Differential Privacy for Regularised Linear Regression (slides)

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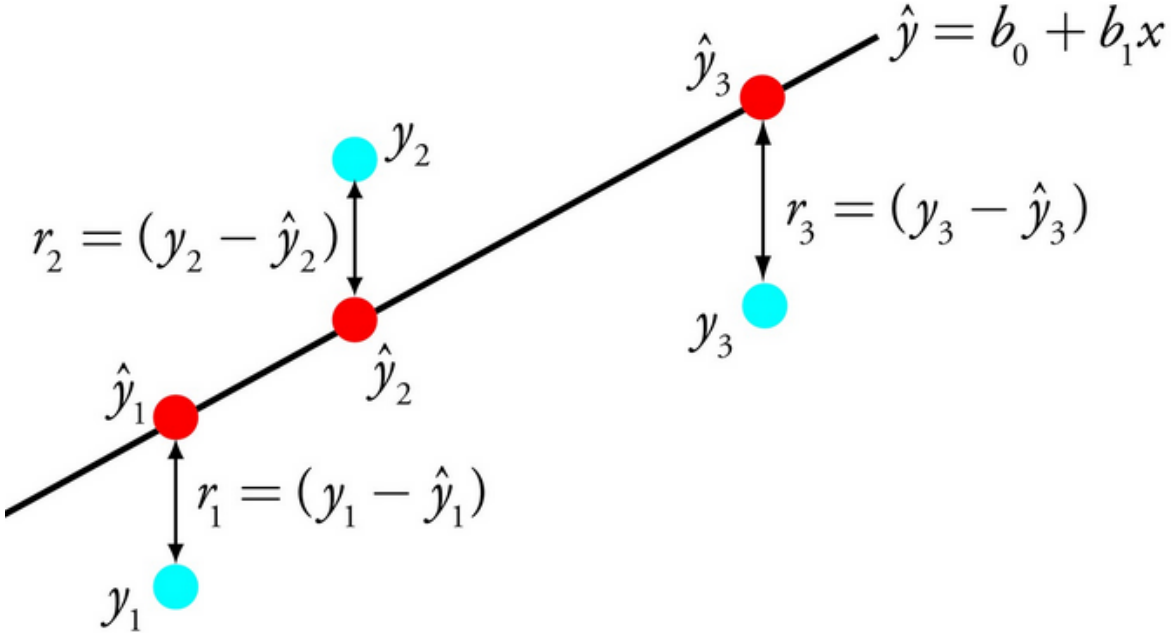
Differential Privacy

A randomised algorithm $\mathcal{M}$ with domain $\mathcal{D}$ is ϵ-differentially private if for all $S \in \mathrm{Range}(\mathcal{M})$ and $D, D' \in \mathcal{D}$ such that D and D' are *neighbouring datasets*

$$\log\left(\left|\frac{Pr(\mathcal{M}(D) \in S)}{Pr(\mathcal{M}(D') \in S)}\right|\right) \leq \epsilon$$

Privacy preserving mechanisms that satisfy the definition of ϵ-differential privacy are ϵ-differentially private mechanisms.

Linear Regression



Linear regression uses a linear function to map predictor attributes, $x_i \in \mathbb{R}^d$, of a data point $t_i = (x_i, y_i)$ to its response attribute, $y_i \in \mathbb{R}$.

Objective function for the regression

Linear Regression

The parameters $\theta \in \mathbb{R}^d$ are estimated by minimising *mean squared loss* over the training dataset, T.

$$\theta^* = \arg\min_{\theta} \left(\frac{1}{|T|} \sum_{t_i \in T} (x_i^t \theta - y_i)^2 \right)$$

Objective function for regularised linear regressions

Ridge

$$\theta^* = \arg\min_\theta \left(\left(\frac{1}{n} (X\theta - Y)^2 \right) + \lambda \|\theta\|_2^2 \right)$$

LASSO

$$\theta^* = \arg\min_\theta \left(\left(\frac{1}{n} (X\theta - Y)^2 \right) + \lambda \|\theta\|_1 \right)$$

Elastic net

$$\theta^* = \arg\min_\theta \left(\left(\frac{1}{n} (X\theta - Y)^2 \right) + \lambda (\alpha \|\theta\|_2^2 + (1-\alpha)\|\theta\|_1) \right)$$

Functional mechanism

The functional mechanism is a privacy preserving mechanism that introduces noise in the objective function that is minimised to estimate the parameters of a machine learning model.

Zhang et al. propose a functional mechanism. [**?**] that satisfies ϵ-differential privacy. It does so by adding a calibrated amount of Laplace noise in the coefficients of the Taylor series expansion of the function of the linear regression.

Satisfying ϵ-differential privacy

Sensitivity

Sensitivity of a function f, denoted as Δ_f, is the maximum change in the output when any pair of neighbouring datasets is given to the function.

$$\Delta_f = \max_{x \sim y} \|f(x) - f(y)\|_1$$

Laplace Mechanism

Dwork et al. [**?**] propose a privacy preserving mechanism that adds Laplace random noise to the output of a function, f. If the scale of the Laplace random variable is set to $\frac{\Delta_f}{\epsilon}$, the mechanism satisfies ϵ-differential privacy.

