On Intuitionistic Fuzzy order of an Element of a Group

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Abstract: *In this paper an attempt has been made to define the notion of intuitionistic fuzzy order of an element in intuitionistic fuzzy subgroup . Here we prove that every element of a group and its inverse have the same intuitionistic fuzzy order . We also define the order of intuitionistic fuzzy subgroup and prove the Lagrange's Theorem in the intuitionistic fuzzy case .Some properties of the intuitionistic fuzzy order of an element has been discussed.*

Keywords: *Intutionistic fuzzy subgroup (IFSG) , Intuitionistic fuzzy normal subgroup (IFNSG) , Intuitionistic fuzzy order,*

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1. Introduction : After the introduction of the concept of fuzzy set by Zadeh [9] several researches were conducted on the generalization of the notion of fuzzy set. The idea of Intuitionistic fuzzy set was given by Atanassov [1]. The concept of fuzzy order of an element in fuzzy subgroup has been defined by Suryansu Ray [8].The latest detail of this topic can be found in [3]. In this paper we introduce the notion of intuitionistic fuzzy order of an element of intuitionistic fuzzy subgroup which is quite different from the fuzzy order of an element in fuzzy subgroup. We also study some of its properties.

2. Preliminaries :

Definition (2.1)[1] Let X be a fixed non-empty set. **An Intuitionistic fuzzy set** (IFS) A of X is an object of the following form $A = \{< x , \mu_A(x) , \nu_A(x) > : x \in X\}$, where $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ define the degree of membership and degree of non-membership of the element $x \in X$ respectively and for any $x \in X$, we have $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Remark (2.2): When $\mu_A(x) + \nu_A(x) = 1$, i.e. when $\nu_A(x) = 1 - \mu_A(x) = \mu_A{}^c(x)$. Then A is called **fuzzy set.**

Definition (2.3)[1] Let $A = \{ < x, \mu_A(x), \nu_A(x) > : x \in X\}$ and $B = \{ < x , \mu_B(x) , \nu_B(x) > : x \in X\}$ be any two IFS's of X , then

(i) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$

(ii) $A = B$ if and only if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ for all $x \in X$

(iii) $A \cap B = \{ < x, (\mu_A \cap \mu_B)(x) , (\nu_A \cap \nu_B)(x) > : x \in X\}$, where

$(\mu_A \cap \mu_B)(x) = \text{Min}\{ \mu_A(x) , \mu_B(x)\}$and $(\nu_A \cap \nu_B)(x) = \text{Max}\{ \nu_A(x) , \nu_B(x) \}$ (iv) $A \cup B = \{ < x, (\mu_A \cup \mu_B)(x) , (\nu_A \cup \nu_B)(x) > : x \in X\}$, where

$(\mu_A \cup \mu_B)(x) = \text{Max}\{ \mu_A(x) , \mu_B(x)$ and $(\nu_A \cup \nu_B)(x) = \text{Min}\{ \nu_A(x) , \nu_B(x) \}$

Definition (2.4)[4 ,5] An IFS $A = \{ < x, \mu_A(x), \nu_A(x) >: x \in G\}$ of a group G is said to be **intuitionistic fuzzy subgroup** of G (In short IFSG) of G if

(i) $\mu_A(xy) \geq \text{Min} \{\mu_A(x), {}_A(y)\}$

(ii) $\mu_A(x^{-1}) = \mu_A(x)$

(iii) $\nu_A(xy) \leq \text{Max} \{\nu_A(x), \nu_A(y)\}$

(iv) $\nu_A(x^{-1}) = \nu_A(x)$, for all $x , y \in G$

Or Equivalently A is IFSG of G if and only if

$\mu_A(xy^{-1}) \geq \text{Min} \{\mu_A(x), {}_A(y)\}$ and $\nu_A(xy) \leq \text{Max} \{\nu_A(x), \nu_A(y)\}$

Definition (2.5)[5] An IFSG $A = \{ < x, \mu_A(x), \nu_A(x) > : x \in G\}$ of a group G said to be **intuitionistic fuzzy normal subgroup** of G (In short IFNSG) of G if

(i) $\mu_A(xy) = \mu_A(yx)$

(ii) $\nu_A(xy) = \nu_A(yx)$, for all $x , y \in G$

Or Equivalently A is an IFNSG A of a group G is **normal** iff
$\mu_A(y^{-1} x y) = \mu_A(x)$ and $\nu_A(y^{-1} x y) = \nu_A(x)$, for all $x , y \in G$

3. Intuitionistic fuzzy order of an element :

Definition(3.1) Let A be IFSG of finite a group G. Let x be any element of G, then the **intuitionistic fuzzy order** of the element x w.r.t A is denoted by $IFO_A(x)$ and is defined as :

$IFO_A(x) = O(H(x))$, where $H(x) = \{ y \in G : \mu_A(y) \geq \mu_A(x) \text{ and } \nu_A(y) \leq \nu_A(x)\}$

Thus $H(e) = \{ y \in G : \mu_A(y) = \mu_A(e) \text{ and } \nu_A(y) = \nu_A(e)\}$. So $IFO_A(e) = O(H(e))$.

