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20.8.18

Graph Theory

Graph $G = (V, E)$

V - non empty set of vertices
E - set of edges.

$e \in E$

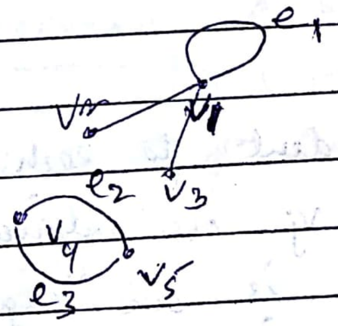
$e = (v_i, v_j)$

end vertices of the edge e

for undirected graph: $e \rightarrow$ unordered pair of vertices.

for directed graph: $e \rightarrow$ order pair

- $e = (v_i, v_i) \rightarrow$ self-loop
  $\rightarrow$ an edge whose end vertices are same.



$e_2 = (v_4, v_5) = e_3$

parallel edges $\rightarrow$ whose end vertices are same.

- Simple Graph - A graph without self-loop & parallel edges.

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- Multigraph – A graph with ~~a~~ parallel edges & without self-loops.
- Pseudo-graph: parallel edges & self loop, both are allowed.

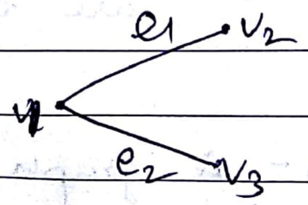
Finite	Infinite
no. of edges & vertices are finite	both are ∞.

$V(G) \rightarrow$ set of vertices in graph G
$E(G) \rightarrow$ " " edges " " "

Cardinality = $|V(G)|$. → order of G
$|E(G)|$ → size of G

* e_i v_i
$e_i = (v_i, v_j)$

- v_i & e_i are incident to each other.
- two vertices v_i & v_j are adjacent to each other if there is an edge b/w them.
- Two edges are adjacent to each other if they are incident on a common vertex.



degree of a vertex: $d(v_i)$ = no. of edges incident

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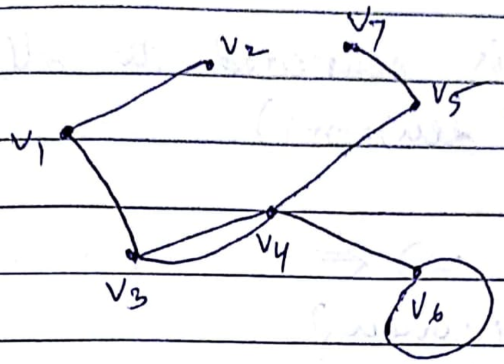
* Self loop counted twice.



$d(v_1) = 2$ $\quad$ $d(v_4) = 4$ $\quad$ $d(v_7) = 1$
$d(v_2) = 1$ $\quad$ $d(v_5) = 2$
$d(v_3) = 3$ $\quad$ $d(v_6) = 3$

Vertex having degree 1 is called a Pendant vertex and a vertex of degree 0 is called an isolated vertex.

Theorem: A simple graph with atleast two vertices has atleast two vertices of same degree.

Proof:

$G=(V,E)$ $\quad$ $|V(G)| = n \geq 2$

$d(v_i) \leq n-1 \quad \forall \; v_i \in V$

Suppose no two vertices of G have same degree.

Let $d(v) = 0, \; d(v') = 1, \; - - - \; d(u) = n-1$

$\Rightarrow$ v is an isolated vertex.

But u is connected to all other vertices ($d(u) = n-1$)

$\Rightarrow \Leftarrow$

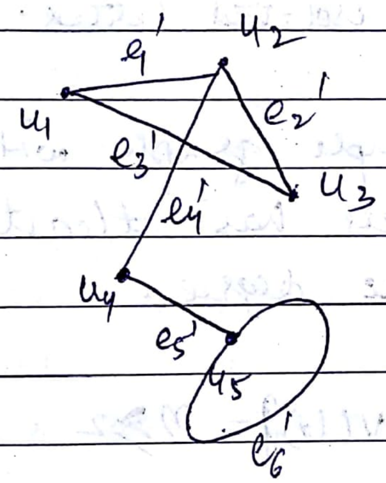
Hence, (Theorem).