Satisfying ϵ-differential privacy

Sensitivity of the mean squared loss

In [**?**], Zhang et al. show that the *sensitivity* of the loss function less than to two times the sum of maximum values of coefficients in the Taylor expansion.

Taylor expansion of the *mean squared loss*

$$loss_T(\theta) = \frac{1}{|T|}\sum_{t_i \in T}(x_i^t\theta - y_i)^2$$

$$= \frac{1}{|T|}\sum_{(x_i,y_i)\in T}(y_i)^2 - \frac{1}{|T|}\sum_{j=1}^{d}\left(2\sum_{d_i\in T} y_i x_{ij}\right)\theta_j$$

$$+ \frac{1}{|T|}\sum_{1\leq j,l\leq d}\left(\sum_{d_i\in T} x_{ij}x_{jl}\right)\theta_j\theta_l$$

Satisfying ϵ-differential privacy

If we normalize all the attributes in any dataset, D, of size n such that they lie in $[-1, 1]$,

$$\Delta_{loss(\theta)} \leq \frac{2}{n} \max_{t_i \in D} \left(y^2 + 2 \sum_{j=1}^{d} y x_{ij} + \sum_{1 \leq j,l \leq d} x_{ij} x_{jl} \right)$$

$$= \frac{2}{n}(1 + 2d + d^2)$$

Summing up...

For a training dataset T with n data points, the predictor attributes can be represented as a matrix $X \in \mathbb{R}^{n \times d}$ and the response attribute as a vector $Y \in \mathbb{R}^n$. In this notation, the loss function is written as:

$$\begin{aligned} loss_T(\theta) &= \frac{1}{n}\left(\theta^t(X^tX)\theta - 2\theta^t(X^tY) + Y^tY\right) \\ &= \frac{1}{n}\left(\theta^t(M)\theta - 2\theta^t(N) + O\right) \end{aligned}$$

Laplace mechanism is used that adds $Laplace(0, \frac{\Delta_f}{\epsilon})$ noise in M, N and O. And the output of the functional mechanism, θ^*, is calculated as:

$$\theta^* = \arg\min_{\theta} \frac{1}{n}\left(\theta^t(M_{noisy})\theta - 2\theta^t(N_{noisy}) + O_{noisy}\right)$$

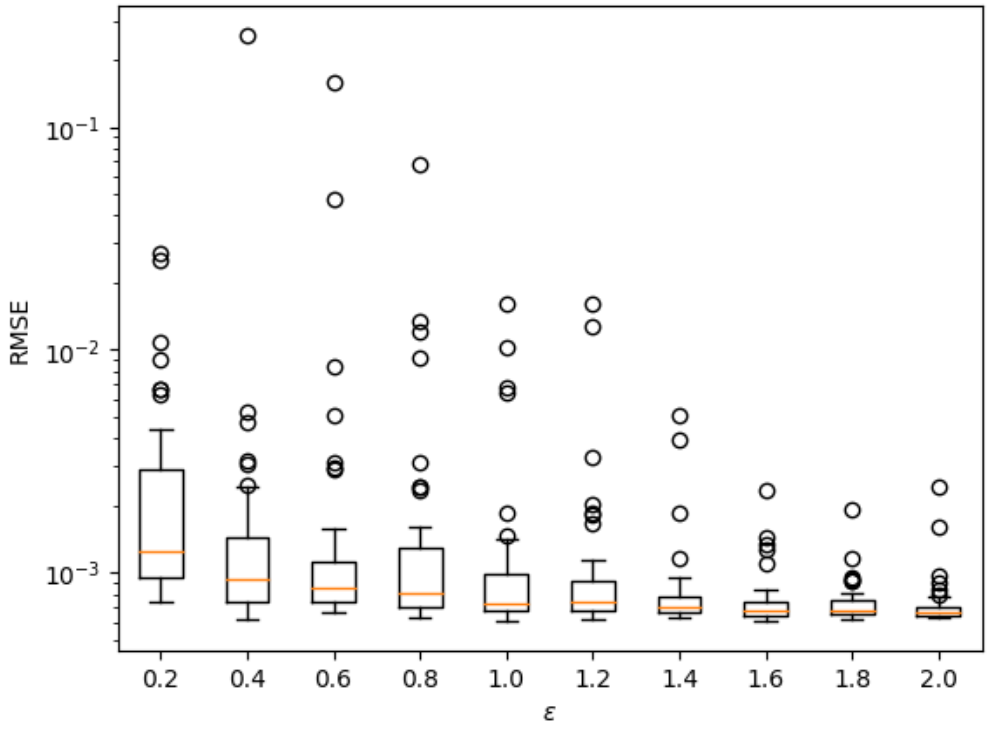
Dataset

Canonical Census Dataset

We conduct experiments on a subset of the 2000 US census dataset provided by Minnesota Population Center in its Integrated Public Use Microdata Series [**?**].We consider $316,276$ records of the heads of households in our dataset.