Theorem(3.2) For any IFSG A of a group G and for any element x of G, H(x) is a subgroup of G .

Proof. Clearly, $H(x) \neq \varnothing$ for $x, e \in H(x)$. Let $a , b \in H(x)$ be any element, then
$\mu_A(a) \geq \mu_A(x)$, $\nu_A(a) \leq \nu_A(x)$ and $\mu_A(b) \geq \mu_A(x)$, $\nu_A(b) \leq \nu_A(x)$
$\mu_A(ab^{-1}) \geq \text{Min}\{\mu_A(a), \mu_A(b)\} \geq \mu_A(x)$ and $\nu_A(ab^{-1}) \leq \text{Max}\{\nu_A(a) , \nu_A(b) \} \leq \nu_A(x)$
Thus $ab^{-1} \in H(x)$. Hence H(x) is subgroup of G.

Theorem(3.3) For any IFSF A of G and for any element $x \neq e$ of G , we have $H(e) \subseteq H(x)$ and so $IFO_A(e) \leq IFO_A(x)$

Proof. Let $z \in H(e)$ be any element, then we have $\mu_A(z) = \mu_A(e)$ and $\nu_A(z) = \nu_A(e)$

i.e. $\mu_A(z) \geq \text{Max}\{ \mu_A(x) : \text{for all } x \in G \}$ and $\nu_A(z) \leq \text{Min}\{ \nu_A(x) : \text{for all } x \in G \}$

$\Rightarrow$ $\mu_A(z) \geq \mu_A(x)$ and $\mu_A(z) \leq \nu_A(x)$ for all $x \in G$

$\Rightarrow$ $z \in H(x)$. Thus $H(e) \subseteq H(x)$ and so $IFO_A(e) \leq IFO_A(x)$

Definition(3.4) Let A be IFSG of a finite group G , then the intuitionistic fuzzy order of IFSG A is denote by IFO(A) and is defined by :

$$IFO(A) = \text{Min}\{ IFO_A(x) : \text{for all } x \in G \} = IFO_A(e)$$

Theorem(3.5): Let A be IFSG of finite group G and $x \in G$ be any element, then

(i) $O(x) \mid IFO_A(x)$

(ii) $IFO_A(x) \mid O(G)$

Proof. (i) Let $O(x) = m$. Then $K = \{ x , x^2 , ..., x^{m-1} , x^m = e \}$ is a subgroup of G.

Also $\mu_A(x^2) \geq \mu_A(x)$ and $\nu_A(x^2) \leq \nu_A(x) \Rightarrow x^2 \in H(x)$

Similarly, $x^3 , x^4 , ..., x^{m-1} , x^m \in H(x)$

$\Rightarrow K = \{ x , x^2 , ..., x^{m-1} , x^m = e \} \subseteq H(x)$. Thus K is a subgroup of H(x)

Therefore by Lagrange's Theorem $O(K) \mid O(H(x))$ i.e. $O(x) \mid IFO_A(x)$

(ii) Since H(x) is a subgroup of G . Therefore by Lagrange's Theorem $O(H(x)) \mid O(G)$ i.e. $IFO_A(x) \mid O(G)$

Theorem (3.6)(Lagrange's theorem for IFSG): Let G be a finite group and A be IFSG of G , then IFO(A) divides O(G).

Proof. By theorem (1.5)(ii) part , we have $IFO_A(x) \mid O(G)$ for all $x \in G$

Therefore $\text{Min}\{ IFO_A(x) : \text{for all } x \in G \} \mid O(G) \Rightarrow IFO(A) \mid O(G)$.

