2

Mathematical Modelling Through Ordinary Differential Equations of First Order

2.1 MATHEMATICAL MODELLING THROUGH DIFFERENTIAL EQUATIONS

Mathematical Modelling in terms of differential equations arises when the situation modelled involves some *continuous* variable(s) varying with respect to some other continuous variable(s) and we have some reasonable hypotheses about the *rates of change* of dependent variable(s) with respect to independent variable(s).

When we have one dependent variable x (say population size) depending on one independent variable (say time t), we get a mathematical model in terms of an *ordinary differential equation* of the *first order*, if the hypothesis is about the rate of change dx/dt. The model will be in terms of an *ordinary differential equation of the second order* if the hypothesis involves the rate of change of dx/dt.

If there are a number of dependent continuous variables and only one independent variable, the hypothesis may give a mathematical model in terms of a *system of first or higher order ordinary differential equations.*

If there is one dependent continuous variable (say velocity of fluid u) and a number of independent continuous variables (say space coordinates x, y, z and time t), we get a mathematical model in terms of a *partial differential equation*. If there are a number of dependent continuous variables and a number of independent continuous variables, we can get a mathematical model in terms of systems of *partial differential equations*.

Mathematical models in terms of ordinary differential equations will be studied in this and the next two chapters. Mathematical models in terms of partial differential equations will be studied in Chapter 7.

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2.2.1 Populational Growth Models

Let $x(t)$ be the population size at time t and let b and d be the birth and death rates, i.e. the number of individuals born or dying per individual

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per unit time, then in time interval $(t, t + \Delta t)$, the numbers of births and deaths would be $bx\,\Delta t + 0(\Delta t)$ and $dx\,\Delta t + 0(\Delta t)$ where $0(\Delta t)$ is an infinitesimal which approaches zero as Δt approaches zero, so that

$$x(t + \Delta t) - x(t) = (bx(t) - dx(t))\Delta t + 0(\Delta t), \tag{1}$$

so that dividing by Δt and proceeding to the limit as $\Delta t \to 0$, we get

$$\frac{dx}{dt} = (b - d)x = ax \quad \text{(say)} \tag{2}$$

Integrating (2), we get

$$x(t) = x(0) \exp (at), \tag{3}$$

so that the population grows exponentially if $a > 0$, decays exponentially if $a < 0$ and remains constant if $a = 0$ (Figure 2.1)

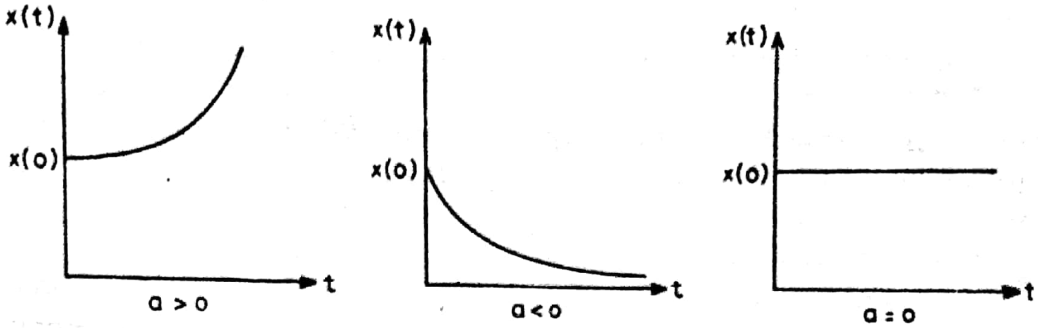

Figure 2.1

(i) If $a > 0$, the population will become double its present size at time T, where

$$2x(0) = x(0) \exp (aT) \quad \text{or} \quad \exp (aT) = 2$$

or
$$T = \frac{1}{a} \ln 2 = (0.69314118)a^{-1} \tag{4}$$

T is called the doubling period of the population and it may be noted that this doubling period is independent of $x(0)$. It depends only on a and is such that greater the value of a (i.e. greater the difference between birth and death rates), the smaller is the doubling period.