* $\sum_{i=1}^{7} d(u_i) = 16$

no. of edges $= 8$

$\therefore \sum d(v_i) = 16 = 2 \times 8 = 2 \times$ no. of edges in G.

ex:



$d(u_1) = 2$
$d(u_2) = 3$
$d(u_3) = 2$
$d(u_4) = 2$
$d(u_5) = 3$

$\sum d(v_i) = 12 = 2 \times 6 = 2 \times$ no. of edges.

* **Fundamental Theorem of Graph Theory:-**

In any graph G, the sum of degree of all the vertices is equal to twice of no. of edges in the graph.

Proof: It is true because every edge contributes to 2 degrees.

Theorem: In any graph G, the no. of vertices of odd degree is always even.

Proof:

$$\sum_{i=1}^{n} d(v_i) = 2 \times \text{no. of edges}$$

$$\sum_{\text{even}} d(v) + \sum_{\text{odd}} d(u) = 2 \times \text{no. of edges} = \text{even no.}$$

$$\sum_{\text{odd}} d(u) = \text{even no.}$$

$\Rightarrow$ no. of odd vertices is even.

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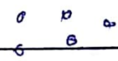
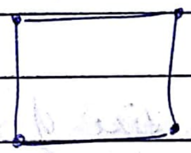
Type of Graphs:

1. Null graph: no edge i.e. every vertex is an isolated vertex.



2. Regular graph: every vertex of equal degree



2 – regular graph

r – regular graph

degree 'r'

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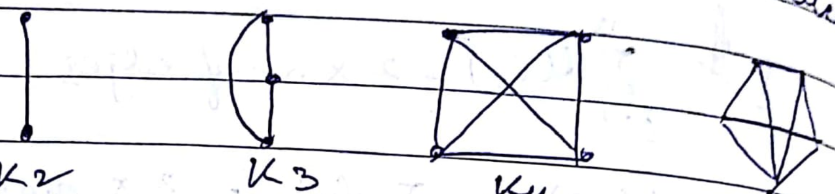
3. Complete graph :- A simple graph in which every two vertices are adjacent to each other i.e. ∃ an edge b/w every pair of vertices

$K_n \rightarrow$ complete graph on n vertices



K_2 K_3 K_4 K_5

Every complete graph is (n-1) reg.

- no. of edges in $K_n = {}^nC_2 = \frac{n(n-1)}{2}$

Proof: for $n=2 \rightarrow$ no. of edges $= 1 = \frac{2(2-1)}{2}$

By induc^n

let result is true for $n=K$

T.P. for $n = K+1$

no. of edges of $K_{K+1} = \frac{K(K-1)}{2} + K$

$= \frac{K(K+1)}{2}$

Que. What is the size of an r-regular graph having p-vertices?

Soln: $\sum_{i=1}^{p} d(v_i) = 2 \times q$

$p \times r = 2q$

$\therefore q = \frac{rp}{2}$.

Que: Does there exist a 4-reg. graph on 6 vertices?

Soln: Yes.

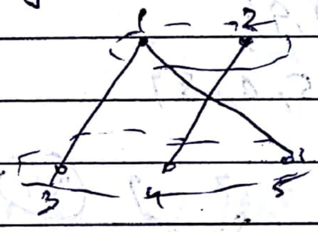
$\because q = \frac{6 \times 4}{2} = 12 \Rightarrow$ exists

- If $p = 5, r = 7$

$\therefore q = \frac{5 \times 7}{2}$ ✗ $\Rightarrow$ doesn't exist.

* Bipartite Graph:

A graph $G = (V, E)$ is said to be bipartite if its vertex set V can be partitioned into 2 subsets V_1 & V_2 s.t. no two vertices in the same are adjacent to each other.



