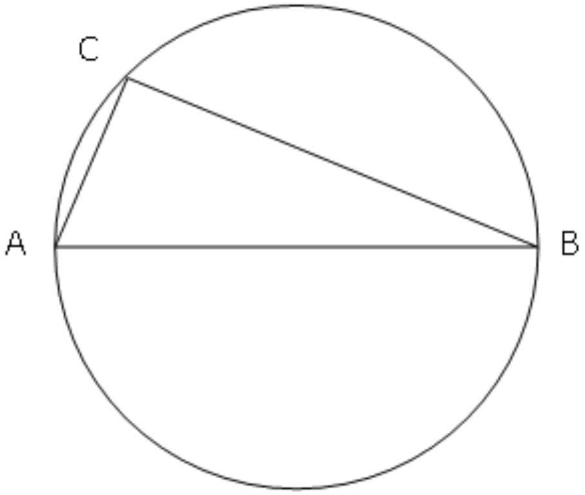
4 Star Problems

15 Degree Triangle

Problem

In the diagram below, AB represents the diameter, C lies on the circumference of the circle, and you are given that (Area of Circle) / (Area of Triangle) = 2π.



Prove that the two smaller angles in the triangle are exactly 15^o and 75^o respectively.

Solution

We shall consider three quite different solutions to this problem.

Method 1

As triangle ABC is in a semi-circle, angle ACB is a right angle.

Let angle CBA = θ, AB = 2*r*, AC = 2*r*sin(θ), and BC = 2*r*cos(θ).

Therefore the area of triangle = 4 r^2 sin(θ)cos(θ) / 2 = 2 r^2 sin(θ) cos(θ), and using the double angle identity: sin(2θ) = 2 sin(θ) cos(θ), area of triangle = r^2 sin(2θ).

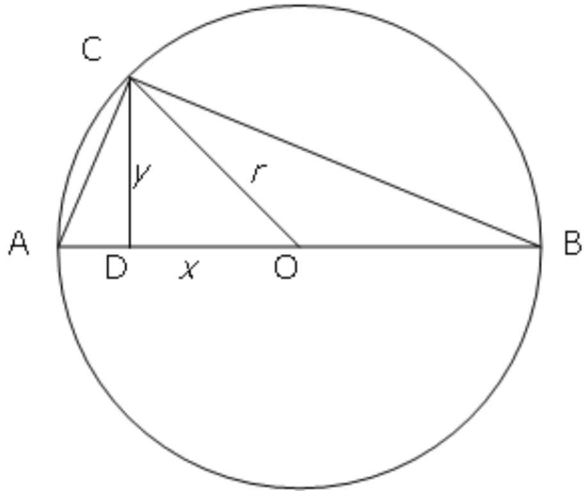
So, (Area of Circle) / (Area of Triangle) = πr^2 / r^2 sin(2θ) = π / sin(2θ) = 2π

$\therefore$ sin(2θ) = 1/2 $\Rightarrow$ 2θ = 30^o $\Rightarrow$ θ = 15^o

Hence the complementary angle must be 75 degrees.

Method 2

In the diagram, CD is perpendicular to AB. Let the diameter, AB = $2r$, OC = r, OD = x, and CD = y.



Therefore the area of triangle = $2r\,y / 2 = r\,y$.

So, (Area of Circle) / (Area of Triangle) = $\pi r^2 / r\,y = \pi r / y = 2\pi \Rightarrow r = 2y$

Using the Pythagorean Theorem, $x^2 + y^2 = r^2$, so $x^2 + y^2 = 4y^2 \Rightarrow x^2 = 3y^2 \Rightarrow x = \sqrt{3}\,y$

Let angle DOC = θ. Therefore $\tan(\theta) = y / x = y / \sqrt{3}\,y = 1 / \sqrt{3} \Rightarrow \theta = 30^o$.

By using the result that the angle at the centre of a circle (AOC) is twice the angle at the circumference we deduce that angle ABC = 15^o, and it follows that the complementary angle must be 75 degrees.

Method 3

In triangle ABC, let BC = a, AC = b, and AB = c.

Because the angle in a semi-circle is a right angle, area of triangle = $ab / 2$, and using the Pythagorean theorem, $a^2 + b^2 = c^2$.

As radius, $r = c / 2$, $r^2 = c^2 / 4 = (a^2 + b^2) / 4$.

$$\therefore \quad \frac{\pi(a^2 + b^2) / 4}{ab / 2} = 2\pi$$

$$\frac{a^2 + b^2}{2ab} = 2$$

$$\therefore \quad a^2 + b^2 = 4ab$$
$$\therefore \quad a / b + b / a = 4$$

But in triangle ABC, tan(A) = a / b and tan(B) = b / a, so tan(A) + tan(B) = 4.

However, tan(B) = tan(90 – A) = 1 / tan(A). Therefore,tan(A) + 1 / tan(A) = 4.

By letting t = tan(A), we get t + 1 / t = 4. This leads to the quadratic, t^2 - 4t + 1 = 0, which has roots t = tan(A) = 2 ± √3.

We will now show that the root corresponding with tan(A) = 2 – √3 is exactly 15 degrees.

Let T = tan(15) and by using the trigonometric identity, tan(2x) = 2tan(x) / (1 – $\tan^2(x)$), we get tan(30) = 1 / √3 = 2T / (1 – T^2). This leads to the quadratic, T^2 + 2 √3 – 1 = 0, which has two roots, T = -√3 ± 2. However, as tan(15) > 0, we take the positive root ⇒ tan(15) = 2 – √3.

Hence the complementary angle must be 75 degrees.

ALGEBRAIC COSINE

Problem

Given that x is a rational multiple of π, prove that $\cos(x)$ is algebraic.

Solution

We shall begin by proving the recurrence relation, $\cos(nx) = 2\cos(x)\cos([n-1]x) - \cos([n-2]x)$.

Starting with the right hand side:

$$
\begin{aligned}
& 2\cos(x)\cos([n-1]x) - \cos([n-2]x) \\
= \; & 2\cos(x)\cos(nx-x) - \cos(nx-2x) \\
= \; & 2\cos(x)[\cos(nx)\cos(x) + \sin(nx)\sin(x)] - [\cos(nx)\cos(2x) + \sin(nx)\sin(2x)] \\
= \; & 2\cos^2(x)\cos(nx) - 2\sin(x)\cos(x)\sin(nx) - [\cos(nx)(2\cos^2(x)-1) - \sin(nx)2\sin(x)\cos(x)] \\
= \; & 2\cos^2(x)\cos(nx) - 2\sin(x)\cos(x)\sin(nx) - 2\cos^2(x)\cos(nx) + \cos(nx) + 2\sin(x)\cos(x)\sin(nx) \\
= \; & \cos(nx)
\end{aligned}
$$

Clearly we can write $\cos(2x)$ in terms of $\cos(x)$: $\cos(2x) = 2\cos^2(x)-1$.

By using the recurrence relation we can then write $\cos(3x)$ in terms of $\cos(x)$, and by repeated use we can write $\cos(ax)$ as a polynomial in terms of $\cos(x)$; that is, $\cos(ax) = \mathrm{P}[\cos(x)]$.

Therefore $\cos((a / b)\pi) = \cos(a(1 / b)\pi)) = \mathrm{P}[\cos((1 / b)\pi)]$.

When $a = b$, $\cos((a / b)\pi) = \cos(\pi) = -1$, which is clearly algebraic.
$\therefore \mathrm{P}[\cos((1 / b)\pi)] = -1 \Rightarrow \cos((1 / b)\pi)$ is algebraic.

And as $\cos((a / b)\pi)$ can be written as a polynomial in terms of $\cos((1 / b)\pi)$ we deduce that $\cos(x)$ is algebraic when x is a rational multiple of π. **Q.E.D.**

Notes

Because $\sin^2(x) + \cos^2(x) = 1$ and $\tan(x) = \sin(x) / \cos(x)$ it follows that $\sin(x)$ and $\tan(x)$ will also be algebraic if x is a rational multiple of π.

As π radians = 180 degrees, 1 degree = $\pi / 180$, so cos(1 degree) = $\cos(\pi / 180)$ is algebraic. So it follows that sin(x), cos(x), and tan(x) will be algebraic for all rational angles given in degrees.

The Hemite-Lindermann Theorem states that $a_1e^{b_1} + a_2e^{b_2} + \ldots \neq 0$ if all a_i and b_i are algebraic. (Additionally the coefficients must be non-zero and the exponents must be distinct.) In other words, no sum of exponential functions with algebraic coefficients and exponents will ever equal zero.

Using the identity $y = \cos(x) = (e^{ix} + e^{-ix}) / 2$ we get the equation, $e^{ix} + e^{-ix} - 2ye^0 = 0$. But according to the theorem this cannot be true if all the coefficients and exponents are algebraic. We can see that when $x = \pi$, $y = \cos(\pi) = -1$ is clearly algebraic. So this theorem proves that $x = \pi$ is the non-algebraic entity, and π is transcendental. It also demonstrates that when x (in radians) is algebraic, $y = \cos(x)$ must be transcendental.

ALMOST EQUILATERAL TRIANGLES

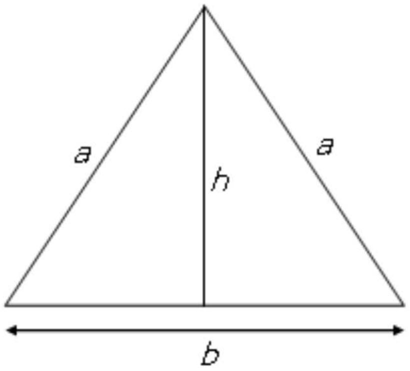
Problem

We shall define an *almost equilateral triangles* to be a triangle for which two sides are equal and the third differs by no more than one unit. The smallest such triangle with integral length sides and area is 5-5-6.

Prove that infintitely many *almost equilateral triangles* with integral length sides and area exist.

Solution

By the definition an *almost equilateral triangles* measuring *a*-*a*-*b* is isosceles.



Using the Pythagorean Theorem: $a^2 = (b/2)^2 + h^2$, so $4a^2 = b^2 + 4h^2$.

As $b = a \pm 1$, $b^2 = a^2 \pm 2a + 1$

$$\begin{aligned}
\therefore\ 4a^2 &= a^2 \pm 2a + 1 + 4h^2 \\
3a^2 \pm 2a - 1 + 4h^2 &= 0 \\
9a^2 \pm 6a - 3 - 12h^2 &= 0 \\
9a^2 \pm 6a + 1 - 12h^2 &= 4 \\
(3a \pm 1)^2 - 12h^2 &= 4 \\
\therefore\ ((3a \pm 1)/2)^2 - 3h^2 &= 1
\end{aligned}$$

By writing $x = (3a \pm 1)/2$ and $y = h$, we get the Pell equation: $x^2 - 3y^2 = 1$. Given one solution, it is well known that Pell equations have infinitely many solutions, and for completeness we shall prove this.

However, we must first show that integer x corresponds to an integer solutions for a;

b being integer follows as $b = a \pm 1$.

As $x = (3a \pm 1)/2$, we get $a = (2x\pm1)/3$

It should be clear that x cannot be divisible by 3, otherwise $x^2 - 3y^2$ would be a multiple of 3 and could not be equal to 1.

So given that $x \equiv \pm1 \bmod 3$, $2x \equiv \pm 1 \bmod 3$, and so one of $2x+1$ or $2x-1$ will be a multiple of 3. Hence for every integer solution of the equation $x^2 - 3y^2 = 1$, we have an integer solution for a and b. Now we shall prove that infinitely many solutions exist.

Given (x,y), a solution pair to the Pell equation $x^2 - 3y^2 = 1$, consider the larger pair $(x^2+3y^2,2xy)$:

$$\begin{aligned}(x^2+3y^2)^2 - 3(2xy)^2 &= x^4 + 6x^2y^2 + 9y^4 - 12x^2y^2\\ &= x^4 - 6x^2y^2 + 9y^4\\ &= (x^2 - 3y^2)^2\\ &= 1\end{aligned}$$

In other words, if (x,y) is a solution then $(x^2+3y^2,2xy)$ will also be a solution, and as (7,4) leads to the first solution 5-5-6, we prove that infinitely many *almost equilateral triangles* with integral length sides and area exist.

Note that although the infinite solution set of the Pell equation $x^2 - 3y^2 = 1$ is in a one-to-one mapping with the set of *almost equilateral triangles* we are seeking, this particular iterative method: $(x,y) \rightarrow (x^2+3y^2,2xy)$, will NOT produce every solution.

COMPOSITE FIBONACCI TERMS

Problem

Let F_n represents the nth term of the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, ...

Prove that for all composite values of $n > 4$, F_n is composite.

Solution

We shall begin by proving Lemma 1: $F_{m+n} = F_{m-1}F_n + F_mF_{n+1}$.

Using this result for the two basis cases: $n = 1$ and $n = 2$ we get the following.

$n = 1$: $F_{m+1} =F_{m-1}F_1 + F_mF_2$
But as $F_1 = F_2 = 1$, $F_{m+1} = F_{m-1} + F_m$, which is clearly true.

$n = 2$: $F_{m+2} = F_{m-1}F_2 + F_mF_3$
As $F_2 = 1$ and $F_3 = 2$, $F_{m+2} = F_{m-1} + 2F_m = F_{m-1} + F_m + F_m = F_{m+1} + F_m$, which is also true.

Let us assume that Lemma 1 is true for $n = k$ and $n = k - 1$.
That is, $F_{m+k} = F_{m-1}F_k + F_mF_{k+1}$ and $F_{m+k-1} = F_{m-1}F_{k-1} + F_mF_k$.

$$\begin{aligned}\therefore F_{m+k+1} &= F_{m+k} + F_{m+k-1}\\ &= F_{m-1}F_k + F_{m-1}F_{k-1} + F_mF_k\\ &= F_{m-1}(F_k + F_{k-1}) + F_m(F_{k+1} + F_k)\\ &= F_{m-1}F_{k+1} + F_mF_{k+2}\end{aligned}$$

And as this is the expected result we prove that Lemma 1 is true for all values of n.

Now we prove Lemma 2: $F_m \mid F_{mn}$.

Clearly the basis case, $n = 1$, is true: $F_1 = 1 \mid F_m$.

Let us assume that $F_m \mid F_{mk}$.

$\therefore F_{m(k+1)} = F_{mk+k} = F_{mk-1}F_m + F_{mk}F_{m+1}$ (using Lemma 1)

By the induction hypothesis, $F_m \mid F_{mk}$, which means that F_m divides both terms on the right hand side. Hence $F_m \mid F_{m(k+1)}$ and we show that $F_m \mid F_{mn}$ for all values of n.

For all composite values of $n > 4$, where $n = ab$, it is clear that $2 \leq a, b < n$.

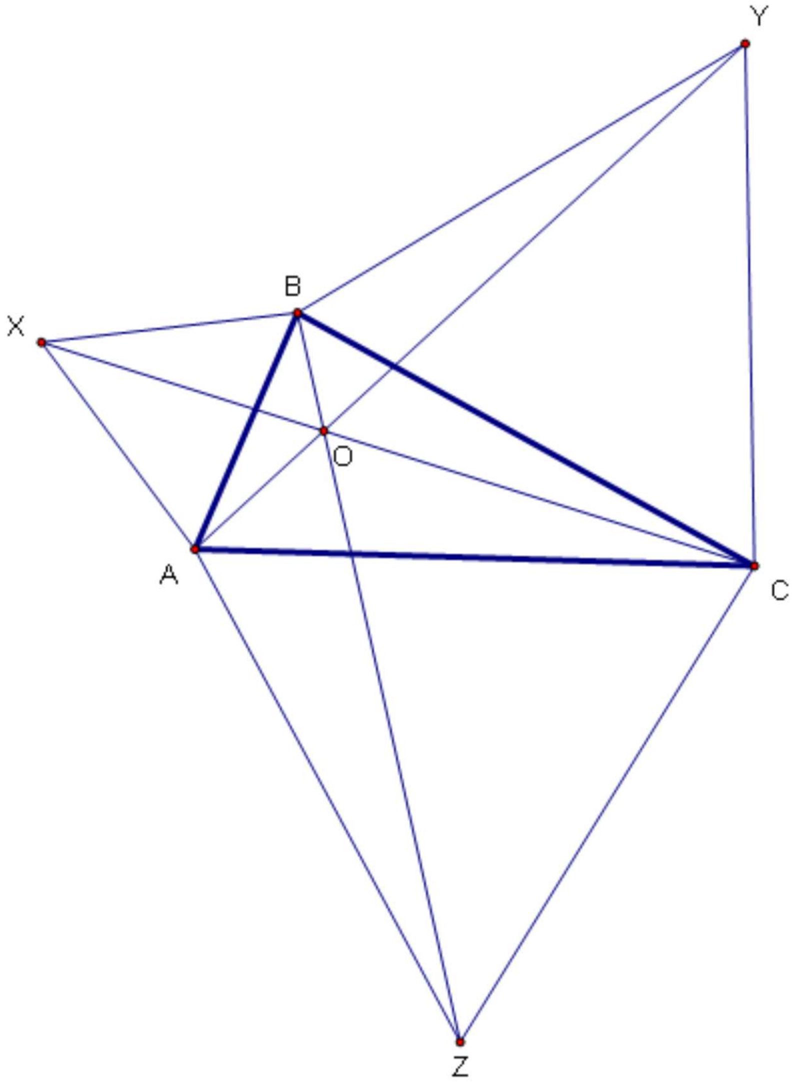
Moreoever, one of *a* or *b* must be greater than 2. Without loss of generality let us assume that $a > 2$.

By Lemma 2: $F_a \mid F_{ab} = F_n$, and because $a > 2$, $F_a > 1$ and we prove that F_n is composite. **Q.E.D.**

CONCURRENT CONGRUENT SEGMENTS

Problem

In triangle ABC equilateral triangles ABX, BCY, and ACZ are constructed.

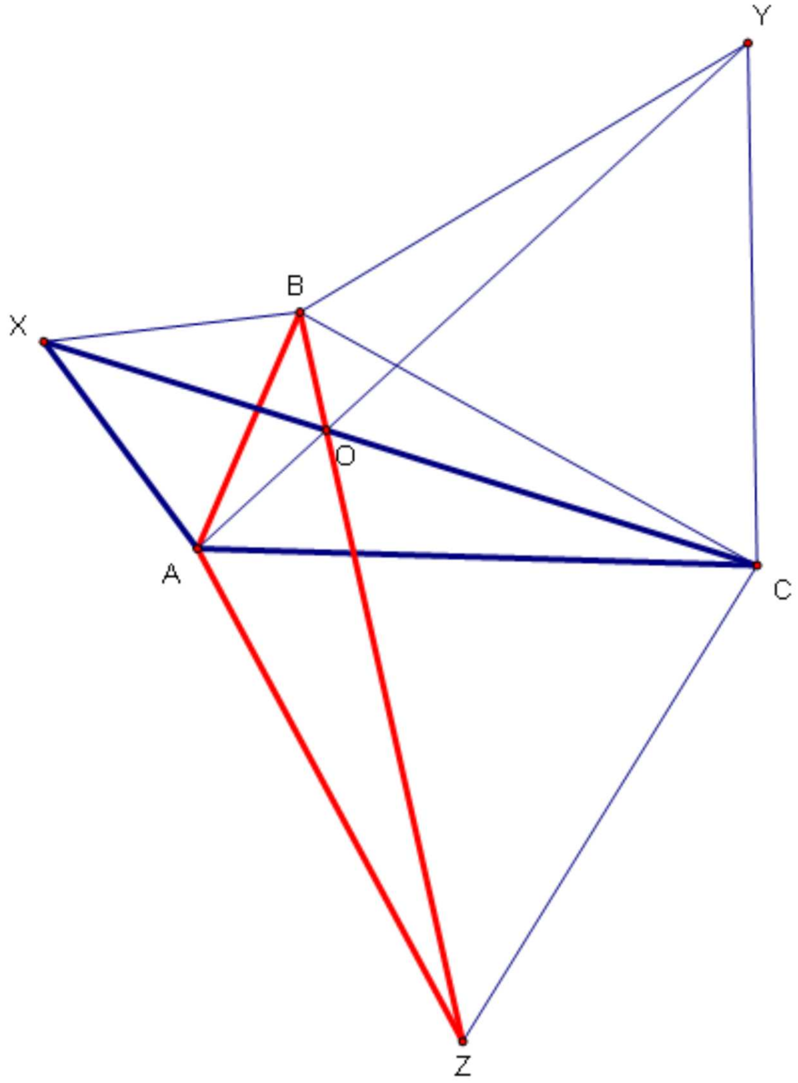


Prove that AY = BZ = CX and that all three lines are concurrent at O.

Solution

We shall begin by showing that the three segments are congruent.

Consider triangle AXC in the following diagram.

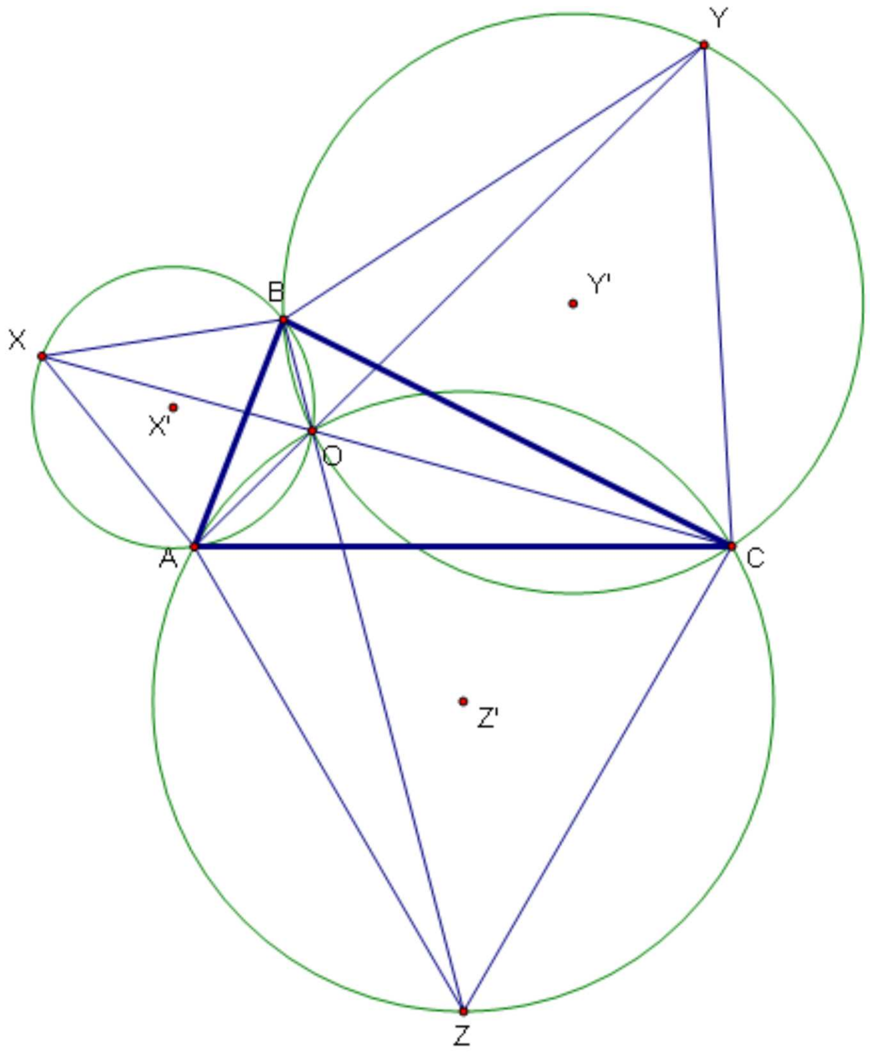


By rotating edge AX through 60 degrees about A we produce edge AB and edge AC coincides with edge AZ (also rotated 60 degrees about A). As angle CAX is preserved through this rotation, angle CAX = angle BAZ. Therefore triangles AXC and ABZ must be congruent and it follows that CX = BZ.

