CHAPTER 7 Consolidation, Extension, and Discussion

TEACHING SUGGESTIONS

For the section on dynamic games of competition, you can begin by asking if anyone in the class has played competitive tennis (club or collegiate or better); there is usually one. If not, a dedicated viewer of major tournaments on TV is a good substitute. Ask such a person which point in a game is the most important; the answer is invariably "break point," where the server is 30–40 or Ad Down. Then ask which game in a set is the most important; the answer is usually the seventh or the eighth. Ask why, and the answer is often, "The coach told me" or "The TV commentators say so." Sometimes they may say, "Lose that point and you've lost the game" or "Lose that game and you've lost the set." The former is trivially true, whereas the latter is not true, and you can recount (or ask them to recall) many counterexamples. Offer to explain the idea of the importance of a point or a set in a more systematic way. When you have done so, and also brought out the parallel with R&D or related competitions in business, for example the current rivalry between Microsoft and Netscape, not only will the specific point have been gotten across more clearly but the confidence of the class in the whole analytical apparatus will have increased.

For those of you who want to focus on solving for mixed-strategy equilibria in larger games than the two-by-two games which were the focus of Chapter 5, this is your chance. You can construct your own examples or add strategies to the two-by-two games you covered earlier. This is a good time to play the rock-scissors-paper game described in the "Game Playing in Class" section of Chapter 5. You can use that game to motivate many of the issues relevant to mixing in larger games.

If you can bring a laptop equipped with Gambit to the class, you can solve even more complex games on the spot. For example, you can take the soccer penalty kick game of Exercise 7.3, ask the students for suggestions on how the strategies or the payoffs might differ from the numbers in the text, and display the resulting equilibrium immediately. If you can hook up the computer so that the monitor screen can be directly shown on the class screen, the effect is more dramatic.

We expect you will teach the more mathematical material in this chapter only if your class has enough familiarity with algebra and the general process of mathematical reasoning. Such a class should not need any storytelling or game playing.

If your class is sufficiently sophisticated in logical reasoning, then you can spend more time on the concept of *rationalizability*. Start with the example of Figure 7.1 in the text, and show how for each player, every strategy can be justified on the basis of a complete set of logically consistent beliefs about the other player's beliefs about your beliefs about . . . what one might choose; that is, every strategy is rationalizable. For example, Row can justify choosing A on the basis of the belief that Column will play B, which in turn can be justified because Row believes that Column believes that Row will play C, which in turn can be justified because Row believes that Column believes that Row believes that Column will play A, and so on. Then move on to a more complex matrix, which comes from Douglas Bernheim's original paper on rationalizability [*Econometrica*, vol. 52 (1984), pp. 1007–1028], and is also discussed in Andreu Mas-Colell, Michael Whinston, and Jerry Green, *Microeconomic Theory* (New York: Oxford University Press, 1995), page 244:

		COLUMN			
		C1	*C2*	*C3*	*C4*
ROW	R1	0, ***7***	2, ***5***	7, ***0***	0, ***1***
	R2	5, ***2***	3, ***3***	5, ***2***	0, ***1***
	R3	7, ***0***	2, ***5***	0, ***7***	0, ***1***
	R4	0, ***0***	0, ***–2***	0, ***0***	10, ***–1***

For Row, the strategies R1, R2, and R3 are all rationalizable on the basis of a chain of beliefs constructed by arguments similar to that given above. But R4 is not rationalizable; for that strategy, Row would have to believe that Column would play C4, but that strategy is dominated for Column by an equal-probability mixture of C1 and C3. Incidentally, this also conveys the idea that a strategy may be dominated by a mixed strategy that combines some of the other pure strategies, something that we have not discussed in the book.

If you are teaching the class at this level, you must have used the example of a quantity-setting duopoly with downward-sloping best-response functions. Now you can discuss rationalizability in its context. Here the logic of successive rounds of beliefs leads to the elimination of more and more ranges of strategies. An example of the thought process proceeds thus: My output must be nonnegative, therefore I can be sure that the other firm will never produce more than its monopoly output (its best response to my 0), therefore it will not believe that I will produce any less than my best response to its monopoly output, and so on. This eventually leaves only the (Cournot) Nash equilibrium as rationalizable.

