3 Star Problems

ABSOLUTE SUM

Problem

For all real numbers, $|x|$ is defined as the absolute value of x; for example $|4.2| = 4.2$ and $|-7| = 7$.

Given that x and y are integer, how many different solutions does the equation $|x| + 2|y| = 100$ have?

Solution

If $|y| > 50$, $2|y| > 100$, and $|x|$ would need to be negative, so $-50 \le y \le 50$.

If $y = \pm 50$, then $2|y| = 100 \Rightarrow x = 0$; that is, one solution for $y = -50$ and one solution for $y = 50$.

But for $-49 \le y \le 49$ there will be two values of x for each value of y; for example, if $y = -20$, $2|y| = 40$, $|x| = 60 \Rightarrow x = \pm 60$.

Therefore from -49 to 49 there are 49 (negative) + 1 (zero) + 49 (positive) = 99 values of y, each of which has two solutions.

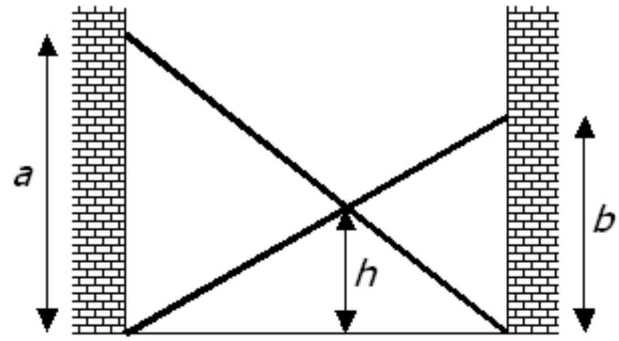
Hence there are $2\times99 + 2 = 200$ distinct solutions to the equation.

If k is a positive integer (odd or even), investigate the number of solutions of the equation $|x| + 2|y| = k$.
What about the equation $|x| + 3|y| = k$?
If the set of solutions are plotted as points, what shape is made?
What if x and y are not restricted to the set of integers?
Investigate the graph $|x|/a + |y|/b = 1$.

ALLEY LADDERS

Problem

Two ladders are placed on opposite diagonals in an alley such that one ladder reaches *a* units up one wall, the other ladder reaches *b* units up the opposite wall and they intersect *h* units above the ground.

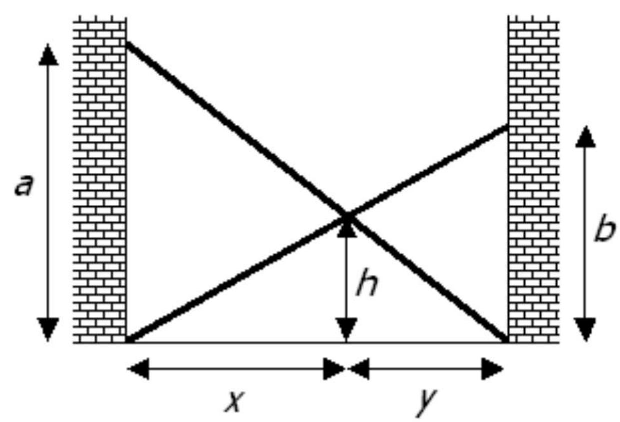


Prove the following result.

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{h}$$

Solution

Consider the following diagram:



By similar triangles:

$$\frac{x+y}{a} = \frac{y}{h} \text{ and } \frac{x+y}{b} = \frac{x}{h}$$

Adding equations:

$$\frac{x + y}{a} + \frac{x + y}{b} = \frac{x + y}{h}$$

Dividing by ($x + y$):

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{h}$$

ALTERNATING SIGN SQUARE SUM

Problem

It can be seen that $6^2 - 5^2 + 4^2 - 3^2 + 2^2 - 1^2 = 21 = T_6$, the sixth triangle number.

Prove that the nth triangle number, $T_n = n^2 - (n-1)^2 + (n-2)^2 - \dots 1^2$.

Solution

The difference between two consecutive squares, $k^2 - (k-1)^2 = 2k - 1$.

Let us consider even n, and group them into $n/2$ pairs:

$$
\begin{aligned}
S &= (n^2-(n-1)^2) + \dots + (6^2-5^2) + (4^2-3^2) + (2^2-1^2) \\
&= 2n-1 + \dots + 11 + 7 + 3 \\
&= 3 + \dots + 2n-5 + 2n-1 \\
\therefore 2S &= (n/2)(2n + 2) \\
&= n(n + 1) \\
\therefore S &= n(n + 1)/2
\end{aligned}
$$

When n is odd we have $(n-1)/2$ pairs plus one odd term:

$$
\begin{aligned}
S &= (n^2-(n-1)^2) + \dots + (5^2-4^2) + (3^2-2^2) + 1^2 \\
&= (2n-1 + \dots + 9 + 5) + 1 \\
\therefore 2S &= ((n-1)/2)(2n + 4) + 2 \\
&= (n-1)(n + 2) + 2 \\
&= n^2 + n \\
&= n(n + 1) \\
\therefore S &= n(n + 1)/2
\end{aligned}
$$

ANONYMOUS AUTHOR

Problem

At school, Bill Speareshake was always the class dunce, but in recent years he has met great success as a poet and playwright. Many people believe that his equally recent involvement with people in high places is a mere coincidence, but others suspect that the real genius behind his works is none other than the esoteric figure Crispin Bacon. In fact, the skeptics believe that his latest masterpiece, in the form of a short poem, contains the identity of the true author.

> a doZen, A grOsS, And a SCorE
> pLUS 3 TIMES ThE squArE RooT of 4
> DiViDED by 7
> PLUS 5 Times 11
> is 9 sqUareD, AnD NOT a BIT more

Can you solve the mystery?

Solution

The clue to solving this was in the pretext, as it is based on one of Francis Bacon's famous cipher systems. We begin by splitting the text into blocks of five and removing spaces and punctuation. We then replace normal characters with zeroes and large characters with ones, to form binary strings. A brief introduction to binary strings can be found in the Code Breaking Library. Alternatively you could do a search for more in-depth information on decimal/binary conversion on the internet, decimal binary conversion (Google search link). The binary strings are converted into decimal numbers and these represent the alphabetical position of the letters that form the secret message.

adoZe	00010	2	B
nAgrO	01001	9	I
sSAnd	01100	12	L
aSCor	01100	12	L
EpLUS	10111	23	W
3TIME	01111	15	O
SThEs	11010	26	Z

quArE	00101	5	E
RooTo	10010	18	R
f4DiV	00101	5	E
iDEDb	01110	14	N
y7PLU	01111	15	O
S5Tim	10100	20	T
es11i	00010	2	B
s9sqU	00001	1	A
areDA	00011	3	C
nDNOT	01111	15	O
aBITm ore	01110	14	N

The secret message reading as, BILL WOZ ERE NOT BACON.

Some people believe that Lord Francis Bacon was the real author of the works of William Shakespeare and the debate will continue, I'm sure, without resolve. There are compelling arguments on both sides, with each camp claiming that only a fool would believe the contrary. A search on the internet with the keywords: baconian shakespeare (Google search link), will produce numerous documents discussing this issue.

A RADICAL PROOF

Problem

The radical of n, rad(n), is the product of distinct prime factors of n. For example, $504 = 2^3 \times 3^2 \times 7$, so rad(504) $= 2 \times 3 \times 7 = 42$.

Given any triplet of relatively prime positive integers (a, b, c) for which $a + b = c$, and with $a < b < c$, it is conjectured, but not yet proved, that the largest element of the triplet, $c < \text{rad}(abc)^2$.

Assuming that this conjecture is true, prove that $x^n + y^n = z^n$ has no integer solutions for $n \geq 6$.

Solution

Let $a = x^n$, $b = y^n$, and $c = z^n$.

If x, y, and z were coprime then the maximum radical of abc would be xyz.

Therefore $\text{rad}(abc) \leq xyz$, but as z is the greatest in value, it follows that $\text{rad}(abc) < z^3$.

By the conjecture, $c < \text{rad}(abc)^2 < (z^3)^2$; that is, $c = z^n < z^6$.

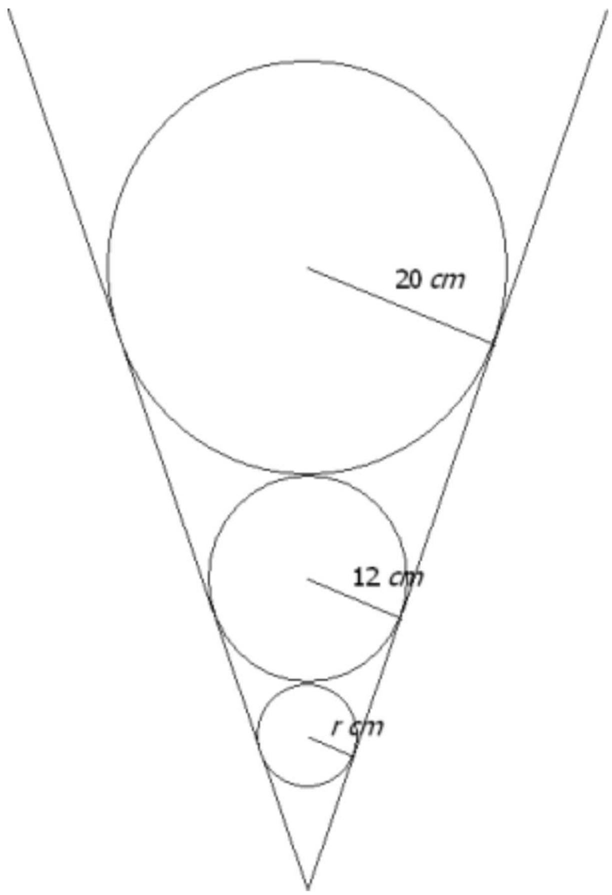
Hence $n < 6$, and we conclude that $x^n + y^n = z^n$ has no integer solutions for $n \geq 6$.

The cases of $n = 3$, 4, and 5 all have elementary proofs, so if this conjecture were true, it would provide for an elegant completion of the proof of FLT (Fermat's Last Theorem). Of course the importance of this conjecture not yet being proved cannot be overstressed. It may turn out that the proof of this conjecture is more difficult than the current proof for FLT. Also note that the proof of FLT does not provide proof of this conjecture. However, mathematicians are still encouraged to find a proof for this conjecture, and other results relating to the radical function, as its usefulness is far reaching into many other areas of current mathematical research.

Bag Of Balls

Problem

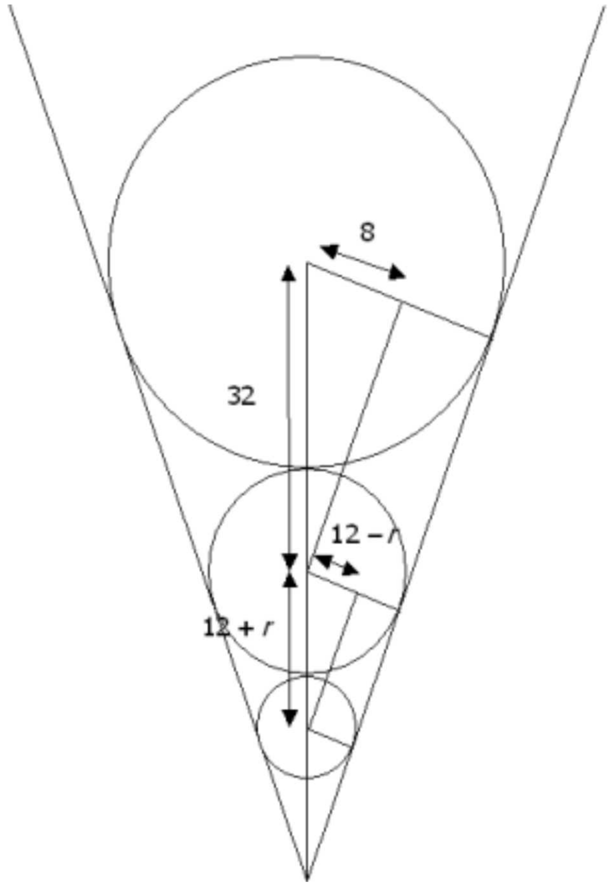
Three balls are placed inside a cone such that each ball is in contact with the edge of the cone and the next ball.



If the radii of the balls are 20 cm, 12 cm, and r cm respectively, what is the value of r?

Solution

Consider the following diagram.



Using similar triangles, $(12 + r)/(12 - r) = 32/8 = 4$.

$\therefore 12 + r = 48 - 4r$

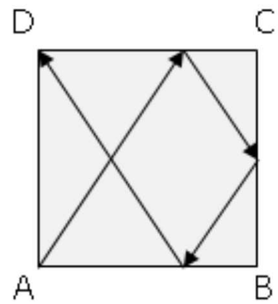
$\therefore 5r = 36 \Rightarrow r = 7.2$ cm.

What if the radii of the three balls were 20 cm, r cm, and 10 cm?
If the top sphere has radius, x, and the bottom sphere has radius, y, find r in terms of x and y.

BOUNCING BALL

Problem

A ball is projected from the bottom left corner of unit square ABCD into its interior. We shall assume that the speed of the ball remains constant and it will continue bouncing off the edges until it arrives at a corner. For example, if the ball strikes 2/3 of the way from D to C it will terminate at D.



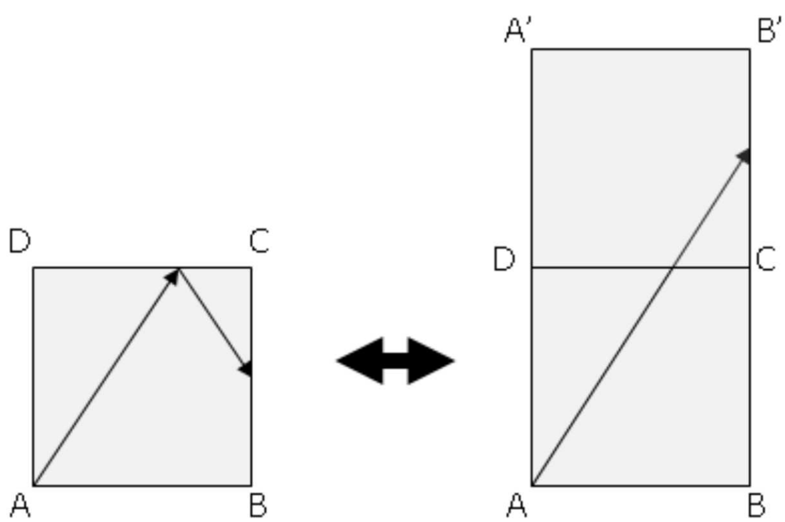
Where must the ball strike on DC to finish at A?

Solution

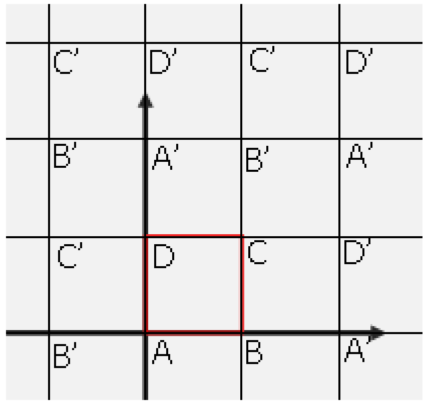
We shall solve this problem in two ways.

Method 1

By allowing the square to be reflected we can consider the reflection to be a continuous straight path. In the example below, the reflected path on the left diagram can be represented by the continuous path in the diagram on the right.



Extending this across the entire plane we can place the original unit square ABCD on co-ordinate axes with A at the origin.

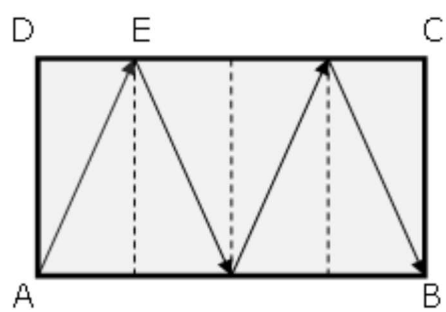


It can be seen that for the ball to finish at A it must terminate at one of the image points of A, represented by A'. As we are dealing with a unit square these points would be represented by the co-ordinate (x, y) where both x and y are even. Let x = 2a and y = 2b.

However, if a line passes through the point (*2a*. *2b*) then it must first have passed through the point (a, b), which means that it is impossible for the ball to return to its corner of origin before first arriving at one of the other corners.

Method 2

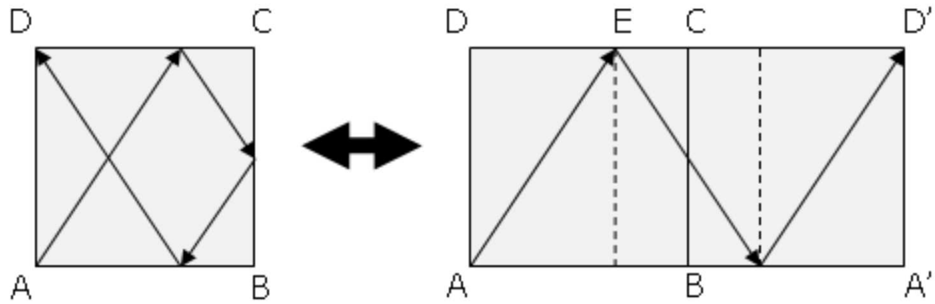
First we shall consider rectangle ABCD. Suppose the ball is projected from A towards E, a point on DC, such that the length DE splits DC into equal sections. In the example below, DE = DC / 4.



It can be seen that the ball will terminate at B if the number of sections is even and it will terminate at C if the number of sections is odd.

We shall now approach the problem in a similar manner to the previous method. We shall allow the ball to bounce off the top and bottom edges: DC and AB, but instead of allowing the ball to bounce off the vertical edges: DA and CB, we shall repeatedly reflect the square in a horizontal direction to consider the path of the ball to be a continuous path.

For example, in the unit square ABCD let DE = 2/3. As 3×2/3 = 2, it will be necessary to have two complete squares for the ball to reach a corner; in this case the ball would terminate at D.



In general, if DE = p/q, where GCD(p, q) = 1, then we shall have $q \times p/q = p$ complete squares and we shall generate q sections.

We have already established that it is necessary for there to be an even number of sections in a rectangle for the ball to terminate in the bottom corner, so q must be even. However, it can be seen that as the square is repeatedly reflected the bottom right corner alternates between the images of A and B Hence there must be an even number of squares for the image of A to be in the bottom right corner.

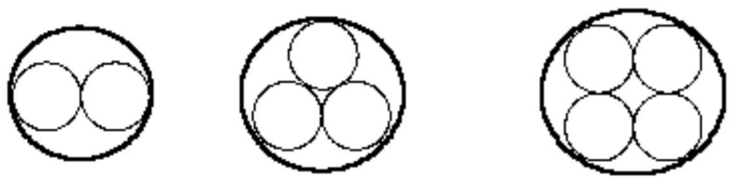
This would require both p and q to be even which is a contradiction. Hence it is impossible for the ball to terminate at A.

Given that DE = p/q where GCD(p, q) = 1, can you deduce which corner the ball will terminate in?
What would happen if DE were irrational?

CABLES

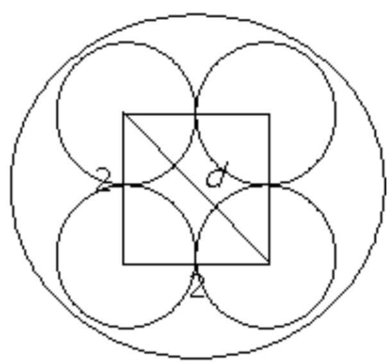
Problem

A telephone company places round cables in round ducts. Assuming the diameter of a cable is 2 cm, what would the diameter of the duct be for two cables, three cables and four cables?



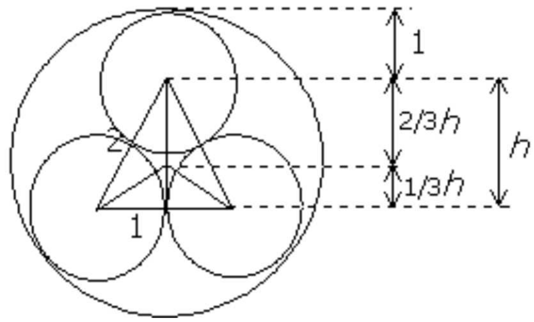
Solution

Clearly the duct for two cables must have a diameter of 4 cm, however, the duct for three and four cables requires a little more effort. Let us first consider the duct for four cables:



Using the Pythagorean Theorem, $d^2 = 2^2 + 2^2$, giving $d = \sqrt{8} \approx 2.83$ cm. Therefore, the duct diameter will be $\sqrt{8} + 2 \approx 4.83$ cm.

Now we consider the three cable duct,



First of all we use the Pythagorean Theorem to find the height of the triangle, $2^2 = h^2 + 1^2$, therefore $h = \sqrt{3} \approx 1.73$ cm.

Using the geometric result that states the centre of an equilateral triangle is 1/3 the height of the triangle, we deduce that the distance from the centre of the triangle to its apex is $(2/3)h = 2\sqrt{3} / 3 \approx 1.15$ cm.

So the duct radius will be $2\sqrt{3} / 3 + 1$, hence the diameter will be $4\sqrt{3} / 3 + 2 \approx 4.31$ cm.

Prove the geometric result that states the distance from the base of an equilateral triangle to the centre is 1/3 the distance from the base to the apex.

Find the diameter of a duct containing five cables.
(Think carefully about the way in which five cables would group.)

Can you generalise for *n* cables?
Note: This is an unsolved problem, but you never know... :o)

Coloured Strings

Problem

In a cuboid shaped room a hook is placed in the centre of each wall, the floor, and the ceiling. Every pair of hooks has either a piece of red or blue string tied between them. As this task is being performed triangles will be formed between sets of three hooks with edges being red or blue.

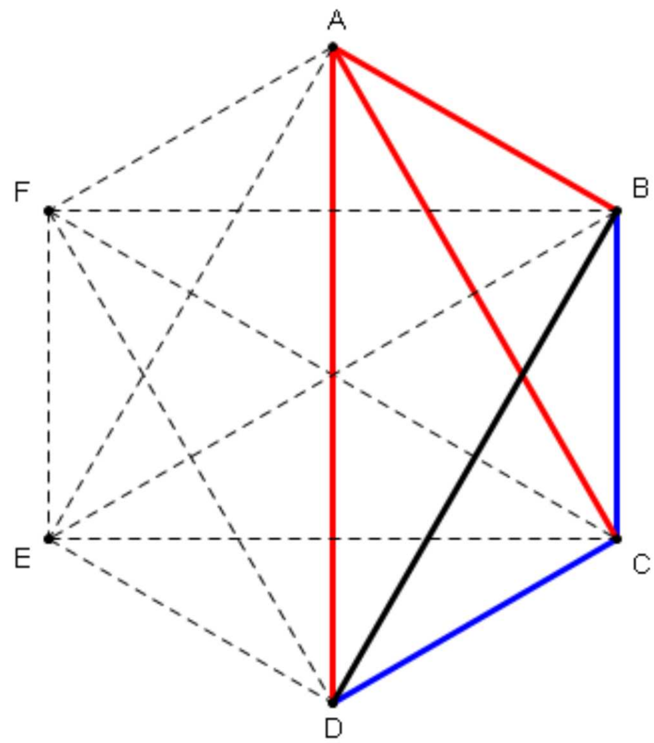
Prove that it is impossible to complete this task without forming at least one triangle of the same colour.

