1 Star Problems

ADD TO ONE-THOUSAND

Problem

There are exactly three different pairs of positive integers that add to make six.

> 1 + 5 = 6
> 2 + 4 = 6
> 3 + 3 = 6

How many different pairs of positive integers add to make one-thousand?

Solution

By writing the sums as: 1 + 999, 2 + 998, 3 + 997, ... , 499 + 501, 500 + 500. It is clear that the number of different pairs is five hundred.

How many different triplets of positive integers add to make one-thousand?

ADD TO SIX

Problem

By using positive integers, how many different ways can you make a sum that is equal to six?

For example you could use:

$3 + 1 + 1 + 1 = 6$
$4 + 2 = 6$
$1 + 2 + 3 = 6$

(Consider 4 + 2 to be the same as 2 + 4)

Solution

Considering the number of 1's used in the sum.

6x1's:	1+1+1+1+1+1
5x1's:	None
4x1's:	1+1+1+1+2
3x1's:	1+1+1+3
2x1's:	1+1+2+2 and 1+1+4
1x1:	1+2+3 and 1+5
0x1's:	2+2+2, 2+4 and 3+3

Giving 10 solutions.

What if 2+4 is considered to be different to 4+2?

Investigate the number of different sums to make all the integers from 1 to 100.

Ancient Riddle

Problem

The end of topic tests in mathematics are looming and you realise that you have spent one too many lunch times surfing the internet and not revising. Just before you log off you suddenly remember that you never actually logged into the terminal. Curious about who left the computer logged in, you discover that it was your mathematics teacher.

You decide to "research" the contents of his private directory and stumble upon a file called, "end_of_topic_test.doc". After checking that no one is looking, you double click on the filename and are greated with a box requesting a password.

Not to be out-done, you look through the files and discover a file called, "password.gif". You cannot believe your luck! Excitedly you double click it and are presented with the following image:

ΑΡΧΙΜΗΔΗΣ

Can you discover the password?

Solution

Although this is not strictly an encryption system, this challenge will have tested the code breaker's ability to research with the information given. Once the characters were recognised as Greek, there should be little difficulty finding a table with Greek/Latin alphabet equivalents.

For reference:

Small Letters	Capital Letters	Letter Name	Latin Equivalent
α	Α	Alpha	a
β	Β	Beta	b
γ	Γ	Gamma	g
δ	Δ	Delta	d
ε	Ε	Epsilon	*short* e
ζ	Ζ	Zeta	z

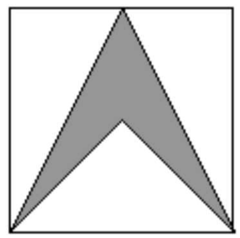
η	Η	Eta	*long* e
θ	Θ	Theta	th
ι	Ι	Iota	i
κ	Κ	Kappa	k
λ	Λ	Lambda	l
μ	Μ	Mu	m
ν	Ν	Nu	n
ξ	Ξ	Xi	x
ο	Ο	Omicron	*short* o
π	Π	Pi	p
ρ	Ρ	Rho	r
σ	Σ	Sigma	s
τ	Τ	Tau	t
υ	Υ	Upsilon	u
φ	Φ	Phi	f, ph
χ	Χ	Chi	ch
ψ	Ψ	Psi	ps
ω	Ω	Omega	*long* o

Hence, the Latin equivalent of the Greek characters, ΑΡΧΙΜΗΔΗΣ, reveals the password: ARCHIMEDES.

Area Of Arrow

Problem

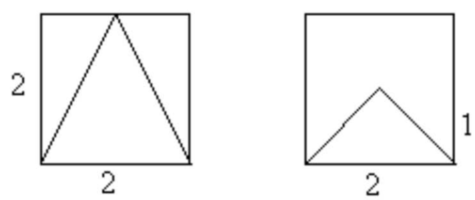
An arrow is formed in a 2 × 2 square by joining the bottom corners to the midpoint of the top edge and the centre of the square.



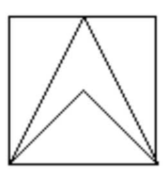
Find the area of the arrow.

Solution

Consider the two diagrams below.



The area of the square is 4, so the area of the large triangle is 2 (half of the square) and the area of the small triangle is 1 (quarter of the square).



Hence the area of the arrow is 2 – 1 = 1 square unit.

What would be the area of a similar arrow, drawn in a 10 × 10 square?
Can you generalise for an $n \times n$ square?
What about arrows in general?

ARITHMETIC RING

Problem

The digits 1, 2, 3, 4, 5, 6, 7 and 8 are placed in the ring below.

1	5	3
8		7
4	6	2

With the exception of 6 and 7, no two adjacent numbers are consecutive.

Show how it is possible to arrange the digits 1 to 8 in the ring so that no two adjacent numbers are consecutive.

Solution

Here is one solution.

1	4	7
6		2
3	8	5

Are there any more solutions?
What if you had to arrange the numbers 1 to 12 in a 4 by 4 ring?
How many numbers would there be in an $n \times n$ ring?

ARITHMETIC VOLUME

Problem

A sequence is arithmetic if the numbers increases by a fixed amount. For example, 2, 5, 8, are in an arithmetic sequence with a common difference of 3, and if these numbers represented the side lengths of a cuboid, the volume, $V = 2\times5\times8 = 80$ units^3.

How many cuboids exist for which the volume is less than 100 units^3 and the integer side lengths are in an arithmetic sequence?

Solution

Although $3\times3\times3 = 27$ is a cuboid with a volume less than 100 units^3, we shall discount cubes on the grounds that 3, 3, 3, is a trivial example of an arithmetic sequence.

By considering the first term, *a*, and the common difference, *d*, we have a method of systematically listing all of the solutions:

a=1, d=1, $1\times2\times3 = 6$
a=1, d=2: $1\times3\times5 = 15$
a=1, d=3: $1\times4\times7 = 28$
a=1, d=4: $1\times5\times9 = 45$
a=1, d=5: $1\times6\times11 = 66$
a=1, d=6: $1\times7\times13 = 91$
a=2, d=1: $2\times3\times4 = 24$
a=2, d=2: $2\times4\times6 = 48$
a=2, d=3: $2\times5\times8 = 80$
a=3, d=1: $3\times4\times5 = 60$

Hence there are exactly ten cuboids.

What is the maximum volume obtainable for a cuboid with side lengths in an arithmetic sequence and having a volume less than 1000 units^3?
Can you generalise for a volume less than V units^3?

As Easy As 1234

Problem

Using each of the digits 1, 2, 3, and 4, once and only once, with the basic rules of arithmetic (+, −, ×, ÷, and parentheses), express all of the integers from 1 to 25.

For example, $1 = 2 \times 3 - (1 + 4)$

Solution

Of course, there are may be other ways of arriving at each of these numbers:

$1 = 2 \times 3 - (1 + 4)$	$14 = 1 \times 4 \times 3 + 2$
$2 = 4 - 3 + 2 - 1$	$15 = 3 \times 4 + 1 + 2$
$3 = 2 \times 3 - (4 - 1)$	$16 = 2(1 + 3 + 4)$
$4 = 2 \times 4 - (1 + 3)$	$17 = 3(2 + 4) - 1$
$5 = 2 \times 4 - 1 \times 3$	$18 = 3(2 + 4) \times 1$
$6 = 2 \times 4 - 3 + 1$	$19 = 3(2 + 4) + 1$
$7 = 3(4 - 1) - 2$	$20 = 1 \times 4 \times (2 + 3)$
$8 = 2 + 3 + 4 - 1$	$21 = 4(2 + 3) + 1$
$9 = 2 \times 3 + (4 - 1)$	$22 = 2(3 \times 4 - 1)$
$10 = 1 + 2 + 3 + 4$	$23 = 3 \times 4 \times 2 - 1$
$11 = 2 \times 3 + (1 + 4)$	$24 = 1 \times 2 \times 3 \times 4$
$12 = 3 \times 4 \times (2 - 1)$	$25 = 2 \times 3 \times 4 + 1$
$13 = 3 \times 4 + 1 + 2$	

Extensions

- If you are now permitted to use square roots, exponents, and factorials, can you produce all of the integers from 1 to 100?
- What is the first natural number that cannot be derived?
- Which is the first number that cannot be obtained if you are only permitted to use the basic rules of arithmetic (+, −, ×, and ÷)?
- What is the largest known prime you can produce?

Notes

Surprisingly it is possible to produce any finite integer using logarithms in a rather ingenious way.

We can see that,

$\sqrt{2} = 2^{1/2}$
$\sqrt{\sqrt{2}} = (2^{1/2})^{1/2} = 2^{1/4}$
$\sqrt{\sqrt{\sqrt{2}}} = ((2^{1/2})^{1/2})^{1/2} = 2^{1/8}$, and so on.

Therefore,

$\log_2(\sqrt{2}) = 1/2 = (1/2)^1$
$\log_2(\sqrt{\sqrt{2}}) = 1/4 = (1/2)^2$
$\log_2(\sqrt{\sqrt{\sqrt{2}}}) = 1/8 = (1/2)^3$, et cetera.

Hence,

$\log_{1/2}(\log_2(\sqrt{2})) = 1$
$\log_{1/2}(\log_2(\sqrt{\sqrt{2}})) = 2$
$\log_{1/2}(\log_2(\sqrt{\sqrt{\sqrt{2}}})) = 3, \ldots$

By using the integer part function, $1/[\sqrt{(3!)}] = 1/[2.449...] = 1/2$, we can obtain the required base 1/2, and using $\sqrt{4}$ to obtain the base 2, we can now produce any finite integer using the digits 1, 2, 3, and 4.

$\log_{(1/[\sqrt{(3!)}])}(\log_{\sqrt{4}}(\sqrt{2})) = 1$
$\log_{(1/[\sqrt{(3!)}])}(\log_{\sqrt{4}}(\sqrt{\sqrt{2}})) = 2$
$\log_{(1/[\sqrt{(3!)}])}(\log_{\sqrt{4}}(\sqrt{\sqrt{\sqrt{2}}})) = 3, \ldots$

AVERAGE PROBLEM

Problem

For a set of five whole numbers, the mean is 4, the mode is 1, and the median is 5. What are the five numbers?

Solution

As the mode is 1, there must be at least two 1's. But because the median is 5, the third number must be 5, and so we have the set of numbers $\{1, 1, 5, x, y\}$.

If the mean is 4, the sum of the numbers must be $4 \times 5 = 20$; that is, $1 + 1 + 5 + x + y = 20 \Rightarrow x + y = 13$.

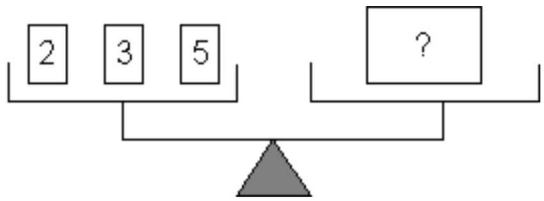
Without loss of generality (WLOG), let $x \le y$, and if $x = y$, we get $x + x = 13$, $2x = 13$, $x = 6.5$. Clearly $x \ge 5$, and so $5 \le x \le 6.5$

However, if $x = 5$, we would have two modal values: 1 and 5. Hence we deduce that $x = 6$, $y = 7$, and the set of five numbers must be $\{1, 1, 5, 6, 7\}$.

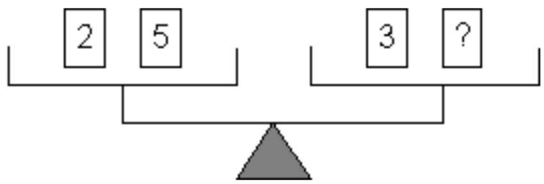
Balancing Scales

Problem

In order for the scales to balance the block on the right side must be 10 kg.



It is possible to use the same weights to weigh, for example, 4 kg.



In fact using 2 kg, 3 kg and 5 kg weights it is possible to weigh all but one value from 1 kg to 10 kg. What is that value?
(Note: You don't have to use all three weights.)

Solution

The following weights can be found, with the desired weight indicated in brackets.

Left Side	Right Side
3	2 (1)
2	(2)
3	(3)
2 5	3 (4)
5	(5)
3 5	2 (6)
2 5	(7)
3 5	(8)
2 3 5	(10)

And so it is not possible to weigh 9 kg.

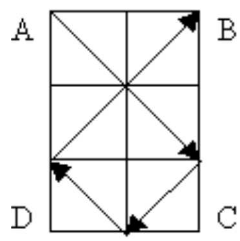
What if you used 1 kg, 3 kg and 6 kg to weigh all the weights from 1 kg to 10 kg?
What about using 1 kg, 2 kg and 7 kg?

If $x + y + z = T$, what must be special about x, y and z so that all the weights from 1 kg to T kg can be weighed?
(Hint: Try $x + y = T$ to begin with.)

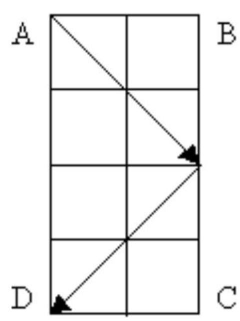
BEAM OF LIGHT

Problem

If a light beam is fired through the top left corner of a 2 by 3 rectangular prism block at a 45 degrees towards the opposite wall, it will emerge from the top right corner.



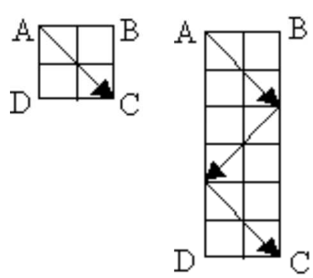
If the light bounces around a 2 by 4 rectangle, it emerges from corner D.



If a beam of light is fired through the top left corner of a 2 by 50 rectangle, which corner will it emerge from?

Solution

Consider the two diagrams.



It can be seen from this that rectangles measuring 2×2, 2×6, 2×10, 2×14, ... , 2×50, that is, even non-multiples of 4, will finish in the bottom right corner, C.

Will the light ever return to A?

Investigate other size grids, $3 \times k$, $4 \times k$, et cetera.

BIGGER DIGIT

Problem

For each of the numbers: 41, 83, 32, the first digit is greater in value than the second digit.

How many 2-digit numbers have this property?

Solution

If we begin to list the numbers in groups: 10; 20,21; 30,31,32; 40,41,42,43; ... ; 90,91,92,93,94,95,96,97,98 ; we can see that the total number of 2-digit numbers, for which the first digit is greater than the second digit, will be 1 + 2 + ... + 9 = 45.

How many 3-digit numbers exist for which the first digit is greater in value than both the second digit and the third digit?
Can you generalise for n-digit numbers?
What about 3-digit numbers for which the first digit is greater than the sum of the second and third digits?

Birds And Bunnies

Problem

A cage contains birds and rabbits. There are sixteen heads and thirty-eight feet. How many birds are there in the cage?

Solution

If we let the number of birds be represented by b and the number of rabbits be represented by r then we get the following two equations:

$b + r = 16$ (1)
$2b + 4r = 38$ (2)

Dividing the second equation by two gives:

$b + 2r = 19$ (3)

If we now subtract equation (1) from equation (3) we get $r = 3$, and as $b + r = 16$ it follows that the number of birds, b, must be 13.

What if you were told that the number of feet was equal to twice the number of heads?
What if there were sixteen heads and twenty-eight feet?

BIRTHDAY PARTY

Problem

At a birthday party, one-half drank only lemonade, one-third drank only cola, fifteen people drank neither, and nobody drinks both.

How many people were at the party?

Solution

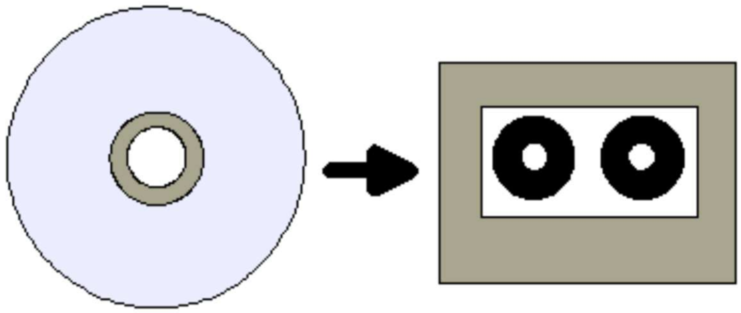
As $1/2 + 1/3 = 3/6 + 2/6 = 5/6$, we know that $1/6$ drank neither.

So there must have been $6 \times 15 = 90$ people at the party.

CD To Tape

Problem

Matilda wanted to transfer her favourite CD to tape.



The CD has six songs and the length of the tracks are 7:55, 9:40, 9:15, 12:45, 8:20 and 11:30; a total playing time of 59:25. After changing the order of the songs, Matilda was able to fit all the songs, without any breaks, on a sixty minute tape. How did she arrange the songs?

Solution

For the songs to fit on the tape, each side must be almost completely filled, as the total playing time will leave 35 seconds unused on a sixty minute tape. If the two longest songs were placed on the same side, 12:45 and 11:30 take up 24:15 together, leaving only 5:45, which is not enough for any other single song. So they must go on opposite sides.

