CHAPTER 3 Games with Sequential Moves

TEACHING SUGGESTIONS

Most students find the idea of rollback very simple and natural, even without drawing or understanding trees. Of course, they start by being able to do only one or two steps. A very simple way to get them to think about carrying the process all the way back to the beginning is to get pairs of students in class to play a game that requires rollback reasoning. One such game was described in the game list for Chapters 1 and 2 (Game 1 Claim a Pile of Dimes); two others are found in Exercise 3.7 and described in the game list for this chapter. The discussion that follows the play of the game can build up to the general notion of rollback. With this understood, the class will be much more receptive to the formalism of trees as a systematic way of practicing rollback.

Once you have succeeded in introducing the idea of rollback and in motivating the usefulness of trees, it is also important to focus on specifics. It is nice to have a sample game (similar in size and format to the Senate race game in the chapter) that you can use to illustrate a variety of concepts and can refer to continually during the first part of the term. One possibility is the two by two tennis-point game, introduced in the text in Section 12 of Chapter 4. While the game most logically fits the description of a simultaneous-move game (with no equilibrium in pure strategies), you can argue that you want to analyze the case in which the player about to hit the ball makes a particular move that indicates in which direction the ball will go; her opponent can read the body language and respond accordingly. Then the game is sequential and amenable to rollback analysis. You could also use a baseball example where the pitcher "tips" his pitch and the batter can be ready for it.

When you first illustrate the tree for a specific game like the sequential-move version of the tennis-point game, you will want to identify and label the various components of the tree: initial node, decision nodes, terminal nodes, and branches. You will also want to introduce the important components of analysis that are considered in any game: players, actions, payoffs, strategies, outcomes, and equilibria. In your simple tree, you will easily be able to identify players, actions, payoffs, and outcomes. Strategies, especially for sequential games, and equilibria take more effort.

One of the hardest ideas for students to grasp is the game-theoretic concept of a strategy as a complete plan of action. We have exhorted many classes of students to think of a strategy as something you can write down on a piece of paper and give to your mother so that she can play the game for you; then you have to write down instructions for mom in such a way that she knows what you want her to do no matter what happens in the game before it is your turn to move. And that means *no matter what happens*—that strategy has to cover every possible contingency. Most students can relate to the need for explaining everything down to the last detail to their mothers. In your sample sequential-move game, you can then show the number of strategies available to the first mover and the larger number available to the second mover; you will want to show how to construct the second mover's contingent strategies and how to describe them.

Once you have discussed the issues surrounding contingent strategies, you can derive the rollback equilibrium and show which strategies are used in equilibrium. To help students appreciate how quickly contingent strategies increase in complexity, you can go on to an example of a three-or-more player game. The text abbreviates the contingent strategies in this game; for example, Big Giant's strategy "try for Urban Mall if Frieda's does and try for Urban Mall if Frieda's tries for Rural Mall" is abbreviated (Urban, Urban) or (UU). Students who have difficulty with the idea of strategies as complete plans of action may be more comfortable with the description of this strategy (and others) in the notation (if U, then U; if R, then U). If you are looking for an example that is similar but not identical to ours, Peter Ordeshook provides a comprehensive analysis of a three-person voting example involving a roll-call (sequential) vote on a pay raise in *Game Theory and Political Theory* [London/New York: Cambridge University Press (1986)]; the tree for his game is the same size and shape as that for our three-store-mall game.

You may want to have some discussion with your students about the increase in complexity that arises when the number of moves, in addition to the number of players, is increased. Most students will have at least heard of the chess-playing computer, Deep Blue, and may be interested in what game theory has to tell them about the ongoing chess-based saga of human versus machine. Others may want to pursue the discrepancies between the predictions of rollback—like immediate pickup in the pile of dimes game—and outcomes in actual play, particularly if they had the opportunity to play a game of this type themselves. Our students have found it interesting to see the tree for the centipede game at this point, and you can use it to show how a full tree is sometimes not necessary for the complete analysis of the game.