GAME PLAYING IN CLASS

GAME 1 Rock-Scissors-Paper

This game described in Chapter 5, can be used when the Chapter 7 material on mixing in larger games is introduced. Many of you will find it natural to move directly from Chapter 5 to Section 4 of Chapter 7. You can then return to cover Chapter 6 and the earlier parts of Chapter 7 in later lectures.

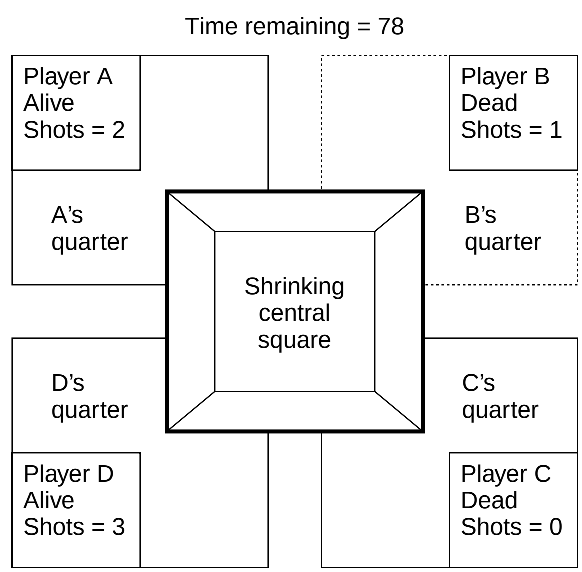
COMPUTER GAME 1 Shoot-out at the Computer Corral

This is a duel with four shooters and three bullets each; see the instructions below for details. This game is probably best played right after the class has studied Chapter 7, where a much simpler duel, with two shooters and one bullet each, is solved analytically. The duel at the complexity of this game has no known analytical solution, and it is interesting to find out which strategies work well.

You can try the game under varying conditions, to see what difference this makes to the strategies and the outcomes. For example: (1) Background music—the Bolero, which gets gradually louder, or the theme from the James Bond movies or from *The Good, the Bad, and the Ugly*, which have shooting connotations. Does this cause the players to shoot sooner? (2) Screens of different colors—do more soothing colors cause them to shoot later? (3) Information conditions other than the three we tried.

INSTRUCTIONS FOR THE SHOOT-OUT GAME

1. The game is in three phases, and each phase consists of a number of rounds. We will first play a practice session, with one round of each of the three phases, and then play for real, with six rounds of each of the three phases.
2. At each play, you will be grouped with three others chosen randomly from the class. Each play (the three times in the practice session and the 3 × 6 = 18 times in the real play) will have different groupings.
3. On your screen you will see a square like that shown in the figure below. The whole large square is divided into four smaller squares, each of which represents the quarter owned by one player. The players are labeled A, B, C, and D. Everyone on his own screen is always labeled A and located in the top left quarter.



4. Each player has three bullets. You can shoot at any of the other players (shooting repeatedly at one player *is* allowed) or shoot in the air. You can shoot at any time, until *either* you are hit by someone else's bullet *or* you run out of bullets.
5. To shoot at a player, use your mouse to move the cursor to anywhere in his or her quarter and click the Left mouse button. To shoot in the air, click in your own quarter.
6. There is a central blue square that overlaps the four quarters. This central square gradually shrinks to

nothing. The total time of this shrinkage process is 2 minutes (120 seconds). The time remaining in seconds is displayed at the top just above the big square. The accuracy of your shooting increases as the central square shrinks. (Think of the players as standing at the corners of this central square and getting closer to one another as the square shrinks.) The probabilities that you as A score a hit when you shoot are different for shots at the players adjacent to you (B and D) than they are for your shots at the diagonally opposite player (C). The probabilities increase continuously through time according to a smooth curve; the following table shows the probabilities at just a few times.