By considering triangle BCX and triangle BAY in the same way we can show that AY = CX.

Thus we show that AY = BZ = CX.

To prove that the segments are concurrent at O we shall consider the circumscribed circles, with X", Y", and Z" representing the circumcentres of each triangle.



Clearly circles X" and Y" intersect at B and O, but we must also show that circle Z" passes through this common point.

As OAXB is a cyclic quadrilateral the sum of opposite angles is 180 degrees. As angle AXB is 60 degrees, angle AOB must be 120 degrees.

Similarly angle BOC is also 120 degrees.

As the angles around O add to 360 degrees, it follows that angle AOC must be 120 degrees.

But as angle AZC is 60 degrees, we can use the converse of the cyclic quadrilateral theorem and deduce that because opposite angles, AOC and AZC, add to 180 degrees, the points O, A, C, and Z must lie on the same circle. That is, circle Z" passes through A and C and also passes through O.

Hence circles X", Y", and Z" are concurrent at O.

Now we are able to show that the segments AY, BZ, and CX are also concurrent at O.

In circle X", angles BAX and BOX share the same chord, BX, so it follows that angle

BAX = angle BOX = 60 degrees.

In the same way we can show that angle BOY and angle COY are both 60 degrees.

As angles BOX, BOY, and COY add to 180 degrees, the points X, O, and C must must lie on the same straight line. Therefore segment CX passes through O.

Similarly we can show that segments AY and BZ pass through O. **Q. E. D.**

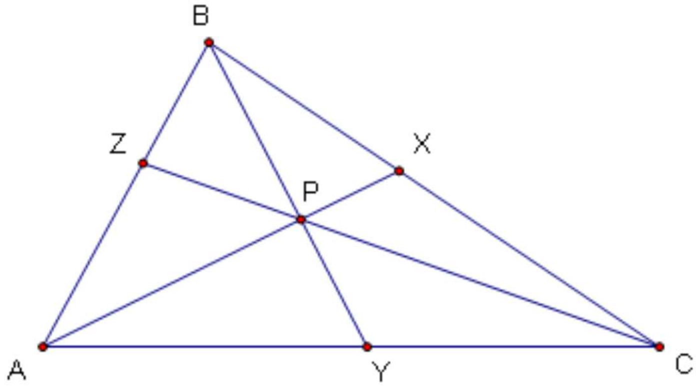
If P represents any point inside triangle ABC, the position of the point that minimises PA + PB + PC is called the Fermat/Torricelli point.

- Show that if any interior angle exceeds 120 degrees then the Fermat point is found at that vertex.
- If no interior angle exceeds 120 degrees, prove that the point of intersection of the three circles produced from the constructed equilateral triangles is the Fermat point.
- Prove that this point is unique.

CONCURRENT SEGMENTS IN A TRIANGLE

Problem

Consider triangle ABC and the segments joining the points X, Y, and Z, on opposite edges.



Prove that the segments AX, BY, and CZ are concurrent at P if and only if (AZ/BZ)(BX/CX)(CY/AY) = 1.

Solution

This result is known as Ceva's Theorem and is a fundamental theorem used in advanced Euclidean geometry. We shall prove it in two different ways. The first method uses area and the second method makes use of adding an auxillary line.

Area Method

Considering AC as the base of triangle ABC, let the altitude be h.

∴ area ΔABY = AY×h/2 ⇒ AY = 2×area ΔABY/h; similarly CY = 2×area ΔCBY/h.

Hence AY/CY = (2×area ΔABY/h)/(2×area ΔCBY/h) = area ΔABY/area ΔCBY.

In the same way, AY/CY = area ΔAPY/area ΔCPY.

Using the fact that $a/b = c/d = (a-c)/(b-d)$ (see Proportional Difference)

AY/CY = (area ΔABY – area ΔAPY)/(area ΔCBY – area ΔCPY) = area ΔABP/area ΔCBP

By considering edges AB and BC in the same way we get:

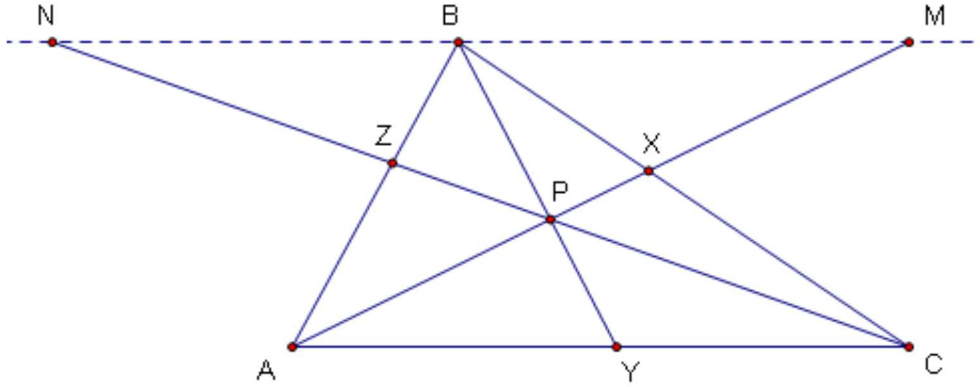
AZ/BZ = area ΔACP/area ΔBCP and BX/CX = area ΔBZP/area ΔCAP

∴ (AZ/BZ)(BX/CX)(CY/AY) = (area ΔACP/area ΔBCP)(area ΔBZP/area ΔCAP)(area ΔABP/area ΔCBP)= 1

Hence we have proved that if AX, BY, and CZ are concurrent then the product of given ratios will be one.

Auxillary Line Method

Consider the following diagram where a line parallel to AC has been added and AX has been extended to meet this line at M and CZ has been extended to N.



As angle BXM = angle AXC (opposite angles), angle CAM = angle BMX (alternate angles), and angle ACX = angle MBX (alternate angles), triangles AXC and BXM are similar; that is, ΔAXC ~ ΔBXM. By considering this pair of similar triangles and three other pairs of similar triangles we can derive four ratios.

ΔAXC ~ ΔBXM ⇛ BX/CX = BM/AC

ΔAZC ~ ΔBZN ⇛ AZ/BZ = AC/BN

ΔCPY ~ ΔBPN ⇛ CY/BN = PY/BP

ΔAPY ~ ΔBPM ⇛ AY/BM = PY/BP

From the last two ratios we get CY/BN = AY/BM ⇛ CY/AY = BN/BM.

Multiplying the first two ratios by this new ratio we get the desired result:

(BX/CX)(AZ/BZ)(CY/AY) = (BM/AC)(AC/BN)(BN/BM) = 1

Now it is necessary to prove the converse: if the product of ratios are equal to one then the segments are concurrent.

Suppose that AX and CZ intersect at a common point P. Construct BY" that passes through P.

As these segments are concurrent by construction it follows that (AZ/BZ)(BX/CX)(CY"/AY") = 1.

However, if we move Y along AC such that (AZ/BZ)(BX/CX)(CY/AY) = 1, then it follows that CY/AY = CY"/AY" and Y must coincide with Y" proving concurrency.

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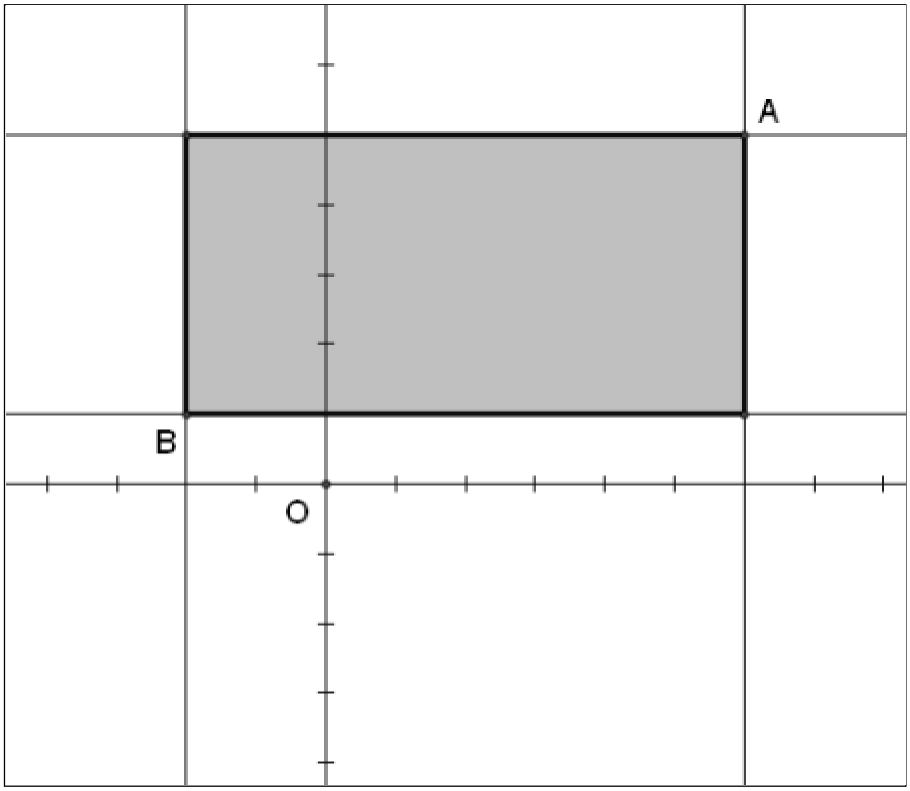
Contains The Origin

Problem

Two points, A and B, are selected at random on a co-ordinate plane. Lines are drawn through each of the points that are parallel to both the *x* and *y* axes so as to form a rectangular region.



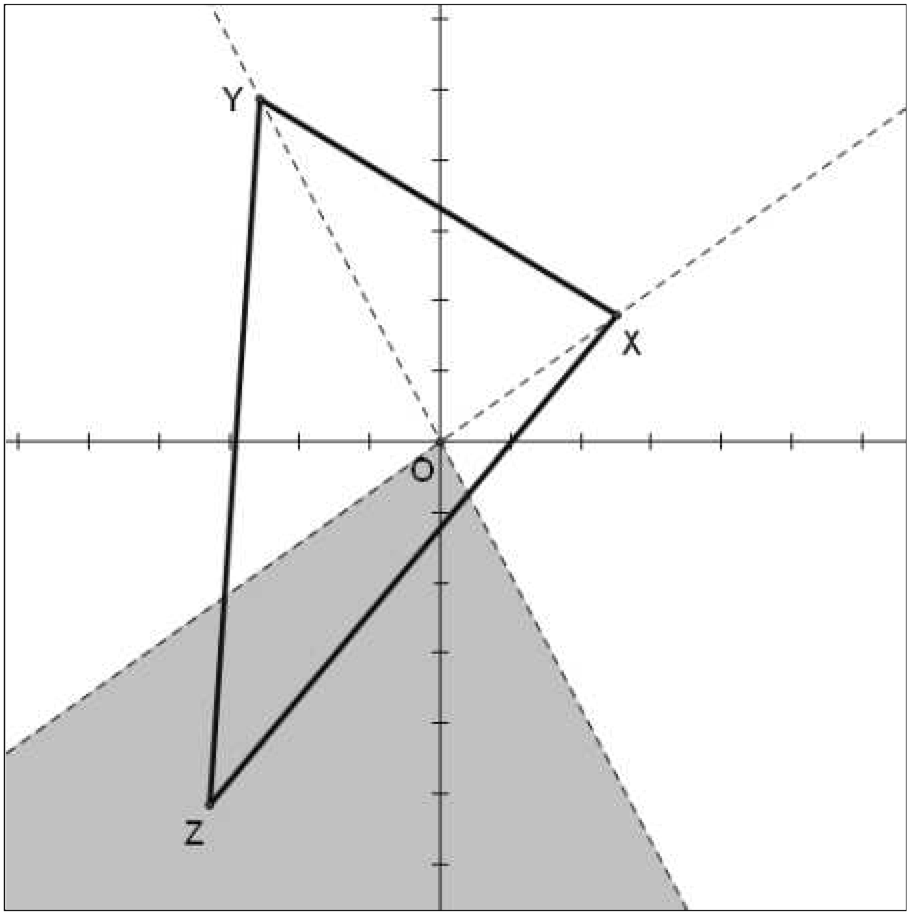
Three points, X, Y, and Z, are selected at random on the same plane so as to form a triangle.

Which shape is most likely to contain the origin?

Solution

Given A, the only way that the rectangle will contain the origin is if B is found in the diagonally opposite quadrant. That is, the probability that the rectangle contains the origin is 1/4.

Given X and Y, we can draw lines XO and YO so as to form four regions.



It is clear that the triangle will contain the origin if Z is chosen within the shaded region, and as X and Y are determined randomly, we shall assume that the area of the shaded region will be uniformly distributed from 0 to 1/2 of the plane. Therefore the expected area will be 1/4.

Hence the probability that each shape contains the origin is 1/4.

Can we assume that the area of the shaded region is uniformly distributed?
What assumptions must be made about the way that X and Y are chosen?
What is the probability that a quadrilateral, formed from the four randomly selected points A, B, C, and D, contains the origin?

Continued Fraction Irreducible Convergents

Problem

Any real number, α, can be represented as a continued fraction which may either terminate or be infinite (repeating or non-repeating).

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

Let p_n/q_n be defined as the nth convergent:

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots + \cfrac{1}{a_n}}}$$

For example, $\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}$

And the first five convergents are:

$p_0/q_0 = 1$
$p_1/q_1 = 3/2$
$p_2/q_2 = 7/5$
$p_3/q_3 = 17/12$
$p_4/q_4 = 41/29$

Assuming that the following recurrence relation holds for $n \ge 2$ for all continued fractions:

$$p_n = a_n p_{n-1} + p_{n-2}$$
$$q_n = a_n q_{n-1} + q_{n-2}$$

Prove that p_n/q_n is irreducible.

Solution

Let us consider the difference between two consecutive convergents:

$$\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} = \frac{p_{k+1}\, q_k - p_k\, q_{k+1}}{q_k\, q_{k+1}}$$
$$= \frac{\mathrm{D}_k}{q_k\, q_{k+1}}$$

Working with the numerator we can use the recurrence relation to replace p_{k+1} and q_{k+1}:

$$\begin{aligned}
\therefore\ \mathrm{D}_k &= p_{k+1}\, q_k - p_k\, q_{k+1} \\
&= (a_{k+1}\, p_k + p_{k-1})\, q_k - p_k\, (a_{k+1}\, q_k + q_{k-1}) \\
&= a_{k+1}\, p_k\, q_k + p_{k-1}\, q_k - a_{k+1}\, p_k\, q_k - p_k\, q_{k-1} \\
&= p_{k-1}\, q_k - p_k\, q_{k-1} \\
&= -(p_k\, q_{k-1} - p_{k-1}\, q_k) \\
&= -\mathrm{D}_{k-1}
\end{aligned}$$

$\therefore \mathrm{D}_k = -\mathrm{D}_{k-1} = \mathrm{D}_{k-2} = \ldots = (-1)^k \mathrm{D}_0$.

But $\mathrm{D}_0 = p_1\, q_0 - p_0\, q_1$.

And by the definition of the continued fraction, $p_0/q_0 = a_0/1 \Rightarrow p_0 = a_0$ and $q_0 = 1$.

Similarly $p_1/q_1 = a_0 + 1/a_1 = (a_0a_1 + 1)/a_1 \Rightarrow p_1 = a_0a_1 + 1$ and $q_1 = a_1$.

$\therefore \mathrm{D}_0 = (a_1a_0 + 1)\times 1 - a_0\times a_1 = 1$

Hence $\mathrm{D}_\mathrm{k} = (-1)^k$.

Let j be the highest common factor of p_n and q_n; that is, $p_n = j\, p_n''$ and $q_n = j\, q_n''$.

$$\begin{aligned}
\therefore\ \mathrm{D}_k &= p_{k+1}\, q_k - p_k\, q_{k+1} \\
&= p_{k+1}\, j\, q_k'' - j\, p_k''\, q_{k+1} \\
&= j\, (p_{k+1}\, q_k'' - p_k''\, q_{k+1})
\end{aligned}$$

But as $\mathrm{D}_\mathrm{k} = (-1)^k$ we get,

$$j\, (p_{k+1}\, q_k'' - p_k''\, q_{k+1}) = (-1)^k$$

Clearly the right hand side is only divisible by 1, and so we conclude that the highest common factor, $j = 1$. Hence p_n/q_n is irreducible.

By considering $\frac{p_{k+1}}{} - \frac{p_k}{}$ prove that p_n/q_n converges as n increases.

q_{k+1} q_k

CONTINUED FRACTION RECURRENCE RELATION

Problem

Any real number, α, can be represented as a continued fraction which may either terminate or be infinite (repeating or non-repeating).

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

Let p_n/q_n be defined as the nth convergent:

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots + \cfrac{1}{a_n}}}$$

For example, $\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}$

And the first five convergents are:

$$\begin{aligned} p_0/q_0 &= 1 \\ p_1/q_1 &= 3/2 \\ p_2/q_2 &= 7/5 \\ p_3/q_3 &= 17/12 \\ p_4/q_4 &= 41/29 \end{aligned}$$

Prove that the following recurrence relation holds for $n \geq 2$ for all continued fractions:

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2} \\ q_n &= a_n q_{n-1} + q_{n-2} \end{aligned}$$

Solution

From the continued fraction, $p_0/q_0 = a_0/1 \Rightarrow p_0 = a_0$ and $q_0 = 1$.

Similarly $p_1/q_1 = a_0 + 1/a_1 = (a_0a_1 + 1)/a_1 \Rightarrow p_1 = a_0a_1 + 1$ and $q_1 = a_1$.

In other words, p_0, p_1, q_0, and q_1 are clearly defined. Now let us consider the convergent p_2/q_2 as a continued fraction expansion.

$$\begin{aligned}
\frac{p_2}{q_2} &= a_0 + \frac{1}{a_1 + 1/a_2} \\
&= a_0 + \frac{1}{\frac{a_1a_2 + 1}{a_2}} \\
&= a_0 + \frac{a_2}{a_1a_2 + 1} \\
&= \frac{a_0(a_1a_2 + 1) + a_2}{a_1a_2 + 1} \\
&= \frac{a_0a_1a_2 + a_0 + a_2}{a_1a_2 + 1} \\
&= \frac{a_2(a_0a_1 + 1) + a_0}{a_1a_2 + 1} \\
&= \frac{a_2(a_0a_1 + 1) + a_0}{a_1a_2 + 1}
\end{aligned}$$

But as $p_1 = a_0a_1 + 1$, $p_0 = a_0$, $q_1 = a_1$, and $q_0 = 1$, we get:

$$\frac{p_2}{q_2} = \frac{a_2p_1 + p_0}{a_2q_1 + q_0}$$

Hence the recurrence relation is true for $n = 2$.

Now suppose that the recurrence relation holds for $n = k$, and consider the continued fraction expansions of the convergents p_k/q_k and p_{k+1}/q_{k+1}.

$$\frac{p_k}{q_k} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots + \cfrac{1}{\boxed{a_k}}}}$$

$$\frac{p_{k+1}}{q_{k+1}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots + \cfrac{1}{\boxed{a_k + \cfrac{1}{a_{k+1}}}}}}$$

It can be seen that going from p_k/q_k to p_{k+1}/q_{k+1} we have replaced a_k with $a_k +$

$1/a_{k+1}$.

$$\therefore \frac{p_k}{q_k} = \frac{\boxed{a_k} p_{k-1} + p_{k-2}}{a_k q_{k-1} + q_{k-2}} \rightarrow \frac{p_{k+1}}{q_{k+1}} = \frac{\boxed{(a_k + 1/a_{k+1})} p_{k-1} + p_{k-2}}{(a_k + 1/a_{k+1}) q_{k-1} + q_{k-2}}$$

$$= \frac{a_{k+1} a_k p_{k-1} + p_{k-1} + a_{k+1} p_{k-2}}{a_{k+1} a_k q_{k-1} + q_{k-1} + a_{k+1} q_{k-2}}$$

$$= \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}}$$

$$= \frac{a_{k+1} p_k + p_{k-1}}{a_{k+1} q_k + q_{k-1}}$$

This is the expected outcome for p_{k+1}/q_{k+1}. As the recurrence relation is true for n = 2, and p_0, p_1, q_0, and q_1 are clearly defined, we have shown that the recurrence relation holds for all values of $n \geq 2$.

However, for practical purposes it is convenient to set $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = a_0$, and $q_0 = 1$. Then the recurrence relation holds for $n \geq 1$.

COUNTING DIGITS

Problem

How many digits does the number 2^{1000} contain?

Solution

As 2^{1000} is not a multiple of 10, it follows that,
$10^m < 2^{1000} < 10^{m+1}$, where 10^m contains $m + 1$ digits.

Solving $2^{1000} = 10^k$, where $m < k < m + 1$

$k = \mathit{log}\ 2^{1000} = 1000\ \mathit{log}\ 2 \approx 301.02999...$, so $m = [1000\ \mathit{log}\ 2] = 301$.

Hence 2^{1000} contains 302 digits.

What is the least value of n, such that 2^n contains exactly one million digits?

DIOPHANTINE CHALLENGE

Problem

Given that x, y, and b are positive integers, prove that the Diophantine equation, $x^2 + (b–x)y = \pm 1$ in x and y, has at least four solutions for all values of b.

Solution

We shall begin by rearranging the equation, $x^2 \pm 1 = (x–b)y$.

When $b=1$, and taking the subtract form of LHS, we get $x^2–1 = (x+1)(x–1) = (x–1)y$, so $y = x+1$. That is, we have infinitely many solutions for (x,y): (1,2), (2,3), (3,4), ...

For $b \geq 2$, let us deal with a slightly more general form, $x^2+a = (x–b)y$.

Clearly $x–b$ divides $x^2–bx$, and as $x–b$ divides the RHS, it follows that it must divide x^2+a. Therefore, $x–b$ divides $(x^2+a)–(x^2–bx) = a+bx$.

Similarly $x–b$ divides $(a+bx)–(bx–b^2) = b^2+a$.

So a solution exists for each value of $x–b$ that divides $b^2 \pm 1$, or rather each factor of $b^2 \pm 1$.

When $x–b = b^2–1$ or $x–b = b^2+1$, we get $x = b^2+b \pm 1$, which are both positive integers. And as we have already established that $x–b$ divides both sides of the Diophantine equation, $y = (x^2 \pm 1)/(x–b)$ will also be positive integers. Thus we have two positive integer solutions for x and y.

