$\sum_{1\le i<j\le n} p(E_i \cap E_j) + \sum_{1\le i<j<k\le n} p(E_i \cap E_j \cap E_k) - \cdots + (-1)^{n+1} p\left(\bigcap_{i=1}^{n} E_i\right)$

Section 7.6

1. 75 **3.** 6 **5.** 46 **7.** 9875 **9.** 540 **11.** 2100 **13.** 1854 **15. a)** $D_{100}/100!$ **b)** $100D_{99}/100!$ **c)** $C(100,2)/100!$ **d)** 0 **e)** $1/100!$ **17.** 2,170,680 **19.** By Exercise 18 we have $D_n - nD_{n-1} = -[D_{n-1} - (n-1)D_{n-2}]$. Iterating, we have $D_n - nD_{n-1} = -[D_{n-1} - (n-1)D_{n-2}] = -[-(D_{n-2} - (n-2)D_{n-3})] = D_{n-2} - (n-2)D_{n-3} = \cdots = (-1)^n(D_2 - 2D_1) = (-1)^n$ because $D_2 = 1$ and $D_1 = 0$. **21.** When n is odd **23.** $\phi(n) = n - \sum_{i=1}^{m} \frac{n}{p_i} + \sum_{1\le i<j\le m} \frac{n}{p_i p_j} - \cdots \pm \frac{n}{p_1 p_2 \cdots p_m} = n \prod_{i=1}^{m}\left(a - \frac{1}{p_i}\right)$ **25.** 4 **27.** There are n^m functions from a set with m elements to a set with n elements, $C(n,1)(n-1)^m$ functions from a set with m elements to a set with n elements that miss exactly one element, $C(n,2)(n-2)^m$ functions from a set with m elements to a set with n elements that miss exactly two elements, and so on, with $C(n, n-1) \cdot 1^m$ functions from a set with m elements to a set with n elements that miss exactly $n-1$ elements. Hence, by the principle of inclusion–exclusion, there are $n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \cdots + (-1)^{n-1}C(n, n-1) \cdot 1^m$ onto functions.

Supplementary Exercises

1. a) $A_n = 4A_{n-1}$ **b)** $A_1 = 40$ **c)** $A_n = 10 \cdot 4^n$ **3. a)** $M_n = M_{n-1} + 160{,}000$ **b)** $M_1 = 186{,}000$ **c)** $M_n = 160{,}000n + 26{,}000$ **d)** $T_n = T_{n-1} + 160{,}000n + 26{,}000$ **e)** $T_n = 80{,}000n^2 + 106{,}000n$ **5. a)** $a_n = a_{n-2} + a_{n-3}$ **b)** $a_1 = 0, a_2 = 1, a_3 = 1$ **c)** $a_{12} = 12$ **7. a)** 2 **b)** 5 **c)** 8 **d)** 16 **9.** $a_n = 2^n$ **11.** $a_n = 2 + 4n/3 + n^2/2 + n^3/6$ **13.** $a_n = a_{n-2} + a_{n-3}$ **15.** $O(n^4)$ **17.** $O(n)$ **19. a)** $18n + 18$ **b)** 18 **c)** 0 **21.** $\Delta(a_n b_n) = a_{n+1}b_{n+1} - a_n b_n = a_{n+1}(b_{n+1} - b_n) + b_n(a_{n+1} - a_n) = a_{n+1}\Delta b_n + b_n \Delta a_n$ **23. a)** Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$. Then $G'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$. Therefore, $G'(x) - G'(x) = \sum_{n=0}^{\infty}[(n+1)a_{n+1} - a_n]x^n = \sum_{n=0} x^n/n! = e^x$, as desired. That $G(0) = a_0 = 1$ is given. **b)** We have $[e^{-x}G(x)]' = e^{-x}G'(x) - e^{-x}G(x) = e^{-x}[G'(x) - G(x)] = e^{-x} \cdot e^x = 1$. Hence, $e^{-x}G(x) = x + c$, where c is a constant. Consequently, $G(x) = xe^x + ce^x$. Because $G(0) = 1$, it follows that $c = 1$. **c)** We have $G(x) = \sum_{n=0}^{\infty} x^{n+1}/n! + \sum_{n=0}^{\infty} x^n/n! = \sum_{n=1}^{\infty} x^n/(n-1)! + \sum_{n=0}^{\infty} x^n/n!$. Therefore, $a_n = 1/(n-1)! + 1/n!$ for all $n \ge 1$, and $a_0 = 1$. **25.** 7 **27.** 110 **29.** 0 **31. a)** 19 **b)** 65 **c)** 122 **d)** 167 **e)** 168 **33.** $D_{n-1}/(n-1)!$ **35.** 11/32

