

Basis and change of basis matrices

Definitions

The *basis* is a coordinate system used to describe vector spaces (sets of vectors). It is a reference that you use to associate numbers with geometric vectors.

To be considered as a basis, a set of vectors must:

- Be linearly independent.
- Span the space.

Every vector in the space is a unique combination of the basis vectors. The dimension of a space is defined to be the size of a basis set.

For instance, there are two basis vectors in \mathbb{R}^2 (corresponding to the x and y -axis in the Cartesian plane), or three in \mathbb{R}^3 .

Vectors can be represented as arrows going from the origin to a point in space. The coordinates of this point can be stored in a list. The geometric representation of a vector in the Cartesian plane implies that we take a reference: the directions given by the two axes x and y .

Basis vectors are the vectors corresponding to this reference. In the Cartesian plane, the basis vectors are orthogonal unit vectors (length of one), generally denoted as \mathbf{i} and \mathbf{j} .

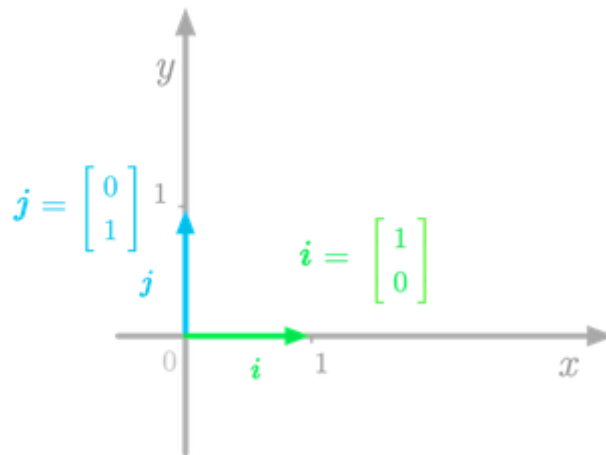


Figure 1: The basis vectors in the Cartesian plane.

For instance, in Figure 1, the basis vectors \mathbf{i} and \mathbf{j} point in the direction of the axis x and y respectively. These vectors give the standard basis. If you put these basis vectors into a matrix, you have the following identity matrix.

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, the columns of \mathbf{I}_2 span \mathbb{R}^2 . In the same way, the columns of \mathbf{I}_3 span \mathbb{R}^3 and so on.

Orthogonal basis

Basis vectors can be orthogonal because orthogonal vectors are independent.

However, the converse is not necessarily true: non-orthogonal vectors can be linearly independent and thus form a basis (but not a standard basis).

The basis of your vector space is very important because the values of the coordinates corresponding to the vectors depend on this basis. By the way, you can choose different basis vectors, like in the ones in Figure 2 for instance.

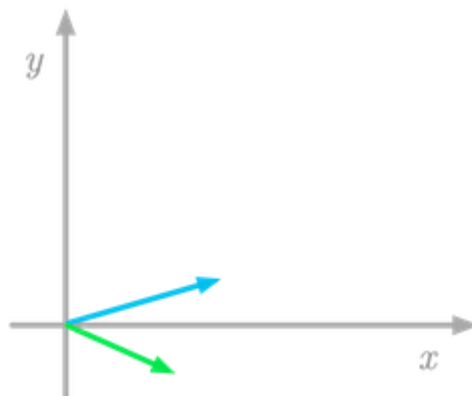


Figure 2. *Another basis in a two-dimensional space.*

Linear Combination of Basis Vectors

You can consider any vector in a vector space as a linear combination of the basis vectors.

For instance, take the following two-dimensional vector \mathbf{v} :

$$\mathbf{v} = \begin{bmatrix} 2 \\ -0.5 \end{bmatrix}$$

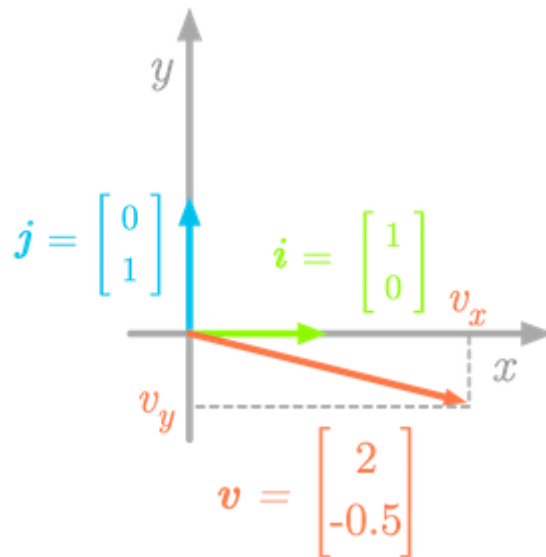


Figure 3. Components of the vector \mathbf{v} .