There is additional material in Chapter 7, on dynamic games of competition, that is relevant to a discussion of sequential-move games. You may want to include some of that material with your discussion of the topics covered in Chapter 3 or wait until later (or simply skip it).

GAME PLAYING IN CLASS

GAME 1 Adding Numbers—Win at 100

This game is described in Exercise 3.7a. In this version, two players take turns choosing a number between 1 and 10 (inclusive), and a cumulative total of their choices is kept. The player to take the total exactly to 100 is the winner.

The first pair starts by choosing numbers more or less at random, until the total drifts into the 90s and the player with the next turn clinches a win. The second (or maybe third) time you play it, when the total gets somewhere in the 80s, one of that pair will realize that she wins if she takes the total to 89. When she does that, the other will (probably) realize that she has lost, and as she concedes, the rest of the class will realize it, too. The next pair will quickly settle into subgame-perfect play.

By the fifth or sixth pair, everyone will have figured out that starting at 0 (being the first mover) guarantees a win: start with 1, and then say 11 *minus* what the other says, thus taking the total successively to 12, 23, . . ., 78, 89, 100. You can hold a brief discussion and build this insight into the general idea of backward induction. You can also point out how the equilibrium strategy is a complete plan of action.

GAME 2 Adding Numbers—Lose If Go to 100 or Over (Win at 99)

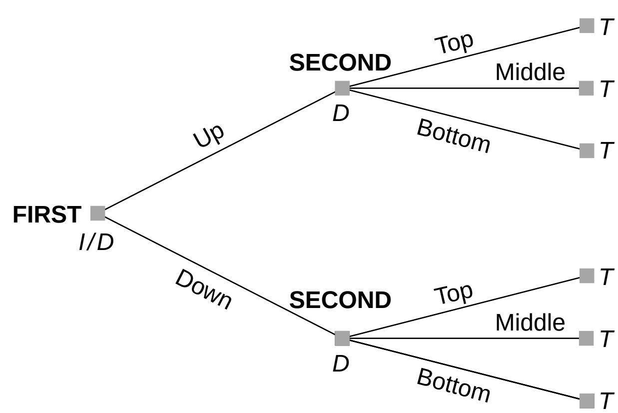
This game is described in Exercise 3.7b. In the second version, two players again take turns choosing a number between 1 and 10 (inclusive), and a cumulative total of their choices is kept. This time, the player who causes the total to equal or exceed 100 is the loser.

The first pair starts by choosing numbers more or less at random, until the total drifts into the 90s and the player with the next turn clinches a win by taking the total to 99. The second (or maybe third) time you play, when the total gets somewhere in the 80s, one of that pair will realize that she wins if she takes the total to 88. When she does that, the other will (probably) realize that she has lost, and as she concedes, the rest of the class will realize it, too. The next pair will quickly settle into subgame-perfect play as in the first version.

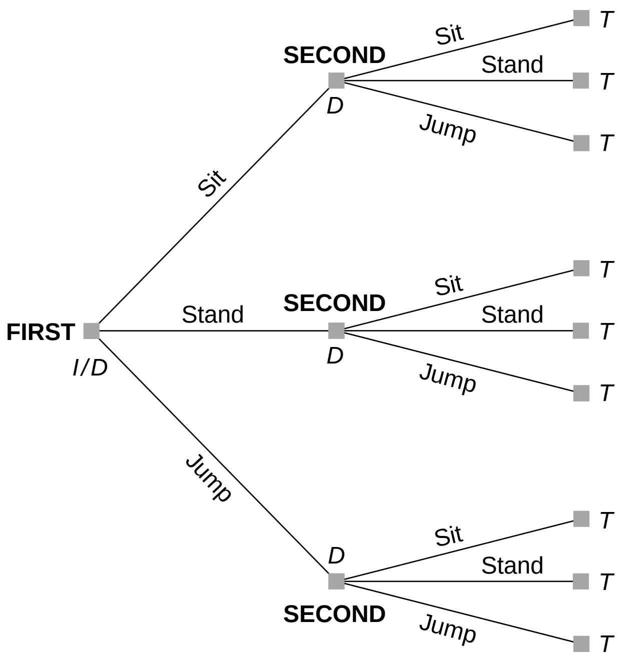
Eventually everyone will have figured out that starting at 0 (being the first mover) guarantees not a win but a loss. In this version of the game, it is better to go second: let the first player choose any number and then say 11 *minus* what the other says. Here, the second player takes the total successively to 11, 22, . . ., 77, 88, 99; the first player must then take the total to 100 (or more) and lose. You can hold a brief discussion comparing the two versions of the game; this helps make the point about order advantages in different games.