CHAPTER 8

Section 8.1

1. a) $\{(0,0), (1,1), (2,2), (3,3)\}$ **b)** $\{(1,3), (2,2), (3,1), (4,0)\}$ **c)** $\{(1,0), (2,0), (2,1), (3,0), (3,1), (3,2), (4,0), (4,1), (4,2), (4,3)\}$ **d)** $\{(1,0), (1,1), (1,2), (1,3), (2,0), (2,2), (3,0), (3,3), (4,0)\}$ **e)** $\{(0,1), (1,0), (1,1), (1,2), (1,3), (2,1), (2,3), (3,1), (3,2), (4,1), (4,3)\}$ **f)** $\{(1,2), (2,1), (2,2)\}$ **3. a)** Transitive **b)** Reflexive, symmetric, transitive **c)** Symmetric **d)** Antisymmetric **e)** Reflexive, symmetric, antisymmetric, transitive **f)** None of these properties **5. a)** Reflexive, transitive **b)** Symmetric **c)** Symmetric **d)** Symmetric **7. a)** Symmetric **b)** Symmetric, transitive **c)** Symmetric **d)** Reflexive, symmetric, transitive **e)** Reflexive, transitive **f)** Reflexive, symmetric, transitive **g)** Antisymmetric **h)** Antisymmetric, transitive **9.** (c), (d), (f) **11. a)** Not irreflexive **b)** Not irreflexive **c)** Not irreflexive **d)** Not irreflexive **13.** Yes, for instance $\{(1,1)\}$ on $\{1,2\}$ **15.** $(a,b) \in R$ if and only if a is taller than b **17.** (a) **19.** None **21.** $\forall a \forall b\,[(a,b) \in R \to (b,a) \notin R]$ **23.** 2^{mn} **25. a)** $\{(a,b) \mid b \text{ divides } a\}$ **b)** $\{(a,b) \mid a \text{ does not divide } b\}$ **27.** The graph of f^{-1} **29. a)** $\{(a,b) \mid a$ is required to read or has read $b\}$ **b)** $\{(a,b) \mid a$ is required to read and has read $b\}$ **c)** $\{(a,b) \mid$ either a is required to read b but has not read it or a has read b but is not required to$\}$ **d)** $\{(a,b) \mid a$ is required to read b but has not read it$\}$ **e)** $\{(a,b) \mid a$ has read b but is not required to$\}$ **31.** $S \circ R = \{(a,b) \mid a$ is a parent of b and b has a sibling$\}$, $R \circ S = \{(a,b) \mid a$ is an aunt or uncle of $b\}$ **33. a)** $\mathbf{R}^2$ **b)** R_6 **c)** R_3 **d)** R_3 **e)** $\emptyset$ **f)** R_1 **g)** R_4 **h)** R_4 **35. a)** R_1 **b)** R_2 **c)** R_3 **d)** $\mathbf{R}^2$ **e)** R_3 **f)** $\mathbf{R}^2$ **g)** $\mathbf{R}^2$ **h)** $\mathbf{R}^2$ **37.** b got his or her doctorate under someone who got his or her doctorate under a; there is a sequence of $n+1$ people, starting with a and ending with b, such that each is the advisor of the next person in the sequence **39. a)** $\{(a,b) \mid a - b \equiv 0, 3, 4, 6, 8, \text{ or } 9 \pmod{12}\}$ **b)** $\{(a,b) \mid a \equiv b \pmod{12}\}$ **c)** $\{(a,b) \mid a - b \equiv 3, 6, \text{ or } 9 \pmod{12}\}$ **d)** $\{(a,b) \mid a - b \equiv 4 \text{ or } 8 \pmod{12}\}$ **e)** $\{(a,b) \mid a - b \equiv 3, 4, 6, 8, \text{ or } 9 \pmod{12}\}$ **41.** 8 **43. a)** 65,536 **b)** 32,768 **45. a)** $2^{n(n+1)/2}$ **b)** $2^n 3^{n(n-1)/2}$ **c)** $3^{n(n-1)/2}$ **d)** $2^{n(n-1)}$ **e)** $2^{n(n-1)/2}$ **f)** $2^{n^2} - 2 \cdot 2^{n(n-1)}$ **47.** There may be no such b. **49.** If R is symmetric and $(a,b) \in R$, then $(b,a) \in R$, so $(a,b) \in R^{-1}$. Hence, $R \subseteq R^{-1}$. Similarly, $R^{-1} \subseteq R$. So $R = R^{-1}$. Conversely, if $R = R^{-1}$ and $(a,b) \in R$, then $(a,b) \in R^{-1}$, so $(b,a) \in R$. Thus R is symmetric. **51.** R is reflexive if and only if $(a,a) \in R$ for all $a \in A$ if and only if $(a,a) \in R^{-1}$ [because $(a,a) \in R$ if and only if $(a,a) \in R^{-1}$] if and only if R^{-1} is reflexive. **53.** Use mathematical induction. The result is trivial for $n = 1$. Assume R^n is reflexive and transitive. By Theorem 1, $R^{n+1} \subseteq R$. To see that $R \subseteq R^{n+1} = R^n \circ R$, let $(a,b) \in R$. By the inductive hypothesis, $R^n = R$ and hence, is reflexive. Thus $(b,b) \in R^n$. Therefore $(a,b) \in R^{n+1}$. **55.** Use mathematical induction. The result is trivial for $n = 1$. Assume R^n is reflexive. Then $(a,a) \in R^n$ for all $a \in A$ and $(a,a) \in R$. Thus $(a,a) \in R^n \circ R = R^{n+1}$ for all $a \in A$. **57.** No, for instance, take $R = \{(1,2), (2,1)\}$.

Section 8.2

1. $\{(1,2,3), (1,2,4), (1,3,4), (2,3,4)\}$ **3.** (Nadir, 122, 34, Detroit, 08:10), (Acme, 221, 22, Denver, 08:17), (Acme, 122, 33, Anchorage, 08:22), (Acme, 323, 34, Honolulu 08:30), (Nadir, 199, 13, Detroit, 08:47), (Acme, 222, 22, Denver, 09:10), (Nadir, 322, 34, Detroit, 09:44) **5.** Airline and flight number, airline and departure time **7. a)** Yes **b)** No **c)** No **9. a)** Social Security number **b)** There are no two people with the same name who happen to have the same street address. **c)** There are no two people with the same name living together. **11.** (Nadir, 122, 34, Detroit, 08: 10), (Nadir, 199, 13, Detroit, 08: 47), (Nadir, 322, 34, Detroit, 09: 44) **13.** (Nadir, 122, 34, Detroit, 08: 10), (Nadir, 199, 13, Detroit, 08: 47), (Nadir, 322, 34, Detroit, 09: 44), (Acme, 221, 22, Denver, 08: 17), (Acme, 222, 22, Denver, 09: 10) **15.** $P_{3,5,6}$

17.

Airline	*Destination*
Nadir	Detroit
Acme	Denver
Acme	Anchorage
Acme	Honolulu

19.

Supplier	*Part_ number*	*Project*	*Quantity*	*Color_ code*
23	1092	1	2	2
23	1101	3	1	1
23	9048	4	12	2
31	4975	3	6	2
31	3477	2	25	2
32	6984	4	10	1
32	9191	2	80	4
33	1001	1	14	8

21. Both sides of this equation pick out the subset of R consisting of those n-tuples satisfying both conditions C_1 and C_2. **23.** Both sides of this equation pick out the set of n-tuples that are in R, are in S, and satisfy condition C. **25.** Both sides of this equation pick out the m-tuples consisting of i_1th, i_2th, ..., i_mth components of n-tuples in either R or S. **27.** Let $R = \{(a, b)\}$ and $S = \{(a, c)\}$, $n = 2$, $m = 1$, and $i_1 = 1$; $P_1(R - S) = \{(a)\}$, but $P_1(R) - P_1(S) = \emptyset$. **29. a)** J_2 followed by $P_{1,3}$ **b)** (23, 1), (23, 3), (31, 3), (32, 4) **31.** There is no primary key.

Section 8.3

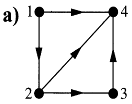
1. a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **b)** $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ **d)** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

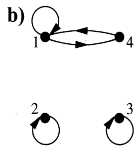
3. a) (1, 1), (1, 3), (2, 2), (3, 1), (3, 3) **b)** (1, 2), (2, 2), (3, 2) **c)** (1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (3, 3) **5.** The relation is irreflexive if and only if the main diagonal of the matrix contains only 0s. **7. a)** Reflexive, symmetric, transitive **b)** Antisymmetric, transitive **c)** Symmetric **9. a)** 4950 **b)** 9900 **c)** 99 **d)** 100 **e)** 1 **11.** Change each 0 to a 1 and each 1 to a 0.