Proposition(3.7): For any IFSF A of G and for any element x of G , we have $IFO_A(x^{-1}) = IFO_A(x)$

Proof. By definition , we have $IFO_A(x^{-1}) = O(H(x^{-1}))$, where

$H(x^{-1}) = \{ y \in G : \mu_A(y) \geq \mu_A(x^{-1})$ and $\nu_A(y) \leq \nu_A(x^{-1})\}$

But $\mu_A(x^{-1}) = \mu_A(x)$ and $\nu_A(x^{-1}) = \nu_A(x)$ for all $x \in G$

So, $H(x^{-1}) = \{ y \in G : m_A(y) \geq m_A(x)$ and $n_A(y) \leq n_A(x)\} = H(x)$

Therefore $O(H(x^{-1})) = O(H(x)) = IFO_A(x)$

Hence $IFO_A(x^{-1}) = IFO_A(x)$

Proposition(3.8) :Let G be a ûnite group and let A be an Intuitionistic fuzzy subgroup of G. If $O(y) \mid O(x)$ and $x, y \in <z>$ for some $z \in G$, then

$$\mu_A(x) \leq \mu_A(y) \text{ and } \nu_A(x) \geq {}_A(y).$$

Proof. Let $O(y) = k$. Then $O(x) = kq$ for some $q \in N$.

Now $x , y \in < z >$ for some $z \in G$, therefore let $y = z^i$ and $x = z^j$ for some $i, j \in Z$. Hence $z^{ik} = e = z^{jkq}$. Thus $y = x^q$.

Hence $\mu_A(y) = \mu_A(x^q)$ e" $\mu_A(x)$ and $\nu_A(y) = \nu_A(x^q) \leq {}_A(x)$

Equality of O(x) and O(y) does not imply the equality of $IFO_A(x)$ and $IFO_A(y)$, as is shown in the following example:

Example (3.9). Let G ={e, a, b, ab} be the Klein four-group. Deûne the Intuitionistic fuzzy subgroup A of G by :

A = {< e , t_0 , s_0 > , < ab , t_0 , s_1 > , < a , t_1 , s_1 > , < b , t_1 , s_1 > }, where $t_0 > t_1$ and $s_0 < s_1$ and t_i , s_i [0,1] and $t_i + s_i \leq 1$, for i = 1, 2. Clearly, O(a)= O(ab)=2, but $IFO_A(a)$=2and $IFO_A(ab)$=1. Also observe that in this example, O(a)|O(ab), but $\mu_A(ab) > \mu(a)$ and $\nu_A(ab) < \nu_A(a)$

Remark : Here Proposition 3.8. does n't hold, since the elements a and ab do not lie in the same cyclic subgroup of G.

Theorem (3.10):.Let A be IFSG of a finite cyclic group G = < a > . Let x , y be any two element of G such that O(x) = O(y) , then $IFO_A(x) = IFO_A(y)$.

Proof. By proposition (3.8) , we get $\mu_A(x)= \mu_A(y)$ and $\nu_A(x) = \nu_A(y)$

Therefore H(x) = H(y) and so O(H(x)) = O(H(y)) and hence $IFO_A(x) = IFO_A(y)$.

Theorem (3.11).Let A be a IFNSG of a group G. Then

$IFO_A(x) = IFO_A(y^{-1} xy)$ for all x, y “ G.

Proof. Let x, y “ G. Then we have $\mu_A(x) = \mu_A(y^{-1} xy)$ and $\nu(x) = \nu(y^{-1} xy)$

H(x) = { z Î G : $m_A(z) \geq m_A(x)$ and $n_A(z) \leq n_A(x)$}

= { z Î G : $m_A(z) \geq m_A(y^{-1} xy)$ and $n_A(z) \leq n_A(y^{-1} xy)$}

= $H(y^{-1} xy)$

Thus $IFO_A(x) = IFO_A(y^{-1} xy)$.

The next example shows that above Theorem is not valid if A is not IFNSG of G.

Example (3.12) .Let D_3 =<a , b | $a^3 = b^2 = e$, ba =a^2 b > be the dihedral group with six elements. Deûne a IFSG A = (μ_A , ν_A) of D_3 by

$$\mu_A(x) = \begin{cases} t_0 & ; \text{if } x \in \langle b \rangle \\ t_1 & ; \text{if otherwise} \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} s_0 & ; \text{if } x \in \langle b \rangle \\ s_1 & ; \text{if otherwise} \end{cases}$$

where $t_0 > t_1$ and $s_0 < s_1$ and t_i , s_i [0,1] and $t_i + s_i \leq 1$, for i = 1, 2.
Clearly $a^{-1}ba \notin \langle b \rangle$
Also , $IFO_A(a^{-1}ba) = 6$ and $IFO_A(b) = 2$. Thus $IFO_A(a^{-1}ba) \neq IFO_A(b)$