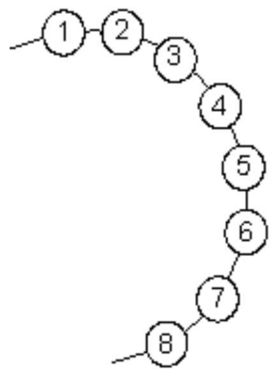
By trial we get:
Side 1 12:45, 7:55 and 9:15 (29:55 total)
Side 2 11:30, 9:40 and 8:20 (29:30 total)

If the total playing time is less than the length of the tape, is a solution always possible?

CHILDREN IN A CIRCLE

Problem

A group of children stand holding hands in a large circle and a teacher walks around the circle giving each child in order a number 1, 2, 3, 4, ...



If number 12 is standing opposite number 30, how many children are there in the circle?

Solution

Let c be the number of children in the circle. Half the children, $c/2$, will be in-between child 12 and child 30.

So $12 + c/2 = 30$.

Therefore $c/2 = 18$.

That is, there are 36 children in the circle.

CHOCOLATE OFFER

Problem

To promote the launch of a new Cradbury's chocolate bar they are offering a, "buy four, get one free" deal. If each chocolate bar costs thirty pence, how much would ten chocolate bars cost?

Solution

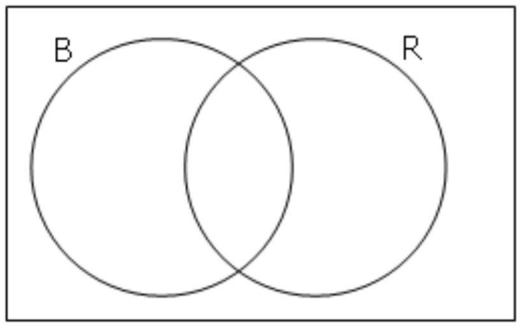
If we get five for the price of four, ten will cost the same as eight. Hence ten chocolate bars will cost $8 \times 0.30 = £2.40$.

How much would seventeen chocolate bars cost?
What if you wanted to buy *n* chocolate bars?
What if it they offered a, "buy three, get one free" deal?
Can you generalise?

CLASS DISTINCTION

Problem

Mr. Venn draws two large overlapping circles on the floor of the sports hall and labels them B and R. He asks all those students with brown hair to stand in the B circle and those that are right handed to stand in the R circle; if they have both brown hair and are right handed, they need to stand in the region where the two circles overlap.



When they return to the classroom he asks his class of thirty two students how many have brown hair: twenty seven put their hands up. He then asks how many students are right handed: twenty four raise their hands.

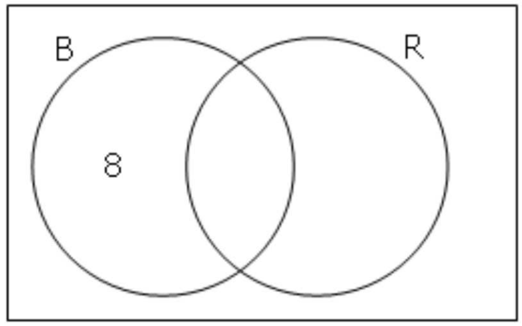
What is the minimum number of students that stood in the overlap?

Solution

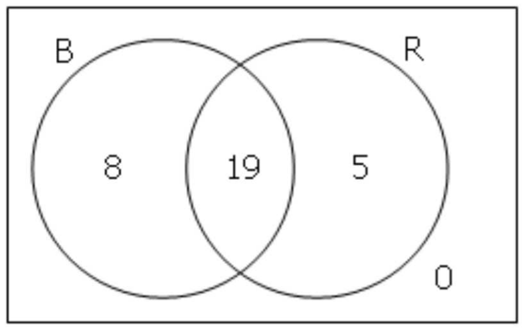
From the information, we can deduce that 32 – 24 = 8 students are left handed.

As we are trying the minimise the number of students with brown hair that are right handed, we would like to maximise the number of brown haired students that are left handed.

We begin by assigning all eight left handed students to the region that is in B but not in R:



As there are twenty seven students in total with brown hair, we know that 27 – 8 = 19 students that have brown hair and are right handed. This is represented by the intersection of the B and R circles. For completion we shall fill in the other values.



That is, the minimum number students that stood in the overlap is 19.

What is the maximum number of right handed students with brown hair?
In terms of being right/left handed and having/not having brown hair, what do the values 5 and 0 on the diagram represent?

COIN PROBLEM

Problem

In my pocket I have exactly 15p, comprising of eight coins made up of 1p, 2p and 5p pieces. How many of each coin do I have?

Solution

As there is at least one of each type of coin, 5 + 2 + 1 = 8 (using three coins), and so the remaining (five) coins must total 15 – 8 = 7 pence.

7p can be made in the following ways:

5	2			**2 coins**
5	11			**3 coins**
2	2	2	1	**4 coins**
2	2	11	1	**5 coins**
2	11	11	1	**6 coins**
11	11	11	1	**7 coins**

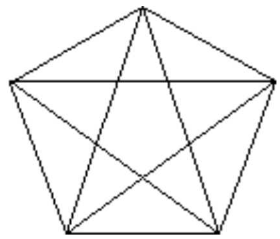
Hence 15p must be made up of one 5p, three 2p and four 1p coins.

Is it possible to have 10p in my pocket comprising of six coins and made up of 1p, 2p and 5p coins?

CONNECTED PENTAGON

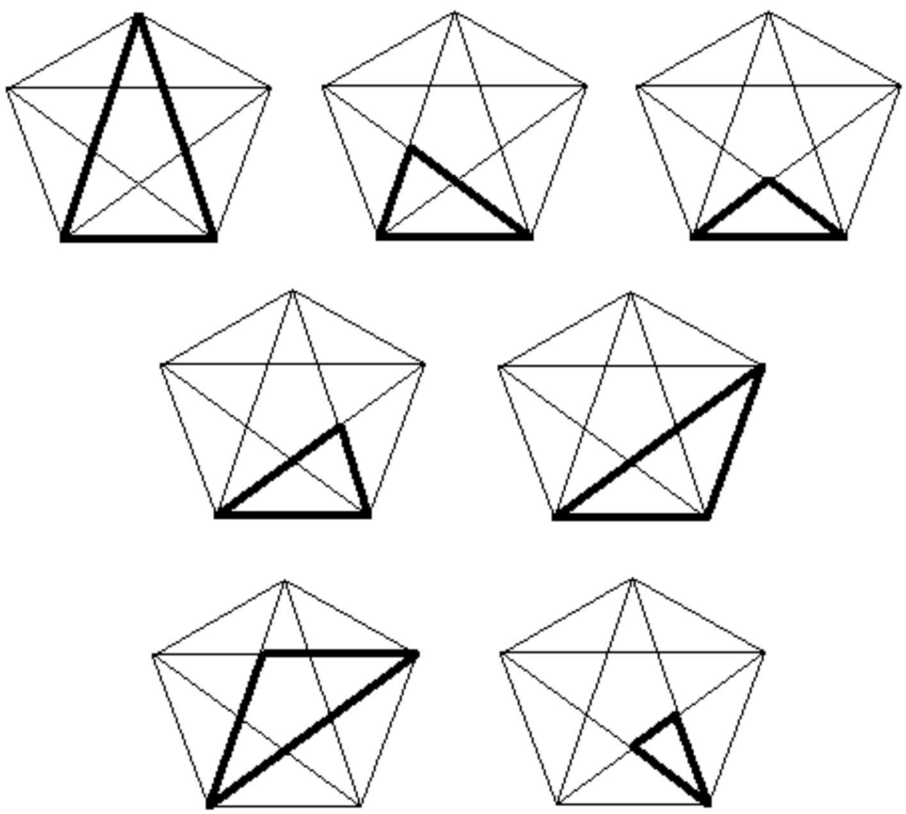
Problem

How many triangles are in a fully connected pentagon?



Solution

By rotational symmetry, all triangles below exist in exactly five other positions.



Making a total of 7 × 5 = 35 triangles in the diagram.

What about a fully connected hexagon?
Can you generalise for n-gons?

Counting Coins

Problem

In the United Kingdom, money is made up of pounds (£) and pence (p). The coins in circulation are 1p, 2p, 5p, 10p, 20p, 50p, £1 (100p), and £2 (200p).

Surprisingly it is possible to have £2.39 worth of coins, made up of 100p + 50p + 4 × 20p + 9 × 1p, and be unable to make exactly £2.

What is the maximum amount of coinage that you could have in your pocket and not able to make exactly £2?

Solution

That maximum amount of money that you can have in your pocket, and be unable to make exactly £2, is £2.43 (243p). This can be achieved in three different ways:

- 100p + 50p + 4 × 20p + 5p + 4 × 2p (11 coins)
- 3 × 50p + 4 × 20p + 5p + 4 × 2p (12 coins)
- 50p + 9 × 20p + 5p + 4 × 2p (15 coins)

Using 1p, 2p, 5p, and 10p coins, what is the maximum amount of money you could have and be unable to make 20p?
What is the maximum amount of coins to achive this?
What about using 1p, 2p, 5p, 10p, and 20p coins, so that you cannot make 50p?
Investigate for different maximum amounts.

COUNTING NINES

Problem

If you wrote all of the numbers from one to one-hundred on a piece of paper, 1, 2, 3, 4, ..., 99, 100. How many number nines would you write in total?

Solution

Let us consider the first and second digit.

Changing the first digit,	?9	10 numbers (i.e. 09, 19, 29, ... , 99)
Changing the second digit,	9?	10 numbers (i.e. 90, 91, 92, ... , 99)

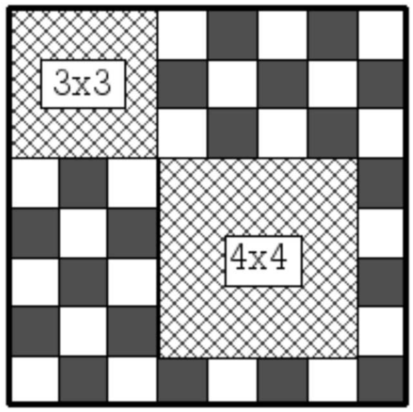
Making a total of 20 number nines being written.

If all the digits on the piece of paper were added together, what would be the total?

COUNTING SQUARES

Problem

A standard chessboard measures 8 x 8 squares, making sixty-four 1 x 1 squares. But there are many more squares, for example 3 x 3, 4 x 4, etc.



How many squares are there altogether on a chessboard?

Solution

There are, 64 × (1x1), 49 × (2x2), 36 × (3x3), 25 × (4x4),
16 × (5x5), 9 × (6x6), 4 × (7x7) and 1 × (8x8) squares.

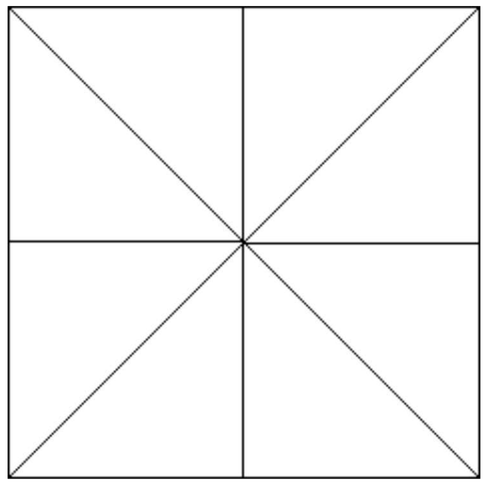
Making 1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 = 204 squares in total.

How many squares are there on a *n* x *n* grid?
What about rectangles on an *m* x *n* grid?

Counting Triangles

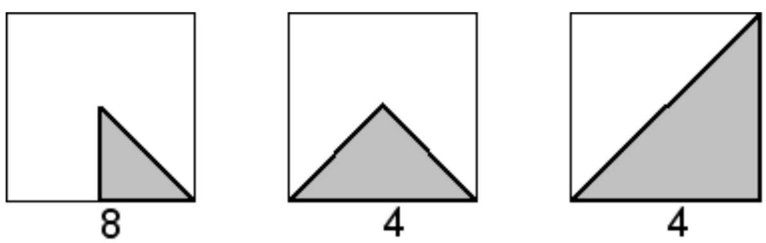
Problem

How many triangles are there in the diagram below?



Solution

Consider the diagrams.

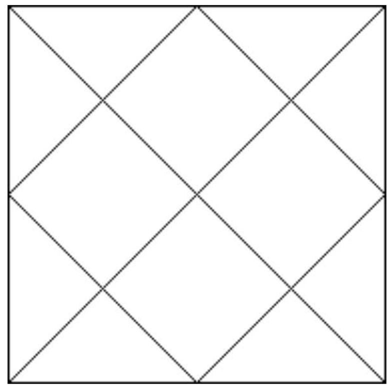


So there are 8 + 4 + 4 = 16 triangles in the diagram.

Counting Triangles Again

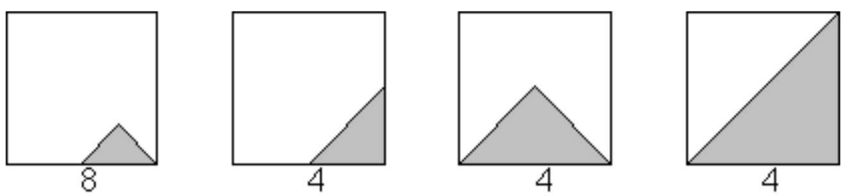
Problem

How many triangles are there in the diagram below?



Solution

Consider the diagrams.



So there are 8 + 4 + 4 + 4 = 20 triangles in the diagram.

COUNTING UP TO ONE THOUSAND

Problem

A boy writes down the counting numbers from 1 to 20 in order and because each of the numbers from 10 to 20 contain two digits he notes that he has actually written down thirty-one digits in total.

i. If his list went from 1 to 1000 how many digits would he have written down?
ii. If one-thousand digits were written down in total, how many complete numbers would have been written down?

Solution

For the first part we consider 1-digit, 2-digit, and 3-digit numbers separately:

> 1 to 9: $9 \times 1 = 9$ digits
> 10 to 99: $90 \times 2 = 180$ digits
> 101 to 999: $900 \times 3 = 2700$ digits

Hence he will have written down $9 + 180 + 2700 + 4 = 2893$ digits in total.

For the second part we note that $9 + 180 = 189$ digits would be written for the numbers 1 to 99. Therefore there would be $1000 - 189 = 811$ more digits written down, and, as $811/3 = 270.333...$, this is equivalent to writing down an additional 270 complete 3-digit numbers. Hence he would have written all the numbers from 1 to 369.

However, a more elegant approach is to assume that each of the numbers from 1 to 999 are written as 3-digit numbers; for example, 2 is written as 002 and 73 is written as 073.

Now we note that each of the numbers from 1 to 99 have an unnecessary zero in the first column and each of the numbers 1 to 9 have an unnecessary zero in the second column, making $99 + 9 = 108$ digits that would not actually be written down.

That is, this system would be equivalent to writing down a list of 3-digit numbers comprising of 1108 digits in total. But $1108/3 = 369.333...$, therefore he would have written all the complete numbers from 1 to 369.

Crossing Trains

Problem

A train travels a distance of 90 miles from A to B in one hour. Another train sets off at the same time and travels from B to A, taking two hours to complete the journey.

How many miles from A did the two trains cross?

Solution

As the first train completed the journey in half the time of the second train it must have travelled twice as fast. So when the two trains crossed the first train would have travelled twice as far and would have completed $^2/_3$ of the journey. Hence both trains were $^2/_3$ of 90 = 60 miles from A.

What if the first train takes 2 hours and the second train takes 3 hours to complete the journey?
Can you generalise?

CROSS COUNTRY RACE

Problem

Alice, Belinda, and Clara were the three representatives for their school in a team cross country race. Alice finished the race in middle position, Belinda finished after Alice, in 19th position, and Clara finished 28th.

How many schools took part in the race?

Solution

As Alice finished in middle position we know that there must be an odd number of runners in the race, but because there are three girls representing each school, the number of runners must be an odd multiple of three. In addition, we know that Clara, who finished 28th, was not last, as there are an odd number of runners. So there must be at least 29 runners.

If there were 33 runners, the middle position would be 17th.
If there were 39 runners, the middle position would be 20th.

However, we are told that Belinda, who finished 19th, finished after Alice.

Hence there must have been 33 runners in the race and we deduce that eleven schools must have taken part.

CUDDLY WUDDLY TRIBE

Problem

A remote Pacific island is occupied by two tribes: the Cuddly-Wuddlies and the Nasty-Wasties. They are utterly indistinguishable by appearance, but differ completely in their character and live at opposite ends of the island. Cuddly-Wuddlies always tell the truth and are reknowned for their kind hospitality, whereas Nasty-Wasties are infamous for their compulsive lying and cannabilism; being especially partial to the taste of visitors.