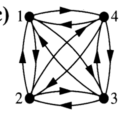
13. a) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

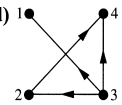
15. a) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **c)** $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

17. $n^2 - k$

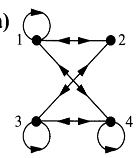
19. a) 

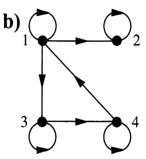
b) 

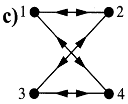
c) 

d) 

21. For simplicity we have indicated pairs of edges between the same two vertices in opposite directions by using a double arrowhead, rather than drawing two separate lines.

a) 

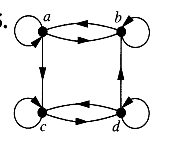
b) 

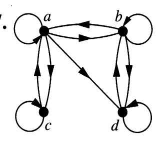
c) 

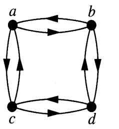
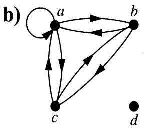
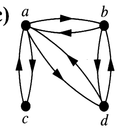
23. $\{(a, b), (a, c), (b, c), (c, b)\}$ **25.** $(a, c), (b, a), (c, d), (d, b)$ **27.** $\{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (d, d)\}$ **29.** The relation is asymmetric if and only if the directed graph has no loops and no closed paths of length 2. **31.** Exercise 23: irreflexive. Exercise 24: reflexive, antisymmetric, transitive. Exercise 25: irreflexive, antisymmetric. **33.** Reverse the direction on every edge in the digraph for R. **35.** Proof by mathematical induction. *Basis step:* Trivial for $n = 1$. *Inductive step:* Assume true for k. Because $R^{k+1} = R^k \circ R$, its matrix is $\mathbf{M}_R \odot \mathbf{M}_{R^k}$. By the inductive hypothesis this is $\mathbf{M}_R \odot \mathbf{M}_R^{[k]} = \mathbf{M}_R^{[k+1]}$.

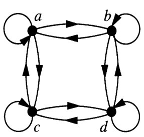
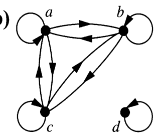
Section 8.4

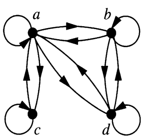
1. a) $\{(0, 0), (0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 0), (3, 3)\}$ **b)** $\{(0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (3, 0)\}$ **3.** $\{(a, b) \mid a \text{ divides } b \text{ or } b \text{ divides } a\}$

5. 

7. 

9. a)  **b)**  **c)** 

11. a)  **b)** 

c) 

13. The symmetric closure of R is $R \cup R^{-1}$. $\mathbf{M}_{R\cup R^{-1}} = \mathbf{M}_R \vee \mathbf{M}_{R^{-1}} = \mathbf{M}_R \vee \mathbf{M}_R^t$. **15.** Only when R is irreflexive, in which case it is its own closure. **17.** *a, a, a, a; a, b, e, a; a, d, e, a; b, c, c, b; b, e, a, b; c, b, c, c; c, c, b, c; c, c, c, c; d, e, a, d; d, e, e, d; e, a, b, e; e, a, d, e; e, d, e, e; e, e, d, e; e, e, e, e* **19. a)** $\{(1,1),(1,5),(2,3),(3,1),(3,2),(3,3),(3,4),(4,1),(4,5),(5,3),(5,4)\}$ **b)** $\{(1,1),(1,2),(1,3),(1,4),(2,1),(2,5),(3,1),(3,3),(3,4),(3,5),(4,1),(4,2),(4,3),(4,4),(5,1),(5,3),(5,5)\}$ **c)** $\{(1,1),(1,3),(1,4),(1,5),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(3,5),(4,1),(4,3),(4,4),(4,5),(5,1),(5,2),(5,3),(5,4),(5,5)\}$ **d)** $\{(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,3),(2,4),(2,5),(3,1),(3,2),(3,3),(3,4),(3,5),(4,1),(4,2),(4,3),(4,4),(4,5),(5,1),(5,2),(5,3),(5,4),(5,5)\}$ **e)** $\{(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2),(2,3),(2,4),(2,5),(3,1),(3,2),(3,3),(3,4),(3,5),(4,1),(4,2),(4,3),(4,4),(4,5),(5,1),(5,2),(5,3),(5,4),(5,5)\}$ **f)** $\{(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2),(2,3),(2,4),(2,5),(3,1),(3,2),(3,3),(3,4),(3,5),(4,1),(4,2),(4,3),(4,4),(4,5),(5,1),(5,2),(5,3),(5,4),(5,5)\}$ **21. a)** If there is a student c who shares a class with a and a class with b **b)** If there are two students c and d such that a and c share a class, c and d share a class, and d and b share a class **c)** If there is a sequence $s_0, \ldots, s_n$ of students with $n \geq 1$ such that $s_0 = a$, $s_n = b$, and for each $i = 1, 2, \ldots, n$, s_i and s_{i-1} share a class **23.** The result follows from $(R^*)^{-1} = \left(\bigcup_{n=1}^{\infty} R^n\right)^{-1} = \bigcup_{n=1}^{\infty}(R^n)^{-1} = \bigcup_{n=1}^{\infty} R^n = R^*$.

25. a) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **d)** $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

27. Answers same as for Exercise 25. **29. a)** $\{(1,1),(1,2),(1,4),(2,2),(3,3),(4,1),(4,2),(4,4)\}$ **b)** $\{(1,1),(1,2),(1,4),(2,1),(2,2),(2,4),(3,3),(4,1),(4,2),(4,4)\}$ **c)** $\{(1,1),(1,2),(1,4),(2,1),(2,2),(2,4),(3,3),(4,1),(4,2),(4,4)\}$ **31.** Algorithm 1: $O(n^{3.8})$; Algorithm 2: $O(n^3)$ **33.** Initialize with $\mathbf{A} := \mathbf{M}_R \vee \mathbf{I}_n$ and loop only for $i := 2$ to $n-1$. **35. a)** Because R is reflexive, every relation containing it must also be reflexive. **b)** Both $\{(0,0),(0,1),(0,2),(1,1),(2,2)\}$ and $\{(0,0),(0,1),(1,0),(1,1),(2,2)\}$ contain R and have an odd number of elements, but neither is a subset of the other.

