Chapter 6: Point Estimation: Exercises Section 6.2 (20–30)
Book Title: *Probability and Statistics for Engineering and the Sciences*, Ninth Edition
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Exercises Section 6.2 (20–30)

20. A diagnostic test for a certain disease is applied to n individuals known to not have the disease. Let $X = \text{the number among the } n \text{ test results that are positive}$ (indicating presence of the disease, so X is the number of false positives) and $P = \text{the probability that a disease-free individual's test result is positive}$ (i.e., p is the true proportion of test results from disease-free individuals that are positive). Assume that only X is available rather than the actual sequence of test results.

    a. Derive the maximum likelihood estimator of p. If $n = 20$ and $x = 3$, what is the estimate?

    b. Is the estimator of part (a) unbiased?

    c. If $n = 20$ and $x = 3$, what is the mle of the probability $(1 - p)^5$ that none of the next five tests done on disease-free individuals are positive?

21. Let X have a Weibull distribution with parameters α and β, so

$$E(X) = \beta \cdot \Gamma(1 + 1/\alpha)$$
$$V(X) = \beta^2 \{\Gamma(1 + 2/\alpha) - [\Gamma(1 + 1/\alpha)]^2\}$$

    a. Based on a random sample $X_1, \ldots, X_n$, write equations for the method of moments estimators of β and α. Show that, once the estimate of α has been obtained, the estimate of β can be found from a table of the gamma function and that the estimate of α is the solution to a complicated equation involving the gamma function.

    b. If $n = 20$, $\overline{x} = 28.0$, and $\Sigma x_i^2 = 16{,}500$, compute the estimates. [*Hint*: $[\Gamma(1.2)]^2 / \Gamma(1.4) = .95$.]

22. Let X denote the proportion of allotted time that a randomly selected student spends working on a certain aptitude test. Suppose the pdf of X is

$$f(x;\theta) = \begin{cases} (\theta+1)x^\theta & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

where $-1 < \theta$. A random sample of ten students yields data $x_1 = .92$, $x_2 = .79$, $x_3 = .90$, $x_4 = .65$, $x_5 = .86$, $x_6 = .47$, $x_7 = .73$, $x_8 = .97$, $x_9 = .94$, $x_{10} = .77$.

a. Use the method of moments to obtain an estimator of θ, and then compute the estimate for this data.

b. Obtain the maximum likelihood estimator of θ, and then compute the estimate for the given data.

23. Let X represent the error in making a measurement of a physical characteristic or property (e.g., the boiling point of a particular liquid). It is often reasonable to assume that $E(X) = 0$ and that X has a normal distribution. Thus the pdf of any particular measurement error is

$$f(x;\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-x^2/2\theta} \qquad -\infty < x < \infty$$

(where we have used θ in place of σ^2). Now suppose that n independent measurements are made, resulting in measurement errors $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$. Obtain the mle of θ.

24. A vehicle with a particular defect in its emission control system is taken to a succession of randomly selected mechanics until $r = 3$ of them have correctly diagnosed the problem. Suppose that this requires diagnoses by 20 different mechanics (so there were 17 incorrect diagnoses). Let $p = P(\text{correct diagnosis})$, so p is the proportion of all mechanics who would correctly diagnose the problem. What is the mle of p? Is it the same as the mle if a random sample of 20 mechanics results in 3 correct diagnoses? Explain. How does the mle compare to the estimate resulting from the use of the unbiased estimator given in Exercise 17?

25. The shear strength of each of ten test spot welds is determined, yielding the following data (psi):

392	376	401	367	389	362	409	415	358	375

a. Assuming that shear strength is normally distributed, estimate the true average shear strength and standard deviation of shear strength using the method of maximum likelihood.

b. Again assuming a normal distribution, estimate the strength value below which 95% of all welds will have their strengths. [*Hint*: What is the 95th percentile in terms of μ and σ? Now use the invariance principle.]

c. Suppose we decide to examine another test spot weld. Let $X = \text{shear strength of the weld}$. Use the given data to obtain the mle of $P(X \leq 400)$. [*Hint*: $P(X \leq 400) = \Phi((400 - \mu)/\sigma)$.]

26. Consider randomly selecting n segments of pipe and determining the corrosion loss (mm) in the wall thickness for each one. Denote these corrosion losses by $Y_1, \ldots, Y_n$. The article **"A Probabilistic Model for a Gas Explosion Due to Leakages in the Grey Cast Iron Gas Mains" (*Reliability Engr. and System Safety* (2013:270–279)** proposes a linear corrosion model: $Y_i = t_i R$, where t_i is the age of the pipe and R, the corrosion rate, is exponentially distributed with parameter λ. Obtain the maximum likelihood estimator of the exponential parameter (the resulting mle appears in the cited article). [*Hint:* If $c > 0$ and X has an exponential distribution, so does cX.]

27. Let $X_1, \ldots, X_n$ be a random sample from a gamma distribution with parameters α and β.

a. Derive the equations whose solutions yield the maximum likelihood estimators of α and β. Do you think they can be solved explicitly?

b. Show that the mle of $\mu = \alpha\beta$ is $\widehat{\mu} = \overline{X}$.

28. Let $X_1, X_2, \ldots, X_n$ represent a random sample from the Rayleigh distribution with density function given in Exercise 15. Determine

a. The maximum likelihood estimator of θ, and then calculate the estimate for the vibratory stress data given in that exercise. Is this estimator the same as the unbiased estimator suggested in Exercise 15?

b. The mle of the median of the vibratory stress distribution. [*Hint*: First express the median in terms of θ.]

29. Consider a random sample $X_1, X_2, \ldots, X_n$ from the shifted exponential pdf

$$f(x; \lambda, \theta) = \begin{cases} \lambda e^{-\lambda(x-\theta)} & x \geq \theta \\ 0 & \text{otherwise} \end{cases}$$

Taking $\theta = 0$ gives the pdf of the exponential distribution considered previously (with positive density to the right of zero). An example of the shifted exponential distribution appeared in Example 4.5, in which the variable of interest was time headway in traffic flow and $\theta = .5$ was the minimum possible time headway.

a. Obtain the maximum likelihood estimators of θ and λ.

b. If $n = 10$ time headway observations are made, resulting in the values 3.11, $.64$, 2.55, 2.20, 5.44, 3.42, 10.39, 8.93, 17.82, and 1.30, calculate the estimates of θ and λ.

30. At time $t = 0$, 20 identical components are tested. The lifetime distribution of each is exponential with parameter λ. The experimenter then leaves the test facility unmonitored. On his return 24 hours later, the experimenter immediately terminates the test after noticing that $y = 15$ of the 20 components are still in operation (so 5 have failed). Derive the mle of λ. [*Hint*: Let $Y = \text{the number that survive 24 hours}$. Then $Y \sim \text{Bin}(n, p)$. What is the mle of p? Now notice that $p = P(X_i \geq 24)$, where X_i is exponentially distributed. This relates λ to p, so the former can be estimated once the latter has been.]

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