The components of the vector \mathbf{v} are the projections on the x -axis and on the y -axis (v_x and v_y , as illustrated in Figure 3). The vector \mathbf{v} corresponds to the sum of its components: $\mathbf{v} = v_x + v_y$, and you can obtain these components by scaling the basis vectors: $v_x = 2\mathbf{i}$ and $v_y = -0.5\mathbf{j}$. Thus, the vector \mathbf{v} shown in Figure 3 can be considered as a linear combination of the two basis vectors \mathbf{i} and \mathbf{j} :

$$\mathbf{v} = 2\mathbf{i} - 0.5\mathbf{j}$$

$$= 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 0.5 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 1 \\ 2 \cdot 0 \end{bmatrix} - \begin{bmatrix} 0.5 \cdot 0 \\ 0.5 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -0.5 \end{bmatrix}$$

Other Bases

The columns of identity matrices are not the only case of linearly independent column vectors. It is possible to find other sets of n vectors linearly independent in \mathbb{R}^n .

For instance, let's consider the following vectors in \mathbb{R}^2 :

$$\mathbf{v} = \begin{bmatrix} 2 \\ -0.5 \end{bmatrix}$$

And

$$\mathbf{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

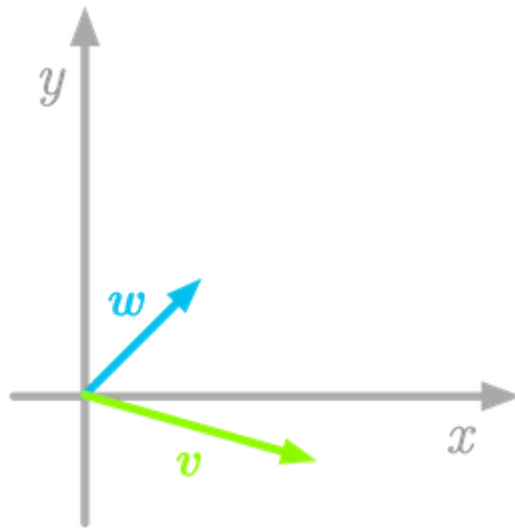


Figure 4: Another basis in a two-dimensional space.

From the definition above, the vectors \mathbf{v} and \mathbf{w} are a basis because they are linearly independent (you can't obtain one of them from combinations of the other) and they span the space (all the space can be reached from the linear combinations of these vectors).

It is critical to keep in mind that, when you use the components of vectors (for instance v_x and v_y , the x and y components of the vector \mathbf{v}), the values are relative to the basis you chose. If you use another basis, these values will be different.

Vectors Are Defined With Respect to a Basis

You saw that to associate geometric vectors (arrows in the space) with coordinate vectors (arrays of numbers), you need a reference. This reference is the basis of your vector space. For this reason, a vector should always be defined with respect to a basis.

Let's take the following vector:

$$\mathbf{v} = \begin{bmatrix} 2 \\ -0.5 \end{bmatrix}$$

The values of the x and y components are respectively 2 and -0.5. The standard basis is used when not specified.

You could write \mathbf{Iv} to specify that these numbers correspond to coordinates with respect to the standard basis. In this case, \mathbf{I} is called the *change of basis matrix*.

$$\mathbf{v} = \mathbf{Iv} = \begin{bmatrix} 2 \\ -0.5 \end{bmatrix}$$

You can define vectors with respect to another basis by using another matrix than \mathbf{I} .

Linear Combinations of the Basis Vectors

Vector spaces (the set of possible vectors) are characterized in reference to a basis. The expression of a geometrical vector as an array of numbers implies that you choose a basis. With a different basis, the same vector \mathbf{v} is associated with different numbers.

Basis is a set of linearly independent vectors that span the space. More precisely, a set of vectors is a basis if every vector from the space can be described as a finite linear combination of the components of the basis and if the set is linearly independent.

Consider the following two-dimensional vector:

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

In the \mathbb{R}^2 Cartesian plane, you can consider \mathbf{v} as a linear combination of the standard basis vectors \mathbf{i} and \mathbf{j} , as shown in Figure 5.