(ii) If $a < 0$, the population will become half its present size in time T' when

$$\frac{1}{2}x(0) = x(0) \exp (aT') \quad \text{or} \quad \exp (aT') = \frac{1}{2}$$

or
$$T' = \frac{1}{a} \ln \frac{1}{2} = -(0.69314118)\, a^{-1} \tag{5}$$

It may be noted that T' is also independent of $x(0)$ and since $a < 0$, $T' > 0$. T' may be called the half-life (period) of the population and it decreases as the excess of death rate over birth rate increases.

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2.2.2 Growth of Science and Scientists

Let $S(t)$ denote the number of scientists at time t, $bS(t)\Delta t + 0(\Delta t)$ be the number of new scientists trained in time interval $(t\ \ t+\Delta t)$ and let $dS(t)\Delta t\ \ + 0(\Delta t)$ be the number of scientists who retire from science in the same period, then the above model applies and the number of scientists should grow exponentially.

The same model applies to the growth of Science, Mathematics and Technology. Thus if $M(t)$ is the amount of Mathematics at time t, then the rate of growth of Mathematics is proportional to the amount of Mathematics, so that

$$dM/dt = aM \qquad \text{or} \qquad M(t) = M(0)\exp(at) \tag{6}$$

Thus according to this model, Mathematics, Science and Technology grow at an exponential rate and double themselves in a certain period of time. During the last two centuries this doubling period has been about ten years. This implies that if in 1900, we had one unit of Mathematics, then in 1910, 1920, 1930, 1940, . . . 1980 we have 2, 4, 8, 16, 32, 64, 128, 256 unit of Mathematics and in 2000 AD we shall have about 1000 units of Mathematics. This implies that 99.9% of Mathematics that would exist at the end of the present century would have been created in this century and 99.9% of all mathematicians who ever lived, would have lived in this century.

The doubling period of mathematics is 10 years and the doubling period of the human population is 30-35 years. These doubling periods cannot obviously be maintained indefinitely because then at some point of time, we shall have more mathematicians than human beings. Ultimately the doubling period of both will be the same, but hopefully this is a long way away.

This model also shows that the doubling period can be shortened by having more intensive training programmes for mathematicians and scientists and by creating conditions in which they continue to do creative work for longer durations in life.

2.2.3 Effects of Immigration and Emigration on Population Size

If there is immigration into the population from outside at a rate proportional to the population size, the effect is equivalent to increasing the birth rate. Similarly if there is emigration from the population at a rate proportional to the population size, the effect is the same as that of increase in the death rate.

If however immigration and emigration take place at constant rate i and e respectively, equation (3) is modified to

$$\frac{dx}{dt} = bx - dx + i - e = ax + k \tag{7}$$

Integrating (7) we get

$$x(t) + \frac{k}{a} = \left(x(0) + \frac{k}{a}\right)e^{at} \tag{8}$$

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The model also applies to growth of populations of bacteria and micro-organisms, to the increase of volume of timber in forest, to the growth of malignant cells etc. In the case of forests, planting of new plants will correspond to immigration and cutting of trees will correspond to emigration.

2.2.4 Interest Compounded Continuously

Let the amount at time t be $x(t)$ and let interest at rate r per unit amount per unit time be compounded continuously then

$$x(t+\Delta t) = x(t) + rx(t)\Delta t + 0(\Delta t),$$

giving

$$\frac{dx}{dt} = xr; \quad x(t) = x(0)e^{rt} \tag{9}$$

This formula can also be derived from the formula for compound interest

$$x(t) = x(0)\left(1+\frac{r}{n}\right)^{nt}, \tag{10}$$

when interest is payable n times per unit time, by taking the limit as $n \to \infty$. In fact comparison of (9) and (10) gives us two definitions of the trancendental number e viz.

(i) e is the amount of an initial capital of one unit invested for one unit of time when the interest at unit rate is compounded continuously

(ii) $$e = \operatorname*{Lt}_{n\to\infty}\left(1+\frac{1}{n}\right)^n \tag{11}$$

Also from (9) if $x(t) = 1$, then

$$x(0) = e^{-rt}, \tag{12}$$

so that e^{-rt} is the present value of a unit amount due one period hence when interest at the rate r per unit amount per unit time is compounded continuously.