$V = \{1, 2, 3, 4, 5\}$

$V_1 = \{1, 2\}$

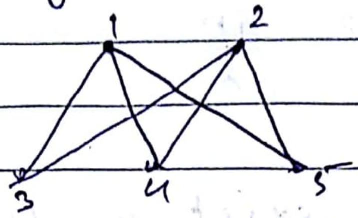
$V_2 = \{3, 4, 5\}$

* Complete bipartite Graph:

A graph $G = (V, E)$ is said to be C.B.P if its vertex set V can be - - - - - and every vertex of V_1

Bipartite def^n

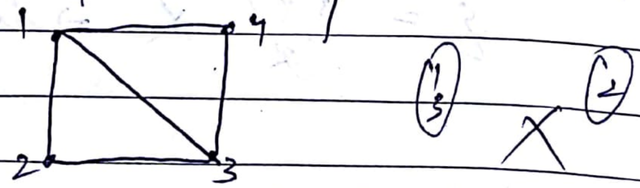
is adjacent to every vertex in V_2.



$K_{m,n} \rightarrow K_{2,3}$

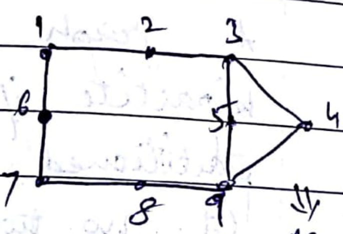
mn edges
& (m+n) vertices.

*. Suppose graph contains a triangle. Check if ~~bar~~ bipartite.



∴ Its not bipartite.

Que. Check if bip.

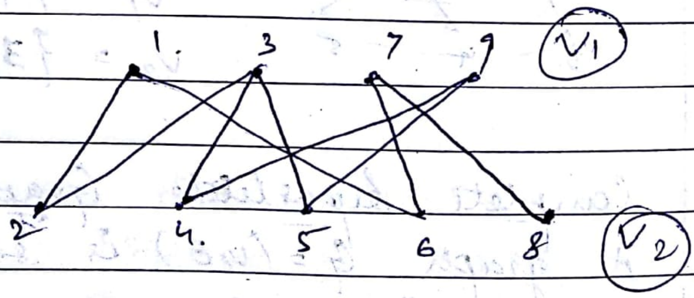


⇓
no Δ.

Partitions:

$V_1 = \{1, 3, 7, 9\}$

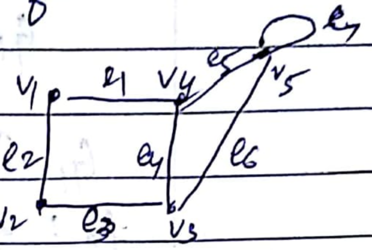
$V_2 = \{2, 4, 5, 6, 8\}$



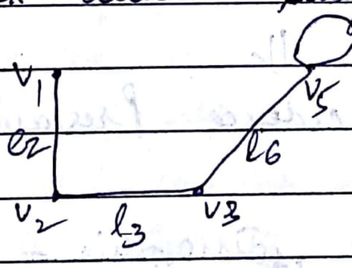
*. <u>Sub</u> - G<u>raph</u> $G = (V, E)$.
& $H = (V', E')$

Then H is subgraph of G if $V' \subseteq V$ & $E' \subseteq E$ s.t. every edge of H has same end vertices as in G.

$H \subseteq_{gh} G$

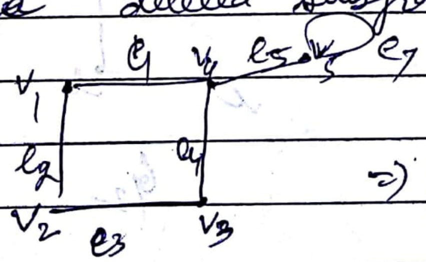


- Null graph $\subseteq_{gh}$ every graph
- $G \subseteq_{gh} G$
- every vertex is a subgraph of G
- every edge with its end vertices is a subgraph of G.
- subgraph of a subgraph of a graph is also a subgraph of G.
- Vertex deleted subgraph of G



$\Rightarrow (G - v_4)$

- Edge deleted subgraph of G



$\Rightarrow (G - e_6)$

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* Two subgraphs of G are said to be edge disjoint if they don't have any edge in common.