Solution

From each hook there are five pieces of string connected to it, which means that the problem is equivalent to colouring all the edges of a fully connected hexagon, ABCDEF, so that no triangle formed between any three vertices has all of its edges the same colour.

If we consider the number of red and blue edges connected to a particular vertex, (R,B), the edges can be coloured in the following way: (0,5), (1,4), (2,3), (3,2), (4,1), (5,0). It is significant to note that at least three edges at each vertex must be the same colour.

Without loss of generality let us suppose that we colour AB, AC, and AD red:



Clearly BC and CD must be both blue, otherwise triangle ABC or triangle ACD would be red. However, it is now impossible to colour BD red or blue without making either triangle ABD red or triangle BCD blue.

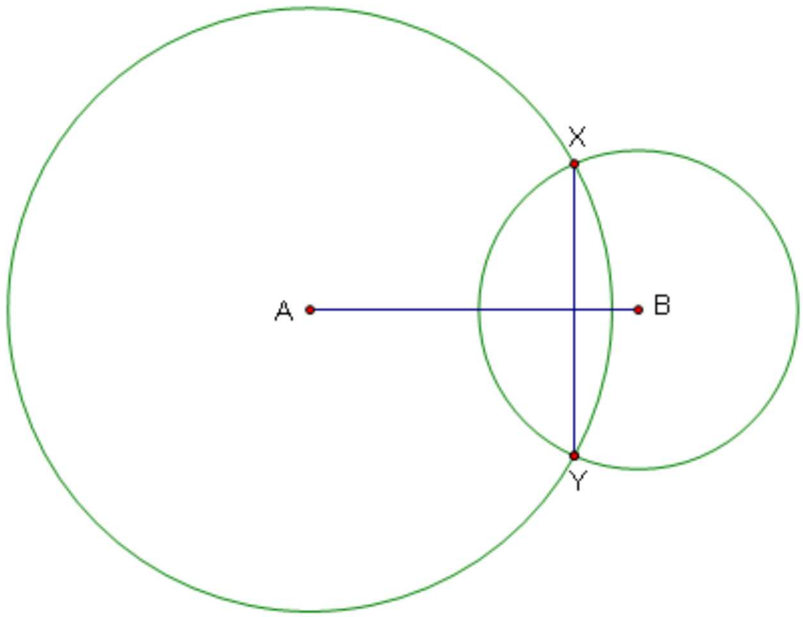
Hence it is impossible to colour the hexagon using two colours so as to not form a triangle with three edges the same colour, which means that the original task involving the pieces of string is also impossible.

What if three colours were allowed?
Investigate using *k* colours to colour a fully connected *n*-gon.

COMMON CHORD

Problem

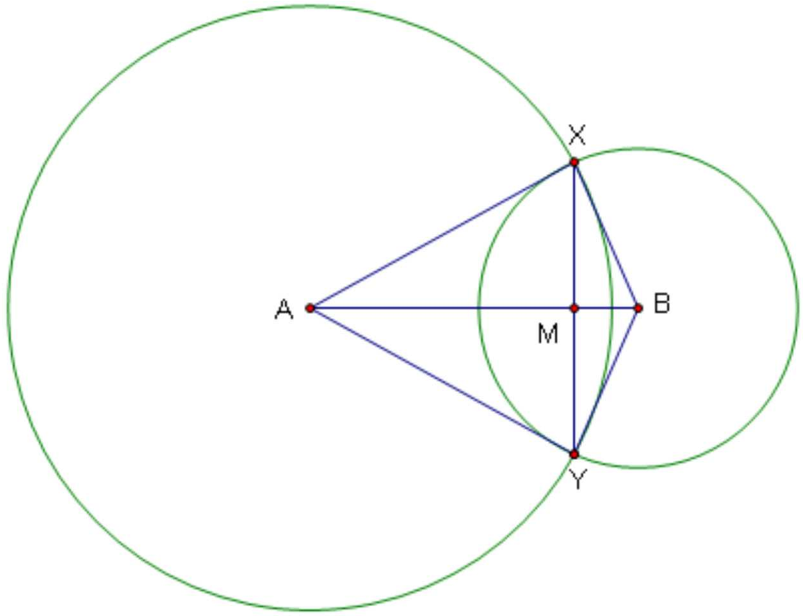
In the diagram below, A and B mark the centres of two circles and XY a chord common to both circles.



Prove that the segment joining the centres, AB, is a perpendicular bisector of the common chord XY.

Solution

We shall add radii AX, AY, BX, and BY to the diagram and mark the intersection of AB with XY as the point M.



Consider triangles AXB and AYB: AX = AY (radii), BX = BY (radii), and AB is common to both. As all the lengths are equal, triangles AXB and AYB are congruent, which means that angle MAX = angle MAY.

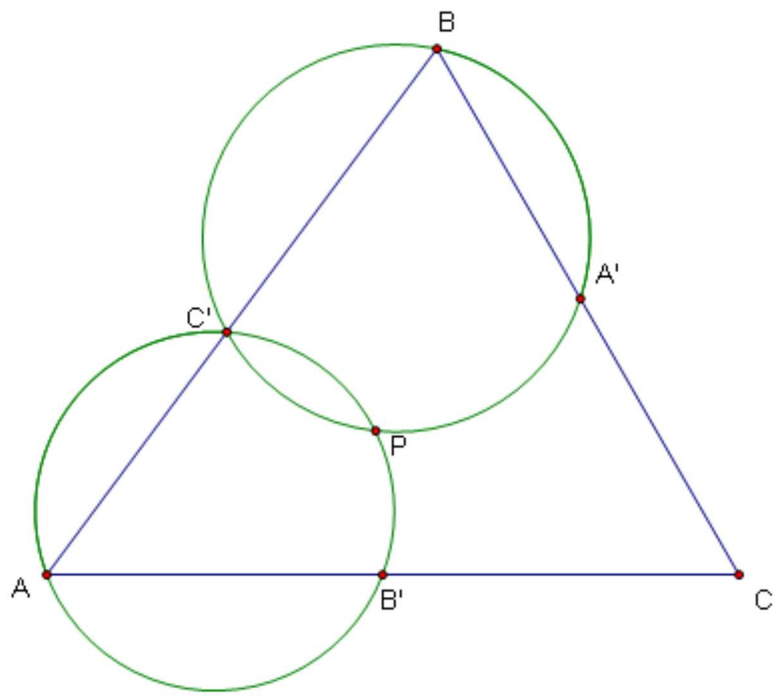
Now consider triangles AMX and AMY: as AM is common to triangles, angle MAX = angle MAY, and AX = AY, by S(ide) A(ngle) S(ide) we deduce that they are congruent. Therefore MX = MY and angle AMX = angle AMY.

But as angles AMX + AMY = 180 degrees, it follows that angle AMX = 90 degrees and we prove that AB is a perpendicular of XY.

CONCURRENT CIRCLES IN A TRIANGLE

Problem

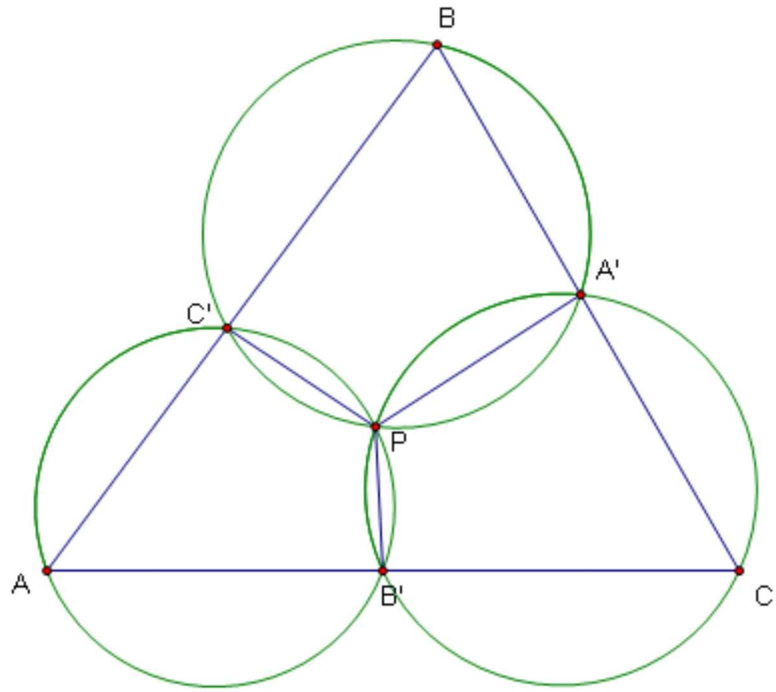
It can be shown that a unique circle passes through three given points. In triangle ABC three points A', B', and C' lie on the edges opposite A, B, and C respectively.



Given that the circle AB'C' intersects circle BA'C' inside the triangle at point P, prove that circle CA'B' will be concurrent with point P.

Solution

Consider the following diagram.



As AB'PC' is a cyclic quadrilateral angle A + angle B'PC' = 180 degrees. Similarly A'PC'B is a cyclic quadrilateral so angle B + angle A'PC' = 180 degrees.

Therefore angle A + angle B = 360 – (angle B'PC' + angle A'PC') = angle A'PB'.

However, in triangle ABC, angle A + angle B = 180 – angle C.

Hence angle A'PB' = 180 – angle C and we show that CA'PB' is a cyclic quadrilateral and the circle passing through C, A', and B' is concurrent at P.

This result is known as Miquel's theorem and remains true if the common point is outside the triangle...

Prove that if circle AB'C' intersect circle BA'C' outside the triangle at point Q that circle CA'B' will still be concurrent at this point Q.

CONSECUTIVE COMPOSITES

Problem

Although there are infinitely many primes it is a most remarkable fact that there can always be found a sequence of n consecutive composite numbers. For example, there are thirteen consecutive composite numbers between the primes 113 and 127.

Prove that there exists a sequence of n consecutive composite numbers for any finite value n.

Solution

Given that n is a positive integer it is clear that $n! = n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1$, is divisible by all the integers 1, 2, 3, ..., n.

So it follows that $n! + 2$ is divisible by 2, $n! + 3$ is divisible by 3, ... , $n! + n$ is divisible by n.

Therefore $(n+1)! + 2$, $(n+1)! + 3$, ... , $(n+1)! + n$, $(n+1)! + (n+1)$ is a sequence of n consecutive composite numbers.

Note that although this proves the existence of a sequence of n consecutive composites it does not find the first such sequence. For example, using this method $n = 5$, $6! = 720$, so 722, 723, 724, 725, and 726 are all composite, but the first sequence of five consecutive composites is 24, 25, 26, 27, and 28.

Also be aware that as n gets larger, the size of the list of consecutive composite numbers increases. However, as n always remains finite, the size of the set remains finite and so this proof does not demonstrate an end to primes.

CONSECUTIVE PRODUCT SQUARE

Problem

We can see that $3 \times 4 \times 5 \times 6 = 360 = 19^2-1$.

Prove that the product of four consecutive integers is always one less than a perfect square.

Solution

Let Q = $n(n + 1)(n + 2)(n + 3) + 1 = n^4 + 6n^3 + 11n^2 + 6n + 1$.

If Q is square, it must be of the form, $(n^2 + an + 1)^2$.

Expanding $(n^2 + an + 1)^2 = n^4 + 2an^3 + (a^2+2)n^2 + 2an + 1$, and comparing coefficients, $2a = 6 \Rightarrow a = 3$; checking: $a^2+2 = 11$.
Therefore, Q = $(n^2 + 3n + 1)^2$.

Hence, the product of four consecutive integers is always one less than a perfect square.

Converging Root

Problem

Consider the iterative formula, $u_{n+1} = \sqrt{u_n + 2}$. By investigating the behaviour it should become clear that all positive starting values converge to the limit 4.

What form must the positive integer, m, take, for the iterative formula, $u_{n+1} = \sqrt{u_n + m}$, to converge to an integral root?

Solution

We shall assume that the limit, L, exists, such that for sufficiently large values of n, $u_{n+1} \approx u_n$, and at limit, $u_{n+1} = u_n = L$.

Therefore, $\sqrt{L + m} = L$, $\sqrt{L} = L - m$.

Squaring both sides, $L = L^2 - 2mL + m^2$, and $L^2 - (2m+1)L + m^2 = 0$.

Using the quadratic formula, $L = (2m+1)\pm\sqrt{(4m+1)}/2$.

As $2m+1$ is odd and $4m+1$ will be an odd square, their sum will be even, so it will always be divisible by 2. That is, the only condition is that $4m+1$ is a perfect square.

Writing, $4m+1 = k^2$, $m = (k^2 - 1)/4$, it is clear that k must be odd,
let $k = 2a - 1$.

$\therefore m = ((4a - 4a + 1) - 1)/4 = a(a - 1)$.

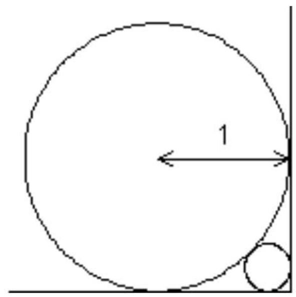
Hence, if m is of the form, $a(a - 1)$, the iterative form will converge to an integral root.

The solution, $L = (2m+1)\pm\sqrt{(4m+1)}/2$, suggests that there are two roots of convergence. Prove that the iterative formula will always converge to the upper limit for all positive starting values.

CORNER CIRCLE

Problem

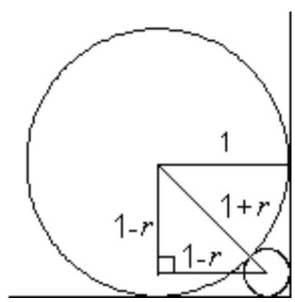
A unit circle is placed against a right angle.



What is the radius of the smaller circle?

Solution

Let the radius of the small circle be r.



Using the Pythagorean Theorem, $(1 + r)^2 = (1 - r)^2 + (1 - r)^2 = 2(1 - r)^2$
$\therefore 1 + 2r + r^2 = 2(1 - 2r + r^2) = 2 - 4r + 2r^2$
$\therefore r^2 - 6r + 1 = 0$

Solving the quadratic we get $r = 3 \pm 2\sqrt{2}$, giving the solution $r = 3 - 2\sqrt{2}$.

If a unit sphere is placed in the corner of a room, what is the largest sphere that can be placed in the gap between the unit sphere and the walls?

CUBES AND MULTIPLES OF 7

Problem

Prove that for any number that is not a multiple of seven, then its cube will be one more or one less than a multiple of 7.

Solution

Given n is any number that is not evenly divisible by 7, let $n = 7a + b$, where $b = 1,2,3,4,5,6$.

$\therefore n^3 = (7a + b)(7a + b)(7a + b)$

Clearly the only term under expansion that does not have at least one multiple of 7 will be b^3 and as $b = 1,2,3,4,5,6$ we get $b^3 = 1,8,27,64,125,216$.

$\therefore b^3 \equiv 1,1,-1,1,-1,-1 \mod 7$, respectively.

Hence $n^3 \equiv \pm 1 \mod 7$, where n is not a multiple of 7.

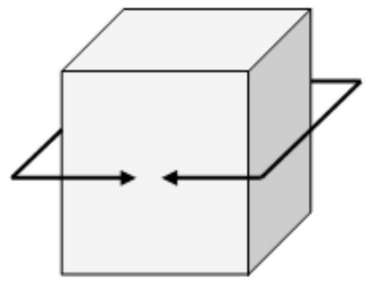
Corollary

$n^3 \equiv -1,0,1 \mod 7$, where $n \in \mathbf{N}$.

CUBOID PERIMETERS TO VOLUME

Problem

For any given cuboid it is possible to measure up to three different perimeters. For example, one perimeter could be measured this way.



Given that cuboid A has perimeters 12, 16, and 20, and cuboid B has perimeters 12, 16, and 24, which cuboid has the greatest volume?

Solution

Let the dimensions of a cuboid be x, y, z. The perimeters will be $2(x + y)$, $2(x + z)$, and $2(y + z)$, so dividing each of the perimeters by 2 will give $x + y$, $x + z$, and $y + z$ respectively.

The sum of these three terms will give $2x + 2y + 2z$, leading to $x + y + z$. Thus by subtracting each of $x + y$, $x + z$, and $y + z$ we will be able to obtain z, y, and x respectively.

Cuboid	**$2(x + y)$**	**$2(x + z)$**	**$2(y + z)$**	**$x + y$**	**$x + z$**	**$y + z$**
A	12	16	20	6	8	10
B	12	16	24	6	8	12

Cuboid	**$2(x + y + z)$**	**$x + y + z$**	**x**	**y**	**z**	**xyz**
A	24	12	2	4	6	48
B	26	13	1	5	7	35

Surprising as it may seem, cuboid A has the greater volume.

DIVISIBLE BY 11

Problem

Consider the following results.

$10^1 + 1 = 11$
$10^2 - 1 = 99 = 9 \times 11$
$10^3 + 1 = 1001 = 91 \times 11$
$10^4 - 1 = 9999 = 909 \times 11$
$10^5 + 1 = 100001 = 9091 \times 11$

Prove that 10^n-1 is divisible by 11 if n is even and 10^n+1 is divisible by 11 if n is odd.

Solution

As $(x-1)^n = (x-1)(x^{n-1} + x^{x-2} + ... + 1)$, we get:

$10^{2k}-1 = 100^k-1 = (100-1)(100^{k-1} + ... + 1) = 99Q \equiv 0 \bmod 11$.

That is, 10^n-1 is divisible by 11 if n is even.

As $10^{2k} \equiv 1 \bmod 11$, we can multiply both sides of the congruence by 10 to obtain $10^{2k+1} \equiv 10 \equiv -1 \bmod 11$; that is, $10^{2k+1}+1 \equiv 0 \bmod 11$.

Hence we prove that 10^n+1 is divisible by 11 if n is odd.

Prove that 100^n+1 is divisible by 101 iff n is odd, 1000^n+1 is divisible by 1001 iff n is odd.
What about 10000^n+1? Generalise.

Divisible By 99

Problem

Find the smallest number that is made up of each of the digits 1 through 9 exactly once and is divisible by 99.

Solution

The first thing we note is that 1 + 2 + ... + 9 = 45 and as the sum of the digits is divisible by 9 then any arrangement of those digits will produce a number that is divisible by 9. So our challenge reduces to finding the smallest number that is divisible by 11.

To test if a number is divisible by 11 we find *a*, the sum of digits in the odd positions, and *b*, the sum of digits in the even positions. If $a - b$ is divisible by 11 then so too is the number; for example, 95953: 9 + 9 + 3 = 21, 5 + 5 = 10, and as 21 − 10 = 11 then we know that 95953 is divisible by 11.

Let us begin with the smallest 9-digit number using each of the digits 1 through 9: 123456789. The first set, {1, 3, 5, 7, 9}, has sum 25 and the second set, {2, 4, 6, 8}, has sum 20. As the difference is 5 we know that the number is not divisible by 11.

However, for the difference to increase from 5 to 11 we need to increase the sum of the first set by 3 and decrease the sum of the second set by 3. This can be achieved by swapping 1 and 4, 3 and 6, or 5 and 8.

Swapping 1 and 4 gives the two sets {4, 3, 5, 7, 9} and {2, 1, 6, 8}. But before we merge these numbers we must place them in ascending order to keep the number as small as possible: {3, 4, 5, 7, 9} and {1, 2, 6, 8}. This produces the number 314256789.

By swapping 3 and 6 we get {1, 5, 6, 7, 9} and {2, 3, 4, 8}, producing the number 125364789.

Finally by swapping 5 and 8 we get {1, 3, 7, 8, 9} and {2, 4, 5, 6}, producing the number 123475869.

Hence the smallest number using the digits 1 through 9 which is divisible by 99 is 123475869.

Using 1, 2, and 3, we can see that 132 is divisible by 11. By considering numbers made up of each of the digits 1 through *n*, which *n*-digit numbers can be made to be divisible by 11?

DOUBLE SQUARE SUM

Problem

Consider the equation, $x^2 + y^2 = 2z^2$; if GCD(x, y) = 1 then the solution is primitive.

For example, $7^2 + 17^2 = 2 \times 13^2$ is a primitive solution, whereas $2^2 + 14^2 = 2 \times 10^2$ is not a primitive solution.

Prove that inifinitely many primitive solutions exist.

Solution

The integral solutions to the equation $a^2 + b^2 = c^2$ are called a Pythagorean Triplet, and it is well known that infinitely many primitive solutions exist.

Let a, b, and c be a set of primitive Pythagorean triplets.

$$\begin{aligned} \therefore (a + b)^2 + (a - b)^2 &= (a^2 + 2ab + b^2) + (a^2 - 2ab + b^2) \\ &= 2(a^2 + b^2) \\ &= 2c^2 \end{aligned}$$

Hence there are infinitely many primitive solutions to the equation $x^2 + y^2 = 2z^2$.

What about the equation $x^2 + y^2 = 3z^2$?
Can you generalise for $x^2 + y^2 = kz^2$?

EQUABLE RECTANGLES

Problem

How many rectangles with integral length sides have an area equal in value to the perimeter?

Solution

Let the rectangle measure x by y so that the area = xy and the perimeter = $2x + 2y$.

$$\begin{aligned} \therefore \quad xy &= 2x + 2y \\ xy - 2y &= 2x \\ y(x - 2) &= 2(x - 2) + 4 \\ \therefore \quad y &= 2 + 4 / (x - 2) \end{aligned}$$

As x and y are integer, it is necessary for $x - 2$ to divide into 4.

Therefore $x - 2 = 1, 2,$ or $4 \Rightarrow x = 3, 4,$ or 6 and $y = 6, 4,$ or 3 respectively.

Hence there are two unique rectangles with area and perimeter equal: an oblong measuring 6 by 3 and a square with side length 4.

Investigate "equable" triangles: equilateral, isosceles, and scalene.

EQUAL CHANCE

Problem

A bag contains n discs, made up of red and blue colours. Two discs are removed from the bag.

If the probability of selecting two discs of the same colour is 1/2, what can you say about the number of discs in the bag?

Solution

Let there be r red discs, so $P(RB) = r/n \times (n-r)/(n-1)$, similarly, $P(BR) = (n-r)/n \times r/(n-1)$.

Therefore, $P(\text{different}) = 2r(n-r)/(n(n-1)) = 1/2$.

Giving the quadratic, $4r^2 - 4nr + n^2 - n = 0$.

Solving, $r = (n \pm \sqrt{n})/2$.

If n is an odd square, $\sqrt{n}$ will be odd, and similarly, when n is an even square, $\sqrt{n}$ will be even. Hence their sum/difference will be even, and divisible by 2.

In other words, n being a perfect square is both a sufficient and necessary condition for r to be integer and the probability of the discs being the same colour to be 1/2.

Prove that $n(n+1)/2$ (a triangle number), must be square, for the probability of the discs being the same colour to be 3/4, and find the smallest n for which this is true.
What does this tell us about n and $n(n+1)/2$ both being square?
Can you prove this result directly?