ANSWERS TO EXERCISES FOR CHAPTER 3

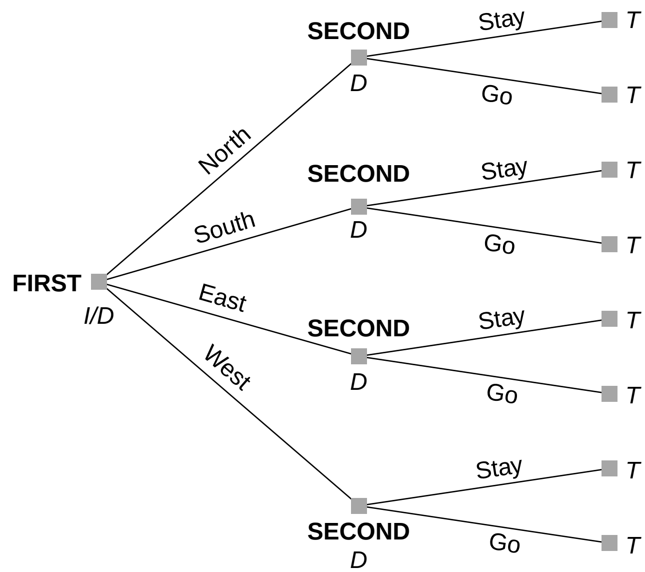
1. (a) One initial node (*I*), three decision nodes (*D*) including the initial node, and six terminal nodes.



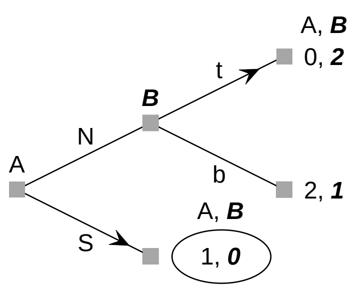
(b) One initial node, four decision nodes including the initial node, and nine terminal nodes.



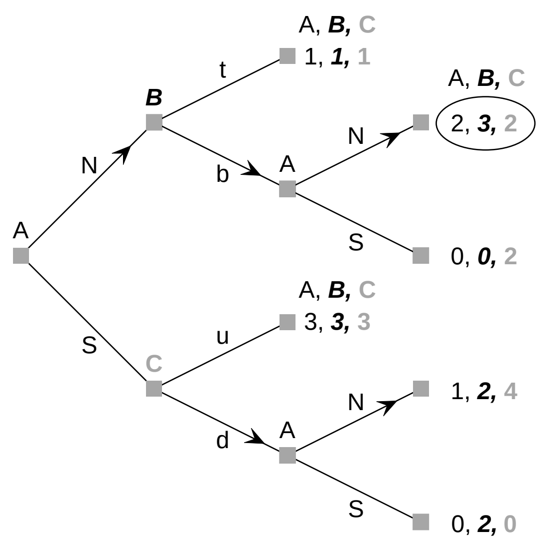
(c) One initial node, five decision nodes including the initial node, and eight terminal nodes.



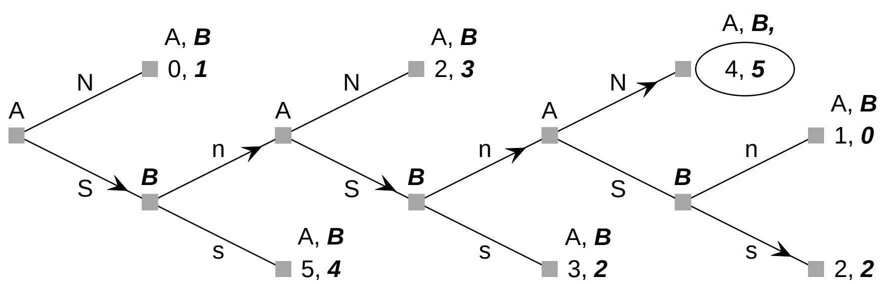
2. (a) Equilibrium payoffs are (1, 0). Player A's equilibrium strategy is S; B's equilibrium strategy is "t if N."