But when $x–b = 1$, we can see that $x = b+1 < b^2+b \pm 1$ for $b \geq 2$, and so this solution in x will be different to the previous two. In addition, by substituting $x–b = 1$ into the Diophantine equation, we get $y = x^2 \pm 1$, which provides two more positive integer solutions for x and y.

Hence we have proved that the Diophantine equation has at least four positive integer solutions all values of b.

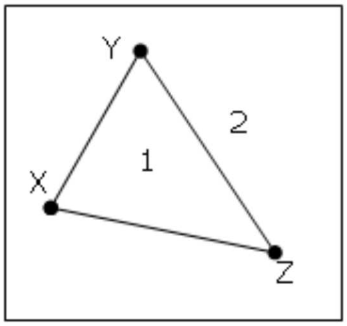
Prove that $b=2$ is the only value of b for which there are exactly four solutions.

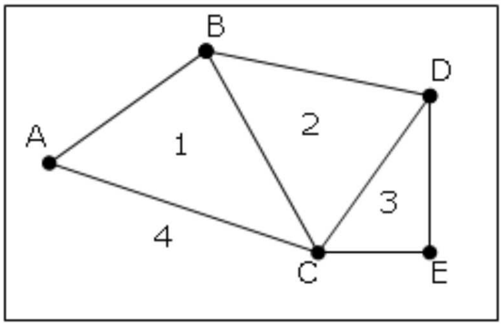
EULER RULES

Problem

A connected graph is said to be planar if it is possible to be drawn with no intersecting edges.

If V represents the number of vertices, E the number of edges, and R the number of regions, it can be verified that the Euler's rule, V + R = E + 2, holds for the two graphs below (the regions are indicated by the numbers on the graphs).



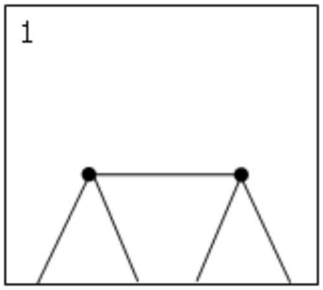


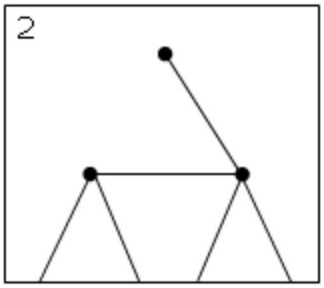
For the graph on the left: V = 3, R = 2, E = 3, so V + R = E + 2 =5.
For the graph on the right: V = 5, R = 4, E = 7, so V + R = E + 2 = 9.

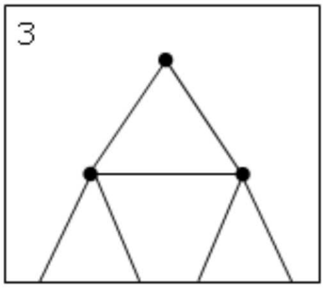
Prove that Euler's rule is true for all planar graphs.

Solution

This simple rule can be proved easily by induction. Consider the following diagrams.







Suppose that diagram 1 represents part of a planar graph for which Euler's rule holds; that is, V + R = E + 2.

By adding one vertex and one edge (diagram 2), we increase both sides of the equation by 1, so the formula remains true.

By adding one extra edge (diagram 3), we introduce one extra region, thus both sides of the equation increase by 1 once more, and the formula still holds.

As it is clearly true for one vertex (V = 1, E = 0, R = 1), it has been proved, inductively, to be true for all planar graphs.

If V represents the number of vertices, E the number of edges, and F the number of faces, prove that Euler's rule, V + F = E + 2, holds for the Platonic solids.
Explain how it is still true for a cylinder.
For which type of polyhedra does it not work?

EVEN PERFECT NUMBERS

Problem

The divisors of a positive integer, excluding the number itself, are called the proper divisors . If the sum of proper divisors is equal to the number we call the number perfect. For example, the divisors of 28 are 1, 2, 4, 7, 14, and 28, so the sum of proper divisors is 1 + 2 + 4 + 7 + 14 = 28.

The first eight perfect numbers are 6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128.

Prove that P is an even perfect number iff it is of the form $2^{n-1}(2^n-1)$ where 2^n-1 is prime.

Solution

Throughout this proof we shall make use of the sum of divisor function, $\sigma(n)$. As this function sums ALL the divisors, including P itself, then P will be perfect iff $\sigma(P) = 2P$.

Given that $P = 2^{n-1}(2^n-1)$, we shall begin by showing that if 2^n-1 is prime then P will be perfect.

$\sigma(P) = \sigma(2^{n-1}(2^n-1)) = \sigma(2^{n-1})\times\sigma(2^n-1)$ due to the multiplicative property of the function.

Now $\sigma(2^{n-1}) = 1 + 2 + 4 + 8 + ... + 2^{n-1} = 2^n-1$.
As 2^n-1 is prime, $\sigma(2^n-1) = 1 + 2^n-1 = 2^n$.

Therefore $\sigma(P) = 2^n(2^{n-1}) = 2\times2^{n-1}(2^n-1) = 2P$, and we prove that P is perfect.

We shall now prove the converse; that is, if P is an even perfect number then it must be of the form $P = 2^{n-1}(2^n-1)$.

Suppose that P is an even perfect number. All evens can be written in the form, $q\times2^{n-1}$, where q is odd and $n \geq 2$.

Therefore, $\sigma(P) = \sigma(q\times2^{n-1}) = \sigma(q)\times\sigma(2^{n-1}) = \sigma(q)\times(2^n-1)$.

But as P is perfect, $\sigma(P) = 2P = q\times2^n = \sigma(q)\times(2^n-1) \Rightarrow \sigma(q) = q\times2^n/(2^n-1)$.

Because $\sigma(q)$ is integer, q must be divisible by 2^n-1, so let $q = r(2^n-1)$.

$\therefore \sigma(q) = q \times 2^n/(2^k-1) = r \times 2^n$.

As q is divisible by r, we know that its sum of divisors will at least be $q + r$.

Therefore, $\sigma(q) = r \times 2^n \geq q + r = r(2^n-1) + r = r(2^n-1+1) = r \times 2^n \Rightarrow \sigma(q) = q + r$.

As q and r are the only divisors, q must be prime and r must be 1.
And as $q + r = r \times 2^n$, $q + 1 = 2^n$, hence q will be a prime of the form 2^n-1.

And so, if P is an even perfect number then it will be of the form $2^{n-1}(2^n-1)$. **Q.E.D.**

Note that no one has yet proved that an odd perfect number exists, but Euler proved that if any do exist they will be of the form $q^a b^2$, where q is a prime of the form $4k+1$. See Odd Perfect Numbers.

EVEN SUM OF TWO ABUNDANT NUMBERS

Problem

The divisors of a positive integer, excluding the number itself, are called the proper divisors . If the sum of proper divisors is equal to the number we call the number perfect. For example, the divisors of 28 are 1, 2, 4, 7, 14, and 28, so the sum of proper divisors is 1 + 2 + 4 + 7 + 14 = 28.

Similarly, if the sum of the proper divisors exceeds the number we call the number abundant. For example, 12 is abundant because the divisors of 12 are 1, 2, 3, 4, 6 12, and the sum of proper divisors $1 + 2 + 3 + 4 + 6 = 14 > 12$.

Prove that every even $n \geq 48$ can be written as the sum of two abundant numbers.

Solution

Let S(n) represent the sum of proper divisors of n.

Suppose that $S(n) = c \geq n$; in other words, n is perfect or abundant.

Let us consider mn where $m \geq 2$.

If d_1, d_2, ... , d_i represents the complete list of proper divisors of n then mn will be divisible by md_1, md_2, ... , md_i. But this list does not represent the complete set of proper divisors of mn because $m \geq 2$ and so it will be missing 1 and possibly more factors.

Therefore $S(mn) = m(d_1 + d_2 + ... + d_i) + q = mc + q$, where $q \geq 1$.

But if $S(n) = c \geq n$ then $mc \geq mn$.

Hence $S(mn) = mc + q > mn$ and we prove that mn will be abundant.

In other words, the multiple of any perfect or abundant number will be abundant.

Now S(6) = 1 + 2 + 3, so 6 is perfect and consequently for $\boldsymbol{k \geq 2}$ all numbers of the form $6k = \{12, 18, 24, ...\}$ will be abundant.

It follows then that all numbers of the form $6k + 12 = \{24, 30, 36, 42, 48, ...\}$ can be written as the sum of two abundant numbers.

As $S(20) = 1 + 2 + 4 + 5 + 10 = 22 > 20$ then 20 is abundant.

Therefore all numbers of the form $6k + 20 = \{32, 38, 44, 50, ...\}$ will be the sum of two abundant numbers.

Similarly, because 20 is abundant then 40 will also be abundant, and so all numbers of the form $6k + 40 = \{52, 58, 64, ...\}$ will be the sum of two abundant numbers.

As $6k + 12 \equiv 0 \bmod 6$, $6k + 20 \equiv 2 \bmod 6$, and $6k + 40 \equiv 4 \bmod 6$ then we have proved that all even numbers, $n \geq 48$, can be written as the sum of two abundant numbers.

EVERY PRIMITIVE TRIPLET

Problem

A Pythagorean triplet, (a, b, c), is defined as a set of positive integers for which $a^2 + b^2 = c^2$. Furthermore, if the set is primitive it means that a, b, and c share no common factor greater than 1.

Prove that the following identities will generate all primitive Pythagorean triplets and determine the conditions for m and n.

$$\begin{aligned} a &= m^2 - n^2 \\ b &= 2mn \\ c &= m^2 + n^2 \end{aligned}$$

Solution

First we note that it is not sufficient to simply show that following identity holds:

$$a^2 + b^2 = (m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2 = c^2$$

Although this is true, and can be verified, it would only demonstrate that the identities will produce Pythagorean triplets. It will not prove that they generate every primitive case. Instead we must analyse the necessary conditions of a, b, and c to derive these identities deductively.

Clearly a and b cannot both be even, otherwise, $a^2 + b^2 = c^2$ would be even, and as a, b, and c would all be even and it would not be primitive.

If a and b were both odd then $a^2 + b^2 \equiv 2 \bmod 4$, but $c^2 \equiv 0, 1 \bmod 4$.

Therefore exactly one of a and b is even, and it follows that c will be odd.

Without loss of generality, let b be even.

$\therefore b^2 = c^2 - a^2 = (c + a)(c - a)$

$$\therefore b^2/4 = (b/2)^2 = \frac{(c+a)}{2}\,\frac{(c-a)}{2}$$

As a and c are both odd, $c + a$ and $c - a$ will be even, and we know that $(b/2)^2$, $(c + a)/2$, and $(c - a)/2$ will all be integer.

As the LHS, $(b/2)^2$, is square, the RHS must be square and as a, b, and c share no common factor greater than 1, both terms on the right must each be square.

Let $m^2 = (c + a)/2$ and $n^2 = (c - a)/2$.

$\therefore b^2/4 = m^2n^2 \Rightarrow b^2 = 4m^2n^2 \Rightarrow b = 2mn$

$$m^2 + n^2 = \frac{(c + a)}{2} + \frac{(c - a)}{2} = c$$

Similarly, $m^2 - n^2 = a$.

As we have deduced these identities from first principles we know that all primitve triplets will be derived from them. Let us now consider the conditions for m and n.

Clearly $m > n$ otherwise $a \le 0$. Also GCD(m, n) = 1, otherwise, a, b, and c would share a common factor greater than 1. If m and n were both odd then $m^2 + n^2$ and $m^2 - n^2$ would be even, and as $b = 2mn$, we would again have a common factor of 2 with a, b, and c.

Hence for every $m > n$, GCD(m, n) = 1, and $m + n \equiv 1 \mod 2$, $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$ will produce a primitive Pythagorean triplet.

NOTE: If we only insist that $m > n$, then the identities will produce non-primitive cases also. However, it is a common misconception to believe that this implies that the identities will produce every triplet (primitive and non-primitive cases). As a single example it can be quickly verified that no values of m and n will ever produce the non-primitive case, (9, 12, 15).

EXPONENTIAL SYMMETRY

Problem

Given that x and y are positive integers, solve $x^y = y^x$.

Solution

For the purpose of this problem we shall assume that $x = y$ is a trivial solution, and it can be seen that if $x = 1$ then $1^y = y^1 \Rightarrow x = y = 1$.

So without loss of generality we can assume that $2 \le x < y$.

Begin by dividing both sides by x^x:

$$x^y/x^x = y^x/x^x$$
$$\therefore x^{y-x} = (y/x)^x$$

As the left hand side is integer it follows that the right hand side is integer.
Let the integer, $m = y/x \Rightarrow y = mx$.

$$\begin{aligned}\therefore \quad x^{mx-x} &= m^x\\ \therefore \quad x^{x(m-1)} &= m^x\\ \therefore (x^{x(m-1)})^{1/x} &= (m^x)^{1/x}\\ \therefore \quad x^{m-1} &= m\end{aligned}$$

As we have ruled out the trivial case $x = y$ and $y = mx$, it is clear that $m \ge 2$.

For $m = 2$, $x^1 = 2 \Rightarrow y = 4$

Let us consider $m \ge 3$:
When $m = 3$, $x^2 = 3$, and as $x \ge 2$, it follows that $x^2 > 3$.
We shall now prove inductively that $x^{m-1} > m$ and thus $x^{m-1} \ne m$ for $m \ge 3$.
If true then $x^{m-2} > m-1$.
$\therefore x^{m-1} = x.x^{m-2} > x(m-1)$
But as $x \ge 2$, $x^{m-1} > 2m-2 > m$, which is the expected result.
Hence $x^{m-1} > m$ for all values of $m \ge 3$.

Therefore we prove that $2^4 = 4^2$ is the only solution in positive integers.

What if we relax the condition of *x* and *y* being positive?
What if we allow *x* and *y* to be rational numbers?

Problem ID: 345 (09 Jul 2008)   Difficulty: 4 Star   [mathschallenge.net]

FACTORIAL AND SQUARE

Problem

Given that a, b, and c are positive integers, solve the following equation.

$$a!b! = a! + b! + c^2$$

Solution

Without loss of generality let us assume that $b \leq a$.

If $b = a$, then, $(a!)^2 = 2(a!) + c^2$.

$\therefore (a!)^2 - 2(a!) + 1 = c^2 + 1$
$(a! - 1)^2 = c^2 + 1$

However, as there exists no square which is one more than another square we deduce that $b < a$.

In addition, $b \neq 1$, otherwise, $a! = a! + 1 + c^2 \Rightarrow c^2 = -1$.

Dividing $a!b! = a! + b! + c^2$ by $b!$ we get, $a! = a!/b! + 1 + c^2/b!$.

Clearly LHS is integer, and as $a!/b!$ is integer, $c^2/b!$ must also be integer. Furthermore, RHS ≥ 3, which means that $a! \geq 3 \Rightarrow a \geq 3$.

From $a!b! = a! + b! + c^2$, we get $a!b! - a! - b! + 1 = c^2 + 1$.

$\therefore (a! - 1)(b! - 1) = c^2 + 1$.

Let us assume that a prime, $p \equiv 3 \bmod 4$ divides the RHS.

If $c^2 + 1 \equiv 0 \bmod p$, it follows that $c^2 \equiv -1 \bmod p$, and it is clear that p does not divide c.

$\therefore (c^2)^{(p-1)/2} = c^{p-1} \equiv (-1)^{(p-1)/2} \bmod p$.

But we are given that $p \equiv 3 \bmod 4$, so $p-1 \equiv 2 \bmod 4$, which means that $(p-1)/2$ will be odd, and $(-1)^{(p-1)/2} = -1$.

Hence $c^{p-1} \equiv -1 \bmod p$, which is a contradiction, because HCF(c, p) = 1 and by

Fermat's Little Theorem, $c^{p-1} \equiv 1 \bmod p$.

In other words, there exists no prime, $p \equiv 3 \bmod 4$, which divides $c^2 + 1$.

However, for $a \geq 4$, $a! \equiv 0 \bmod 4$, and $a! - 1 \equiv 3 \bmod 4$; that is, LHS is divisible by a prime, $p \equiv 3 \bmod 4$.

Hence $1 < b < a \leq 3$, but we have already established that $a \geq 3$, so we deduce that $a = 3$ and $b = 2$.

From $a!b! = a! + b! + c^2$, we get, $12 = 6 + 2 + c^2 \Rightarrow c = 2$.

That is, the original equation has a unique solution: $a = 3$, $b = c = 2$.

Related problems:

Factorial Symmetry: $a!b! = a! + b!$

Factorial And Power Of 2: $a!b! = a! + b! + 2^c$

Factorial Equation: $a!b! = a! + b! + c!$

Factorial Equation

Problem

Given that a, b, and c are positive integers, solve the following equation.

$$a!b! = a! + b! + c!$$

Solution

Without loss of generality let us assume that $a \geq b$ and divide through by $b!$: $a! = a!/b! + 1 + c!/b!$. As we have integers throughout, $c \geq b$.

As RHS ≥ 3, $a! \geq 3 \Rightarrow a \geq 3$.

Clearly $a = b = c$ would give $a! = 3$, which has no solutions. Therefore at least one of a, c must exceed b.

But if $a > b$ and $c > b$ then $b+1$ will divide $a!$ and $c!$ but not 1, so both a and c cannot exceed b.

If $a > b$ and $c = b$ we get $a! = a!/c! + 2$, and then $c+1$ would divide $a!$ but not 2.

So we conclude that $a = b$ and $c > b$, giving $a! = 2 + c!/a!$.

If $c \geq a+3$, then 3 divides $a!$ and $c!/a!$ but not 2, so c cannot exceed a by more than 2.

Writing $a! = 2 + c!/a!$ as $a!(a! - 2) = c!$, and noting that $a < c < a+3$, we get $a! < c! < (a + 3)!$. Hence $a! - 2$ is equal to $(a + 1)(a + 2)$ or $(a + 1)$.

If $a! - 2 = (a + 1)(a + 2)$, we get $a! = a^2 + 3a + 4$. As LHS is divisible by a, RHS will only divide by a if $a = 4$, but this does not lead to a solution.

If $a! - 2 = a + 1$, we get $a! = a + 3$. As LHS is divisible by a, RHS will only divide by a if $a = 3 \Rightarrow b = 3$ and $c = 4$.

That is, $3!3! = 3! + 3! + 4!$ is the only solution.

Related problems:

Factorial Equation: $a!b! = a! + b!$

Factorial And Power Of 2: $a!b! = a! + b! + 2^c$

Factorial And Square: $a!b! = a! + b! + c^2$

Problem ID: 217 (09 Mar 2005)   Difficulty: 4 Star   [mathschallenge.net]

FAREY SEQUENCE

Problem

Starting with F_1={0/1, 1/1}, F_n is generated by comparing adjacent fractions in F_{n-1}: *a/b* and *c/d*. If *b*+*d*=*n*, then the mediant, (*a*+*c*)/(*b*+*d*), is placed between them. Consider the first six sets:

F_1={0/1, 1/1}
F_2={0/1, **1/2**, 1/1}
F_3={0/1, **1/3**, 1/2, **2/3**, 1/1}
F_4={0/1, **1/4**, 1/3, 1/2, 2/3, **3/4**, 1/1}
F_5={0/1, **1/5**, 1/4, 1/3, **2/5**, 1/2, **3/5**, 2/3, 3/4, **4/5**, 1/1}
F_6={0/1, **1/6**, 1/5, 1/4, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5, **5/6**, 1/1}

Remarkably, F_n represents the ordered list of reduced fraction for which the denominator does not exceed *n*, and is called the Farey sequence.

1. What fraction is the immediate predecessor of 2/5 in F_{100}?
2. Given that *a/b* and *c/d* are two consecutive fractions in F_n, prove,
   i. *bc*–*ad*=1.
   ii. when a new fraction of the form, (*a*+*c*)/(*b*+*d*), is inserted, then *a/b* < (*a*+*c*)/(*b*+*d*) < *c/d*.

Solution

1. We note that 2/5 first makes its appearance in F_5, with 1/3 being its immediate predecessor. The next fraction to appear between them will be (1+2)/(3+5)=3/8. As we are considering fractions to the left of 2/5, it can be seen that for each new fraction, the numerator continues to increase by 2 and the denominator increases by 5. Moreover, because we know that 1/3 was to the left, each new fraction will be of the form (1+2*m*)/(3+5*m*).

   In F_{100} no denominator exceeds 100, and as 3+5×19=98, we can deduce that *m*=19. Hence the immediate predecesor of 2/5 in F_{100} will be 39/98.

   What will be the successor of 2/5 in F_{100}?

2. Given two consecutive fractions, *a/b* and *c/d*, we are hoping to prove that *bc*–*ad*=1.

   Clearly it is true for F_1 = {0/1, 1/0}: 1×1–0×0 = 1.

   Let us assume that *bc*–*ad*=1 is true for all elements in F_k, and let us consider

F_{k+1}. If no fraction is inserted then it follows that $bc-ad=1$ is still true.

If a new fraction, of the form $(a+c)/(b+d)$, is inserted, then we must consider two cases.

Case 1 [a/b and $(a+c)/(b+d)$]:
$b(a+c)-a(b+d) = ab+bc-ab-ad = bc-ad = 1$

Case 2 [$(a+c)/(b+d)$ and c/d]:
$c(b+d)-d(a+c) = bc+cd-ad-cd = bc-ad = 1$

In other words, if it is true for F_k it is true for F_{k+1}, and as it is true for F_1, inductively we show that it is true for all sets.

As we have just shown that $b(a+c)-a(b+d) = 1$, it follows that $b(a+c)-a(b+d) > 0$.

Therefore, $b(a+c) > a(b+d)$, $(a+c)/(b+d) > a/b$.

Similarly, from $c(b+d)-d(a+c) = 1$, we get, $c/d > (a+c)/(b+d)$.

Hence, $a/b < (a+c)/(b+d) < c/d$.

FIBONACCI SEQUENCE

Problem

Given the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, ... defined by the second order recurrence relation, $F_{n+2} = F_n + F_{n+1}$. Prove that the nth term, F_n, is given by:

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Solution

We shall prove this by induction, starting by showing that $F_1 = F_2 = 1$.

$$\begin{aligned}
F_1 &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right)\\
&= \frac{1}{2\sqrt{5}}\left(\left(1+\sqrt{5}\right) - \left(1-\sqrt{5}\right)\right)\\
&= \frac{1}{2\sqrt{5}}\left(2\sqrt{5}\right)\\
&= 1\\
F_2 &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right)\\
&= \frac{1}{\sqrt{5}}\left(\frac{6+2\sqrt{5}}{4} - \frac{6-2\sqrt{5}}{4}\right)\\
&= \frac{1}{\sqrt{5}}\left(\frac{3+\sqrt{5}}{2} - \frac{3-\sqrt{5}}{2}\right)\\
&= \frac{1}{\sqrt{5}}\left(\sqrt{5}\right)\\
&= 1
\end{aligned}$$

Now we shall consider $F_n + F_{n+1}$.

$$\begin{aligned}
&\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) + \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)\\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)\\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n\left(1+\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^n\left(1+\frac{1-\sqrt{5}}{2}\right)\right)
\end{aligned}$$

$$= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^n\left(\frac{3-\sqrt{5}}{2}\right)\right)$$

But in calculating F_2 we have seen that, $\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2}$

Similarly, $\left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{3-\sqrt{5}}{2}$

$$\begin{aligned}\therefore F_n + F_{n+1} &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^n\left(\frac{1-\sqrt{5}}{2}\right)^2\right)\\ &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right)\\ &= F_{n+2}\end{aligned}$$

Which is consistent with the result expected; that is, $F_{n+2} = F_n + F_{n+1}$. As it works for F_1 and F_2 it must work for all n.