Time Remaining, *S*	Probability Hitting **B** or **D**, %	Probability Hitting **C**, %
120	37	24
90	44	31
60	61	49
30	78	70
0	100	100

7. The other players may similarly shoot at you at any time they choose, with the same hit probabilities rising over time in the same way as yours, so long as they are alive and have bullets left.
8. Your score in one round equals the number of seconds you survive, plus a bonus of 50 points if you survive the full 2 minutes, plus another bonus of 100 points if you are the only person in your group of four to survive the full 2 minutes. Your score for the whole exercise is the average of your score over the 18 rounds of real play.
9. The three phases differ in the information you have about the other players. The information is displayed in a corner of the quarter for each player; the figure shows the information structure for phase 1. In the first phase, you know who else is alive and how many bullets they have left. In the second phase, you know who else is alive but not how many bullets they have left. In the third phase, you do not even know whether any of the others are alive or dead. (You know your own status and bullet count in all three phases.)

ANSWERS TO EXERCISES FOR CHAPTER 7

1. From the table given in the answer to Question 4.11, it is clear that buying a \$30 ticket is a (weakly) dominated strategy. Thus, any equilibrium mixture will only involve the contestants buying either no ticket or a \$15 ticket. Let Contestant A's probability of buying a \$15 ticket be p, and the probability of his not buying a ticket be $1 - p$. Similarly, Contestant B buys a \$15 ticket with probability q (and doesn't buy with probability $1 - q$); Contestant C buys with probability r (and doesn't buy with probability $1 - r$). The payoff table is then:

C

Don't Buy Ticket (Probability $1 - r$)

		B: *Don't Buy*	B: *Buy*	B: *q-mix*
A	Don't Buy	0, ***0,*** 0	0, ***15,*** 0	0, ***15(1 – r)q,*** 0
	Buy	15, ***0,*** 0	0, ***0,*** 0	$15(1 - r)(1 - q)$, ***0,*** 0

C

Buy \$15 Ticket (Probability r)

		B: *Don't Buy*	B: *Buy*	B: *q-mix*
A	Don't Buy	0, ***0,*** 15	0, ***0,*** 0	0, ***0,*** 15(1 – q)r
	Buy	0, ***0,*** 0	–5, **–5,** –5	–5*qr*, **–5qr,** –5qr

This is a non-constant-sum game, so the probabilities of Contestants B and C should be chosen to keep Contestant A indifferent between buying a \$15 ticket and not buying a ticket. Thus, q and r should be chosen so that $0 = 15(1 - r)(1 - q) - 5qr$, or $15 - 15r - 15q + 10qr = 0$. Since we are looking for a symmetric equilibrium, with $r = q$, the previous equation can be rewritten as $0 = 15 - 30q + 10q^2 = q^2 - 3q + 3/2$. Applying the quadratic formula produces $q = r\ (= p) = 0.634$. Each contestant thus buys a \$15 ticket with probability 0.634 and doesn't buy a ticket with probability 0.366.

2.

		VENDOR 2				
		A	*B*	*C*	*D*	*E*
VENDOR 1	A	85, ***85***	100, ***170***	125, ***195***	150, ***200***	160, ***160***
	B	170, ***100***	110, ***110***	150, ***170***	175, ***175***	200, ***150***
	C	195, ***125***	170, ***150***	120, ***120***	170, ***150***	195, ***125***
	D	200, ***150***	175, ***175***	150, ***170***	110, ***110***	170, ***100***
	E	160, ***160***	150, ***200***	125, ***195***	100, ***170***	85, ***85***

For both vendors, locations A and E are dominated. Thus, for a fully mixed equilibrium, we need only consider each vendor's choice among locations B, C, and D. Let Vendor 1's mixture probabilities be p_B, p_C, and $(1 - p_B - p_C)$. Similarly, let Vendor 2's mixture probabilities be q_B, q_C, and $(1 - q_B - q_C)$. After simplifying, the p-mix and q-mix payoffs are as shown below:

		VENDOR 2			
		B	*C*	*D*	*q*-mix
VENDOR 1	B	110, ***110***	150, ***170***	175, ***175***	$175 - 65q_B - 25q_C$, $\boldsymbol{175 - 65q_B - 5q_C}$
	C	170, ***150***	120, ***120***	170, ***150***	$170 - 50q_C$, $\boldsymbol{150 - 30q_C}$
	D	175, ***175***	150, ***170***	110, ***110***	$110 + 65q_B + 40q_C$, $\boldsymbol{110 + 65q_B + 60q_C}$
	p-mix	$175 - 65p_B - 5p_C$, $\boldsymbol{175 - 65p_B - 25p_C}$	$150 - 30p_C$, $\boldsymbol{170 - 50p_C}$	$110 + 65p_B + 60p_C$, $\boldsymbol{110 + 65p_B + 40p_C}$	