(b) Equilibrium payoffs are (2, 3, 2). Player A's equilibrium strategy is "N and then N if b follows N or *N* if d follows N" or "Always *N*." Player B's equilibrium strategy is "b if N" (or just b). Player C's equilibrium strategy is "d if S" (or just d).



(c) Equilibrium payoffs are (4, 5). Player A's equilibrium strategy is "S then S if n and then N if n again." Player B's equilibrium strategy is "n if S and then n if S again and then s if S again."

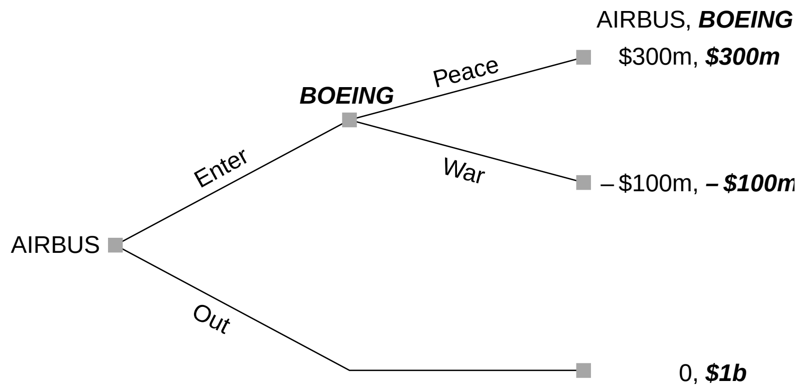


3. For 2(a): Player A has two strategies: (1) N or (2) S. Player B has two strategies: (1) "t if N" or (2) "b if N."

   For 2(b): Player A has eight strategies: (1) "N and N if b or N if d," (2) "N and N if b or S if d," (3) "N and S if b or N if d," (4) "N and S if b or S if d," (5) "S and N if b or N if d," (6) "S and N if b or S if d," (7) "S and S if b or N if d," or (8) "S and S if b or S if d." Player B has two strategies: (1) "t if N," or (2) "b if N."

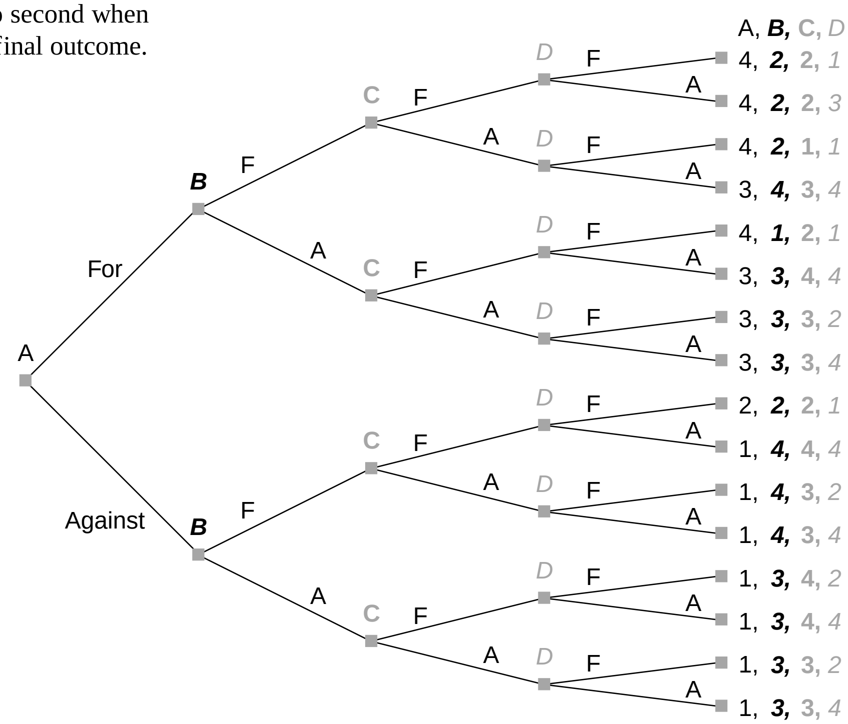
   For 2(c): Player A has four strategies: (1) N, (2) "S and N if n," (3) "S and S if n and N if n again," or (4) "S and S if n and S if n again." Player B has four strategies: (1) "s if S," (2) "n if S and s if S again," (3) "n if S and n if S again and s if S again," or (4) "n if S and n if S again and n if *S* again."