Section 8.5

1. a) Equivalence relation **b)** Not reflexive, not transitive **c)** Equivalence relation **d)** Not transitive **e)** Not symmetric, not transitive **3. a)** Equivalence relation **b)** Not transitive **c)** Not reflexive, not symmetric, not transitive **d)** Equivalence relation **e)** Not reflexive, not transitive **5.** Many answers are possible. (1) Two buildings are equivalent if they were opened during the same year; an equivalence class consists of the set of buildings opened in a given year (as long as there was at least one building opened that year). (2) Two buildings are equivalent if they have the same number of stories; the equivalence classes are the set of 1-story buildings, the set of 2-story buildings, and so on (one class for each n for which there is at least one n-story building). (3) Every building in which you have a class is equivalent to every building in which you have a class (including itself), and every building in which you don't have a class is equivalent to every building in which you don't have a class (including itself); there are two equivalence classes—the set of buildings in which you have a class and the set of buildings in which you don't have a class (assuming these are nonempty). **7.** The statement "p is equivalent to q" means that p and q have the same entries in their truth tables. R is reflexive, because p has the same truth table as p. R is symmetric, for if p and q have the same truth table, then q and p have the same truth table. If p and q have the same entries in their truth tables and q and r have the same entries in their truth tables, then p and r also do, so R is transitive. The equivalence class of $\mathbf{T}$ is the set of all tautologies; the equivalence class of $\mathbf{F}$ is the set of all contradictions. **9. a)** $(x,x) \in R$ because $f(x) = f(x)$. Hence, R is reflexive. $(x,y) \in R$ if and only if $f(x) = f(y)$, which holds if and only if $f(y) = f(x)$ if and only if $(y,x) \in R$. Hence, R is symmetric. If $(x,y) \in R$ and $(y,z) \in R$, then $f(x) = f(y)$ and $f(y) = f(z)$. Hence, $f(x) = f(z)$. Thus, $(x,z) \in R$. It follows that R is transitive. **b)** The sets $f^{-1}(b)$ for b in the range of f **11.** Let x be a string of length 3 or more. Because x agrees with itself in the first three bits, $(x,x) \in R$. Hence, R is reflexive. Suppose that $(x,y) \in R$. Then x and y agree in the first three bits. Hence, y and x agree in the first three bits. Thus, $(y,x) \in R$. If (x,y) and (y,z) are in R, then x and y agree in the first three bits, as do y and z. Hence, x and z agree in the first three bits. Hence, $(x,z) \in R$. It follows that R

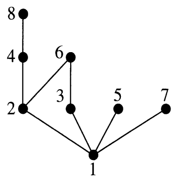
is transitive. **13.** This follows from Exercise 9, where f is the function that takes a bit string of length 3 or more to the ordered pair with its first bit as the first component and the third bit as its second component. **15.** For reflexivity, $((a, b), (a, b)) \in R$ because $a + b = b + a$. For symmetry, if $((a, b), (c, d)) \in R$, then $a + d = b + c$, so $c + b = d + a$, so $((c, d), (a, b)) \in R$. For transitivity, if $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$, then $a + d = b + c$ and $c + e = d + f$, so $a + d + c + e = b + c + d + f$, so $a + e = b + f$, so $((a, b), (e, f)) \in R$. An easier solution is to note that by algebra, the given condition is the same as the condition that $f((a, b)) = f((c, d))$, where $f((x, y)) = x - y$; therefore by Exercise 9 this is an equivalence relation. **17. a)** This follows from Exercise 9, where the function f from the set of differentiable functions (from **R** to **R**) to the set of functions (from **R** to **R**) is the differentiation operator. **b)** The set of all functions of the form $g(x) = x^2 + C$ for some constant C **19.** This follows from Exercise 9, where the function f from the set of all URLs to the set of all Web pages is the function that assigns to each URL the Web page for that URL. **21.** No **23.** No **25.** R is reflexive because a bit string s has the same number of 1s as itself. R is symmetric because s and t having the same number of 1s implies that t and s do. R is transitive because s and t having the same number of 1s, and t and u having the same number of 1s implies that s and u have the same number of 1s. **27. a)** The sets of people of the same age **b)** The sets of people with the same two parents **29.** The set of all bit strings with exactly two 1s. **31. a)** The set of all bit strings that begin 010 **b)** The set of all bit strings that begin 101 **c)** The set of all bit strings that begin 111 **d)** The set of all bit strings that begin 010 **33.** Each of the 15 bit strings of length less than four is in an equivalence class by itself: $[\lambda]_{R_4} = \{\lambda\}$, $[0]_{R_4} = \{0\}$, $[1]_{R_4} = \{1\}$, $[00]_{R_4} = \{00\}$, $[01]_{R_4} = \{01\}, \ldots,$ $[111]_{R_4} = \{111\}$. The remaining 16 equivalence classes are determined by the bit strings of length 4: $[0000]_{R_4} = \{0000,$ $00000, 00001, 000000, 000001, 000010, 000011, 0000000, \ldots\}$, $[0001]_{R_4} = \{0001, 00010, 00011, 000100, 000101, 000110, 000111, 0001000, \ldots\}, \ldots, [1111]_{R_4} = \{1111, 11110, 11111, 111100, 111101, 111110, 111111, 1111000, \ldots\}$ **35. a)** $[2]_5 = \{i \mid i \equiv 2 \pmod 5\} = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$ **b)** $[3]_5 = \{i \mid i \equiv 3 \pmod 5\} = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$ **c)** $[6]_5 = \{i \mid i \equiv 6 \pmod 5\} = \{\ldots, -9, -4, 1, 6, 11, \ldots\}$ **d)** $[-3]_5 = \{i \mid i \equiv -3 \pmod 5\} = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$ **37.** $\{6n + k \mid n \in \mathbf{Z}\}$ for $k \in \{0, 1, 2, 3, 4, 5\}$ **39. a)** $[(1, 2)] = \{(a, b) \mid a - b = -1\} = \{(1, 2), (3, 4), (4, 5), (5, 6), \ldots\}$ **b)** Each equivalence class can be interpreted as an integer (negative, positive, or zero); specifically, $[a, b]$ can be interpreted as $a - b$. **41. a)** No **b)** Yes **c)** Yes **d)** No **43.** (a), (c), (e) **45.** (b), (d), (e) **47. a)** $\{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5)\}$ **b)** $\{(0, 0), (0, 1), (1, 0), (1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$ **c)** $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5)\}$ **d)** $\{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$ **49.** $[0]_6 \subseteq [0]_3$, $[1]_6 \subseteq [1]_3$, $[2]_6 \subseteq [2]_3$, $[3]_6 \subseteq [0]_3$, $[4]_6 \subseteq [1]_3$, $[5]_6 \subseteq [2]_3$ **51.** Let A be a set in the first partition. Pick a particular element x of A. The set of all bit strings of length 16 that agree with x on the last eight bits is one of the sets in the second partition, and clearly every string in A is in that set. **53.** We claim that each equivalence class $[x]_{R_{31}}$ is a subset of the equivalence class $[x]_{R_8}$. To show this, choose an arbitrary element $y \in [x]_{R_{31}}$. Then y is equivalent to x under R_{31}, so either $y = x$ or y and x are each at least 31 characters long and agree on their first 31 characters. Because strings that are at least 31 characters long and agree on their first 31 characters perforce are at least 8 characters long and agree on their first 8 characters, we know that either $y = x$ or y and x are each at least 8 characters long and agree on their first 8 characters. This means that y is equivalent to x under R_8, so $y \in [x]_{R_8}$. **55.** $\{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c), (d, d), (d, e), (e, d), (e, e)\}$ **57. a)** **Z** **b)** $\{n + \frac{1}{2} \mid n \in \mathbf{Z}\}$ **59. a)** R is reflexive because any coloring can be obtained from itself via a 360-degree rotation. To see that R is symmetric and transitive, use the fact that each rotation is the composition of two reflections and conversely the composition of two reflections is a rotation. Hence, (C_1, C_2) belongs to R if and only if C_2 can be obtained from C_1 by a composition of reflections. So if (C_1, C_2) belongs to R, so does (C_2, C_1) because the inverse of the composition of reflections is also a composition of reflections (in the opposite order). Hence, R is symmetric. To see that R is transitive, suppose (C_1, C_2) and (C_2, C_3) belong to R. Taking the composition of the reflections in each case yields a composition of reflections, showing that (C_1, C_3) belongs to R. **b)** We express colorings with sequences of length four, with r and b denoting red and blue, respectively. We list letters denoting the colors of the upper left square, upper right square, lower left square, and lower right square, in that order. The equivalence classes are: $\{rrrr\}$, $\{bbbb\}$, $\{rrrb, rrbr, rbrr, brrr\}$, $\{bbbr, bbrb, brbb, rbbb\}$, $\{rbbr, brrb\}$, $\{rrbb, brbr, bbrr, rbrb\}$. **61.** 5 **63.** Yes **65.** R **67.** First form the reflexive closure of R, then form the symmetric closure of the reflexive closure, and finally form the transitive closure of the symmetric closure of the reflexive closure. **69.** $p(0) = 1$, $p(1) = 1$, $p(2) = 2$, $p(3) = 5$, $p(4) = 15$, $p(5) = 52$, $p(6) = 203$, $p(7) = 877$, $p(8) = 4140$, $p(9) = 21147$, $p(10) = 115975$