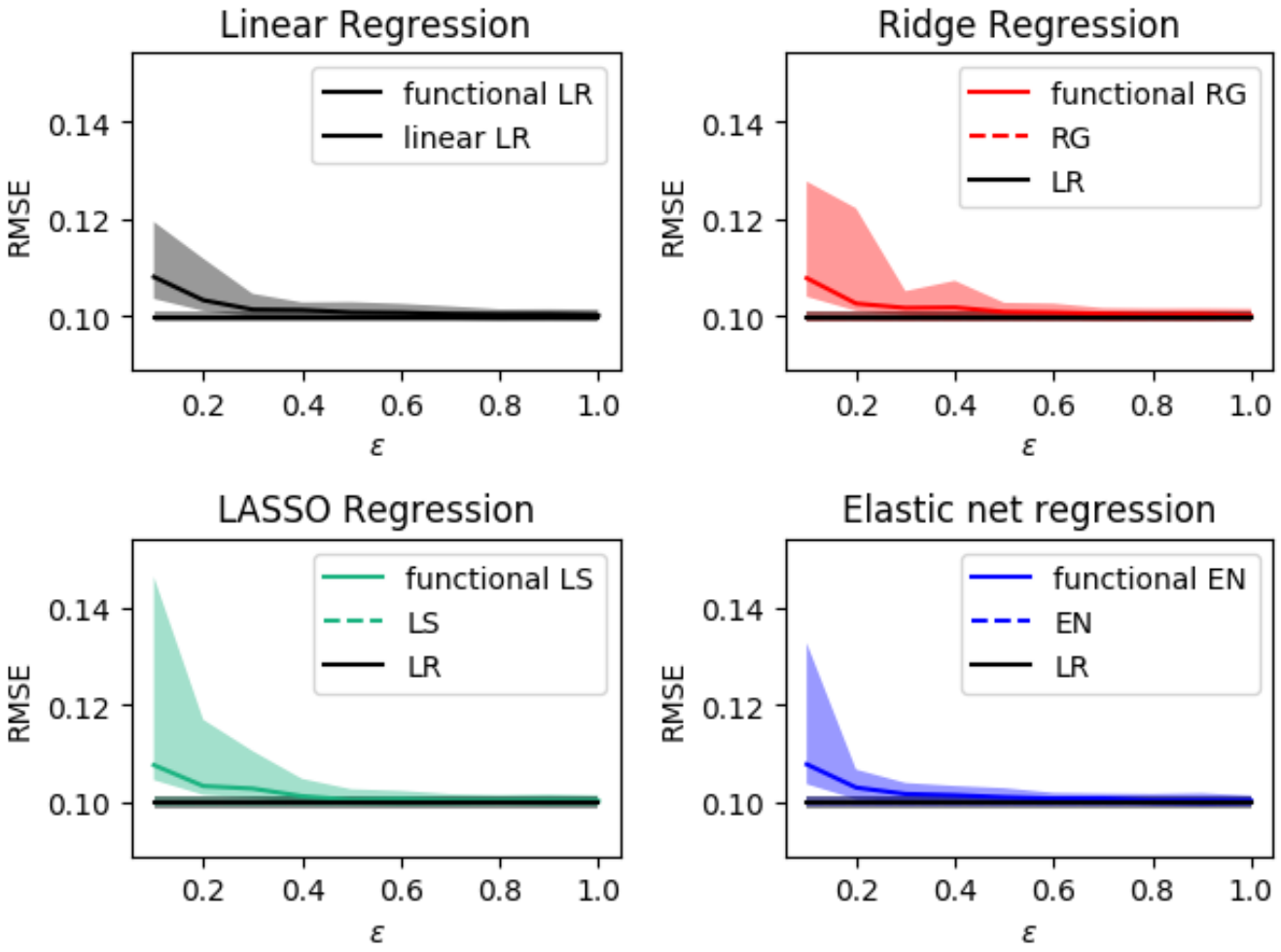
Each record has 6 attributes, namely, *Age, Gender, Race, Marital Status, Education, Income* out of which *Age* and *Income* are numerical attributes and rest of them are categorical attributes. Regression analysis is performed using *Income* as the response attribute and the rest of the attributes as predictor attributes.

Results



Boxplot of RMSE of elastic net regression with functional mechanism for different values of ϵ for the census dataset.

Results


Linear Regression
functional LR
linear LR
RMSE
0.14
0.12
0.10
0.2
0.4
0.6
0.8
1.0
ε
Ridge Regression
functional RG
RG
LR
LASSO Regression
functional LS
LS
Elastic net regression
functional EN
EN

Discussion

Instability in the result

Addition of noise to the quadratic term in the Taylor expansion leads to loss in the convexity of the objective function. Non-convex objective functions do not possess a global minimum. Hence, the local optima give rise to unstable results.

Validity of the privacy guarantee

In [], functional mechanism proves the privacy guarantee of the loss function. Authors say that the same proof naturally leads to the privacy guarantee of the parameters of the regression.

Conclusion