EQUAL COLOURS

Problem

A box contains a mixture of black and white discs, with more black than white discs. A game is played by taking one disc at a time, at random, and without replacement. If an equal number of each colour have been removed the game stops and the player wins.

It is found that the player has an equal chance of winning or losing.

If the box contains twelve discs in total, find the number of black discs.

Solution

Let the total number of blacks discs in the box be b and the number of white discs be w.

Clearly if $b=w$, the player would win every time; and for this problem we are given that $b>w$.

We shall express a win by the use of strings; for example, the sequence:
W, WW, WWB, WWBW, WWBWB, WWBWBB produces the winning string, WWBWBB, that contains three of each colour.

For each winning string, it will only contain an equal number of black and white discs for the first time when the final disc in the string is taken. For example, WBBW does not represent a valid winning string, as it contained an equal number of black and white discs after the second disc was taken. Therefore, given a winning string, we can safely invert the string to obtain a different winning string. For example, WWBWBB and BBWBWW are dual winning strings. Hence there are an equal number of winning strings that start with B and W.

As there are more black discs than white discs, any string starting with W must be a winning string; there will always be enough B's to eventually match the number of W's before the final disc in the box is taken.

P(1st disc W) = $w/(b+w)$, hence P(winning) = $2w/(b+w)$, where $b \geq w$.

Therefore solving $2w/(b+w) = 1/2$, leads to $b = 3w$.

As $b+w = 3w+w = 4w = 12 \Rightarrow w = 3$. Hence there are 3 white discs and 9 black discs in the box.

FACTORIAL AND POWER OF 2

Problem

Given that a, b, and c are positive intgers, solve the following equation.

$$a!b! = a! + b! + 2^c$$

Solution

Without loss of generality let us assume that $a \geq b$ and divide through by $b!$: $a! = a!/b! + 1 + 2^c/b!$, giving integers throughout.

As each term on the RHS is integer, RHS $\geq 3 \Rightarrow a! \geq 3 \Rightarrow a \geq 3$.

For $b > 2$, $b!$ will contain a factor of 3, and so cannot divide 2^c. Thus b=1 or b=2.

If b=1, we get $a! = a! + 1 + 2^c$, which leads to $2^c + 1 = 0$: no solution.

If b=2, we get $a! = a!/2 + 1 + 2^{c-1}$, which leads to $a!/2 = 1 + 2^{c-1}$.

If $a > 3$, $a!/2$ is even, so $2^{c-1} = 1$. But then we get $a!/2 = 2$: no solution.

If $a = 3$, $3 = 1 + 2^{c-1} \Rightarrow c = 2$.

Hence we obtain the only solution: $3!2! = 3! + 2! + 2^2$.

Related problems:

Factorial Symmetry: $a!b! = a! + b!$

Factorial Equation: $a!b! = a! + b! + c!$

Factorial And Square: $a!b! = a! + b! + c^2$

Factorial Divisibility

Problem

Given that n is a positive integer, prove that $(2^n)!$ is divisible by 2^{2n-1}

Solution

Firstly we write, $2^{2n-1} = 2^{n+n-1} = 2^n \, 2^{n-1}$

Then, $(2^n)! = 1 \times 2 \times 3 \times \ldots \times (2^n - 1) \times 2^n$

Clearly, $(2^n)!$ is divisible by 2^n, but as $2^{n-1} < 2^n$, one of the earlier factors of $(2^n)!$ must be 2^{n-1}.

Hence $(2^n)!$ is divisible by 2^{2n-1}.

FALLING SOUND

Problem

A boy drops a stone down a well and hears the splash from the bottom after three seconds. Given that sound travels at a constant speed of 300 m/s and the acceleration of the stone due to gravity is 10 m/s^2, how deep is the well?

Solution

From the moment the stone is dropped to the splash heard three seconds later two distinct events occur: the stone takes t seconds to hit the water below and the sound takes $3 - t$ seconds to travel back up the well.

If s is the depth of the well then we can use $s = ut + at^2 / 2$, where u is initial velocity and a is acceleration.

During the stone's descent, $s = 5t^2$, as the initial velocity is zero and we can ignore the direction of acceleration as we are only concerned with distance rather than displacement.
As the sound of the splash travels back up the well, $s = 300(3 - t)$ (acceleration of sound is zero).

Therefore $5t^2 = 300(3 - t)$, leading to the quadratic, $t^2 + 60t - 180 = 0$.

Solving this we take the positive solution, $t = 6\sqrt{30} - 30$ seconds.

Using $s = 300(3 - t)$ we get the depth of the well, $s = 300(33 - 6\sqrt{30}) \approx 41.0$ metres.

Generalise for a time of x seconds between the stone being released and the sound of the splash being heard.
If the time taken for the sound to travel back is considered to be instantaneous then what depth would be estimated?
At what depth would this error be considered significant?

FIBONACCI RATIO

Problem

The Fibonacci sequence is defined by the second order recurrence relation $F_{n+2} = F_{n+1} + F_n$, where $F_1 = 1$ and $F_2 = 1$.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

Assuming that ratio of adjacent terms in the Fibonacci sequence F_{n+1}/F_n tends to a limit, φ, as *n* increases, prove that $\varphi = (\sqrt{5}+1)/2$.

Solution

If the limit exists for $n \geq L$, it follows that $\varphi = F_{L+1}/F_L = F_{L+2}/F_{L+1}$.

But $F_{L+2}/F_{L+1} = (F_{L+1} + F_L)/F_{L+1} = 1 + F_L/F_{L+1}$.

Therefore $\varphi = 1 + 1/\varphi$, and multiplying through by φ, we get the quadratic equation $\varphi^2 = \varphi + 1$ or $\varphi^2 - \varphi - 1 = 0$.

Solving this quadratic, and ignoring the negative root as the ratio of adjacent terms will be positive, we get $\varphi = (\sqrt{5} + 1)/2$. **Q. E. D.**

Using the following identity prove that the limit exists and $\varphi = (\sqrt{5} + 1)/2$.

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

(See Fibonacci Sequence for proof of identity.)

FIBONACCI SERIES

Problem

Let $S = 1/2 + 1/4 + 2/8 + 3/16 + 5/32 + \ldots + F_k/2^k + \ldots$, where F_k represents the kth term of the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, ...

Find the value of S.

Solution

$$S = F_1/2 + F_2/4 + F_3/8 + F_4/16 + \ldots$$

$$2S = F_1 + F_2/2 + F_3/4 + F_4/8 + \ldots$$

$$4S = 2F_1 + F_2 + F_3/2 + F_4/4 + \ldots$$

$$\therefore 4S - 2S - S = 2F_1 + F_2 - F_1 + (F_3 - F_2 - F_1)/2 + (F_4 - F_3 - F_2)/4 + \ldots$$

But as $F_k = F_{k-1} + F_{k-2}$, it follows that $F_k - F_{k-1} - F_{k-2} = 0$.

$$\begin{aligned} \therefore S &= 2F_1 + F_2 - F_1 \\ &= 2 \end{aligned}$$

Find the value of x for which $a(x) = xF_1 + x^2F_2 + x^3F_3 + \ldots = 1$.
Prove that $a(x) = G$, where G is a positive integer, always has a solution.

FISHY PROBLEM

Problem

Given X follows a Poisson distribution, with mean L, prove that P(X = L) = P(X = L–1) for all positive integer values of L.

For example, if X~Po(5) then P(X=5) = P(X=4) = 0.175 (3 s.f.).

Solution

If X~Po(L) then $P(X = r) = e^{-L} L^r / r!$.

$$\begin{aligned} \therefore P(X = L) &= e^{-L} L^L / L! \\ &= e^{-L} L\, L^{L-1} / (L(L-1)!) \\ &= e^{-L} L^{L-1} / (L-1)! \\ &= P(X = L-1) \end{aligned}$$

What if L is non-integer?

FRACTION RECIPROCAL SUM

Problem

Prove that the sum of a proper fraction and its reciprocal can never be integer.

Solution

Given a fraction, x/y, we can divide top and bottom by HCF(x,y), to get a fraction in its lowest common terms, a/b. As $x/y = a/b$, it follows that $y/x=b/a$.

Therefore, $x/y + y/x = a/b + b/a = k$, where HCF($a,b$)=1.

Multiplying through by ab, $a^2 + b^2 = kab$.

We can see that the right hand side is divisible by a, so the left hand side must be divisible by a. Clearly a^2 is divisible by a, so b^2 must be divisible by a. However, as HCF(a,b)=1, this is impossible, unless $a=b=1$, and this would only happen if $x=y$.

Hence the sum of a proper fraction and its reciprocal can never be integer.

This is an interesting alternative perspective on the original problem. That is, showing that the sum of a proper fraction and its reciprocal cannot be integer is equivalent to showing that the sum of squares of two positive integers cannot be a multiple of their product.

Investigate the solutions to the equation, $a^2 + b^2 + c^2 = kabc$.

HIGHEST ROLL WINS

Problem

In deciding who should pay for lunch, Jane challenges John and James to a game of chance, "I shall take two ordinary 6-sided dice, roll them, and add their scores together. Then one of you shall do the same. If the second total is higher, John pays for lunch, if it is lower, James pays, or if it is same, I will pay. As there are three equally likely outcomes, the game is fair."

Is the game fair?

Solution

We begin by constructing a sample space to list the possible totals.

+	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Now we construct a probability table listing the probabilities of the total being 1, 2, 3, ..., 12:

Total	2	3	4	5	6	7	8	9	10	11	12
Probability	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Finally we calculate the probability that the second score is the same as the first score:

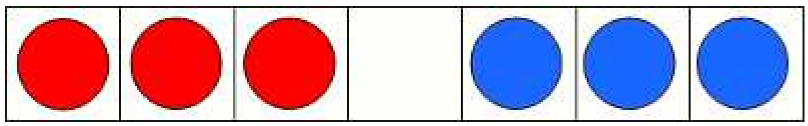
$$\begin{aligned} P(\text{same}) &= P(2 \text{ AND } 2) + P(3 \text{ AND } 3) + \ldots P(12 \text{ AND } 12) \\ &= (1/36)(1/36) + (2/36)(2/36) + \ldots + (1/36)(1/36) \\ &= (1/1296)(1 + 4 \ldots + 1) \\ &= 146/1296 = 73/648 \approx 11.3\% \end{aligned}$$

P(John OR James pays) = 575/648, therefore P(John pays) = P(James pays) = 575/1296 ≈ 44.4%. Hence the game of chance is definitely not fair!

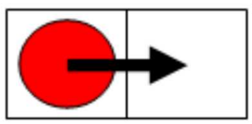
Hops and Slides but Never Square

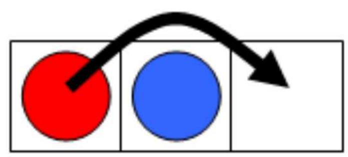
Problem

A horizontal row comprising of $2n + 1$ squares has n red counters placed at one end and n blue counters at the other end, being separated by a single empty square in the centre. For example, when $n = 3$.



A counter can move from one square to the next (slide) or can jump over another counter (hop) as long as the next square is unoccupied.





Let M(n) represent the minimum number of moves/actions to completely reverse the positions of the coloured counters; that is, move all the red counters to the right and all the blue counters to the left.

Prove that M(n) can never be square.

Solution

Experimentally the following results can be obtained.

n	M(n)
1	3
2	8
3	15
4	24
5	35

The results seem to suggest that the formula, $M(n) = n(n + 2) = n^2 + 2n$.

We shall prove this result by considering the total distance travelled by all of the counters. Note that a slide involves travelling a distance of one square whereas a hop accounts for a distance of two squares.

Once a complete reversal of positions has taken place each of the red counters will have moved a total distance of $n + 1$ squares to the right and each blue counter will have moved $n + 1$ squares to the left. So the minimum distance that all of the counters travelled will be a distance of $2n(n + 1) = 2n^2 + 2n$ squares.

However, during this process each counter of one colour must hop over every counter of the other colour. Therefore a total of $n \times n = n^2$ hops (actions) will take place, but as each hop moves a distance of two squares, the hops will account for $2n^2$ squares of the total distance travelled.

Therefore the slides will account for $2n^2 + 2n - 2n^2 = 2n$ squares of the total distance travelled, which also represents the total number of slides (actions).

Hence the total number of actions (made up of hops and slides) will be $n^2 + 2n$.

Therefore $M(n) = n^2 + 2n = (n^2 + 2n + 1) - 1 = (n + 1)^2 - 1$.

But as $n \geq 1$ and $M(n)$ is one less than a perfect square we prove that it can never be square itself.

With the exception of $M(2) = 8$, is it possible for $M(n)$ to be cube?
Given that there are x red counters and y blue counters find a formula for $M(x, y)$.

IMPERFECT SQUARE SUM

Problem

Prove that $n^4 + 3n^2 + 2$ is never square.

Solution

By factoring $n^4 + 3n^2 + 2 = (n^2 + 1)(n^2 + 2)$.

We can see that $(n^2 + 1)^2 < (n^2 + 1)(n^2 + 2) < (n^2 + 2)^2$.

Hence $n^4 + 3n^2 + 2$ lies between two consecutive squares, so it cannot be a square itself.

Prove that $n^4 + 2n^3 + 2n^2 + 2n + 1$ can never be square.

IMPOSSIBLE SOLUTION

Problem

Given that a and b are positive integers, find the conditions for which the equation $\sqrt{a} - b = \sqrt{c}$ has a solution.

Solution

From $\sqrt{a} = b + \sqrt{c}$, square both sides, $a = b^2 + 2b\sqrt{c} + c$.

Rearranging we get, $\dfrac{a - b^2 - c}{2b} = \sqrt{c}$.

As the left hand side is rational, $\sqrt{c}$ must be rational.

Let $\sqrt{c}=x/y$, where HCF(x, y)=1.

Squaring, $c=x^2/y^2$, $cy^2=x^2$.

As the left hand side divides by y^2 and HCF(x^2, y^2)=1, the right hand side will only divide by y^2 if $y^2=1$. Hence $c=x^2$ must be a perfect square.

Furthermore, if c is a perfect square, $\sqrt{a} = b + \sqrt{c}$ will be integer, so a must also be a perfect square.

INCREASING DIGITS

Problem

Working from left to right in a number, if the next digit is greater in value than the preceeding digit, we say that the digits are strictly increasing: 15689, 35679, and 13478 are examples of 5-digit numbers with this property.

Given a number has strictly increasing digits, what is the probability that it contains 5-digits?

Solution

Each non-empty subset taken from {1,2,3,4,5,6,7,8,9} represents each of the possible numbers with strictly increasing digits, and in turn each of these subsets can be represented by a 9-digit binary string. For example, 124789 has strictly increasing digits and can be represented by the string 110100111.

In this way we can see that there are 2^9=512 binary strings, but as one of these strings would be the empty set, there are 511 numbers containing strictly increasing digits that exist in total.

Next we note that a 5-digit number with strictly increasing digits can be represented by selecting a subset of size 5 from {1,2,3,4,5,6,7,8,9}; there are ${}^9C_5$ = 126 ways this can be done.

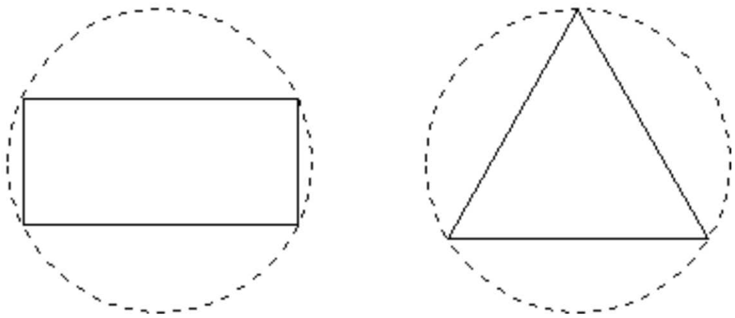
Hence the probability of a number with strictly increasing digits containing exactly 5-digits is 126/511=18/73.

Check out the related Never Decreasing Digits problem.

INSCRIBED RECTANGLE

Problem

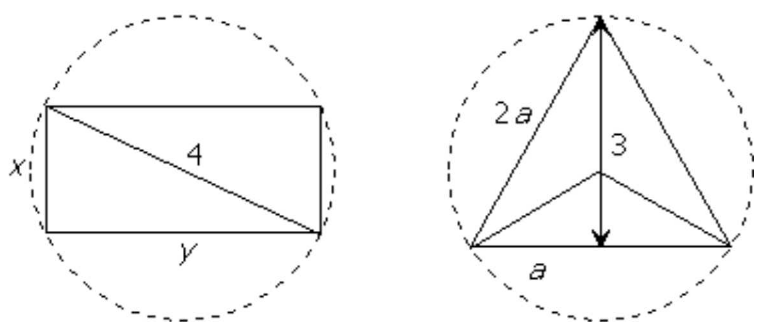
A rectangle and an equilateral triangle with equal area are inscribed in a circle with radius, 2.



Find the dimensions of the rectangle.

Solution

By drawing a diagonal in the rectangle, we form a diameter. In the case of the triangle, we use the geometrical result which states that the centre of an inscribed equilateral triangle is 1/3 the height of the triangle. Hence the radius is 2/3 the height of the triangle, and we deduce that the height of the triangle is 3.



Applying the Pythagorean theorem to the rectangle, $x^2 + y^2 = 16$ [1].

Similarly with the triangle, $4a^2 = a^2 + 9$, so $a = \sqrt{3}$.

Hence the area of the triangle is $3\sqrt{3}$.

As the areas are equal, the area of the rectangle, $xy = 3\sqrt{3}$,
so $x^2y^2 = 27$ [2].

From [1] we get, $y^2 = 16 - x^2$.

Substituting this into [2] we get, $x^2(16 - x^2) = 27$.

Leading to the quartic, $x^4 - 16x^2 + 27 = 0$.

Solving this as a quadratic in x^2 we obtain the roots, $x^2 = 8 \pm \sqrt{37}$.

Using $y^2 = 16 - x^2$ we observe that $y^2 = 16 - (8 \pm \sqrt{37})$.
That is, when $x^2 = 8 + \sqrt{37}$, $y^2 = 8 - \sqrt{37}$, and vice versa.

Therefore the rectangle has dimensions, $\sqrt{(8 + \sqrt{37})}$ by $\sqrt{(8 - \sqrt{37})}$.

Can you generalise for a circle with radius, r?

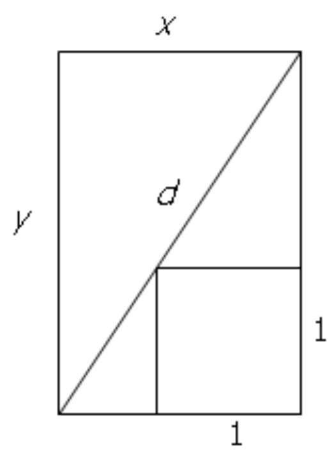
Find the side length of the triangle if the rectangle measures, x by y.

What if the triangle has side length, a?

INTEGRAL AREA

Problem

A rectangle measuring x by y has a unit square placed in the bottom right corner. The diagonal, d, joining the bottom left to the top right of the rectangle passes through the vertex of the square.



If the area of the rectangle is integer, what can you deduce about d?

Solution

By the Pythagorean Theorem, $x^2 + y^2 = d^2$.

By similar triangles,

$(x-1)/1 = 1/(y-1)$

$(x-1)(y-1) = 1$

$xy-x-y+1 = 1$

$\therefore xy = x + y$

Squaring both sides, $(xy)^2 = x^2 + y^2 + 2(xy) = d^2 + 2(xy)$

$\therefore (xy)^2 - 2(xy) = d^2$

$(xy)^2 - 2(xy) + 1 = d^2 + 1$

$(xy-1)^2 = d^2 + 1$

$xy-1 = \pm\sqrt{(d^2 + 1)}$

$\therefore xy = 1 \pm \sqrt{(d^2 + 1)}$

But as the area, $xy > 0$, we only need take the positive root.

$xy = 1 + \sqrt{(d^2 + 1)}$

So for xy to be integer, d^2+1 must be square. Let $d^2 + 1 = k^2$, where $k > 1$, hence $d = \sqrt{(k^2-1)}$.

Given that $d = \sqrt{15}$, find the dimensions of the rectangle.
What if $d = \sqrt{8}$?
For which values of d are both the area and the dimensions integer?

Inverted Logarithm

Problem

Prove that $\log_a(x)\log_b(y) = \log_b(x)\log_a(y)$.

Solution

We shall begin by proving two fundamental properties of logarithms:

1. $\log_a(x) = \log_b(x)/\log_b(a)$
2. $\log_b(a) = 1/\log_a(b)$

Clearly the first result is well known, as it is the method often used by students to evaluate logarithms in different bases on calculators; for example, $\log_2(32) = \log_{10}(32)/\log_{10}(2) = 5$. The second result is perhaps less familiar and is called the reciprocal property of logarithms.

1. Let $\log_a(x) = c \Rightarrow x = a^c$.

   $\therefore \log_b(x) = \log_b(a^c) = c\log_b(a)$

   $\therefore c = \log_a(x) = \log_b(x)/\log_b(a)$ [1]

2. Let $\log_a(b) = c \Rightarrow b = a^c$.

   $\therefore b^{1/c} = a \Rightarrow 1/c = \log_b(a)$

   $\therefore \log_b(a) = 1/c = 1/\log_a(b)$ [2]

Now we shall prove that $\log_a(x)\log_b(y) = \log_b(x)\log_a(y)$.

Using [1], $\log_a(x) = \log_b(x)/\log_b(a)$ and $\log_b(y) = \log_a(y)/\log_a(b)$.

$\therefore \log_a(x)\log_b(y) = (\log_b(x)\log_a(y))/(\log_b(a)\log_a(b))$

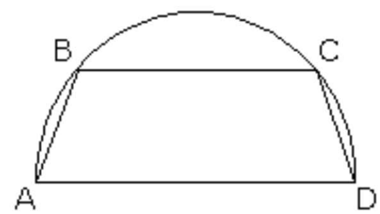
Using [2], $\log_b(a) = 1/\log_a(b) \Rightarrow \log_b(a)\log_a(b) = 1$.

Hence $\log_a(x)\log_b(y) = \log_b(x)\log_a(y)$ **Q. E. D.**

ISOSCELES TRAPEZIUM

Problem

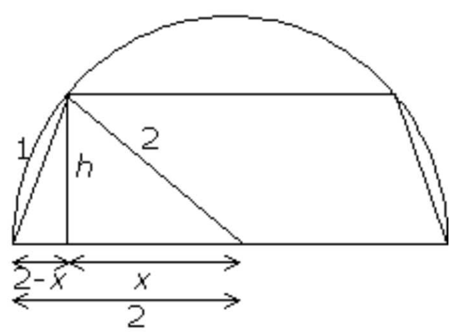
An isosceles trapezium *ABCD* is placed inside a semicircle such that they share the same base, $AD = 4$, and the lengths $AB = DC = 1$ are chords.



Find the length *BC*.

Solution

Consider the following diagram.



By applying the Pythagorean theorem to right-angle triangles on the right,
$h^2 + x^2 = 4$ (1)

In the left hand triangle,
$(2 - x)^2 + h^2 = 1$
$4 - 4x + x^2 + h^2 = 1$ (2)

By substituting (1) into (2), we get,
$4 - 4x + 4 = 1$

Hence $4x = 7 \Rightarrow x = 7/4$.