By the time you arrive the sun is already setting and being out so late is not a good idea. You stand at a fork in the pathway: one direction heads to the Cuddly-Wuddly village and the other leads to the Nasty-Wasty village. However, you don't know which way to go.

A man is standing nearby and as you are in a hurry you have time to ask one simple question.

What question could you ask to ensure that you receive correct directions to the Cuddly-Wuddly village?

Solution

The question is surprisingly simple: "Which way do you live?"

If the man is a Cuddly-Wuddly then he will point to his own village.
If the man is a Nasty-Wasty then he will lie and instead of pointing to his own village he will point to the Cuddly-Wuddly village.

In other words, it doesn't matter which tribe they belong to, either will point to the Cuddly-Wuddly village.

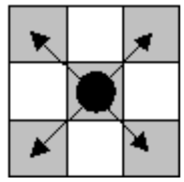
Note: This question will always work with a Cuddly-Wuddly, but it will only work with a Nasty-Wasty if they choose to co-operate with your clever question. If they are as sneaky as they are dishonest then they could say something like, "That way!", pointing back the way you came, which is a blatant lie. In fact, with an unco-operative liar there is no question that you could ever ask to catch them out.

What question could you ask if there were more than two pathways?

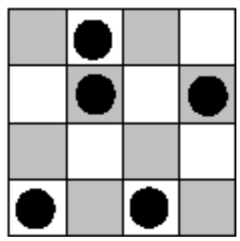
DEFENSIVE BISHOP

Problem

In a game of chess, the Bishop can take any piece that lies on a diagonal.



By placing five Bishop pieces on a 4 by 4 grid every square is protected in such a way that no Bishop threatens any other Bishop.

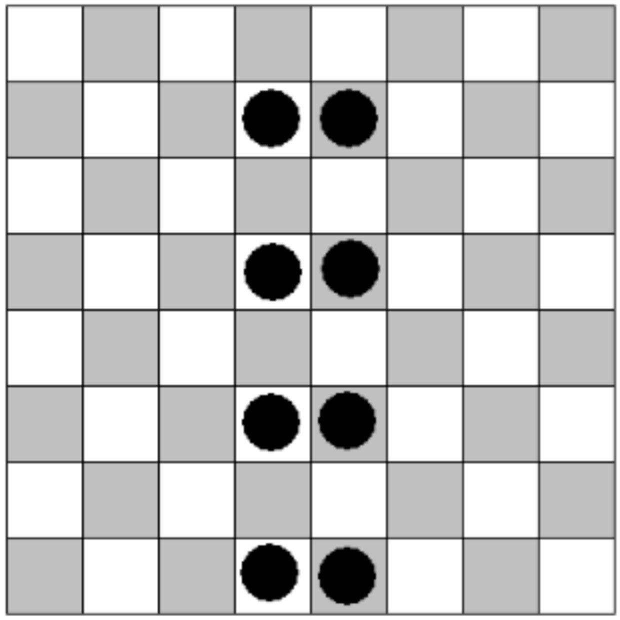


However, it is possible to achieve the same with just four bishops.

What is the least number of Bishop pieces required to protect every square on a standard 8 by 8 chessboard?

Solution

It is possible to protect every square on a chessboard by using just eight Bishop pieces; this is one way it can be done.

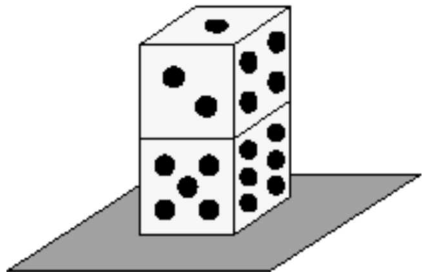


How many different ways can this be achieved?
What about an *n* by *n* grid?
What is the maximum number of non-threatening Bishops pieces that can be placed on a four by four grid?
What about different size grids?

DICE PROBLEM

Problem

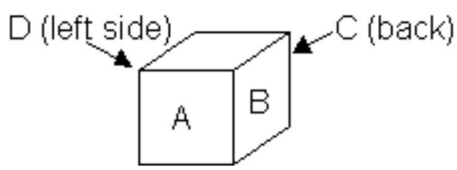
Two ordinary six-sided dice are stacked on top of each other and placed on a table top.



What is the sum of the dots on all the visible faces?

Solution

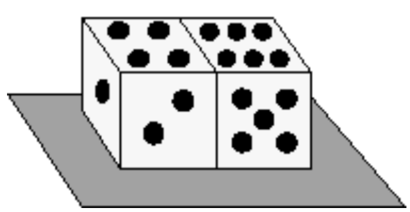
On an ordinary 6-sided die opposite sides add to 7.



As A + C = 7 and B + D = 7, A + B + C + D = 14.

So sum of visible dots will be 2 x 14 + 1 = 29.

If the dice are placed on the table in the following way, what can you determine about the sum of the visible faces?



DIFFERENT SPEEDS

Problem

A tortoise and a hare race against each other.

A hare runs at a constant speed of 36 km per hour for exactly ten seconds and waits for the tortoise to catch up.

If the tortoise takes two hours to move 1 km, how long will it take to catch up?

Solution

36 km per hour is equivalent to 0.6 km per minute, or 600 m per minute, which is the same as 10 m per second. Hence the hare has run a distance of 100 m in 10 s.

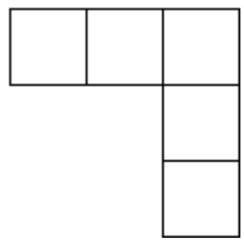
If the tortoise takes 2 hours = 120 minutes to move 1 km = 1000 m, it will take 12 minutes to move 100 m.

How long would the tortoise take to run 1 mile?
Assuming the hare can maintain its speed, how many miles would the hare have run in the same time?

DIFFERENT TOTALS

Problem

Using each of the digits 1 to 5 once, it is possible to place them in the grid so that the row and column have the same total.



How many different totals can this be done with?

Solution

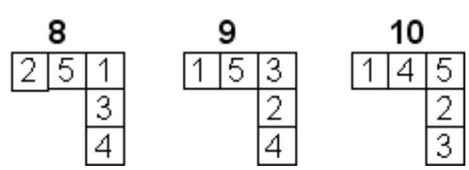
Begin by listing all possible totals using three digits from 1, 2, 3, 4, 5:

	Total
1 2 3	**6**
1 2 4	**7**
1 2 5	**8**
1 3 4	**8**
1 3 5	**9**
1 4 5	**10**
2 3 4	**9**
2 3 5	**10**
2 4 5	**11**
3 4 5	**12**

So it is possible to fill the grid with three different totals:

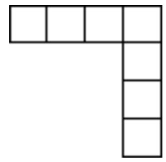
$8 = 2 + 5 + 1 = 1 + 3 + 4$
$9 = 5 + 1 + 3 = 3 + 2 + 4$
$10 = 1 + 4 + 5 = 5 + 2 + 3$

i.e.



Alternatively, note that the common square in the top right corner is always odd... The sum of all five digits, $1 + 2 + 3 + 4 + 5 = 15$. If an even digit is placed in the common square: 15 – 2 = 13 or 15 – 4 = 11, the remaining total will be odd, and this cannot be split between the two legs equally. Hence the only way in which this can be completed is to place an odd digit in the common square: 15 – 1 = 14 (7 on each leg), 15 – 3 = 12 (6 on each leg) and 15 – 5 = 10 (5 on each leg).

What about using the digits 1 to 7 with the following grid?



DIGITAL SQUARE SUM

Problem

The digital sum of the year 2007 is 2 + 0 + 0 + 7 = 9, which is a square number.

How many years during the twenty-first century have a square digital sum?

Solution

The twenty-first century runs from the year 2001 to 2100 inclusive, so with the exception of the year 2100, for which the digital sum is three anyway, the first two digits of all the other years are fixed as 20. Hence for the sum of all the digits to be square the sum of the last two digits number be two less than a perfect square.

As the maximum sum of the last two digits is 18, the greatest sum will be 20. Therefore we can obtain digital sums of 4, 9, or 16, with the last two digits adding to 2, 7, or 14 respectively.

```
 4: 2002, 2011, 2020
 9: 2007, 2016, 2025, 2034, 2043, 2052, 2061, 2070
16: 2059, 2068, 2077, 2086, 2095
```

That is, there are sixteen years during the twenty-first century for which the digital sum is square.

Which century since 1 A.D. has the most square digital sums?

DIGIT PRODUCT

Problem

The product of the digits in the number 126, is $1 \times 2 \times 6 = 12$.

How many other three digit numbers have a product that is equal to 12?

Solution

The three digits must be (126), (134) or (223) and there are fifteen different 3-digit numbers made up of these digits:
126,162,216,261,612,621
134,143,314,341,413,431
223,232,322.

Using three digit numbers, how many ways can you make the product 16?

Investigate different product targets.

What is the first product that you cannot make?

Which product can be made in the most number of ways?

DIGIT SUM

Problem

The digits in the number, 42, add to six. There are exactly six 2-digit numbers with this property: 15, 24, 33, 42, 51, and 60.

How many 3-digit numbers exist for which the sum of the digits is six?

Solution

There are exactly twenty-one 3-digit numbers for which the sum of the digits is six:

105, 150, 501, 510
114, 141, 411
123, 132, 213, 231, 312, 321
204, 240, 402, 420
222
303, 330
600

Can you see how this list was produced?
What about 4-digit numbers with this property?
What about numbers below 10000?
Can you generalise for up to *n*-digit numbers?

Dividing 2 And 3

Problem

How many numbers below one hundred are divisible by both 2 and 3?

Solution

If the number is divisible by both 2 and 3 it will be divisible by 6.
$16 \times 6 = 96$ ($17 \times 6 = 102$)

So there are 16 numbers under 100 that are divisible by both 2 and 3.

How many numbers under 100 are divisible by 2, 3 and 4?

What about being divisible by 2, 3, 4 and 5?

How many numbers under *n* are divisible by the integers 1 to m?

DIVISIBLE CONSECUTIVE SUMS

Problem

By adding four consecutive integers it is possible to make different totals. For example, 16 + 17 + 18 + 19 = 70, which is also divisible by 10.

How many of the numbers under 100 that are divisible by 10 can you make by adding four consecutive integers?

Solution

Taking four consecutive integers, starting from n,

$$n + (n + 1) + (n + 2) + (n + 3) = 4n + 6$$

Clearly for the total ($4n + 6$) to be divisible by 10, $4n$ must end in a 4. So the units digit of n must be 1 or 6,

1 + 2 + 3 + 4 = 10
6 + 7 + 8 + 9 = 30
11 + 12 + 13 + 14 = 50
16 + 17 + 18 + 19 = 70
21 + 22 + 23 + 24 = 90

Giving 5 numbers under 100 that are divisible by 10 and can be made from the sum of four consecutive integers.

How many numbers under 1000, that are divisible by 5, can be made from the sum of four consecutive integers?

EASTER EGGS

Problem

Mr. and Mrs. Roberts have two daughters and three sons. At Easter time every member of the family buys one chocolate Easter egg for each other member. How many Easter eggs will be bought in total?

Solution

There are seven members of the family and each person must buy six Easter eggs.

Therefore, total number of Easter eggs is $7 \times 6 = 42$.

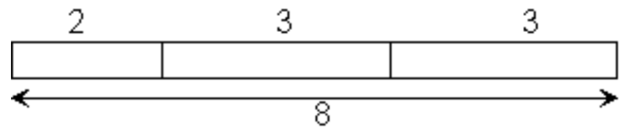
Use this idea to work out the following,

> (i) In a season, ten schools play home and away matches in a hockey league. How many matches that take place in a season.
> (ii) In a netball tournament there are five teams and each team plays each other team once. How many matches will take place?
> (iii) In a form there are thirty-two girls and two are picked at random. How many different pairs can be picked?

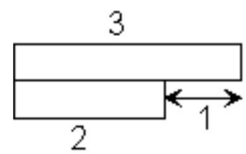
EFFICIENT MEASURES

Problem

Using a 2 metre length and two 3 metre lengths, 8 metres can be measured.



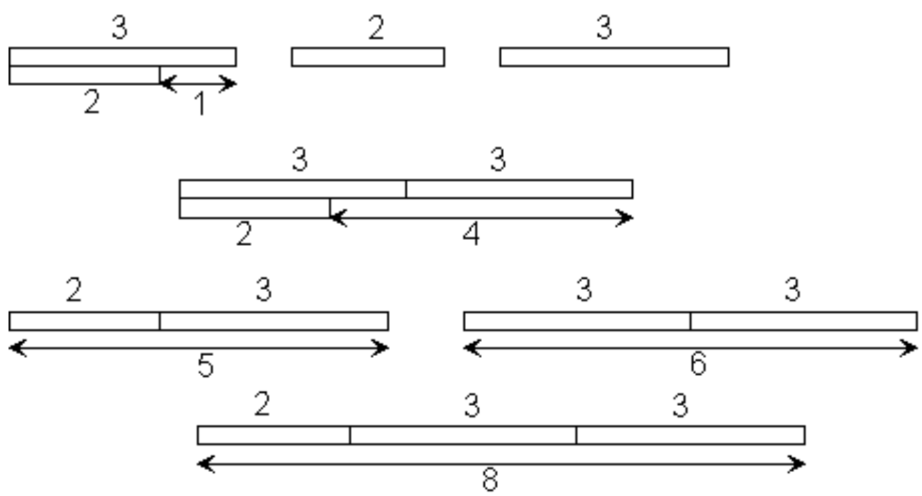
It is possible to measure other lengths, for example, to measure 1 metre.



Which of the lengths from 1 to 8 metres can be measured directly?

Solution

Consider the following diagrams.



The only length not possible is 7 metres.

Even Digits Multiple Of Nine

Problem

Find the smallest multiple of nine containing only even digits.

Solution

If a number is divisible by nine then the sum of digits must be a multiple of nine.

But if all of the digits are even then they must sum to an even multiple of nine, and the smallest even multiple of nine is eighteen.

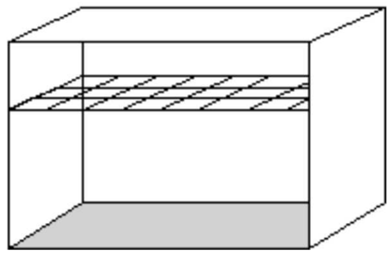
Hence we are looking for the most efficient way to write eighteen as the sum of even digits: 2 + 8 + 8. That is, the smallest multiple of nine containing only even digits is 288.

Find the first five multiples of nine containing only even digits.

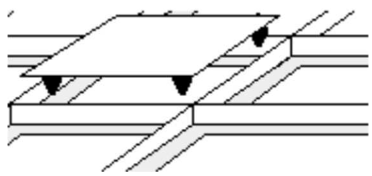
FALSE CEILINGS

Problem

A company that specialises in fitting false ceilings uses a design based on $1m^2$ ceiling tiles arranged in a grid.



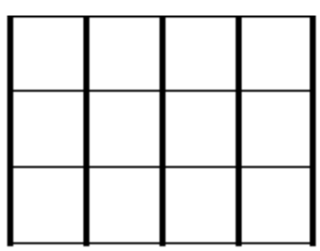
Each tile is supported by metal strips that slot together.



Find the total length of metal strips required to fit this design in a room measuring 3 m by 4 m.

Solution

Consider the 4x3 grid.



The grid consists of 5 vertical strips of 3 units and 4 horizontal strips of 4 units.
So total length of metal strips = $5 \times 3 + 4 \times 4 = 15 + 16 = 31$ m.

What if the room measured m x n?
What if m or n are not integers?

FATHER AND CHILD

Problem

When I was 14 years old my father was 42 years old, which was three times my age. Now he is twice my age, how old am I?

Solution

Let f be the father's age and his child's age be c.

As the father is $42 - 14 = 28$ years older than the child, $f = c + 28$.

At the moment $f = 2c$, hence $2c = c + 28$, and so $c = 28$. That is, the father's child is now 28 years old.

FAULTY SCALES

Problem

A boy stands on a set of faulty scales that record his weight as 52 kg. His sister's weight is recorded as 56 kg and their combined weight is recorded as 111 kg.

What is the true weight of the girl?

Solution

To solve this problem, we shall assume that the scales add or subtract a constant amount from the true weight.

The boy's weight, including the error, is recorded as 52 kg. When his sister steps on to the scales, the increase in weight, recorded by the scales, will be equal to her true weight. Hence, the girl weighs 111 – 52 = 59 kg, and we deduce that the scales subtracts 3 kg from the true weight.

How can we be certain that the scales, instead of adding/subtracting a fixed amount, don't increase/decrease the weight by a particular scaling factor?

FOUR FOURS

Problem

Using exactly four fours and the simple rules of arithmetic, express all of the integers from 1 to 25.

For example, $1 = \frac{4}{4} + 4 - 4$, $2 = \frac{4}{4} + \frac{4}{4}$ and $3 = \frac{4 + 4 + 4}{4}$

Solution

This problem is ambiguous in its wording, "simple rules of arithmetic"; that is, are we to interpret it as only using +, −, ×, and ÷, or will we need to employ a few tricks? Unfortunately it is not possible to obtain all of the integers from 1 to 25 otherwise.