To find the equilibrium p, set Vendor 2's payoffs equal to each other:

$$175 - 65p_B - 25p_C = 170 - 50p_C = 110 + 65p_B + 40p_C$$

$$65p_B - 25p_C = 5 \quad 60 - 90p_C = 65p_B$$

$$60 - 90p_C - 25p_C = 5$$

Then $p_C = 55/115 = 11/23$, $p_B = (25p_C + 5)/65 = (275/23 + 115/23)/65 = (390/23) * 65 = 6/23$, and $p_D = (1 - p_B - p_C) = 1 - 11/23 - 6/23 = 6/23$. Similarly, $p_B = 6/23$, $p_C = 11/23$, and $p_D = 6/23$.

One way to explain why A and E are unused in the equilibrium is to point out that they are (as noted above) dominated. This also implies that A and E are unused because they result in a payoff against the opponent's equilibrium mixture that is lower than that produced by choices B, C, and D. Specifically, when Vendor 2 uses the equilibrium mixture probabilities of (6/23, 11/23, 6/23), Vendor 1's payoff from choosing:

A is 100(6/23) + 125(11/23) + 150(6/23) = 2,875/23.
B is 110(6/23) + 150(11/23) + 175(6/23) = 3,360/23.
C is 170(6/23) + 120(11/23) + 170(6/23) = 3,360/23.
D is 175(6/23) + 150(11/23) + 110(6/23) = 3,360/23.
E is 150(6/23) + 125(11/23) + 100(6/23) = 2,875/23.

Clearly, A and E are inferior choices.

[An alternate possibility is a *partially mixed* equilibrium in which one player plays pure C and the other player mixes, using strategies B and D with probabilities 1/13 = 0.076 and 12/13 = 0.923, respectively. The expected payoff to the player using only C (the pure player) is 170; the expected payoff to the player using a mixture of B and D (the mixer) is 150. The equilibrium can arise in the following way: If Vendor 2 is playing pure C, then Vendor 1 gets equal highest payoffs from B and D and therefore is willing to mix between them in any proportions. Suppose Vendor 1 chooses B with probability p and D with probability $(1 - p)$. To make this an equilibrium, pure C should be Vendor 2's best response to this mixture. A and E are clearly bad for Vendor 2, as we established above. Vendor 2 does not switch to B if

$$110p + 175(1 - p) \le 170, \quad \text{or} \quad 5 \le 65p,$$
$$\text{or} \quad p \ge 1/13 = 0.07692$$

Similarly, Vendor 2 does not switch to D if

$$175p + 110(1 - p) \le 170, \quad \text{or} \quad 65p \ge 60,$$
$$\text{or} \quad p \ge 12/13 = 0.9231$$

Thus there is a whole continuum of mixed-strategy equilibria, in which Vendor 2 plays pure C and Vendor 1 mixes between B and D in any proportions between 1/13 and 12/13. This answer describes the equilibrium that entails at the extreme points of this range.]

3. To find the answers as stated, we look for a *fully mixed* equilibrium in which James's mixture probabilities are p_1 on Left, p_2 on Straight, and $(1 – p_1 – p_2)$ on Right. Similarly, in such an equilibrium, Dean's mixture probabilities would be q_1, q_2, and $(1 – q_1 – q_2)$. Then the complete payoff table is as shown below:

		DEAN			
		Left	*Straight*	*Right*	*q*-mix
JAMES	Left	0, ***0***	–1, ***1***	–2, ***–2***	$-q_2 - 2(1 - q_1 - q_2)$, $\boldsymbol{q_2 - 2(1 - q_1 - q_2)}$
	Straight	1, ***–1***	–2, ***–2***	1, ***–1***	$q_1 - 2q_2 + (1 - q_1 - q_2)$, $\boldsymbol{-q_1 - 2q_2 - (1 - q_1 - q_2)}$
	Right	–2, ***–2***	–1, ***1***	0, ***0***	$-2q_1 - q_2$, $\boldsymbol{-2q_1 + q_2}$
	p-mix	$p_2 - 2(1 - p_1 - p_2)$, $\boldsymbol{-p_2 - 2(1 - p_1 - p_2)}$	$-p_1 - 2p_2 - (1 - p_1 - p_2)$, $\boldsymbol{p_1 - 2p_2 + (1 - p_1 - p_2)}$	$-2p_1 + p_2$, $\boldsymbol{-2p_1 - p_2}$	