4. Tree shown below. Rollback shows that Boeing chooses peace over war if Airbus enters, so Airbus will enter. Rollback equilibrium entails Airbus playing "Enter" and Boeing playing "Peace if entry"; each firm earns $300 million profit in equilibrium.



5. False. The second mover can have the advantage of flexibility, having seen the first player's action (which reveals information), and can use this advantage to make her best response. It is better to go second when the ability to react is beneficial to one's final outcome.

6. (a) See the diagram below.



   (b) In the rollback equilibrium, A and B vote For while C and D vote Against; this leads to payoffs of (3, 4, 3, 4). The complete equilibrium strategies are A votes For, B always votes For, C votes For unless both A and B voted For in which case C votes Against, and D always votes Against.

   (c) The rollback equilibrium in any situation like this, where only one vote needs to change to affect the outcome, will entail the *last* voter's (of the group willing to change) being the one to change her vote. Observers seeing a player like B changing her vote could use this rollback equilibrium to show that if

B had not done so, C would have. This argument is similar to claiming that B did not use her equilibrium strategy, since there was a voter to follow who would have changed her vote if B had not done so.

(d) This argument builds on the analysts' views of the payoff structure for the legislators. Those who had opposed the death penalty originally but switched before the preliminary vote could not afford to switch their votes a second time; such an action presumably would give the impression of indecision or lack of conviction which the electorate might find unappealing. Legislators hoping to maintain their seats would not want to switch positions so often that their constituents could not identify their true beliefs.

7. (a) The reward system changes payoffs for Player A, but does not change the equilibrium strategies in the game. Player A still takes the money at the first opportunity and payoffs are (10, 0).

(b) Round 2 of this game leads to the familiar "take immediately" equilibrium. Given that outcome, there are no credible promises or deals that could be made in round 1, so that round stands on its own. Some of the payoffs in round 1 differ from those in the standard version of the game (e.g., once the pile has seven dimes and A takes the pile, the payoffs are (50, 0) instead of (70, 0)), but the equilibrium is the same. A takes the pile on her first move; payoffs are (10, 0).

(c) As in part b, the second round equilibrium is "take immediately" and the first round stands alone although some of the payoffs differ from the standard version of the game. None of the changes affects the equilibrium outcome. Player A takes the pile immediately and the payoffs are (10, 0).

8. (a) A player wins if she takes the total to 100 and additions of any value from 1 through 10 are allowed. Thus, if you take the sum to 89, you are guaranteed to win; your opponent must take the sum to at least 90 but can take it no higher than 99. In either case you can get to 100 on the next move. Using rollback, you can show that you can win if you can get the sum to 78 or to 67 . . . or to 12 or to 1. Thus, being the first mover and using a strategy that entails choosing 1 on the first move and then saying 11 *minus* whatever your opponent says allows you to win; you take the sum successively to 12, 23, . . ., 78, 89, and 100. Technically, the full equilibrium strategy is (i) if you are the first player, start with 1; (ii) if the current total is not $(100 - 11n)$ for some n, then choose the number that will bring the total to this form; or (iii) if the current total is of the form $(100 - 11n)$, then choose any number (all choices are equally bad).

