STAT604-Assignment-2

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Assignment-2

Part 1

Section 6.2 Problem#22

Solution:(a) Let , 'X' be random variable has pdf.

$$u_1 = E(x) = \int_0^1 x(\theta+2)x^\theta \, dx$$

$$(\theta+1)\int_0^1 x^{\theta+1} \, dx$$

$$\theta+1\left(\frac{x^{\theta+2}}{\theta+2}\right)$$

$$(\theta+1/(\theta+2))$$

$$(\theta+1+1-1/(\theta+2))$$

$$1-\frac{1}{\theta+2}$$

$$1-u_1 = \frac{1}{\theta+2}$$

$$(\theta+2) = \frac{1}{1-u_1}$$

$$\theta = \frac{1}{(1-u_1)^{-2}}$$

It is a moment estimate or of

$$\theta = \bar{X}$$

mean of the sample data

Solution: (b)

Sample

$$\bar{X} = \frac{x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9+x_{10}}{10}$$

$$\bar{X} = \frac{0.92+0.79+0.9+0.65+0.86+0.47+0.73+0.97+0.94+0.77}{10}$$

$$\bar{X} = 0.8$$

The estimate of

$$\theta = \frac{1}{(1-u_1)^{-2}} = 5 - 2 = 3$$

Section 6.2 Problem#23

Given the probability density function (pdf) of the measurement error: $f(x;\theta) =$

$$\frac{1}{\sqrt{2\pi\theta}} \cdot e^{-\frac{x^2}{2\theta}}$$

We want to find the maximum likelihood estimate (MLE) of

$$\theta$$

based on (n) independent measurements $(x_1, x_2, x_3, \ldots, x_n)$ The likely hood function

$$L(\theta)$$

is the product of the pdf for each measurement:

$$\ln(L(\theta)) = \sum_{i=1}^{n} \ln(f(x;\theta))$$

The log-likelihood function is given by:

$$\ln(L(\theta)) = \sum_{i=1}^{N} \ln\left(\frac{1}{\sqrt{2\pi\theta}} \cdot e^{-\frac{x_i^2}{2\theta}}\right)$$

Now, let's differentiate this with respect to θ step by step:

$$\frac{d}{d\theta}\ln(L(\theta)) = \sum_{i=1}^{N} \frac{d}{d\theta}\ln\left(\frac{1}{\sqrt{2\pi\theta}} \cdot e^{-\frac{x_i^2}{2\theta}}\right)$$

Applying the chain rule and sum rule:

$$\frac{d}{d\theta}\ln(L(\theta)) = \sum_{i=1}^{N} \frac{d}{d\theta}\left(-\frac{1}{2}\ln(2\pi\theta) - \frac{x_i^2}{2\theta}\right)$$

Differentiating each term:

$$\frac{d}{d\theta}\ln(L(\theta)) = \sum_{i=1}^{N}\left(-\frac{1}{2\theta} + \frac{x_i^2}{2\theta^2}\right)$$

Combine terms:

$$\frac{d}{d\theta}\ln(L(\theta)) = \sum_{i=1}^{N} \frac{-1 + x_i^2}{2\theta^2}$$

Setting the derivative equal to zero to find the MLE:

$$\sum_{i=1}^{N} \frac{-1 + x_i^2}{2\theta^2} = 0$$

The equation obtained from setting the derivative of the log-likelihood equal to zero is:

$$\sum_{i=1}^{N} \frac{-1 + x_i^2}{2\theta^2} = 0$$

To solve for θ, rearrange the equation:

$$\sum_{i=1}^{N} -1 + x_i^2 = 0$$

Now, isolate θ:

$$\sum_{i=1}^{N} x_i^2 = \sum_{i=1}^{N} 1$$

$$\sum_{i=1}^{N} x_i^2 = N$$

Finally, solve for θ:

$$\theta = \frac{\sum_{i=1}^{N} x_i^2}{N}$$

So, the maximum likelihood estimate (MLE) for θ in this context is $\frac{\sum_{i=1}^{N} x_i^2}{N}$.

Section 6.2 Problem#27

Solution(a): The probability density function (PDF) of the gamma distribution is given by:

$$f(x; a, b) = \frac{1}{\Gamma(a) b^a} x^{(a-1)} e^{-\frac{x}{b}}$$

where $\Gamma(a)$ is the gamma function of a.