Section 8.6

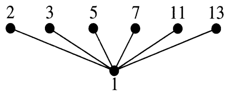
1. a) Is a partial ordering **b)** Not antisymmetric, not transitive **c)** Is a partial ordering **d)** Is a partial ordering **e)** Not antisymmetric, not transitive **3. a)** No **b)** No **c)** Yes **5. a)** Yes **b)** No **c)** Yes **d)** No **7. a)** No **b)** Yes **c)** No **9.** No **11.** Yes **13. a)** $\{(0, 0), (1, 0), (1, 1), (2, 0), (2, 1), (2, 2)\}$ **b)** $(\mathbf{Z}, \leq)$ **c)** $(P(\mathbf{Z}), \subseteq)$ **d)** $(\mathbf{Z}^+,$ "is a multiple of") **15. a)** {0} and {1}, for instance **b)** 4 and 6, for instance **17. a)** $(1, 1, 2) < (1, 2, 1)$ **b)** $(0, 1, 2, 3) < (0, 1, 3, 2)$ **c)** $(0, 1, 1, 1, 0) < (1, 0, 1, 0, 1)$ **19.** $0 < 0001 < 001 < 01 < 010 < 0101 < 011 < 11$

21.

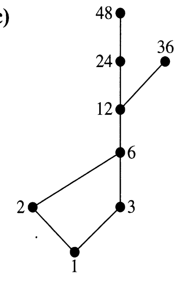
23. a)



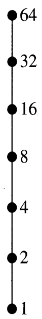
b)



c)



d)