Useful building blocks are square roots and factorial, however, as $\sqrt{4} = 4^{1/2}$ and $4! = 4 \times 3 \times 2 \times 1$, they both use numbers that we do not have. What about the use of place positions, as in 44 or .4? I personally object to this type.

Wherever possible, I have tried to use methods that are as close to the basic rules of arithmetic as possible: add, subtract, multiply and divide, submitting to the use of square root and factorial only where necessary.

$1 = \frac{4}{4} + 4 - 4$	$14 = 4 + 4 + 4 + \sqrt{4}$
$2 = \frac{4}{4} + \frac{4}{4}$	$15 = 4 \times 4 - \frac{4}{4}$
$3 = \frac{4 + 4 + 4}{4}$	$16 = 4 \times 4 + (4 - 4)$
$4 = 4 + 4(4 - 4)$	$17 = 4 \times 4 + \frac{4}{4}$
$5 = \frac{4 \times 4 + 4}{4}$	$18 = 4 \times 4 + 4 - \sqrt{4}$
$6 = \frac{4 + 4}{4} + 4$	$19 = 4! - 4 - \frac{4}{4}$
$7 = 4 + 4 - \frac{4}{4}$	$20 = 4 \times 4 + \sqrt{4} + \sqrt{4}$
$8 = 4 + 4 + (4 - 4)$	$21 = 4! - 4 + \frac{4}{4}$

$9 = 4 + 4 + \frac{4}{4}$

$10 = 4 + 4 + \frac{4}{\sqrt{4}}$

$11 = \frac{4!}{\sqrt{4}} - \frac{4}{4}$

$12 = 4 + 4 + \sqrt{4} + \sqrt{4}$

$13 = \frac{4!}{\sqrt{4}} + \frac{4}{4}$

$22 = 4 \times 4 + 4 + \sqrt{4}$

$23 = 4! - \sqrt{4} + \frac{4}{4}$

$24 = 4 \times 4 + 4 + 4$

$25 = 4! + \sqrt{4} - \frac{4}{4}$

Can you produce all of the integers from 1 to 100?

The integer part function, [x], which has the effect of stripping away the decimal fraction of a number and leaving the integer part, can be used to derive some of the more stubborn numbers. Hence, $[\sqrt{(\sqrt{4})}] = [\sqrt{2}] = [1.41...] = 1$. Similarly, $[4\cos 4] = [3.99...] = 3$ (working in degrees) and $[\log(4)] = [0.602...] = 0$ (working in base 10), and this could be used to eliminate a surplus 4.

Once you"ve given the integers 1 through 100 a good shot, you may like to check out David Wheeler's rather outstanding, *Definitive Four Fours Answer Key*; found at: http://www.dwheeler.com/fourfours/. Wherever he employs contentious functions: for example, square(n)=n^2, which makes use of a 2 in the exponent, he has gone to great lengths to provide an "impurity" index admitting the use of functions which some people may object to using.

Extensions

- What is the first natural number that cannot be derived?
- Which is the first number that cannot be obtained if you are only permitted to use the basic rules of arithmetic (+, −, ×, and ÷)?
- The maximum integer is not a sensible question, as we could apply factorial any finite number of times. But, what is the largest known prime you can produce?
- Using three fours (or threes), which integers can you make?
- Using any number of fours and only addition, subtraction, multiplication and division, produce all of the integers from 1 to 25 in the most efficient way possible.

Notes

Surprisingly it is possible to produce any finite integer using logarithms in a rather ingenious way.

We can see that,

$$\sqrt{4} = 4^{1/2}$$

$\sqrt{\sqrt{4}} = (4^{1/2})^{1/2} = 4^{1/4}$
$\sqrt{\sqrt{\sqrt{4}}} = ((4^{1/2})^{1/2})^{1/2} = 4^{1/8}$ and so on.

By definition,

$\log_4(4^1) = 1$
$\log_4(4^2) = 2$
$\log_4(4^3) = 3$, leading to $\log_4(4^x) = x$.

Therefore,

$\log_4(\sqrt{4}) = 1/2$
$\log_4(\sqrt{\sqrt{4}}) = 1/4$
$\log_4(\sqrt{\sqrt{\sqrt{4}}}) = 1/8$ and so on.

In the same way,

$\log_{1/2}(1/2) = 1$
$\log_{1/2}(1/4) = 2$
$\log_{1/2}(1/8) = 3, \ldots$

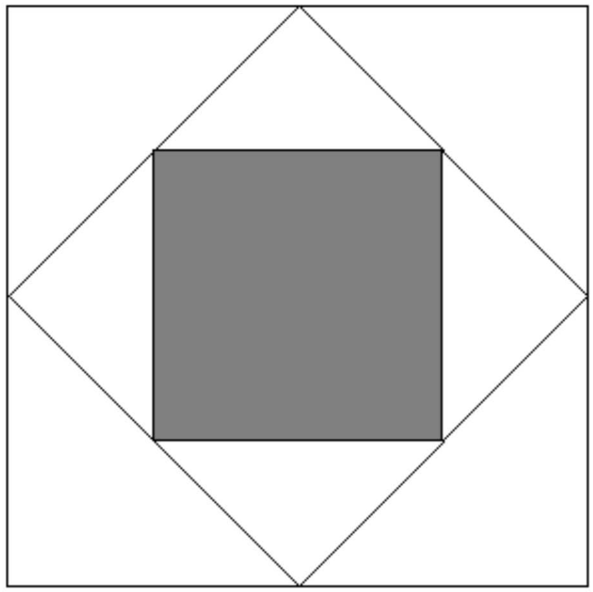
By writing $\log_{1/2}(x)$ as $\log_{(\sqrt{4}/4)}(x)$ we can now produce any finite integer using four fours.

$\log_{(\sqrt{4}/4)}(\log_4(\sqrt{4})) = 1$
$\log_{(\sqrt{4}/4)}(\log_4(\sqrt{\sqrt{4}})) = 2$
$\log_{(\sqrt{4}/4)}(\log_4(\sqrt{\sqrt{\sqrt{4}}})) = 3, \ldots$

Fraction Of A Square

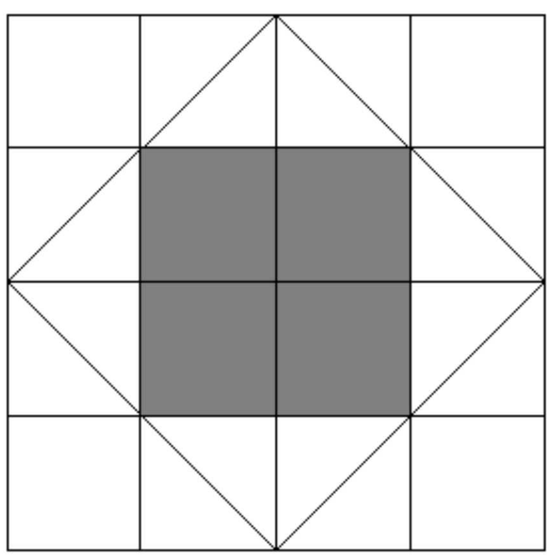
Problem

What fraction of the diagram is shaded?



Solution

Consider the diagram below.



So $^4/_{16} = {^1/_4}$ of the diagram is shaded.

FRACTION PRODUCT

Problem

Work out the exact value of $^1/_2 \times {}^2/_3 \times {}^3/_4 \times {}^4/_5 \times \ldots \times {}^9/_{10}$

Solution

By cancelling adjacent numerators and denominators,
$^1/_{\mathbf{2}} \times {}^{\mathbf{2}}/_3 \times {}^3/_{\mathbf{4}} \times {}^{\mathbf{4}}/ \times \ldots \times /_{\mathbf{8}} \times {}^{\mathbf{8}}/_9 \times {}^9/_{10} = {}^1/_{10}$

What about $^1/_1 \times {}^2/_4 \times {}^3/_9 \times \ldots \times {}^9/_{81} \times {}^{10}/_{100}$?

HEAVY BAGGAGE

Problem

When travelling by aircraft, passengers have a maximum allowable weight for their luggage. They are then charged £10 for every kilogram overweight. If a passenger carrying 40 kg of luggage is charged £50, how much would a passenger carrying 80 kg be charged?

Solution

A £50 charge represents 5 kg overweight (5 × £10 = £50).
So 40 – 5 = 35 kg must be the maximum allowable weight for baggage.

Hence 80 kg is 80 – 35 = 45 kg overweight and the passenger would be charged 45 × £10 = £450.

Can you find a formula to calculate the charge for a given weight of baggage, W kg? What would your formula produce if W = 30 kg?

Hidden Palindrome

Problem

A palindrome is a number which reads the same forwards and backwards. For example, the number 232 is a 3-digit palindrome.

Can you find a square 3-digit palindrome, which is also palindromic when divided by 2?

Solution

If it is palindromic when divided by 2, it must be even. For a square number, n^2, to be even, n must be even.

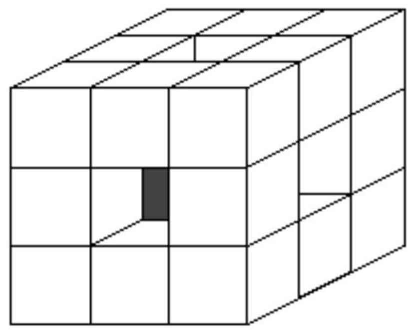
Listing even 3-digit squares:
$10^2 = 100$, $12^2 = 144$, $14^2 = 196$, $16^2 = 256$,
$18^2 = 324$, $20^2 = 400$, $22^2 = 484$, $24^2 = 576$,
$26^2 = 676$, $28^2 = 784$, $30^2 = 900$.

The two candidates are 484 and 676, but as 484/2=242 and 676/2=338, the palindrome we seek is 484.

HOLLOW CUBE

Problem

A cube measuring 3 × 3 × 3 is made up of 27 smaller cubes and has a 1 × 1 square hole pushed right through the centre of each face so that you can see straight through the cube from every side.



The number of small cubes remaining is 20.

If a 5 × 5 × 5 cube has 3 × 3 square holes pushed through the centre of each face, how many smaller cubes would remain?

Solution

A solid cube measuring 5 × 5 × 5 consists of 125 cubes.

A cube measuring 3 × 3 × 3 = 27 cubes is removed from the centre and six faces of 3 × 3 = 9 cubes are removed.

That is, 125 – (27 + 6 × 9) = 125 – (27 + 54) = 125 – 81 = 44 cubes remaining.

What about a 4 × 4 × 4 cube having 2 × 2 squares pushed through each face?

Can you generalise for any sized cube?

Incomplete Fractions

Problem

In the equation,

$$\frac{2}{*} - \frac{*}{5} = \frac{1}{15}$$

The * symbol stands for the same whole number value. Find its value.

Solution

By trial,

$$\frac{2}{3} - \frac{3}{5} = \frac{10}{15} - \frac{9}{15} = \frac{1}{15}$$

Is there a more efficient way to solve this problem?

What about $\frac{*}{9} - \frac{7}{*} = \frac{1}{72}$?

IN THE BEGINNING

Problem

One of the methods used to encrypt a message is to use the message itself. We begin by providing a secret number that is used for the offset of the first letter and is known only to sender and receiver. This starting value is called a seed. For example, if we encode the word, CAT, we begin by finding the alphabetic value of each letter: 3, 1 and 20; and then use the value of the previous letter as an offset, with the first letter using the seed value.

Suppose the seed is 5:

C (3) → (3 + 5[seed] = 8) H
A (1) → (1 + 3[1st letter] = 4) D
T (20) → (20 + 1[2nd letter] = 21) U

So the encoded message is HDU. It should be clear how, using the seed 5, it is possible to decode it easily.

However, your task is to decode the following message without being told the seed:

X B M W W N B G C U A O O M W N W H C V R Y L S W A Q

Good luck!

Solution

It should have been clear that there can be no more than 25 possible seeds to try. Despite the apparent polyalphabetic nature, it is, in fact, a monoalphabetic cipher with a clever twist and is similar in principle to cracking a Caesar-shift cipher. In addition, it seems that there is no easy method of decoding other than trying each combination.

It helps to write the alphabetic values for each of the encoded letters:

```
X  B  M  W  W  N  B  G  C  U  A  O  O  M
24 02 13 23 23 14 02 07 03 21 01 15 15 13
W  N  W  H  C  V  R  Y  L  S  W  A  Q
23 14 23 08 03 22 18 25 12 19 23 01 17
```

By starting with a seed of 1 and working upwards, we can analyse the first, say 5, letters to look for anything promising.

Seed 1:

X (24) → (24 − 1 = 23) W
B (2) → (2 − 23 = 5) E [Note that it was necessary to wrap around the alphabet]
M (13) → (13 − 5 = 8) H
W (23) → (23 − 8 = 15) O
W (23) → (23 − 15 = 8) H
Doesn't look very promising.

Seed 2:
X (24) → (24 − 2 = 22) V
B (2) → (2 − 22 = 6) F
No need to proceed with this one!

Seed 3:
X (24) → (24 − 3 = 21) U
B (2) → (2 − 21 = 7) G
M (13) → (13 − 7 = 6) F
No need to continue with this one either!

Seed 4:
X (24) → (24 − 4 = 20) T
B (2) → (2 − 20 = 8) H
M (13) → (13 − 8 = 5) E
W (23) → (23 − 5 = 18) R
W (23) → (23 − 18 = 5) E
This looks very encouraging and fortunately for us the seed turns out to be 4; revealing the message:

THERE IS NO FUTURE IN TIME TRAVEL

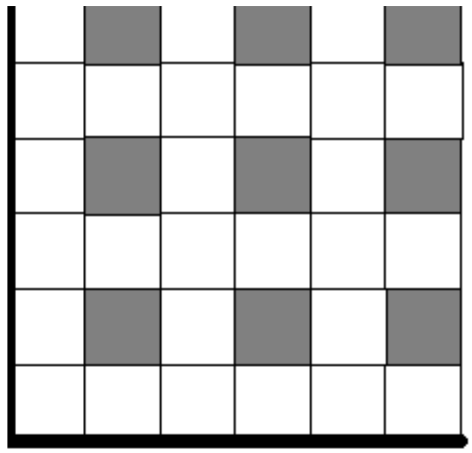
Obviously this trial and improvement method is a little tedious, so once you"ve tried a seed of 1, can you find a more efficient way of decoding with subsequent seeds?

By adapting the method, can you find a more secure system?

KITCHEN FLOOR

Problem

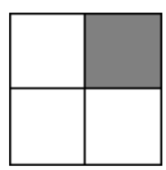
A kitchen floor is to be tiled in the following way,



If the kitchen measures 15 tiles x 25 tiles, how many white tiles will be needed?

Solution

The floor is made up of a 2x2 repeating pattern,



A 15 $\times$ 25 floor will fit 7x12 repeating blocks (14x24 tiles). The "end" lines will be all white.

So there will be 7 $\times$ 12 = 84 grey tiles.

Hence there will be 15 $\times$ 25 – 84 = 374 – 84 = 291 white tiles.

What about a kitchen measuring $m \times n$ tiles?

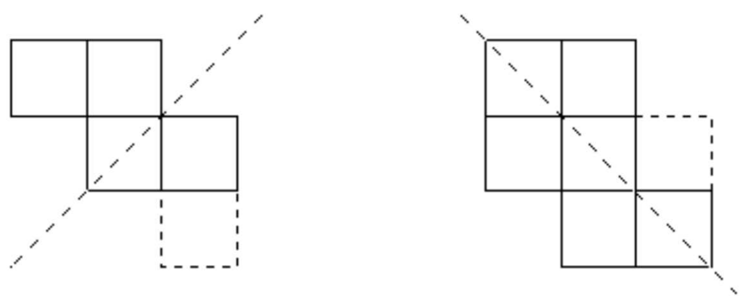
Lines Of Symmetry

Problem

Show how it is possible to make each of the diagrams below have a line of reflective symmetry, by adding a single square to each diagram.

Solution

It can be done as follows:



What is the simplest shape made from unit squares with no reflective symmetry that would require two single squares to be added to make a line of reflection?

Logically Addressed Question

Problem

Truthful Tina and Lying Leah are identical twins whose parents have separated; Tina lives with her mother and Leah lives with her father.

Susan the school secretary, who is known to be particularly clever and a master of logic, wishes to send letters to each of their parents. Although she has two addresses in the school file she doesn't know which address belongs to which parent.

Susan finds one of the girls in the playground but has no idea if she is talking to Tina or Leah. She shows the girl one of the addresses and asks her a question. She thanks the girl and returns to the office able to correctly address both envelopes.

What question could Susan have asked?

Solution

Susan shows the girl the address of one of the parents and asks the question: "Is this your address?"

It it were the father's address then both girls would say, no.

If it were the mother's address then both girls would say, yes.

So from the response Susan is able to logically deduce which address belongs to which parent. However, she has no idea which girl she spoke to.