Finishing With 99

Problem

Consider the following results.

$99^1 = \mathbf{99}$
$99^2 = 9801$
$99^3 = 9702\mathbf{99}$
$99^4 = 96059601$
$99^5 = 9509900 4\mathbf{99}$

Prove that 99^n ends in 99 for odd n.

Solution

Given that $a \equiv b \bmod k$, $a^n \equiv b^n \bmod k$.

As $99 \equiv -1 \bmod 100$, $99^n \equiv (-1)^n \bmod 100$

For odd n, $(-1)^n = -1$.

Therefore $99^n \equiv -1 \bmod 100$ for odd n; that is, one less than a multiple of 100, which means it will end in 99.

Prove that, for odd n, (i) 9^n ends in 9, (ii) 999^n ends in 999, (iii) Generalise.

If $a \equiv b \bmod k$, then prove that $a^n \equiv b^n \bmod k$.

FIREWORK ROCKET

Problem

A firework rocket is fired vertically upwards with a constant acceleration of 4 ms^{-2} until the chemical fuel expires. Its ascent is then slowed by gravity until it reaches a maximum height of 138 metres.

Assuming no air resistance and taking g=9.8 ms^{-2}, how long does it take to reach its maximum height?

Solution

During acceleration phase (as fuel burns), a=4, u=0, let v=w.

$v=u+at$: $w=4t_1$ (1)
$v^2=u^2+2as$: $w^2=8s_1$ (2)

During deceleration phase (fuel expired), a=-9.8, u=w, v=0.

$v=u+at$: $0=w-9.8t_2$, $w=9.8t_2$ (3)
$v^2=u^2+2as$: $0=w^2-19.6s_2$, $w^2=19.6s_2$ (4)

Equating (1) and (3), $4t_1=9.8t_2$, $t_2=(20/49)t_1$, so total time to reach maximum height, $t_1+t_2=(69/49)t_1$.

Equating (2) and (4), $8s_1=19.6s_2$, $s_2=(20/49)s_1$, and as $s_1+s_2=(69/49)s_1=138$, we get $s_1=98$.

Using $s=ut+½at^2$ during acceleration phase, $s_1=2t_1^2$, $t_1=\sqrt{}(s_1/2)=7$.

Hence time to reach maximum height is $(69/49)\times7=69/7$ seconds.

FOURTH POWER PLUS FOUR PRIME

Problem

Given that n is a natural number, when is $n^4 + 4$ prime?

Solution

$$\begin{aligned} n^4 + 4 &= n^4 + 4n^2 + 4 - 4n^2 \\ &= (n^2 + 2)^2 - (2n)^2 \quad \text{[difference of two squares]} \\ &= ((n^2 + 2) + 2n)((n^2 + 2) - 2n) \\ &= (n^2 + 2n + 2)(n^2 - 2n + 2) \end{aligned}$$

As this represents a product it can only be prime if one of the factors is equal to 1. Clearly $n^2 + 2n + 2 > 1$ for $n \geq 1$.

$$\begin{aligned} &\therefore\ n^2 - 2n + 2 = 1 \\ &\therefore\ n^2 - 2n + 1 = 0 \\ &\therefore\ (n - 1)^2 = 0 \Rightarrow n = 1 \end{aligned}$$

When $n = 1$ we get the prime, $n^4 + 4 = 5$. Hence this is the only value of n for which $n^4 + 4$ is prime.

Explain why $n^4 + 4$ will always divide by two distinct numbers of the form $k^2 + 1$ for $n > 1$. For examle, $5^4 + 4 = 629$, which is divisible by $6^2 + 1 = 37$ and $4^2 + 1 = 17$.

GENERAL FACTORIAL

Problem

The Gamma function is defined as, $\Gamma(x) = \int_0^\infty t^{x-1}\, e^{-t}\, dt$.

Given that $\Gamma(x)$ is continuous for $x > 0$ prove that $\Gamma(x + 1) = x!$ for all positive integer values, and consequently is a candidate for extending the factorial function to non-integer positive values.

Solution

We shall integrate $\Gamma(x + 1)$ by parts (integrating e^{-t} and differentiating t^x):

$$\Gamma(x + 1) = \int_0^\infty t^x\, e^{-t}\, dt = [-t^x\, e^{-t}]_0^\infty + x\int_0^\infty t^{x-1}\, e^{-t}\, dt$$

For $x > 0$, we can see that $t^x = 0$ when $t = 0$ and $e^{-t} = 0$ when $t = \infty$, hence we get $\Gamma(x + 1) = x\Gamma(x)$.

Therefore for integers, $n \geq 1$, we get:

$$\begin{aligned}\Gamma(n + 1) &= n\Gamma(n)\\ &= n(n-1)\Gamma(n-1)\\ &= n(n-1)(n-2)\Gamma(n-2)\\ &= \ldots\\ &= n(n-1)(n-2)(n-3)\times\ldots\times3\times2\times1\times\Gamma(1)\end{aligned}$$

$$\Gamma(x) = \int_0^\infty t^{x-1}\, e^{-t}\, dt \Rightarrow \Gamma(1) = \int_0^\infty e^{-t}\, dt = -[e^{-t}]_0^\infty = -(0-1) = 1$$

Therefore $\Gamma(n + 1) = n!$ for $n \geq 1$.

Although we are not permitted to substitute $n = 0$ into the equation, we are pleased to see that $\Gamma(1) = 0! = 1$, as expected.

As $\Gamma(x + 1)$ maps to all the expected values of $x!$ for integer values and is continuous for $x > 0$, it is a suitable candidate for non-integer values.

It is interesting to note that for $x > 0$ the function can also be shown to be holomorphic (everywhere differentiable) and logarithmic convex. In fact, it turns out to be the ONLY function which produces a smooth curve through all the points of the graph $y = x!$ for $x > 0$.

Geometric Division

Problem

A positive integer, n, is divided by d and the quotient and remainder are q and r respectively. In addition d, q, and r are consecutive positive integer terms in a geometric sequence, but not necessarily in that order.

For example, 58 divided by 6 has quotient 9 and remainder 4. It can also be seen that 4, 6, 9 are consecutive terms in a geometric sequence (common ratio 3/2).

Prove that n cannot be prime.

Solution

We are solving $n = dq + r$ and clearly the remainder must be less than the divisor; that is, $r < d$.

If $r > q$, then the terms will be ordered, $d > r > q$.

Let $r = qx$ and $d = qx^2$, where x is the common ratio.

$\therefore n = dq + r = q^2x^2 + r = r^2 + r = r(r + 1)$

But $r(r + 1)$ can only be prime if $r = 1$, and as $r > q$, q would be non-integer or zero.

Hence $q > r$, and without loss of generality we shall suppose that $d > q > r$.

Let $q = rx$ and $d = rx^2$.

$\therefore n = dq + r = r^2x^3 + r = r(x^3r + 1)$

If $r = 1$ then $n = x^3 + 1$. Therefore x^3 must be integer, and as x is the common ratio of a geometric progression $x > 1$.

But $x^3 + 1 = (x + 1)(x^2 - x + 1) = (x + 1)(x(x - 1) + 1)$, and as $x \geq 2$, it follows that $x(x - 1) + 1 \geq 3$. Hence p would be the product of two numbers greater than two and could not be prime.

So let us consider the case where $r > 1$.

If x is integer then $p = r(x^3r + 1)$ will be the product of two numbers greater than one and cannot be prime.

Next, consider x being non-integer; let $x = a/b$ where GCD(a, b) = 1.

$\therefore n = r(x^3r + 1) = r(a^3r/b^3 + 1) = (r/b)(a^3r/b^2 + b)$

But $d = rx^2 = ra^2/b^2$ and as d is integer it follows that b^2 divides r and we can see that $a^3r/b^2 + b$ will be integer greater than one in value.

Let $r/b^2 = c \Rightarrow r/b = bc$, and as $b \geq 2$, $r/b \geq 2$.

Hence n would again be the product of two integers greater than one and cannot be prime. **Q. E. D.**

Harmonic Sum Approximation

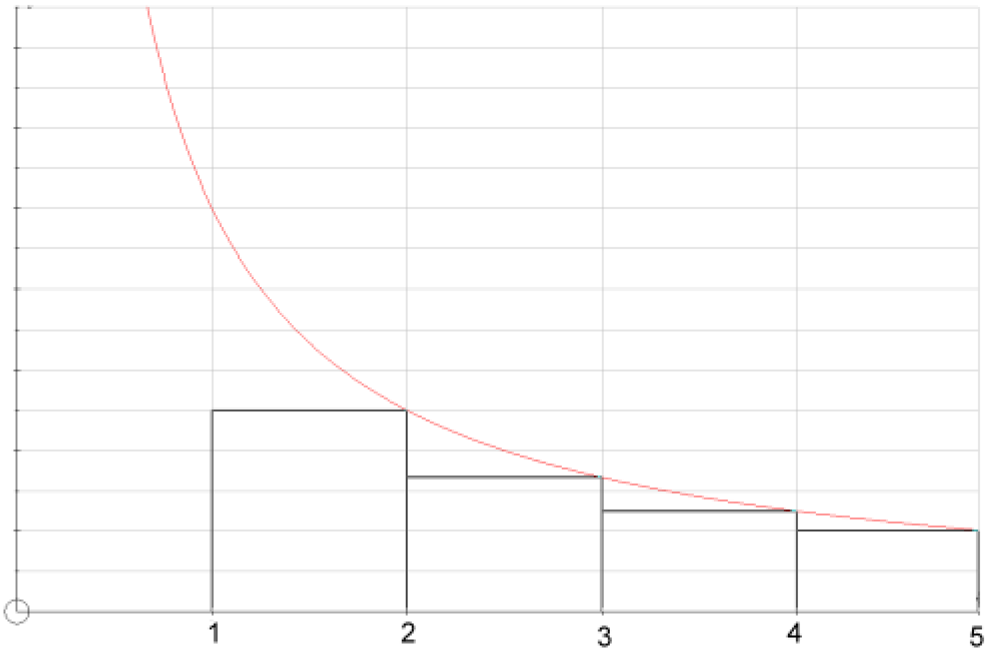
Problem

$H_n = 1 + 1/2 + 1/3 + \ldots + 1/n$ is defined as the nth Harmonic number.

1. Prove that $H_n \approx \ln(n) + 1/(2n) + k$, where $0.5 < k < 1$.
2. By using $k = 0.5772$, estimate the sum of the first one hundred Harmonic numbers.

Solution

Consider the graph $y = 1/x$ for $0 \le x \le 5$:



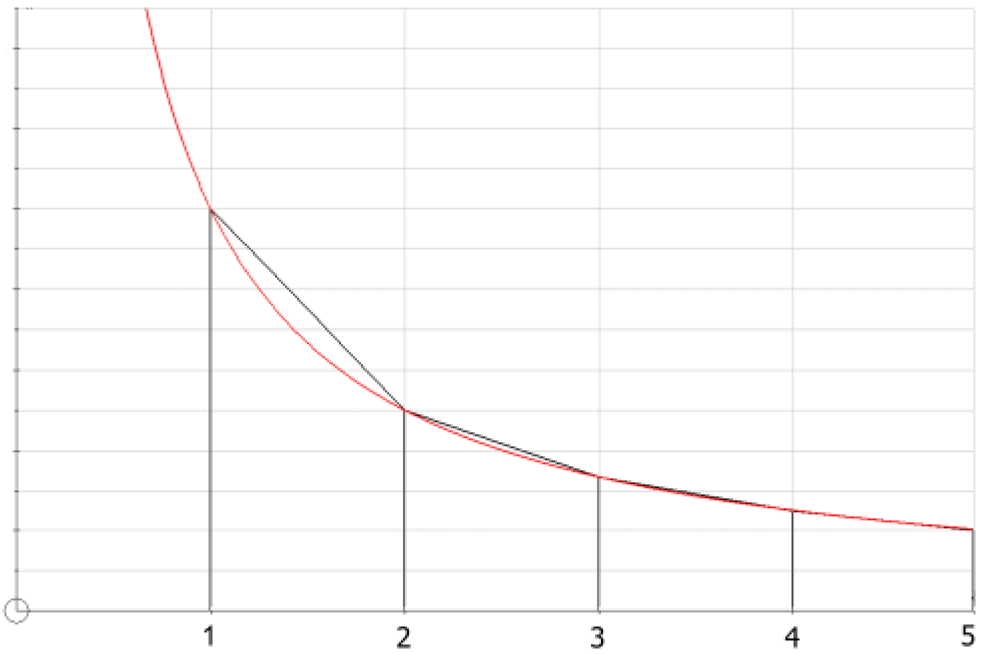
The exact area below the curve from 1 to n is given by $\int_1^n 1/x\,dx = \ln(n)$

By using rectangles, it can be seen that the approximate area is given by, $1/2 + 1/3 + 1/n = H_n - 1$. And as this under-estimates the area:

$$H_n - 1 < \ln(n)$$

$$\therefore H_n < \ln(n) + 1$$

By using the Trapezium rule:



$$\begin{aligned}\ln(n) &\approx (1/2)[1 + 2(1/2 + 1/3 + \ldots + 1/(n-1)) + 1/n] \\ &= 1 + 1/2 + 1/3 + \ldots + 1/n - 1/(2n) - 1/2 \\ &= H_n - 1/(2n) - 1/2\end{aligned}$$

But as the Trapezium rule is over-estimating the area in this case.

$$H_n - 1/(2n) - 1/2 > \ln(n)$$

$$\therefore H_n > \ln(n) + 1/(2n) + 1/2.$$

Hence we establish upper and lower limits:

$$\ln(n) + 1/(2n) + 1/2 < H_n < \ln(n) + 1$$

$$\therefore H_n \approx \ln(n) + 1/(2n) + k, \text{ where } 1/2 < k < 1 - 1/(2n)$$

Although this by no means proves that $k \approx 0.5772$, using a spreadsheet, and working to 10 d.p., let us consider the error between H_n and the approximation $\ln(n) + 1/(2n)$:

n	H_n	$\ln(n) + 1/(2n)$	Error
1	1	0.5	0.5
2	1.5	0.943147181	0.556852819
3	1.833333333	1.265278955	0.568054378
4	2.083333333	1.511294361	0.572038972

5	2.283333333	1.709437912	0.573895421
6	2.45	1.875092803	0.574907197
7	2.592857143	2.01733872	0.575518422
8	2.717857143	2.141941542	0.575915601
9	2.828968254	2.252780133	0.576188121
10	2.928968254	2.352585093	0.576383161
20	3.597739657	3.020732274	0.577007384
50	4.499205338	3.922023005	0.577182333
100	5.187377518	4.610170186	0.577207332
1000	7.485470861	6.908255279	0.577215582

In fact, $k = 0.5772156649$ (10 d.p.) is called the Euler-Mascheroni constant and appears in many other contexts; for example, the average number of divisors of all the numbers from 1 to n is approximately $\ln(n) + 2k - 1$. Although k is suspected to be transcendental, no one so far has even established if it is irrational. Care to prove it?

At this stage it may be tempting to use this approximation to sum the first one hundred Harmonic numbers:

$$
\begin{aligned}
\sum H &= H_1 + H_2 + \ldots + H_{100} \\
&\approx \ln(1)+1/2+0.5772 + \ln(2)+1/4+0.5772 + \ldots \\
&\quad + \ln(100)+1/200+0.5772 \\
&= \ln(1)+\ln(2)+\ldots+\ln(100) + (1/2)(1+1/2+\ldots+1/100) \\
&\quad + 100\times 0.5772 \\
&= \ln(100!) + (1/2)H_{100} + 57.72 \\
&\approx \ln(100!) + (1/2)(\ln(100)+1/200+0.5772) + 57.72 \\
&= \ln(100!) + \ln(10) + 58.0411
\end{aligned}
$$

However, not many calculators can evaluate 100!, and we would expect the error accumulated in one hundred and one approximations to be significant. It turns out that this compound approximation gives an answer of 424.083 (3 d.p.), which, as we shall see, is only correct to the nearest whole number.

Instead we shall consider the general sum of the first n Harmonic numbers:

$$
\begin{aligned}
\sum H &= H_1 + H_2 + \ldots + H_n \\
&= 1 + (1 + 1/2) + (1 + 1/2 + 1/3) + \ldots + (1 + 1/2 + \ldots + 1/n) \\
&= n + (n-1)/2 + (n-2)/3 + \ldots + 2/(n-1) + 1/n
\end{aligned}
$$

Writing $n = 1 + 2/2 + 3/3 + \ldots + (n-1)/(n-1) + n/n$

$$\begin{aligned}\sum H + n &= (n+1) + (n+1)/2 + (n+1)/3 + \ldots + (n+1)/n \\ &= (n+1)(1 + 1/2 + 1/3 + \ldots + 1/n) \\ &= (n+1)H_n\end{aligned}$$

Therefore $\sum H = (n+1)H_n - n$.

This is a much better approach as we need only make one approximation to find the sum: $101(\ln(100) + 1/200 + 0.5772) - 100 \approx 423.924$; incredibly the actual sum, without using the approximation, is 423.925 (3 d.p.).

IMPOSSIBLE QUADRATIC

Problem

Given that p and q are prime, prove that $px^2 - qx + q = 0$ has no rational solutions.

Solution

From $px^2 - qx + q = 0 \Rightarrow x > 0$.

Suppose that $x = m/n$, where HCF(m,n) = 1.

Therefore, $pm^2/n^2 - qm/n + q = 0$, so $pm^2 - qmn + qn^2 = 0$.

Dividing through by m, we get $pm - qn + qn^2/m = 0$. Clearly $m|qn^2$, as all the other terms are integer. But HCF(m,n) = 1, so $m|q$. But because q is prime, we deduce that $m = 1, q$; in the same way $n = 1, p$.

In which case, $x = m/n = 1, 1/p, q, q/p$.

If $x = 1$: $p - q + q = p \neq 0$.

If $x = 1/p$: $1/p - q/p + q = 0$. Multiplying by p/q gives $1/q - q + p = 0$, which is impossible, as all but the first term are integer.

If $x = q$: $pq^2 - q^2 + q = 0$. Dividing by q^2 gives $p - 1 + 1/q = 0$, which is impossible.

If $x = q/p$: $pq^2 - p + q = 0$. Dividing by q gives $pq - p/q + 1 = 0$, which is impossible.

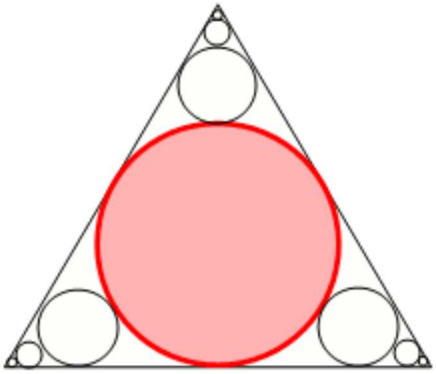
Hence no rational solutions to the quadratic equation $px^2 - qx + q = 0$ exist.

What about the equation, $px^2 - qx + p = 0$?

INFINITE CIRCLES

Problem

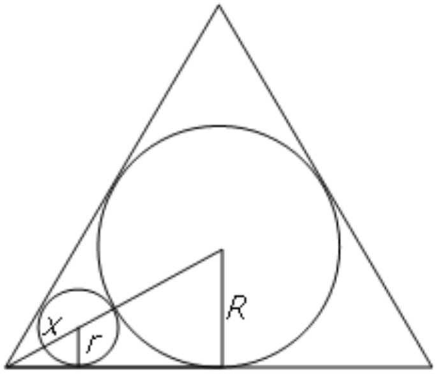
A circle is inscribed inside an equilateral triangle and an infinite set of circles are nested inside such that each circle touches the previous circle and the edges of the triangle act as tangents.



What fraction of the large red circle do the infinite set of smaller circles represent?

Solution

We begin by considering the relationship between one circle and the next.



Let the radius of the large circle be R, the radius of the small circle be r, and the hypotenuse of the small right angled triangle be x.

$\therefore \sin(30^o) = 1/2 = r / x \Rightarrow x = 2r$

In other words, for a 30^o right angle triangle the hypotenuse is twice the height. As the hypotenuse of the big right angle triangle is given by $x + r + R$ we get the following.

$$x + r + R = 2R$$
$$\therefore 2r + r + R = 2R$$
$$\therefore 3r = R$$

That is, the radius of each subsequent circle is 1/3 of the radius of the previous circle.

$$\therefore \pi(R/3)^2 + \pi(R/9)^2 + \ldots = \pi R^2(1/9 + 1/81 + \ldots)$$

Using the sum to infinity for a geometric progression: $a / (1 - r)$, where the first term, $a = 1/9$, and the common ratio, $r = 1/9$, we get $1/9 + 1/81 + \ldots = (1/9)(8/9) = 1/8$.

Hence the infinite set of smaller circles must represent exactly 3/8 the area of the large red circle.

Show that all of the circles occupy approximately 83% of the triangle.

INTEGER INTEGRAL

Problem

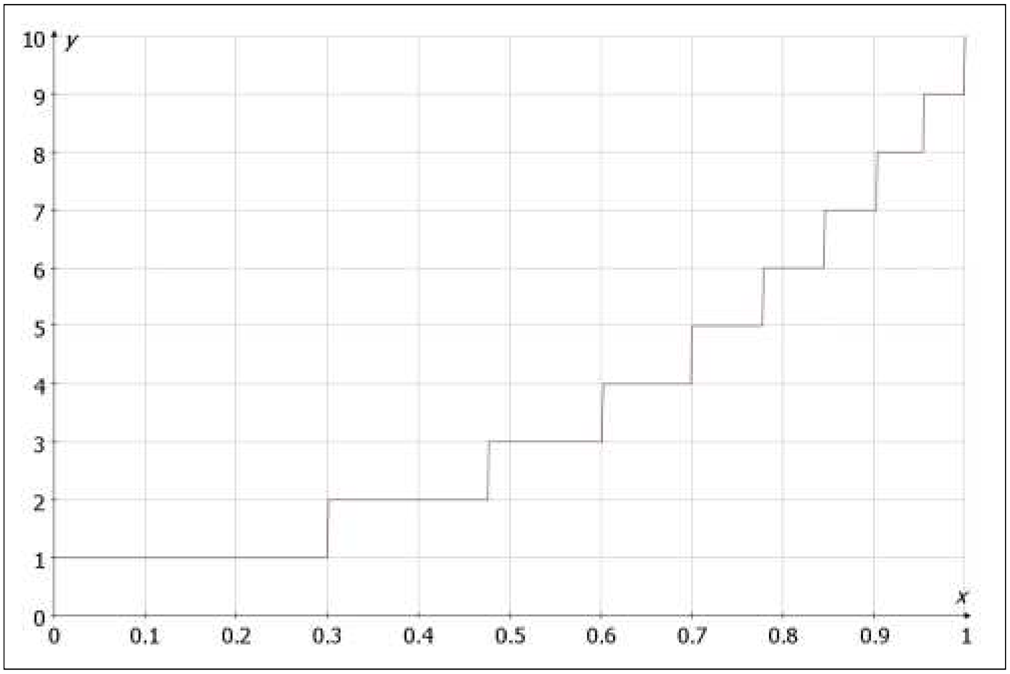
Solve the integral,

$$I=\int_0^1 [10^x]\, dx= 9 - \log(m).$$

Where [ ] is the integer part function and *m* is an integer to be determined.

