Uncertainty Analysis

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Reference:

Textbook: *An introduction to error analysis: the study of uncertainties in physical measurements* by J. R. Taylor (QA275.T38 1982)

Presentation Outline

- Objective
  - Why Worry About Uncertainty?

- How to Calculate and Express Uncertainty
  - Error Presentation
    - Significant Figure, Rounding
  - Error Origination
    - Human, Fixed, Random
    - Fixed Error: Propagation
    - Random Error: Calibration
    - Experimental Procedure

Why Worry About Uncertainty?

- No measurement is free from error
- Engineers are responsible for reporting reliability of data
- Ethical consequences

How?

Answer Three Questions

1. What is a rational way of estimating uncertainty in measurements?
2. How do you calculate propagation of this uncertainty into the measurand?
3. How do you present the results?

Expressing Uncertainty

- Significant Figures
- Rounding

Significant Figures

- Significant Figures
  - 6.02, 0.596, 0.000610
- Multiplication & Division
  - Retain the significant figures of the lesser value
- Addition + Subtraction
  - Retain the precision of the lesser value

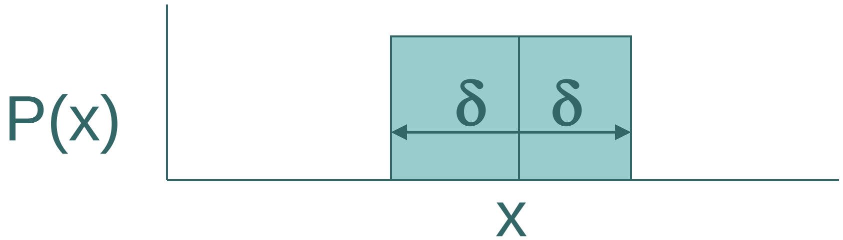
Rounding

- Significant figures limited by the resolution or precision of the measurement
- When the figure ends in ‘5’ choose the even round value (up or down)
- Fixed uncertainty estimated as $\pm\frac{1}{2}$ the precision

Least Count vs Readability

- Least Count?
  - 10 psi
- Readability?
  - 1 to 2 psi

Systematic (Fixed) Uncertainty
Input Precision



For an analog readout,
unless other data is available,
uncertainty is the readability
divided by square root of 12

readability

$$u_z = \frac{\delta_x}{\sqrt{12}}$$

Analog vs Digital Meters

Readability does not apply to digital meters.
Unless other data is available, precision is the least count.
Uncertainty is half of precision.
± 0.05 psi

Estimating Uncertainty

Origin of Uncertainty

1. Human Error
2. Systematic or Fixed Error
3. Random Error

Output Precision

Output = Y, Inputs are: A, B, C

e.g.: Output, Re $= D u \rho / \mu$

$= (Q/t) D \rho / \mu$

3 Cases of Output Precision

Propagation of error

$$Y = A \pm B \pm C \qquad \Delta Y = \sqrt{(\Delta A)^2 + (\Delta B)^2 + (\Delta C)^2}$$

$$Y = ABC \qquad \frac{\Delta Y}{Y} = \sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2 + \left(\frac{\Delta C}{C}\right)^2}$$

$$Y = A^{\alpha} \qquad \frac{\Delta Y}{Y} = \left|\alpha\left(\frac{\Delta A}{A}\right)\right|$$

Combination Cases
of Output Precision

$$Y = AB^2 = AC; \quad C = B^2$$

$$\frac{\Delta Y}{Y} = \sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta C}{C}\right)^2} \qquad \frac{\Delta C}{C} = 2\left(\frac{\Delta B}{B}\right)$$

$$\frac{\Delta Y}{Y} = \sqrt{\left(\frac{\Delta A}{A}\right)^2 + 4\left(\frac{\Delta B}{B}\right)^2}$$

Uncertainty for Complex Equations

$$Y = f\left(A, B\right)$$

$$\Delta Y = \sqrt{\left(\frac{\partial f}{\partial A}\Delta A\right)^2 + \left(\frac{\partial f}{\partial B}\Delta B\right)^2}$$

- Previous examples are derived from this
- Work by hand or
- MathCAD can help solve