MAKING PRIMES

Problem

Find two prime numbers that between them use each of the four digits 1, 2, 3 and 5 exactly once. For example, you could make 53 and 21, however, $21 = 3 \times 7$ is composite (non-prime).

Solution

As no 2-digit prime ends in 5, there can be no 2-digit solution. The only combinations would be: 21 (3×7) + 53 and 23 + 51 (3×17), in which case at least one number in the pair is composite.

So the only possibility is that the solution comprises a single digit prime and a 3-digit prime.

The single digit cannot be 2 or 5, as the sum of the remaining digits 135 and 123, respectively, are divisible by 3; hence any combination of them would be divisible by 3.

Therefore, the single digit prime must be 3 and the 3-digit prime could be either 251 or 521.

Meeting Trains

Problem

Two railway stations, P and Q, are 279 miles apart.
A train departs from P at 2pm and travels at a constant speed of 51 mph towards Q.
At 3pm a second train begins a journey from Q towards P at a constant speed of 60 mph.

How far apart are the two trains twenty minutes before they pass each other?

Solution

As twenty minutes is one third of an hour, each train will have travelled a distance of 51/3 = 17 and 60/3 = 20 miles respectively in the twenty minutes before they pass each other.

Hence they must have been 17 + 20 = 37 miles apart.

MOWING THE LAWN

Problem

Matilda's father takes 20 minutes to mow the back garden lawn, but Matilda takes 30 minutes to do the same job. If they worked together, how long would it take to cut the lawn?

Solution

If Matilda takes 30 minutes to mow 1 lawn, she could mow 2 lawns in 60 minutes. Her father, who takes 20 minutes to mow 1 lawn, could mow 3 lawns in 60 minutes. So both of them could mow 5 lawns in 60 minutes.

Hence they would take 12 minutes to mow the lawn between them.

If Matilda took 25 minutes by herself and her father took 15 minutes, how long would it take them if they worked together?
Can you generalise?

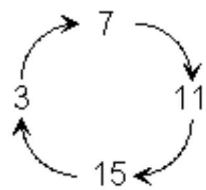
NUMBER CHAIN

Problem

A chain of numbers is made by using the following rule:

Divide by 5 if the number is divisible by 5 otherwise add 4.

For example, starting with the number 7,



It can be seen that the chain has returned to 7.

Which positive integer below 10 will never come back to itself?

Solution

Listing the chain for each starting number:

1 → 5 → 1
2 → 6 → 10 → 2
3 → 7 → 11 → 15 → 3
4 → 8 → 12 → 16 → 20 → 4

Having worked through the starting numbers 1-4, we can see what happens to 5-8.
The only number that has not appeared is 9:

9 → 13 → 17 → 21 → 25 → 5 → 1 → 5

9 is unusual as it works its way into the 1,5 chain, never returning to itself.

Which numbers do not come back to themselves in general?
What happens if you change the rule from +4/÷5?

Hint: Try +3/÷4 and +2/÷3 to start with.

ODD PRODUCT

Problem

It can be seen that the product of the digits in 38, $3 \times 8 = 24$, is even and the product of the digits in 57, $5 \times 7 = 35$, is odd.

How many 2-digit numbers have an odd product?

Solution

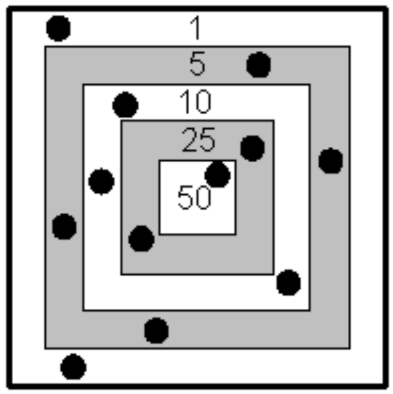
The only way the product of the digits in a 2-digit number can be odd, is if the two digits are odd. The first digit can be 1, 3, 5, 7 and 9, as can be the second digit. Hence there are $5 \times 5 = 25$ combinations of 2-digit numbers comprising two odd digits.

For how many numbers less than one thousand is the product of their digits odd?

ON TARGET

Problem

Two players take it in turns to fire at a target. They get six shots each.



Both players scored exactly the same total. If the second player scored 26 points with her first two shots, which player hit the centre?

Solution

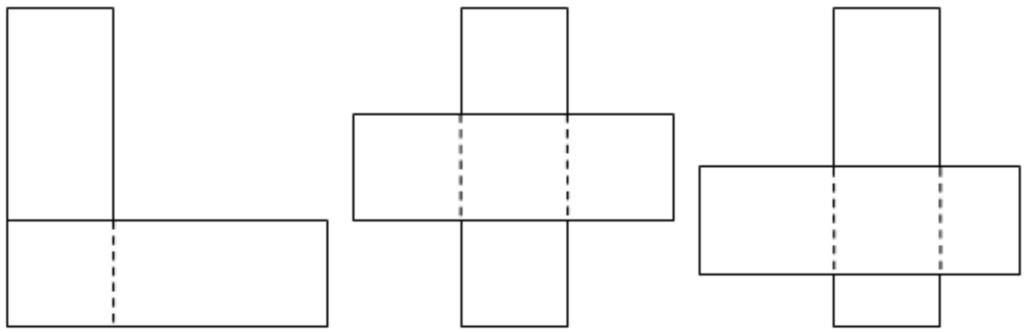
The total points on the board for the twelve shots is 152, so each player must have scored 152 ÷ 2 = 76 points with their six shots. If the second player scored 26 points with her first two shots, she must have scored 76 – 26 = 50 points with her remaining four shots. As both players scored at least one point with each of their shots, it must be player one that hit the centre.

Can you work out precisely how each player scored their 76 points?

OVERLAPPING RECTANGLES

Problem

Pairs of identical rectanglular strips, each measuring 3 by 1, are overlapped in a number of different ways to form three different shapes, shown in the diagram below.

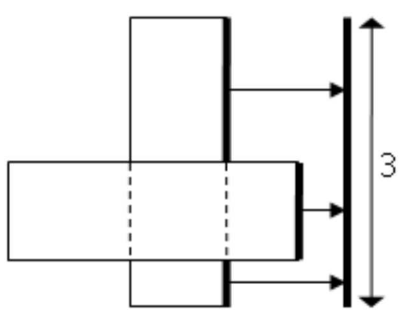


Which shape has the greatest perimeter?

Solution

It should be clear the the L-shape on the left hand side has a perimeter of 12, and assuming symmetry of the cross-shape in the middle, we can see that the perimeter is also 12. But what if the cross-shape is not perfectly symmetrical, and what can we conclude about the shape on the right hand side?

Consider the following diagram.



If the vertical edges on the right hand side of the shape are moved to the right we can see that they add to 3. The same is true of the vertical edges on the left hand side, and the horizontal edges on the top and bottom.

In other words, as long as the rectangles completely overlap the resulting perimeter will always be $4 \times 3 = 12$.

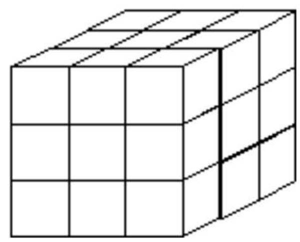
Alternatively, it can be seen that the perimeter of the overlap shape is equal to the perimeter of two rectangles minus the lengths of the four concealed edges, and indeed, this value will be constant. Note that the diagram on the left hand side demonstrates an extreme case: there are two (internal) concealed edges, the other two edges coincide with the external edges and so it is necessary to subtract four concealed edges from the perimeter of the two rectangles.

What if the dimensions of the rectangles were 5 by 2? 9 by 7? *a* by *b*?
What if the rectangles do not overlap at right angles?

PAINTED FACES

Problem

Twenty-seven small red cubes are connected together to make a larger cube that measures 3 x 3 x 3. All of its external faces are painted white and the cube is dismantled.



How many of the small cubes will have exactly two faces painted white?

Solution

The large cube will have:

- One cube with no faces painted (in the centre of the cube).
- Six cubes with one face painted (in the centre of each face).
- Eight cubes with three faces painted (on the corners of the cube).

As $1 + 6 + 8 = 15$, there must be $27 - 15 = 12$ cubes with two faces painted.

Alternatively we can consider the number of edges on the cube, 12, and realise that there is one cube on each edge that has two faces painted.

If the same exercise was performed on a 5 x 5 x 5 cube, how many cubes would have exactly two faces painted?
What about an $n \times n \times n$ cube?

PALINDROMIC DISTANCE

Problem

Julie is a careful driver and never breaks the speed limit. One day as she set off on a journey in her car she noticed that the odometer was displaying 13931 miles, which was palindromic (the same forwards and backwards). After completing the journey, which took two hours, Julie could not believe that it was palindromic once more.

What was her average speed during the journey?

Solution

The next palindrome after 13931 is 14041. Completing 14041 – 13931 = 110 miles in two hours is an average speed of 55 mph. If we considered the next palindromic example, 14141 – 13931 = 210, which means she drove at 105 mph, but we were told that she never breaks the speed limit.

PALINDROMIC YEARS

Problem

A palindromic year is one which reads the same forwards and backwards. There were no other palindromic years between 1991 and 2002, which is a gap of 11 years.

Since year 1 A.D., what was the largest gap between two consecutive palindromic years?

Solution

Changing from 1-digit to 2-digit: 11 – 9 = 2; 2-digit to 3-digit: 101 – 99 = 2; and 3-digit to 4-digit: 1001 – 999 = 2.

That is, the difference is always 2 years.

For 2-digit palindromic years, the gap will always be 11:
E.g. 77 – 66 = 11.

For 3-digit years, the gap will be 10 within the same century or 11 years when a century changes:
E.g. 686 – 676 = 10
E.g. 505 – 494 = 11.

For 4-digit years, the gap is 11 years as the millenium changes otherwise it will be 110 years as centuries change:
E.g. 2002 – 1991 = 11
E.g. 1441 – 1331 = 110.

As there is no other possible cases, the maximum gap between consecutive palindromic years is 110 and it happened 9 times.

In general, what is the largest gap you can have between two consecutive palindromic numbers?

PANDIGITAL MINIMUM DIFFERENCE

Problem

The two 5-digit numbers: 71482 and 30956, are called pandigital, because between them they use all the digits from 0 to 9; their difference is 40526.

What is the minimum difference between two 5-digit pandigitals?

Solution

The challenge is to make the bigger of the two numbers as close as possible above some multiple of ten thousand and the smaller just below. For example, 20345 and 19876 are just above and below 20000; their difference being 469.

With a little thought, it can be seen that the minimum difference is achieved with $50123 - 49876 = 247$.

What about the minimum difference between two 4-digit pandigitals (using the digits 0 to 7)?
What is the greatest sum of two 5-digit pandigitals?

PATHED PATHWAYS

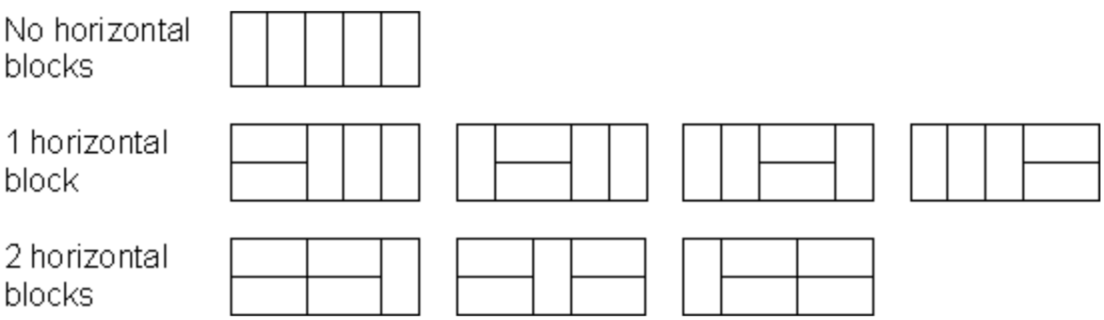
Problem

A pathway measuring 5 m by 2 m is paved with stones measuring 2 m by 1 m. One way in which the pathway could be paved is as follows.

How many different ways can the path be laid?

Solution

The pathways can be paved as follows.



There are 8 ways of paving a 5x2 pathway with 2x1 paving stones.

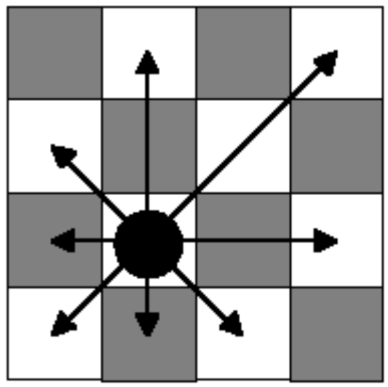
How many ways can you pave a 2 x n pathway using 2x1 stones?
What about different width pathways?

(Hint: Keep the length of the stones equal to the width of the pathway.)

PEACEFUL QUEENS

Problem

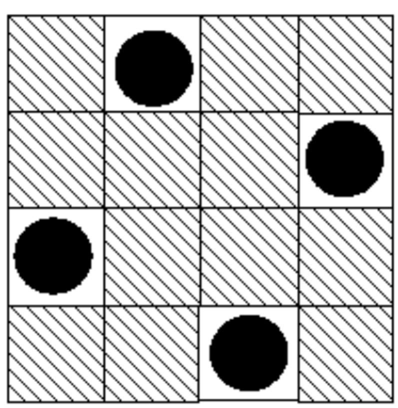
In a game of chess, a queen is able to move any number of squares in an orthogonal direction (up, down, left or right) or diagonally and can capture any piece that blocks its line of movement.



What is the largest number of peaceful queens you can place on a 4x4 chessboard, so that no queen threatens any other queen?

Solution

It is possible to place four peaceful queens on a 4x4 grid.

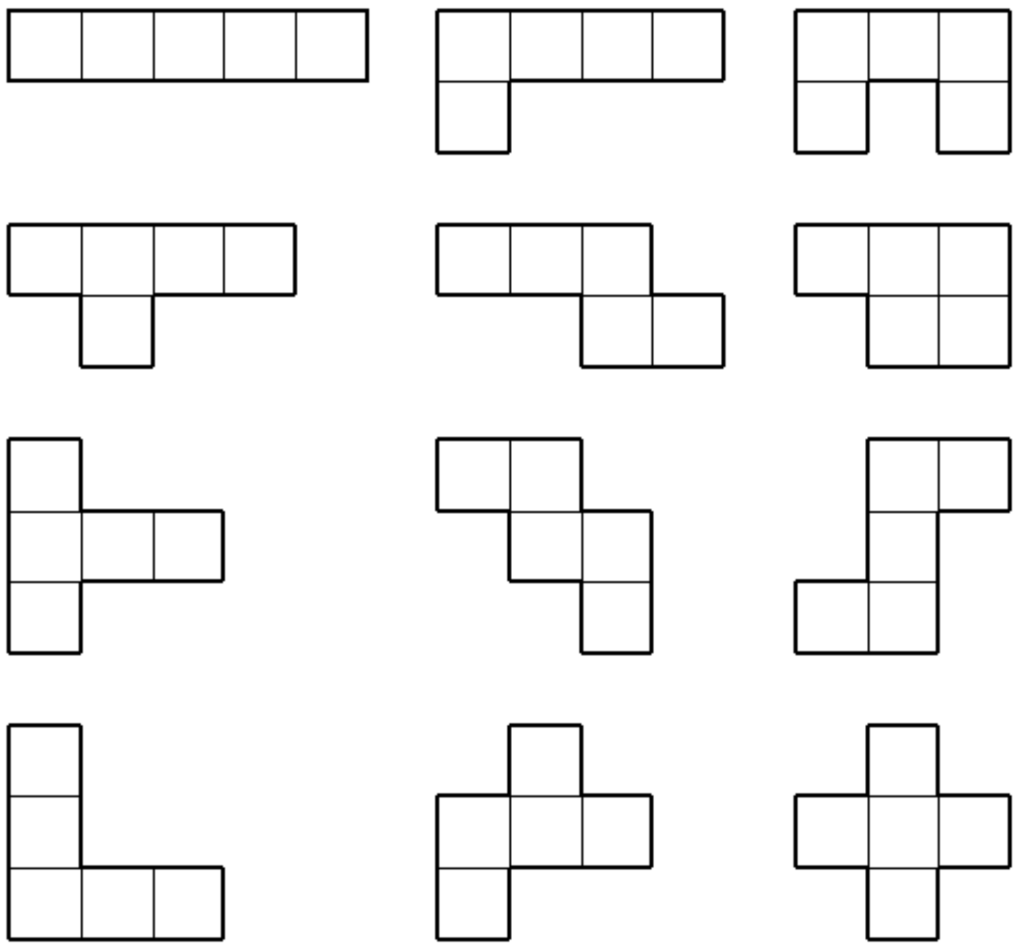


How many peaceful queens can you place on a standard 8x8 chessboard?
What about other pieces, like a castle (rook), a bishop or a knight?