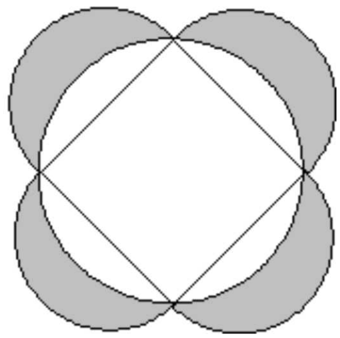
So $BC = 2x = 7/2$.

Given the side length of the trapezium, a, and the diameter of the semicircle, d, can you find an expression for the length of the top?

LUNES

Problem

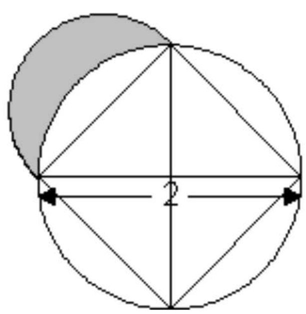
A square is inscribed in a circle with diameter 2. Four smaller circles are then constructed with their diameters on each of the sides of the square.



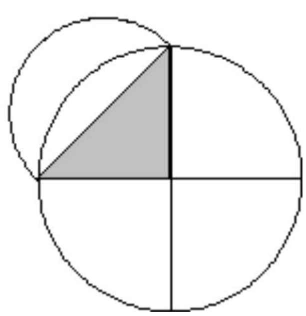
Find the shaded area.

Solution

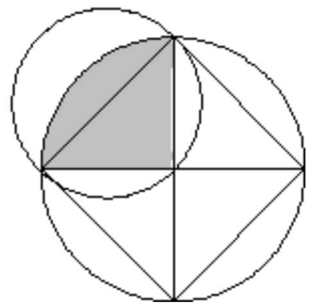
Consider the diagram:



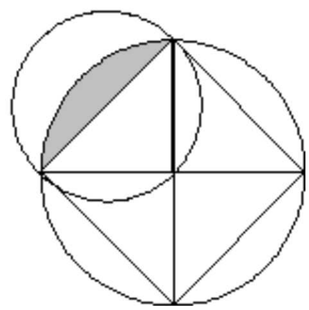
This is best tackled by working towards the area of a single lune, but this needs to be done in a series of careful additive and subtractive steps.



The area of the right angle triangle is 1/2.

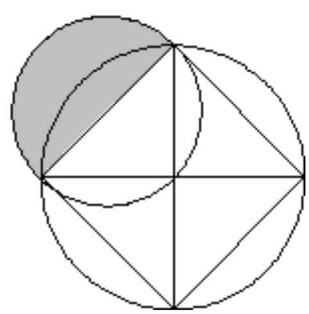


The area of the large circle is $\pi \times 1^2 = \pi$, so the area of the (shaded) quarter circle is $\pi/4$.



Therefore the area of the shaded segment is $\pi/4 - 1/2$.

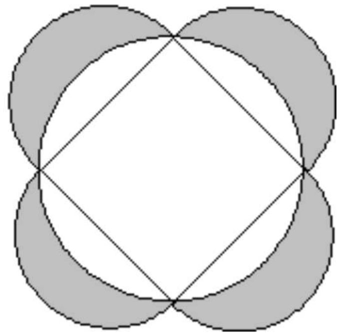
Using the Pythagorean Theorem, the length of the square's diagonal, $d = \sqrt{2}$. So the radius of the smaller circle is $\sqrt{2}/2 = 1/\sqrt{2}$.



Therefore the area of one small circle is $\pi \times (1/\sqrt{2})^2 = \pi/2$ and so the area of the shaded semi-circle will be $\pi/4$.

Hence the area of one lune is $\pi/4 - (\pi/4 - 1/2) = 1/2$ (surprising eh?)

So the shaded area of the original shape is 2.



What would be the shaded area if the same construction was performed on the edges of an inscribed equilateral triangle?

What about other inscribed regular polygons?

MEAN CLAIM

Problem

The contents of twelve boxes of matches were recorded as:

34, 31, 29, 35, 33, 30, 31, 28, 27, 35, 32, 31

On the box it stated, "Average contents 32 matches", but the sample mean can be shown to be about 31.3.

When the company received a complaint that the average contents is less than 32, they claimed that approximately one in five samples of size twelve would have a mean less than 31.3.

By using the sample data, test this claim and determine the minimum sample mean before the claim can be rejected at a 5% significance level.

Solution

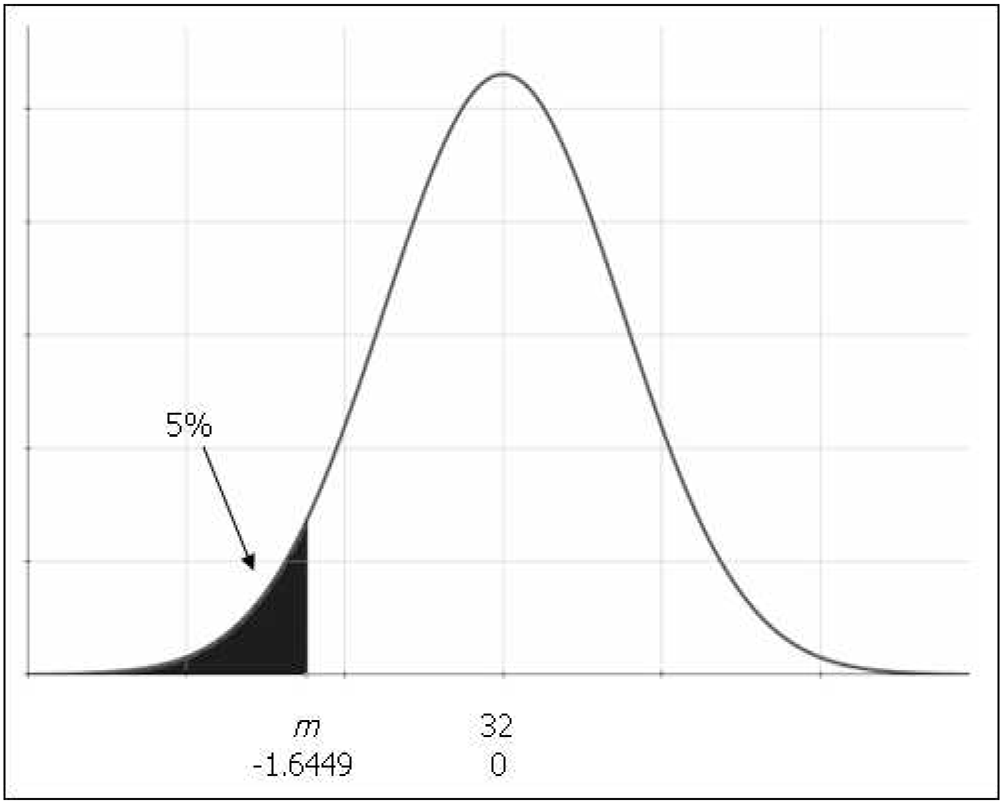
Sample mean, $\bar{x} = \sum x/n = 376/12$.
Sample variance, $s^2 = \sum x^2/n - \bar{x}^2 = 11856/12-(376/12)^2 = 56/9$.

We are told that the population mean, $\mu = 32$, but because we do not know the population variance we must use the sample variance. As $n < 30$, we will use the best estimator, $\hat{\sigma}^2 = ns^2/(n-1) = 12(56/9)/11 = 224/33$.

By the central limit theorem, the distribution of sample means will be approximately normal: $\bar{X} \approx N(\mu, \sigma^2/n)$. As $\hat{\sigma}^2/n = (224/33)/12 = 56/99$, we shall assume that $\bar{X} \approx N(32, 56/99)$.

If the sample mean is $376/16 \approx 31.3$, then the standardised normal score, $z = (376/12 - 32)/\sqrt{(56/99)} \approx -0.886$. Using normal probability distribution tables, $P(\bar{X} < 376/12) = P(z < -0.89) = 0.18783 \approx 19\%$, which agrees with the claim that approximately one in five such samples would have a mean less than 31.3.

Given the claim that the population mean is 32, let us establish the lower bound for the sample mean if 5% is considered to be statistically significant; from the tables we obtain a critical z-score of -1.6449.



As $z = (\overline{X} - \mu)/\sqrt{(\sigma^2/n)}$, $-1.6449 = (m - 32)/\sqrt{(56/99)} \Rightarrow m \approx 30.8$. In other words, the sample mean would have to be less than 30.8 before the result is considered statistically significant at a 5% level and the claim that the population mean is 32 could be rejected. Hence we would accept a sample mean of 31.3.

Technically the central limit theorem states that the distribution of sample means is approximately normal and that approximation improves as n increases. For small samples, with $n < 30$, we should use the Student's t-distribution. With $n = 12$, the number of degrees of freedom are 11 and so the critical t-score for a 5% significance level is 1.796.

Solving $(m - 32)/\sqrt{(56/99)} = -1.796$ we get $m \approx 30.6$. So we would still draw the same conclusion that a sample mean of 31.3 is not statistically significant at a 5% level and we can accept the manufacturer's claim that the average contents is 32 matches.

Mean Proof

Problem

Prove that $(a + b)/2 \geq \sqrt{(ab)}$, where a, b are non-negative real numbers.

Solution

Without loss of generality, assume that $a \leq b$; let $b = a + k$, where $k \geq 0$.

Therefore, $(a + b)/2 = (a + a + k)/2 = (2a + k)/2 = a + k/2$.

Also, $ab = a(a + k) = a^2 + ak = (a + k/2)^2 - k^2/4$.

Clearly $(a + k/2)^2 \geq (a + k/2)^2 - k^2/4 = ab$; that is, $((a + b)/2)^2 \geq ab$.

Hence $(a + b)/2 \geq \sqrt{(ab)}$.

Alternatively we begin with the observation that $(\sqrt{a} - \sqrt{b})^2 \geq 0$.

Expanding we get, $a + b - 2\sqrt{(ab)} \geq 0$, which leads to $(a + b)/2 \geq \sqrt{(ab)}$.

Prove that $(a + b + c)/3 \geq \sqrt[3]{(abc)}$.
Can you generalise?

MEAN SEQUENCE

Problem

A second order recurrence relation is defined by $u_{n+1} = (u_n+u_{n-1})/2$; that is, each new term is the mean of the previous two terms.

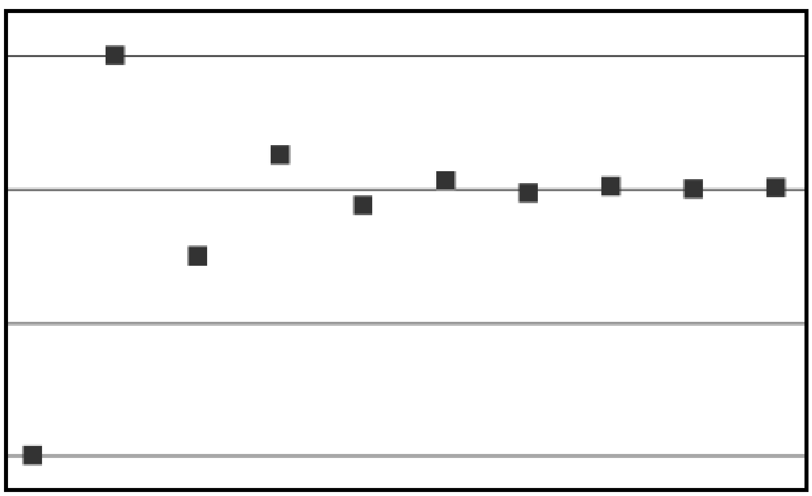
For example, when $u_1=2$ and $u_2=5$, we generate the following sequence:

$$2, 5, 3.5, 4.25, 3.875, \ldots .$$

Find the limit of the sequence for different starting values.

Solution

Consider the following diagram.



By the definition of the sequence, the next term, u_{k+1} will lie between the two previous terms, u_k and u_{k-1}. Thus, as this process continues, each iteration will reduce the upper and lower bounds, demonstrating that the sequence converges towards a value.

Let the distance between the first and second term be defined as one unit, so that, from the first term u_1, the terms follow the following series:

$$S = 1 - 1/2 + 1/4 - 1/8 + \ldots$$
$$2S = 2 - 1 + 1/2 - 1/4 + 1/8 - \ldots$$
$$\therefore 3S = 2 \Rightarrow S = 2/3$$

That is, the sequence converges towards a value, L, which is 2/3 above the first term: $L = u_1+2(u_2-u_1)/3 = (u_1+2u_2)/3$.

If $u_1 > u_2$ then the sequence follows the series:

$$S = -1 + 1/2 - 1/4 + \ldots$$
$$2S = -2 + 1 - 1/2 + 1/4 - \ldots$$
$$\therefore 3S = -2 \Rightarrow S = -2/3$$

And we get the same limit, $L = u_1 - 2(u_1 - u_2)/3 = (u_1 + u_2)/3$.

What about the third order recurrence relation: $u_{n+1} = (u_n + u_{n-1} + u_{n-2})/3$?
Investigate the *k* th order recurrence relation.

Multiple Of Six Difference

Problem

Given that x and y are integer, prove that the difference between the expressions $x^3 + y$ and $x + y^3$ is a multiple of six.

Solution

Without loss of generality suppose that $x^3 + y > x + y^3$.

$$\begin{aligned} \therefore (x^3 + y) - (x + y^3) &= x^3 - x - y^3 + y \\ &= x(x^2 - 1) - y(y^2 - 1) \\ &= x(x - 1)(x + 1) - y(y - 1)(y + 1) \end{aligned}$$

As $(x - 1)$, x, and $(x + 1)$ are three consecutive integers, at least one of the terms must be even and exactly one of the terms must be a multiple of three, so the product will be a multiple of six. In the same way $y(y - 1)(y + 1)$ will be a multiple of six.

Hence the difference will always be a multiple of six.

MULTIPLICATIVELY PERFECT

Problem

The proper divisors of a positive integer, n, are all the divisors excluding n itself. For example, the proper divisors of 6 are 1, 2, and 3.

A number, n, is said to be multiplicatively perfect if the product of its proper divisors equals n. The smallest such example is six: $6 = 1 \times 2 \times 3$; the next such example is eight: $8 = 1 \times 2 \times 4$.

Determine the nature of all multiplicatively perfect numbers.

Solution

We shall begin by exploring numbers of the form, $n = p^k$, where p is prime.

Thus we are looking for $n = p^k = 1 \times p \times p^2 \times \ldots \times p^{k-1} = p^{1+2+\ldots+(k-1)}$.

$$\therefore 1 + 2 + \ldots + (k-1) = k(k-1)/2 = k$$
$$\therefore k^2 - k = 2k$$
$$\therefore k(k-3) = 0 \Rightarrow k = 3$$

Hence the cube of all primes will be multiplicatively perfect.

It should be clear that any number of the form, $n = pq$, where p and q are distinct prime, will also be multiplicatively perfect, as the proper divisors will be 1, p, and q.

Moreover we can prove that where there are at least two prime factors it is only numbers that have exactly two distinct prime factors that are multiplicatively perfect.

Suppose that $n = mpq$, where m is some integer greater than 1 which could be prime or composite.

The proper divisors of n will be at least 1, m, p, q, but also mp, mq, pq, and possibly more if m is composite. Therefore the product of proper divisors will exceed n and it cannot be multiplicatively perfect.

Hence n will be multiplicatively perfect iff it is of the form p^3 or pq.

ODD POWER DIVISIBILITY

Problem

Prove that $6^n + 8^n$ is divisible by 7 iff n is odd.

Solution

We shall prove this in two different ways. The first proof, which is the simplest, makes use of congruences and the second proof makes use of the binomial expansion .

First Proof

As $6 \equiv -1 \bmod 7$ and $8 \equiv 1 \bmod 7$ it follows that $6^n \equiv (-1)^n$ and $8^n \equiv 1$.

$\therefore S = 6^n + 8^n \equiv (-1)^n + 1 \bmod 7$.

If n is even, $(-1)^n = 1 \Rightarrow S \equiv 2 \bmod 7$.
If n is odd, $(-1)^n = -1 \Rightarrow S \equiv 0 \bmod 7$.

Thus S is divisible by 7 iff n is odd.

Second Proof

Let $x = 6^n = (7-1)^n$ and $y = 8^n = (7+1)^n$.

$$\begin{aligned} \therefore y &= 7^n + C(n,n-1)7^{n-1} + C(n,n-2)7^{n-2} + \ldots + C(n,2)7^2 + C(n,1)7 + 1 \\ x &= 7^n - C(n,n-1)7^{n-1} + C(n,n-2)7^{n-2} - \ldots \qquad \pm 1 \end{aligned}$$

$$\therefore x + y = 2\times 7^n + 2C(n,n-2)7^{n-2} + 2C(n,n-4)7^{n-4} + \ldots$$

We note that the last term of the series for x will be -1 if n is odd and $+1$ if n is even. Therefore the series for $x + y$ will end $2C(n,1)7$ if n is odd and 2 if n is even. In other words, all the terms will divide by 7 except the last term when n is odd.

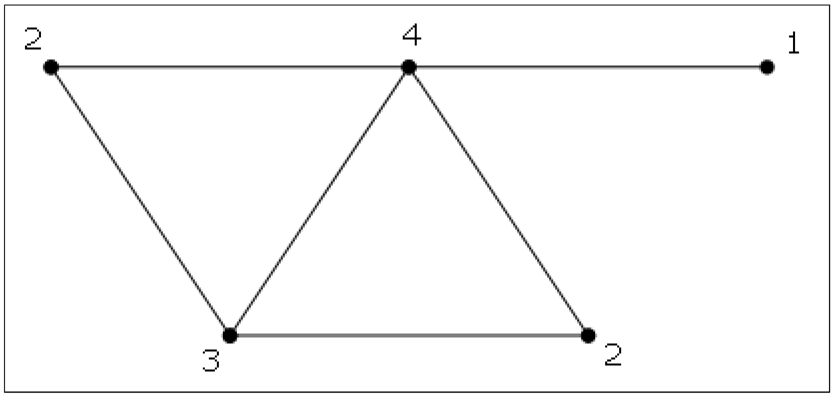
Hence $x + y \equiv 0 \bmod 7$ iff n is odd.

What can you deduce about $x + y \bmod 49$?
Investigate the remainder when $(a-1)^n + (a+1)^n$ is divided by a or a^2.

ODD VERTICES

Problem

A graph consists of vertices and edges. The order of a vertex is defined to be the number of connected edges. For example, in the graph below the order of each vertex is identified.



It can be seen that there are two odd vertices and three even vertices.

Prove that in any graph there will always be an even number of odd vertices.

Solution

Let us suppose that we already have a graph for which there are an even number of odd vertices.

First we note that adding an unconnected vertex does no change the order of any existing vertex, and as the new vertex will have order zero, we do not change the number of odd vertices in the graph.

However, if we add a new edge it must connect between two existing vertices and there are precisely three possibilities depending on the current order of the two vertices it will be connected to:

i. If they both have an even order the new edge will make them both odd, increasing the number of odd vertices by two.
ii. If they are both odd the new edge will make them both even, reducing the number of odd vertices by two.
iii. If one is odd and the other is even the new edge will make one vertex odd and the other even, not changing the total number of odd vertices.

Whichever way we can see that IF the graph had an even number of odd vertices before we added the new edge it will have an even number after we add it.

It should be clear that all graphs can be created by starting with a collection of unconnected vertices and then adding the edges one at a time.

As a collection of unconnected vertices each have order zero, the graph satisfies the requirement that there be an even number of odd vertices.

We have just shown that adding new edges will not violate this balance. Hence we prove inductively that all graphs have an even number of odd vertices.

PANDIGITAL PRIMES

Problem

The digits 0 to 9 are written on ten pieces of card and, by arranging them into five blocks of two, it is possible to form five primes. For example,

0	5	2	3	4	7	6	1	8	9

How many different sets of five primes can you form this way?

Solution

Clearly zero can only be used as a leading digit: 02, 03, 05, and 07. It is also useful to note that, with the exception of 02, no prime ends in an even digit, so evens must be used as leading digits.

02, 03, 05, 07
23, 29
41, 43, 47
61, 67
83, 89

If we use 02 as one of our numbers, it will be necessary to use a prime made up of two odd digits, as 4, 6, and 8 will be used as leading digits for three out of the other four 2-digit numbers. However, as 5 can no longer be used as a final digit, we know that 5 must be used as a leading digit: 53 or 59.

We shall consider two cases:

<u>Case 1</u> (all leading digits are even)

We can see that 9 can only be used in 29 or 89. But if we use the 9 in 29, we must use the 3 with 83; similarly, if we use 9 with 89, we must the 3 with 23. That is, 3 and 9 must be used with 2* and 8*.

We can now eliminate 03 and 43, and using the same reasoning, we can see that as 1 and 7 must be used with 4* and 6*.

The only possible remaining combination is 05.

This gives the solution sets: {05, 23, 41, 67, 89}, {05, 23, 47, 61, 89}, {05, 29, 41, 67, 83}, and {05, 29, 67, 41, 83}.

Case 2 (using 02 and 5*)

By similar reasoning to the first case, we can see that the 3 and 9 can only be used with 5* and 8*.

This leaves 1, 4, 6, and 7, and 41, 47, 61, and 67 are all prime.

Giving the solution sets: {02, 41, 53, 67, 89}, {02, 41, 59, 67, 83}, {02, 47, 53, 61, 89}, and {02, 47, 59, 61, 83}.

Hence there are 8 solutions in total.

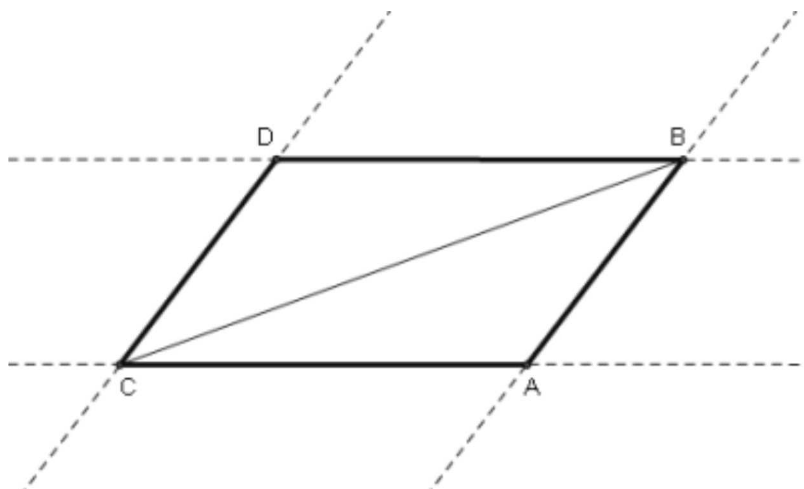
PARALLELOGRAM PROPERTY

Problem

Given that a parallelogram is a quadrilateral with two pairs of parallel sides, prove that a quadrilateral has opposite sides of equal length if and only if it is a parallelogram.

