2 Star Problems

3x3 Magic Square

Problem

A 3x3 magic square is a square grid containing the numbers 1 to 9 in such a way that the sum of each row, column, and diagonal has the same "magic total".

4	9	2
3	5	7
8	1	6

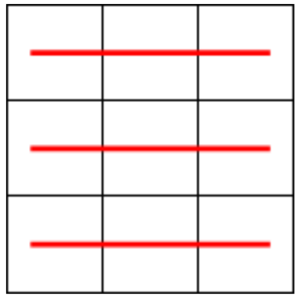
By considering rotations and reflections to be equivalent, prove that this 3x3 magic square is the only solution.

Solution

We shall begin by assigning letters to each of the cells in the 3x3 grid.

a	b	c
d	e	f
g	h	i

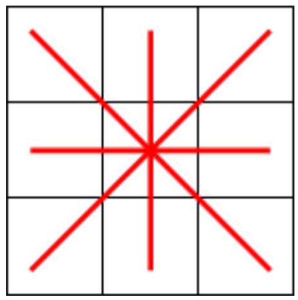
Although we do not know the individual value of each letter we do know that each of the digits 1 through 9 is assigned to the letters in some order. If we let the "magic total" for each row, column, and diagonal be T then it can be seen that adding the following rows is equivalent to 3T.



But this is also equivalent to adding every cell.

$\therefore a + b + c + d + e + f + g + h + i = 1 + 2 + ... + 9 = 45 = 3T \Rightarrow T = 15$

Suppose instead we add the following lines together.



$\therefore (a + e + i) + (d + e + f) + (g + e + c) + (b + e + h) = 4T$
$\therefore (a + b + c + d + e + f + g + h + i) + 3e = 60$
$\therefore 45 + 3e = 60 \Rightarrow e = 5$

Hence we know that the "magic total" is 15 and the central cell must be 5. So in order for each line to have the same total of 15 it will be necessary for the cells either side of the central cell to be of the form $5 - x$ and $5 + x$.

Take a moment to verify that the following arrangement is necessary and that each row, column, and diagonal adds to 15.

$5-x$	$5+x+y$	$5-y$
$5+x-y$	5	$5-x+y$
$5+y$	$5-x-y$	$5+x$

It can be seen that the cell with greatest value, $5 + x + y = 9 \Rightarrow x + y = 4$. Clearly x and y must be different values, otherwise the cells with $5 + x$ and $5 + y$ would have the same value; it should also be clear that $x, y > 0$, because if, for example, $x = 0$, then $5 + x = 5 - x = 5$.

Without loss of generality, let $x < y$, and as their sum is 4 it follows that $x = 1$ and $y = 3$. By substituting these values into the grid above we obtain the solution given in the problem and hence prove that this solution is unique.

Prove that the "magic total" for an n x n grid is given by $n(n^2 + 1) / 2$.

ADDING DIGITS

Problem

Imagine writing down every possible combination of the digits 1, 2, 3, and 4. What would be the sum of all these combinations?

For example, with the digits 1, 2 and 3; there are six combinations and the sum of these combinations is
123 + 132 + 213 + 231 + 312 + 321 = 1332.

Solution

Considering combinations,

1234	Each digit appears in each column 6 times.
1243	So each column must add up to,
1324	$6(1 + 2 + 3 + 4) = 60$.
1342	
1423	$\therefore$ Total = $60(1 + 10 + 100 + 100) = 66660$.
1432	
2134 , etc.	

What about the digits 1, 2, 3, ..., *n* ?

ALTERNATE FIBONACCI RATIO

Problem

The Fibonacci sequence is defined by the second order recurrence relation $F_{n+2} = F_{n+1} + F_n$, where $F_1 = 1$ and $F_2 = 1$.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

Consider the ratio of adjacent terms, F_{n+1}/F_n:

1/1 = 1
2/1 = 2
3/2 = 1.5
5/3 = 1.666...
8/5 = 1.6
13/8 = 1.625
21/13 = 1.615...
34/21 = 1.619...
55/34 = 1.617...
89/55 = 1.618...

As n increases it is well known that the ratio of adjacent terms F_{n+1}/F_n tends towards $\varphi = (\sqrt{5}+1)/2 \approx 1.618$ (see Fibonacci Ratio).

What does F_{n+2}/F_n tend towards as n increases?

Solution

By definition, $F_{n+2} = F_{n+1} + F_n$.

Therefore $R = F_{n+2}/F_n = (F_{n+1} + F_n)/F_n = F_{n+1}/F_n + 1$.

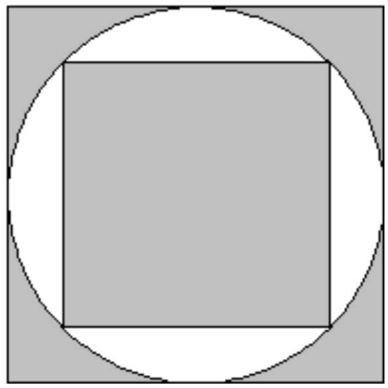
Hence as $n \to \infty$, $R \to \varphi + 1 = (\sqrt{5}+1)/2 + 1 = (\sqrt{5}+3)/2$.

Show that $F_{n+3}/F_n \to 2\varphi + 1$ and $F_{n+4}/F_n \to 3\varphi + 2$.
Prove in general that $F_{n+k}/F_n \to F_k\varphi + F_{k-1}$.

ALTERNATING SQUARES

Problem

A grey square has a white circle inscribed and a grey square inscribed in the circle.



What fraction of the diagram is white?

Solution

Area of outside square = $2r \times 2r = 4r^2$
Area of inside square = $½ \times 4r^2 = 2r^2$
Area of circle = πr^2

$\therefore$ Area white = $\pi r^2 - 2r^2 = r^2(\pi - 2)$

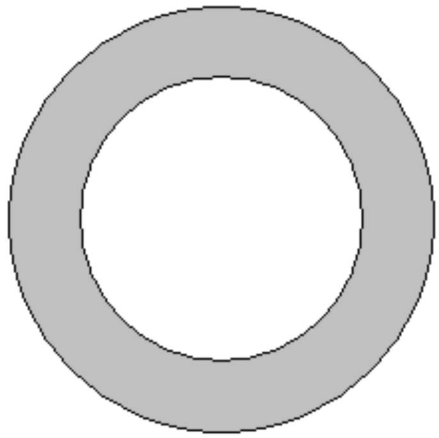
Hence, fraction of the diagram white = $r^2(\pi - 2)/4r^2 = (\pi - 2)/4 \approx 0.285$

If another white circle was inscribed in the inside square, what fraction would be white now?
What would happen if white circles and grey squares were recursively inscribed?
Would the white area tend towards a limit?