Solution

Consider the graph, $y = [10^x]$.



The first step occurs when $10^x = 2 \Rightarrow x = \log 2 \approx 0.301$, the second step occurs when $10^x = 3 \Rightarrow x = \log 3 \Rightarrow 0.477$, and so on.

$$\begin{aligned}I&=1(\log 2) + 2(\log 3 - \log 2) + 3(\log 4 - \log 3) + \ldots + 9(\log 10 - \log 9)\\&=9\log 10 - (\log 2 + \log 3 + \ldots + \log 8)\\&=9 - \log(8!)\end{aligned}$$

Evalute, $\int_{0}^{\log(n)} [10^{x}]\, dx$.

IRRATIONALITY OF PI

Problem

Using "school level" mathematics prove that $\pi \approx 3.1415926535...$ is irrational.

Solution

This problem is extremely difficult and "five stars" would have perhaps been a more appropriate difficulty rating. So for that reason the solution should be used as a learning experience. In fact, using theorems from higher mathematics it is trivial to prove the irrationality of π, as it becomes a direct consequence of those theorems. But as those results are far from trivial to prove we shall utilise elementary concepts throughout, albeit pushing the boundaries on "school level" mathematics. We shall prove this by contradiction.

Let us suppose that π is rational and can be written as the ratio of two positive integers: $\pi = a / b$.

We now define the function, $f(x) = x^n(a - bx)^n / n!$, of which we shall make extensive use.

As $a = \pi b$ we can write $f(x) = x^n(b\pi - bx)^n / n! = (bx)^n(\pi - x)^n / n!$

$$\begin{aligned} \therefore f(\pi - x) &= (b(\pi - x))^n(\pi - (\pi - x))^n / n! \\ &= (b(a / b - x))^n x^n / n! \\ &= (a - bx)^n x^n / n! \\ &= f(x) \end{aligned}$$

So when $x = 0$ we get $f(0) = f(\pi) = 0$, which is integer.

Now let us define $G(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - ... + (-1)^n f^{(2n)}(x)$, where $f^{(k)}(x)$ represents the *kth* derivative of $f(x)$.

$$\begin{aligned} f(x) &= x^n(a - bx)^n / n! \\ &= x^n(C(n,0)a^n + C(n,1)a^{n-1}(-bx) + C(n,2)a^{n-2}(-bx)^2 + ... + C(n,n)(-bx)^n) / n! \\ &= (C(n,0)a^n x^n - C(n,1)a^{n-1}bx^{n+1} + C(n,2)a^{n-2}b^2x^{n+2} - ... + (-1)^n C(n,n)b^n x^{2n}) / n! \\ &= (m_n x^n + m_{n+1}x^{n+1} + m_{n+2}x^{n+2} + ... + m_{2n}x^{2n}) / n!, \text{ where } m_k \text{ is integer} \end{aligned}$$

In other words, f(x) is a degree 2n polynomial where the coefficient of x^k is given by m_k / n!, and the lowest power is x^n.

It should be clear that if we differentiate f(x) less than n times we will have a polynomial with no constant term. Hence $f^{(k)}(0) = 0$ for $k < n$. Similarly if we differentiate more than 2n times we will have no terms left and so $f^{(k)}(x) = 0$ for $k > 2n$.

For $n < k < 2n$ there will be a single constant term, and as the remaining terms will contain an "x", it follows that $f^{(k)}(0)$ will be equal to this constant term. If x^k is differentiated k times then it becomes $k\times(k-1)\times(k-2)\times...\times2\times1 = k!$.

Therefore the constant term will be $k!m_k / n!$ and as $k > n$ this will be integer.

Thus $f^{(k)}(0)$ is integer for all k.

As we have already shown that $f(x) = f(\pi - x)$ we get the following by repeated use of the chain rule.

$$f^{(1)}(x) = -f^{(1)}(\pi - x)$$
$$f^{(2)}(x) = f^{(2)}(\pi - x)$$
$$f^{(3)}(x) = -f^{(3)}(\pi - x)$$
$$...$$
$$f^{(k)}(x) = (-1)^k f^{(k)}(\pi - x)$$

Therefore $f^{(k)}(0) = (-1)^k f^{(k)}(\pi)$ and we can see that $f^{(k)}(\pi)$ must also be integer for all k.

Hence G(0) and G(π) by definition are integer.

From,

$$G(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - ... + (-1)^n f^{(2n)}(x)$$

Differentiate twice to get,

$$G^{(2)}(x) = f^{(2)}(x) - f^{(4)}(x) + f^{(6)}(x) - ... + (-1)^n f^{(2n+2)}(x)$$

As we have already noted, $f^{(2n+2)}(x) = 0$,

$$\therefore G(x) + G^{(2)}(x) = f(x)$$

Let $z = G^{(1)}(x)\sin(x) - G(x)\cos(x)$ and if we differentiate, using the product rule, we get,

$$dz / dx = (G^{(2)}\sin(x) + G^{(1)}\cos(x)) - (G^{(1)}\cos(x) - G(x)\sin(x))$$

$$\begin{aligned} &= (G^{(2)}\sin(x) + G(x)\sin(x)) \\ &= (G^{(2)} + G(x))\sin(x) \\ &= f(x)\sin(x) \end{aligned}$$

Hence by using the Fundamental Theorem of Calculus it follows that the integral of $f(x)\sin(x)$ must be $z = G^{(1)}(x)\sin(x) - G(x)\cos(x)$.

$$\begin{aligned} \therefore \int_0^\pi f(x)\sin(x)\,dx &= (G^{(1)}(\pi)\sin(\pi) - G(\pi)\cos(\pi)) - (G^{(1)}(0)\sin(0) - G(0)\cos(0)) \\ &= G(0) - G(\pi), \text{ which is integer.} \end{aligned}$$

Recall that $f(x) = x^n(a - bx)^n / n!$, so for $0 < x < \pi$:

$$\begin{aligned} &0 < \sin(x) < 1 \\ &0 < x^n < \pi^n \\ &a = b\pi \Rightarrow 0 < a - bx < a \Rightarrow 0 < (a - bx)^n < a^n \end{aligned}$$

$\therefore 0 < f(x)\sin(x) < (a\pi)^n / n!$

In other words, no point on the curve $y = f(x)\sin(x)$ between 0 and π is higher than $(a\pi)^n / n!$, so the area below the curve must be less than the rectangular region given by $\pi \times (a\pi)^n / n!$.

$$\therefore 0 < \int_0^\pi f(x)\sin(x)\,dx < a^n\pi^{n+1} / n!$$

Consider the sequence given by $u_n = a^n / n!$.
Therefore, the next term, $u_{n+1} = a^{n+1} / (n+1)! = (a.a^n) / ((n+1)!\ n!) = (a / (n+1))\ u_n$.
However, it can be seen that as $n \to \infty$, $a / (n+1) \to 0$.

Therefore $\lim_{n\to\infty}(a^n / n!) = 0$, and for sufficiently large n, $a^n\pi^{n+1} / n! < 1$.

$$\therefore 0 < \int_0^\pi f(x)\sin(x)\,dx < 1$$

But we have already shown that the value of the integral is integer. Thus by *reductio ad absurdum* we prove that our initial assumption that π could be written as the ratio of two integers is false and π must be irrational. **Q.E.D.**

IRRATIONAL COSINE

Problem

Prove that $\cos(1^o)$ is irrational.

Solution

Using Euler's formula:

$$\cos(5x) + i\text{sinx}(5x) = e^{5xi} = (e^{xi})^5 = (\cos(x) + i\sin(x))^5$$

Let $c = \cos(x)$ and $s = \sin(x)$.

$$\begin{aligned}\therefore \cos(5x) + i\sin(5x) &= (c + si)^5 \\ &= c^5 + 5c^4si - 10c^3s^2 - 10c^2s^3i + 5cs^4 + s^5i\end{aligned}$$

Equating real coefficients: $\cos(5x) = c^5 - 10c^3s^2 + 5cs^4$.

Using the identity $s^2 = 1 - c^2$ throughout:

$$\begin{aligned}\cos(5x) &= c^5 - 10c^3(1-c^2) + 5c(1-c^2)^2 \\ &= 11c^5 - 10c^3 + 5c(1-2c^2+c^4) \\ &= 16c^5 - 20c^3 + 5c\end{aligned}$$

In the same way we can show that $\cos(3x) = 4c^3 - 3c$.

Using the identity $\cos(A-B) = \cos A \cos B + \sin A \sin B$:

$$\begin{aligned}\cos(15) &= \cos(45-30) \\ &= \cos45 \cos30 + \sin45 \sin30 \\ &= 1/\sqrt{2} \times \sqrt{3}/2 + 1/\sqrt{2} \times 1/2 \\ &= (\sqrt{3} + 1)/2\sqrt{2}\end{aligned}$$

So clearly $\cos(15)$ is irrational.

Using $\cos(5x) = 16c^5 - 20c^3 + 5c$:

$$\cos(15) = 16\cos^5(3) - 20\cos^3(3) + 5\cos(3) = (\sqrt{3} + 1)/2\sqrt{2}$$

Now if $\cos(3)$ were rational, then so too would $\cos(15)$. But as we know that $\cos(15)$ is irrational it follows that $\cos(3)$ is also irrational.

In the same way by using cos(3x) = $4c^3 - 3c$:

$$\cos(3) = 4\cos^3(1) - 3\cos(1)$$

And as cos(3) is irrational it follows that cos(1) is indeed irrational. **Q.E.D.**

LOAN REPAYMENTS

Problem

If £2000 is borrowed, interest is charged at an annual rate of 12%, and the loan is to be fully repaid after thirty-six fixed monthly payments, how much should each monthly repayment be?

Solution

We shall begin by finding a general formula for fixed payments also receiving compound interest.

If the rate of growth for each period is r and the fixed payment is p then the balance after n periods is given by the recurrence relation: $u_n = r\,u_{n-1} + p$.

If the initial balance is u_0, then we can recursively apply this relation.

$$u_1 = r\,u_0 + p$$

$$\begin{aligned} u_2 &= r\,u_1 + p \\ &= r(r\,u_0 + p) + p \\ &= r^2\,u_0 + rp + p \end{aligned}$$

$$\begin{aligned} u_3 &= r\,u_2 + p \\ &= r(r^2\,u_0 + rp) + p \\ &= r^3\,u_0 + r^2p + rp + p \end{aligned}$$

This leads to, $u_n = r^n\,u_0 + p(r^{n-1} + r^{n-2} + \ldots + r^2 + r + 1)$.

Writing $S_n = r^{n-1} + r^{n-2} + \ldots + r^2 + r + 1$, $rS_n = r^n + r^{n-1} + \ldots + r^2 + r$.

Therefore $S_n(r - 1) = r^n - 1$, giving $S_n = (r^n - 1)/(r - 1)$.

Hence $u_n = r^n\,u_0 + p(r^n - 1)/(r - 1)$.

For example, if we start with a balance of £100, receive interest at a rate of 5% per annum, and make regular yearly payments of £50 for ten years, the balance after ten years, $u_{10} = 1.05^{10}\times 100 + 50(1.05^{10} - 1)/(1.05 - 1) \approx$ £791.78. As £100 + 10×£50 = £600 has been invested, this would represent a growth of about 32% in our original investment.

Now we return to the original problem...

As the annual interest is 12%, the monthly rate, $r = 1.12^{1/12}$, $r^{36} = 1.12^3$, $u_0 = 2000$, and we are aiming for $u_{36} = 0$.

Solving $0 = 1.12^3 \times 2000 + p(1.12^3 - 1)/(1.12^{1/12} - 1)$, we get $p \approx$ £65.84.

Thirty-six fixed payments of £65.84 is £2370.24, which means that the interest charged on the full loan is about 18.5%.

Lucky Dip

Problem

A bag contains one red and one blue disc. In a game a player pays £2 to play and takes a disc at random. If the disc is blue then it is returned to the bag, one extra red disc is added, and another disc is taken. This game continues until a red disc is taken.

Once the game stops the player receives winnings equal in pounds to the number of red discs that were in the bag on that particular turn.

Find the exact value of the expected return on this game.

Solution

Let X be the number of red discs in the bag when play stops.

$$\begin{aligned}
&P(X = 1) = 1/2\\
&P(X = 2) = 1/2 \times 2/3\\
&P(X = 3) = 1/2 \times 1/3 \times 3/4\\
&\ldots\\
&P(X = k) = 1/2 \times 1/3 \times \ldots \times 1/k \times k/(k+1) = k/(k+1)!
\end{aligned}$$

$$\begin{aligned}
\therefore E(X) &= 1/2! \times 1 + 2/3! \times 2 + 3/4! \times 4 + \ldots\\
&= 1^2/2! + 2^2/3! + 3^2/4! + \ldots
\end{aligned}$$

To evaluate this series we need to consider the general term, $k^2/(k+1)!$, but before this we will evaluate a different series.

$$k/(k+1)! = (k+1 - 1)/(k+1)! = 1/k! - 1/(k+1)!$$

Hence the original series becomes a telescoping series:

$$\begin{aligned}
\therefore 1/2! + 2/3! + \ldots &= (1/1!-1/2!) + (1/2!-1/3!) + (1/3!-1/4!) + \ldots\\
&= 1
\end{aligned}$$

Now we write, $k^2/(k+1)! = (k(k+1) - k)/(k+1)! = 1/(k-1)! - k/(k+1)!$.

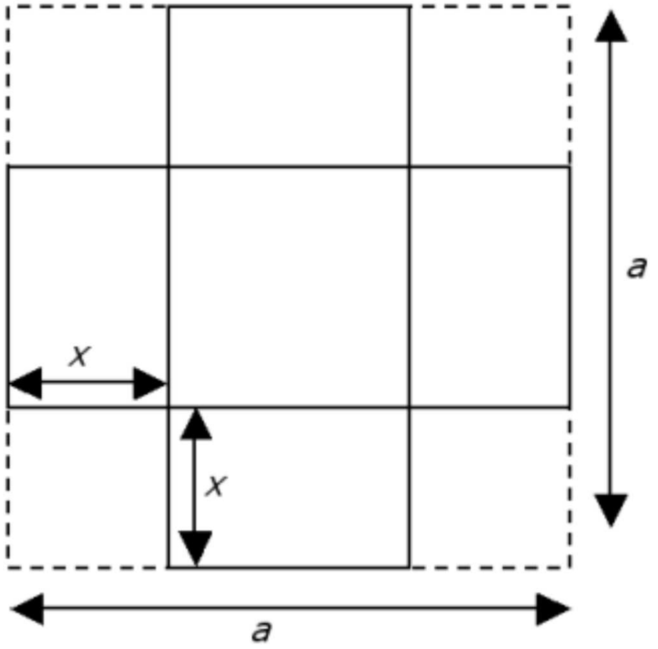
$$\begin{aligned}
\therefore E(X) &= 1^2/2! + 2^2/3! + 3^2/4! + \ldots\\
&= (1 + 1 + 1/2! + 1/3! + \ldots) - (1/2! + 2/3! + \ldots)\\
&= e - 1
\end{aligned}$$

Hence the return on each game will be $2 - e-1 = 3 - e \approx 28$ pence in the "bankers" favour.

MAXIMISED BOX

Problem

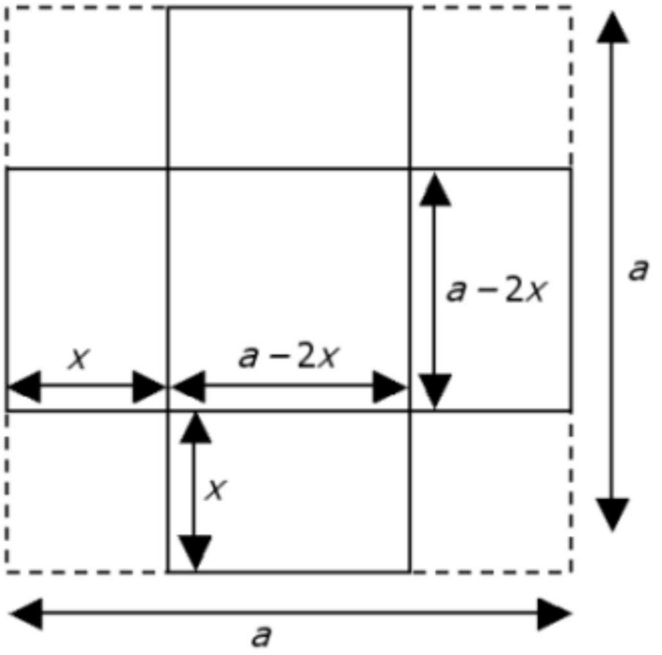
Four corners measuring *x* by *x* are removed from a sheet of material that measures *a* by *a* to make a square based open-top box.



Prove that the volume of the box is maximised iff the area of the base is equal to the area of the four sides.

Solution

Consider the diagram.



$A_{\text{base}} = (a - 2x)^2 = a^2 - 4ax + 4x^2$

$\therefore$ Volume of box, $V = a^2x - 4ax^2 + 4x^2$

$^{dV}/_{dx} = a^2 - 8ax + 12x^2$

At turning point, $^{dV}/_{dx} = 0$

$\therefore a^2 - 8ax + 12x^2 = 0$

$(a - 2x)(a - 6x) = 0$

$\therefore x = a/2, a/6.$

Clearly $x = a/2$ is a trivial solution, as $A_{\text{base}} = 0$, so we need only consider the solution $x = a/6$.

$^{d^2V}/_{dx^2} = -8a + 24x.$

When $x = a/6$, $^{d^2V}/_{dx^2} = -4a < 0 \Rightarrow$ V is at a maximum value.

Note that it is not sufficient to show that the area of the base equals the area of the sides when $x = a/6$, as we are attempting to prove that the volume is maximised if and only if this condition is true.

$A_{\text{sides}} = 4x(a - 2x)$

Solving $A_{\text{base}} = A_{\text{sides}}$, $(a - 2x)^2 = 4x(a - 2x)$

$\therefore (a - 2x)^2 - 4x(a - 2x) = 0$

$(a - 2x)((a - 2x) - 4x) = 0$

$(a - 2x)(a - 6x) = 0$

$\therefore x = a/2,\ a/6.$

We reject $x = a/2$, as $A_{\text{base}} = 0$.

Hence $A_{\text{base}} = A_{\text{sides}}$ has a unique non-trivial solution, $x = a/6$, which is when the volume of the box is maximised.

Prove the same holds for an open-top box with a rectangular base.

Note: Despite its apparent similarities with the last problem this is very difficult to prove.

MAXIMUM PRODUCT

Problem

For any positive integer, k, let $S_k = \{x_1, x_2, \dots , x_n\}$ be the set of real numbers for which $x_1 + x_2 + \dots + x_n = k$ and $P = x_1 x_2 \dots x_n$ is maximised.

For example, when $k = 10$, the set $\{2, 3, 5\}$ would give P = 30 and the set $\{2.2, 2.4, 2.5, 2.9\}$ would give P = 38.25. In fact, $S_{10} = \{2.5, 2.5, 2.5, 2.5\}$, for which P = 39.0625.

Prove that P is maximised when all the elements of S are equal in value and rational.

Solution

This proof will make use of the AM-GM inequality, which states that for any set of real numbers their arithmetic mean is greater than or equal to their geometric mean.

$$(x_1 + x_2 + \dots + x_n)/n \geq (x_1 x_2 \dots x_n)^{1/n}$$

In particular, equality is given if and only if $x_1 = x_2 = \dots = x_n$.

$\therefore (x_1 + x_2 + \dots + x_n)/n = k/n \geq (x_1 x_2 \dots x_n)^{1/n}$
$\therefore (k/n)^n \geq x_1 x_2 \dots x_n = P$

As $P \leq (k/n)^n$ we demonstrate that P will be maximised when the terms are all equal.

We now consider the maximum value of $P = (k/n)^n = e^{n \ln(k/n)}$.

$\therefore P' = e^{n \ln(k/n)}(1.\ln(k/n) - 1) = (k/n)^n(\ln(k/n) - 1)$

Solving $P' = 0$ we get $\ln(k/n) = 1 \Rightarrow k/n = e \Rightarrow n = k/e$.

As n represents the number of elements in S it is necessary for n to be integer. That is, P will be maximised when $n = [k/e]$ or $n = [k/e] + 1$, where [ ] is the integer part function.

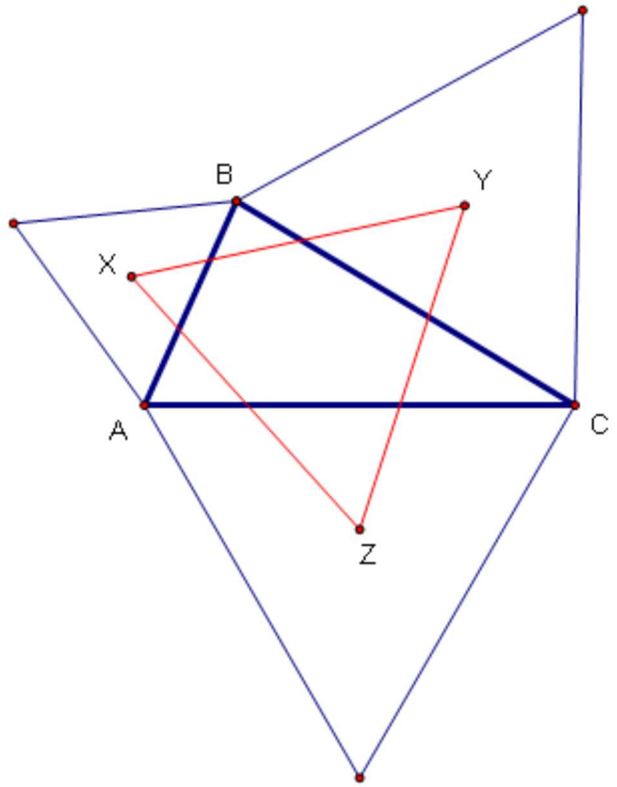
Whichever produces the maximum, $x_1 = x_2 = \dots = x_n = k/n$ will be rational.

NAPOLEON TRIANGLE

Problem

One of the greatest military leaders in history, Napoleon Bonaparte, was also an amateur mathematician and is credited for the following result.

