What is an Orthogonal Matrix?

Orthogonal Matrix is a square matrix in which all its rows and columns are mutually orthogonal unit vectors, meaning that each row and column of the matrix is perpendicular to every other row and column, and each row or column has a magnitude of 1.

Orthogonal Matrix Definition

Mathematically, an n x n matrix A is considered orthogonal if

$$A^T = A^{-1}$$

OR

$$AA^T = A^TA = I$$

Where,

- A^T *is the transpose of the square matrix,*
- A^{-1} *is the inverse of the square matrix, and*
- ***I*** *is the identity matrix of same order as A.*

$A^T = A^{-1}$ (Condition for an Orthogonal matrix)…(i)

Pre-multiply by A on both sides,

We get, $AA^T = AA^{-1}$,

We know this relation of the identity matrix, $AA^{-1} = I$, (of the same order as A).

So we can also write it as $AA^T = I$. (From (i))

Similarly, we can derive the relation $A^TA = I$.

So, from the above two equations, we get $AA^T = A^TA = I$.

Condition for Orthogonal Matrix

For any matrix to be an orthogonal Matrix, it needs to fulfil the following conditions:

- Every two rows and two columns have a dot product of zero, and
- Every row and every column has a magnitude of one.

Orthogonal Matrix in Linear Algebra

The condition of any two vectors to be orthogonal is when their dot product is zero.

Similarly, in the case of an orthogonal matrix, every two rows and every two columns are orthogonal. Also, one more condition is that the length of every row (vector) or column (vector) is 1.

$$A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

For, let's consider a 3×3 matrix, i.e.,

Here, the dot product between vector 1 and vector 2 i.e. between row 1 and row 2

Row 1 · Row 2 = (1/3)(-2/3)+(2/3)(2/3)+(-2/3)(1/3) =0

So, **Row 1** and **Row 2** are Orthogonal.

Also, the **Magnitude of Row 1** = $((1/3)^2+(2/3)^2+(-2/3)^2)^{0.5}$ = 1

Similarly, we can check for all other rows.

Thus, this matrix A is an example of Orthogonal Matrix.

How to Identify Orthogonal Matrices?

If the transpose of a square matrix with real numbers or elements equals the inverse matrix, the matrix is said to be orthogonal. Or, we may argue that a square matrix is an orthogonal matrix if the product of the square matrix and its transpose results in an identity matrix.

Suppose A is a square matrix with real elements and of n x n order and A^T is the transpose of A. Then according to the definition, if, $A^T = A^{-1}$ is satisfied, then,

$$A \cdot A^T = I$$

Properties of an Orthogonal Matrix

Some of the properties of Orthogonal Matrix are:
- Inverse and Transpose are equivalent. i.e., $A^{-1} = A^T$.
- An identity matrix is the outcome of A and its transpose. That is, $AA^T = A^TA = I$.
- In light of the fact that its determinant is never 0, an orthogonal matrix is always non-singular.
- An orthogonal diagonal matrix is one whose members are either 1 or -1.
- A^T is orthogonal as well. A^{-1} is also orthogonal because $A^{-1} = A^T$.
- The eigenvalues of A are ±1 and the eigenvectors are orthogonal.
- As $I \times I = I \times I = I$, and $I^T = I$. Thus, I an identity matrix (I) is orthogonal.

Orthogonal Matrix Applications

Some of the most common applications of Orthogonal Matrix are:
- Used in multivariate time series analysis.
- Used in multi-channel signal processing.
- Used in QR decomposition.