PENTOMINOES

Problem

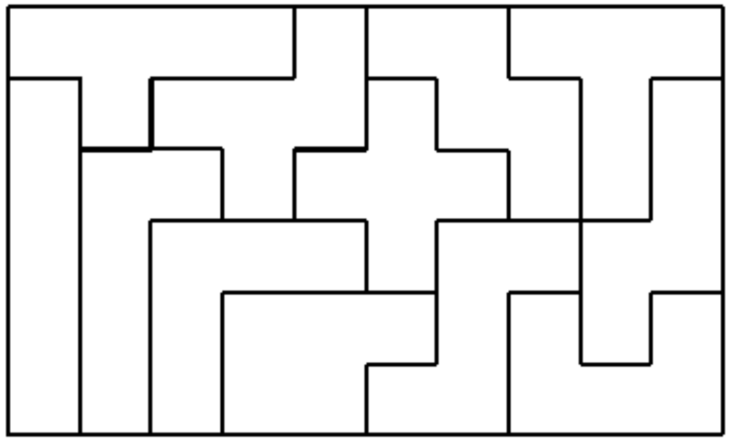
A pentominoe is a shape made up of five unit square and there there are exactly twelve different pentominoes.



Arrange the pentominoes to form a rectangle measuring 10 by 6.

Solution

Incredibly, there are 2339 solutions to this problem. Despite this, it still remains very difficult to find a solution; here is one solution:



Extensions

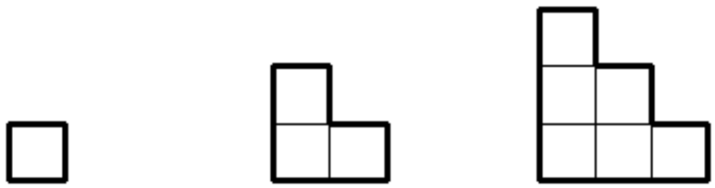
- Arrange the pentominoes to form a 12 by 5 (1010 solutions), 15 by 4 (368 solutions) or a 20 by 3 rectangle (2 solutions).
- By using the set of pentominoes and an additional 2×2 tetrominoe, form a square measuring 8 by 8.
- It is possible to take any pentominoe and, using nine of the remaining eleven pieces, form a figure that is mathematically similar – an enlargement, scale factor 3. Try to create similar figures for each of the twelve pentominoes.

For the definitive guide to dissection problems (and solutions) I would refer you to Stewart Coffin's amazing *The Puzzling World of Polyhedral Dissections*; found at: http://www.johnrausch.com/PuzzlingWorld/. In particular, there is an entire section dedicated to Pentominoe problems: http://www.johnrausch.com/PuzzlingWorld/chap02d.htm.

PERIMETER SEQUENCE

Problem

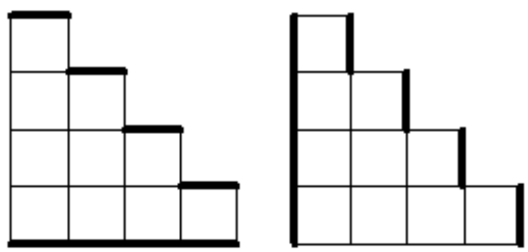
The first three stages of a sequence are shown.



What is the perimeter of the tenth in the sequence?

Solution

By considering the next in the sequence.

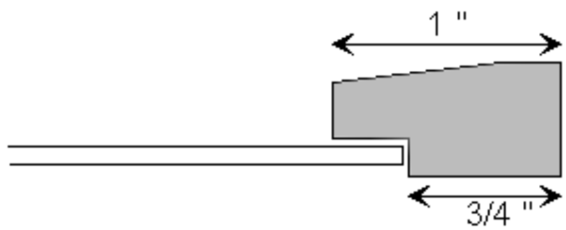


It should be clear that the *n* th in the sequence will be *n* units tall and n units wide, so there will be $2n$ horizontal units and $2n$ vertical units, making the total perimeter $4n$. That is, the perimeter of the tenth in the sequence will be 40 units.

PICTURE FRAME

Problem

A photograph measuring 12" × 16" is to be mounted in a frame in the following way.

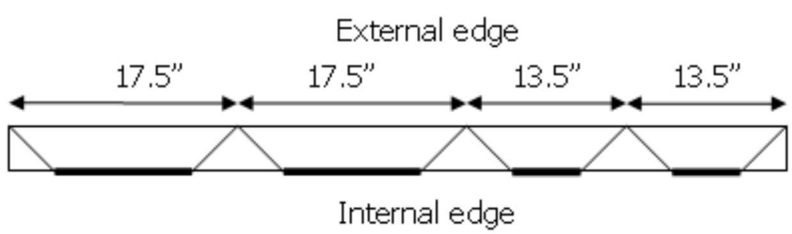


What length of frame material will be needed to construct the frame?

Solution

The finished frame will be 3/4" larger than the photograph at each edge. Hence the external dimensions of the frame will be 13.5" × 17.5".

If the frame material had a simple rectangular cross-section cut we could alternate directions to use less material, but this type of frame has a recess on one side for the picture to slide in to, so it is necessary to make cuts in the following way.



Hence the minimum length required for the photograph will be 17.5 + 17.5 + 13.5 + 13.5 = 62".

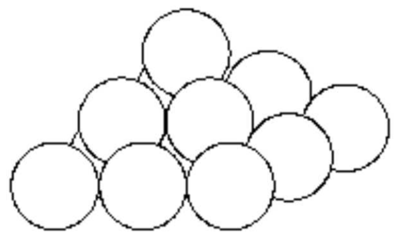
However, when making picture frames it is alway a good idea to allow an additional 1/8" both vertically and horizontally in the recess to give the picture room to be seated properly. Also the width of the saw cut should be taken into account.

By allowing for the additional room and assuming that the saw removes 1/8" of material with each cut, what length of frame material should be used?

PILE OF ORANGES

Problem

A pile of oranges are arranged to make a square based pyramid by having one orange on the top layer, four oranges on the second layer, nine oranges on the third layer, and so on. Such that consecutive layers will have a number of oranges equal to consecutive square numbers: 1, 4, 9, 16, 25, ...



If there were one-thousand oranges in the pile used to make the pyramid not all of them would be needed. How many oranges would be left over?

Solution

$1^2 + 2^2 + 3^2 + \dots + 13^2 = 819$ and $1000 - 819 = 181$ ($14^2 = 196$)

So there will be 181 oranges left over.

Can you find a better way to add together square numbers?

What if there was one million oranges to build a pyramid from?

POSTAGE STAMPS

Problem

A small village post office is selling 1st class stamps and 2nd class stamps in a book costing £1. If a 1st class stamp is 9p and a 2nd class stamp is 7p, how many ways can a £1 book be filled?

Solution

We are attempting to solve the Diophantine equation $9a + 7b = 100$, of which there are two positive solutions:
$1 \times 9 + 13 \times 7 = 100$
$8 \times 9 + 4 \times 7 = 100$

How many solutions are there to the equation $4a + 3b = 100$?
Try different equations and see if you can spot any patterns.

PRODUCT OF ONES

Problem

Thirty-seven is the smallest number we can multiply three by to produce a product consisting entirely of ones, $37 \times 3 = 111$.

What is the smallest number you can multiply seven by to produce a product consisting entirely of ones?

Solution

One method is to divide numbers in the sequence 111, 1111, 11111, ... , searching for an integral quotient.

Leading to $\frac{111111}{7} = 15873$ being the first such value.

Thirty-seven was the smallest number to multiply three to produce a product consisting of ones. What is the next such number?

Can you generalise?

What about for seven?

Product Of Zero

Problem

Take any 2-digit number and multiply the digits together. If this process is continued, all 2-digit numbers will become a single digit number.

For example, 75: $7 \times 5 = 35$, $3 \times 5 = 15$, $1 \times 5 = 5$
68: $6 \times 8 = 48$, $4 \times 8 = 32$, $3 \times 2 = 6$
45: $4 \times 5 = 20$, $2 \times 0 = 0$

How many 2-digit numbers will finish on zero?

Solution

Obviously any number ending with 0 will give zero directly.

10, 20, 30, 40, 50, 60, 70, 80 and 90

But indirectly, numbers like 25 require two steps: $2\times 5 = 10$, $1\times 0 = 0$.

25, 52=>10	45, 54=>20	56, 65=>30	58, 85=>40

Less obviously 69 needs three steps: $6\times 9 = 54$, $5\times 4 = 20$, $2\times 0 = 0$.

55=>55	59, 95=>45	69, 96=>54	78, 87=>56

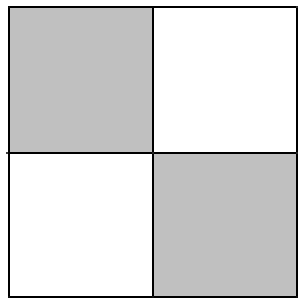
Giving 9 + 8 + 7 = 24 solutions.

Which numbers under 1000 have this property?

Quarter Square

Problem

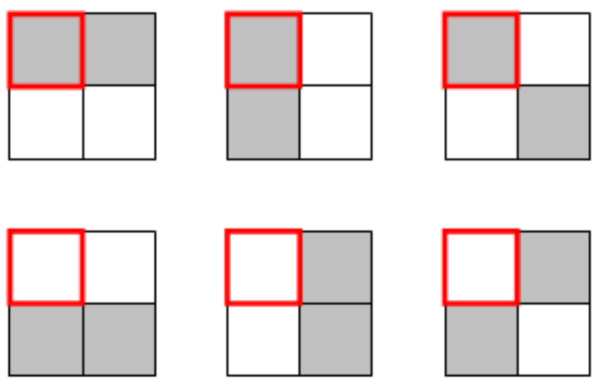
A square is split into four smallers squares and exactly two of these smaller squares are shaded. For example, the top left and bottom right squares could be shaded.



How many distinct ways can exactly two smaller square be shaded?

Solution

It can be seen that there are exactly six ways:



However, we could have arrived at this answer by a quite different method. Let us consider two distinct cases: the top left square is either shaded or unshaded.

(i) If the top left square is shaded then there are three remaining squares that could be shaded (see the top three diagrams).
(ii) If the top left square is unshaded then there are three remaining squares that could also be left unshaded, which means that the other two must be shaded (see the bottom three diagrams).

Therefore there are 3 + 3 = 6 ways of shading exactly two smaller square.

We can use this method to explain why there are six ways of picking two numbers from {1,2,3,4}. We pick 1 and one other number from {2,3,4} or we do not pick 1 and do not pick one other from {2,3,4}.

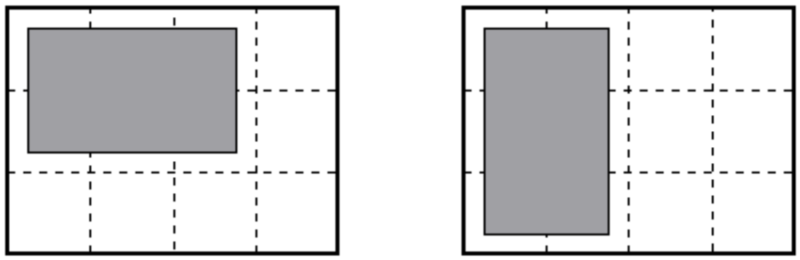
However, when we consider picking two numbers from {1,2,3,4,5} we need to adapt the strategy slightly. We pick 1 and one other number from {2,3,4,5}: 4 ways, or we do not pick 1 and do not pick two from {2,3,4,5}. This second case is equivalent to picking two from {1,2,3,4}: 6 ways. Hence there are 4 + 6 = 10 ways of picking two numbers from {1,2,3,4,5}.

Use the same method to explain why there are 15 ways of picking two from {1,2,3,4,5,6}.
How many ways can you pick two numbers from {1,2,3,4,5,6,7}?

RECTANGULAR ARRANGEMENTS

Problem

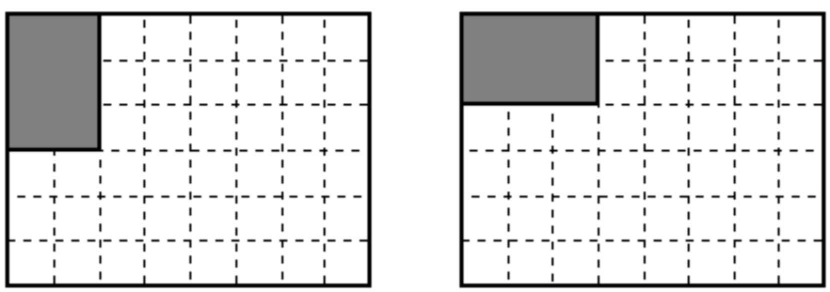
There are seven ways a 3x2 block can be placed inside a rectangle measuring 4x3.



How many ways can a 3x2 block be placed inside a rectangle measuring 8x6?

Solution

Consider the diagrams.



1st diagram: $7 \times 4 = 28$ positions for the block.
2nd diagram: $6 \times 5 = 30$ positions.

Giving 28 + 30 = 58 different positions for 3x2 block on an 8x6 rectangle.

How many positions can a 3x2 block be placed on an m x n rectangle?
What about different sized blocks?

Reverse Difference

Problem

Find the value of the digit *c* in the following calculation.

$$\begin{array}{r} a\ b \\ -\ b\ a \\ \hline c\ 4 \end{array}$$

Solution

Writing $(10a + b) - (10b + a) = 9a - 9b = 9(a - b)$, we can see that the difference must be a multiple of nine.

The only 2-digit multiple of nine ending with the digit 4 is 54, hence $c = 5$.

What about the following subtraction?

$$\begin{array}{r} a\ b \\ -\ b\ a \\ \hline c\ 0 \end{array}$$

REVERSE PRIME

Problem

The number 13 is prime and so too is its reverse, 31. How many two digit primes can you find for which their reverse is also prime?

Solution

It should be clear that the primes cannot have an even number of tens; for example, 23, as its reverse will have even units and will be divisible by 2. Similarly, we cannot have 5 tens.

By considering the list of primes with 1, 3, 7, or 9 tens, we obtain the following list of nine 2-digit primes for which their reverse is also prime:

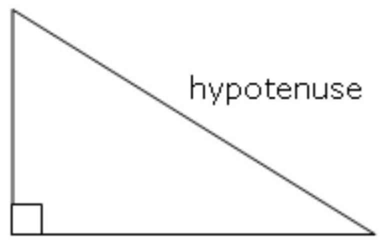
11, 13, 17, 31, 37, 71, 73, 79, and 97.

How many 3-digit primes have this property?

Right Angle Reasoning

Problem

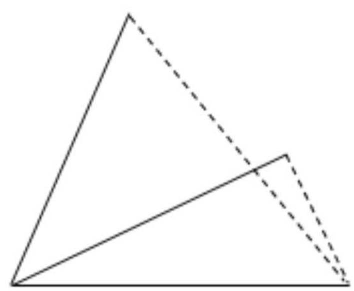
In a right angle triangle the side opposite the right angle is called the hypotenuse.



Explain why the hypotenuse must be the longest side.

Solution

It should be clear that as the size of the angle increases, length of the opposite side also increase.



We know that the sum of angles in any (planar) triangle is 180 degrees. So if one of the angles is a right angle then the other two angles must add to 90 degress between them and neither of them can equal or exceed 90 degrees.

Hence the biggest angle in a right angle triangle must be the right angle and it follows that the longest side is opposite the right angle; that is, the hypotenuse is the longest side.

ROTATIONAL YEARS

Problem

If you write the year using the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, the year 1961 was a special year because it read the same upside-down. How many years since the birth of Christ have had this property?

Solution

First we consider 1-digit and 2-digit years.

1-digit: 1, 8
2-digit: 11, 69, 88, 96

3-digit years include all 2-digit combinations with 0,1 and 8 in-between.

3-digit: 101, 609, 808, 906
111, 619, 818, 916
181, 689, 888, 986

4-digit years are made of 1**1 with all 2-digit (including 00) combinations in-between.

4-digit: 1001, 1111, 1691, 1881, 1961

So there are 23 years which have this property.

Which numbers under one million have this property?

SEEING CLEARLY

Problem

Looking through the shelves in a second hand book shop you stumble upon a book containing mathematical puzzles. What particularly catches your attention is that someone has written a message, in pencil, on the back page of the book:

```
NAELVHESRHAERTGQUWEKWJIATG
HZATMXATNHCSAQRXRZYLIONGGT
ARWBAGTSENRUBKUKFCFGAWLVOQ
```

Can you discover the meaning of the secret message?

Solution

Begin by removing every second letter:

```
N E V E R A R G U E W I T
H A M A N C A R R Y I N G
A W A T E R B U F F A L O
```

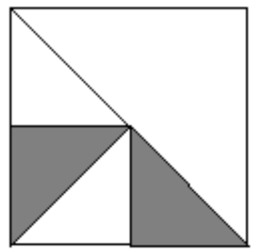
Reading across the rows, reveals the excellent advice:

NEVER ARGUE WITH A MAN CARRYING A WATER BUFFALO

SHADED AREA

Problem

A square measuring 5 cm by 5 cm is dissected in the following way.