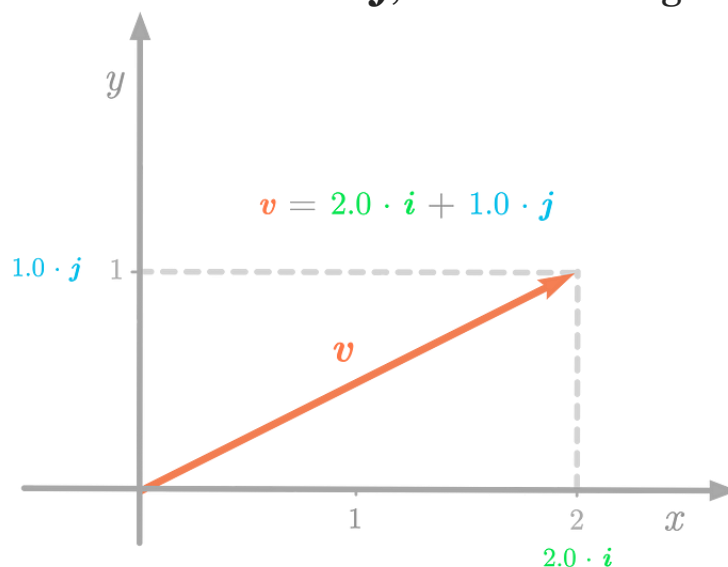


Figure 5: The vector \mathbf{v} can be described as a linear combination of the basis vectors \mathbf{i} and \mathbf{j} .

But if you use another coordinate system, \mathbf{v} is associated with new numbers. Figure 6 shows a representation of the vector \mathbf{v} with a new coordinate system (\mathbf{i}' and \mathbf{j}').

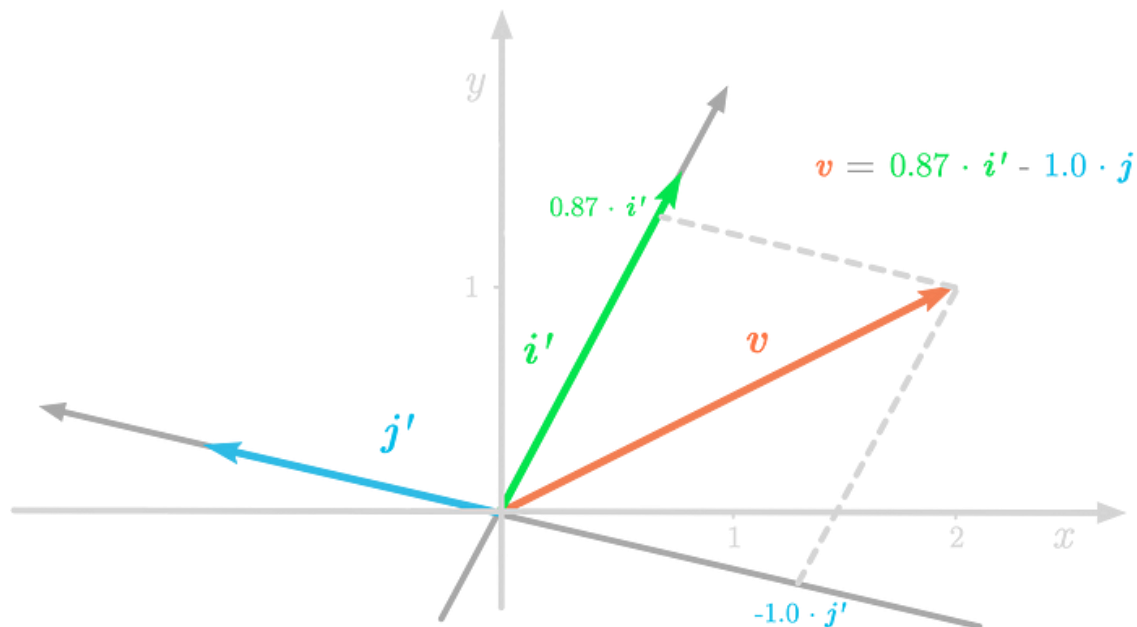


Figure 6: The vector \mathbf{v} with respect to the coordinates of the new basis.