James's *p*-mix must keep *Dean* indifferent: $–p_2 – 2(1 – p_1 – p_2) = p_1 – 2p_2 + (1 – p_1 – p_2) = –2p_1 – p_2$ or $–2 + 2p_1 + p_2 = 1 – 3p_2 = –2p_1 – p_2$. The first two parts yield $2p_1 = 3 – 4p_2$, and the last two parts yield $2p_1 = 2p_2 – 1$. Then $3 – 4p_2 = 2p_2 – 1$; this leads to $p_2 = 4/6 = 2/3$. Plug in to find $2p_1 = 2(2/3) – 1 = 1/3$, so $p_1 = 1/6$. Then $1 – p_1 – p_2 = 1/6$ also. Similarly, q_1 and q_2 are chosen so that $–2 + 2q_1 + q_2 = 1 – 3q_2 = –2q_1 – q_2$; this yields $q_2 = 2/3$, $q_1 = 1/6$, and $1 – q_1 – q_2 = 1/6$ also.

In the standard version of Chicken, a player guarantees at least a safe outcome (and a payoff of –1) by choosing to swerve. In this variant of chicken, however, swerving either Left or Right is not a guarantee of safety; there may still be a collision. This makes any sort of swerve a less attractive action and thereby increases the chances that a driver will choose Straight.

(Note that there are additional mixed-strategy equilibria that can be found for this game. These equilibria are only *partially mixed*.)

The first possibility: suppose James mixes only with probability p for L and $1 – p$ for S. Then Dean gets

$0p – 1(1 – p)$ from L, $1p – 2(1 – p)$ from S, and $–2p – 1(1 – p)$ from R

Thus, R is dominated and not used in Dean's mix. Dean's indifference between L and S gives $x = 1/2$, and Dean's expected payoff is then –1/2. This is really just like the two-by-two chicken game. The fully mixed equilibrium has higher probability of S, and lower probabilities of L and R together, than the partially mixed equilibrium, because with full mixing the two swerving strategies are no longer so safe.

The second possibility: only one player mixes. There are equilibria in which one player chooses pure S and the other mixes with arbitrary probabilities between L and R.]

4. As usual for sequential-move games, we begin at the end. Suppose both aristocrats have walked five steps and are right next to each other, but neither has yet fired. The payoffs are shown below, and it is easy to see that Shoot is the dominant strategy at this step for each player; the resulting payoffs are (0, 0).

		CHAGRIN	
		Shoot	*Not*
RENARD	Shoot	0, ***0***	1, ***–1***
	Not	–1, ***1***	0, ***0***

Knowing the equilibrium at the five-step subgame, we use rollback to consider the payoffs after each player has walked four steps:

		CHAGRIN	
		Shoot	*Not*
RENARD	Shoot	0, ***0***	0.6, ***– 0.6***
	Not	– 0.6, ***0.6***	0, ***0***

Payoffs above are derived as follows:

1. Top right cell: If Renard shoots and Chagrin does not, then Renard hits with probability 0.8 and gets 1, and misses with probability 0.2, in which case Chagrin has a sure shot after five moves, so Renard gets –1. Renard's expected payoff is $0.8 \times 1 + 0.2 \times (–1) = 0.6$. Chagrin's expected payoff is $0.8 \times (–1) +$

$0.2 \times 1 = -0.6$. Bottom left cell calculations are similar.

2. Top left cell: If both shoot, Chagrin gets $0.8 \times 0.8 \times 0 + 0.8 \times 0.2 \times 1 + 0.2 \times 0.8 \times (-1) + 0.2 \times 0.2 \times 0 = 0$.
3. Bottom right cell: If neither shoots on step 4, the game goes to step 5, where we know the equilibrium payoff is (0, 0). Thus, we make use of the analysis of step 5 performed above. Now analyzing the payoff table for step 4, we see Shoot is the dominant strategy for both, and the resulting payoff is (0, 0).