Combined uncertainties

- Random uncertainty with systematic uncertainty

$$u_i = \sqrt{u_{ir}^2 + u_{is}^2}$$

Calibration and Experimental Uncertainty

- Single Point Calibration

- Multi-point Calibration

  Linear Regression

Standard Error

Single-point Calibration

i.e., measuring a single point multiple times

- Sample mean $$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$
- Standard Deviation $$\sigma = \left[\frac{1}{n-1}\cdot\sum_i \left(x_i - \bar{x}\right)^2\right]^{1/2}$$
- Standard error of mean $$\sigma_x = \frac{\sigma}{n^{1/2}}$$
- Estimate of x = $$y = \bar{x} \pm \sigma_x$$

Multi-point Line Fit

Linear Regression

$$\hat{y}_i = m \cdot x_i + c$$

$$m = \frac{\sum_i \left[(x_i - \bar{x}) \cdot (y_i - \bar{y}) \right]}{\sum_i (x_i - \bar{x})^2}$$

$$c = \bar{y} - (m \cdot \bar{x})$$

$$R = \frac{\sum_i \left[(x_i - \bar{x}) \cdot (y_i - \bar{y}) \right]}{\left\{ \left[\sum_i (x_i - \bar{x})^2 \right] \cdot \left[\sum_i (y_i - \bar{y})^2 \right] \right\}^{1/2}}$$

Multi-point Line Fit, cont.

Linear Regression

- Estimates for uncertainty (Standard Error of regression) for the fit of y upon x:

$$S_{yx} = \left[\left(\frac{1}{n-2} \right) \cdot \sum_i (y_i - \hat{y}_i)^2 \right]^{1/2}$$

the slope:

$$S_m = \frac{S_{yx}}{\left[\sum_i (x_i - \bar{x})^2 \right]^{1/2}}$$

and the intercept:

$$S_c = S_{yx} \cdot \left[\frac{\sum_i x_i^2}{n \cdot \sum_i (x_i - \bar{x})^2} \right]^{1/2}$$

Multi-point Calibration

- Uncertainty in average measurement determined from linear regression calibration

$$U_y = \frac{S_{yx}}{m}\sqrt{\frac{1}{k} + \frac{1}{n} + \frac{(\bar{y}_0 - \bar{y})^2}{m^2 s^2 (n-1)}}$$

- Where n is the number of calibration readings, k is the number of measurements, m is the slope, $\bar{y}_0$ is the average of the measurements, $\bar{y}$ is the average of the calibration readings, and s^2 is the sample variance of the x-variable.

Weighted Multi-point Line Fit with Uncertainty

Linear Regression when the error distribution changes with *y*
-error distribution is weighted

$$m = \frac{\left(\sum w_i\right)\left(\sum w_i x_i y_i\right) - \left(\sum w_i x_i\right)\left(\sum w_i y_i\right)}{\left(\sum w_i\right)\left(\sum w_i x_i^2\right) - \left(\sum w_i x_i\right)^2}$$

$$c = \frac{\left(\sum w_i x_i^2\right)\left(\sum w_i y_i\right) - \left(\sum w_i x_i\right)\left(\sum w_i x_i y_i\right)}{\left(\sum w_i\right)\left(\sum w_i x_i^2\right) - \left(\sum w_i x_i\right)^2}$$