We know that if x , y $\in$ G , then O(ab) = O(ba) . The following example explains that this fact is not true in Intuitionistic fuzzy order of element in a group

Example(3.13) Let G = S_4 be the symmetric group of 4 elements . Let i be the identity element of S_4 , a = (13)(24) , b = (14)(23) , c = (12)(34) , d = (234) , g = (134) are the elements of A_4 and let A be the IFS of G defined by

$$\mu_A(x) = \begin{cases} 1 & ; \text{if } x = i, a \\ 0.75 & ; \text{if } x = b, c \\ 0.5 & ; \text{if } x \in A_4 - \{i,a,b,c\} \\ 0.3 & ; \text{if } x \in S_4 - A_4 \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} 0 & ; \text{if } x = i, a \\ 0.2 & ; \text{if } x = b, c \\ 0.4 & ; \text{if } x \in A_4 - \{i,a,b,c\} \\ 0.6 & ; \text{if } x \in S_4 - A_4 \end{cases}$$

It is easy to verify that :A is IFSG of S_4 .and $IFO_A(dg) = IFO_A(a) = 2$ and $IFO_A(gd) = IFO_A(b) = 4$. Thus $IFO_A(dg) \neq IFO_A(gd)$.

Theorem (3.14).Let A be a IFNSG of a group G . Then

$$IFO_A(ab) = IFO_A(ba) \text{ for all } a, b \text{ “ } G.$$

Proof. Since A be IFNSG of group G .Therefore by theorem (3.11) , we have

$$IFO_A(x) = IFO_A(y^{-1}xy) \text{ for all } x, y \in G.$$

Therefore $IFO_A(ab) = IFO_A((b^{-1}b)(ab)) = IFO_A(b^{-1}(ba)b) = IFO_A(ba)$.

Remark : Note that in Example (3.13) A is not IFNSG of G.

Theorem(3.15): Let A be IFNSG of a group G , then the set G /A = { xA : x$\in$G} is a group with the operation (xA)(yA) = (xyA).

Proof . Let xA , yA be any two element of G/A , where x , y $\in$ G

Therefore xA , $y^{-1}A \in G/A \Rightarrow (xA)(y^{-1}A) = (xy^{-1})A \in G/A$

Hence G/A is a group.

Definition (3.16): Let A be IFSG of a group G , then the number of elements in the set G /A is called the index of the IFSG of A in G. and is denoted by [G : A].

Theorem(3.17) : Let A be IFNSG of a group G , then there exist a natural homomorphism *f* : G -> G /A defined by *f* (x) = xA , for all x $\in$G with kernel { x $\in$ G : $m_A(x) = m_A(e)$ and $n_A(x) = n_A(e)$ }

Proof. By Theorem (3.4) of [6] . It is enough to show that

$$\text{Ker } f = \{x \in G : m_A(x) = m_A(e) \text{ and } n_A(x) = n_A(e)\}$$

Now Ker *f* = {x $\in$ G : *f*(x) = A }= {x $\in$ G : xA = A }

Thus xA = A (xA)(y) = A(y) , for all y $\in$G

$\Rightarrow$ $(m_{xA}(y), n_{xA}(y)) = (m_A(y), n_A(y))$, for all y $\in$G

$\Rightarrow$ $(m_A(x^{-1}y), n_A(x^{-1}y)) = (m_A(y), n_A(y))$, for all y $\in$G

$\Rightarrow$ $m_A(x^{-1}y) = m_A(y)$ and $n_A(x^{-1}y) = n_A(y)$, for all y $\in$G

Also $m_A(e) = m_A(x^{-1}x) = m_A(x)$. Similarly $n_A(e) = n_A(x^{-1}x) = n_A(x)$

Thus Ker *f* = {x $\in$ G : $m_A(x) = m_A(e)$ and $n_A(x) = n_A(e)$}

Remark (3.18): Note that O(Ker *f*) = IFO(A)

Theorem(3.19): Let A be IFNSG of a finite group G , then

$$[G : A] = O(G) / IFO(A).$$

Proof. By fundamental Theorem of Homomorphism , we have

$$G / \text{Ker } f \cong G/A$$

Therefore O(G /A) = O(G / Ker *f*) = O(G) / O(Ker *f*) = O(G) / IFO(A)

Remark (3.20): The above theorem don't hold in fuzzy group (see [3].)

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