In the new basis, \mathbf{v} is a new set of numbers:

$$\begin{bmatrix} 0.86757991 \\ -1.00456621 \end{bmatrix}$$

The Change of Basis Matrix

You can use a *change of basis matrix* to go from a basis to another. To find the matrix corresponding to new basis vectors, you can express these new basis vectors (\mathbf{i}' and \mathbf{j}') as coordinates in the old basis (\mathbf{i} and \mathbf{j}).

Let's take again the preceding example. You have:

$$\mathbf{i}' = \begin{bmatrix} 0.8 \\ 1.5 \end{bmatrix}$$

And

$$\mathbf{j}' = \begin{bmatrix} -1.3 \\ 0.3 \end{bmatrix}$$

This is illustrated in Figure 7.

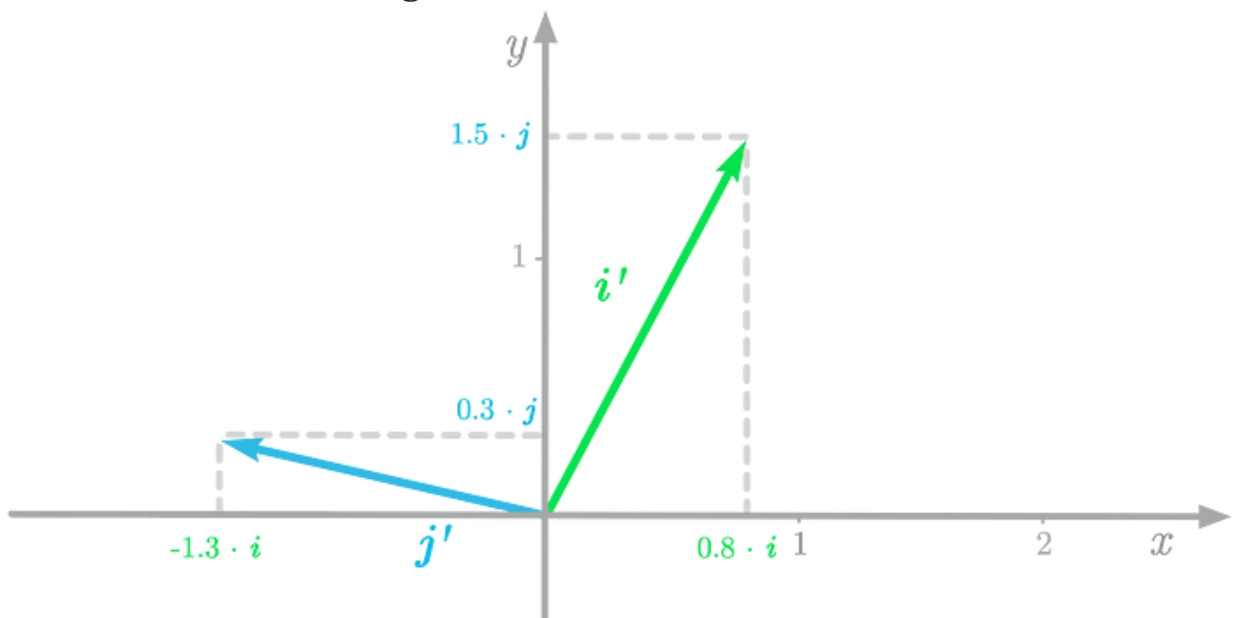


Figure 7: The coordinates of the new basis vectors with respect to the old basis.

Since they are basis vectors, \mathbf{i}' and \mathbf{j}' can be expressed as linear combinations of \mathbf{i} and \mathbf{j} :

$$\mathbf{i}' = 0.8 \cdot \mathbf{i} + 1.5\mathbf{j}$$

$$\mathbf{j}' = -1.3 \cdot \mathbf{i} + 0.3\mathbf{j}$$

Let's write these equations under the matrix form:

$$\begin{bmatrix} \mathbf{i}' \\ \mathbf{j}' \end{bmatrix} = \mathbf{i} \begin{bmatrix} 0.8 \\ -1.3 \end{bmatrix} + \mathbf{j} \begin{bmatrix} 1.5 \\ 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 1.5 \\ -1.3 & 0.3 \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \end{bmatrix}$$