25. (a, b), (a, c), (a, d), (b, c), (b, d), (a, a), (b, b), (c, c), (d, d) **27.** (a, a), (a, g), (a, d), (a, e), (a, f), (b, b), (b, g), (b, d), (b, e), (b, f), (c, c), (c, g), (c, d), (c, e), (c, f), (g, d), (g, e), (g, f), (g, g), (d, d), (e, e), (f, f) **29.** $(\emptyset, \{a\})$, $(\emptyset, \{b\})$, $(\emptyset, \{c\})$, $(\{a\}, \{a, b\})$, $(\{a\}, \{a, c\})$, $(\{b\}, \{a, b\})$, $(\{b\}, \{b, c\})$, $(\{c\}, \{a, c\})$, $(\{c\}, \{b, c\})$, $(\{a, b\}, \{a, b, c\})$, $(\{a, c\}, \{a, b, c\})$ $(\{b, c\}, \{a, b, c\})$ **31.** Let $(S, \preccurlyeq)$ be a finite poset. We will show that this poset is the reflexive transitive closure of its covering relation. Suppose that (a, b) is in the reflexive transitive closure of the covering relation. Then $a = b$ or $a \prec b$, so $a \preccurlyeq b$, or else there is a sequence $a_1, a_2, \ldots, a_n$ such that $a \prec a_1 \prec a_2 \prec \cdots \prec a_n \prec b$, in which case again $a \preccurlyeq b$ by the transitivity of $\preccurlyeq$. Conversely, suppose that $a \prec b$. If $a = b$ then (a, b) is in the reflexive transitive closure of the covering relation. If $a \prec b$ and there is no z such that $a \prec z \prec b$, then (a, b) is in the covering relation and therefore in its reflexive transitive closure. Otherwise, let $a \prec a_1 \prec a_2 \prec \cdots \prec a_n \prec b$ be a longest possible sequence of this form (which exists because the poset is finite). Then no intermediate elements can be inserted, so each pair $(a, a_1), (a_1, a_2), \ldots, (a_n, b)$ is in the covering relation, so again (a, b) is in its reflexive transitive closure. **33. a)** 24, 45 **b)** 3, 5 **c)** No **d)** No **e)** 15, 45 **f)** 15 **g)** 15, 5, 3 **h)** 15 **35. a)** $\{1,2\}$, $\{1,3,4\}$, $\{2,3,4\}$ **b)** $\{1\}$, $\{2\}$, $\{4\}$ **c)** No **d)** No **e)** $\{2, 4\}$, $\{2, 3, 4\}$ **f)** $\{2, 4\}$ **g)** $\{3, 4\}$, $\{4\}$ **h)** $\{3, 4\}$ **37.** Because $(a, b) \preccurlyeq (a, b)$, $\preccurlyeq$ is reflexive. If $(a_1, a_2) \preccurlyeq (b_1, b_2)$ and $(a_1, a_2) \neq (b_1, b_2)$, either $a_1 \prec b_1$, or $a_1 = b_1$ and $a_2 \prec b_2$. In either case, (b_1, b_2) is not less than or equal to (a_1, a_2). Hence, $\preccurlyeq$ is antisymmetric. Suppose that $(a_1, a_2) \prec (b_1, b_2) \prec (c_1, c_2)$. Then if $a_1 \prec b_1$ or $b_1 \prec c_1$, we have $a_1 \prec c_1$, so $(a_1, a_2) \prec (c_1, c_2)$, but if $a_1 = b_1 = c_1$, then $a_2 \prec b_2 \prec c_2$, which implies that $(a_1, a_2) \prec (c_1, c_2)$. Hence, $\preccurlyeq$ is transitive. **39.** Because $(s, t) \preceq (s, t)$, $\preccurlyeq$ is reflexive. If $(s, t) \preccurlyeq (u, v)$ and $(u, v) \preccurlyeq (s, t)$, then $s \preccurlyeq u \preccurlyeq s$ and $t \preccurlyeq v \preccurlyeq t$; hence, $s = u$ and $t = v$. Hence, $\preccurlyeq$ is antisymmetric. Suppose that $(s, t) \preccurlyeq (u, v) \preccurlyeq (w, x)$. Then $s \preccurlyeq u$, $t \preccurlyeq v$, $u \preccurlyeq w$, and $v \preccurlyeq x$. It follows that $s \preccurlyeq w$ and $t \preccurlyeq x$. Hence, $(s, t) \preccurlyeq (w, x)$. Hence, $\preccurlyeq$ is transitive. **41. a)** Suppose that x is maximal and that y is the largest element. Then $x \preccurlyeq y$. Because x is not less than y, it follows that $x = y$. By Exercise 40(a) y is unique. Hence, x is unique. **b)** Suppose that x is minimal and that y is the smallest element. Then $x \succcurlyeq y$. Because x is not greater than y, it follows that $x = y$. By Exercise 40(b) y is unique. Hence, x is unique. **43. a)** Yes **b)** No **c)** Yes **45.** Use mathematical induction. Let $P(n)$ be "Every subset with n elements from a lattice has a least upper bound and a greatest lower bound." *Basis step:* $P(1)$ is true because the least upper bound and greatest lower bound of $\{x\}$ are both x. *Inductive step:* Assume that $P(k)$ is true. Let S be a set with $k + 1$ elements. Let $x \in S$ and $S' = S - \{x\}$. Because S' has k elements, by the inductive hypothesis, it has a least upper bound y and a greatest lower bound a. Now because we are in a lattice, there are elements $z = \text{lub}(x, y)$ and $b = \text{glb}(x, a)$. We are done if we can show that z is the least upper bound of S and b is the greatest lower bound of S. To show that z is the least upper bound of S, first note that if $w \in S$, then $w = x$ or $w \in S'$. If $w = x$ then $w \preccurlyeq z$ because z is the least upper bound of x and y. If $w \in S'$, then $w \preccurlyeq z$ because $w \preccurlyeq y$, which is true because y is the least upper bound of S', and $y \preccurlyeq z$, which is true because $z = \text{lub}(x, y)$. To see that z is the least upper bound of S, suppose that u is an upper bound of S. Note that such an element u must be an upper bound of x and y, but because $z = \text{lub}(x, y)$, it follows that $z \preccurlyeq u$. We omit the similar argument that b is the greatest lower bound of S. **47. a)** No **b)** Yes **c)** (*Proprietary*, {*Cheetah*, *Puma*}), (*Restricted*, {*Cheetah*, *Puma*}), (*Registered*, {*Cheetah*, *Puma*}), (*Proprietary*, {*Cheetah*, *Puma*, *Impala*}), (*Restricted*, {*Cheetah*, *Puma*, *Impala*}), (*Registered*, {*Cheetah*, *Puma*, *Impala*}) **d)** (*Nonproprietary*, {*Impala*, *Puma*}), (*Proprietary*, {*Impala*, *Puma*}), (*Restricted*, {*Impala*, *Puma*}), (*Nonproprietary*, {*Impala*}), (*Proprietary*, {*Impala*}), (*Restricted*, {*Impala*}), (*Nonproprietary*, {*Puma*}), (*Proprietary*, {*Puma*}), (*Restricted*, {*Puma*}), (*Nonproprietary*, $\emptyset$), (*Proprietary*, $\emptyset$), (*Restricted*, $\emptyset$) **49.** Let Π be the set of all partitions of a set S with $P_1 \preccurlyeq P_2$ if P_1 is a refinement of P_2, that is, if every set in P_1 is a subset of a set in P_2. First, we show that $(\Pi, \preccurlyeq)$ is a poset. Because $P \preccurlyeq P$ for every partition P, $\preccurlyeq$ is reflexive. Now suppose that $P_1 \preccurlyeq P_2$ and $P_2 \preccurlyeq P_1$. Let $T \in P_1$. Because $P_1 \preccurlyeq P_2$, there is a set $T' \in P_2$ such that $T \subseteq T'$. Because $P_2 \preccurlyeq P_1$ there is a set $T'' \in P_1$ such that $T' \subseteq T''$. It follows that $T \subseteq T''$. But because P_1 is a partition, $T = T''$, which implies that $T = T'$ because $T \subseteq T' \subseteq T''$. Thus, $T \in P_2$. By reversing the roles of P_1 and P_2 it follows that every set in P_2 is also in P_1. Hence, $P_1 = P_2$ and $\preccurlyeq$ is antisymmetric. Next, suppose that $P_1 \preccurlyeq P_2$ and $P_2 \preccurlyeq P_3$. Let $T \in P_1$. Then there is a set $T' \in P_2$ such that $T \subseteq T'$. Because $P_2 \preccurlyeq P_3$ there is a set $T'' \in P_3$ such that $T' \subseteq T''$. This means that $T \subseteq T''$. Hence, $P_1 \preccurlyeq P_3$. It follows that $\preccurlyeq$ is transitive. The greatest lower bound of the partitions P_1 and P_2 is the partition P whose subsets are the nonempty sets of the form $T_1 \cap T_2$ where $T_1 \in P_1$ and $T_2 \in P_2$. We omit the justification of this statement here. The least upper bound of the partitions P_1 and P_2 is the partition that corresponds to the equivalence relation in which $x \in S$ is related to $y \in S$ if there is a sequence $x = x_0, x_1, x_2, \ldots, x_n = y$ for some nonnegative integer n such that for each i from 1 to n, x_{i-1} and x_i are in the same element of P_1 or of P_2. We omit the details that this is an equivalence relation and the details of the proof that this is the least upper bound of the two partitions. **51.** By Exercise 45 there is a least upper bound and a greatest lower bound for the entire finite lattice. By definition these elements are the greatest and least elements, respectively. **53.** The least element of a subset of $\mathbf{Z}^+ \times \mathbf{Z}^+$ is that pair that has the smallest possible first coordinate, and, if there is more than one such pair, that pair among those that has the smallest second coordinate. **55.** If x is an integer in a decreasing sequence of elements of this poset, then at most $|x|$ elements can follow x in the sequence, namely, integers whose absolute values are $|x| - 1, |x| - 2, \ldots, 1, 0$. Therefore there can be no infinite decreasing sequence. This is not a totally ordered set, because 5 and -5, for example, are incomparable. **57.** To find which of two rational numbers is larger, write them with a positive common denominator and compare numerators. To show that this set is dense, suppose that $x < y$ are two rational numbers. Then their average, i.e., $(x + y)/2$, is a rational number between them. **59.** Let $(S, \preccurlyeq)$ be a partially ordered set. It is enough to show that every nonempty subset of S contains a least element if and only if there is no infinite decreasing sequence of elements $a_1, a_2, a_3, \ldots$ in S (i.e., where $a_{i+1} \prec a_i$ for all i). An infinite decreasing sequence of elements clearly has no least element. Conversely, let A be any nonempty subset of S that has no least element. Because A is nonempty, choose $a_1 \in A$. Because a_1 is not the least element of A, choose $a_2 \in A$ with $a_2 \prec a_1$. Because a_2 is not the least element of A, choose $a_3 \in A$ with $a_3 \prec a_2$. Continue in this manner, producing an infinite decreasing sequence in S. **61.** $a \prec_t b \prec_t c \prec_t d \prec_t e \prec_t f \prec_t g \prec_t h \prec_t i \prec_t j \prec_t k \prec_t l \prec_t m$ **63.** $C \prec A \prec B \prec D \prec E \prec F \prec G$ **65.** Determine user needs $\prec$ Write functional requirements $\prec$ Set up test sites $\prec$ Develop system requirements $\prec$ Write documentation $\prec$ Develop module A $\prec$ Develop module B $\prec$ Develop module C $\prec$ Integrate modules $\prec$ α test $\prec$ β test $\prec$ Completion