Annulus

Problem

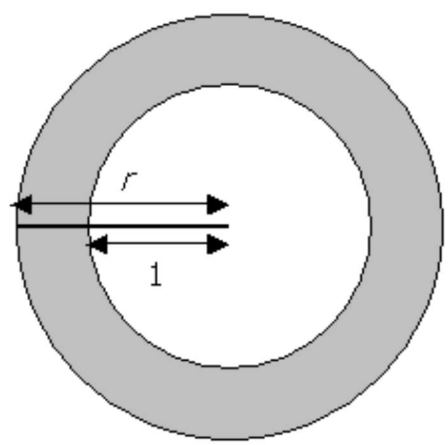
A circle has a unit circle (radius, $r = 1$) removed from its centre to produce an annulus.



If the area remaining is the same as the unit circle removed, find the width of the annulus.

Solution

Let the radius of the original circle be, r.



If the area remaining is equal to the area removed (a unit circle), the area of the original circle will be twice a unit circle.

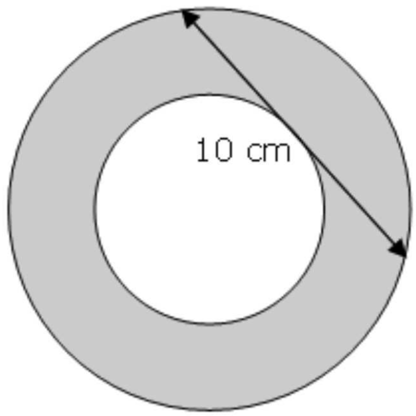
$\therefore \pi r^2 = 2\pi \Rightarrow r^2 = 2$.

Hence, the radius of the original circle, $r = \sqrt{2}$ and so the width of the annulus is $\sqrt{2} - 1$.

Area Of Annulus

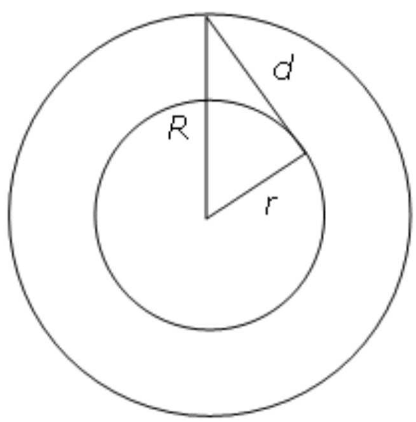
Problem

Find the area of the annulus.



Solution

Surprisingly it is unnecessary to know the radius of either circle to determine the area of the annulus. Consider the following diagram, where R is the radius of the large circle and r is the area of the small circle.



As a tangent meets a radius at right angles, the triangle is right angled, so $R^2 = r^2 + d^2$.

$\therefore \pi R^2 = \pi r^2 + \pi d^2$

So the area of the annulus, $\pi R^2 - \pi r^2 = \pi d^2 = 25\pi$.

AVERAGE MATCHES

Problem

The contents of twelve boxes of matches were recorded as:

34, 31, 29, 35, 33, 30, 31, 28, 27, 35, 32, 31

On the box it stated, "Average contents 32 matches." Is this correct?

Solution

The statement does not mention which average was used so we shall calculate the mean, median, and mode.

The mean is 376/12 ≈ 31.3.
By arranging in order: 27, 28, 29, 30, 31, 31, 31, 32, 33, 33, 34, 35, we can see that the middle two numbers are 31, so the median is 31.
As 31 occurs most often the mode is also 31.

As all three averages are below 32 it is tempting to dismiss the claim. However, this is a classic error of sampling. If a coin were spun twelve times and exactly six heads and six tails were not obtained, would we suspect the coin of being biased? Of course not!

How far below the stated "average" must the sample mean be in order to invalidate the claim? (see Mean Claim)

A Number And Its Reciprocal

Problem

Consider the function, $f(x) = x + 1/x$.

$f(2) = 2 + 1/2 = 2.5$
$f(0.4) = 0.4 + 1/0.4 = 2.9$
$f(1.25) = 1.25 + 1/1.25 = 2.05$

Prove that $f(x) \geq 2$ for all real values of $x > 0$.

Solution

Clearly $(x - 1)^2 \geq 0$ for all real values of x.

$\therefore x^2 - 2x + 1 \geq 0$
$x^2 + 1 \geq 2x$

As $x > 0$ we can divide through by x without changing the sign of the inequality.

$\therefore x + 1/x \geq 2$ **Q.E.D.**

i. Prove that $(x + 1)^2 / x \geq 4$ for $x > 0$.
ii. Determine the domain of x for which $x^2 \geq 4(x-1)$ is true.

BLONDE HAIR BROWN EYES

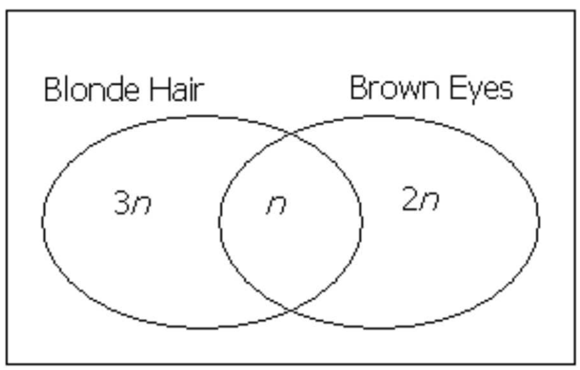
Problem

In a particular class, each student has blonde hair or brown eyes; 1/4 of students with blonde hair have brown eyes and 1/3 of the students with brown eyes have blonde hair.

What fraction of the class in total have brown eyes?

Solution

This is best solved with a Venn diagram. Let the number of students with blonde hair and brown eyes be n. So the number of students with blonde hair will be $4n$ (as 1/4 have both) and the number of students with brown eyes will be $3n$ (as 1/3 have both).

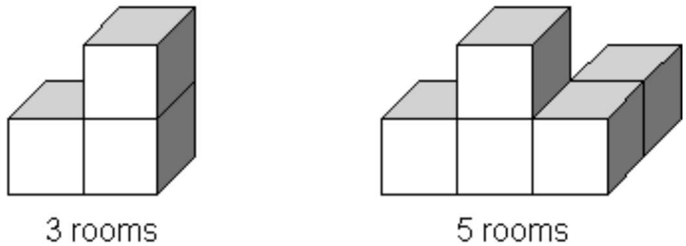


From this we can determine that 3/6 = 1/2 of the class have brown eyes.

BOX WORLD

Problem

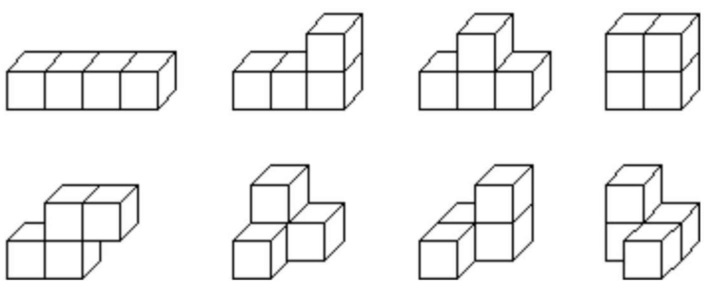
In box world all the buildings are made from cubic box sections.