Solution

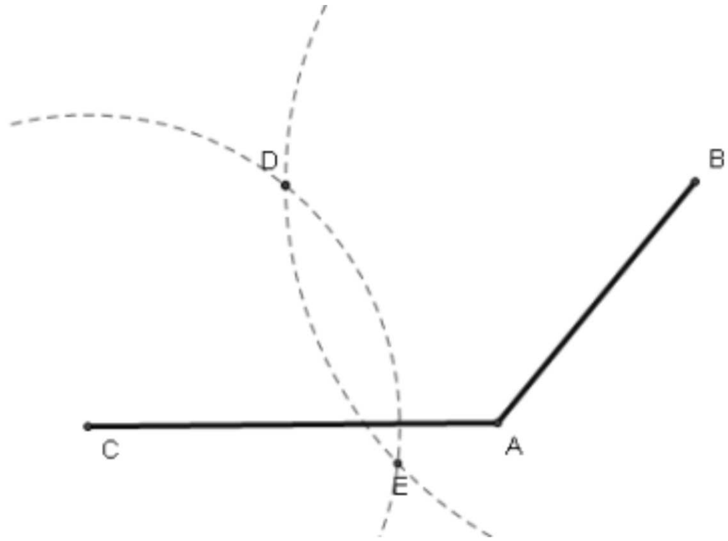
The first part of the proof is to show that a parallelogram has opposite sides equal length.



As AC is parallel with BD we know that the alternate angles, ACB and DBC, are equal. Similarly angles ABC and DCB are equal.

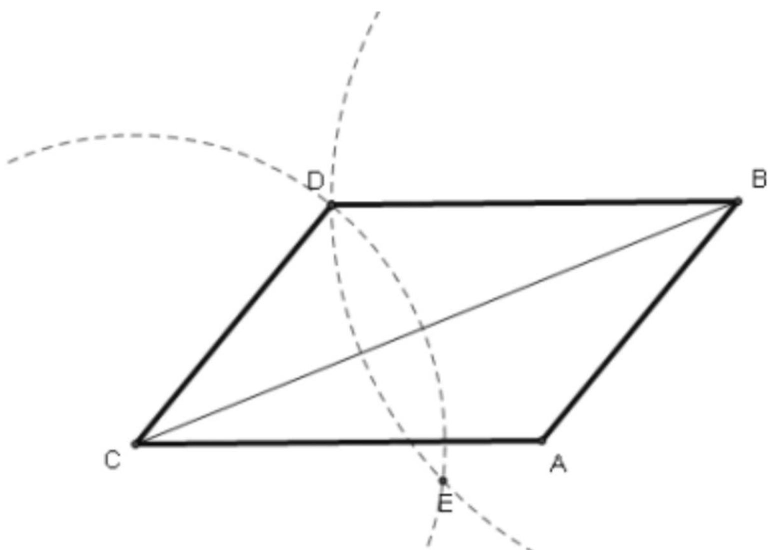
In triangles ABC and BCD they share the diagonal BC and by the angle/side/angle property we determine that they are congruent. Hence length AB = CD and AC = BD.

Now we shall prove the converse, that a quadrilateral with opposite sides equal length must be a parallelogram.



We begin with the line segments AB and AC, and without loss of generality let us suppose that AB ≤ AC. Construct two circles: centre B, radius AC; and centre C, radius AB. The intersection of these two circles generates two points, D and E.

It can be seen that these points of intersection, D and E, lie either side of AC. So it is only by connecting B (and C) to D will we produce a simple polygon (connecting B to E will cross the segment AC producing a complex polygon).



By the method of construction we know that AB = CD and AC = BD. Therefore the triangles ABC and BCD have three common lengths and must be congruent.

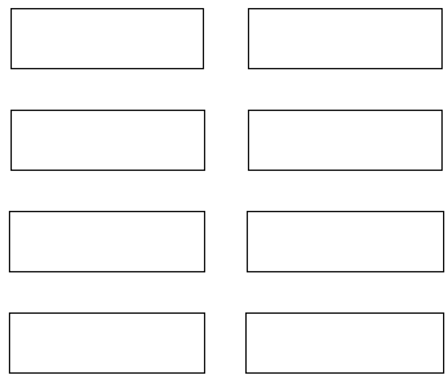
Because angles ACB and DBC are equal, segments AC and BD must be parallel. Similarly AB is parallel with CD and we prove that ABCD is a parallelogram.

Hence we have proved that a quadrilateral has opposite sides of equal length if and only if it is a parallelogram.

Pathway Arrangements

Problem

A pathway measuring four units in length can be paved in exactly eight different ways using any combination of paving stones measuring one to four units in length.



Find the number of ways that a pathway measuring n units in length can be paved if paving stones of any whole unit length size can be used.

Solution

We begin by showing that each pathway arrangement can be represented by a binary string with the alternating lengths of 0's and 1's corresponding with the length of each paving stone.

Using this system, the arrangements given in the problem are demonstrated.

0 101	⇔	1	1	1	1
0 010	⇔	2	1	1	
0 110	⇔	1	2	1	
0 100	⇔	1	1	2	
0 011	⇔	2	2		
0 001	⇔	3	1		
0 111	⇔	1	3		
0 000	⇔	4			

Clearly for a pathway length n, the string must comprise of n bits. However, note that all the strings begin with zero, otherwise a string like 1010 would be a duplicate of its "inverted" string, 0101. In other words, we need only concern ourselves with the last $n-1$ bits; it is also significant to note that every combination of these $n-1$ bits corresponds precisely with a unique pathway arrangement.

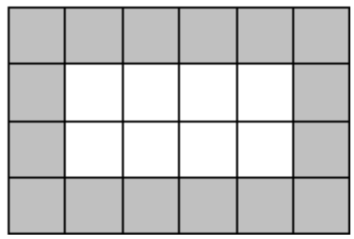
Hence there are exactly 2^{n-1} different pathway arrangements.

What if the maximum length stone is two units in length?
Investigate for different maximum length stones.

PECULIAR PERIMETER

Problem

From the diagram below we can see that the number of tiles on the perimeter, 16, exceeds the number of tiles on the inside, 8.



How many rectangles exist for which the number of tiles on the perimeter are equal to the number of tiles on the inside?

Solution

If the rectangle measures $m \times n$, the number of tiles on the perimeter is $2m + 2n - 4$.

For the number of tiles inside to equal the number of tiles on the perimeter, they must both be equal to half the area.

Therefore $\frac{mn}{2} = 2m + 2n - 4$, which gives $mn - 4m - 4n + 8 = 0$

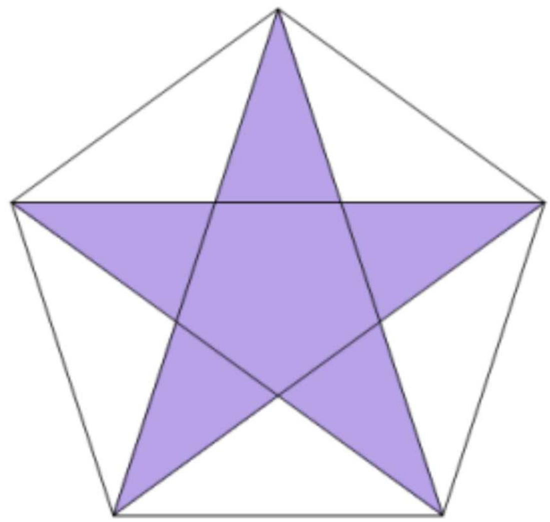
By writing this as $mn - 4m - 4n + 16 = 8$, we get $(m - 4)(n - 4) = 8$.

As m and n are both integer we obtain the factor pairs 1×8 and 2×4. Hence the only two solutions are rectangles measuring 5×12 and 6×8.

PENTAGON STAR

Problem

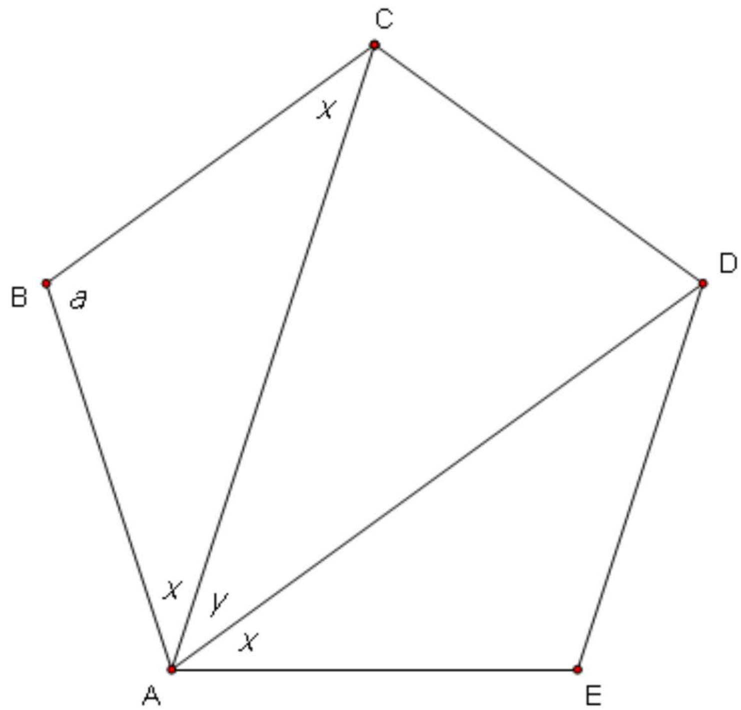
In a regular pentagon the diagonals are joined to form a star.



What fraction of the pentagon does the star occupy?

Solution

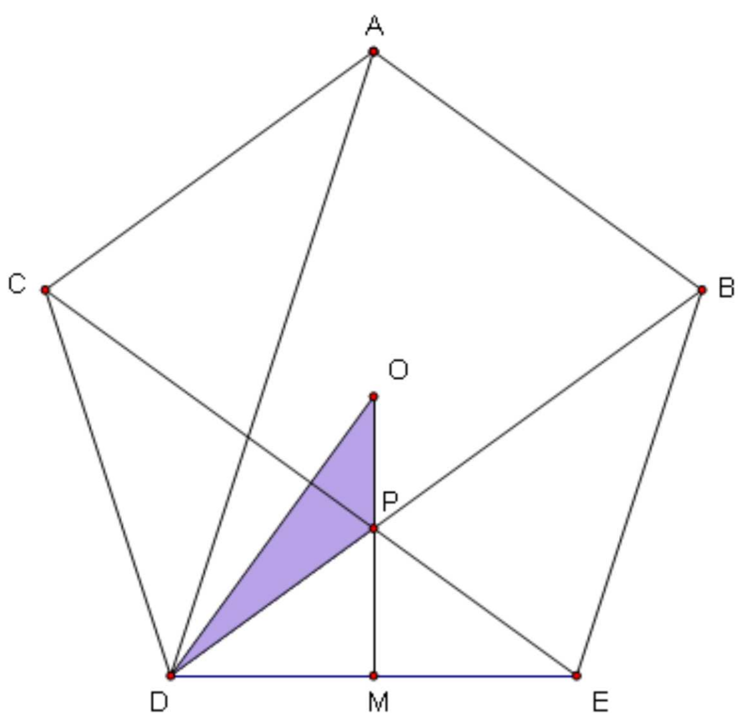
Consider the following diagam.



As the interior angle of the pentagon, $a = 108^{o}$, $2p = 180 - 108 = 72$, therefore $p = 36$.

But $a = 2p + q$, $108 = 72 + q \Rightarrow q = 36$; that is, the two diagonals in a regular pentagon trisect the interior angle.

In the diagram below, M is the midpoint of DE and O is the centre.



It should be clear that we only need consider the fraction of triangle ODM that is shaded in order to determine the fraction of the whole pentagon that is occupied by the star.

In addition, as triangle ODM and triangle PDM share the same base we need only consider the ratio of their respective heights, PM and OM.

As OD bisects angle ADB, angle ODM = 18 + 36 = 54, so OM = DM tan(54), and PM = DM tan(36).

Therefore triangle PDM occupies tan(36)/tan(54) ≈ 53% of triangle ODM, and as this represents the fraction that is not shaded, we deduce that the star occupies approximately 47% of pentagon.

Prove without the use of a calculator that tan(36)/tan(54) > 1/2.

PERFECT TRIANGLES

Problem

The divisors of a positive integer, excluding the number itself, are called the proper divisors . If the sum of proper divisors is equal to the number we call the number perfect. For example, the divisors of 28 are 1, 2, 4, 7, 14, and 28, so the sum of proper divisors is 1 + 2 + 4 + 7 + 14 = 28.

The first eight perfect numbers are 6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128.

It can be shown that P is an even perfect number iff it is of the form $2^{n-1}(2^n-1)$ where 2^n-1 is prime.

Prove that all even perfect numbers are triangle numbers.

Solution

By definition the mth triangle number, $T_m = 1 + 2 + 3 + ... + m = m(m + 1)/2$.

Let $m = 2^n-1$.

$$\begin{aligned} \therefore T_m &= (2^n-1)(2^n-1+1)/2 \\ &= (2^n-1)(2^n)/2 \\ &= 2^{n-1}(2^n-1) \end{aligned}$$

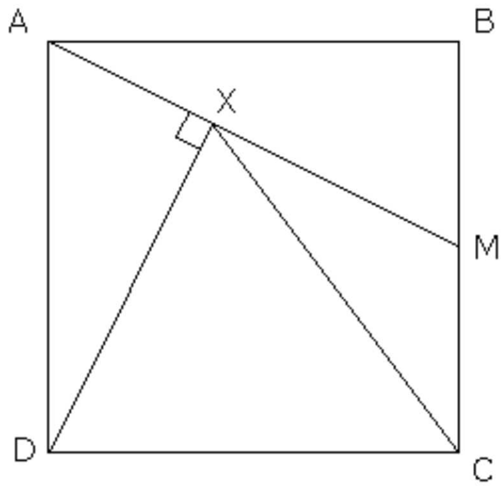
Hence all even perfect numbers are triangle numbers.

Furthermore, T_m is a perfect number if m is a prime of the form 2^n-1. For example, $2^2-1 = 3$ and $T_3 = 6$; $2^3-1 = 7$ and $T_7 = 28$; $2^5-1 = 31$ and $T_{31} = 496$; and so on.

PERPENDICULAR CONSTRUCTION

Problem

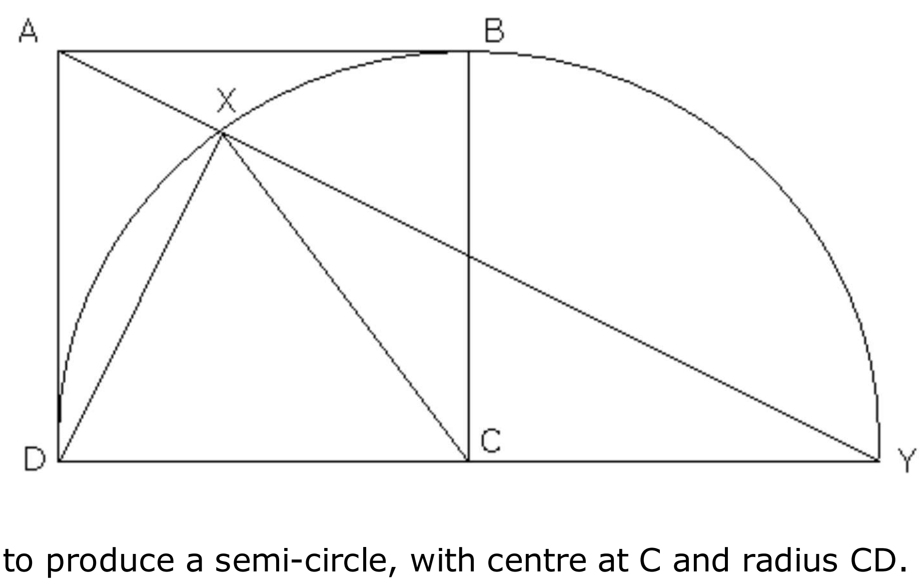
ABCD is a unit square, M is the midpoint of BC, and DX is perpendicular to AM.



Prove that triangle DXC is isosceles but not equilateral.

Solution

Consider the following diagram.



It is possible to produce a semi-circle, with centre at C and radius CD. As X lies on the semi-circle, triangle DXY will be right angle, and length CX will be equal to the radius, 1 unit. Hence lengths CD and CX are equal and we have proved that the triangle is, at least, isosceles.

Using the Pythagorean Theorem, $AY^2 = 2^2 + 1^2 \Rightarrow AY = \sqrt{5}$.

By similar triangles, DX/DY = AD/AY.

Therefore $DX/2=1/\sqrt{5} \Rightarrow DX = 2/\sqrt{5}$.

Hence triangle DXC is isosceles (CD=CY=1), but not equilateral ($DX=2/\sqrt{5}$).

By labelling D as the origin,

i. find the equation of the line passing through A and X.
ii. use the result that the product of gradients is -1 iff two lines are perpendicular to find the equation of the line through D and X.
iii. find the point of intersection of the two lines, X.
iv. use this information to solve the problem.

PLATONIC SOLIDS

Problem

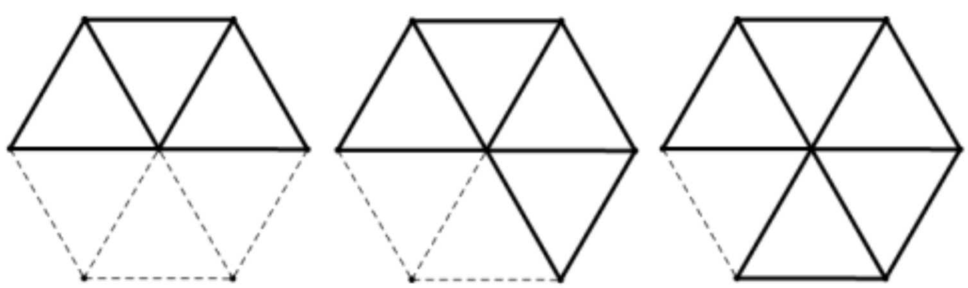
It is well known that the five Platonic solids are the regular tetrahedron (four equilateral triangle faces), cube (six square faces), regular octahedron (eight equilateral triangle faces), regular dodecahedron (twelve regular pentagon faces), and the regular icosahedron (twenty equilateral triangle faces).

Prove that no more than five regular (convex) polyhedra exist.

Solution

We begin by noting that for a polyhedron to be regular each face must be a regular polygon, also we note that at least three faces must meet at a common vertex. It shall also be assumed that we already know that a regular tetrahedron is formed from four equilateral triangles, similarly a cube is formed from six squares, a regular octahedron is formed from eight equilateral triangles, a regular dodecehedron is formed from twelve regular pentagons, and a regular icosahedron is formed from twenty equilateral triangles.

It can be seen that no more than five equilateral triangles can be placed around a central vertex before we form a (flat) planar shape.

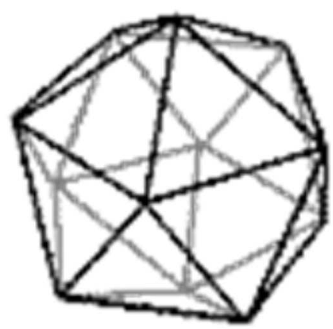


Consider the diagram on the left: as we pull together the two radii on the horizontal diagonal we form an equilateral triangle base pyramid. That is, we create the first of the Platonic solids: the regular tetrahedron.

Four equilateral triangles "pull together" to form a square base pyramid and two of these combine to create a regular octahedron.

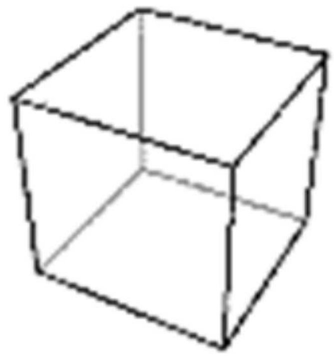
(Note that we have not proved that a regular octahedron is actually being formed, rather that a solid consisting of eight equilateral triangles is made.)

By using five equilateral triangles we form a pentagon base pyramid and four of these combine to produce a regular icosahedron.



It can be seen that six equilateral triangles will tesselate perfectly around the central vertex to a regular hexagon, and therefore cannot be a candidate for creating a regular polyhedron.

Next we consider placing squares around a vertex. It should be clear that it is only possible to fit three squares together (four would tessalate perfectly). Two of these would combine to form a cube.



In a similar way it should be clear that only three pentagons can be placed around a central vertex (three interior angles of 108^{o} add to 324^{o}); four of these "crowns" fit togther to form a regular dodecahedron.

As the interior angle of a regular hexagon is 120^{o}, three of these will tesselate perfectly around a central vertex, and clearly for any polygon containing more edges we will not be able to fit at least three around a central vertex at all.

Hence there can be no more than five regular (convex) polyhedra.

Prove that eight equilateral triangles must necessarily form a regular octahedron. What about the other Platonic solids?

POLYNOMIAL ROOTS

Problem

Let $P(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_2x^2 + c_1x + c_0$, where each of the coefficients, c_k, are integer.
Prove that the roots of the equation $P(x) = 0$ will either be irrational or integer.

Solution

It is clear that irrational roots exist. For example, $P(x) = x^2 - 2 = 0 \Rightarrow x = \pm\sqrt{2}$.

So let us suppose that x is rational and can be written as the ratio of two integers, a / b, such that GCD(a, b) = 1. By letting $x = a / b$ we get:

$$\frac{a^n}{b^n} + c_{n-1}\frac{a^{n-1}}{b^{n-1}} + \ldots + c_2\frac{a^2}{b^2} + c_1\frac{a}{b} + c_0 = 0$$

Multiplying through by b^n gives:

$$a^n + c_{n-1}a^{n-1}b + \ldots + c_2a^2b^{n-2} + c_1 ab^{n-1} + c_0b^n = 0$$

It can be seen that all terms are integer.

If we divide through by b then the right hand side will be integer, but the left side can only be integer if $b \mid a^n$. As a and b are relatively prime, b must equal 1, which means that if x is rational then it must be integer.

Hence the roots of the equation $P(x) = 0$ will either be irrational or integer. **Q.E.D.**

POWERFUL DIVISIBILITY

Problem

Given that n is a positive integer, prove that $21^n - 5^n + 8^n$ is always divisible by 24.

Solution

By using the result,
$x^n - y^n = (x - y)(x^{n-1} + \ldots + y^{n-1})$

$21^n - 5^n + 8^n = (21 - 5)(21^{n-1} + \ldots + 5^{n-1}) + 8^n = 16X + 8^n \equiv 0 \; mod \; 8$

And by writing the expression differently,
$8^n - 5^n + 21^n = (8 - 5)(8^{n-1} + \ldots + 5^{n-1}) + 21^n = 3Y + 21^n \equiv 0 \; mod \; 3$

As the expression is divisible by both 3 and 8, it must be divisible by 24.

POWERFUL DIVISOR

Problem

Consider the expression $x^x + 1$, where x be a positive integer.

It can be verified that $x = 7$ is the least value for which $x^x + 1$ divides by 2^3.

Given that n is a positive integer, find the least value of x for which $x^x + 1$ is divisible by 2^n.

Solution

As $n \geq 1$, 2^n will be even and for $x^x + 1$ to be even it is clear that x must be odd.