(b) In this version, you lose if you force the total to equal or exceed 100, so you can win if you take the total to 99. Using the same type of analysis as above, you see that you can win if you can get the sum to 88, 77, . . ., 22, or 11. This time you want to be the second mover. Your strategy should be to say 11 *minus* whatever your opponent says; this strategy takes you successively to 11, 22, . . ., 77, 88, 99, and a win. The full equilibrium strategy is (i) if you are the first player, choose any number (all choices are equally bad); (ii) if the current total is a multiple of 11, choose any number (all choices are equally bad); or (iii) if the current total is not a multiple of 11, choose the number that will make the total a multiple of 11 (this is equivalent to choosing 11 minus the number just chosen by your opponent).

ADDITIONAL EXERCISES WITH ANSWERS

1. Consider a (simplified) game played between a pitcher (who chooses between throwing a fastball or a curve) and a batter (who chooses which pitch to expect). The batter has an advantage if he guesses the pitch that is actually thrown. This is a constant-sum game, where the payoff is measured by the probability that the batter will get a base hit (a high probability of a hit is good for the batter and bad for the pitcher).

   If a pitcher throws a fastball, and the batter guesses fastball, the probability of a hit is 0.300.

   If the pitcher throws a fastball, and the batter guesses curve, the probability of a hit is 0.200.

   If the pitcher throws a curve, and the batter guesses curve, the probability of a hit is 0.350.

   If the pitcher throws a curve, and the batter guesses fastball, the probability of a hit is 0.150.

   (a) Suppose that the pitcher is "tipping" his pitches (*tipping* means that the pitcher is holding the ball, positioning his body, or doing something else in a way that reveals to the opposing team which pitch he is going to throw). For our purposes, this means that the pitcher-batter game is now a sequential game in which the pitcher announces his pitch choice before the batter has to choose his strategy. Draw this situation using a game tree.

   (b) Suppose that the pitcher knows he is tipping his pitches but can't stop himself from doing so. Thus, the pitcher and batter are playing the game you just drew. Find the rollback equilibrium of this game.

   (c) Now change the timing of the game, so that the batter has to reveal his action (perhaps by altering his batting stance) before the pitcher chooses which pitch to throw. Draw the game tree for this situation, and find the rollback equilibrium.

ANSWER (b) Pitcher throws a fastball; batter guesses fastball. (c) Batter guesses curve; pitcher throws a fastball.

2. The most basic version of a LIV allows the executive office holder (Governor or President) to accept part of a bill passed by the legislature (so that part becomes law) and to veto (or reject) other parts of the bill. Without LIV power, the executive can only accept or reject the whole bill. Obtaining LIV power thus appears to give a governor more power than he or she had previously.

   Dixit and Nalebuff's *Thinking Strategically* analyzes a situation like the following. Suppose that the U.S. President gets LIV authority. Suppose also that there are two distinct proposals (A and B) being debated in Washington. Congress likes proposal A; the President likes proposal B. These proposals are not mutually exclusive; either or both (or none) may become law. There are thus four possible outcomes. The following table shows how the President and Congress rank the possible outcomes (where a larger number represents a more favorable outcome).

Becomes law	Congress	President
A only	4	1
B only	1	4
A and B	3	3
Nothing	2	2

   The timing of the game between the Congress and the President is that Congress (may) pass a bill, which is then sent on to the President for him either to accept or to veto, in full or (if possible) in part.

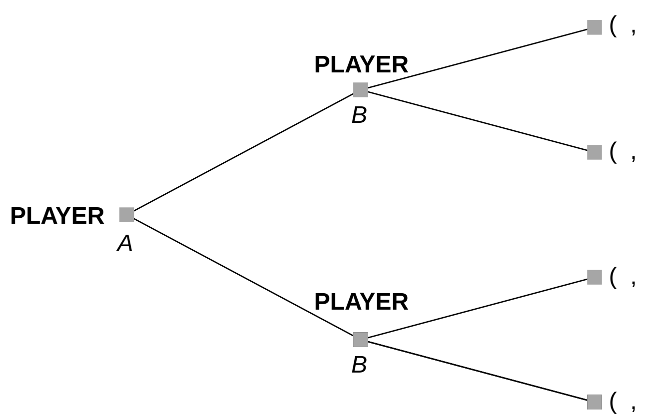
   If the President does not have LIV power, this game can be illustrated with the tree on the top of page 13; with LIV power, with the tree below it.