2.2.5 Radio-Active Decay

Many substances undergo radio-active decay at a rate proportional to the amount of the radioactive substance present at any time and each of them has a half-life period. For uranium 238 it is 4.55 billion years. For potassium it is 1.3 billion years. For thorium it is 13.9 billion years. For rubidium it is 50 billion years while for carbon 14, it is only 5568 years and for white lead it is only 22 years.

In radiogeology, these results are used for radioactive dating. Thus the ratio of radio-carbon to ordinary carbon (carbon 12) in dead plants and animals enables us to estimate their time of death. Radioactive dating has also been used to estimate the age of the solar system and of earth as 45 billion years.

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2.2.6 Decrease of Temperature

According to Newton's law of cooling, the rate of change of temperature of a body is proportional to the difference between the temperature T of the body and temperature T_s of the surrounding medium, so that

$$\frac{dT}{dt} = k(T - T_s), \quad k < 0 \tag{13}$$

Cooling

and

$$T(t) - T_s = (T(0) - T_s)e^{kt} \tag{14}$$

and the excess of the temperature of the body over that of the surrounding medium decays exponentially.

2.2.7 Diffusion

According to Fick's law of diffusion, the time rate of movement of a solute across a thin membrane is proportional of the area of the membrane and to the difference in concentrations of the solute on the two sides of the membrane.

If the area of the membrane is constant and the concentration of solute on one side is kept fixed at a and the concentration of the solution on the other side initially is $c_0 < a$, then Fick's law gives

$$\frac{dc}{dt} = k(a - c), \quad c(0) = c_0, \tag{15}$$

K < 0 ?

so that

$$a - c(t) = (a - c(0))e^{-kt} \tag{16}$$

and $c(t) \to a$ as $t \to \infty$, whatever be the value of c_0.

2.2.8 Change of Price of a Commodity

Let $p(t)$ be the price of a commodity at time t, then its rate of change is proportional to the difference between the demand $d(t)$ and the supply $s(t)$ of the commodity in the market so that

$$\frac{dp}{dt} = k(d(t) - s(t)), \tag{17}$$

where $k > 0$, since if demand is more than the supply, the price increases. If $d(t)$ and $s(t)$ are assumed linear functions of $p(t)$, i.e. if

$$d(t) = d_1 + d_2 p(t), \quad s(t) = s_1 + s_2 p(t), \quad d_2 < 0, s_2 > 0 \tag{18}$$

we get

$$\frac{dp}{dt} = k(d_1 - s_1 + (d_2 - s_2)p(t)) = k(a - \beta p(t)), \quad \beta > 0 \tag{19}$$

or

$$\frac{dp}{dt} = K(p_e - p(t), \tag{20}$$

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where p_e is the equilibrium price, so that

$$p_e - p(t) = (p_e - p(0))e^{-kt} \tag{21}$$

and

$$p(t) \to p_e \quad \text{as} \quad t \to \infty$$

EXERCISE 2.2

1. Suppose the population of the world now is 4 billion and its doubling period is 35 years, what will be the population of the world after 350 years, 700 years, 1050 years? If the surface area of the earth is 1,860,000 billion square feet, how much space would each person get after 1050 years?
2. Find the relation between doubling, tripling and quadrupling times for a population.
3. In an archeological wooden specimen, only 25% of original radio carbon 12 is present. When was it made?
4. The rate of change of atmospheric pressure p with respect to height h is assumed proportional to p. If $p = 14.7$ psi at $h = 0$ and $p = 7.35$ at $h = 17{,}500$ feet, what is p at $h = 10{,}000$ feet?
5. What is the rate of interest compounded continuously if a bank's rate of interest is 10% per annum?
6. A body where temperature T is initially 300°C is placed in a large block of ice. Find its temperature at the end of 2 and 3 minutes?
7. The concentration of potassium in kidney is 0.0025 milligrammes per cubic centimetre. The kidney is placed in a large vessel in which the potassium concentration is 0.0040 mg/cm³. In 1 hour the concentration in the kidney increases to 0.0027 mg/cm³. After how much time will the concentration be 0.0035 mg/cm³?
8. A population is decaying exponentially. Can this decay be stopped or reversed by an immigration at a large constant rate into the population?