To have the basis vectors as columns, you need to transpose the matrices. You get:

$$\begin{bmatrix} i' \\ j' \end{bmatrix}^T = \left(\begin{bmatrix} 0.8 & 1.5 \\ -1.3 & 0.3 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} \right)^T$$

$$\begin{bmatrix} i' & j' \end{bmatrix} = \begin{bmatrix} i \\ j \end{bmatrix}^T \begin{bmatrix} 0.8 & 1.5 \\ -1.3 & 0.3 \end{bmatrix}^T$$

$$\begin{bmatrix} i' & j' \end{bmatrix} = \begin{bmatrix} i & j \end{bmatrix} \begin{bmatrix} 0.8 & -1.3 \\ 1.5 & 0.3 \end{bmatrix}$$

This matrix is called the change of basis matrix. Let's call it **C**:

$$\mathbf{C} = \begin{bmatrix} 0.8 & -1.3 \\ 1.5 & 0.3 \end{bmatrix}$$

As you can notice, **each column of the change of basis matrix is a basis vector of the new basis.**

Now we can use the change of basis matrix \mathbf{C} to convert vectors from the output basis to the input basis.

Change of basis vs linear transformation

The main technical difference between the two is that change of basis must be invertible, which is not required for linear transformations.

Finding the Change of Basis Matrix

A change of basis matrix maps an input basis to an output basis. Let's call the input basis \mathbf{B}_1 with the basis vectors \mathbf{i} and \mathbf{j} , and the output basis \mathbf{B}_2 with the basis vectors \mathbf{i}' and \mathbf{j}' . You have:

$$\mathbf{B}_1 = [\mathbf{i} \quad \mathbf{j}]$$

And

$$\mathbf{B}_2 = [\mathbf{i}' \quad \mathbf{j}']$$

From the equation of the change of basis, you have:

$$[\mathbf{i}' \quad \mathbf{j}'] = [\mathbf{i} \quad \mathbf{j}] \mathbf{C}$$

$$\mathbf{B}_2 = \mathbf{B}_1 \mathbf{C}$$

If you want to find the change of basis matrix given \mathbf{B}_1 and \mathbf{B}_2 , you need to calculate the inverse of \mathbf{B}_1 to isolate \mathbf{C} :

$$\mathbf{B}_2 = \mathbf{B}_1 \mathbf{C}$$

$$\mathbf{B}_1^{-1} \mathbf{B}_2 = \mathbf{B}_1^{-1} \mathbf{B}_1 \mathbf{C}$$

$$\mathbf{B}_1^{-1} \mathbf{B}_2 = \mathbf{C}$$

$$\mathbf{C} = \mathbf{B}_1^{-1} \mathbf{B}_2$$

In words, you can calculate the change of basis matrix by multiplying the inverse of the input basis matrix (\mathbf{B}_1^{-1} , which contains the input basis vectors as columns) by the output basis matrix (\mathbf{B}_2 , which contains the output basis vectors as columns).

Converting vectors from the output to the input basis

Note that this change of basis matrix allows you to convert vectors from \mathbf{B}_2 to \mathbf{B}_1 and not the opposite. Intuitively, this is because moving an object is the opposite of moving the reference. Thus, to go from \mathbf{B}_1 to \mathbf{B}_2 , you must use the inverse of the change of basis matrix \mathbf{C}^{-1} .

Note that if the input basis is the standard basis ($\mathbf{B}_1 = \mathbf{I}$), then the change of basis matrix is simply the output basis matrix:

$$\mathbf{C} = \mathbf{B}_1^{-1} \mathbf{B}_2 = \mathbf{I}^{-1} \mathbf{B}_2 = \mathbf{I} \mathbf{B}_2 = \mathbf{B}_2$$

Invertible Change of Basis Matrix

Since the basis vectors are linearly independent, the columns of \mathbf{C} are linearly independent, and thus, \mathbf{C} is invertible.

Example: Changing the Basis of a Vector

Let's change the basis of a vector \mathbf{v} , using again the geometric vectors represented in Figure 6.

Notation

You'll change the basis of \mathbf{v} from the standard basis to a new basis. Let's denote the standard basis as \mathbf{B}_1 and the new basis as \mathbf{B}_2 . Remember that the basis is a matrix containing the basis vectors as columns. You have:

$$\mathbf{B}_1 = [\mathbf{i} \quad \mathbf{j}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

And

$$\mathbf{B}_2 = [\mathbf{i}' \quad \mathbf{j}'] = \begin{bmatrix} 0.8 & -1.3 \\ 1.5 & 0.3 \end{bmatrix}$$

Let's denote the vector \mathbf{v} relative to the basis \mathbf{B}_1 as $[\mathbf{v}]_{\mathbf{B}_1}$:

$$[\mathbf{v}]_{\mathbf{B}_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The goal is to find the coordinates of \mathbf{v} relative to the basis \mathbf{B}_2 , denoted as $[\mathbf{v}]_{\mathbf{B}_2}$.

