 9 of their monograph *Pick interpolation and Hilbert function spaces* [AM02]. It is interesting to note that the proof used an equivalent form of the Kadison Singer problem, called the Paving Conjecture (as demonstrated in [And79]). The long standing Kadison Singer problem was resolved by Marcus, Spielman, and Srivastava in [MSS15] in 2013.

Finally, the authors explore interpolating sequences for multipliers of pairs of reproducing kernel Hilbert spaces. Tsikalas resolved some of the problems asked by the authors in his paper [Tsi23].

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