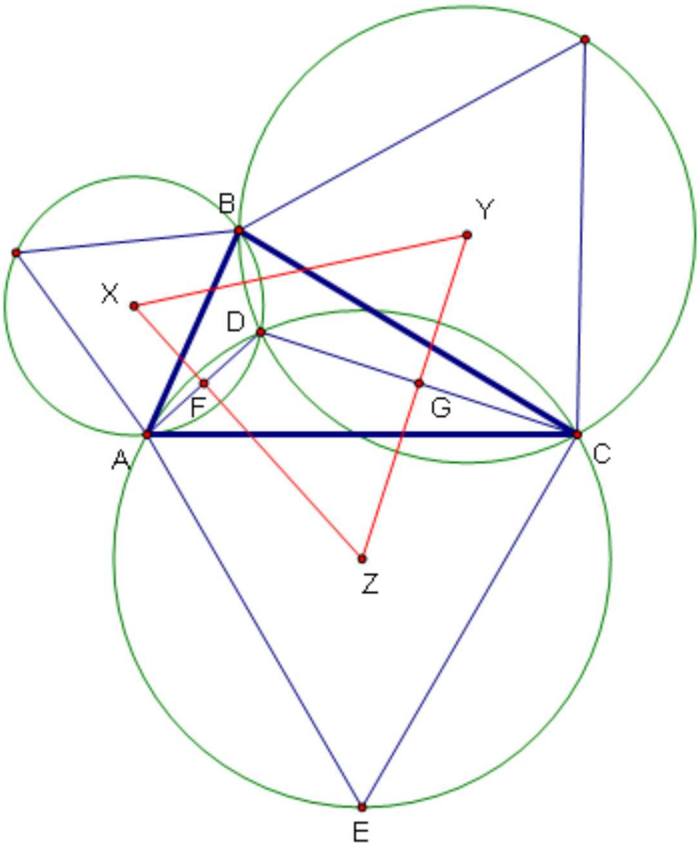
If equilateral triangles are constructed on the sides of any triangle then the centres joining the constructed triangles will always form an equilateral triangle.



Prove "Napoleon's Theorem".

Solution

We shall begin by drawing circumcircles of each constructed equilateral triangle with centres X, Y, and Z respectively.



First we shall show that all three circles pass through the common point, D. Consider circle Z.

Because quadrilateral ADCE is cyclic, angles AEC + ADC = 180 degrees, and as angle AEC = 60 degrees by construction, it follows that angle ADC = 120 degrees.

Similarly we can show that each of the angles ADB and BDC are 120 degrees.

Therefore angles ADC + ADB + BDC = 360 degrees, and we demonstrate that the three circles are concurrent at D.

Now as AD is a common chord to circles X and Z the segment joining their centres, XZ, is a perpendicular bisector of AD (see Common Chord).

Therefore angle DFZ = 90 degrees; and in the same way angle DGZ = 90 degrees.

Hence in quadrilateral FDGZ, we can see that angle FZG = 60 degrees.

In the same way we can show that angle ZXY = angle XYZ = 60 degrees and we prove that the constructed triangle is indeed equilateral.

NEVER DECREASING DIGITS

Problem

A number has strictly increasing digits if each digit exceeds the digit to its left and is said to be a strictly increasing number; for example, 1357.
A number has increasing digits if each digit is not exceeded by the digit to its left and is said to be an increasing number; for example, 45579.

There are exactly 219 increasing numbers below one-thousand.

How many numbers below one million are increasing?

Solution

We will begin by deriving, from first principles, the information given in the problem relating to numbers below one-thousand.

It is possible to use binary strings to produce increasing numbers. Consider the following algorithm.

```
Let S be binary string
Let C = 0
Let K = 1
Label A
   Let D be value of Kth digit of S
   If D = 1 then output C
   If D = 0 then increment c by 1
   Increment K by 1
If K has not exceeded the length of S then goto A
Stop
```

For example, the binary string 001101001 would produce 2235.

In general it can be seen that an *n*-digit increasing number will be produced by a binary string containing *n* 1's, and the maximum digit will be represented by the number of 0's, as each zero causes the "counter" to increase by 1.

Using this idea we can represent all the 3-digit increasing numbers made up of the digits 0, 1, and 2 using 5-digit binary strings consisting of three 1's and two 0's:

11100 = 000
11010 = 001
11001 = 002
10110 = 011
10101 = 012

10011 = 022
01110 = 111
01101 = 112
01011 = 122
00111 = 222

But we must subtract one to remove 000, so there are C(5,3)–1 = 9 such numbers.

In the same way a 3-digit increasing number using each of the digits 0 to 9 requires a binary string containing three 1's and nine 0's; that is, there are exactly C(12,3) – 1 = 219 increasing numbers below one-thousand.

Now we can return to the problem: increasing numbers below one million.

As the numbers contain six digits the binary string will contain six 1's and nine 0's. Hence there are C(15,6) – 1 = 5004 increasing numbers below one million.

Check out the related (strictly) Increasing Digits problem.

Problem ID: 264 (29 Jan 2006) Difficulty: 4 Star [mathschallenge.net]

Never Prime

Problem

Prove that $14^n + 11$ is never prime.

Solution

In problems of this nature it is often helpful to substitute values for n to see if anything useful can be pertained. Thus the first seven terms are 25, 207, 2755, 38427, 537835, 7529547, respectively. Although we cannot be certain, it seems when n is odd, $14^n + 11$ is divisible by 5, and, although not entirely obvious, when n is even it is divisible by 3.

Let us consider n being even: $14^{2k} = 196^k$. As $196 \equiv 1 \bmod 3$, it follows that $196^k \equiv 1 \bmod 3$. Therefore $14^{2k} + 11$ is divisible by 3.

When n is odd: $14^{2k+1} = 14\times14^{2k} = 14\times196^k$. As $196 \equiv 1 \bmod 5$, it follows that $196^k \equiv 1 \bmod 5$, and $14\times196^k \equiv 14 \equiv 4 \bmod 5$. Therefore $14^{2k+1} + 11$ is divisible by 5.

Hence $14^n + 11$ is divisible by 5 and 3 alternately, and can never be prime.

Of course, with this insight we can approach it far more efficiently by noting that $14 \equiv -1 \bmod 15$, hence 14^n will be alternately -1/1.

> When n is odd: $14^n + 11 \equiv -1 + 11 = 10 \bmod 15 \Rightarrow$ divisible by 5
> When n is even: $14^n + 11 \equiv 1 + 11 = 12 \bmod 15 \Rightarrow$ divisible by 3

Prove that $(14^n + 4)/2$ is never prime.

ODD PERFECT NUMBERS

Problem

The divisors of a positive integer, excluding the number itself, are called the proper divisors . If the sum of proper divisors is equal to the number we call the number perfect. For example, the divisors of 28 are 1, 2, 4, 7, 14, and 28, so the sum of proper divisors is 1 + 2 + 4 + 7 + 14 = 28.

The first eight perfect numbers are 6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128.

It is well known that P is an even perfect number iff it is of the form $2^{n-1}(2^n-1)$ where 2^n-1 is prime.

No one has yet discovered an odd perfect number, and the existence of them is in doubt.

However, if Z is an odd perfect, prove that it must be of the form $c^2\ q^{4k+1}$ where $q \equiv 1 \bmod 4$ is prime.

Solution

We shall work modulo 4 throughout this proof, and we begin by writing Z as a product of distinct prime factors, noting that as Z is odd, all prime factors must be odd:

Let $Z = p_1^{a_1}\ p_2^{a_2} \ldots p_m^{a_m}\ q_1^{b_1}\ q_2^{b_2} \ldots q_n^{b_n}$, where $p \equiv 3$ and $q \equiv 1$.

If a is even then $p^a \equiv 1$.
If a is odd then $p^a \equiv 3$.
If b is even or b is odd then $q^b \equiv 1$.

So $Z \equiv 1,3$ and it follows that $2Z \equiv 2$.

If Z is perfect then the sum of divisors, $\sigma(Z) = 2Z \equiv 2$.

Using the multiplicative property of the function, $\sigma(ab) = \sigma(a)\ \sigma(b)$, we need consider the possible values of each $\sigma(p^a)$ and $\sigma(q^b)$.

By definition, $\sigma(p^a) = 1 + p + \ldots + p^a$.

If a is even then $\sigma(p^a) \equiv 1$.

If a is odd then $\sigma(p^a) \equiv 0$.

But if a is odd then the product of these residuals will be zero and we know that $\sigma(Z) \equiv 2$. Hence a must be even and the product of all $\sigma(p^a)$ terms will be congruent with 1.

We shall now consider $\sigma(q^b) = 1 + q + ... + q^b$ and recall that $q^b \equiv 1$ for all values of b.

If $b \equiv 0$ then $\sigma(q^b) \equiv 1$.
If $b \equiv 1$ then $\sigma(q^b) \equiv 2$.
If $b \equiv 2$ then $\sigma(q^b) \equiv 3$.
If $b \equiv 3$ then $\sigma(q^b) \equiv 0$.

In the same way that a cannot be odd, b cannot be of the form $4k + 3$.

If b is even, that is, $b = 4k$ or $b = 4k + 2$, then the product of those $\sigma(q^b)$ terms will be congruent with 1 or 3.

Therefore Z can comprise any number of primes, p and q raised to an even power, which will form a square number.

But it is necessary to have exactly one sigma term with $b = 4k + 1$ so that the product of all sigma terms will be congruent with 2.

Hence if Z is an odd perfect then it must be of the form $c^2\ q^{4k+1}$ where $q \equiv 1$ is prime. **Q. E. D.**

ORDER OF A PRIME

Problem

If p is prime and GCD(a, p) = 1, we know by Fermat's Little Theorem that $a^{p-1} \equiv 1 \bmod p$.

Consider 3^k mod 7:

$3^1 \equiv 3$
$3^2 \equiv 2$
$3^3 \equiv 6$
$3^4 \equiv 4$
$3^5 \equiv 5$
$3^6 \equiv 1$

However, $p-1$ is not necessarily the least value of k for which the residue is one. For example, 2^k mod 7:

$2^1 \equiv 2$
$2^2 \equiv 4$
$2^3 \equiv 1$
$2^4 \equiv 2$
$2^5 \equiv 4$
$2^6 \equiv 1$

We call k = ord(a, p) the least value for which $a^k \equiv 1 \bmod p$; for example ord(3, 7) = 6 and ord(2, 7) = 3.

Prove that $p-1$ is always a multiple of ord(a, p).

Solution

By definition k = ord(a, p) is the least value for which $a^k \equiv 1 \bmod p$, and by FLT we know that $k \le p-1$.

Let $p-1 = bk + r$, where $0 \le r < k$.

If $r = 0$ then we are done, as $p-1$ is a multiple of k, but for $r > 0$, k will not a multiple of $p-1$.

$\therefore a^{p-1} = a^{bk+r} = a^{bk}\, a^r$.

But $a^{bk} = (a^k)^b \equiv 1$, because by definition $a^k \equiv 1$.

Hence $a^{p-1} \equiv a^r \equiv 1$.

However, $r < k$, and by definition k is the least value for which the residue is one. So we conclude that $r = 0$, and we prove that $p-1$ is a multiple of $k = \text{ord}(a, p)$.

PAIRWISE PRODUCTS

Problem

For any set of real numbers, R = $\{x, y, z\}$, let sum of pairwise products,
S = $xy + xz + yz$.

Given that $x + y + z = 1$, prove that S $\leq 1/3$.

Solution

Let $x = 1/3 + a$, $y = 1/3 + b$, and $z = 1/3 + c$.

$\therefore x + y + z = 1/3 + a + 1/3 + b + 1/3 + c = 1 + a + b + c$.

But as $x + y + z = 1$, we deduce that $a + b + c = 0$.

$\therefore (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc) = 0$
$2(ab + ac + bc) = -(a^2 + b^2 + c^2)$
$\therefore ab + ac + bc = -(a^2 + b^2 + c^2)/2 = -d$, where $d \geq 0$

So $xy + xz + yz$
$= (1/3 + a)(1/3 + b) + (1/3 + a)(1/3 + c) + (1/3 + b)(1/3 + c)$
$= 1/9 + a/3 + b/3 + ab + 1/9 + a/3 + c/3 + ac + 1/9 + b/3 + c/3 + bc$
$= 1/3 + (2/3)(a + b + c) + ab + ac + bc$

As $a + b + c = 0$ and $ab + ac + bc = -d$, we get,
S = $xy + xz + yz = 1/3 - d \leq 1/3$ **Q.E.D.**.

For a set of real numbers, R = $\{x_1, x_2, \ldots , x_n\}$ where $x_1 + x_2 + \ldots +x_n = 1$, prove that the sum of pairwise products, S $\leq (n-1)/(2n)$.

PERFECT DIGIT

Problem

The divisors of a positive integer, excluding the number itself, are called the proper divisors . If the sum of proper divisors is equal to the number we call the number perfect. For example, the divisors of 28 are 1, 2, 4, 7, 14, and 28, so the sum of proper divisors is 1 + 2 + 4 + 7 + 14 = 28.

The first eight perfect numbers are 6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128.

Prove that the last digit of an even perfect numbers will be 6 or 8.

Solution

It can be shown that P is an even perfect number iff it is of the form $2^{n-1}(2^n-1)$ where 2^n-1 is prime.

Moreover it is necessary for n to be prime which can be easily demonstrated.

Suppose that n is composite; let $n = xy$.

$\therefore 2^{xy} - 1 = (2^x - 1)(2^{(x-1)y} + 2^{(x-2)y} + ... + 2^{2y} + 2^y + 1)$

Clearly the second bracket is greater than 1, so for $2^n - 1$ to be prime it is necessary for $2^x - 1 = 1 \Rightarrow x = 1$ and $y = n$, which must be prime.

Now with the exception of $n = 2$, for which P = 6 anyway, all other primes are odd. Hence we must deal with two cases where $n \equiv 1,3 \bmod 4$:

(i) If $n \equiv 1 \bmod 4$; let $n = 4k + 1$:

$$\begin{aligned} \therefore P &= 2^{n-1}(2^n-1) \\ &= 2^{4k}(2^{4k+1} - 1) \\ &= (2^4)^k(2.(2^4)^k - 1) \\ &= 16^k(2.16^k - 1) \end{aligned}$$

If $16^i \equiv 6 \bmod 10$ then $16.16^i = 16^{i+1} \equiv 16.6 = 96 \equiv 6 \bmod 10$, and as $16 \equiv 6 \bmod 10$ it follows that $16^i \equiv 6 \bmod 10$ for all values of $i \geq 1$.

Therefore P $\equiv$ 6(2.6 – 1) = 66 $\equiv$ 6 mod 10.

In other words, if $n \equiv 1$ mod 4 then the last digit of P will be 6.

(ii) If $n \equiv 3$ mod 4; let $n = 4k + 3$:

$$\begin{aligned}\therefore \text{P} &= 2^{4k+2}(2^{4k+3} - 1)\\ &= 2^2\cdot(2^4)^k(2^3.(2^4)^k - 1)\\ &= 4.16^k(8.16^k - 1)\end{aligned}$$

Therefore P $\equiv$ 4.6(8.6 – 1) = 1128 $\equiv$ 8 mod 10.

Hence the last digit of an even perfect number will be 6 or 8. **Q.E.D.**

Prove that if $n \equiv 3$ mod 4 then the last two digits of P will be 28.

PERFECT POWER SUM

Problem

Let x and y be positive whole numbers, and let p be any odd prime.

It is well known that $x^3 + y^3$ is never equal to an odd prime.

But given that n is a positive integer which contains an odd factor greater than one, prove that $x^n + y^n = p$ has no solutions.

Solution

Let V = $x^n + y^n$.

As V is a prime greater than two it is clear that $x \neq y$, otherwise the sum would be $2x^n$. In addition $x > 1$, in which case $x^n > x$.

So without loss of generality let $x > y \geq 1$. Therefore $1 < x + y <$ V.

It can be verified for odd values of n that:

$$\text{V} = (x + y)(x^{n-1} - x^{n-2}y + \ldots + x^2y^{n-3} - xy^{n-2} + y^{n-1})$$

In other words, V is divisible by $x + y$, which lies between 1 and V. Hence for odd values of n, V cannot be prime.

Suppose that $n = ab$, where $a = 2^k$ and b is an odd greater than one.

$$\begin{aligned}\therefore \text{V} &= x^n + y^n \\ &= (x^a)^b + (y^a)^b \\ &= (x^a + y^a)((x^a)^{b-1} - (x^a)^{b-2}(y^a) + \ldots - (x^a)(y^a)^{b-2} + (y^a)^{b-1})\end{aligned}$$

In the same way as before, $1 < x^a + y^a <$ V, and as V is divisible by $x^a + y^a$ it cannot be prime.

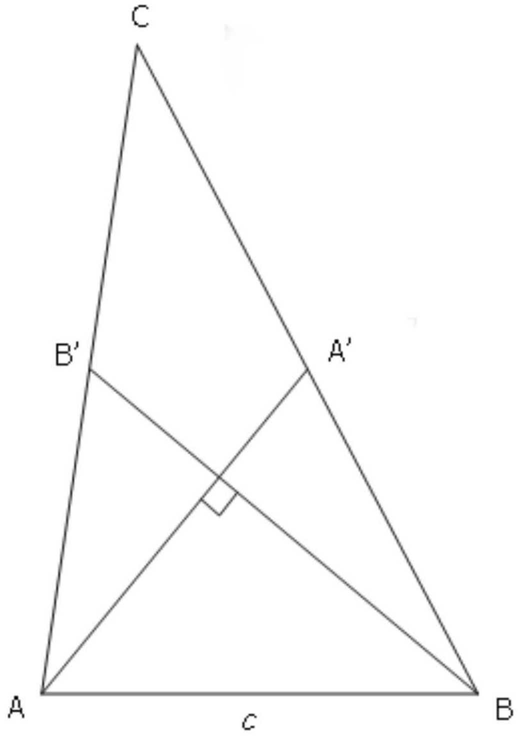
Hence we prove that $x^n + y^n$ is never equal to an odd prime unless n is of the form 2^k.

Given that $n = 2^k$, investigate when V is prime.

PERPENDICULAR MEDIANS

Problem

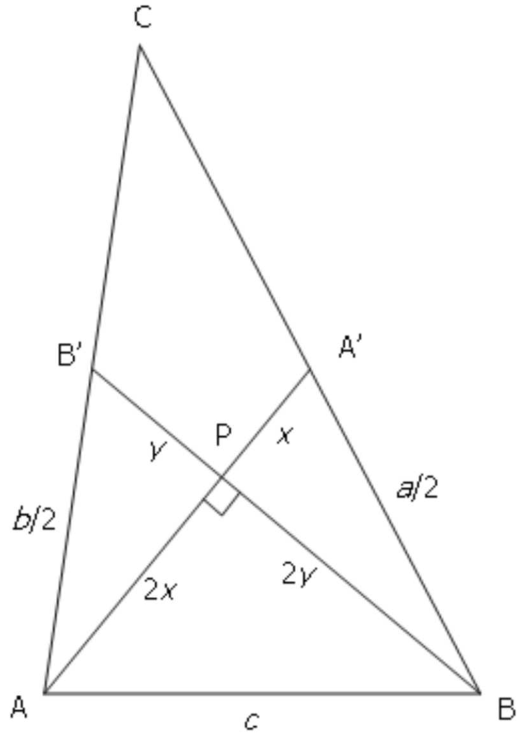
In triangle ABC, A" is the midpoint of BC, B" is the midpoint of AC, and the line segments AA" and BB" are perpendicular. Let AC = *b*, BC = *a*, and AB = *c*.



1. Find *c* in terms of *a* and *b*.
2. Determine the values of *b* in terms of *a* for which the triangle exists.
3. Calculate the size of the maximum interior angle at vertex C.

Solution

As any median splits each other median in the ratio 2:1 (see Triangle Median), we can label the following lengths.



Using the Pythagorean Theorem:

$$x^2 + (2y)^2 = (a/2)^2 \Rightarrow x^2 + 4y^2 = a^2/4 \quad \text{(triangle BPA")}$$
$$(2x)^2 + y^2 = (b/2)^2 \Rightarrow 4x^2 + y^2 = b^2/4 \quad \text{(triangle APB")}$$

$\therefore 5x^2 + 5y^2 = (a^2 + b^2)/4 \Rightarrow x^2 + y^2 = (a^2 + b^2)/20$

In triangle APB, $(2x)^2 + (2y)^2 = c^2 \Rightarrow 4x^2 + 4y^2 = c^2$

$\therefore c^2 = 4(x^2 + y^2) = (a^2 + b^2)/5 \Rightarrow c = \sqrt{((a^2 + b^2)/5)}$.

Using the cosine rule in triangle ABC:

$$\begin{aligned}\cos(C) &= (a^2 + b^2 - c^2)/(2ab)\\ &= (a^2 + b^2 - (a^2 + b^2)/5)/(2ab)\\ &= (5a^2 + 5b^2 - (a^2 + b^2))/(10ab)\\ &= (4a^2 + 4b^2)/(10ab)\\ &= (2a^2 + 2b^2)/(5ab)\end{aligned}$$

As a and b are both positive, $0 < \cos(C) < 1$.

$\therefore (2a^2 + 2b^2)/(5ab) < 1$

Solving at equality with 1:

$$2a^2 + 2b^2 = 5ab$$

$\therefore 2a^2 - 5ab + 2b^2 = 0$

Solving the quadratic in b:

$$\begin{aligned} b &= (5a \pm \sqrt{(25a^2 - 16a^2})/4 \\ &= (5a \pm 3a)/4 \\ &= a/2,\ 2a \end{aligned}$$

That is, $a/2 < b < 2a$ for the triangle to exist.

By writing $\cos(C) = r$, we get $2a^2 - 5abr + 2b^2 = 0$. Solving this quadratic in b, we get $b = (5ar \pm \sqrt{(25a^2r^2 - 9a^2)})/4$. However, for a real solution the discriminant, $25a^2r^2 - 9a^2 \geq 0$. Solving equal to zero, $25a^2r^2 = 9a^2 \Rightarrow r^2 = 9/25 \Rightarrow r = 4/5$. Hence the greatest interior angle at C will be $\cos^{-1}(4/5) \approx 36.9^o$.

PRIME PARTNER

Problem

Given that p is an odd prime and n is a positive integer, prove that there always exists a value of n for which the expression $n^2 + np$ is a perfect square.

For example, when $p = 7$, $9^2 + 9\times7 = 144 = 12^2$.

Furthermore, prove that this value of n is unique.

Solution

Let $n^2 + np = n(n + p) = a^2$.

We will begin by showing that n and $n + p$ must be relatively prime.

If the prime q were a common factor of both n and $n + p$, then q would divide $n + p - n = p \Rightarrow p = q$. Hence n must be a multiple of p.

Let us suppose that $n = kp$.

$\therefore kp(kp + p) = a^2$
$\quad k(k + 1) = (a/p)^2$

As k is integer, a/p must also be integer.

But as $k(k + 1) = (a/p)^2$ it follows that $k^2 < (a/p)^2$ and $(k + 1)^2 > (a/p)^2$.

$\therefore k < a/p < k + 1$

Clearly the integer a/p cannot lie between two consecutive integers, hence we conclude that n is not a multiple of p. Furthermore we demonstrate that GCD(n, $n + p$) = 1.

In which case, for $n(n + p)$ to be a perfect square, both n and $n + p$ must be perfect squares.

Let $n = u^2$ and $n + p = v^2$.

$\therefore n + p - n = p = v^2 - u^2 = (v - u)(v + u)$

For this product to be prime, $v - u = 1$, and $v + u = p$. Subtracting we get $2u = p -$

1, hence $n = u^2 = (p - 1)^2/4$.