Not counting different orientations, how many distinct ways can you connect four rooms together?

Solution

There are 8 distinct ways of connecting four unit cubes.



How many distinct ways can you connect together five unit cubes?
What if you count different orientations?
Can you generalise?

CALENDAR CUBES

Problem

Two wooden cubes have digits painted on each face in such a way that it is possible to display any date of the month.

For example, 

Show clearly how you would label each face to achieve this.

Solution

Clearly each cube must have 0, 1 and 2 painted on it. Leading to the solution.

Cube 1	Cube 2
0	0
1	1
2	2
3	6
4	7
5	8

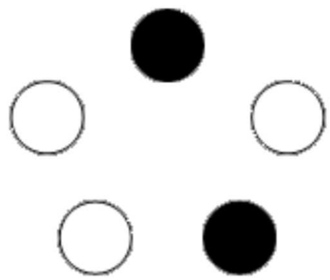
The trick being, the 6 can be turned around to make 9.

How would two wooden cubes be labelled to make all 2-digit primes?
(This time the 6 is different from the 9!)

CANDELABRA

Problem

Three white candles and two black candles can be arranged in a number of ways in a pentagon shaped candelabra.

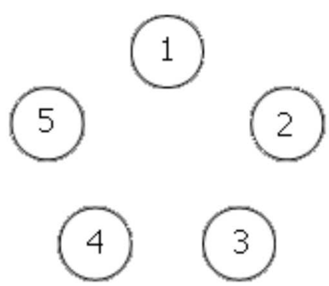


If the candles are placed at random, find the probability that the three white candles will be adjacent.

Solution

This can be solved in two different ways.

Method 1
Begin by numbering each of the candle holders:



We can use these labels to indicate which candle goes in which place:

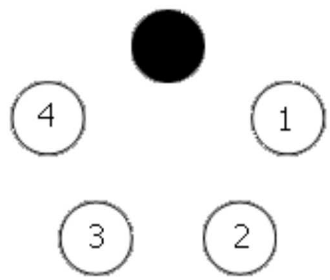
```
B B W W W *
B W B W W
B W W B W
B W W W B *
W B B W W *
W B W B W
W B W W B
```

```
W W B B W *
W W B W B
W W W B B *
```

Due to the circular arrangement we need to be careful in identifying the arrangements for which whites are adjacent (marked with *).

As there are ten possible ways of filling the five candle holders, the probability of three whites being adjacent is 5/10 = 1/2.

<u>Method 2</u>
Without loss of generality we can consider a black candle being placed in the top circle; if this is not the case, it is always possible to rotate the arrangement so that this is true.



It can be seen that if the other black candle is placed in 1 or 4, three whites will be adjacent.

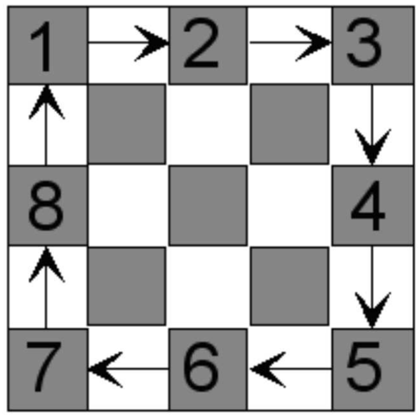
Therefore, the probability of three whites being adjacent is 2/4 = 1/2.

If four white candles and two black candles were placed in a hexagonal candelabra, what would be the probability of the whites being adjacent?
Investigate for different numbers of black and white candles.

CHEQUERED FLOOR

Problem

A room measuring 5x5 is tiled with a chequered design.



If someone stood in the top left corner of the room and walked around the outside edge, they would step on 8 grey tiles in total.

If they walked around the outside edge of a larger square shaped room, with the same chequered tile floor design, and stood on 148 grey tiles in total, what are the dimensions of the room?

Solution

By working out the number of grey tiles on the perimeter of different sized square rooms.

Room Size	**Grey Tiles (around perimeter)**
2x2	2
3x3	4
4x4	6
5x5	8

Leading to an n x n room having $2(n - 1)$ grey tiles around outside edge.

Solving $2(n - 1) = 148$, we get $n = 75$.

So the room measures 75x75 tiles.

How many grey tiles in total are there on the floor of a room measuring 75x75?
Can you prove that the number of grey tiles on the perimeter of an $n \times n$ room will be $2(n - 1)$?
What about a room measuring m by n?

CHRISTMAS TREES

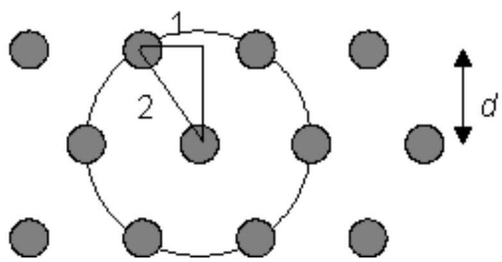
Problem

When Christmas trees are planted they should stand at least 2 metres away from one another whilst growing.

What is the maximum number of trees that can be planted in one square kilometre?

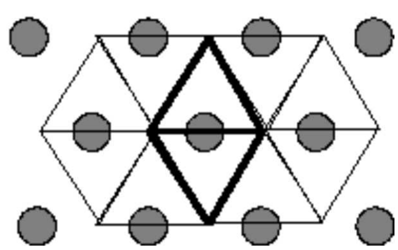
Solution

The best arrangement is a triangular matrix:



By applying the Pythagorean Theorem to the triangle, $2^2 = d^2 + 1^2$, hence the smallest allowable distance between rows, $d = \sqrt{3} \approx 1.73$ metres.

Each tree occupies the same area as two equilateral triangles.



That is, each tree occupies $2\sqrt{3}$ m^2. As 1 km^2 = 1000 × 1000 = 1,000,000 m^2, the maximum number of trees that can be planted is $1000000/(2\sqrt{3}) = 500000/\sqrt{3} \approx 288673$ trees.

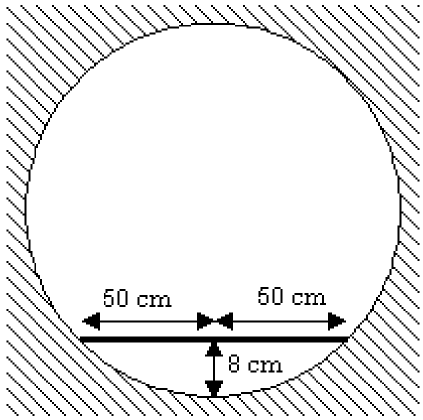
If the trees were planted in a square matrix, what is the maximum number of trees that could be planted in 1 km^2?