What area is shaded?

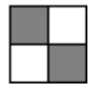
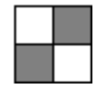
Solution

The two shaded triangles combine to make a small square which is one-quarter the size of the large square.

So area shaded is $\frac{5 \times 5}{4} = 6.25$

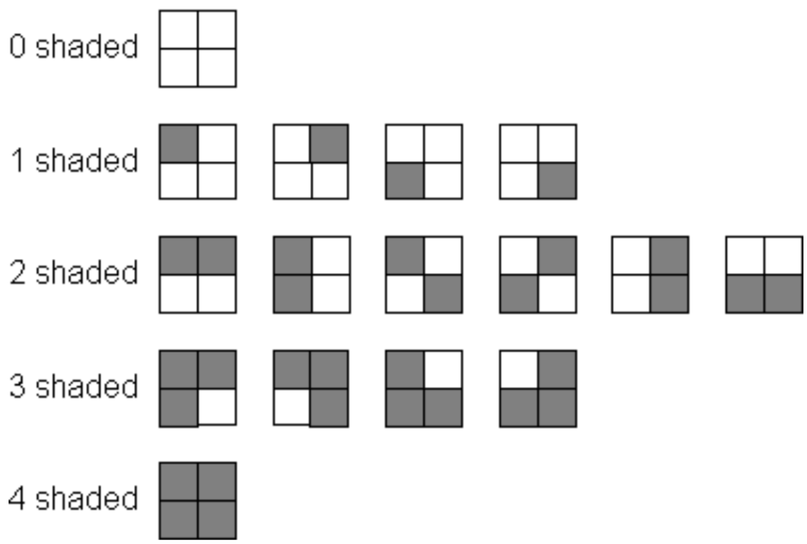
SHADED GRID

Problem

By only shading whole squares and given that  is considered different to , how many different ways can a 2x2 grid be shaded?

Solution

Consider the following diagrams.



Therefore the number of possible ways to shade a 2x2 square is
1 + 4 + 6 + 4 + 1 = 16.

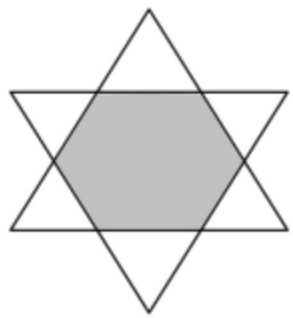
What about other sized grids?

Hint: Consider the total number of squares making up the grid and don't just think about square grids; a square is either shaded or not shaded... ;)
What if you used three colours: white, grey and black?
Can you generalise for any size grid and any number of colours?

SHADED HEXAGON

Problem

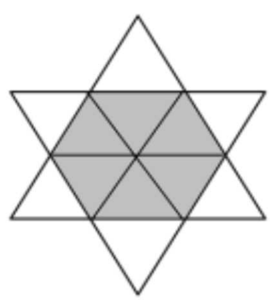
Two congruent equilateral triangles, each with an area equal to 36 cm^2, are placed on top of each other so that they form a regular hexagonal overlap.



Find the area of the hexagon.

Solution

The triangles can be split up as follows:



It can be seen that the shaded area is $^6/_9$ of one triangle, that is $^2/_3$ of 36 = 24 cm^2.

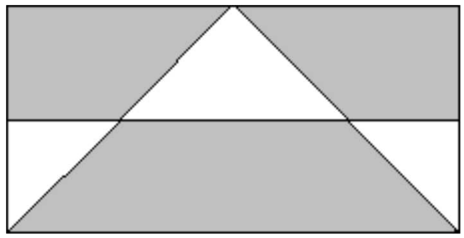
Must the triangles be equilateral?

What will the area of the overlap be if two squares, each with an area of 36 cm^2, are overlapped in the same way to form an octagon?
Is the octagon regular?

SHADED RECTANGLE

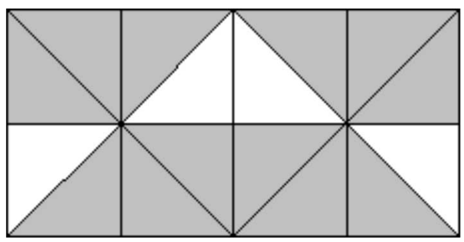
Problem

What fraction of the diagram is shaded?



Solution

We can split the diagram as follows.

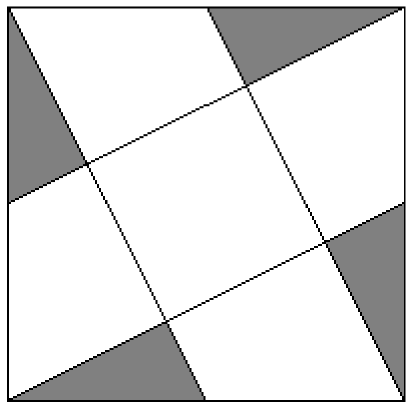


Hence $^{12}/_{16} = {}^{3}/_{4}$ of the diagram is shaded

SHADED SQUARE

Problem

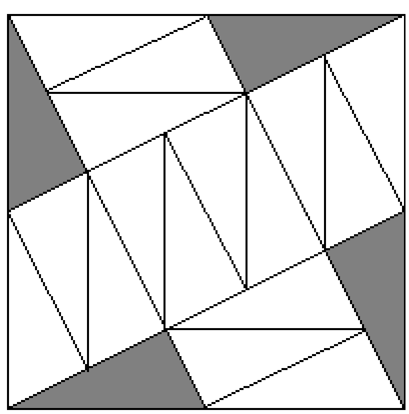
The top left corner of a square is joined to the midpoint of the bottom edge, the midpoint of the top edge is joined to the bottom right corner; the top right corner is joined to the midpoint of the left edge, and the midpoint of the right edge is joined to the bottom left corner.



What fraction of the square is shaded?

Solution

Consider the following diagram.

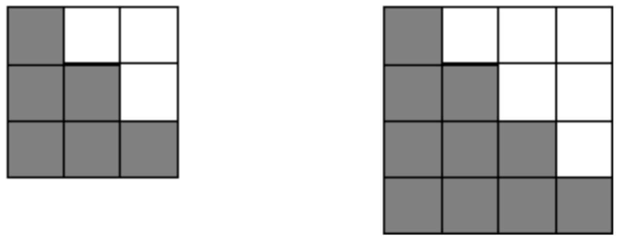


By symmetry it should be clear that each triangle is congruent. Hence 4/20 = 1/5 of the square is shaded.

Shading Pattern

Problem

In the 3x3 grid, $^{6}/_{9}$ is shaded and in the 4x4 grid, $^{10}/_{16}$ is shaded.



If a 20x20 grid was shaded in the same way, what fraction would be shaded?

Solution

By collecting data for the first few grids and choosing denominators carefully.

Size of Grid	Fraction Shaded
2x2	$^{3}/_{4}$
3x3	$^{6}/_{9} = ^{2}/_{3} = ^{4}/_{6}$
4x4	$^{10}/_{16} = ^{5}/_{8}$
5x5	$^{15}/_{25} = ^{3}/_{5} = ^{6}/_{10}$

It can be seen that $^{(n+1)}/_{2n}$ is shaded.

So in a 20x20 grid, $^{21}/_{40}$ is shaded.

Can you prove that $^{(n+1)}/_{2n}$ is shaded on an grid?
(Hint: $1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n + 1)$.)

SIMPLE FRACTIONS

Problem

A fraction whose numerator (top number) is less than its denominator (bottom number) is called a simple fraction.

$$\frac{1}{9} \quad \frac{2}{9} \quad \frac{4}{9} \quad \frac{5}{9} \quad \frac{7}{9} \quad \frac{8}{9}$$

There are six simple fractions involving ninths that cannot be cancelled down. How many simple fractions with a denominator equal to 24 cannot be cancelled down?

Solution

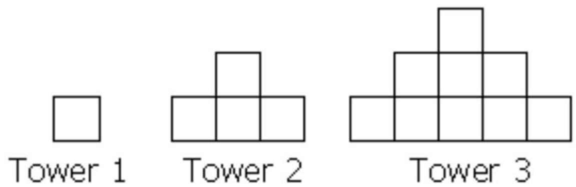
We are looking for the set of integers less than 24 that are co-prime (have no factors in common) with 24: 1, 5, 7, 11, 13, 17, 19 and 23.
So there are 8 simple fractions with a denominator of 24 that cannot be cancelled down.

Investigate difference fractions.
(Hint: look at the numbers that are not co-prime to begin with.)

SKELETON TOWERS

Problem

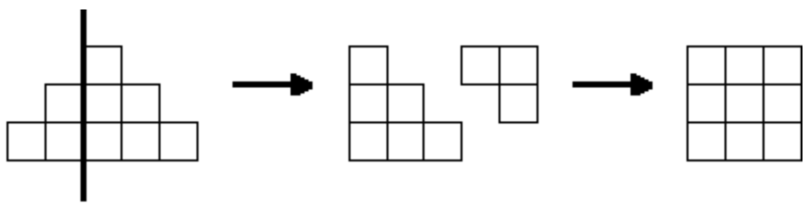
Skeleton towers are generated as follows.



How many blocks would be needed to construct the 100th tower?

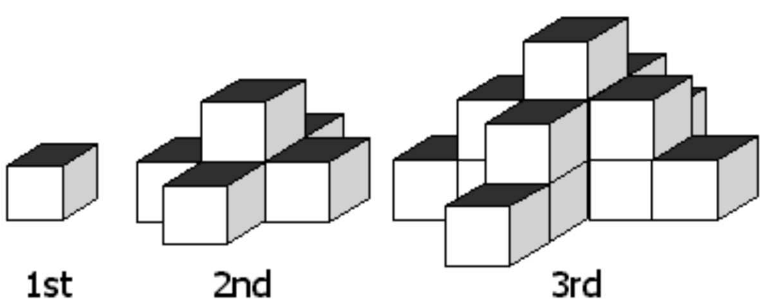
Solution

It is possible to split a tower in the following way.



And so the 100th tower can be made into a 100×100 square, requiring 10,000 blocks.

One way of constructing three dimensional skeleton towers is as follows.



How many blocks would be needed to construct the 100th tower now?

SQUARE AGE

Problem

If you were 35 years old in the year 1225 it would be a very special time mathematically, because $35^2 = 1225$. That is, the square of your age at that moment is equal to the year. This does not happen very often.

Augustus de Morgan, a famous mathematician, was one of those lucky people and in 1864 he wrote:

"At some point in my life the square of my age was the same as the year."

When was he born?

Solution

$42^2=1764$
If he was 42 in 1764, he would have been born in 1764 – 42 = 1722. As he wrote it in 1864, he would have been 1864 – 1722 = 142 years old at the time!!!

$43^2=1849$
If he was 43 in 1849, he would have been born in 1849 – 42 = 1806, making him 1864 – 1806 = 58 years old when he wrote the statement.

$44^2=1936$ (not happened by 1864)
So Augustus De Morgan must have been born in 1806.

It is likely that you know somebody with the same special property with their age. When were they born and when is their special year?

STRAWBERRY MILK

Problem

A strange cult, with a belief that strawberries contain the Elixir of life, have succesfully isolated the gene responsible for producing the flavour in strawberry plants. However, it seems that they have been able to combine this material with bovine DNA to produce a highly infectious virus that affects cattle. Although the virus causes no harm to the cow, it will cause the milk that it produces to be tainted with a strawberry flavour. Once infected there is no cure and all future generations will produce stawberry flavoured milk.

We were able to intercept an encoded message, sent to all members of the cult, regarding a final meeting to discuss the release of the virus. We know that the meeting has been arranged to take place in Athens, Greece. It is imperative that you decode this cipher so that we can uncover the identity of the mysterious leader of this cult and arrange his arrest.

```
LFI TIVZG OVZWVI, WLMZOW NXILMZOW, RMERGVH ZOO NVNYVIH GL GSV
URMZO NVVGRMT RM ZGSVMH, TIVVXV.
```

Solution

A little analysis of the punctuation should lead us to the suspicion that the message ends with the location of the meeting: ATHENS, GREECE. We also note the coincidence of the number of letters and the double E in GREECE. If this is the case we are dealing with a monoalphabetic substitution code and we could then set about replacing the known letters:

```
LFI TIVZG OVZWVI, WLMZOW NXILMZOW, RMERGVH ZOO
??R GREAT ?EA?ER, ??NAL? ?C?ONAL?, ?N??TE? A??

NVNYVIH GL GSV URMZO NVVGRMT RM ZGSVMH, TIVVXV.
?E??ERS T? THE ??NA? ?EET?NG ?N ATHENS, GREECE.
```

After this, it takes little effort to establish the missing letters:

```
OUR GREAT LEADER, DONALD MCRONALD, INVITES ALL MEMBERS TO THE
FINAL MEETING IN ATHENS, GREECE.
```

Although not based on the Caesar cipher (moving each letter by a fixed amount), the system employed is not based a random translation table. Can you discover the rule?

If you write out a table of known letters:

```
A B C D E F G H I J K L M N O P Q R S T U V W X Y Z
```

Z ? ? W V U T S R ? ? O N M L ? ? I H G F E ? ? ? ?

You should be able to understand, XLWV YIVZPVMT.

STRING FRACTIONS

Problem

A piece of string measures 2/3 m. Without the use of any other measuring tools, how would you use the string to measure 1/2 m?

Solution

Begin by folding the string in half along its length; this will be 1/3 m.
Fold the 1/3 m length in half, giving 1/6 m.
Finally fold the 1/6 m length back along the length of the string.
The remaining length will be 2/3 – 1/6 = 1/2 m.

What if you wanted to measure 1/4 m using the same piece of string?
What about 1/5 m?

Sum Digital Sum And Product

Problem

The number 49 is unusual in that $4 \times 9 + (4+9) = 36 + 13 = 49$. In other words, if we add the product and sum of the digits we get the number itself.

Find all 2-digit solutions.

Solution

Let the number, $N = 10a+b = ab + a+b$.

$\therefore 9a = ab$, $9a - ab = 0$, $a(9 - b) = 0 \Rightarrow a = 0$ or $b = 9$.

But as N is a 2-digit number, $a \neq 0$.

Hence the 2-digit solutions are: 19, 29, 39, ..., 99.

What about 3-digit solutions?

Sum Of Three

Problem

Three different positive integers add to make sixteen. The two smallest numbers add to make the biggest number. How many different solutions can you find?

Solution

Let the integers be a, b and c such that $a < b < c$.

Therefore $a + b + c = 16$.

But $a + b = c$, so $c + c = 2c = 16$, hence $c = 8$.

So solutions are: (1,7,8), (2,6,8) and (3,5,8)

System Upgrade

Problem

During an automated daily task a computer spends 20% of the time importing the raw data from thousands of text files, 70% of the time processing it, and 10% of the time exporting the results to spreadsheet files and archiving the data.

To help speed-up the task a new chip is purchased that will double the speed of the processing task and the newly purchased hard drives allow both import and export times to be reduced by 30%.

How much time will be saved overall and what percentage of the time will be spent doing each of importing, processing, and exporting the data?

Solution

When we are not given actual times it is often convenient to work with an initial value of 100 so that final percentage changes can be compared more easily.

On the old system, the computer took 70 units of time to complete the processing part of the task; the new chip doubles the processing speed, so it will now complete in 35 units of time.

It took 30 units of time to import and export the data. But because the new drives reduce import and export time by 30%, it will complete in 70% of the previous time. And as 10% of 30 is 3, it will now take 21 units of time.

Hence the new system will take 35 + 21 = 56 units of time to complete the task, which represents an overall reduction of 44% in the time taken to complete the task.