Supplementary Exercises

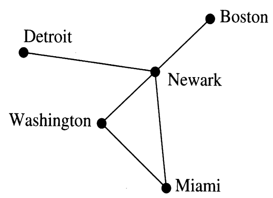
1. a) Irreflexive (we do not include the empty string), symmetric **b)** Irreflexive, symmetric **c)** Irreflexive, antisymmetric, transitive **3.** $((a, b), (a, b)) \in R$ because $a + b = a + b$. Hence, R is reflexive. If $((a, b), (c, d)) \in R$ then $a + d = b + c$, so that $c + b = d + a$. It follows that $((c, d), (a, b)) \in R$. Hence, R is symmetric. Suppose that $((a, b), (c, d))$ and $((c, d), (e, f))$ belong to R. Then $a + d = b + c$ and $c + f = d + e$. Adding these two equations and subtracting $c + d$ from both sides gives $a + f = b + e$. Hence, $((a, b), (e, f))$ belongs to R. Hence, R is transitive. **5.** Suppose that $(a, b) \in R$. Because $(b, b) \in R$ it follows that $(a, b) \in R^2$. **7.** Yes, yes **9.** Yes, yes **11.** Two records with identical keys in the projection would have identical keys in the original. **13.** $(\Delta \cup R)^{-1} = \Delta^{-1} \cup R^{-1} = \Delta \cup R^{-1}$ **15. a)** $R = \{(a, b), (a, c)\}$. The transitive closure of the symmetric closure of R is $\{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$ and is different from the symmetric closure of the transitive closure of R, which is $\{(a, b), (a, c), (b, a), (c, a)\}$. **b)** Suppose that (a, b) is in the symmetric closure of the transitive closure of R. We must show that (a, b) is in the transitive closure of the symmetric closure of R. We know that at least one of (a, b) and (b, a) is in the transitive closure of R. Hence, there is either a path from a to b in R or a path from b to a in R (or both). In the former case, there is a path from a to b in the symmetric closure of R. In the latter case, we can form a path from a to b in the symmetric closure of R by reversing the directions of all the edges in a path from b to a, going backward. Hence, (a, b) is in the transitive closure of the symmetric closure of R. **17.** The closure of S with respect to property **P** is a relation with property **P** that contains R because $R \subseteq S$. Hence, the closure of S with respect to property **P** contains the closure of R with respect to property **P**. **19.** Use the basic idea of Warshall's algorithm, except let $w_{ij}^{[k]}$ equal the length of the longest path from v_i to v_j using interior vertices with subscripts not exceeding k, and equal to -1 if there is no such path. To find $w_{ij}^{[k]}$ from the entries of $\mathbf{W}_{k-1}$, determine for each pair (i, j) whether there are paths from v_i to v_k and from v_k to v_j using no vertices labeled greater than k. If either $w_{ik}^{[k-1]}$ or $w_{kj}^{[k-1]}$ is -1, then such a pair of paths does not exist, so set $w_{ij}^{[k]} = w_{ij}^{[k-1]}$. If such a pair of paths exists, then there are two possibilities. If $w_{kk}^{[k-1]} > 0$, there are paths of arbitrary long length from v_i to v_j, so set $w_{ij}^{[k]} = \infty$. If $w_{kk}^{[k-1]} = 0$, set $w_{ij}^{[k-1]} = \max(w_{ij}^{[k-1]}, w_{ik}^{[k-1]} + w_{kj}^{[k-1]})$. (Initially take $\mathbf{W}_0 = \mathbf{M}_R$.) **21.** 25 **23.** Because $A_i \cap B_j$ is a subset of A_i and of B_j, the collection of subsets is a refinement of each of the given partitions. We must show that it is a partition. By construction, each of these sets is nonempty. To see that their union is S, suppose that $s \in S$. Because P_1 and P_2 are partitions of S, there are sets A_i and B_j such that $s \in A_i$ and $s \in B_j$. Therefore $s \in A_i \cap B_j$. Hence, the union of these sets is S. To see that they are pairwise disjoint, note that unless $i = i'$ and $j = j'$, $(A_i \cap B_j) \cap (A_{i'} \cap B_{j'}) =$