Can you explain why a triangular arrangement is the most efficient method of planting trees?

CIRCULAR PIPES

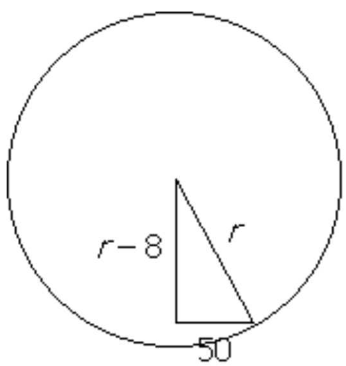
Problem

A surveyor places a metre stick against the base of a large circular underground pipe and finds that the midpoint of the stick is 8 cm from the pipe wall.



Use this information to find the inside diameter of the pipe.

Solution



Using the Pythagorean Theorem, $r^2 = (r - 8)^2 + 50^2 = r^2 - 16r + 64 + 2500$

$\therefore 16r = 2564 \Rightarrow r = 160.25$

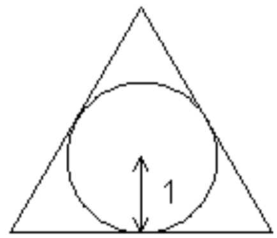
So the internal diameter is 320.5 cm.

What if the stick was 10 cm from the wall?
Can you generalise for any distance x?

CIRCUMSCRIBED TRIANGLE

Problem

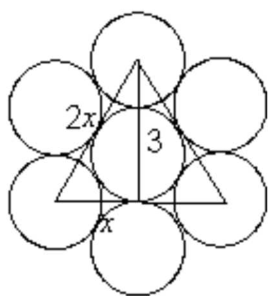
An equilateral triangle is circumscribed.



Find the area of the triangle.

Solution

Consider the following diagram.



Using the Pythagorean Theorem, $4x^2 = x^2 + 9$
$\therefore 3x^2 = 9 \Rightarrow x = \sqrt{3}$.

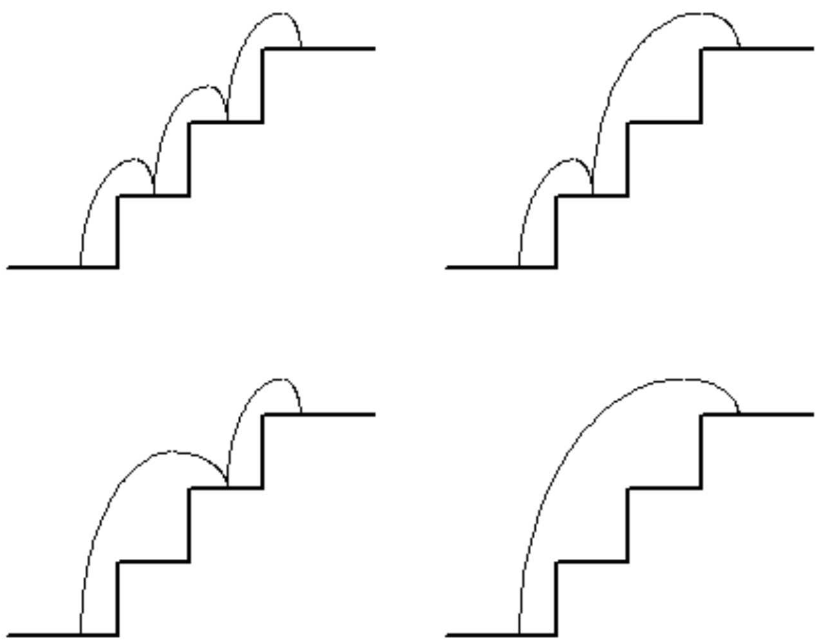
Area of triangle is $3\sqrt{3}$.

It can be seen that the perimeter of the triangle is $6\sqrt{3}$, i.e. double the area. What radius circle would have a circumscribed equilateral triangle with an area the same size as the perimeter?

CLIMBING STAIRS

Problem

It is possible to climb three steps in exactly four different ways.



How many ways can you climb ten steps?

Solution

If there are ten steps, it is necessary to arrive at the 10th step, but all the other steps in between are optional.

Step Number	1	2	3	4	5	6	7	8	9	10
	0	0	0	1	0	1	0	0	0	1

So the binary sequence, 0001010001, represents a journey using steps 4, 6 and 10. To find the total number of ways of climbing 10 steps we need to consider how many binary sequences can be made using 9 binary digits (as step 10 must always be a 1).

Hence there are $2^9 = 512$ ways of climbing ten steps.

What if the maximum number of steps you can climb at a time is two?

COLOURED DICE

Problem

Two dice have each face coloured either red or blue. The first die has five red faces and one blue face.

The two dice are rolled.

How many faces on the second die need be red so that there is an equal chance of getting two faces of the same colour?

Solution

If P(Red on 2nd die)=p, then P(Blue on 2nd die)=$1-p$.

P(same colour)=P(RR)+P(BB)=$(5/6)p+(1/6)(1-p)=(4p+1)/6$.

But as we are trying to get, P(same colour)=$1/2=3/6$, it follows that $4p+1=3$, $4p=2$, and $p=1/2$; that is, three red and three blue faces.

This can be solved in a surprisingly simple way. Regardless of obtaining red or blue on the first die, the only way we would have equal chance of getting two faces of the same colour is if the second die has an equal number of red and blue faces.

CONSECUTIVE CUBE

Problem

Three consecutive integers are multiplied together and the middle number is added. E.g. $3 \times 4 \times 5 = 60$ and $60 + 4 = 64 = 4^3$

Will this always produce a cube number?

Solution

Taking three consecutive integers, n, $n + 1$ and $n + 2$ we get.

$$
\begin{aligned}
n(n+1)(n+2) + (n+1) &= (n+1)(n(n+2)+1) \\
&= (n+1)(n^2+2n+1) \\
&= (n+1)(n+1)^2 \\
&= (n+1)^3
\end{aligned}
$$

Consecutive Prime Sum

Problem

If we add the primes 11 and 23 we get 34, which is twice the prime 17.

Give that p and q are consecutive primes, show that the equation $p + q = 2r$, where r is prime, has no solutions.

Solution

If we rearrange $p + q = 2r$, we get $r = (p + q)/2$. That is, r is the mean of p and q. But as they are consecutive primes there can be no prime in-between them. Hence the equation has no solutions if r is prime.

Counting Sequence

Problem

Consider the infinite sequence:, 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, ...

What is the 1000th term?

Solution

Without having the nature of the sequence explicitly stated we must make an assumption about its behaviour. It seems reasonable to assume that the sequence is made up of one 1, two 2's, three 3's, and so on.