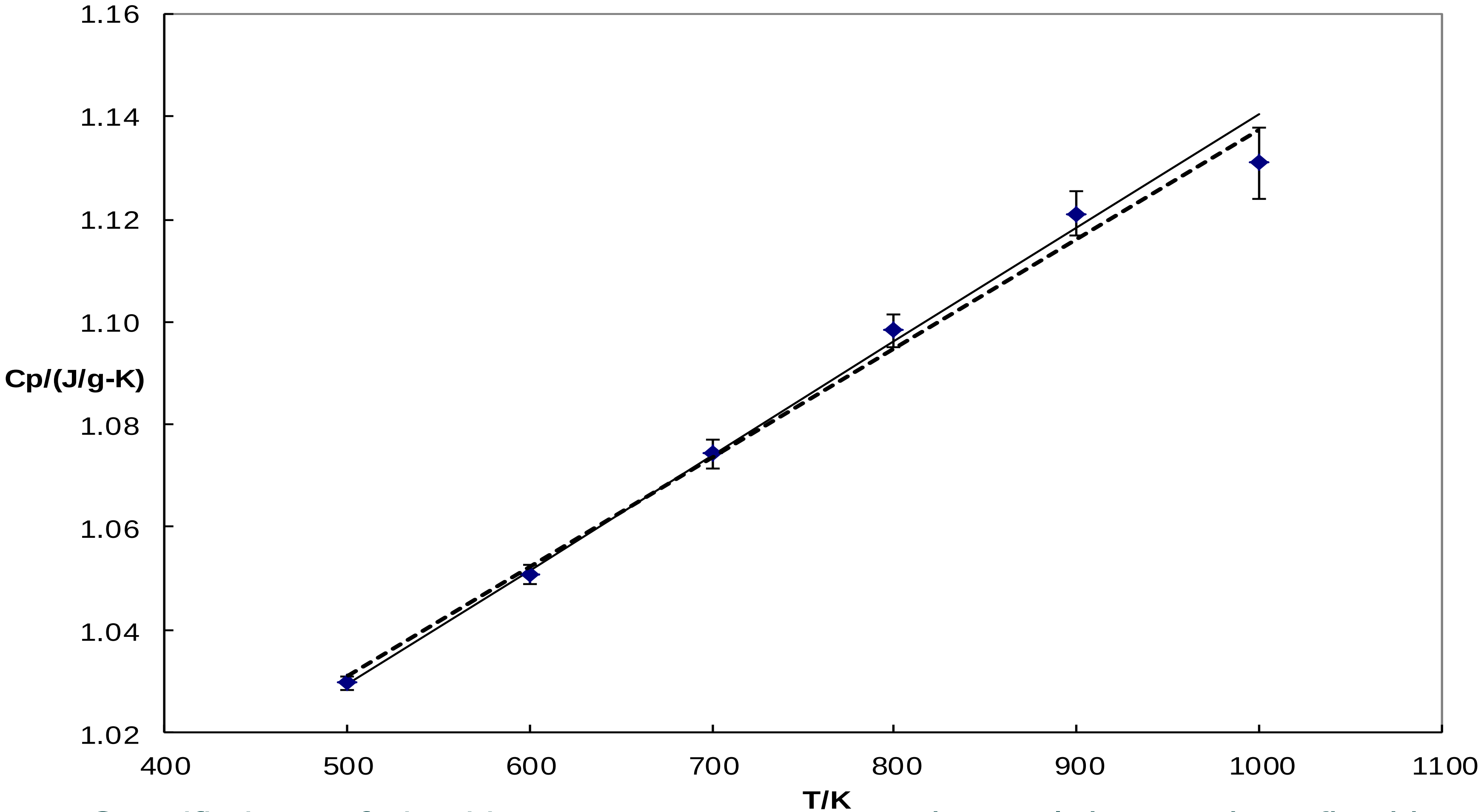
$$w_i = 1/u^2 = 1/\left(u_r^2 + u_f^2\right)$$

Example

$$\sigma_x = \frac{\sigma}{n^{1/2}} = s$$

Experimental results for the specific heat of air.

T/K	500	600	700
c_p / (J g^{-1} K^{-1})	1.0296	1.0507	1.0743
s / (J g^{-1} K^{-1})	0.0022	0.0032	0.0046
u_r / (J g^{-1} K^{-1})	0.0013	0.0018	0.0027
u_f / (J g^{-1} K^{-1})	0.0001	0.0001	0.0001
u / (J g^{-1} K^{-1})	0.0013	0.0019	0.0027
w	620000	290000	140000



Specific heat of air with temperature: · ♦ experimental data; — best fit with weighting: --- best fit without weighting

Experimental Procedure

1. Identify the data reduction equation or sequence of calculation steps.
2. Collect (or estimate) a data point: x_1, x_2, x_3, ... x_n.
3. Estimate the uncertainties in each variable: Δx_1, Δx_2, Δx_3, … Δx_n.
4. Calculate the predicted mean value, y.

Experimental Procedure

5. Calculate the uncertainty of the predicted mean value according to the error propagation equation, Δy.
6. Identify the variables with the largest contribution to uncertainty
7. Modify your experimental technique to reduce the larger uncertainties.
8. Take the rest of the data and repeat these steps if necessary.

Present the Results

- Predicted value ± Uncertainty (odds)
  - $y \pm \Delta y$ (95% confidence)
- Significant figures
  - Report uncertainty with <u>only</u> one significant figure .
  - Precision in the predicted mean value is determined from the uncertainty
  - Example, compare
    - 123.456 ± 0.123 should be reported as follows
    - 123.4 ± 0.1

Summary

- How to Calculate and Express Uncertainty
  - Error Presentation
    - Significant Figures, Rounding
  - Error Origination
    - Human, Fixed, Random
    - Fixed Error: Propagation
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The End

- Next week's lecture
  - Technical Writing
  - Peer Proofreading
  - Each lab partner brings a hard copy of a draft report, *i.e.*, two for each group.