$(A_i \cap A_{i'}) \cap (B_j \cap B_{j'}) = \emptyset$. **25.** The subset relation is a partial ordering on any collection of sets, because it is reflexive, antisymmetric, and transitive. Here the collection of sets is $\mathbf{R}(S)$. **27.** Find recipe $\prec$ Buy seafood $\prec$ Buy groceries $\prec$ Wash shellfish $\prec$ Cut ginger and garlic $\prec$ Clean fish $\prec$ Steam rice $\prec$ Cut fish $\prec$ Wash vegetables $\prec$ Chop water chestnuts $\prec$ Make garnishes $\prec$ Cook in wok $\prec$ Arrange on platter $\prec$ Serve **29. a)** The only antichain with more than one element is $\{c, d\}$. **b)** The only antichains with more than one element are $\{b, c\}$, $\{c, e\}$, and $\{d, e\}$. **c)** The only antichains with more than one element are $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{a, b, c\}$, $\{d, e\}$, $\{d, f\}$, $\{e, f\}$, and $\{d, e, f\}$. **31.** Let $(S, \preccurlyeq)$ be a finite poset, and let A be a maximal chain. Because $(A, \preccurlyeq)$ is also a poset it must have a minimal element m. Suppose that m is not minimal in S. Then there would be an element a of S with $a \prec m$. However, this would make the set $A \cup \{a\}$ a larger chain than A. To show this, we must show that a is comparable with every element of A. Because m is comparable with every element of A and m is minimal, it follows that $m \prec x$ when x is in A and $x \neq m$. Because $a \prec m$ and $m \prec x$, the transitive law shows that $a \prec x$ for every element of A. **33.** Let aRb denote that a is a descendant of b. By Exercise 32, if no set of $n + 1$ people none of whom is a descendant of any other (an antichain) exists, then $k \leq n$, so the set can be partitioned into $k \leq n$ chains. By the pigeonhole principle, at least one of these chains contains at least $m + 1$ people. **35.** We prove by contradiction that if S has no infinite decreasing sequence and $\forall x\,(\{\forall y[y \prec x \rightarrow P(y)]\} \rightarrow P(x))$, then $P(x)$ is true for all $x \in S$. If it does not hold that $P(x)$ is true for all $x \in S$, let x_1 be an element of S such that $P(x_1)$ is not true. Then by the conditional statement already given, it must be the case that $\forall y[y \prec x_1 \rightarrow P(y)]$ is not true. This means that there is some x_2 with $x_2 \prec x_1$ such that $P(x_2)$ is not true. Again invoking the conditional statement, we get an $x_3 \prec x_2$ such that $P(x_3)$ is not true, and so on forever. This contradicts the well-foundedness of our poset. Therefore, $P(x)$ is true for all $x \in S$. **37.** Suppose that R is a quasi-ordering. Because R is reflexive, if $a \in A$, then $(a, a) \in R$. This implies that $(a, a) \in R^{-1}$. Hence, $a \in R \cap R^{-1}$. It follows that $R \cap R^{-1}$ is reflexive. $R \cap R^{-1}$ is symmetric for any relation R because, for any relation R, if $(a, b) \in R$ then $(b, a) \in R^{-1}$ and vice versa. To show that $R \cap R^{-1}$ is transitive, suppose that $(a, b) \in R \cap R^{-1}$ and $(b, c) \in R \cap R^{-1}$. Because $(a, b) \in R$ and $(b, c) \in R$, $(a, c) \in R$, because R is transitive. Similarly, because $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$, $(b, a) \in R$ and $(c, b) \in R$, so $(c, a) \in R$ and $(a, c) \in R^{-1}$. Hence, $(a, c) \in R \cap R^{-1}$. It follows that $R \cap R^{-1}$ is an equivalence relation. **39. a)** Because $\mathrm{glb}(x, y) = \mathrm{glb}(y, x)$ and $\mathrm{lub}(x, y) = \mathrm{lub}(y, x)$, it follows that $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$. **b)** Using the definition, $(x \wedge y) \wedge z$ is a lower bound of x, y, and z that is greater than every other lower bound. Because x, y, and z play interchangeable roles, $x \wedge (y \wedge z)$ is the same element. Similarly, $(x \vee y) \vee z$ is an upper bound of x, y, and z that is less than every other upper bound. Because x, y, and z play interchangeable roles, $x \vee (y \vee z)$ is the same element. **c)** To show that $x \wedge (x \vee y) = x$ it is sufficient to show that x is the greatest lower bound of x, and $x \vee y$. Note that x is a lower bound of x, and because $x \vee y$ is by definition greater than x, x is a lower bound for it as well. Therefore, x is a lower bound. But any lower bound of x has to be less than x, so x is the greatest lower bound. The second statement is the dual of the first; we omit its proof. **d)** x is a lower, and an upper, bound for itself and itself, and the greatest, and least, such bound. **41. a)** Because 1 is the only element greater than or equal to 1, it is the only upper bound for 1 and therefore the only possible value of the least upper bound of x and 1. **b)** Because $x \preccurlyeq 1$, x is a lower bound for both x and 1 and no other lower bound can be greater than x, so $x \wedge 1 = x$. **c)** Because $0 \preccurlyeq x$, x is an upper bound for both x and 0 and no other bound can be less than x, so $x \vee 0 = x$. **d)** Because 0 is the only element less than or equal to 0, it is the only lower bound for 0 and therefore the only possible value of the greatest lower bound of x and 0. **43.** $L = (S, \subseteq)$ where $S = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$ **45.** Yes **47.** The complement of a subset $X \subseteq S$ is its complement $S - X$. To prove this, note that $X \vee (S - X) = 1$ and $X \wedge (S - X) = 0$ because $X \cup (S - X) = S$ and $X \cap (S - X) = \emptyset$. **49.** Think of the rectangular grid as representing elements in a matrix. Thus we number from top to bottom and within that from left to right. The partial order is that $(a, b) \preceq (c, d)$ iff $a \leq c$ and $b \leq d$. Note that $(1, 1)$ is the least element under this relation. The rules for Chomp as explained in Chapter 1 coincide with the rules stated in the preamble here. But now we can identify the point (a, b) with the natural number $p^{a-1}q^{b-1}$ for all a and b with $1 \leq a \leq m$ and $1 \leq b \leq n$. This identifies the points in the rectangular grid with the set S in this exercise, and the partial order $\preceq$ just described is the same as the divides relation, because $p^{a-1}q^{b-1} \mid p^{c-1}q^{d-1}$ if and only if the exponent on p on the left does not exceed the exponent of p on the right, and similarly for q.

CHAPTER 9

Section 9.1

1. a)



b)