* Isomorphism:

$G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$

$G_1 \cong G_2$ iff $\exists$ one-one onto fn : $G_1 \to G_2$ s.t. $v_1 \in V_1 \Rightarrow f(v_1) \in V_2$, $e \in E_1 \Rightarrow f(e) \in E_2$ and if v_1 & v_2 are end vertices of an edge e in G_1 then $f(v_1)$ & $f(v_2)$ will be end vertexes of $f(e)$ in G_2.

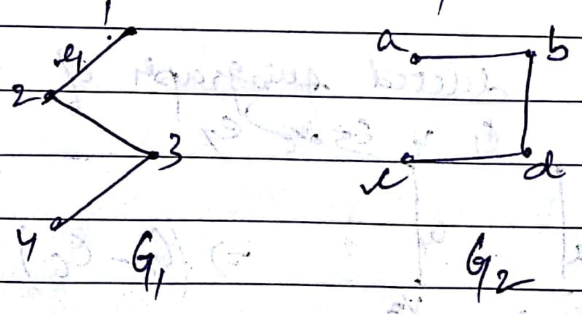
v $f(v) \to f(e)$

v is an end vertex of an edge e then $f(v)$ is an end vertex of $f(e)$

$\Downarrow$

Incidence Preserving Property.

Ans: Check if isomorphic -



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soln: $f : G_1 \to G_2$ s.t.

$f(1) = a$, $f(2) = b$, $f(3) = d$, $f(4) = c$

$\left.\begin{array}{l} f(1) = a \\ f(2) = b \end{array}\right\}$ edge b/w them.

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2) **Isomorphism**: $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$
$f : G_1 \to G_2$ 1-1 onto

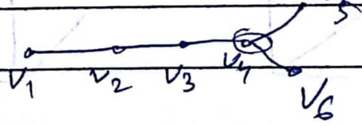
s.t. $v_1 \in V_1 \Rightarrow f(v_1) \in V_2$
& $e_1 \in E_1 \Rightarrow f(e_1) \in E_2$

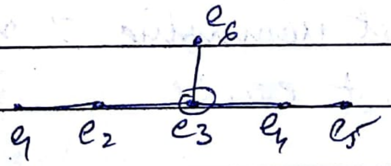
If v is an end pt. of an edge e then $f(v)$ is an end pt. of $f(e)$.

observations ⇒

necessary but not suff.
1. The no. of vertices in both the graphs should be same.
2. The no. of edges " " " " "
3. Both the graphs should have equal no. of vertices of given degree.

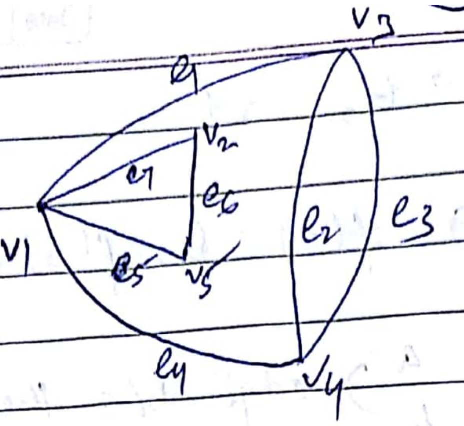
ex: G_1 :



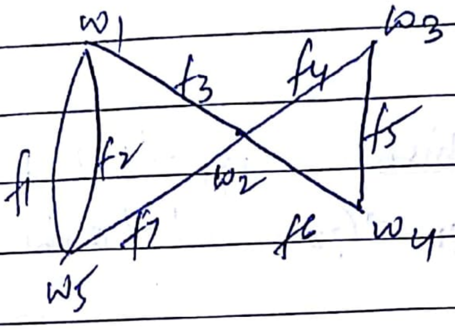


Not isomorphic although above cond^ns are satisfied.