For step 3, the duelists are six steps apart and the probability of hitting if you shoot is 0.6. Similar calculations as those for step 4 (for instance, in top right and bottom left cells, use $0.6 \times 1 + 0.4 \times (-1) = 0.2$ and get the following payoff table:

		CHAGRIN	
		Shoot	*Not*
RENARD	Shoot	0, ***0***	0.2, ***– 0.2***
	Not	– 0.2, ***0.2***	0, ***0***

Again, shoot is dominant for each player and yields the payoffs (0, 0).

Next move back to step 2, with the pair eight paces apart and the probability of hitting down to 0.4. The payoff table is now:

		CHAGRIN	
		Shoot	*Not*
RENARD	Shoot	0, ***0***	– 0.2, ***0.2***
	Not	0.2, ***– 0.2***	0, ***0***

Things have changed: Not shoot is now the dominant strategy for both. The resulting equilibrium payoff is still (0, 0).

Finally, at step 1, with the pair in their starting positions and the probability of hitting only 0.2, the payoff matrix is as shown below:

		CHAGRIN	
		Shoot	*Not*
RENARD	Shoot	0, ***0***	– 0.6, ***0.6***
	Not	0.6, ***– 0.6***	0, ***0***

Not shoot is again the dominant strategy with payoffs of (0, 0).

Actual play of the game will result in both players shooting on step 3. The full equilibrium strategy for either player is: "Do not shoot on steps 1 and 2 no matter what. At any step, if the other player has shot and missed while you have yet to shoot, then wait until step 5 to shoot. If you arrive at step 3 (or later) and the other player has not yet shot, then shoot at once."

5. Two firms are trying to decide how much time to spend developing their product before releasing it into the market. The longer a firm waits, the better the product it will be selling, and (other things equal) the better its chance of taking over the market. However, the longer one firm waits, the greater the chance that the other firm will release its product first; if that product is successful, the firm that delayed may be shut out of the market. Thus, each firm has to choose between waiting to "shoot" (thus increasing its chance of hitting the "bullseye") and "shooting" early (in order to be sure of getting off a shot before the other player hits the target).
6. For the shooter playing against the goalie's strategy of L and R 42.2% and C 15.6%, payoffs from each of his possible strategies are:

HL 42.2(0.50) + 15.6(0.85) + 42.2(0.85) = 70.23
LL 42.2(0.40) + 15.6(0.95) + 42.2(0.95) = 71.79
HC 42.2(0.85) + 15.6(0) + 42.2(0.85) = 71.74
LC 42.2(0.70) + 15.6(0) + 42.2(0.70) = 59.08
HR 42.2(0.85) + 15.6(0.85) + 42.2(0.50) = 70.23
LR 42.2(0.95) + 15.6(0.95) + 42.2(0.40) = 71.79

Thus, the shooter should be using low side shots (LL and LR) and high centered shots (HC).

For the goalie playing against the shooter using LL 37.8%, HC 24.4%, and LR 37.8% of the time, payoffs from each of his possible strategies are:

L 0.378(0.40) + 0.244(0.85) + 0.378(0.95) = 71.77
C 0.378(0.95) + 0.244(0) + 0.378(0.95) = 71.82
R 0.378(0.95) + 0.244(0.85) + 0.378(0.40) = 71.77

Accounting for rounding error, these payoffs are identical.

7. The payoff table is reproduced below:

		CHAGRIN				
		1	*2*	*3*	*4*	*5*
RENARD	1	0.00	–0.12	–0.28	–0.44	–0.60
	2	0.12	0.00	0.04	–0.08	–0.20
	3	0.28	–0.04	0.00	0.28	0.20
	4	0.44	0.08	–0.28	0.00	0.60
	5	0.60	0.20	–0.20	–0.60	0.00

We provide the derivation for a few of the cells; other calculations are similar. If Renard (R) and Chagrin (C) both shoot on step 1 (top left cell), then R hits with probability 0.2 and C hits with probability 0.2 (R gets 0), R hits with probability 0.2 and C misses with probability 0.8 (R gets 1), R misses with probability 0.8 and C hits with probability 0.2 (R gets – 1), or R misses with probability 0.8 and C misses with probability 0.8 (R gets 0). Then R's expected payoff = (0.2)(0.2)(0) + (0.2)(0.8)(1) + (0.8)(0.2)(–1) + (0.8)(0.8) = 0.