Consider the following factorisations:

$$x^3 + 1 = (x + 1)(x^2 - x + 1)$$
$$x^5 + 1 = (x + 1)(x^4 - x^3 + x^2 - x + 1)$$
$$x^7 + 1 = (x + 1)(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1)$$

That is, for odd values of x, $x^x + 1 = (x + 1)(x^{x-1} - x^{x-2} + x^{x-3} - \ldots + x^2 - x + 1)$.

Each term in the second bracket will be odd and as x is odd there will be an odd number of terms creating an odd sum. Hence $x^x + 1$ will be divisible by 2^n if and only if it divides $x + 1$.

Thus $x = 2^n - 1$ is the least value for which $x^x + 1$ is divisible by 2^n.

Power Divisibility

Problem

Consider the following results.

$8^1-1 = 7 = 7\times1$
$8^2-1 = 63 = 7\times9$
$8^3-1 = 511 = 7\times73$
$8^4-1 = 4095 = 7\times585$
$8^5-1 = 32767 = 7\times4681$

Prove that 8^n-1 is always divisible by 7.

Solution

Clearly it is true for $n=1$: $8^1-1 = 7$.

Assume that it is true for $n=k$: 8^k-1 is divisible by 7.

Consider the next case, $n=k+1$.

$8^{k+1}-1 = 8^k8-1 = 8(8^k-1)+8-1 = 8(8^k-1)+7$.

That is, if 8^k-1 is divisible by 7, $8^{k+1}-1$ will also divide evenly by 8. As it works for $n=1$, it must be true for all n.

Prove that a^n-1 is always divisible by $a-1$.

PRIMES AND SQUARE SUMS

Problem

Prove that there exists no prime which is one less than a multiple of four that can be written as the sum of two squares.

Solution

Let $p = a^2 + b^2$.

With the exception of $p = 2$ (which is not one less than a multiple of 4 anyway) all primes are odd. Hence one square must be even and one square must be odd for their sum to be odd.

So W.L.O.G. (without loss of generality) let $a = 2x$ (even) and
$b = 2y + 1$ (odd).

$\therefore a^2 = (2x)^2 = 4x^2$
$b^2 = (2y + 1)^2 = 4y^2 + 4y + 1$

$\therefore p = 4x^2 + 4y^2 + 4y + 1 = 4(x^2 + y^2 + y) + 1 = 4k + 1$

That is, an odd square and an even square always adds to a number which is one more than a multiple of 4. Hence we prove that no prime which is one less than a multiple of 4 can be written as the sum of two squares.

Note

It is important to realise that the proof given shows that an odd square added to an even square will produce a number of the form $4k + 1$. It does not show that every number of the form $4k + 1$ can be obtained by adding an odd and even square. For example, the number 9 is of the form $4k + 1$, and this cannot be obtained by adding two squares. However, Pierre de Fermat (1601-1665) was able to prove that ALL primes of the form $4k + 1$ can be written as the sum of two squares, but the proof for this is far beyond the scope of an elementary approach.

PRIME DIFFERENCE

Problem

Given that a and b are positive integers and the difference between the expressions $a^2 + b$ and $a + b^2$ is prime, find the values of a and b.

Solution

Without loss of generality suppose that $a^2 + b > a + b^2$.

$\therefore p = (a^2 + b) - (a + b^2) = (a - b)(a + b - 1)$

As $a - b < a + b - 1$ we have $a - b = 1$ and $a + b - 1 = p$.

Adding both expressions gives $2a - 1 = p + 1 \Rightarrow a = p/2 + 1$.

Clearly p cannot be odd. Hence there is a unique solution when $p = 2$, for which $a = 2$ and $b = 1$.

Prove that $(a^2 + b) - (a + b^2)$ is always even.

Prime Exponent And Fourth Power Sum

Problem

Given that p is prime, when is $4^p + p^4$ prime?

Solution

For any prime, $p > 5$, $p \equiv 1, 3, 7, 9 \bmod 10$.

If $p \equiv 1, 9 \bmod 10$, then $p^2 \equiv 1 \bmod 10$, and $p^4 \equiv 1 \bmod 10$.

If $p \equiv 3, 7 \bmod 10$, then $p^2 \equiv -1 \bmod 10$, and $p^4 \equiv 1 \bmod 10$.

Therefore, for $p > 5$, $p^4 \equiv 1 \bmod 5$.

We note that $4^1 \equiv 4 \bmod 5$. Now if $4^m \equiv 4 \bmod 5$, then multiplying both sides by 4^2 gives $4^{m+2} \equiv 4 \bmod 5$. That is, if n is odd then $4^n \equiv 4 \bmod 5$.

Hence, for $p > 5$, $4^p + p^4 \equiv 0 \bmod 5$, and cannot be prime.

When $p = 2$, $4^p + p^4 = 32$.
When $p = 3$, $4^p + p^4 = 145$.
When $p = 5$, $4^p + p^4 = 1649 = 17 \times 97$.

Hence the expression $4^p + p^4$ is never prime.

See Prime Exponent And Square Sum.

Prime Exponent And Square Sum

Problem

Given that p is prime, when is $2^p + p^2$ prime?

Solution

For $p > 3$, all primes can be written in the form, $p = 6k \pm 1$.

Therefore $p^2 = 36k^2 \pm 12k + 1 \equiv 1 \bmod 3$.

If $2^m \equiv 2 \bmod 3$, then $2^{m+1} \equiv 1 \bmod 3$, and $2^{m+2} \equiv 2 \bmod 3$.

As $2^1 \equiv 2 \bmod 3$, it follows that $2^n \equiv 2 \bmod 3$ for all odd n.

Hence when $p > 3$, $2^p + p^2 \equiv 0 \bmod 3$, and cannot be prime.

When $p = 2$, $2^p + p^2 = 8$.
When $p = 3$, $2^p + p^2 = 17$.

That is, $2^p + p^2$ is prime only when $p = 3$.

See Prime Exponent And Fourth Power Sum.

PRIME FORM

Problem

Given that $n \geq 2$ and $a^n - 1$ is prime, prove that $a = 2$, and n must be prime.

Solution

First we note that $a^n - 1 = (a - 1)(a^{n-1} + a^{n-2} + \ldots + a^2 + a + 1)$.

As $n \geq 2$, the second bracket on the RHS (right hand side) will be greater than or equal to $a + 1 \geq 2$.

If the LHS is prime then the first bracket on the RHS, $a - 1$, must equal 1, in which case $a = 2$.

Let $n = xy$.

$\therefore 2^{xy} - 1 = (2^x - 1)(2^{(x-1)y} + 2^{(x-2)y} + \ldots + 2^{2y} + 2^y + 1)$

That is, $2^n - 1$ is divisible by $2^x - 1$, where x is a factor of n.

But as $2^n - 1$ is prime then $2^x - 1 = 1 \Rightarrow x = 1$.

Similarly if y had factors x' and y' we could show that $x' = 1$, and so on *ad infinitum*. In other words, $y = n$, which must be prime. **Q.E.D.**

Does it follow that $2^n - 1$ is prime if n is prime?

PRIME POWER

Problem

Given that p is prime it can be seen that $p + 1$ is a perfect cube when $p = 7$. What is most surprising is that this is the only value of p for which $p + 1$ is a perfect cube.

Prove that there only ever exists one prime value p for which $p + 1$ is a perfect power and determine the condition for this perfect power to exist.

Solution

Let $p + 1 = a^k$.

$\therefore p = a^k - 1 = (a - 1)(a^{k-1} + a^{k-2} + ... + a^2 + a + 1)$

For RHS to be prime, $a - 1 = 1 \Rightarrow a = 2$.

Hence for $p + 1$ to be a perfect power it must be of the form 2^k. Moreoever, as $p = 2^k - 1$ it can be seen that there will only exist a perfect k^{th} power if $2^k - 1$ is prime.

For example, when $k = 4$, $2^4 - 1 = 15 = 3\times5$, hence there will not exist a prime value of p for which $p + 1$ is a perfect fourth power.

In fact, it turns out that $p + 1$ can only be a perfect k^{th} power if the value of k is itself prime.

Suppose that k is composite; let $k = xy$.

$\therefore 2^{xy} - 1 = (2^x - 1)(2^{(x-1)y} + 2^{(x-2)y} + ... + 2^{2y} + 2^y + 1)$

But if $2^k - 1$ is prime then $2^x - 1 = 1 \Rightarrow x = 1 \Rightarrow k = y$.

Similarly if y had factors x" and y" we could show that x" = 1, and so on *ad infinitum*. In other words, $y = k$, which must be prime.

Does it follow that $p + 1$ can always be written as a perfect k^{th} power if k is prime?

PRIME POWER SUM

Problem

For any given prime, p, prove that $2^p + 3^p$ can never be a perfect square.

Solution

With the exception of p=2, for which $2^2 + 3^2 = 13$ and is not square anyway, the sum, $2^p + 3^p$, will always be odd.

All odd squares are congruent with 1 mod 4: $(2k+1)^2 = 4k^2 + 4k + 1$.

For $p \geqq 2$, $2^p \equiv 0 \mod 4$. So if the sum is to be a perfect square, 3^p must be congruent with 1 mod 4.

However, $3^1 \equiv 3 \mod 4$, and as p will be odd (the case for p = 2 has been excluded), we note that multiplying both sides of the congruence by 3^2 to get the next odd power, $3^3 \equiv 27 \equiv 3 \mod 4$. Similarly, $3^5 \equiv 3 \mod 4$. That is, each time we multiply by 3^2 it continues to be congruent with 3 mod 4. Therefore, $2^p + 3^p$ can never be a perfect square.

PYTHAGOREAN TRIPLET PRODUCT

Problem

A Pythagorean triplet, (a, b, c), is defined as a set of positive integers for which $a^2 + b^2 = c^2$.

Prove that for every triplet abc is a multiple of sixty.

Solution

If we can show that it is true for all primitive cases, then it must be true for all non-primitive cases (multiples of primitive triplets).

We shall use the fact that for $m > n$, all primitive triplets can be generated with the following identities (see Every Primitive Triplet):

$$a = m^2 - n^2$$
$$b = 2mn$$
$$c = m^2 + n^2$$

We shall complete this proof by showing that abc must contain a factor of 4, 3, and 5.

- Factor of 4:
  If either m or n are even then $b = 2mn$ will be a multiple of 4. If both m and n are odd, then $m^2 + n^2$ will be even, and bc will be a multiple of 4. Hence there will always be a factor of 4 present.

- Factor of 3:
  If either m or n are multiples of 3 then $b = 2mn$ will be a multiple of 3. Next we note that for each $m \equiv 1, 2 \bmod 3$, that m^2 can only be congruent with 1. Therefore $a = m^2 - n^2 \equiv 0 \bmod 3$. Hence there will always be a factor of 3 present.

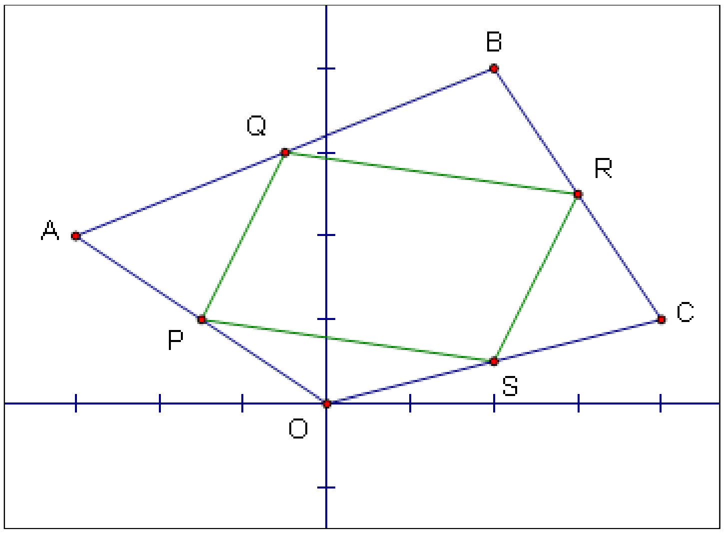
- Factor of 5:
  If either m or n are multiples of 5 then $b = 2mn$ will be a multiple of 5. Next we note that for each $m \equiv 1, 2, 3, 4 \bmod 5$, that m^2 can only be congruent with 1 or 4. Therefore, either $a = m^2 - n^2 \equiv 0 \bmod 5$ or $c = m^2 + n^2 \equiv 1 + 4 \equiv 0 \bmod 5$. Hence there will always be a factor of 5 present.

Thus we demonstrate that $abc \equiv 0 \bmod 60$ for all Pythagorean triplets.

Quadrilateral Parallelogram

Problem

Three points, A, B, and C, are chosen at random such that OABC forms a quadrilateral. The midpoints of each edge, P, Q, R, and S, are joined.



Prove that the quadrilateral PQRS will always be a parallelogram.

Solution

We shall prove this result by consideration of vectors.

Let $\overline{OA} = 2\boldsymbol{a}$, $\overline{OB} = 2\boldsymbol{b}$, and $\overline{OC} = 2\boldsymbol{c}$.

So $\overline{AB} = \overline{AO} + \overline{OB} = -2\boldsymbol{a} + 2\boldsymbol{b}$ and $\overline{CB} = \overline{CO} + \overline{OB} = -2\boldsymbol{c} + 2\boldsymbol{b}$.

The position vectors of P, Q, R, and S respectively will be:

$$\overline{OP} = (1/2)\overline{OA} = \boldsymbol{a}$$

$$\overline{OS} = (1/2)\overline{OC} = \boldsymbol{c}$$

$$\overline{OQ} = \overline{OA} + (1/2)\overline{AB} = 2\boldsymbol{a} + -\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{a} + \boldsymbol{b}$$

$$\overline{OR} = \overline{OC} + (1/2)\overline{CB} = 2\boldsymbol{c} + -\boldsymbol{c} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{c}$$

Using these we can obtain vectors for each side of the quadrilateral PQRS:

$$\overline{PQ} = \overline{PO} + \overline{OQ} = -\boldsymbol{a} + \boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b}$$

$\overline{SR} = \overline{SO} + \overline{OR} = -\boldsymbol{c} + \boldsymbol{b} + \boldsymbol{c} = \boldsymbol{b}$

$\overline{PS} = \overline{PO} + \overline{OS} = -\boldsymbol{a} + \boldsymbol{c}$

$\overline{QR} = \overline{QO} + \overline{OR} = -(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{b} + \boldsymbol{c} = -\boldsymbol{a} + \boldsymbol{c}$

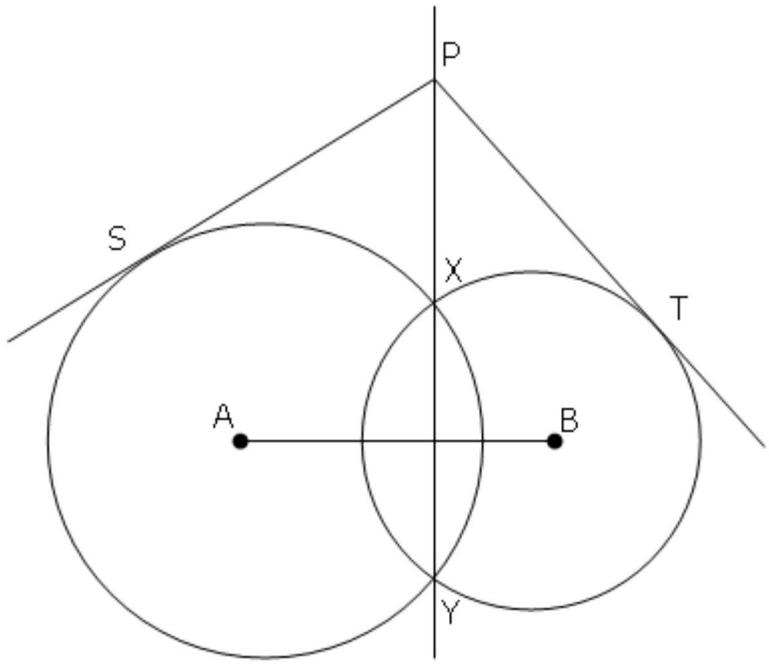
As $\overline{PQ}$ is parallel with and equal to $\overline{SR}$, similarly with $\overline{PS}$ and $\overline{QR}$, we prove that the quadrilateral PQRS is a parallelogram.

Was it necessary to consider $\overline{PS}$ and $\overline{QR}$ to prove the result?

RADICAL AXIS

Problem

Two circles, centred at A and B, intersect at X and Y. From a point, P, on the common secant through XY, tangents are drawn to each circle at S and T.

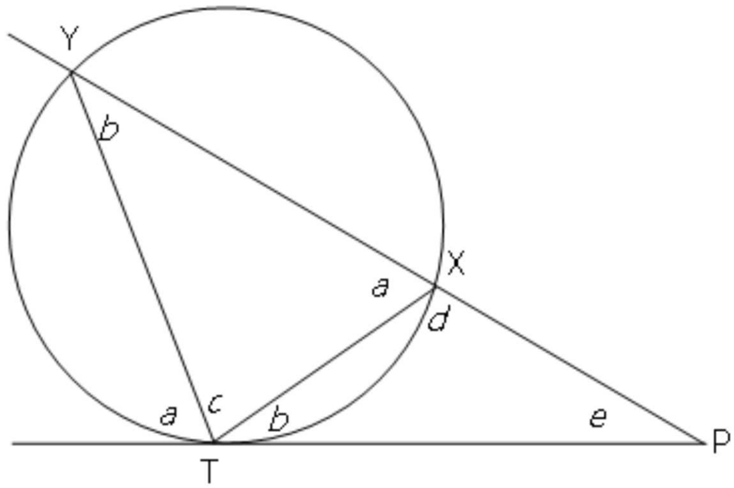


The locus of points, P, for which PT = PS is called the radical axis.

Prove that the common secant is the radical axis.

Solution

Consider the diagram below.



We note that the size of the two angles marked *a* are equal because of the alternate segment theorem (from chord TY); similarly with the size of the angles marked *b* (chord TX).

As *b*+*c* and *d* are complementary angles to *a* (with the straight line), it follows that *d* = *b*+*c*.

Therefore triangle PTY is similar to triangle PXY.

$\therefore$ PT/PY = PX/PT, which gives, $(PT)^2 = PX \times PY$.

As a consequence of this result we can see in the original diagram that
$(PT)^2 = PA \times PB$ and $(PS)^2 = PX \times PY$, hence PT = PS.

Prove that the radical axis for two non-intersecting circles is a line perpendicular to their centres.

RATIONAL QUADRATIC

Problem

Given that a and p are positive integers and p is prime, find the values of a for which the following quadratic has rational roots.

$$ax^2 - px - p = 0$$

Solution

Begin by dividing through by p: $(a/p)x^2 - x - 1 = 0$.

To have rational roots, the discriminant, $4a/p + 1 = k^2$, where $k = r/s$ and let us suppose that HCF(r,s) = 1.

$\therefore 4a/p = k^2 - 1 = r^2/s^2 - 1 = (r^2 - s^2)/s^2$, giving $4a = p(r^2 - s^2)/s^2$

Clearly $4a$ is integer which can only happen if $s = 1$, as neither p nor r^2 is divisible by s^2 for any other value; that is, k is integer.

So $4a/p = k^2 - 1 = (k + 1)(k - 1)$.

This leads to $a = p(k + 1)(k - 1)/4 \Rightarrow k$ must be odd, as a is integer.

Let $k = 2m + 1$, so that $a = p(2m + 2)(2m)/4 = pm(m + 1)$.

That is, for the quadratic to have rational roots, a must be a multiple of p, moreoever it must be of the form $pm(m + 1)$.

Prove that the roots will be of the form $1/c$ and $-1/(c + 1)$.

Rational Roots Quadratic

Problem

In the quadratic equation $ax^2 + bx + c = 0$, the coefficients a, b, c are non-zero integers.

If $b = -5$, by making $a = 2$ and $c = 3$, the equation $2x^2 - 5x + 3 = 0$ has rational roots. But what is most remarkable is that it is possible to interchange these coefficients in any order and the quadratic will still have rational roots.

Suppose that b is chosen at random. Prove that there always exist coefficients a and c that will produce rational roots. Moreover, once determined, no matter how these three coefficients are shuffled, the quadratic equation will still yield rational roots.

Solution

We shall prove this in two different ways.

Proof 1:
Although the first proof is elegant and provides a method for determining one set of values of a and c, given b, it tells us nothing about the true nature of the problem, nor does it reveal that there are infinitely many different sets of values of a and c that can be determined for any given b.

It can be verified that the quadratic equation, $2x^2 + x - 3 = 0$ has rational roots and every arrangement of coefficients will yield rational roots. But the important observation is to note that any integral multiple of this "base" equation, with $b = 1$, will lead to another quadratic with "universal" rational roots. For example, if we multiply by 7, we get $14x^2 + 7x - 21 = 0$. This is equivalent to making $b = 7$ and determining that $a = 14$ and $c = -21$.

Proof 2:
This proof is perhaps the most revealling and makes use of the fact that the discriminant, $b^2 - 4ac$, must be the square of a rational if the roots of the equation are to be rational. In fact because all the coefficients are integer we can go further by saying that the discriminant must be square.

We also note that interchanging the positions of a and c has not effect on the rationality of the discriminant. Hence we only need consider the three cases of a, b, and c being the coefficient of x. We shall initially consider the cases where b and c are the coefficient of x.

Let $b^2 - 4ac = r^2$ and $c^2 - 4ab = s^2$, where r and s are integer.

$\therefore r^2 - s^2 = b^2 - c^2 + 4ac - 4ac$

$\therefore (r + s)(r - s) = (b - c)(b + c + 4a)$

Let $r + s = b + c + 4a$ and $r - s = b - c$.

Adding both equations we get $2r = 2b + 4a \Rightarrow r = b + 2a$, and subtracting gives, $2s = 2c + 4a \Rightarrow s = c + 2a$.

Squaring r and equating with the respective discriminant we get,
$b^2 - 4ac = b^2 + 4ab + 4a^2 \Rightarrow 4a^2 + 4ab + 4ac = 0$.

Using the same procedure with s leads to exactly the same equation,
$4a(a + b + c) = 0$, and as $a \neq 0$ we deduce that $a + b + c = 0$.

We must now show that the third discriminant, $a^2 - 4bc$, is also square.

As $a = -(b + c)$, squaring both sides gives, $a^2 = b^2 + c^2 + 2bc$.

$\therefore a^2 - 4bc = b^2 + c^2 + 2bc - 4bc = b^2 + c^2 - 2bc = (b - c)^2$ **QED**

Is it possible that a quadratic equation exists for which any combination of the coefficients will yield rational roots and $a + b + c \neq 0$?

RECIPROCAL SYMMETRY

Problem

Given that a and b are positive integers, for which values is the following expression integer?

$$a/b + b/a$$

Solution

Let $c = a/b + b/a$.

Suppose that HCF(a,b) = k, such that $a = kd$ and $b = ke$.

Therefore, $c = d/e + e/d$ and HCF(d,e) = 1.

Multiplying through by d gives $cd = d^2/e + e$. Clearly cd and e are integer, so d^2/e must be integer. But as HCF(d,e) = 1, we deduce that $e = 1$; and by symmetry $d = 1$.

Hence $c = d/e + e/d = 1 + 1 = 2$, and as $a/b = d/e = 1$, we further deduce that $a = b$ can take on any non-zero integer value.