ex: G_1:



G_2:

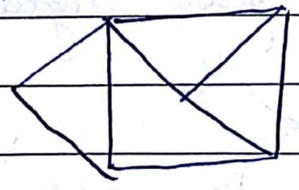


They are isomorphic defined by:

$g : G_1 \to G_2$

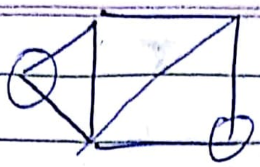
$g(v_1) = w_2$	$g(e_1) = f_3$	$g(e_6) = f_5$
$g(v_2) = w_3$	$g(e_2) = f_1$	$g(e_7) = f_4$
$g(v_3) = w_1$	$g(e_3) = f_2$	
$g(v_4) = w_5$	$g(e_4) = f_7$	
$g(v_5) = w_4$	$g(e_5) = f_6$	

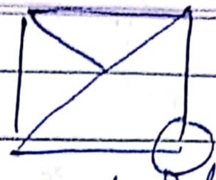
ex:



Not isomorphic $\because$ no. of edges are not equal.

Ex!





no. of vertices of degree 2 = 2 | no. of 2 degree vertices = 1.

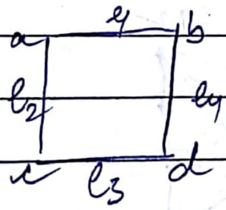
∴ Not isomorphic.

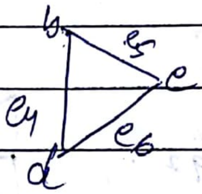
* Operations on Graph:-

1. **Union**: G_1 & G_2 be 2 graphs

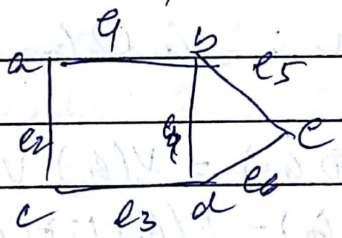
$G_1 \cup G_2$

$$V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$$
$$E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$$





$\Downarrow \cup$



2. Intersection : $G_1 \cap G_2$

$$V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$$
$$E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$$

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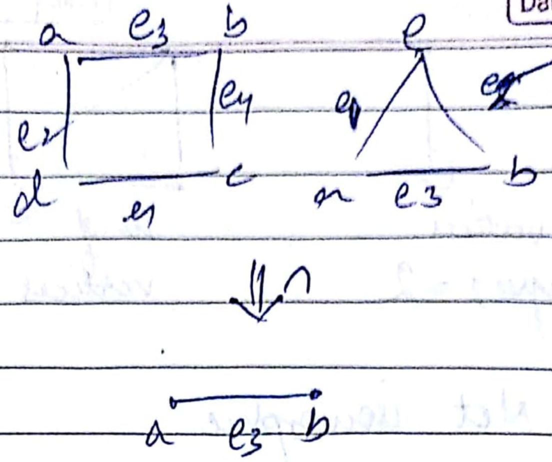
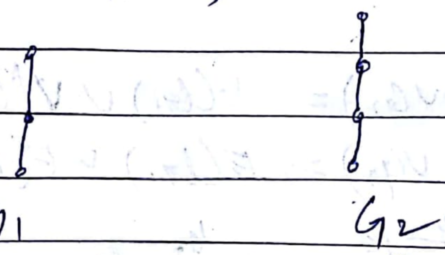


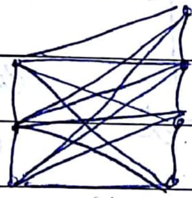
3. Sum of ~~two~~ graphs : $G_1 + G_2$

$V(G_1 + G_2) = V(G_1) + V(G_2)$

$E(G_1 + G_2)$ ⇒ Draw edge b/w every vertex of G_1 & G_2.