That is, a value of n exists for every odd prime, and we have deductively shown that this value is necessarily of the form, $n = (p - 1)^2/4$.

Prove that for $k > 2$ there only exists a value of n such that $n^k + n^{k-1}p$ is a perfect power of k for some values of p and when n exists it is indeed unique; investigate the conditions for p to have such a "partner".

PRIME RECIPROCALS

Problem

Given that P = $\{p_1, p_2, \ldots, p_k\}$ is a set of distinct, not necessarily consecutive primes, prove that $1/p_1 + 1/p_2 + \ldots 1/p_k$ is never integer.

Solution

$$\begin{aligned} S &= 1/p_1 + 1/p_2 + \ldots 1/p_k \\ &= (p_2p_3\ldots p_k)/(p_1p_2\ldots p_k) + (p_1p_3\ldots p_k)/(p_1p_2\ldots p_k) + \ldots \\ &= (p_2p_3\ldots p_k + p_1p_3\ldots p_k + \ldots + p_1p_2\ldots p_{k-1})/(p_1p_2\ldots p_k) \end{aligned}$$

Therefore, $p_2p_3\ldots p_k + p_1p_3\ldots p_k + \ldots + p_1p_2\ldots p_{k-1} = S\ p_1p_2\ldots p_k$.

As all but the last term on the left hand side contains the factor, p_k, we can write, $p_kQ + p_1p_2\ldots p_{k-1} = S\ p_1p_2\ldots p_k$.

Dividing both sides by p_k: $Q + p_1p_2\ldots p_{k-1}/p_k = S\ p_1p_2\ldots p_{k-1}$.

But as none of $p_1, p_2, \ldots, p_{k-1}$ divide by p_k, the RHS, and more specifically, S, cannot be integer. **Q.E.D.**

What can you determine about the possible values of the fraction, S = a/b?

PRIMITIVE PYTHAGOREAN TRIPLETS

Problem

Given that (x, y, z) is a primitive Pythagorean triplet, prove that the following transformation will produce another primitive Pythagorean triplet.

$$(x'', y'', z'') = (x, y, z)\begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & -2 \\ 2 & 2 & 3 \end{pmatrix}$$

Solution

By multiplying, we get the following three linear equations:

$$\begin{aligned} x'' &= x-2y+2z \\ y'' &= 2x-y+2z \\ z'' &= 2x-2y+3z \end{aligned}$$

Without loss of generality we shall say that $x < y < z$.

The proof shall be done in three parts. Firstly we shall show that (x'', y'', z'') is different to (x, y, z), then we shall prove that it is a Pythagorean triplet, finally showing that it will be primitive.

i. From each of the equations above:

$$\begin{aligned} x'' &= x-2y+2z > x-2z+2z = x \Rightarrow x'' > x \\ y'' &= 2x-y+2z > 2x-y+2y = 2x+y \Rightarrow y'' > y \\ z'' &= 2x-2y+3z > 2x-2z+3z = 2x+z \Rightarrow z'' > z \end{aligned}$$

That is, each new triplet generated by this transformation will be strictly increasing.

ii. $$\begin{aligned} (x'')^2 &= x^2+4y^2+4z^2-4xy+4xz-8yz \\ (y'')^2 &= 4x^2+y^2+4z^2-4xy+8xz-4yz \\ (z'')^2 &= 4x^2+4y^2+9z^2-8xy+12xz-12yz \end{aligned}$$

$$\begin{aligned} \text{Therefore } (x'')^2+(y'')^2 &= 5x^2+5y^2+8z^2-8xy+12xz-12yz \\ &= x^2+y^2-z^2+4x^2+4y^2+9z^2-8xy+12xz-12yz \\ &= x^2+y^2-z^2+(z'')^2 \end{aligned}$$

But as it is given that (x,y,z) is a Pythagorean triplet, $x^2+y^2-z^2 = 0$.

Hence $(x'')^2+(y'')^2 = (z'')^2$, and we prove the first two parts: (x'',y'',z'') is a different Pythagorean triplet to (x, y, z).

iii. By using the inverse matrix we are able to transform $(x''\ y'', z'')$ back to (x, y, z):

$$(x, y, z) = (x'', y'', z'')\begin{pmatrix} 1 & -2 & -2 \\ 2 & -1 & -2 \\ -2 & 2 & 3 \end{pmatrix}$$

This gives the following linear equations:

$x = x''+2y''-2z''$
$y = -2x''-y''+2z''$
$z = -2x''-2y''+3z''$

If HCF$(x'', y'', z'') = h$, then each of x, y, and z will share the same common factor. But as we are given that (x, y, z) is primitive, HCF$(x, y, z) = 1$, hence $h = 1$, and we prove that (x'', y'', z'') is also primitve; our proof is complete.

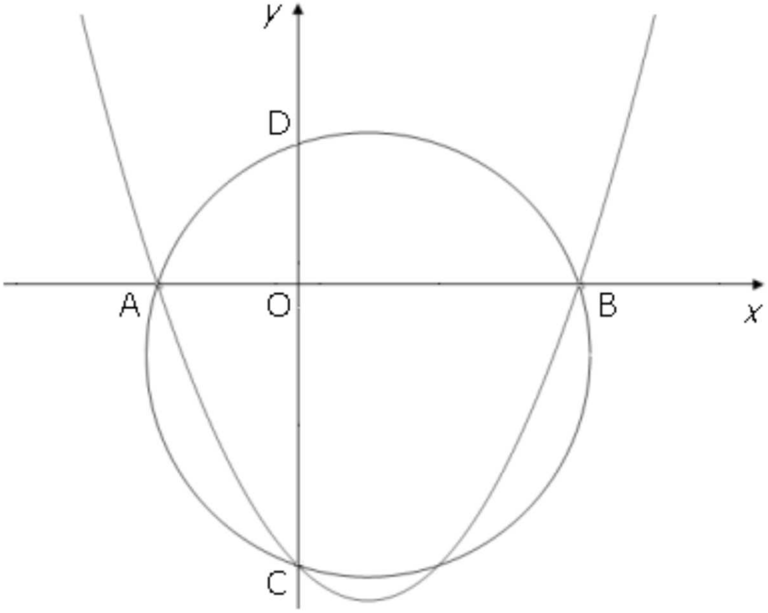
We have just proved that a primitive Pythagorean triplet transformed by the matrix M_1 produces a different primitive Pythagorean triplet. Prove that M_2 and M_3 also produce new primitive Pythagorean triplets.

$$M_1 = \begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & -2 \\ 2 & -2 & 3 \end{pmatrix} \quad M_2 = \begin{pmatrix} -1 & -2 & -2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix} \quad M_3 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

QUADRATIC CIRCLE

Problem

The quadratic curve, $y = x^2 – bx – c$, where $b,c \in \mathbb{R}^{\oplus}$, is drawn and the points of intersection with the axes are labelled, A, B, and C. A circle is drawn through the points A, B, and C, and a fourth point D is created on the y-axis.



Find the co-ordinate of D.

Solution

As the discriminant of the quadratic equation, $x^2 – bx – c = 0$, is $b^2 + 4c$, which is positive, there will be two roots. The product of the roots, $\alpha_1\alpha_2 = -c$, so one root will be negative and one will be positive. Let the three co-ordinates be A $(\alpha_1,0)$, B $(\alpha_2,0)$ and C $(0,c)$. Hence the lengths, AO = $-\alpha_1$ (as the root will be negative), OB = α_2, and CO = c.

In a circle with two intersecting chords, the product of the segments are equal: AO× OB = CO×OD, therefore, $-\alpha_1\alpha_2 = c \times \text{OD}$.

As $\alpha_1\alpha_2 = -c$, we get, OD = 1.

That is, the co-ordinate of D is (0,1), which, amazingly, is independent of both *b* and *c*.

What if *b* was allowed to be positive or negative?
What about *c*?

Investigate the general quadratic curve, $y = ax^2+bx+c$.
Prove the result that in a circle with two intersecting chords, the product of the segments are equal.

QUADRATIC DIFFERENCES

Problem

The positive integers, x, y, and z are consecutive terms in an arithmetic progression. Given that n is also a positive integer, for how many values of n below one-thousand does the equation $x^2 - y^2 - z^2 = n$ have no solutions?

Solution

Let $x = a + d$, $y = a$, and $z = a - d$.

$\therefore (a + d)^2 - a^2 - (a - d)^2 = n$
$\quad a^2 + 2ad + d^2 - a^2 - a^2 + 2ad - d^2 = n$
$\therefore 4ad - a^2 = a(4d - a) = n$

Let $u = a$ and $v = 4d - a \Rightarrow u + v = 4d \equiv 0 \bmod 4$. In other words, for a solution to exist the factors of n must add to a multiple of four.

We shall deal with n being of the form $2^m r$, where r is odd, and for increasing values of m.

- $m = 0$ (n is odd):
  If $n = r$ then the factors u and v must both be odd. But if they are both congruent with 1 or both congruent with -1 modulo 4 then $u + v \equiv 2 \bmod 4$, and there will be no solution; if they are different then $u + v \equiv 0 \bmod 4$, and there will always be a solution. Hence $n = uv \equiv 1 \bmod 4$, or n being of the form $4k + 1$, will have no solutions.

- $m = 1 \Rightarrow n = 2r = a(4d - a)$:
  If $a = 2r$, $4d - a = 1 \Rightarrow 4d = 2r + 1$. But as the RHS is odd, this is impossible.
  If $a = r$, $4d - a = 2 \Rightarrow 4d = r + 2$. Impossible, as RHS is odd.
  In other words if $n = 2(2k + 1) = 4k + 2$, there will be no solutions.

- $m = 2 \Rightarrow n = 4r$:
  If $a = 2r$, $4d - a = 2 \Rightarrow 4d = 2r + 2 = 2(r + 1)$.
  And as $r + 1$ is even, we will always have at least one solution if $n = 4r$.

- $m = 3 \Rightarrow n = 8r$:
  If $a = 8r$, $4d - a = 1 \Rightarrow 4d = 8r + 1$. Impossible.
  If $a = 4r$, $4d - a = 2 \Rightarrow 4d = 4r + 2$. Impossible.
  If $a = 2r$, $4d - a = 4 \Rightarrow 4d = 2r + 4$. Impossible.
  If $a = r$, $4d - a = 8 \Rightarrow 4d = r + 8$. Impossible.

Hence if $n = 8(2k + 1) = 16k + 8$ there will be no solutions.

- $m \geq 4$:
  If $a = 4r$, $4d - a = 2^{m-2} \Rightarrow 4d = 4(r + 2^{m-4})$.
  Hence for $m \geq 4$ there will always be at least one solution.

Thus there will be no solutions for numbers of the form $4k + 1$, $4k + 2$, and $16k + 8$. As the first and second cases are odd and even respectively, they are mutually exclusive, and although the the second and third are both even, the third is divisible by 4, whereas the second is not divisible by 4. Hence all three forms are mutually exclusive.

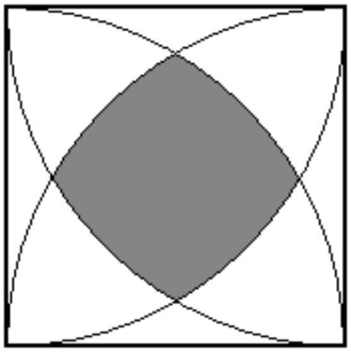
As $4 \times 249 + 1 = 997$, $4 \times 249 + 2 = 998$, and $16 \times 61 + 8 = 984$, there are exactly $249 + 249 + 61 = 559$ values of n below one-thousand that have no solution.

For which values of n will there be exactly one solution?

QUARTER CIRCLES

Problem

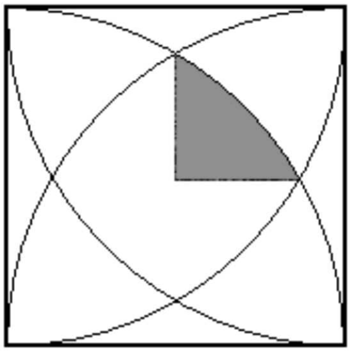
Four quarter circles are drawn from each vertex in a unit square.



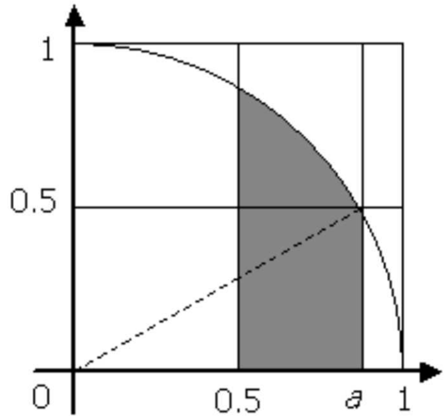
Find the area of the shaded region.

Solution

Consider the diagram below.



By using the unit circle (see diagram below), we can find the area between 0.5 and a by integrating under the curve, $x^2 + y^2 = 1$.
That is, $y = \sqrt{(1 - x^2)}$; taking positive root, because we only want the top part of the circle.



Using the Pythagorean Theorem, $1^2 = a^2 + \tfrac{1}{2}^2 \Rightarrow a = \sqrt{3}/2$.

$$\text{Area under curve} = \int_{1/2}^{\sqrt{3}/2} \sqrt{(1 - x^2)}\, dx$$

Using the substitution, $x = \sin(u)$, $dx = \cos(u)\, du$, and we get:

$$\text{Area under curve} = \int_{\pi/6}^{\pi/3} \cos^2(u)\, du$$

$$= \int_{\pi/6}^{\pi/3} \tfrac{1}{2}(1 - \sin(2u))\, du$$

∴ Area under curve $= \tfrac{1}{2}[u + \tfrac{1}{2}\cos(2u)]_{\pi/6}^{\pi/3} = \pi/12$

Area of rectangle $= \tfrac{1}{2}(\sqrt{3}/2 - 1/2) = (\sqrt{3} - 1)/4$.

∴ Area of ¼ shaded region $= \pi/12 - (\sqrt{3} - 1)/4$.

Hence the area of the shaded region is, $\pi/3 - \sqrt{3} + 1$.

RADICAL CONVERGENCE

Problem

Consider the Pell equation, $x^2 - 2y^2 = 1$, where x and y are positive integers.

The smallest solution is (3,2), with subsequent solutions being (17,12), (99,70), (577,408), ...

What is most interesting is that the sequence of solutions produce convergents for $\sqrt{2} = 1.414213...$

$$\begin{aligned} 3/2 &= 1.5 \\ 17/12 &= 1.416666... \\ 99/70 &= 1.414285... \\ 577/408 &= 1.414215... \end{aligned}$$

Assuming that at least one solution exists, prove that the ordered solutions of the equation $x^2 - dy^2 = 1$ produce convergents for $\sqrt{d}$.

Solution

From $x^2 - dy^2 = 1$ complete the square to get $(x - \sqrt{d}\,y)(x + \sqrt{d}\,y) = 1$, and as both side must be positive we deduce that $x > \sqrt{d}\,y$.

This leads to $x - \sqrt{d}\,y = 1 / (x + \sqrt{d}\,y)$.

Therefore $x/y - \sqrt{d} = 1 / (y(x + \sqrt{d}\,y))$; note also that $x/y - \sqrt{d}$ will always be positive.

But as $x > \sqrt{d}\,y$ it follows that writing $\sqrt{d}\,y + \sqrt{d}\,y$ in the denominator will decrease the value of the denominator and hence increase the value of the fraction.

That is, $x/y - \sqrt{d} < 1 / (y(\sqrt{d}\,y + \sqrt{d}\,y)) < \sqrt{d}\,y / (y(\sqrt{d}\,y + \sqrt{d}\,y))$.

Hence $0 < x/y - \sqrt{d} < 1 / (2y^2)$, or $\sqrt{d} < x/y < \sqrt{d} + 1 / (2y^2)$.

It can be clearly seen that x/y will always be greater than $\sqrt{d}$, but as y increases the error, $1 / (2y^2)$, decreases.

To complete the proof we must show that infinitely many increasing solutions in x and y can be found in order to approach the limit.

Consider the mapping $(x,y) \rightarrow (x^2+dy^2,2xy)$:

$$\begin{aligned}(x^2+dy^2)^2 - d(2xy)^2 &= x^4 + 2dx^2y^2 + d^2y^4 - 4dx^2y^2\\ &= x^4 - 2dx^2y^2 + d^2y^4\\ &= (x^2 - dy^2)^2\\ &= 1\end{aligned}$$

That is, if (x,y) is a solution to $x^2 - dy^2 = 1$ then $(x^2+dy^2,2xy)$ is also a solution, and as $2xy > y$ we prove that infinitely many solution exist which converge to the limit $\sqrt{d}$.

RANDOM CHORDS

Problem

A random chord is drawn by connecting two independent randomly selected points on the circumference of a circle.

If r random chords are drawn, find the probability that none of them intersect.

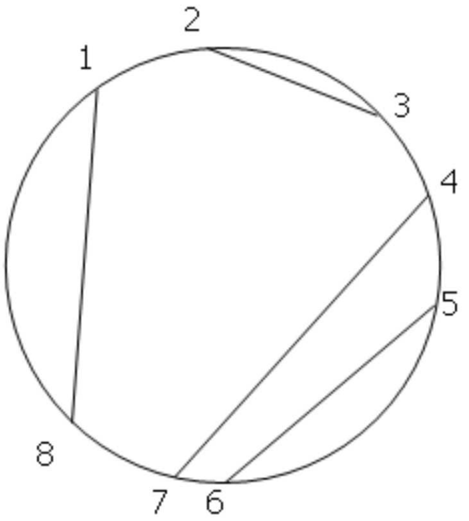
Solution

First we shall consider the number of ways in which r chords can be connected. Imagine a circle with $n=2r$ points that are numbered sequentially 1 to n around the circumference of the circle.

The "first" chord is connected to 1 and ($n-1$) other possible end points.
The start point of the "second" chord will be connected to the next free point: 2 unless first chord connected from 1 to 2. It will be able to connect to ($n-3$) other end points.
In a similar way, the other chords will be able to connect to ($n-5$), ($n-7$), ..., 5, 3, 1 other end points.

Therefore, the number of ways of drawing r chords, D, is given by,
$D = (n-1)(n-3)...5.3.1 = (2r-1)(2r-3)...5.3.1 = (2r)!/(r!\ 2^r)$

Next we shall consider the number of ways that no chords can intersect.



This diagram can be represented by the bracket string: 1(2(3)4(5(6)7)8), or more concisely as, (() (())), with a left bracket signifying a start point and a right bracket signifying an end point.

For $n = 2r$ points there will be r pairs of brackets, meaning there are $C(2r,r)$ different bracket strings in total that can be formed. However, not all strings will represent non-intersecting chords; for example, ())(() does not.

By considering the number of non-intersecting bracket strings, C, for the first three cases of r, we get:

r=1: C=1   ()
r=2: C=2   () (), (())
r=3: C=5   () () (), () (()), (())(), (() ()), ((()))

It turns out the number of non-intersecting chords are the sequence of Catalan numbers: 1, 2, 5, 14, 42, ..., and $C = C(2r,r)/(r+1)$. For completeness we shall derive this from first principles.

The trick to counting the number of brackets that correspond to non-intersecting chords is to subtract the number of "bad" bracket strings from the total number of bracket strings. We note that "bad" bracket strings occur when, working from left to right, the number of right brackets exceeds the number left brackets. For example, ())(().

The first time we encounter a right bracket out of place, we shall invert every bracket thereafter: left becomes right and right becomes left. For example, the "bad" string, ())((), becomes, ())))(.

The result of this exercise is that we end up with $(r-1)$ left brackets and $(r+1)$ right brackets, and as each one of these strings corresponds to a unique "bad" bracket string, we can determine that there are $C(2r,r+1)$ "bad" bracket strings in total.

Hence there are $C = C(2r,r)-C(2r,r+1)$ non-intersecting chords.

$$\begin{aligned} C = C(2r,r)-C(2r,r+1) &= \frac{(2r)!}{r!\,r!} - \frac{(2r)!}{(r-1)!(r+1)!} \\ &= \frac{(r+1)(2r)!}{(r+1)\,r!\,r!} - \frac{r(2r)!}{(r+1)\,r!\,r!} \\ &= \frac{(2r)!}{(r+1)\,r!\,r!} \\ &= \frac{C(2r,r)}{r+1} \end{aligned}$$

$$\therefore \text{P(no intersecting chords)} = C/D = \frac{(2r)!/((r+1)\,r!\,r!)}{(2r)!/(r!\,2^r)}$$

$$= \frac{(2r)!/((r+1)!\ r!)}{(2r)!/(r!\ 2^r)}$$

$$= 2^r/(r+1)!$$

RECIPROCAL RADICAL SUM

Problem

Consider the following series.

$$S(n) = \frac{1}{\sqrt{1} + \sqrt{2}} + \frac{1}{\sqrt{2} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{4}} + \ldots + \frac{1}{\sqrt{n} + \sqrt{(n+1)}}$$

For which values of *n* is S(*n*) rational?

Solution

Let us consider a general term in the series and the effect of multiplying top and bottom by its conjugate, $\sqrt{(k+1)} - \sqrt{k}$.

$$\begin{aligned}\frac{1}{\sqrt{k} + \sqrt{(k+1)}} &= \frac{\sqrt{(k+1)} - \sqrt{k}}{(\sqrt{k} + \sqrt{(k+1)})(\sqrt{(k+1)} - \sqrt{k})} \\ &= \frac{\sqrt{(k+1)} - \sqrt{k}}{\sqrt{k}\sqrt{(k+1)} - k + k + 1 - \sqrt{k}\sqrt{(k+1)}} \\ &= \sqrt{(k+1)} - \sqrt{k}\end{aligned}$$

$$\begin{aligned}\therefore S(n) &= \frac{1}{\sqrt{1} + \sqrt{2}} + \frac{1}{\sqrt{2} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{4}} + \ldots + \frac{1}{\sqrt{n} + \sqrt{(n+1)}} \\ &= (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + (\sqrt{4} - \sqrt{3}) + \ldots + (\sqrt{(n+1)} - \sqrt{n}) \\ &= \sqrt{(n+1)} - 1\end{aligned}$$

Hence S(*n*) is rational iff *n* is one less than a perfect square.

RECIPROCAL SUM

Problem

Consider the equation,

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z} \quad \text{where x, y, z are positive integers}$$

For a given *x*, determine the number of solutions.

Solution

From $1/x + 1/y = 1/z$, we get $yz+xz = xy$, $xz = xy-yz = y(x-\mathrm{z})$, therefore, $y = xz/(x-z)$, which means $z < x$.

Let $k = x-z$, so $z = z-k$, giving, $y = x(x-k)/k$.

Therefore, $y = x^2/k-x$. But if $k = x$ then $y = 0$, so in order to have $y > 0$ we must have $k < x$; in addition, *y* will only be integer if x^2 divides by *k*.

That is, the number of solutions for a given *x* is the same as the number of divisors of x^2 less than *x*.

For example, when $x = 12$, $x^2 = 144$, $k = \{1, 2, 3, 4, 6, 8, 9\}$.