   (a) Enter the payoffs for Congress and for the President (in that order) for all of the possible outcomes of both versions of this game.
   (b) Using backward induction, describe a rollback equilibrium of the no-LIV game.
   (c) Using backward induction, describe a rollback equilibrium of the with-LIV game.
   (d) In this situation has the additional power given by line-item-veto authority helped or hurt the President? In your own words, explain why.
   (e) For whichever of the two outcomes is worse for the President, describe (in your own words) what he or she might do to improve that outcome.
   (f) Suppose that outcomes were ranked in the way shown below. In this situation, would the President gain or lose from having LIV authority?

Becomes law	Congress	President
A only	2	4
B only	3	1
A and B	4	3
Nothing	1	2

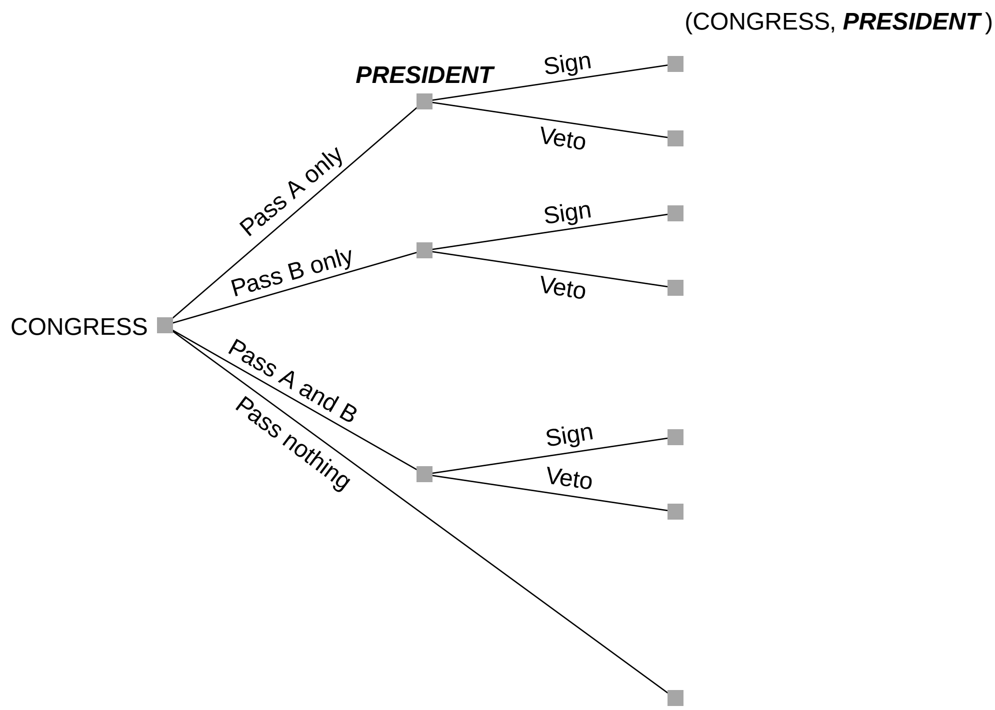
   ANSWER (b) Congress passes A and B together; the President signs the bill. Both A and B become law. (c) Either Congress passes A only and the President vetoes it, or Congress passes nothing. Neither proposal becomes law. (d) Hurt the president. Congress can't count on him to sign the whole bill. (e) The President could promise (see Chapter 9) to sign both bills. (f) LIV authority helps the President. With no LIV, A and B both become law. With LIV, only A becomes law.

3. Use the sequential game tree shown below to answer this question. Suppose we wish to list all the possible *strategies* that Player B *could* use in this game. How many strategies are there on this list?