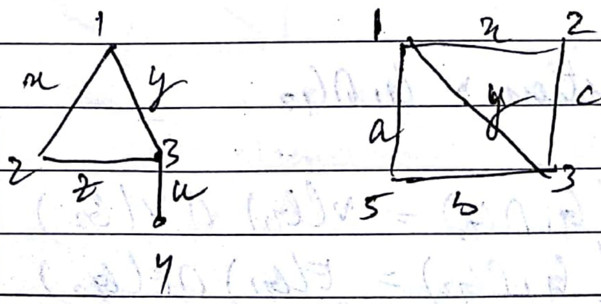
$G_1 + G_2$:



4. Ring sum : $G_1 \oplus G_2$

$V(G_1 \oplus G_2) = V(G_1) \cup V(G_2)$

$E(G_1 \oplus G_2) = E(G_1) \cup E(G_2) - [E(G_1) \cap E(G_2)]$



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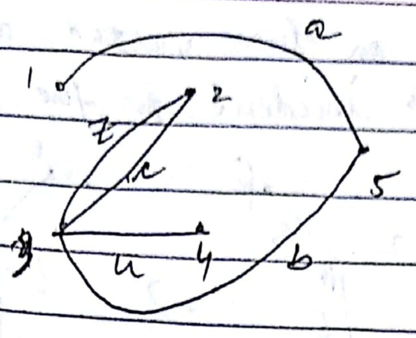
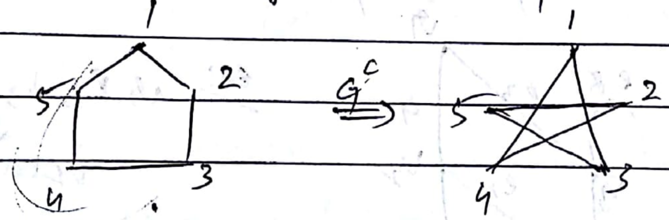


5. Complement of a graph : G^c or G'

G - any graph

$$V(G^c) = V(G)$$

Two vertices are adjacent in G^c iff they are not adjacent in G.



6. Decomposition :- G is decomposed into two graphs G_1 & G_2 iff:

$$G_1 \cup G_2 = G$$

& $G_1 \cap G_2$ = null graph

7. Fusion : A pair of vertices a, b in a graph are said to be fused (merged) if the two vertices are replaced by a single new vertex say a, b s.t. every edge that was

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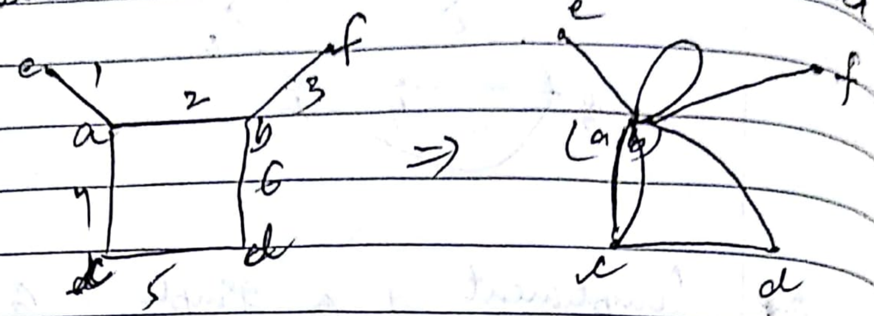
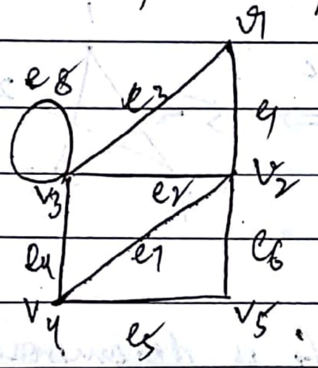
incident on the vertex a or b or on both is incident on the new vertex



* ~~WALK!~~

A walk is an alternating seq. of ~~edg.~~ vertices & edges, starting and ending with vertices s.t. each edge is incident on the vertices preceding and following it.