If Renard shoots on step 1 and Chagrin on step 2, R hits with probability 0.2 (R gets 1), R misses with probability 0.8 and C hits with probability 0.4 (R gets – 1), or both miss with probability (0.8)(0.6). R's expected payoff = (0.2)(1) + (0.8)(0.4)(–1) + (0.8)(0.6)(0) = –0.12.

If Renard shoots on step 1 and Chagrin on step 3, R hits with probability 0.2 (R gets 1), R misses with probability 0.8 and C hits with probability 0.6 (R gets –1), or both miss with probability (0.8)(0.4). R's expected payoff = (0.2)(1) + (0.8)(0.6)(–1) + (0.8)(0.4)(0) = –0.28.

If Renard shoots on step 2 and Chagrin on step 3, then R hits with probability 0.4 (R gets 1), R misses with probability 0.6 and Chagrin hits with probability 0.6 (R gets –1), or both miss with probability (0.6)(0.4). R's expected payoff = (0.4)(1) + (0.6)(0.6)(–1) + (0.6)(0.4)(0) = 0.04.

Other cells are calculated in a similar fashion; entries below the main diagonal are symmetric (Renard's payoff is the negative of that calculated for the matching cell above the diagonal).

Given the payoff table, we look for equilibria. For each player, the pure strategy 1 is dominated by the pure strategy 2. To verify that the equilibrium in the remaining four-by-four game entails strategy 4 unused and strategies 2, 3, and 5, used in the proportions 5/11, 5/11, and 1/11, find Renard's payoffs from each of his pure strategies against Chagrin's equilibrium mix:

From 2: $(5/11) \times 0 + (5/11) \times 0.04 + 0 \times (-0.08) + (1/11) \times (-0.2) = 0$

From 3: $(5/11) \times (-0.04) + (5/11) \times 0 + 0 \times 0.28 + (1/11) \times 0.2 = 0$

From 4: $(5/11) \times 0.08 + (5/11) \times (-0.28) + 0 \times 0 + (1/11) \times 0.6 = -0.36$

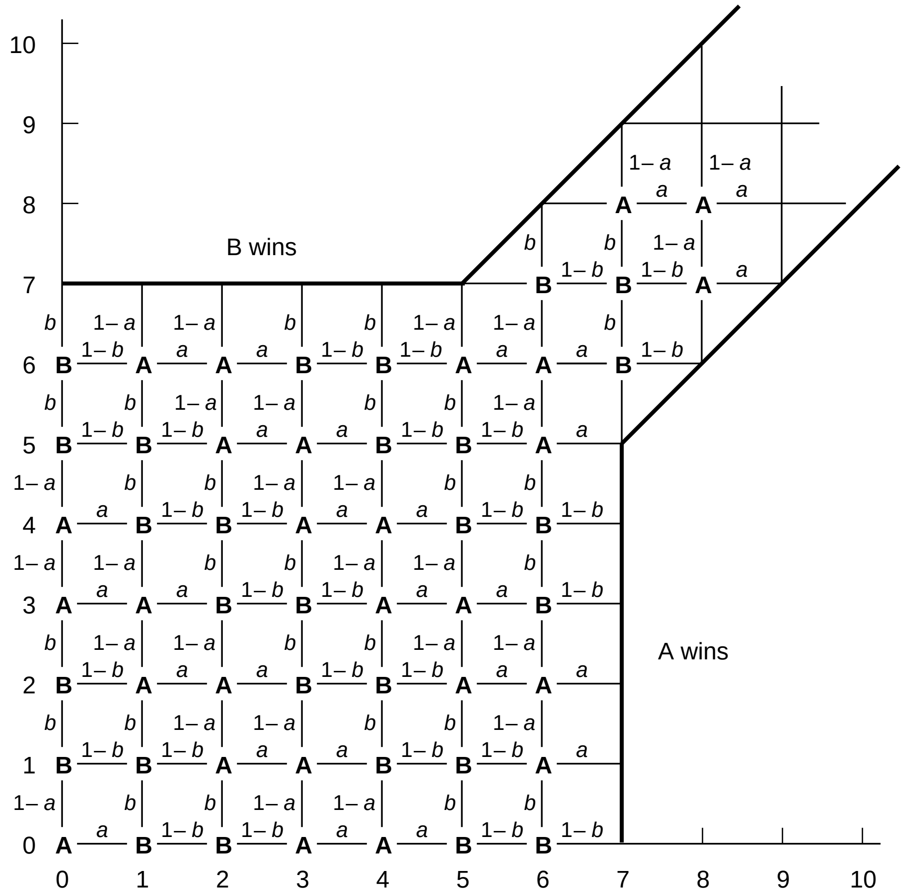
From 5: $(5/11) \times 0.2$ 3+ $(5/11) \times (-0.2) + 0 \times (-0.6) + (1/11) \times 0 = 0$