   (a) 1
   (b) 2
   (c) 4
   (d) 8

   ANSWER (c)


(CONGRESS, PRESIDENT)
PRESIDENT
CONGRESS
Pass A only
Pass B only
Pass A and B
Pass nothing
Sign
Veto
Sign
Veto
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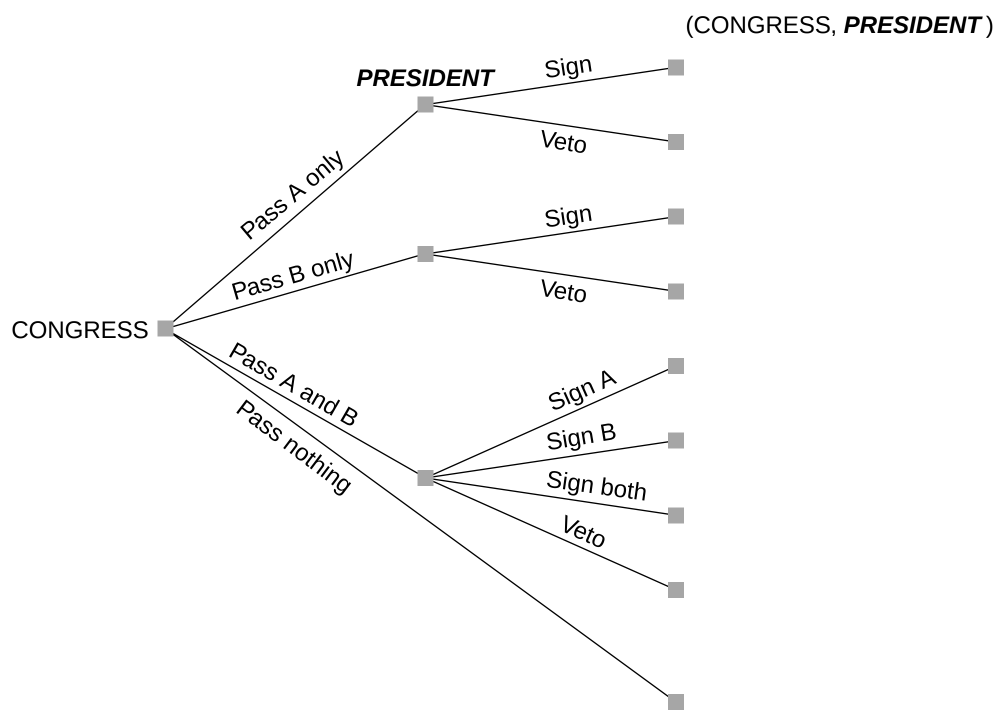
Veto


(CONGRESS, PRESIDENT)
PRESIDENT
CONGRESS
Pass A only
Pass B only
Pass A and B
Pass nothing
Sign
Veto
Sign
Veto
Sign A
Sign B
Sign both
Veto

4. Before the Allied invasion of France in 1944, the Germans had to decide where to place their defenses. They had three choices: they could concentrate their defenses at Calais (GC), concentrate at Normandy (GN), or split their defenses between both locations (GS). The Allies had two choices: they could attack at Calais (AC) or at Normandy (AN). Assume that this is a zero-sum game and that the possible outcomes are ranked as in the following matrix (where larger numbers represent outcomes more favorable for the Allies).

		GERMANS		
		GN	*GC*	*GS*
ALLIES	AN	1	4	3
	AC	6	2	5

Assume that this game is played sequentially, with the Germans' having to move first. In the rollback equilibrium outcome, the Germans _____ and the Allies _____.

(a) defend Calais; attack Calais
(b) defend Calais; attack Normandy
(c) split their defenses; attack Calais
(d) split their defenses; attack Normandy
(e) defend Normandy; attack Calais
(f) defend Normandy; attack Normandy

ANSWER (b)

5. In the previous question, suppose the payoff matrix were changed to the following.

		GERMANS		
		GN	*GC*	*GS*
ALLIES	AN	1	5	3
	AC	6	2	4

The rest of the game is unchanged. Use the possible answers (a) to (f) in Exercise 4 and the new payoff matrix. In the rollback equilibrium outcome, the Germans _____ and the Allies _____.

ANSWER (c)