2.3 NON-LINEAR GROWTH AND DECAY MODELS

2.3.1 Logistic Law of Population Growth

As population increases, due to overcrowding and limitations of resources, the birth rate b decreases and the death rate d increases with the population size x. The simplest assumption is ~~to take~~ original death rate

$$b = b_1 - \boxed{b_2}x, \quad d = d_1 + \boxed{d_2}x, \quad b_1, b_2, d_1, d_2 > 0, \tag{22}$$

so that (2) becomes original b.r.

$$\frac{dx}{dt} = ((b_1 - d_1) - (b_2 + d_2)x)x = x(a - bx), \quad a > 0,\ b > 0 \tag{23}$$

Integrating (23), we get

b/a b.f

$$\frac{x(t)}{a - bx(t)} = \frac{x(0)}{a - bx(0)} e^{at} \tag{24}$$

Equations (23) and (24) show that

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(i) $x(0) < a/b \Rightarrow x(t) < a/b \Rightarrow dx/dt > 0 \Rightarrow x(t)$ is a monotonic increasing function of t which approaches a/b as $t \to \infty$.

(ii) $x(0) > a/b \Rightarrow x(t) > a/b \Rightarrow dx/dt < 0 \Rightarrow x(t)$ is a monotonic decreasing function of t which approaches a/b as $t \to \infty$.

Now from (23)

$$\frac{d^2x}{dt^2} = a - 2bx, \tag{25}$$

so that $d^2x/dt^2 \gtreqless 0$ according as $x \lesseqgtr a/2b$. Thus in case (i) the growth curve is convex if $x < a/2b$ and is concave if $x > a/2b$ and it has a point of inflexion at $x = a/2b$. Thus the graph of $x(t)$ against t is as given in Figure 2.2.

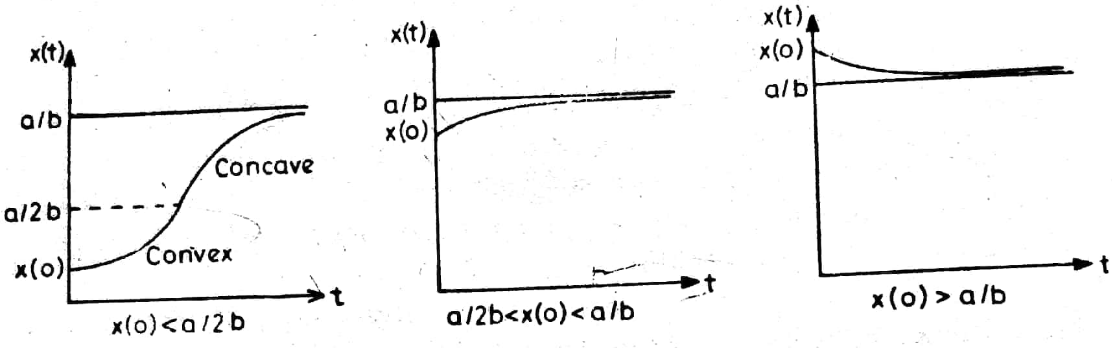

Figure 2.2

—If $x(0) < a/2b$, $x(t)$ increases at an increasing rate till $x(t)$ reaches $a/2b$ and then it increases at a decreasing rate and approaches a/b at $t \to \infty$

—If $a/2b < x(0) < a/b$, $x(t)$ increases at a decreasing rate and approaches a/b as $t \to \infty$

—If $x(0) = a/b$, $x(t)$ is always equal to a/b

—If $x(0) > a/b$, $x(t)$ decreases at a decreasing absolute rate and approaches a/b as $t \to \infty$

2.3.2 Spread of Technological Innovations and Infectious Diseases

Let $N(t)$ be the number of companies which have adopted a technological innovation till time t, then the rate of change of the number of these companies depends both on the number of companies which have adopted this innovation and on the number of those which have not yet adopted it, so that if R is the total number of companies in the region

$$\frac{dN}{dt} = kN(R - N), \tag{26}$$

which is the logistic law and shows that ultimately all companies will adopt this innovation.