RELATIVELY PRIME PERMUTATIONS

Problem

Euler's Totient function, $\varphi(n)$, is used to determine the number of numbers less than n that are relatively prime to n. For example, $\varphi(6) = 2$, because only 1 and 5 are relatively prime with 6.

Interestingly, $\varphi(63) = 36$, and this is the first example of a number which produces a permutation of the value of its Totient function.

Given that p is prime, prove that p will not be a permutation of $\varphi(p)$, and prove that p^2 will not be a permuation of $\varphi(p^2)$.

Solution

As 1, 2, 3, ... , $p-1$ are relatively prime to p, it can be seen that $\varphi(p) = p-1$. Except for $p = 2$, for which $\varphi(2) = 1$ anyway, all other primes are odd, which means the final digit will be 1, 3, 5, 7, or 9. As $p-1$ will only change the final digit (tens and above will be unaffected), and will change it to 0, 2, 4, 6, or 8, respectively, we can see that $\varphi(p)$ cannot be a permutation of p.

For p^2 we can see that it will divide by $1\times p$, $2\times p$, ... , $p\times p$, hence $\varphi(p^2) = p^2-p = p(p-1)$.

With the exception of 3, for which $\varphi(3) = 2$ anyway, $p \equiv 1,2 \bmod 3$, so $p-1 \equiv 0,1 \bmod 3$, and $p(p-1) \equiv 0,2 \bmod 3$. Similarly, $p^2 = p\times p \equiv 1 \bmod 3$.

It is well known that a number and the sum of its digits are congruent mod 3, which means that two numbers that are permutations of one another will be congruent mod 3. As we have shown that $p^2 \equiv 1 \bmod 3$ and $\varphi(p^2) \equiv 0,2 \bmod 3$, we prove that they can never be permutations of each other.

Prove that p^k can never be a permutation of $\varphi(p^k)$.

REVERSE EQUIVALENCE

Problem

By adding the different 2-digit numbers 12 and 32 we get 44. If the digits in each number are reversed we get two different 2-digit numbers, and 21 + 23 also equals 44.
The same is true of 42 + 35 = 24 + 53 = 77.

Prove that the sum of two 2-digit numbers with this property will always be divisible by 11.

Solution

Let the 2-digit numbers be (ab) and (cd).

We shall ignore cases like 11 + 11 = 11 + 11 and 12 + 21 = 21 + 12, as proving that (ab) + (ba) = $10a + b + 10b + a = 11a + 11b$ is divisibly by 11 is trivial.

The problem requires the sum, $S = (ab) + (cd) = (ba) + (dc)$, such that (ab) ≠ (cd) ≠ (ba)

$S = (10a + b) + (10c + d) = (10b + a) + (10d + c)$.

Therefore $9a + 9c = 9b + 9d$ and so $a + c = b + d$.

Hence we obtain a pair of numbers with the required property if the sum of the first digits of each number is equal to the sum of the second digits. For example, 41 + 36 = 14 + 63 = 77, because 4 + 3 = 1 + 6.

Now $S = (10a + b) + (10c + d) = 10(a + c) + b + d$

As $a + c = b + d$, $S = 10(a + c) + a + c = 11(a + c)$.

Hence the sum, S, is always divisible by 11.

What adding three 2-digit numbers?
Can you find two 3-digit numbers with this property?

Sequence Divisibility

Problem

Prove that every term in the infinite sequence 18, 108, 1008, 10008, 100008, ... , is divisible by 18.

Solution

The n th term, u_n , of the sequence 18, 108, 1008, 10008, ... is given by
$u_n = 10^n + 8$.

$$\begin{aligned} \therefore u_{n+1} &= 10^{n+1} + 8 \\ &= 10.10^n + 8 \\ &= (9 + 1).10^n + 8 \\ &= 9.10^n + 10^n + 8 \\ &= 9.10^n + 10^n + 8 \end{aligned}$$

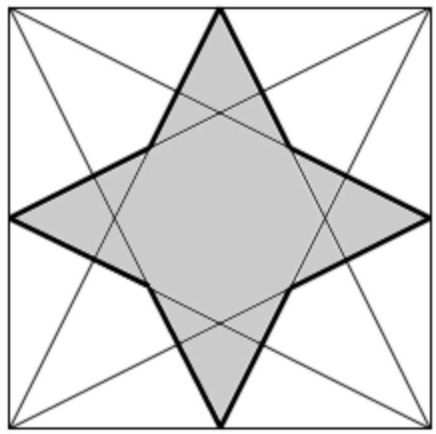
If $u_n = 10^n + 8$ is divisible by 18, then so too will be u_{n+1},
as $9 \times$ [even] will be divisble by 18.

We can see that $u_1 = 18$, hence $10^n + 8$ must be divisible by 18 for all n.

SHADED OCTAGON

Problem

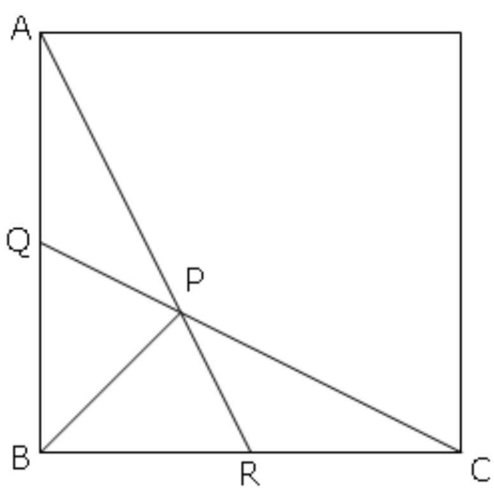
In the square below, each vertex is joined to the midpoint of the opposite sides and a star shape is formed.



What fraction of the square is shaded?

Solution

Consider the following unit.



As R is the midpoint of BC, length BR = length RC, and as ΔBPR and ΔRPC have the same altitude they must have the same area. In the same way, ΔAPQ and ΔQPB have the same area.

Therefore, A = Area ΔAPQ = Area ΔQPB = Area ΔBPR = Area ΔRPC.

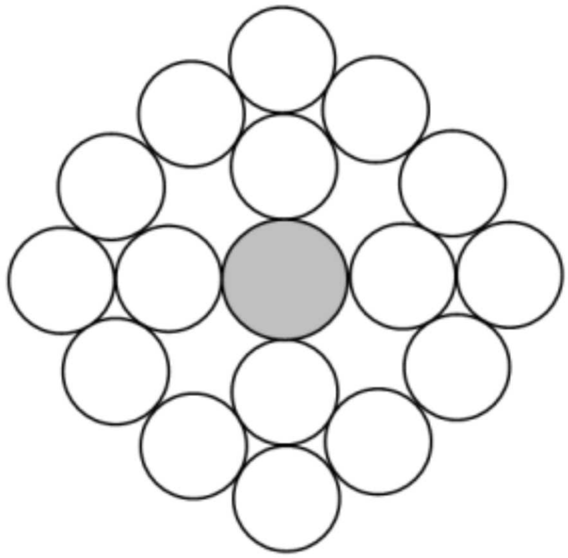
As Area $\Delta BAR = 1/4$ of the square, it follows that $3A = 1/4 \Rightarrow A = 1/12$.

Thus the area of quadrilateral BQPR = 2A, and we can see that in the original diagram $4 \times 2A = 8A = 2/3$ is unshaded. Hence the shaded "star" represents 1/3 of the area of the square.

SIXTEEN CIRCLES

Problem

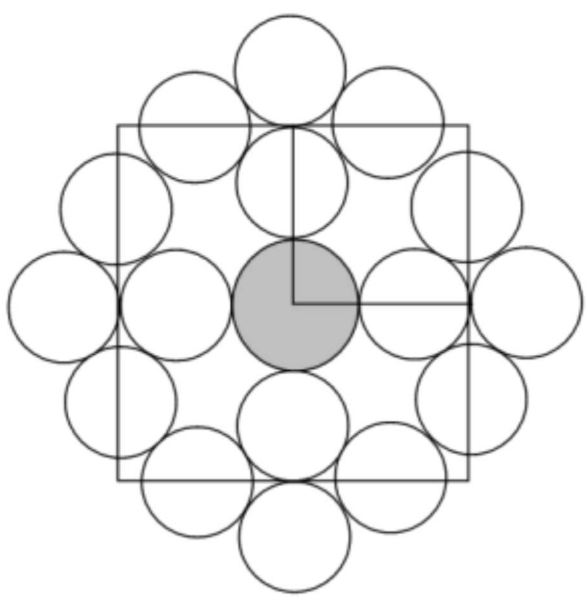
Sixteen unit circles are placed together as shown in the diagram.



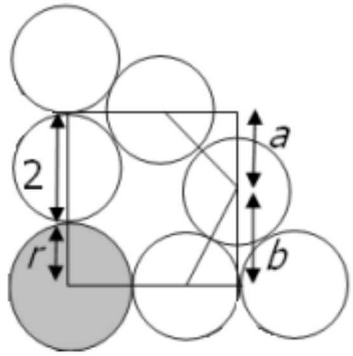
Find the radius of the shaded circle in the centre.

Solution

We begin be drawing a large square and a quarter square on the diagram.



Let us consider the top right quadrant.



Using the Pythagorean Theorem, $2^2 = a^2 + a^2 \Rightarrow a = \sqrt{2}$.

Similarly, $2^2 = b^2 + 1^2 \Rightarrow b = \sqrt{3}$.

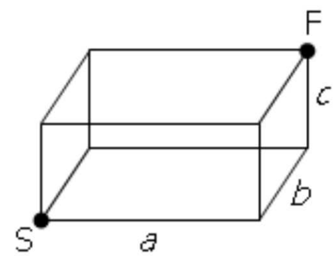
As $r + 2 = \sqrt{2} + \sqrt{3}$, we get $r = \sqrt{2} + \sqrt{3} - 2$.

If the centres of the circles in the outside ring of the original diagram are joined, will it form a regular dodecagon?

SPIDER FLY DISTANCE

Problem

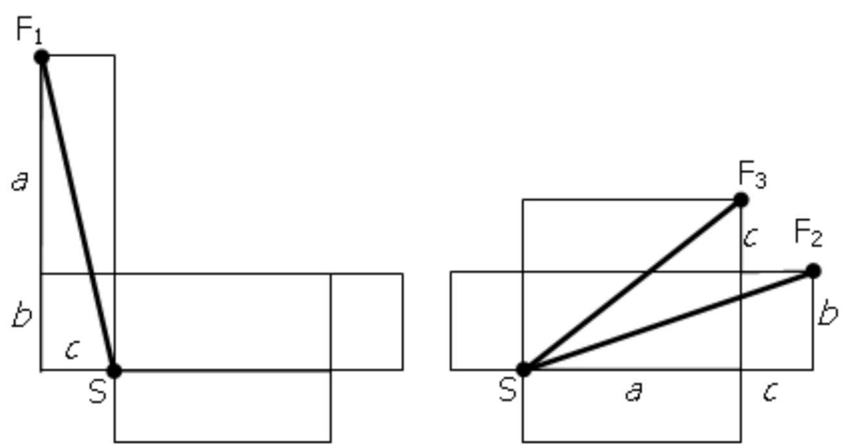
A spider, S, is in one corner of a cuboid room, with dimensions a by b by c, and a fly, F, is in the opposite corner.



Find the shortest distance from S to F.

Solution

There are three straight line routes from S to F.



Let the distances from S to F_1, F_2, and F_3, be d_1, d_2, and d_3 respectively.

Using the Pythagorean Theorem we get:

$$d_1^2 = (a+b)^2 + c^2 = a^2 + b^2 + c^2 + 2ab$$
$$d_2^2 = (a+c)^2 + b^2 = a^2 + b^2 + c^2 + 2ac$$
$$d_3^2 = (b+c)^2 + a^2 = a^2 + b^2 + c^2 + 2bc$$

Without loss of generality, let us assume that $a \geq b \geq c$.

As $b \geq c$, $ab \geq ac$, and it follows that $d_1 \geq d_2$.
Similarly, as $a \geq c$, $ab \geq bc$, and $d_1 \geq d_3$.
And finally, as $a \geq b$, $ac \geq bc$, giving $d_2 \geq d_3$.

Hence, $d_1 \geq d_2 \geq d_3$ and, of the three routes, the shortest distance would be from S to F_3; that is, the journey from S to the longest edge.

What is the smallest cuboid for which the shortest route is integer?
What about the smallest cuboid for which all three routes are integer?

SQUARE SEARCH

Problem

Given that p is prime, when is $8p+1$ square?

Solution

Let $8p + 1 = k^2$. As LHS is odd, k must be odd; let $k = 2m + 1$.

Therefore $8p + 1 = 4m^2 + 4m + 1$, leading to $2p = m^2 + m = m(m + 1)$.

As m and $m + 1$ are consecutive integers, one of them must be even. However, after dividing through by 2 we can see that LHS is prime, so RHS must be prime. This can only happen when $m = 2$; that is, $p = 3$.

Hence $8p + 1$ can only be square when $p = 3$.

Related problem:

> Triangle Search: When is $8p+1$ a triangle number?

Sum Of Squares And Multiple Of Product

Problem

Given that x, y are positive integers and $x \neq y$, consider the following two results:

$5^2 + 13^2 = 194$, $5 \times 13 \times 3 = 195$
$11^2 + 41^2 = 1802$, $11 \times 41 \times 4 = 1804$

We can see that in both cases that the sum of the squares of x and y are *almost* a multiple of their product.

Prove that $x^2 + y^2$ can never be multiple of xy.

Solution

We are considering solutions to the equation $x^2 + y^2 = kxy$, where k is some positive integer.

Suppose that GCD(x, y) = h; let $x = hm$ and $y = hn$.

Therefore $h^2m^2 + h^2n^2 = kh^2mn \Rightarrow m^2 + n^2 = kmn$, where GCD($m$, n) = 1.

So without loss of generality we can consider the primitive $m^2 + n^2 = kmn$.

As $m \neq n$ they cannot both be equal to 1, so let us suppose that $m > 1$. In addition, m must contain a prime factor that n does not contain; call this prime factor, p.

Clearly m^2 and kmn are divisible by p, but n^2 does not contain this factor. Hence no integer solution exists and we prove that $x^2 + y^2$ can never be multiple of xy.

If x, y, and z are positive integers, does $x^2 + y^2 + z^2 = xyz$ contain solutions?

SUM PRODUCT NUMBERS

Problem

A positive integer, N, is an sum-product number if there exists a set of positive integers:
$S = \{a_1, a_2, ..., a_k\}$, such that $N = a_1 \times a_2 \times ... \times a_k = a_1 + a_2 + ... + a_k$. In order for the set to be a sum and product, it is necessary that S contains at least two elements.

Prove that N is a sum-product number iff it is composite.

Solution

If N is prime, the two factors must be 1 and p. However, $1+p > 1\times p$, hence N cannot be prime.

If N is composite it can be written as $N = ab$, where $a,b \geq 2$.

$N = ab = (a-1)b-a+a+b$.

Let $m = (a-1)b-a$, so that $N = ab = m+a+b$.

As $a,b \geq 2$, let a=2 (the minimum value), so $m = (a-1)b-a \geq b-2 \geq 0$.

As $N = ab = m+a+b$, and m is never negative, we demonstrate that the sum will be equal to or less than the product.

In which case, the difference, m, can be made up of a sum of ones, for which adding as many ones as necessary will not affect the product.

For example, $N = 10 = 2\times5 = 10$, and as $2+5=7$, we can write,
$10 = 2\times5\times1\times1\times1 = 2+5+1+1+1$.

Thus N is a sum-product number iff it is composite.

If each element of S can be any integer (positive or negative), prove that all positive integers are sum-product numbers.

THE FIBONACCI SERIES

Problem

Given that F_n represents the *n*th term of the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, ..., and $S_n = F_1 + F_2 + \ldots + F_n$. Prove that $S_n = F_{n+2} - 1$.

For example, $S_5 = F_1 + \ldots + F_5 = 1 + 1 + 2 + 3 + 5 = 12 = 13 - 1 = F_7 - 1$.

Solution

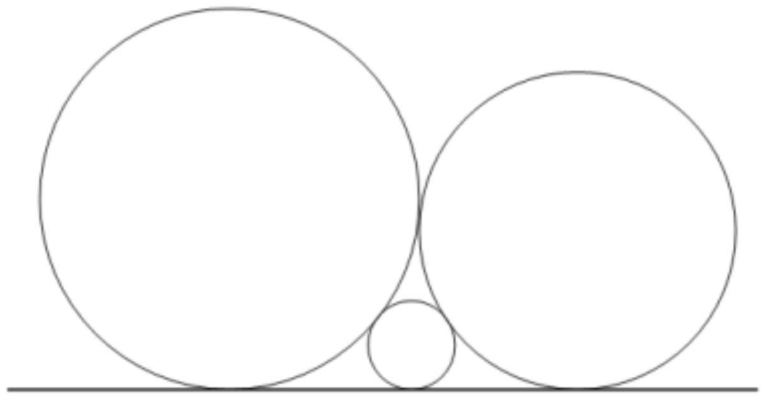
From $F_n = F_{n-1} + F_{n-2}$ we get $F_{n-2} = F_n - F_{n-1}$.

$$\begin{aligned}
\therefore S_n &= F_1 + F_2 + \ldots + F_n \\
&= (F_3 - F_2) + (F_4 - F_3) + \ldots + (F_{n+1} - F_n) + (F_{n+2} - F_{n+1}) \\
&= F_{n+2} - F_2 \\
&= F_{n+2} - 1
\end{aligned}$$

THREE CIRCLES

Problem

Three touching circles have a common tangent.

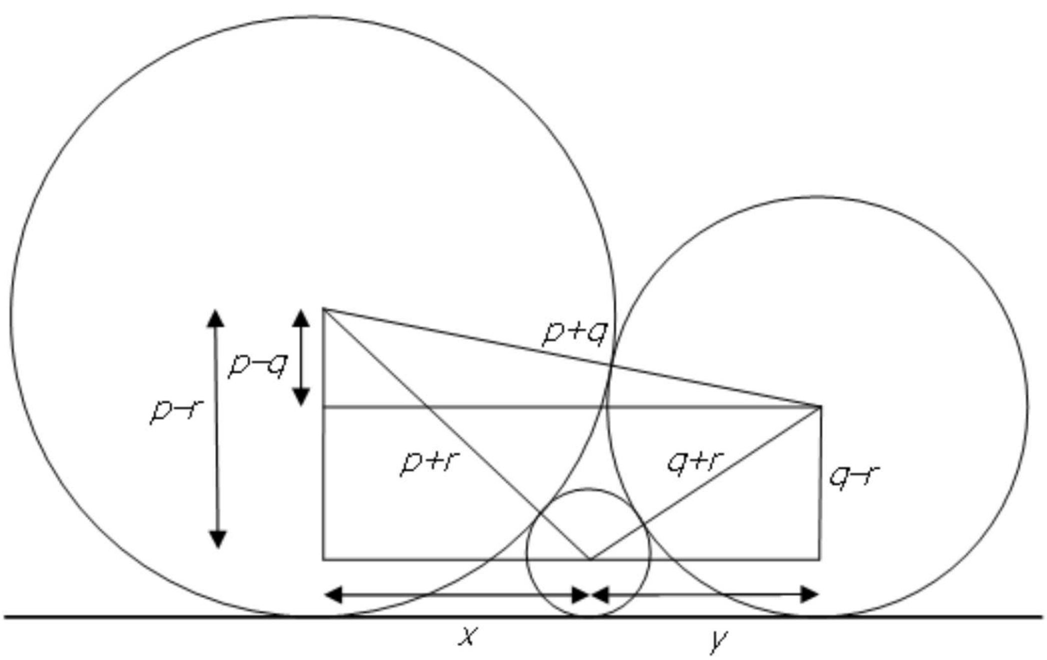


If the radii of the circles in decreasing order of size are p, q, and r, prove that the following relationship holds:

$$\frac{1}{\sqrt{p}} + \frac{1}{\sqrt{q}} = \frac{1}{\sqrt{r}}$$

Solution

Consider the diagram.



Using the Pythagorean Theorem we get:

$(p+q)^2 = (p-q)^2 + (x+y)^2$
$p^2 + 2pq + q^2 = p^2 - 2pq + q^2 + (x+y)^2$
$(x+y)^2 = 4pq \Rightarrow x + y = 2\sqrt{(pq)}$ (*)

$(p+r)^2 = (p-r)^2 + x^2$
$p^2 + 2pr + r^2 = p^2 - 2pr + r^2 + x^2$
$x^2 = 4pr \Rightarrow x = 2\sqrt{(pr)}$

Similarly, $y = 2\sqrt{(qr)}$.

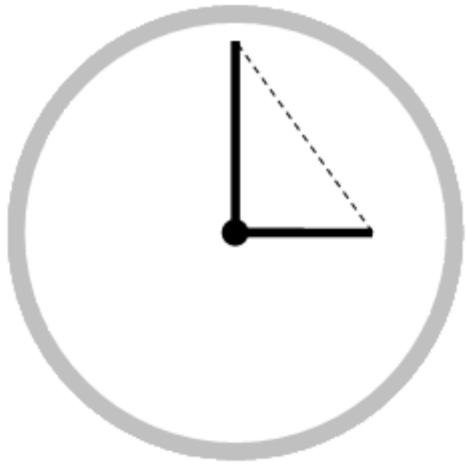
From (*), $2\sqrt{(pr)} + 2\sqrt{(qr)} = 2\sqrt{(pq)}$.

Dividing by $2\sqrt{(pqr)}$ gives the desired result, $\dfrac{1}{\sqrt{p}} + \dfrac{1}{\sqrt{q}} = \dfrac{1}{\sqrt{r}}$

TICK TOCK TRIANGLE

Problem

The hour hand on a wall clock is 3 inches long and the minute hand is 4 inches long. At three o"clock, the hands form a 3-4-5 right angle triangle.



At a random time during the day it is noted that the triangle formed by the hands has integer length sides. What is the probability that the triangle is isosceles?

Solution

Due to symmetry we can consider the hour hand to be on the right side of the clock. There is additional symmetry with the minute hand being the same distance before and after the hour hand, so we shall consider the cases where the minute hand is after the hour hand.

At 12 o"clock the minute and hour hand coincide and the distance between them is 1 inch. Similarly, at 6 o"clock, the distance is 7 inches. In both these cases the points are collinear, so a triangle is not formed. So there are precisely five possible integer distances between the ends of the minute and hour hand: 2, 3, 4, 5, and 6; the triangle will be isoscles when the distance is 3 or 4.

Therefore, P(isosceles|integer distance)=2/5.

At precisely what times does this happen?
It is interesting that P(integer distance)=0, P(distance 3 or 4)=0, but P(isosceles|integer distance)=P(3 or 4)/P(integer distance)=2/5. How do you account for this?
Given that the length of the minute hand is *m* and the length of the hour hand is *h*, find P(isosceles|integer distance).

What about P(right-angle|integer distance)?

TRAILING ZEROES

Problem

$13! = 13 \times 12 \times 11 \times ... \times 2 \times 1 = 6227020800$, and it can be seen that 13! contains two trailing zeroes.

How many trailing zeroes does 1000! contain?