Using Linear Combinations

Let's express the vector \mathbf{v} as a linear combination of the input and output basis vectors:

$$\begin{cases} \mathbf{v} = c_1 \mathbf{i} + c_2 \mathbf{j} \\ \mathbf{v} = d_1 \mathbf{i}' + d_2 \mathbf{j}' \end{cases}$$

The scalars c_1 and c_2 are weighting the linear combination of the input basis vectors, and the scalars d_1 and d_2 are weighting the linear combination of the output basis vectors. You can merge the two equations:

$$c_1 \mathbf{i} + c_2 \mathbf{j} = d_1 \mathbf{i}' + d_2 \mathbf{j}'$$

Now, let's write this equation in matrix form:

$$\begin{bmatrix} i & j \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} i' & j' \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$\mathbf{B}_1 \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{B}_2 \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

The vector containing the scalars c_1 and c_2 corresponds to $[\mathbf{v}]_{\mathbf{B}_1}$ and the vector containing the scalars d_1 and d_2 corresponds to $[\mathbf{v}]_{\mathbf{B}_2}$. You have:

$$\mathbf{B}_1 [\mathbf{v}]_{\mathbf{B}_1} = \mathbf{B}_2 [\mathbf{v}]_{\mathbf{B}_2}$$

This is an equation with the term you want to find: $[\mathbf{v}]_{\mathbf{B}_2}$. You can isolate it by multiplying each side by \mathbf{B}_2^{-1} :

$$\mathbf{B}_1 [\mathbf{v}]_{\mathbf{B}_1} = \mathbf{B}_2 [\mathbf{v}]_{\mathbf{B}_2}$$

$$\mathbf{B}_2^{-1} \mathbf{B}_1 [\mathbf{v}]_{\mathbf{B}_1} = \mathbf{B}_2^{-1} \mathbf{B}_2 [\mathbf{v}]_{\mathbf{B}_2}$$

$$[\mathbf{v}]_{\mathbf{B}_2} = \mathbf{B}_2^{-1} \mathbf{B}_1 [\mathbf{v}]_{\mathbf{B}_1}$$

You have also:

$$\mathbf{B}_1[\mathbf{v}]_{B_1} = \mathbf{B}_2[\mathbf{v}]_{B_2}$$

$$\mathbf{B}_1^{-1}\mathbf{B}_1[\mathbf{v}]_{B_1} = \mathbf{B}_1^{-1}\mathbf{B}_2[\mathbf{v}]_{B_2}$$

$$[\mathbf{v}]_{B_1} = \mathbf{B}_1^{-1}\mathbf{B}_2[\mathbf{v}]_{B_2}$$

The term $\mathbf{B}_2^{-1}\mathbf{B}_1$ is the inverse of $\mathbf{B}_1^{-1}\mathbf{B}_2$, which is the change of basis matrix \mathbf{C} described before. This shows that \mathbf{C}^{-1} allows you to convert vectors from an input basis \mathbf{B}_1 to an output basis \mathbf{B}_2 and \mathbf{C} from \mathbf{B}_2 to \mathbf{B}_1 .

In the context of this example, since \mathbf{B}_1 is the standard basis, it simplifies to:

$$[\mathbf{v}]_{B_2} = \mathbf{B}_2^{-1}\mathbf{I}[\mathbf{v}]_{B_1}$$

$$[\mathbf{v}]_{B_2} = \mathbf{B}_2^{-1}[\mathbf{v}]_{B_1}$$

This means that, applying the matrix \mathbf{B}_2^{-1} to $[\mathbf{v}]\mathbf{B}_1$ allows you to change its basis to \mathbf{B}_2 .

```
v_B1 = np.array([2, 1])
B_2 = np.array([
    [0.8, -1.3],
    [1.5, 0.3]
])

v_B2 = np.linalg.inv(B_2) @ v_B1
v_B2
```

These values are the coordinates of the vector \mathbf{v} relative to the basis \mathbf{B}_2 . This means that if you go to $0.86757991\mathbf{i}' - 1.00456621\mathbf{j}'$ you arrive to the position (2, 1) in the standard basis, as illustrated in Figure 6.