If the sequence continued up to the last n, there would be $1 + 2 + ... + n = \frac{1}{2}n(n+1)$ terms.

We can verify that, $\frac{1}{2} \times 44 \times 45 = 990$.

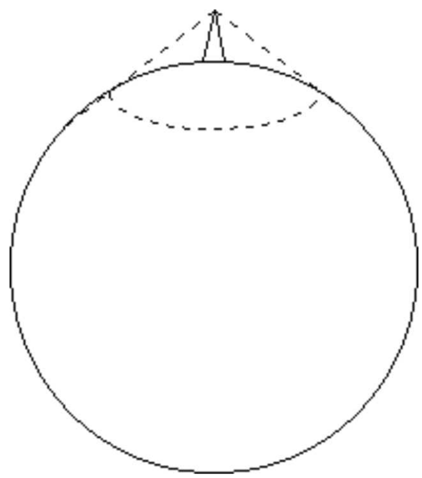
In other words, the 990th term is the last 44, the next 45 terms will be 45, hence the 1000th term must be 45.

What is the k^{th} term?

CURVATURE OF THE EARTH

Problem

From how far on a clear day, and assuming there are no obstructions, could you see from the top of the Eiffel tower?

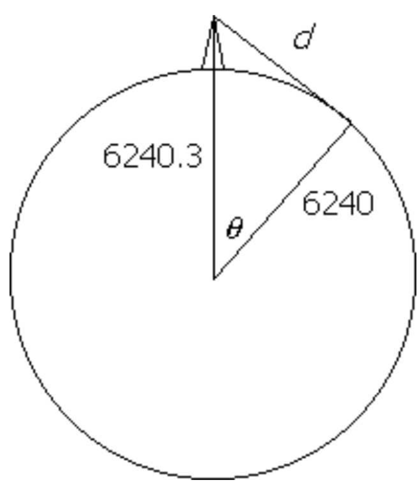


Take the height of the Eiffel tower as 300 metres and the radius of the Earth as 6240 km.

Solution

We shall answer this question in two ways:

Let us begin by considering the distance from the top of the tower to furthest point on the surface of the earth (the tangental length).



Applying the Pythagorean Theorem, $6240.3^2 = d^2 + 6240^2$. Hence $d \approx 61.2$ km.

However, this does not represent the distance along the curvature of the earth (the arc length). To do this we apply trigonometry to the right angle triangle above, $\cos\theta = 6240/6240.3$, thus $\theta \approx 0.562$ degrees.

As the radius of the earth is 6240 km, the circumference is $2\pi \times 6240 \approx 39187$ km. So the arc length will be $0.562/360 \times 39187 \approx 61.2$ km, which is the same answer as the other method... coincidence?

Does the tangental length always equal the arc length?

What fraction of the earth is visible from the top of the Eiffel tower?

To simplify this problem, assume that the viewing range is a flat circle with a radius equal to the maximium viewing distance (61.2 km).

From how far away, on a clear day and with no obstructions, can the Eiffel tower be seen?

Note: This is not the same as the original question. Consider the following question first...

What is the maximum distance that two 2 metre tall people can see each other?

What is the typical range of a geostationary satellite?

Their orbit is about 35900 km (yep, that's right!) above the surface of the earth.

What fraction of the globe does a geostationary satellite typically cover?

WARNING: This problem is VERY difficult!

DISCO RATIOS

Problem

The ratio of boys to girls at a school disco is 9:10.

An extra 17 boys arrive and the ratio changes to 8:7.

How many girls are there at the disco?

Solution

Originally there were $9n$ boys and $10n$ girls. With an extra 17 boys, we have $9n+17$ boys and we know the ratio changes to 8:7.

Therefore $(9n+17)/10n=8/7$, $7(9n+17)=8\times10n$, $63n+119=80n$.

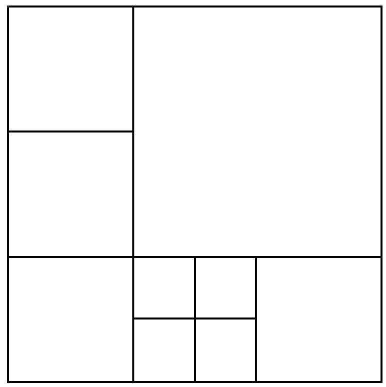
Solving this, $17n=119$, hence $n=7$.

So we can determine that there are $10n=70$ girls at the disco.

DIVIDED SQUARE

Problem

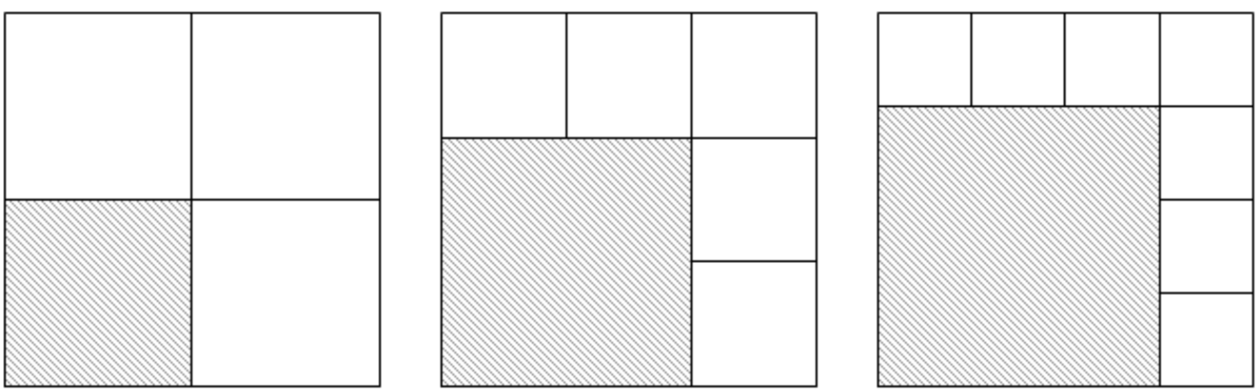
The diagram below shows how you could split a unit square up into nine non-overlapping squares.



Prove that the only number of non-overlapping squares you cannot split a unit square into are 2, 3, or 5 smaller squares.

Solution

Consider the following geometrical sequence.



It can be seen that each new "term" in the sequence will have two more squares than the previous. So it is possible to split the unit square into 4, 6, 8, 10, ... non-overlapping squares. That is, all the even terms above and including four.

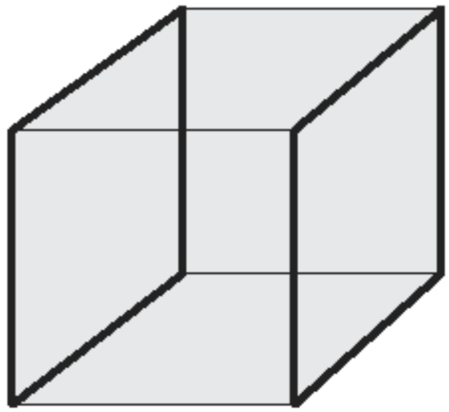
By splitting each of the shaded grey square into 2 by 2 smaller squares we lose one square and gain four, adding three squares to any term in this sequence. Therefore we can split a unit square into 7, 9, 11, 13, ... non-overlapping squares. That is, all the odd terms above and including seven.