Similarly if $N(t)$ is the number of infected persons, the rate at which the number of infected persons increases depends on the product of the numbers of infected and susceptible persons. As such we again get (26), where R is the total number of persons in the system.

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It may be noted that in both the examples, while $N(t)$ is essentially an integer-valued variable, we have treated it as a continuous variable. This can be regarded as an idealisation of the situation or as an approximation to reality.

2.3.3 Rate of Dissolution

Let $x(t)$ be the amount of undissolved solute in a solvent at time t and let c_0 be the maximum concentration or saturation concentration, i.e. the maximum amount of the solute that can be dissolved in a unit volume of the solvent. Let V be the volume of the solvent. It is found that the rate at which the solute is dissolved is proportional to the amount of undissolved solute and to the difference between the concentration of the solute at time t and the maximum possible concentration, so that we get

$$\frac{dx}{dt} = kx(t)\left(\frac{x(0) - x(t)}{V} - c_0\right) = \frac{kx(t)}{V}((x_0 - c_0V) - x(t)) \quad (27)$$

2.3.4 Law of Mass Action: Chemical Reactions

Two chemical substances combine in the ratio $a:b$ to form a third substance Z. If $z(t)$ is the amount of the third substance at time t, then a proportion $az(t)/(a + b)$ of it consists of the first substance and a proportion $bz(t)/(a + b)$ of it consists of the second substance. The rate of formation of the third substance is proportional to the product of the amount of the two component substances which have not yet combined together. If A and B are the initial amounts of the two substances, then we get

$$\frac{dz}{dt} = k\left(A - \frac{az}{a+b}\right)\left(B - \frac{bz}{a+b}\right) \quad (28)$$

This is the non-linear differential equation for a second order reaction. Similarly for an nth order reaction, we get the non-linear equation

$$\frac{dz}{dt} = k(A_1 - a_1z)(A_2 - a_2z)\ldots(A_n - a_nz), \quad (29)$$

where $a_1 + a_2 + \ldots + a_n = 1.$

EXERCISE 2.3

1. If in (24), $a = 0.03134$, $b = (1.5887)(10)^{-10}$, $x(0) = 39 \times 10^6$, show that

$$x(t) = \frac{313,400,000}{1.5887 + 78,7703^{-0.03134t}}$$

This is Verhulst model for the population of USA when time zero corresponds to 1790. Estimate the population of USA in 1800, 1850, 1900 and 1950. Show that the point of inflexion should have occurred in about 1914. Find also the limiting population of USA on the basis of this model.

2. In (26) $k = 0.007$, $R = 1000$, $N(0) = 50$, find $N(10)$ and find when $N(t) = 500$.

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3. Obtain the solution of (27) when $x_0 > c_0 V$ and $x_0 < c_0 V$ and interpret your results.

4. Obtain the solutions of (28) and (29).

5. Substances X and Y combine in the ratio 2 : 3 to form Z. When 45 grams of X and 60 grams of Y are mixed together, 50 gms of Z are formed in 5 minutes. How many grams of Z will be found in 210 minutes? How much time will it like to get 70 gms of Z?

6. Cigarette consumption in a country increased from 50 per capita in 1900 AD to 3900 per capita in 1960 AD. Assuming that the growth in consumption follows a logistic law with a limiting consumption of 4000 per capita, estimate the consumption per capita in 1950.

7. One possible weakness of the logistic model is that the average growth rate $1/x\, dx/dt$ is largest when x is small. Actually some species may become extinct if this population becomes very small. Suppose m is the minimum viable population for such a species, then show that

$$dx/dt = rx\left(1 - \frac{x}{k}\right)\left(1 - \frac{m}{x}\right)$$

has the desired property that x becomes extinct if $x_0 < m$. Also solve the differential equations in the two cases when $x_0 > m$ and $x_0 < m$.