Ex! $v_1 e_3 v_3 e_8 v_3 e_4 v_4$
↓
open walk

$v_2 e_2 v_3 e_4 v_4 e_7 v_2$
↓
closed walk

By default → open walk.

* Path ! It is a walk in which no vertex appears more than once

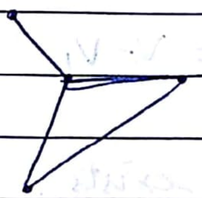
$v_1 e_1 v_2 e_6 v_5 e_5 v_4$

$v_1\ e_1\ v_2\ e_6\ v_5\ e_5\ v_4 e_4 v_3 e_3 v_1$ → closed path.

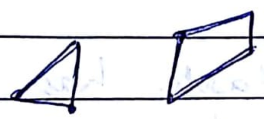
cycle.

Closed Path: It is a walk in which no vertex appears more than once except the terminal vertices.

*. Connected Graph:-
A graph is connected if there is at least one path b/w every pair of vertices
Otherwise Disconnected.



⇒ Connected but not complete.



⇒ Disconnected

Every disconn. has atleast two $\underbrace{\text{connected subgraphs}}_{\text{components}}$.

* Theorem: A graph G is disconnected iff its vertex set V can be partitioned into 2 non-empty disjoint subsets V_1 & V_2 s.t. ∄ no edge in G whose one vertex is in V_1 & another in V_2.

Proof: Suppose such a partition exists

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let $a \in V_1$ & $b \in V_2$
There's no edge b/w a & b $\Rightarrow$ no path
$\Rightarrow$ Disconnected.

Converse: Suppose G is disconnected.
let $a \in V$

$V_1 \rightarrow$ set of all vertices of G which are connected to 'a' through a path

$\Rightarrow \phi \neq V_2 = V - V_1$

$\therefore$ Partition exists.

* Theorem: If A graph has exactly 2 vertices of odd degree, there must be a path joining them.

Proof: Let G be any graph having exactly two vertices of odd degree.

1. G is connected ✓
2. G is disconnected

Both vertices are in same component ✓

Both vertices are in diff. comp. X
+
no. of odd degree vertices is always even.

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Theorem: A simple graph with n vertices and k components can have at most $\frac{1}{2}(n-k)(n-k+1)$ edges.

Proof: Let the no. of vertices in each component of G be $n_1, n_2, \ldots n_k$.

$$n_1 + n_2 + \ldots + n_k = n$$

$$\sum_{i=1}^{k}(n_i - 1) = n - k$$

S.b.s.

$$\left[\sum_{i=1}^{k}(n_i - 1)\right]^2 = n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k}(n_i^2 - 2n_i) + k + \text{non-ve cross terms} = n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k} n_i^2 \leq n^2 + k^2 - 2nk - k + 2n$$

$$\leq n^2 - k(k-2n) - (k-2n)$$

$$\sum n_i^2 \leq n^2 - (k-1)(k-2n)$$

max. no. of edges in i^{th} comp. $= \frac{1}{2} n_i(n_i - 1)$

$\therefore$ The max. no. of edges in the graph

$$= \frac{1}{2}\sum_{i=1}^{k}(n_i - 1)n_i = \frac{1}{2}\sum(n_i^2 - n_i)$$

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$$= \frac{1}{2}\left[\sum n_i^2 - n\right]$$

$$\leq \frac{1}{2}\left[n^2 - (k-1)(k-2n) - n\right].$$

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Circuit: A closed walk without repetition of edges. Vertex may be repeated.

Eulerian Circuit: A circuit which contains every edge of the graph exactly once.

A graph containing Eulerian ckt. is known as Eulerian graph.

Theorem: A connected graph G is Eulerian iff every vertex of G is of even degree.

Proof:

Suppose G is Eulerian

$\therefore \exists$ an Eulerian ckt. in G.