As $\mathrm{z} = x-k$ and $y = xz/k$, we get the seven solutions: (12,132,11) (12,60,10) (12,36,9) (12,24,8) (12,12,6), (12,6,4) and (12,4,3).

Repunit Divisibility

Problem

A number consisting entirely of ones is called a repunit. We shall define R(*k*) to be a repunit of length *k*; for example, R(6) = 111111.

Given that *n* is a positive integer and GCD(*n*, 10) = 1, prove that there always exists a value, *k*, for which R(*k*) is divisible by *n*.

Solution

Consider R(*k*) mod *n* for $1 \le k \le n$.

Under modulo *n* the residues can be 0, 1, 2, ... , $n-1$; that is, there are up to *n* different possible remainders when R(*k*) is divided by *n*.

If each residue is different then one of them must be zero, in which case there exists a value of *k* such that *n*|R(*k*) and the proof is complete.

However, if they are not all different, there must be at least two residues that are equal (by the Pigeon Hole Principle). Suppose that R(*i*) = R(*j*), where $i > j$, then it should be clear that R(*i*) – R(*j*) ≡ 0 mod *n*.

Now consider R(7) – R(4) = 1111111 – 1111 = 1110000.

$\therefore$ R(*i*) – R(*j*) = 10^j R(*i* – *j*) ≡ 0 mod *n*

In other words, 10^j R(*i*–*j*) is a multiple of *n*, and as GCD(*n*, 10) = 1, it must be R(*i* – *j*) that is a multiple of *n*.

But as $1 \le j < i \le n$, it follows that $i - j < n$, and so there must exist a value of $k = i - j$ such that *n*|R(*k*). **Q.E.D.**

SLIDING BOX

Problem

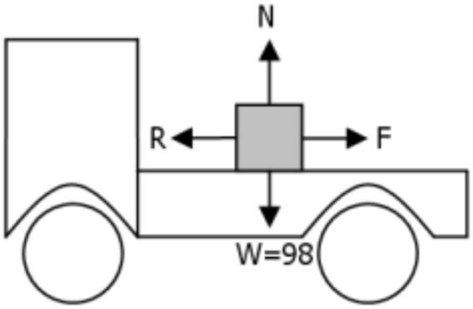
A 10kg box is placed on the back of a lorry, and as the lorry accelerates uniformly from 0 to 36 kmh^{-1} in 5 seconds, the box begins to slide. The lorry then continues to drive at a constant speed and the box comes to rest.

If the box slides 2 metres in total, and assuming g=9.8 ms^{-2}, find μ, the coefficient of friction between the box and the surface.

Solution

If truck accelerates from 0 to 36 kmh^{-1}=10 ms^{-1} in 5 seconds, acceleration is 2 ms^{-2}.

Consider the diagram.



The limiting equilibrium, R=98μ.

For simplicity, we shall assume throughout that the coefficient of friction remains constant and that the dynamic friction (once the box begins sliding) is the same as the static (limiting) friction.

As truck accelerates, F = ma = 10×2 = 20N, and as we are told that the box slides, we know that F must exceed R and the box will accelerate.

$\therefore$ F – R = ma: $20 - 98\mu = 10a_1$, $a_1 = 2 - 9.8\mu$
$s=ut+½at^2$: $s_1=0+½(2 - 9.8\mu)\times5^2=(25/2)(2 - 9.8\mu)$
$v=u+at$: $v=5(2 - 9.8\mu)$

When truck reaches 36 kmh^{-1} there is no accelerating force acting on the box by the motion of the truck, so F = 0 and R will act against the motion of the box, bringing it to rest.

$F - R = ma$: $0-98\mu=10a_2$, $a_2=-9.8\mu$
$v^2=u^2+2as$: $0=(5(2-9.8\mu))^2-19.6\mu s_2$, $s_2=25(2-9.8\mu)^2/(19.6\mu)$

Therefore, $s_1+s_2=(25/2)(2-9.8\mu)+25(2-9.8\mu)^2/(19.6\mu)=2$.

Multiply through by 98μ.

$1225\mu(2-9.8\mu)+125(2-9.8\mu)^2=196\mu$
$(2-9.8\mu)[1225\mu+125(2-9.8\mu]=196\mu$
$500-2450\mu=196\mu$

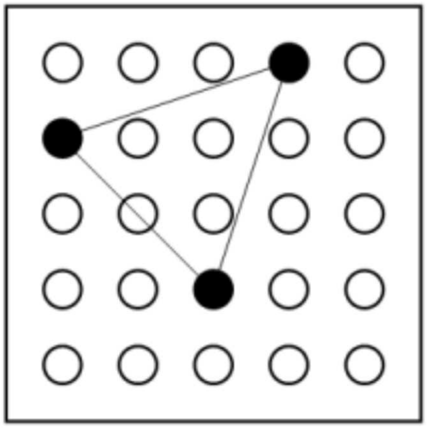
Hence, $\mu = 500/2646 = 250/1323$.

If the lorry accelerated uniformly to 36 kmh^{-1} in 10 seconds, how far would the box slide?
Given that the lorry accelerates uniformly to a speed of w ms^{-1} in t seconds, before continuing at a constant speed, how far will the box slide in total?

SQUARE LATTICE TRIANGLES

Problem

A 5 by 5 square lattice is formed by drilling holes in a piece of wood. Three pegs are placed in this lattice at random.



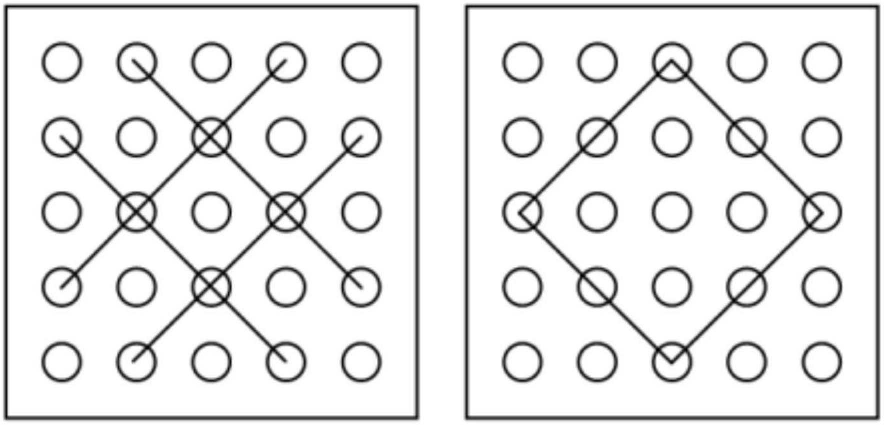
Find the probability that three randomly chosen points of a 5 by 5 lattice will form a triangle.

Solution

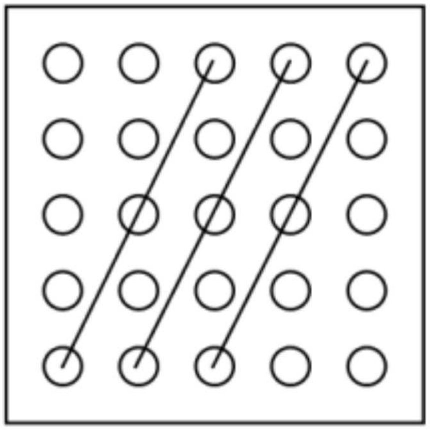
There are ${}^{25}C_3$ = 2300 ways of picking three points from twenty-five. However, some of the sets will be collinear, and this can happen in a number of ways.

Along each vertical, horizontal, and main diagonal:
$12\,{}^{5}C_3 = 12\times10 = 120$

Along the other diagonals:
$4({}^{4}C_3+{}^{3}C_3)=4(4+1)= 20$



Then, we have twelve more:



2300 – (120+20+12) = 2148, hence the probability of three random points forming a triangle will be 2148/2300 = 537/575 ≈ 0.934.

What about a 6 by 6 lattice?
Although there is no general formula, investigate other sized lattices to find an algorithm.

SQUARE SUM IS SQUARE

Problem

It is well known that there exist pairs of distinct squares that add to make another square.
For example, $5^2 + 12^2 = 25 + 144 = 169 = 13^2$.

But this can be extended to any number of squares. For example, $2^2 + 5^2 + 14^2 = 15^2$.

Prove that there exists a sum of *n* distinct squares that is also square.

Solution

Let S denote the sum of $n-1$ arbitrarily chosen distinct even squares; clearly S will be even. Thus the problem reduces to showing that $S + a^2 = b^2$.

$\therefore S = b^2 - a^2 = (b + a)(b - a)$

As S is even, let $b - a = 2 \Rightarrow b + a = 2a + 2$.

$\therefore S = (b + a)(b - a) = 2(2a + 2) \Rightarrow a = S/4 - 1$

For *a* to be integer requires S to be divisible by 4. But as each of the $n - 1$ squares are even then S will be divisible by 4.

However, for the squares to be distinct we must ensure that *a* has not appeared among the $n - 1$ even squares. This can be achieved by ensuring that $a^2 > S$; that is, if the value of a^2 is greater than the sum of the $n - 1$ squares then it must be greater than each of the squares making the sum.

Solving $a^2 = (S/4 - 1)^2 = S$ leads to the quadratic $S^2 - 24S + 16 = 0$.
The largest root, $S = 12 + 8\sqrt{2} \approx 23.3$.
So for any value of $S \geq 24$ we will have $a^2 > S$.

Clearly for $n \geq 4$ the sum of the $n - 1$ distinct even squares must exceed 24, as $2^2 + 4^2 + 6^2 = 56$.
For $n \leq 3$ we need only ensure that one of the chosen squares is 6^2.

Hence there exists a sum of *n* distinct squares that is also square.

What would happen if the sum of the n – 1 arbitrarily chosen distinct squares were odd?

STRING OF ONES AND ZEROES

Problem

Prove that there always exists a number made up of ones and zeroes that is divisible by the positive integer, n.

Solution

Euler's Totient Theorem states that, $a^{\varphi(n)} \equiv 1 \mod n$, if HCF($a$,$n$)=1.

We shall begin by considering the case where n is prime, and for brevity, we shall write $\varphi(n)=\text{P}$.

Clearly $10^{\text{P}} \equiv 1 \mod n$ will be true, as n is prime and we know that HCF(10,n)=1. Raising both sides of the congruence to the power, k, we get $10^{k\text{P}} \equiv 1 \mod n$.

Hence $\text{N} = 10^{\text{P}} + 10^{2\text{P}} + \ldots + 10^{n\text{P}} \equiv 0 \mod n$.

In other words, there exists a number, N, made up of ones and zeroes that is divisible by n, if n is prime.

For example, when n=7, $\varphi(7)=6$, and as $10^6 \equiv 1 \mod 7$, it follows that $\text{N} = 10^6 + 10^{12} + \ldots + 10^{42} \equiv 0 \mod 7$.

If n is composite, HCF(10,n) may not be 1, and we cannot make use of Euler's Totient Theorem. However, if HCF(10,n)$>$1, this factor must be exclusively made up of 2's and 5's. Therefore, there exists some number, h, of the form 2^a5^b such that HCF(10,n/h)=1.

Clearly $10^{\varphi(n/h)} \equiv 1 \mod n/h$, and so there must exist a number, N, made up of ones and zeroes that is divisible by n/h.

In which case, we can multiply N by 10 as many times as necessary so that 10^mN is still made up of ones and zeroes, but will be divisible by n. In fact, $m=\max(a,b)$; can you see why?

For example, when n=60, $60/(2^2 \times 5)=3$. As HCF(10,3)=1, $\varphi(3)=2$, we know that $\text{N} = 10^2 + 10^4 + 10^6 \equiv 0 \mod 3$. Hence 100N will be made up of ones and zeroes and, being divisible by 300, must be divisible by 60.

Note that this method only proves the existence of such a number, but it does not provide the smallest such case. In the example given it turns out that N = $10^2 + 10^4 + 10^6$ = 1010100 is already divisible by 60 before multiplying by 100.

Can you find a more optimal approach?

Hint: think about $10^{\varphi(n/n)}$.

SUM OF TWO ABUNDANT NUMBERS

Problem

The divisors of a positive integer, excluding the number itself, are called the proper divisors . If the sum of proper divisors is equal to the number we call the number perfect. For example, the divisors of 28 are 1, 2, 4, 7, 14, and 28, so the sum of proper divisors is 1 + 2 + 4 + 7 + 14 = 28.

Similarly, if the sum of the proper divisors exceeds the number we call the number abundant. For example, 12 is abundant because the divisors of 12 are 1, 2, 3, 4, 6 12, and the sum of proper divisors $1 + 2 + 3 + 4 + 6 = 14 > 12$.

By first showing that $315p$ is abundant for all primes, $p \leq 103$, prove that all integers greater than 28123 can be written as the sum of two abundant numbers.

Solution

Let $S(n)$ represent the sum of proper divisors of n.

$S(315) = S(3^2 \times 5 \times 7) = 1 + 3 + 5 + 7 + 9 + 15 + 21 + 35 + 45 + 63 + 105 = 309$.

When considering $S(315p)$ we must deal with two cases.

Case 1: p is coprime with 315 ($p \neq 3, 5, 7$)

$\therefore S(315p) = p(1 + 3 + ... + 105) + (1 + 3 + ... + 105) + 315 = 309p + 624$

For $315p$ to be abundant, $315p < 309p + 624 \Rightarrow p < 104$.

Case 2: $p = 3, 5, 7$

$S(315p) = p(1 + 3 + ... + 105) + q$, where q is the sum of divisors not containing a factor of 3, 5, or 7 respectively.

So for $315p$ to be abundant it is sufficient to show $315p < 309p + q \Rightarrow 6p < q$.

When $p = 3$, $6p = 18$, $q = 1 + 5 + 7 + 35 = 48 \Rightarrow 6p < 48$
When $p = 5$, $6p = 30$, $q = 1 + 3 + 7 + 9 + 21 + 63 = 104 \Rightarrow 6p < 104$
When $p = 7$, $6p = 42$, $q = 1 + 3 + 5 + 9 + 15 + 45 = 48 \Rightarrow 6p < 78$

Hence $315p$ is abundant for all primes, $p \leq 103$.

It can be shown that multiples of abundant numbers are also abundant (see Even Sum Of Two Abundant Numbers). Thus all values of m from 2 to 103 will either be prime or contain a prime in that domain. So although 315 is deficient, 315*m* is guaranteed to be abundant for $2 \le m \le 103$.

We now search for the smallest abundant number that is coprime with 315. Considering numbers of the form $2^k \times 11$ we find that 88 is the smallest such example. Therefore the expression $315a + 88b$ will be the sum of two abundant numbers for $2 \le a \le 103$ and $b \ge 1$.

Clearly the expression produces integers congruent with 315*a* mod 88, and although $0 \le a \le 87$ will produce all possible congruences we require $a \ge 2$ to ensure 315*a* is abundant; that is, $2 \le a \le 89$ will produce all possible congruences.

It is necessary to have at least one multiple of 88, so $315 \times 89 + 88$ represents that last integer before the congruences repeat. Hence every integer $n > 315 \times 89 + 88 = 28123$ can be written in the form $315a + 88b$, which will be the sum of two abundant numbers. **Q.E.D.**

Tangent Sum And Product

Problem

Prove that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$ for any non-right angle triangle.

Solution

As $A + B + C = 180^{\circ}$, it follows that $\tan(A + B + C) = \tan 180^{\circ} = 0$.

Using the addition formula:

$$\tan(A + (B + C)) = \frac{\tan A + \tan(B + C)}{1 - \tan A \tan(B + C)} = 0$$

$\therefore \tan A + \tan(B + C) = 0$.

Using the addition formula again:

$$\tan \mathrm{A} + \frac{\tan B + \tan C}{1 - \tan B \tan C} = 0$$

$\therefore \tan A(1 - \tan B \tan C) + \tan B + \tan C = 0$

$\therefore \tan \mathrm{A} - \tan A \tan B \tan C + \tan B + \tan C = 0$

Hence, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

As $\mathrm{A} + \mathrm{B} = 180^{\circ} - \mathrm{C}$, use $\tan(\mathrm{A} + \mathrm{B}) = \tan(180^{\circ} - \mathrm{C})$ to prove the same result by a slightly simpler route.

TIME LOSES INTEGRITY

Problem

The velocity of a body increases with constant acceleration. Its motion is recorded between two points, A and B, that are one metre apart, with M being the midpoint of AB.

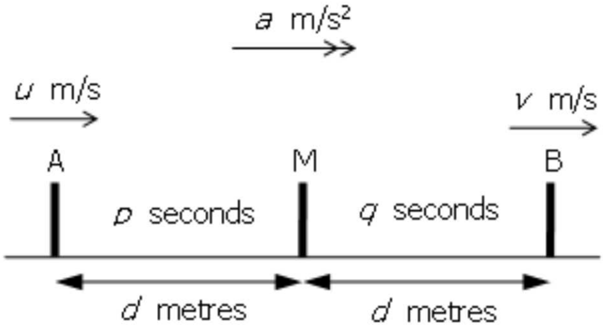
The body takes *p* seconds to travel from A to M and *q* seconds to travel from M to B.

Given that the change in speed from A to B, measured in metres per second, is integer, prove that *q* cannot be integer.

Solution

Let *a* represent the constant acceleration, *u* represents the initial velocity as it passes A, and *v* represents the final velocity as it passes B. Although we know that AM = MB = 0.5, we shall solve the general case initially, and let AM = MB = *d*.

Consider the following diagram.



We shall prove this in two different ways.

First proof:

Using the equation of motion, $s = ut + 0.5at^2$, from A and M and A to B we get the following two equations.

From A to M: $d = up + 0.5ap^2$ [1]

From A to B:
$$\begin{aligned} 2d &= u(p + q) + 0.5a(p + q)^2 \\ &= up + 0.5ap^2 + uq + 0.5a(q^2 + 2pq) \\ &= d + uq + 0.5a(q^2 + 2pq) \\ \therefore d &= uq + 0.5aq(q + 2p) \text{ [2]} \end{aligned}$$

Multiplying equation [1] through by q and equation [2] through by p gives the following.

$$dq = upq + 0.5ap^2q \Rightarrow upq = dq - 0.5ap^2q$$

$$dp = upq + 0.5apq(2p + q) \Rightarrow upq = dp - 0.5apq(2p + q)$$

$$\begin{aligned} \therefore dp - 0.5apq(2p + q) &= dq - 0.5ap^2q \\ dp - dq &= 0.5apq(2p + q) - 0.5ap^2q \\ d(p - q) &= 0.5apq(2p + q - p) \\ &= 0.5apq(p + q) \\ 2d(p - q) &= apq(p + q) \end{aligned}$$

$$\therefore a = \frac{2d(p - q)}{pq(p + q)}$$

But as $v = u + at$, the change in speed, $c = v - u = at = a(p + q)$.

$$\therefore c = \frac{2d(p - q)}{pq(p + q)}(p + q) = \frac{2d(p - q)}{pq} = \frac{p - q}{pq} \quad \text{(as } d = 0.5 \text{ metres)}$$

$$\begin{aligned} \therefore \qquad cpq &= p - q \\ cpq + q &= p \\ q(cp + 1) &= p \end{aligned}$$

Hence the time taken to travel from M to B, $q = \dfrac{p}{cp + 1}$.

We were told that the body is increasing in speed, so c is a positive integer. Therefore $cp + 1 > p$, and we prove that q cannot be integer.

Alternative proof:
Let T = $p + q$. As $c = v - u = a$T, the constant acceleration, $a = c$ / T.

Using $s = ut + 0.5at^2$, we get $1 = uT + 0.5(c / \mathrm{T})\mathrm{T}^2 = \mathrm{T}(u + 0.5c)$.

Hence T = 1 / (u + 0.5c).

Clearly T is maximised when the denominator is minimised; that is, $u = 0$ and $c = 1$, and we get T = 2 seconds.

As this is the maximum time the body can take to travel from A to B and speed is increasing, it must take less than 1 second to pass from M to B, and cannot be integer.

Extension:
We note from the first proof that $cpq = p - q$, and this can rearranged to give the equation, $p = q / (1 - cq)$. If we let $c = 2$ and let $q = 0.4$, we get $p = 2$, which means that T = 2.4 seconds.

How can this be so?
Explain why *p* cannot be integer.
What about the total length of time to travel from A to B?

Unique Square Sum

Problem

You are given that ALL primes that are one more than a multiple of 4 can be written as the sum of two squares. For example, $13 = 2^2+3^2$.

Assuming that a prime is expressible as the sum of two squares, prove that it can be done in only one way.

Solution

Suppose that the prime, $p = a^2+b^2 = c^2+d^2$.

Clearly HCF(a,b)=1, otherwise a^2+b^2 would be composite; similarly HCF(c,d)=1.

We will begin by establishing a result that will be used later.

If $ad = bc$ then $a|bc$, but as HCF(a,b)=1, $a|c$. Let $c = ka$.

$\therefore ad = kab \Rightarrow d = kb$.

So $p = c^2+d^2 = (ka)^2+(kb)^2 = k^2(a^2+b^2)$.

And because of the initial supposition, it is necessary that $k = 1$. Hence $c = ka = a$ and $d = kb = b$.

By the symmetry of our initial supposition, from $ad = bc$ we can freely interchange c and d to get $ac = bd$. And in the same way we can show that $c = b$ and $d = a$.

Hence our proof will be complete if we can show that either $ad = bc$ or $ac = bd$.

From the initial supposition, $p = a^2+b^2 = c^2+d^2$, we can write:

$$a^2 = p-b^2 \text{ and } d^2 = p-c^2.$$

$\therefore a^2d^2 = (p-b^2)(p-c^2) = p^2-p(b^2+c^2)+b^2c^2$

$\therefore a^2d^2 \equiv b^2c^2 \mod p$

$\therefore ad \equiv \pm bc \mod p$

So either (i) $ad-bc \equiv 0 \mod p$, or (ii) $ad+bc \equiv 0 \mod p$.

As $0 < a^2, b^2, c^2, d^2 < p$

$0 < a, b, c, d < \sqrt{p}$

$\therefore 0 < ad, bc < p$

Hence $-p < ad-bc < p$ and $0 < ad+bc < 2p$.

If (i) is true then $ad-bc \equiv 0 \bmod p$, but as $ad-bc$ lies between $-p$ and p, it follows that $ad-bc = 0 \Rightarrow ad = bc$. However, we have shown that if this is true, then $c = a$ and $d = b$, and the square sum must be unique.

Let now consider (ii), $ad+bc \equiv 0 \bmod p$. We have shown that $ad+bc$ lies between 0 and $2p$, so if this is true then $ad+bc = p$. This case requires a useful, and tricky, identity to be obtained.

$$\begin{aligned} p^2 &= (a^2+b^2)(c^2+d^2) \\ &= a^2c^2+a^2d^2+b^2c^2+b^2d^2 \\ &= (ad^2+2abcd+b^2c^2)+(a^2c^2-2abcd+b^2d^2) \\ &= (ad+bc)^2+(ac-bd)^2 \end{aligned}$$

Hence if $ad+bc = p$, then $p^2 = p^2+(ac-bd)^2$.

This leads to $ac-bd = 0 \Rightarrow ac = bd$. However, we have shown that if this is true, then $c = b$ and $d = a$, and once again we show that the square sum is unique.

Hence we prove that p can be written as the sum of two squares in only one way.