It should be quickly evident that the unit square cannot be split into 2, 3, or 5 smaller squares. Hence we prove the result.

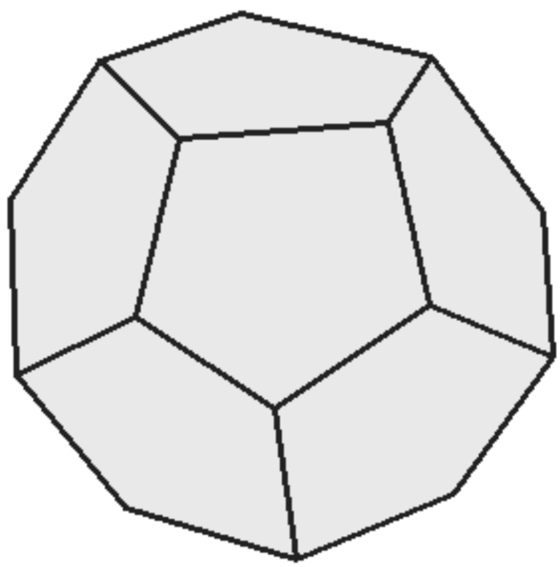
DODECAGON EDGES

Problem

A cube (regular hexahedron) consists of 6 faces and 12 edges.

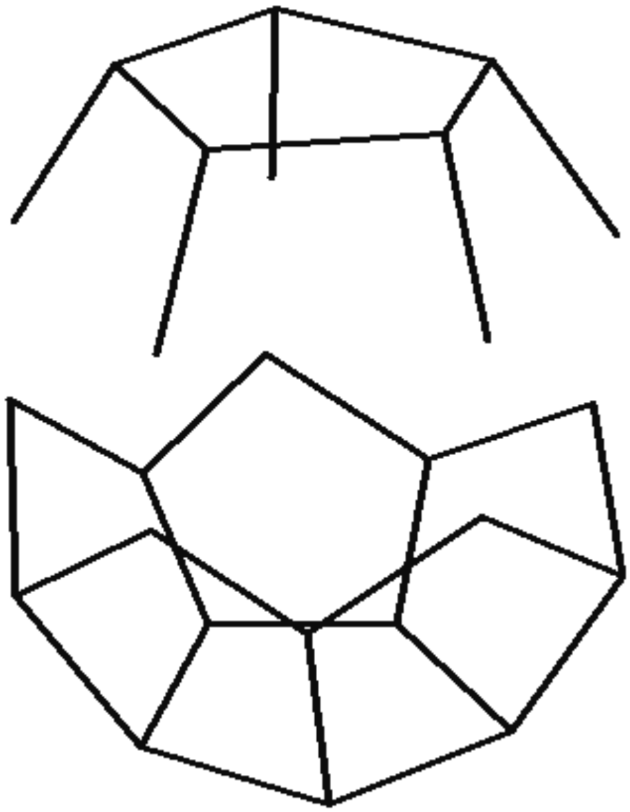


A dodecahedron has 12 faces, how many edges does it have?



Solution

If we dismantle a wire frame dodecahedron in the following way:



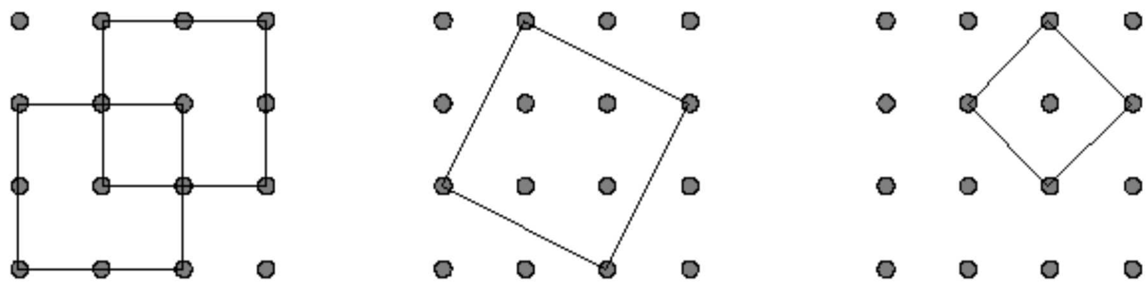
The top section is made up of 10 edges; a pentagon (5 edges) plus five. The bottom section is made up of exactly the same, plus middle ring of 10 edges. That is, a dodecagon has 30 edges.

How many edges does an icosahedron (20 faces) have?
What about the other Platonic solids?
Can you find a connection between the number of faces, edges and vertices?
Investigate other solids.

DOTTY SQUARES

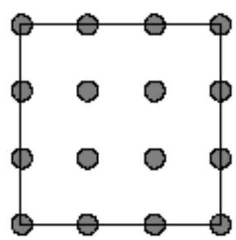
Problem

By joining dots together, how many squares can you draw on a grid measuring 4 dots by 4 dots?

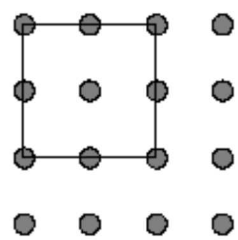


Solution

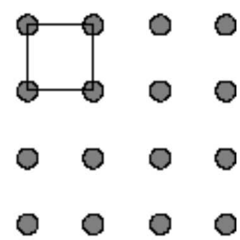
Consider the following diagrams.



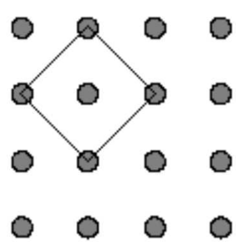
1



4



9



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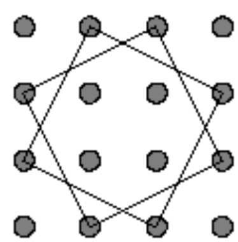


2

So there would be 1 + 4 + 9 + 4 + 2 = 20 squares that can be drawn on a 4 × 4 grid.

What about 5 × 5 grids?

Can you generalise for any size grid?

DOUBLE AN ODD SUM

Problem

Consider the following results:

$$4\,(1) = 1 + 3$$
$$4\,(1 + 3) = 1 + 3 + 5 + 7$$
$$4\,(1 + 3 + 5) = 1 + 3 + 5 + 7 + 9 + 11$$
$$4\,(1 + 3 + 5 + 7) = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15$$

If S_n represents the sum of the first n odd numbers, prove that $4S_n = S_{2n}$.