$v_0 e_1 v_1 e_2 v_2 \; - - \; e_{n-1} v_n e_n v_0$

degree 2

converse - Suppose all the vertices are of even degree

$v_0 e_1 v_1 e_2 v_2 - - - e_{n-1} v_n e_n v_0 \rightarrow H_1$

$H_2 = G - H_1$

$v_i e_1' v_{i+1} e_2' --- v_i$

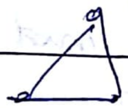
<u>Defn:</u> A path in a graph is Eulerian if it contains every edge of G exactly once.

$K_n \to d(v_i) = n-1 \ \forall i$

for Eulerian, $n-1$ = even

$\Rightarrow n$ = odd

K_3



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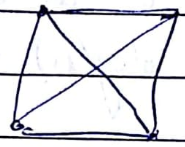
K_4



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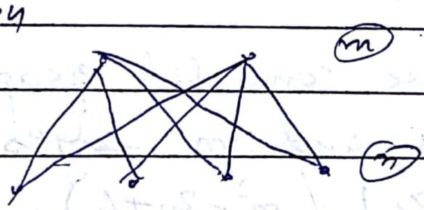
• $K_{m,n} \to$ both m, n should be even for Eulerian.

$K_{2,4}$



* Hamiltonian path & circuit :-

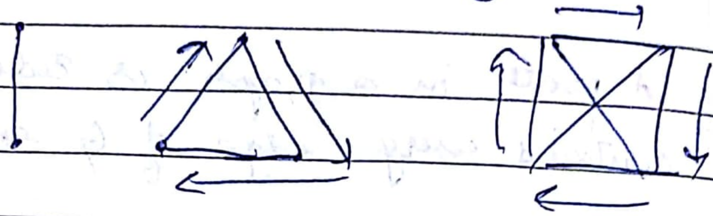
H. path is a path that contains every vertex of the graph exactly once.

except terminal

H. circuit is a ckt. that contains every vertex exactly once.

A graph is Hamiltonian if it contains a H. circuit.

$K_n \rightarrow$ H. for every n.



Theorem 1: Let G be a simple connected graph with n vertices, $n > 2$. G has a H. ckt. if for every two vertices u & v in G that are not adjacent, $deg(u) + deg(v) \geq n$.

2: A simple connected graph G with n vertices, $n \geq 3$ is Ham. if $deg(v) \geq n/2 \ \forall \ v \in V(G)$.

3: A simple connected graph with n vertices and m edges is Ham. if $m \geq \frac{1}{2}(n^2 - 3n + 6)$.

Do proofs.

4. length of path = no. of edges in that path

$v_0 e_1 v_1 e_2 v_2 \rightarrow$ length 2.

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*- shortest path in a non-weighted graph
↓
min. no. of edges reqd.

*: Breadth first search (BFS) Algorithm :-

→ To find the shortest path in a non-weighted graph.

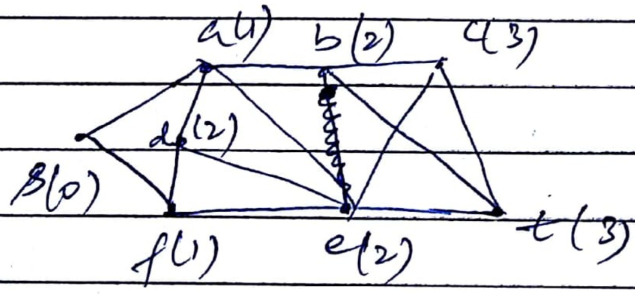
~~Let~~ to find shortest path b/w s & t.

step 1: Label vertex s with 0, set $i = 0$.

step 2: find all unlabelled vertices in G which are adjacent to vertices labelled i. If there are no such vertices, then i is not connected to s. If there are such vertices, label them $i+1$.

step 3: If t is labelled, go to step 4. If not, increase i to $i+1$ and go to step 2.

step 4: The length of the shortest path from s to t is $(i+1)$.



∴ length of shortest path = 3 = label of t.

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Back-tracking algo:-

s-f-e-t
~~s~~ s-a-e-t
s-a-b-t