8. Show that the logistic model can be written as

$$\frac{1}{N}\frac{dN}{dt} = r\left(\frac{K-N}{K}\right)$$

Deduce that K is the limiting size of the population and the average rate of growth is proportional to the fraction by which the population is unsaturated.

9. If $F(t)$ is the food consumed by population $N(t)$ and S is the food consumed by the population K, Smith replaced $(K-N)/N$ in Ex. 8 by $(S-F)/S$. He also argued that since a growing population consumes food faster than a saturated population, we should take $F(t) = c_1 N + c_2\, dN/dt$, $c_1, c_2 > 0$. Use this assumption to modify the logistic model and solve the resulting differential equation.

9. A generalisation of the logistic model is

$$\frac{1}{N}\frac{dN}{dt} = \frac{r}{\alpha}\left(1 - \left(\frac{N}{K}\right)^{\alpha}\right), \quad \alpha > 0$$

Solve this differential equation. Show that the limiting population is still K and the point of inflexion occurs when the population in $K(\alpha+1)^{1/2\alpha}$. Show that this increases monotonically from $K/2$ to K as α increases from unity to ∞. What is the model if $\alpha \to 0$? What happens if $\alpha \to -1$?

10. A fish population which is growing according to logistic law is harvested at a constant rate H. Show that

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - H$$

Show that if $D = kH/r - K^2/4 = a^2 > 0$, $N(t)$ approaches a constant limit as $t \to \pi/2\, K/r^2$, but is discontinuous there and cannot predict beyond this

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value of t. If $D = 0$, show that the limiting population is $K/2$. If $D < 0$, show that the ultimate population size is $K/2(1 + \sqrt{1 - 4H/rK})$.

11. For each of the models discussed in this subsection, state explicitly the assumptions made. Try to extend the model when one or more of these assumptions are given up or modified. Obtain some critical results which may be different between the original and modified models and which may be capable of being tested through observations and experiments.

2.4 COMPARTMENT MODELS

In the last two sections, we got mathematical models in terms of ordinary differential equations of the first order, in all of which variables were separable. In the present section, we get models in terms of linear differential equations of first order.

We also use here the principle of continuity i.e. that the gain in amount of a substance in a medium in any time is equal to the excess of the amount that has entered the medium in the time over the amount that has left the medium in this time.

2.4.1 A Simple Compartment Model

Let a vessel contain a volume V of a solution with concentration $c(t)$ of a substance at time t (Figure 2.3) Let a solution with constant concentration C in an overhead tank enter the vessel at a constant rate R and after mixing thoroughly with the solution in the vessel, let the mixture with concentration $c(t)$ leave the vessel at the same rate R so that the volume of the solution in the vessel remains V.

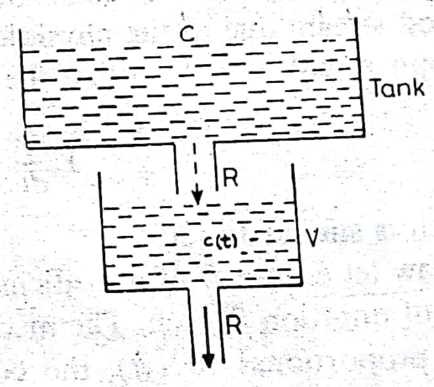

Figure 2.3

Using the principle of continuity, we get

$$V(c(t + \Delta t) - c(t)) = RC\Delta t - Rc(t)\Delta t + 0(\Delta t)$$

giving

$$V\frac{dc}{dt} + Rc = RC \tag{30}$$

Integrating

$$c(t) = c(0)\exp\left(-\frac{R}{V}t\right) + C\left(1 - \exp\left(-\frac{R}{V}t\right)\right) \tag{31}$$

As $t \to \infty$, $c(t) \to C$, so that ultimately the vessel has the same concentration as the overhead tank. Since

$$c(t) = C - (C - c_0)\exp\left(-\frac{R}{V}t\right), \tag{32}$$

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