Solution

We shall prove the result using two informal methods.

Method 1
Begin by representing the series 1 + 3 + 5 + 7 + 9 + 11 diagramatically.

O

O O O

O O O O O

O O O O O O O

O O O O O O O O O

O O O O O O O O O O O

In general, S_{2n} is represented by a triangle with $2n$ rows. It is clear that the top half of triangle, which represents S_n, is one quarter of the triangle. Hence $S_{2n} = 4S_n$.

Method 2
Consider the following diagram which represents the sum of the first four odd numbers: 1 + 3 + 5 + 7.

O O O O

O O O O

O O O O

O O O O

This leads to the result, $S_n = n^2$.

Therefore $S_{2n} = (2n)^2 = 4n^2 = 4S_n$.

EGYPTIAN DIVISIBILITY

Problem

A group of Archaeologists discovered some simple hieroglyphs on the stone lid of a tomb in Egypt. When they translated them they realised that it was a four digit number, but more remarkably it is the smallest number that can be divided by all of the numbers from 1 to 10 without any remainder. What was that number?

Solution

By considering the prime factors of each of the numbers from 1 to 10:

2 = 2
3 = 3
4 = 2x2
5 = 5
6 = 2x3
7 = 7
8 = 2x2x2
9 = 3x3
10 = 2x5

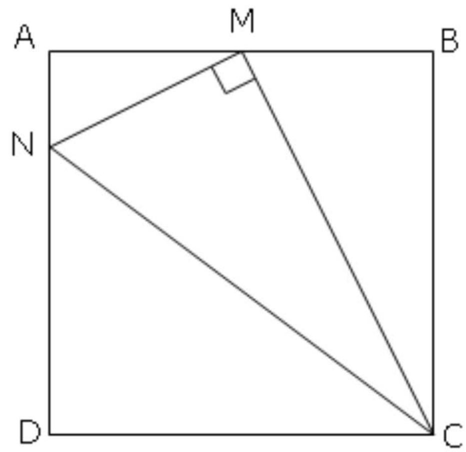
We can deduce that the smallest number which evenly divides 2, 3, 4, ... , 10, must be $2 \times 3 \times 2 \times 5 \times 7 \times 2 \times 3 = 2520$.

Alternatively we can consider which factors from 1 to 10 are necessary. A factor of 8 deals with 2, 4 and 8. 9 deals with 3 and 9. So $9 \times 8 = 72$, which immediately deals with 6 (shared factors of 2 and 3), but we still need this number to be divisible by 5, 7 and 10. As 72 contains a factor of 2, using a factor of 5 makes it divisible by 10. Hence the smallest number must be $72 \times 5 \times 7 = 2520$.

EQUAL ANGLES

Problem

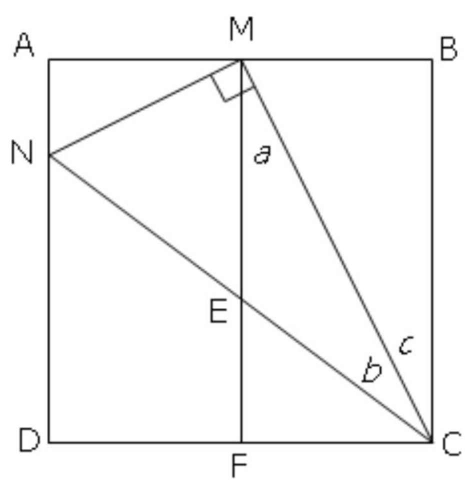
In the square ABCD, M is the midpoint of AB.
A line is drawn through M perpendicular to CM to locate N.



Prove that the size of angle BCM is equal to the size of angle MCN.

Solution

We begin by drawing a line through M and parallel to BC to produce E on CN and F on CD.



As right angle triangle CMN is half of a rectangle, CN is one diagonal, and because E is the midpoint of CN, CE = EN = EM.

In which case, triangle CEM is isosceles, $a = b$ (base angles equal).

Because MF is parallel to BC, $a = c$ (alternate angles).

Hence $b = c$, and we prove that the size of angle BCM is equal to the size of angle MCN.

If AB=4, find the perimeter of triangle CDN.

EXPRESSING DIVISIBILITY

Problem

The sum of the first n squares, $1^2 + 2^2 + ... + n^2 = n(n + 1)(2n + 1) / 6$.

For example, $1^2 + 2^2 + ... + 10^2 = 10 \times 11 \times 21 / 6 = 385$.

Prove that $n(n + 1)(2n + 1)$ is divisible by six for all integer values, n.

Solution

To prove that $n(n + 1)(2n + 1)$ is divisible by 6 we must show that it is divisible by both 2 and 3.

As n and $n + 1$ are consecutive integers then exactly one of them will be even and thus divisible by 2.

If neither n nor $n + 1$ is divisible by 3 then $n + 2$ will be. Therefore $2(n + 2) = 2n + 4$ and $2n + 4 - 3 = 2n + 1$ will be divisible by 3.

Hence $n(n + 1)(2n + 1)$ is divisible by both 2 and 3. **Q.E.D.**

FACTORIAL SYMMETRY

Problem

Given that a and b are positive integers, solve the following equation.

$$a!b! = a! + b!$$

Solution

If we divide through by $b!$ we get, $a! = a!/b! + 1$.

As LHS is integer, $a!/b!$ must be integer, and it follows that $b! \leq a!$.

If, instead, we divide through by $a!$ we get, $b! = 1 + b!/a!$, and in the same way we deduce that $a! \leq b!$.

Hence $a! = b!$, giving the unique solution $a! = 2 \Rightarrow a = b = 2$.

Alternatively, divide the original equation by $a!b!$ to give $1 = 1/b! + 1/a!$. Clearly $a! = b! = 2$.

Related problems:

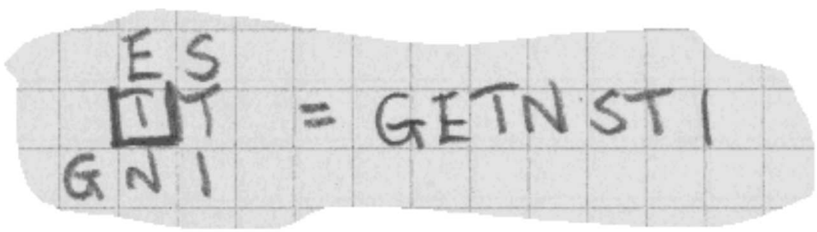
> Factorial And Power Of 2: $a!b! = a! + b! + 2^c$
>
> Factorial Equation: $a!b! = a! + b! + c!$
>
> Factorial And Square: $a!b! = a! + b! + c^2$