Assuming $\Gamma(a)$ is the gamma function of a, the likelihood function for a sample of n independent observations $X_1, X_2, ..., X_n$ from a gamma distribution is given by:

$$L(a, b) = f(X_1; a, b) \cdot f(X_2; a, b) \cdot ... \cdot f(X_n; a, b)$$

$$= \frac{1}{\Gamma(a)^n b^{a \cdot n}} \prod_{i=1}^{n} X_i^{(a-1)} e^{-\frac{X_i}{b}}$$

The log-likelihood function is:

$$\ln L(a, b) = -n \ln \Gamma(a) - na \ln b + (a - 1) \sum \ln(X_i) - \sum \frac{X_i}{b}$$

To find the maximum likelihood estimators of a and b, we take the partial derivatives of the log-likelihood function with respect to a and b and set them equal to zero:

$$\frac{\partial \ln L(a,b)}{\partial a} = -n\psi(a) + \sum \ln(X_i) = 0$$

where $\psi(a)$ is the digamma function (derivative of the logarithm of the gamma function).

$$\frac{\partial \ln L(a,b)}{\partial b} = -\frac{na}{b} + \sum \frac{X_i}{b^2} = 0$$

The above equations cannot be solved explicitly. However, numerical methods can be used to find the values of a and b that maximize the log-likelihood function.

Solution(b): The probability density function (PDF) of the gamma distribution is given by:

$$f(x;a,b) = \frac{1}{\Gamma(a)b^a} x^{(a-1)} e^{-\frac{x}{b}}$$

where $\Gamma(a)$ is the gamma function of a.

The mean of a gamma distribution is given by:

$$E(X) = a \cdot b$$

Therefore, we have:

$$a \cdot b = \mu$$

Since the Maximum Likelihood Estimators (MLE) of a and b are:

$$a = \left(\frac{n}{\psi(a)}\right)^{-1}$$

$$b = \frac{1}{n}\sum_{i=1}^{n} X_i$$

where $\psi(a)$ is the gamma function, which is the first derivative of the natural logarithm of the gamma function. Substituting these values of a and b in the equation for μ, we get:

$$\mu = a \cdot b = \left[\left(\frac{n}{\psi(a)}\right)^{-1}\right]\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \bar{X}$$

Here, $\bar{X}$ represents the sample mean.

Part 2

To find MLE for d:

Part (i)

The probability function of x is given by:

$$P(x,\lambda) = \frac{e^{-\lambda} \cdot \lambda^x}{x!} \quad \text{for } x = 0, 1, 2, \ldots$$

The likelihood function is:

$$L = e^{-nd} \cdot \frac{\lambda^{x_1} \cdot x_2 \cdot \ldots \cdot x_n}{x_1! \cdot x_2! \cdot \ldots \cdot x_n!}$$

So,

$$\ln L = -nd + \sum_i x_i \ln \lambda - \ln(x_1! \cdot x_2! \cdot \ldots \cdot x_n!)$$

Taking the partial derivative with respect to d and equating it to 0, we get:

$$\frac{\partial}{\partial d} \ln L = -n + \frac{\sum_i x_i}{d} = 0$$

Differentiating (1) again partially with respect to d, we get:

$$\frac{\partial^2}{\partial d^2} \ln L = -\frac{\sum x_i}{d^2} < 0$$

This shows that equation (1) will indeed give the maximum value. Solving for d, we get:

$$d = \frac{\sum x_i}{n} = \bar{x}$$

So, we prove that the MLE for d is given by $\bar{d} = \bar{x}$.

Part (ii)

To find the probability function of x, it is given by:

$$P(x, d) = \frac{e^{-\lambda} \cdot d^x}{x!}$$

The first moment about the origin is given by:

$$M_1^1 = E(x) = \sum_{x=0}^{\infty} x \cdot P(x) = \sum_{x=0}^{\infty} x \left(\frac{e^{-d} \cdot d^x}{x!} \right)$$

After simplifying, we get:

$$M_1^1 = \lambda$$

But $\bar{x} = \frac{x_1+x_2+\ldots+x_n}{n}$, so we get:

$$\bar{\lambda} = (\lambda)$$

Thus, we prove that the moment estimator of λ is given by $\bar{\lambda} = \bar{x}$.

Part (iii)

They are the same: the maximum likelihood estimate of d is given by $\frac{\sum x_i}{n} = \bar{x}$. The maximum likely hood of

$$\lambda$$

=

$$0 \times 4 + 1 \times 12 + 2 \times 22 + 3 \times 14 + 4 \times 9 = 112$$

Therefore,

$$\frac{112}{50} = 2.24$$