Solution

Every time a number is multiplied by 10 an extra trailing zero is added, and as $10 = 2 \times 5$, we need only consider the number of factors of 5 present in 1000!; there are an abundance of factors of 2.

There are 1000/5=200 factors of 5, but we must also ensure that we take into account $25 = 5 \times 5$, which contains two factors of 5; hence there will be 1000/25=40 extra factors of 5. Similarly, $125 = 5^3$, contains three factors of 5, and $625 = 5^4$.

1000/5 = 200, 1000/25 = 40, 1000/125 = 8, and 1000/625 = 1.6.

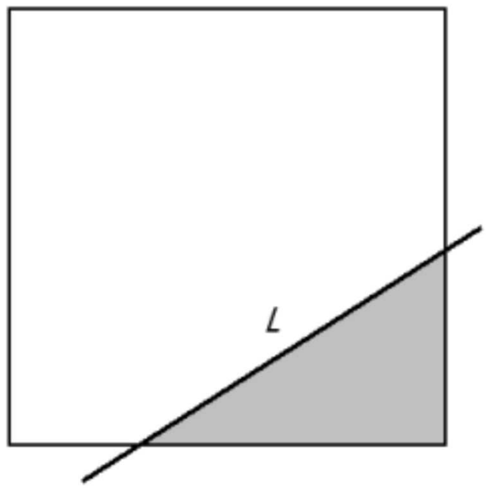
Therefore, 1000! contains, 200 + 40 + 8 + 1 = 249 trailing zeroes.

How many trailing zeroes does *n*! contain?

TRIANGLE IN SQUARE

Problem

A line segment is placed on top of a unit square so as to form a triangle region.

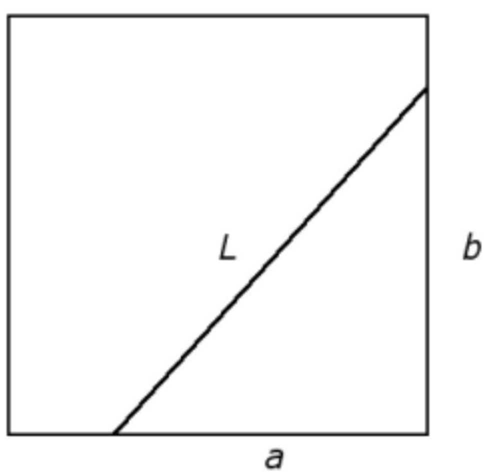


Given the length of the line segment, L, find the maximum area of the triangle.

Solution

Clearly for $L \geq \sqrt{2}$, the maximum area will be ½, when the segment is placed along the diagonal of the square.

For $L < \sqrt{2}$, consider the following diagram.



$b = \sqrt{(L^2 - a^2)}$.

So area of triangle, $A = \frac{1}{2}ab = \frac{1}{2}a\sqrt{(L^2 - a^2)} = \frac{1}{2}a(L^2 - a^2)^{1/2}$.

We may proceed from here via a calculus or non-calculus approach:

<u>Calculus Method</u>

$$\begin{aligned} dA/da &= \tfrac{1}{2}(L^2 - a^2)^{1/2} + \tfrac{1}{2}a\tfrac{1}{2}(L^2 - a^2)^{-1/2}(-2a) \\ &= \tfrac{1}{2}(L^2 - a^2)^{1/2} - \tfrac{1}{2}a^2(L^2 - a^2)^{-1/2} \end{aligned}$$

At turning point, $dA/da = 0$.

Therefore, $\frac{1}{2}(L^2 - a^2)^{1/2} = \frac{1}{2}a^2(L^2 - a^2)^{-1/2}$

$\sqrt{(L^2 - a^2)} = a^2/\sqrt{(L^2 - a^2)}$

Hence, $L^2 - a^2 = a^2$, leading to, $a = L/\sqrt{2}$.

As area of triangle, $A = \frac{1}{2}a\sqrt{(L^2 - a^2)}$,

$$\begin{aligned} A_{max} &= \tfrac{1}{2}(L/\sqrt{2})\sqrt{(L^2 - L^2/2)} \\ &= \tfrac{1}{2}(L/\sqrt{2})\sqrt{(L^2/2)} \\ &= \tfrac{1}{2}(L/\sqrt{2})(L/\sqrt{2}) \\ &= L^2/4 \end{aligned}$$

<u>Non-calculus Method</u>

As the area of the triangle, $A = \frac{1}{2}a\sqrt{(L^2 - a^2)}$ is defined for positive values, the value of a for which it maximises is the same for A^2.

Therefore $4A^2 = a^2(L^2 - a^2) = a^2L^2 - a^4 = L^2/4 - (\frac{1}{2}L - a^2)^2$.

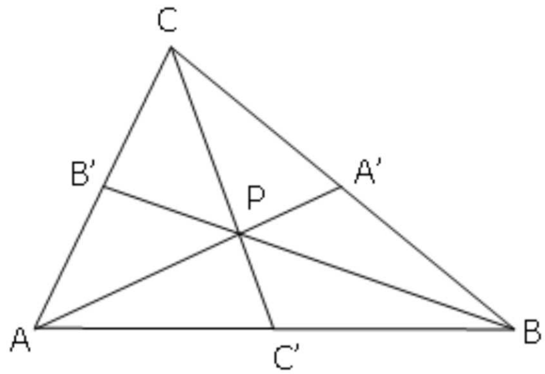
Hence, $4A^2$ will maximise when $(\frac{1}{2}L - a^2)^2 = 0 \Rightarrow a = L/\sqrt{2}$.

Proceeding as in previous method we deduce that $A_{max} = L^2/4$.

TRIANGLE MEDIAN

Problem

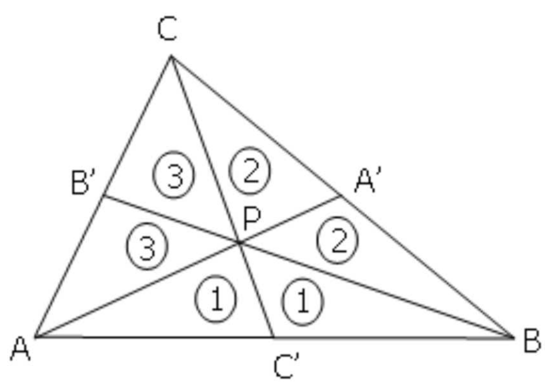
The line segments joining the vertices of triangle ABC to the midpoints of the opposite edges are called medians and are concurrent at P.



Prove that P splits each median in the ratio 2:1.

Solution

Consider the following diagram.



As C" is the midpoint of AB, AC"=C"B, so the area of triangle APC" = area of triangle C"PB = A_1. Similarly area BPA" = area A"PC = A_2 and area CPB" = area B"PA = A_3.

As BB" bisects the area of the triangle, $A_3+2A_1 = A_3+2A_2$; hence $A_1=A_2$.

Let us consider the way that PB splits triangle ABA": triangles ABP and PBA" have the same altitude, and as area ABP = 2 × area PBA", it follows that AP = 2(PA"), proving that P splits the median AA" in the ratio 2:1.

Prove that AA", BB", and CC" are concurrent.

TRIANGLE RECIPROCALS

Problem

Find the value of the series of reciprocals of triangle numbers:

$$1/1 + 1/3 + 1/6 + 1/10 + 1/15 + \ldots$$

Solution

Let S = 1/1 + 1/3 + 1/6 + 1/10 + …

$$\therefore S/2 = 1/2 + 1/6 + 1/12 + 1/20 + \ldots$$
$$= 1/(1\times2) + 1/(2\times3) + 1/(3\times4) + \ldots + 1/(n(n + 1)) + \ldots$$

But $1/(n(n + 1)) = (n + 1 - n)/(n(n + 1))$
$$= (n + 1)/(n(n + 1)) - n/(n(n + 1))$$
$$= 1/n - 1/(n + 1)$$

Therefore S/2 = (1/1 – 1/2) + (1/2 – 1/3) + (1/3 – 1/4) + … = 1.

Hence S = 1/1 + 1/3 + 1/6 + … = 2.

Triangle Search

Problem

Given that p is prime, when is $8p+1$ a triangle number?

Solution

Let $8p + 1 = k(k + 1)/2$, so $16p + 2 = k(k + 1)$

Therefore $16p = k^2 + k - 2 = (k - 1)(k + 2)$.

If $k - 1$ is odd, then $k + 2$ will be even, and vice versa. In other words, only one of these factors is even, and so it must be a multiple of 16.

As $k - 1$ and $k + 2$ differ by 3, we can write $16p = 16m(16m \pm 3)$, leading to $p = m(16m \pm 3)$.

Clearly $m = 1$, otherwise we would be dealing with a composite number, and $p = 13$ or $p = 19$.

When $p = 13$, $8p + 1 = 105 = t_{14}$ and when $p = 19$, $8p + 1 = 153 = t_{17}$.

Related problem:

> Square Search: When is $8p+1$ square?

TWO SQUARES

Problem

Given that n is a positive integer and $2n + 1$ and $3n + 1$ are perfect squares, prove that n is divisible by 40.

Solution

Clearly $2n + 1$ is odd, so let $2n + 1 = (2k + 1)^2 = 4k^2 + 4k + 1$.
$\therefore n = 2k(k + 1)$

As exactly one of k or $k + 1$ is even, we deduce that n is divisible by 4.

Because n is even, $3n + 1$ must be odd; let $3n + 1 = (2j + 1)^2 \Rightarrow 3n = 4j(j + 1)$.

Similarly, as exactly one of j or $j + 1$ is even, we further deduce that n must be divisible by 8.

Next we note that all square numbers have remainders 0, 1, or 4 when divided by 5.

Proof: if $a \equiv 0, 1, 2, 3, 4 \mod 5$ then $a^2 \equiv 0, 1, 4, 4, 1 \mod 5$.

Now we consider $2n + 1$ and $3n + 1$ under modulo 5:

mod 5		
n	$2n + 1$	$3n + 1$
0	1	1
1	3	4
2	0	2
3	2	0
4	4	3

From this we can see that it is only possible for both $2n + 1$ and $3n + 1$ to be squares when n is divisible by 5.

As n is divisible by 8 and 5 it must be divisible by 40.

It must be noted that $n \equiv 0 \bmod 40$ is a necessary condition but is not sufficient. For example, when $n = 40$, we can see that $2n + 1 = 81 = 9^2$ and $3n + 1 = 121 = 11^2$. However, when $n = 80$, neither $2n + 1 = 161$ nor $3n + 1 = 241$ are square.

Does it follow that if $n \equiv 0 \bmod 40$ and $2n + 1$ is square then $3n + 1$ is square? Find the next two cases of n for which both are square; can you deduce anything further about n?

UNEXPECTED SUM

Problem

Find the exact value of the following infinite series:

$$1/2! + 2/3! + 3/4! + 4/5! + \ldots$$

Solution

First we note that the general term in this series can be written differently:

$$k/(k+1)! = (k+1 - 1)/(k+1)! = 1/k! - 1/(k+1)!$$

Hence the original series becomes a telescoping series:

$$\begin{aligned} \therefore 1/2! + 2/3! + \ldots &= (1/1! - 1/2!) + (1/2! - 1/3!) + (1/3! - 1/4!) + \ldots \\ &= 1 \end{aligned}$$

UNINVITED GUEST

Problem

Given that $x^2 + x + 1 = 0$, find the value of x^3.

Solution

Begin by multiplying through by x:

$$x^3 + x^2 + x = 0$$

Adding one to both sides:

$$x^3 + x^2 + x + 1 = 1$$

But as $x^2 + x + 1 = 0$ we deduce that $x^3 = 1$.

What is interesting about this problem is that knowing too much could make it more difficult to solve. If you try to solve the original equation you will encounter the square root of a negative number. For most students this is the end of the road. But eventually students learn how to solve these and would obtain the complex roots: $x = (-1\pm\sqrt{3}\, i)/2$. But even then cubing these roots is rather tedious and would only lead to the result we established in a couple of simple steps.

However, it must be noted that we cannot simply use the result we obtained above: $x^3 = 1$, to claim that $x = \sqrt[3]{1} = 1$ is a root of the original equation; try it out and you'll see that it doesn't "work". It turns out (using Advanced Mathematics) that the cube root of unity has three values: the complex roots given above and 1. But in this case it is only the two complex roots that satisfy the original equation.

The reason this happens is related to a common trick used in proving 1 = 0 or similar absurd results. The secret is to watch out for the point in the "proof" where the extra root (which doesn't belong to the original equation) is introduced. If the original equation is a quadratic then it has two roots. If as part of our algebraic manipulation we produce a cubic then that cubic may have up to three distinct roots of which only two can belong to the original equation.

See if you can spot the flaw in the following related fallacious proof.

Suppose that $x^2 + x + 1 = 0$.

We can rearrange differently to get $x + 1 = -x^2$ and $x(x + 1) = -1$

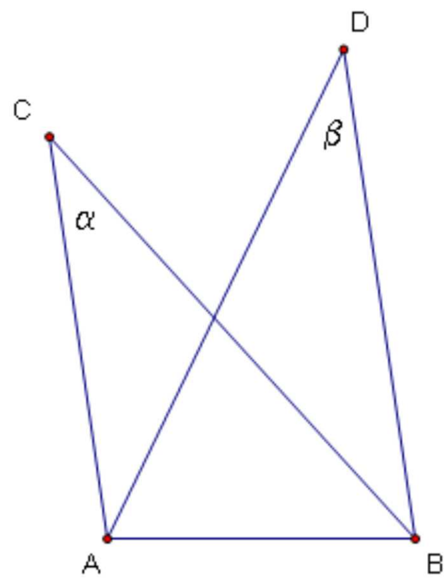
Substituting $-x^2$ for $x + 1$ in the second equation we get $-x^3 = -1 \Rightarrow x = 1$.

But try substituting $x = 1$ into the original equation... it seems that $3 = 0$?!

Unique Circle Equal Angles

Problem

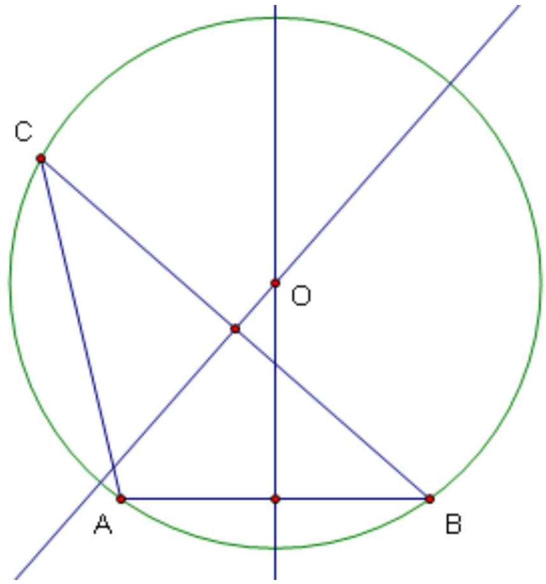
Triangles ABC and ABD share the base AB.



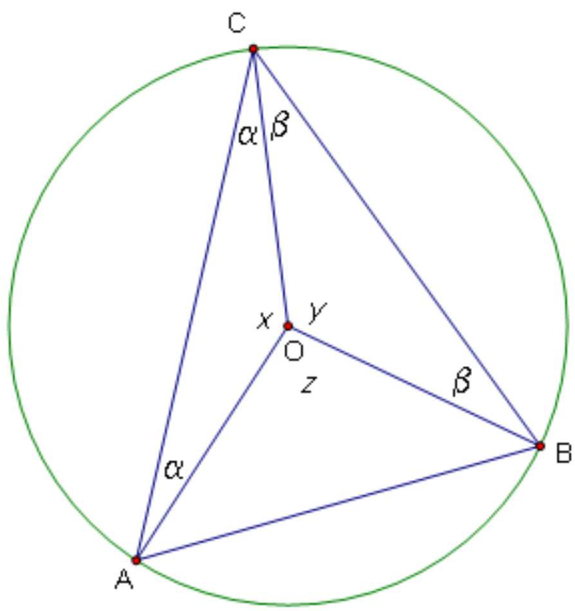
Prove that a unique circle passes through all four points A, B, C, and D, iff (if and only if) $\alpha = \beta$.

Solution

First of all we shall show that a unique circle exists that will pass through three vertices of any given triangle. Consider a general triangle ABC. We know that all points on the perpendicular bisector of AB will be equidistant from A and B. Similarly all points on the perpendicular bisector of BC will be equidistant from B and C. Hence the intersection of these two perpendicular bisectors, O, will be equidistant from A, B, and C, and will mark the centre of the circle passing through all three vertices, and clearly two lines intersect at a unique point.



Now we shall prove that in a given circle all angles on the same circumference from a common chord will be equal. We shall achieve this by showing that the centre angle is twice the angle at the circumference. Consider the following diagram.



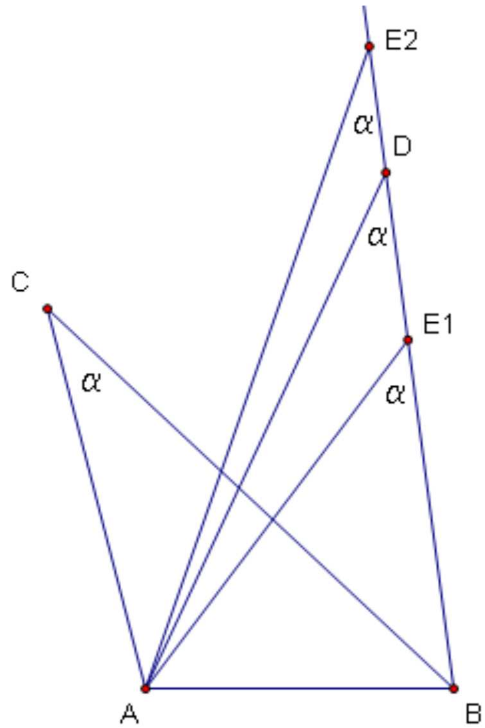
As OA, OB, and OC are all radii, triangles OAC and OBC are isosceles. Hence $2\alpha + x = 180$ and $2\beta + y = 180$. Also $x + y + z = 360$.

Therefore $x + y + z = 2\alpha + x + 2\beta + y$, so $z = 2\alpha + 2\beta = 2(\alpha + \beta)$. That is, the angle at the centre if twice the angle at the circumference.

As the point on the circumference, C, was arbitrarily chosen it should be clear that all angles on the circumference will be half the centre angle, and thus equal.

Now we are able to show in the original diagram that if $\alpha = \beta$ then a unique circle must pass through all four points, A, B, C, and D.

Given triangle ABC we know that a unique circle passes through each of the vertices. Now suppose that the circle does not pass through D, but passes through some other point on the line through BD: E1 or E2.



We have just shown that all angles on the circumference are equal, so whichever point the circle passes through on the line through BD will also produce an angle α. It should be clear that there can be only one point on this line for which the angle will be α. Hence if the angle at D is equal to α we know that the unique circle which passes through A, B, and C, must also pass through D.

Prove that the centre of the circle will be found on the edge of the triangle iff it is a right angle triangle.

XOR CHALLENGE

Problem

Sorting through your emails, and deleting the abundance of junk mail that seems to arrive on a daily basis, your attention is drawn to one particular mail:

```
----- Original Message -----
From: <comments@mathschallenge.net>
To: <mastermind@coolmail.co.uk>
Sent: Saturday, March 1, 2003 1:00 AM
Subject: Only the best need try...

Hi mastermind

So you think you're good at code breaking? I'll tell you how
I encoded the message and all you need to do is decode it!

I started by using the table below to change the message into
a binary string, then XOR"ed it with the alternating string
101010101..., finally I converted it back into readable text.
I guess you'll have to research how XOR works. ;)

Here's the table:
```

Bin	**Chr**	**Bin**	**Chr**	**Bin**	**Chr**	**Bin**	**Chr**
00000	<spc>	01000	H	10000	P	11000	X
00001	A	01001	I	10001	Q	11001	Y
00010	B	01010	J	10010	R	11010	Z
00011	C	01011	K	10011	S	11011	.
00100	D	01100	L	10100	T	11100	,
00101	E	01101	M	10101	U	11101	?
00110	F	01110	N	10110	V	11110	!
00111	G	01111	O	10111	W	11111	-

```
Here's the encoded message:

                    ,GEXPYFCCOK

Good luck!
```

mathschallenge.net

Can you decode the message?

Solution

Let us begin with an example. We start by converting the word, CAT, in to binary: 00011 00001 10100. The XOR function works by comparing corresponding bits (0 or 1) in the original binary string with another binary string, called the encryption key. The table below shows what happens when we do **A** XOR **B** to produce **C**.

A	B	C
0	0	0
0	1	1
1	0	1
1	1	0

In other words, if the two bits we are comparing are the same we get 0 otherwise we get 1.

It often helps to write the original binary string above the encryption key:

```
000110000110100 (Original)
101010101010101 (Encryption Key)
101100101100001 (Encoded)
```

Breaking this down we get 10110 01011 00001, which gives the encoded word, VKA.

The amazing feature of the XOR function is that it is self-inversing. That is, if we re-apply the encryption key to the encoded message we return to the original message. So using this system on our encoded message we get:

Text (Encoded)	,	G	E	X	P	Y	F	C	C	O	K
Binary (Encoded)	11100	00111	00101	11000	10000	11001	00110	00011	00011	01111	01011
Encryption Key	10101	01010	10101	01010	10101	01010	10101	01010	10101	01010	10101
Binary (Decoded)	01001	01101	10000	10010	00101	10011	10011	01001	10110	00101	11110
Text (Decoded)	I	M	P	R	E	S	S	I	V	E	!

Hence the decoded message reads: IMPRESSIVE!

As you can see, the XOR function provides a powerful means of encoding messages. Notice how the alternating binary string we used presents two encoded forms for each character; SS in the original message became YF. A more secure method is to use a phrase as a key. For example, we could use the word CAT in binary (see above) to produce the encryption key 000110000110100, which can be repeated throughout the length of the original message.

ZIP CODES

Problem

In America they make use of ZIP (Zoning Improvement Plan) codes to help direct mail. Each city/town makes use of a 5-digit code (we shall ignore ZIP+4 codes); for example, Hillsgrove, Pennslyvania, is 18619.

However, due to rotational symmetry certain codes, like the one used for Hillsgrove are detour-prone, as they could be read as a different code upside down. As a result, 61981 (Hillsgrove's ZIP code read upside down) is not used.

How many 5-digit ZIP codes are detour-prone?

Solution

We shall consider 18619 and 61981 (its rotational partner) as two separate detour-prone codes.

All detour prone ZIP codes will exclusively contain the digits 0, 1, 6, 8, and 9; that is, $5^5 = 3125$ 5-digit numbers.

However, codes like, 66899, are not detour prone, as it reads the same upside down. In fact, any ZIP code with the following criteria will not be detour prone:

i. 0, 1, or 8, as middle digit.
ii. First/second digit is 0, 1, 6, 8, or 9 (with corresponding last/fourth digit will being 0, 1, 9, 8, and 6, respectively.)

This accounts for $5 \times 5 \times 3 = 75$.

Hence there are 3125 – 75 = 3050 detour prone 5-digit ZIP codes.

How many *n*-digit detour-prone ZIP codes are there?