All the used strategies perform equally well, while the unused strategy performs worse. Renard is willing to mix among strategies 2, 3, and 5 in a way that also keeps Chagrin willing to mix. The situation is identical when considering Renard's equilibrium mix. The expected payoff for each player in equilibrium is 0.

8. Let players be called A and B; A will be the player to serve the first point. Then,

a = Probability{A wins any one point which he serves}
b = Probability{B wins any one point which *he* serves}

Below is the schematic picture of possible situations and probabilities of transition to next situation:



Now we must distinguish six end-game situations. Call them (5, 5), (6, 5), (5, 6), (6, 6), (7, 6), and (6, 7). Then (7, 7) is equivalent to (5, 5), and so on. And, if we write $P(i, j)$ = Probability{A wins tiebreak starting at (i, j)},

$P(5, 5) = bP(5, 6) + (1 - b)P(6, 5)$
$P(6, 5) = a + (1 - a)P(6, 6)$
$P(5, 6) = aP(6, 6) + (1 - a)(0)$
$P(6, 6) = aP(7, 6) + (1 - a)P(6, 7)$
$P(7, 6) = bP(5, 5) + (1 - b)(1)$
$P(6, 7) = b(0) + (1 - b)P(5, 5)$

Substituting and solving, we find

$$P(5, 5) = P(6, 6) = \frac{a(1 - b)}{a(1 - b) + b(1 - a)}$$

$$P(5, 6) = \frac{a^2(a - b)}{a(1 - b) + b(1 - a)}$$

$$P(6, 5) = \frac{a(1 - ab)}{a(1 - b) + b(1 - a)}$$

$$P(6, 7) = \frac{a(1 - b)^2}{a(1 - b) + b(1 - a)}$$

$$P(7, 6) = \frac{(1 - b)(a + b - ab)}{a(1 - b) + b(1 - a)}$$

Then everything else can be calculated using the standard recursion method, as done in the chapter. Some sample numerical solutions using a Fortran program are shown in the following tables; the numbers in the cells are the $P(i, j)$.

For the case in which $a = 0.65$ and $b = 0.65$:

7							0.175	
6	0.000	0.017	0.026	0.040	0.114	0.325	0.500	0.675
5	0.019	0.054	0.123	0.176	0.249	0.500	0.825	
4	0.078	0.111	0.215	0.387	0.500	0.635	0.886	
3	0.186	0.243	0.315	0.500	0.711	0.824	0.926	
2	0.264	0.410	0.500	0.600	0.785	0.922	0.974	
1	0.347	0.500	0.667	0.757	0.841	0.946	0.991	
0	0.500	0.582	0.735	0.863	0.920	0.963	0.994	
	0	*1*	*2*	*3*	*4*	*5*	*6*	*7*

For the case in which $a = 0.70$ and $b = 0.60$:

7							0.249	
6	0.000	0.033	0.048	0.068	0.170	0.426	0.609	0.782
5	0.040	0.099	0.198	0.262	0.346	0.609	0.883	
4	0.144	0.189	0.324	0.512	0.620	0.737	0.930	
3	0.302	0.370	0.447	0.631	0.810	0.892	0.958	
2	0.405	0.561	0.643	0.727	0.869	0.959	0.987	
1	0.505	0.653	0.792	0.856	0.911	0.974	0.996	
0	0.663	0.731	0.847	0.930	0.962	0.983	0.998	
	0	*1*	*2*	*3*	*4*	*5*	*6*	*7*