After the upgrade, the ratio of times to complete each part of the task is,

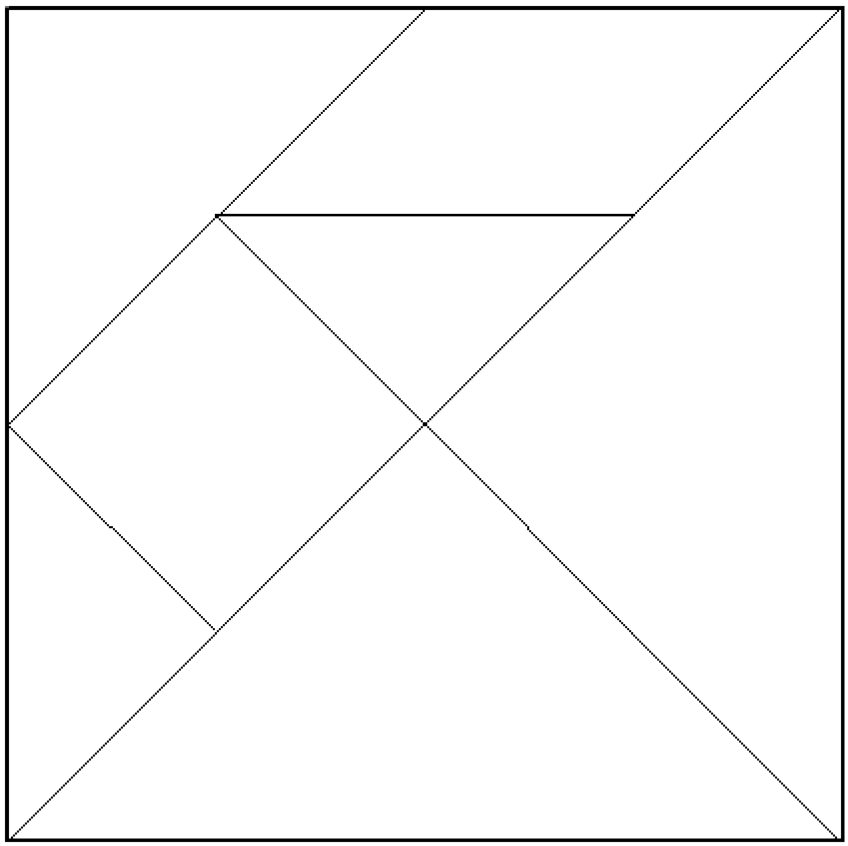
$$\text{import} : \text{process} : \text{export} = 14 : 35 : 7 = 2 : 5 : 1$$

That is, the computer spends 2 + 5 + 1 = 8 units of time to complete the task of which it spends 2/8 = 25% of the time importing data, 5/8 = 62.5% of the time processing data, and 1/8 = 12.5% of the time exporting data.

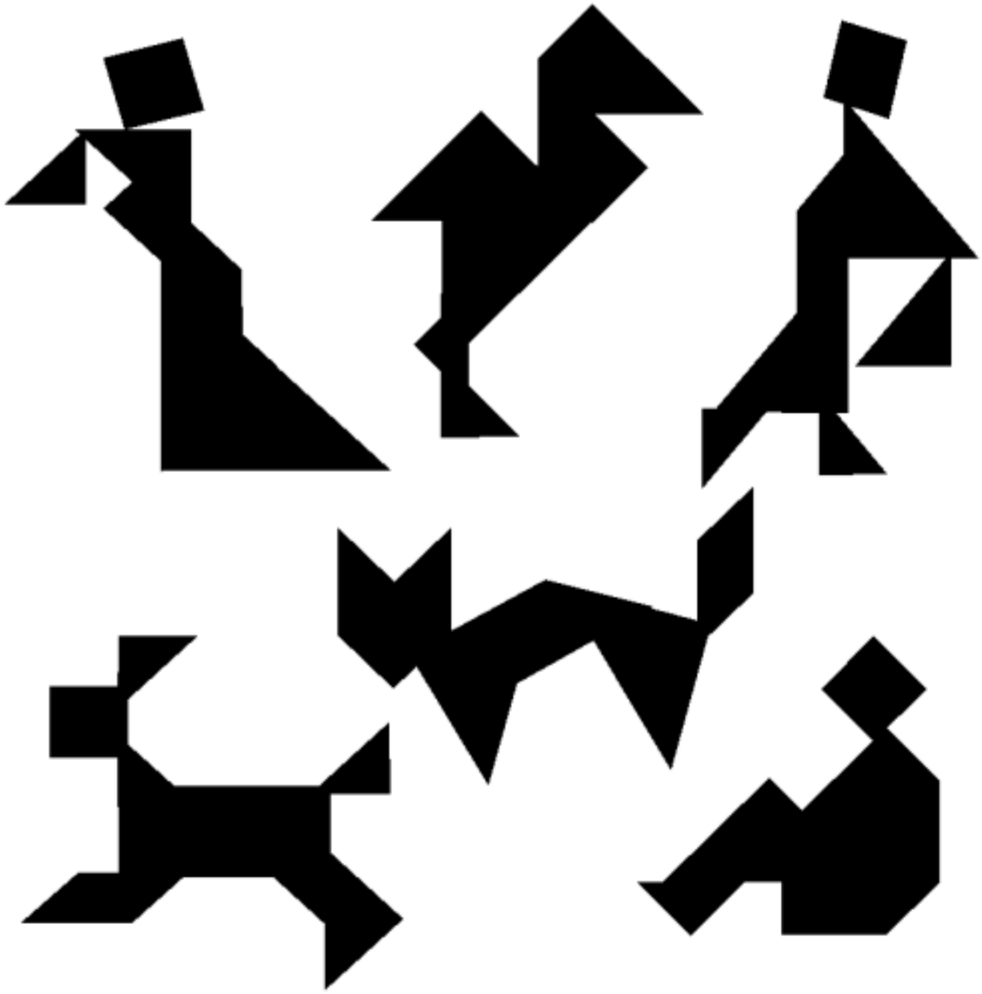
TANGRAMS

Problem

A traditional Chinese tangram consists of a square split up to form seven shapes.

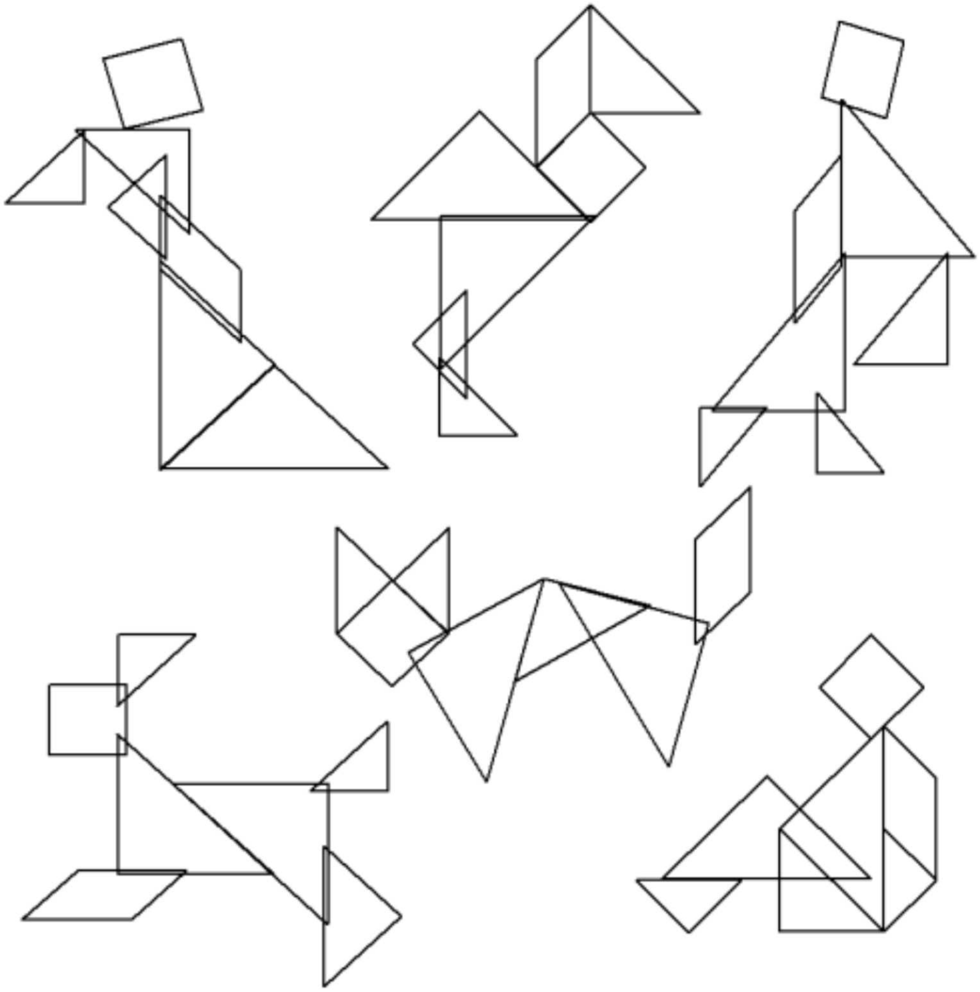


By rearranging ALL seven pieces see how many of the following pictures you can make.



Solution

Each of the figures can be formed in the following way:

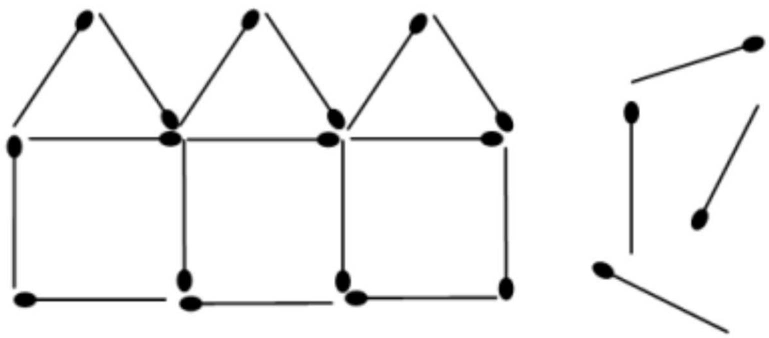


Investigate to see what other pictures can you make.

TERRACED HOUSES

Problem

Using the following pattern, a maximum of three "terraced houses" can be made from a pile of twenty matchsticks.



As only sixteen matchsticks were required there would be four left.

Using one-thousand matchsticks and the same design to make the maximum number of "terraced house" possible, how many matchsticks will be left over?

Solution

To build 1, 2, 3, ... , n "terraced houses", requires 6, 11, 16, ... , $5n+1$ matchsticks.

Solving $5n+1 = 1000$, $5n = 999 \Rightarrow n = 199.8$; that is, 199 complete houses can be made. As this requires $5\times199+1=996$ matchsticks, there will be $1000-996 = 4$ matchsticks left over.

If the number of matchsticks used is a multiple of 10, how many matchsticks will be left over?
Can you prove this?

TEST AVERAGE

Problem

A student has an average score of 85% after completing four tests.

What is the lowest possible percentage score in any one of the tests?

Solution

If the average score after four tests is 85, the total of all four test scores must be $85 \times 4 = 340$. As the maximum score in any one test would be 100, we shall assume that they scored 100 in three tests, making a total of 300.

Hence the minimum score in any one test could be $340 - 300 = 40\%$.

What would the student need to score on the next test to take the average to 90%?

The Age Of Her Life

Problem

Sarah started school at the age of five. She spent one quarter of her life being educated, and went straight into work. After working for one half of her life, she lived for fourteen happy years after retiring.

How old was she when she retired?

Solution

As Sarah spent $1/4 + 1/2 = 3/4$ of her life between school and work, the five years before school and fourteen years after retiring must be $1/4$ of her life.

That is, she lived for $4 \times 19 = 76$ years, and so she retired when she was $76 - 14 = 62$ years old.

To Catch A Liar

Problem

In talking to three children you ask A if they always tell lies. Although A fully understands you she answers in a language that only B understands.

B says that A just denied being a liar.

C says that although she doesn't know what A said, she is a liar and cannot be trusted.

Given that each of the children will always lie or always tell the truth, how many liars are there?

Solution

It should be clear that a compulsive liar cannot admit to being a liar, so A would deny being a liar whether or not they always lied or always told the truth. In which case B told the truth about what A said and we establish that B is always truthful.

If A is a liar then C told the truth, but if A is not a liar then C lied.

Although it is impossible to establish whether or not A is a liar or a truth teller, we can be certain that only one of A or C is a liar and the other must always tell the truth.

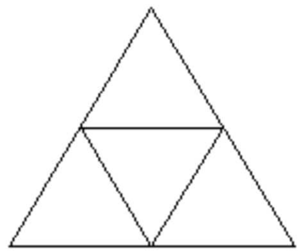
Hence there is exactly one liar amongst the three children.

What if instead C said that A always tells the truth and a fourth child, D, says that B is the only liar out of the three of them?

TRIANGLE DISSECTION

Problem

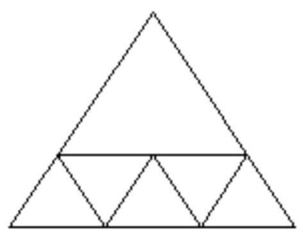
It is possible to split up an equilateral triangle to make four smaller equilateral triangles.



How would you split up an equilateral triangle to make six smaller equilateral triangles, that need not all be the same size?

Solution

It can be done as follows.



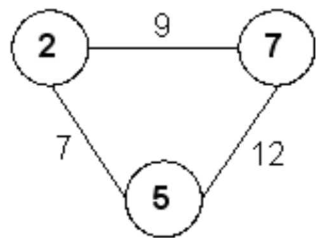
Can you split the triangle to form two smaller equilateral triangles?

What number of smaller equilateral triangles can you make and which ones are not possible?

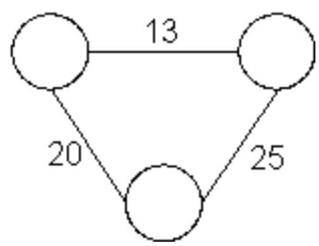
TRIANGULAR ARITHMETIC

Problem

In the diagram below, the number on the connecting line is the sum of the two numbers in the circles.

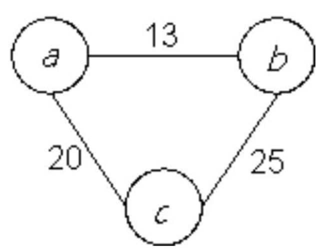


What numbers are missing from the circles in the diagram below?



Solution

By labelling each vertex as follows.



$a + b = 13$, $b + c = 25$ and $a + c = 20$.

Adding all three equations, $2a + 2b + 2c = 58 \Rightarrow a + b + c = 29$

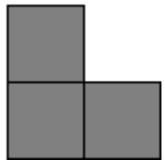
As $a + b = 13$, we get $13 + c = 29 \Rightarrow c = 16$

Similarly, $b + 20 = 29 \Rightarrow b = 9$ and $a + 25 = 29 \Rightarrow a = 4$

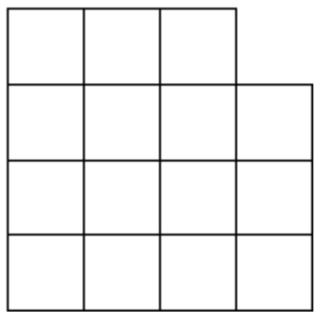
TRIOMINOES

Problem

A triomino is an L-shaped figure made up of three unit squares.

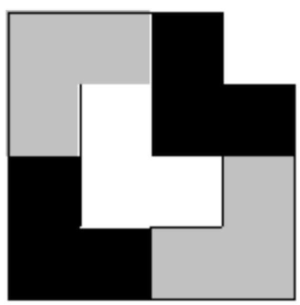


Show how it is possible to cover the 4×4 board with five triominoes.



Solution

The grid can be filled in the following way:



Prove that is it always possible to cover a grid measuring 8×8, 16×16, 32×32, ... , $2^n \times 2^n$, with triominoes always leaving one square.

TWO-DIGIT SUM AND PRODUCT

Problem

If you multiply together the digits of the number 42, $4 \times 2 = 8$, but if you add the digits together, $4 + 2 = 6$. For the number 31, the product of the digits, $3 \times 1 = 3$ and the sum of the digits, $3 + 1 = 4$.

Can you find a two-digit number for which the product of its digits is the same as the sum of its digits?

Solution

Given the 2-digit number. (ab), we are solving: $a + b = ab$.

Therefore $ab - b = a$, $b(a - 1) = a$, giving $b = a/(a - 1)$.

Considering the possible values of the digit, a: $b = 2/1, 3/2, 4/3, \ldots, 9/8$ and the only integer solution is 2/1; that is, $a = 2$ and $b = 2$.

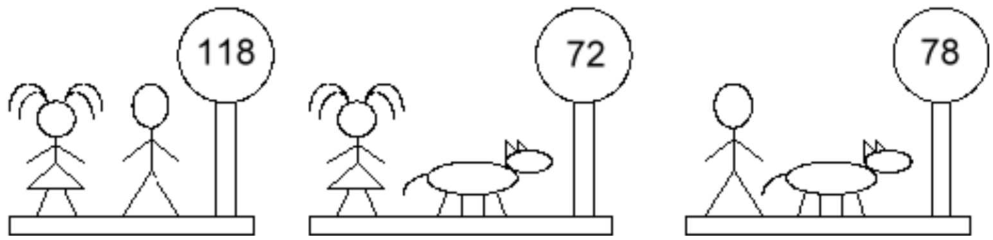
Hence there is only one two-digit solution, 22.

For which 2-digit numbers do the product of their digits exceed their sum?
Can you find any 3-digit number for which the sum of the digits is equal to the product of its digits?
What about n-digit numbers?

WEIGHING SCALES

Problem

A set of weighing scales measures weight in kilograms. A boy, a girl, and a dog stand on the scales in three different ways.



How heavy would the boy, girl, and dog weigh together?

Solution

Let the weight of the boy, girl, and dog be b, g and d respectively.

$b + g = 118$, $g + d = 72$, $b + d = 78$

Adding all three equations: $2b + 2g + 2d = 268$.

Hence the weight of all three, $b + g + d = 134$ kg.

How heavy would each be individually?