FAVOURITE MATHEMATICIAN

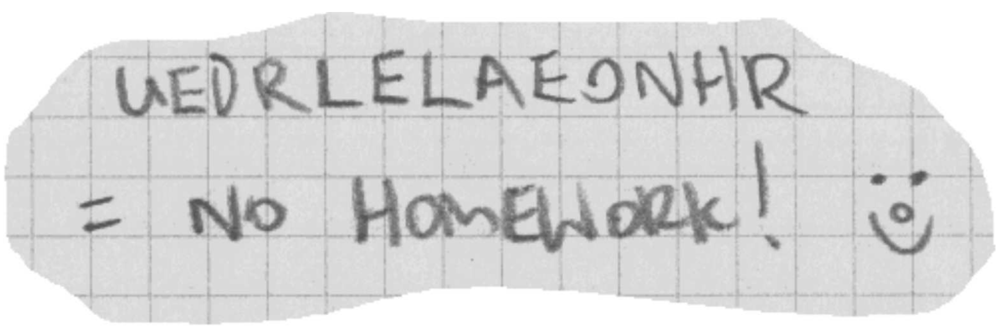
Problem

During the first lesson with your new and rather eccentric mathematics teacher he continues his tradition of telling the class that if anyone can guess his favourite mathematician they will be let off homework for the rest of the year. However, if you make a wrong guess you are duty bound to wash his car every week for the rest of the year. In all the years he has taught, the sparkle on his car has been the envy of every other teacher. Despite this seemingly impossible challenge, one person has managed to work it out... your older brother!

Of course he is not going to tell you the answer, but after a little "research" you find two scraps of paper hidden inside one of his books. The first piece seems to hold a clue.



The second piece is obviously the answer you are seeking.



Solution

The encryption process is simple. Starting inside the centre box, the plain text is written up and outward in a clockwise spiral. Then the cipher text is formed by reading down the columns.

The secret of decryption is to produce a spiral of numbers, starting at 1 and counting up to the number of letters in the cipher text. In our cipher, the number of letters is 13:

10	11	12	13
9	2	3	
8	**1**	4	
7	6	5	

This provides a map to overlay the cipher, UEDRLELAEONHR, and it is then a simple matter of writing it down the columns:

U	L	E	R
E	E	O	
D	**L**	N	
R	A	H	

Hence the name we seek is, of course, *Leonhard Euler*.

Why is this system unsuitable for longer messages?
E.g. HTIWMETACUOLHNAMYBISYBEOSEEAPRSYSTEM.

What could you do to develop it into a more secure system?

Finishing On 150

Problem

In order to win a game of darts, a player must finish on exactly zero and their last dart must land in a double or hit the bull's eye.

For example, if a player hit treble twenty (60 points), double twenty (40 points) and the bull's eye (50 points) they would score 150 points and could use this combination to finish.

How many ways can a player finish from 150 points with three darts?

Solution

If we finish on a bull's eye: 150 – 50 = 100, we must make 100 with the first two darts: 40,60,50 / 50,50,50 / 60,40,50

The highest double to finish on is $2\times 20 = 40$ and 150 – 40 = 110; which can be made in two ways: 50,60,40 / 60,50,40

The highest score possible with first two darts is $2\times 60 = 120$, so the lowest double is 150 – 120 = 30, i.e. double 15.

With the exception of finishing on a bull's eye the first two darts must score over 100, which means they must both be trebles and so the sum of their scores will be divisible by 3. As 150 is divisible by 3, the double must also be divisible by 3.

Double 15, 150 – 30 = 120: 60,60,30
Double 18, 150 – 36 = 114: 54,60,36 / 60,54,36 / 57,57,36

And so there are 9 ways of finishing on 150.

What is the highest score it is possible to finish on in a game of darts?

Investigate the number of ways of finishing on different scores.

FIVE-DIGIT DIVISIBILITY

Problem

Using each of the digits 1, 2, 3, 4, and 5 exactly once to form 5-digit numbers, how many are divisible by 12?

Solution

First we note that $12 = 3 \times 4$; in other words, if a number is divisible by 12 it will be divisible by 3 and 4.

For a number to be divisible by 3, the sum of its digits must be divisible by 3: $1 + 2 + 3 + 4 + 5 = 15$. As 15 is divisible by 3, ALL 5-digit numbers made up of the digits 1, 2, 3, 4, and 5 will be divisible by 3.

For a number to be divisible by 4, the last two digits must divide by 4.

In our case, the number must end in: 12, 24, 32, or 52.

If the number ends in 12, the first three digits will be 3, 4, and 5, and there are exactly six ways of arranging these numbers: 345, 354, 435, 453, 534, 543.

Similarly, if the number ends in 24, 32, or 52, the first three digits will be 135, 145, and 134 respectively.

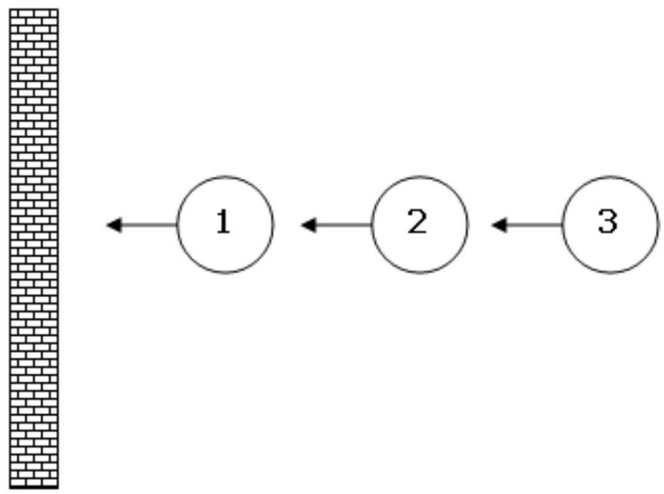
As there are six ways of arranging three digits digits: abc, acd, bac, bca, cab, cba, there are $4 \times 6 = 24$ different 5-digit numbers that are divisible by 12.

Investigate the number of *n*-digit numbers, made up of the digits 1 to *n*, that are divisible by 12.
What about being divisible by 15, or 18?
Investigate different divisors.

Four Hats

Problem

A teacher shows three clever boys a pile of four hats: two blue and two red. They are all blind-folded, each given a hat to wear at random, and lined-up in a line so that they are all facing towards a wall. When the blind-folds are removed, the first boy can only see the wall, the second boy can see the first boy, and the third boy can see the first two boys. None of the boys can see the colour of his own hat.



The first boy to correctly shout out the colour of his own hat will have no homework that evening. However, if he guesses incorrectly, he will have to complete the other boy's homework.

Which of the boys, 1 to 3, is most likely to correctly deduce by logic alone the colour of his hat?

Solution

There are exactly six ways in which the four hats can be allocated:

Option	Boy 1	Boy 2	Boy 3	Not Used
1	B	B	R	R
2	B	R	B	R
3	B	R	R	B

4	R	B	B	R
5	R	B	R	B
6	R	R	B	B

If boy 3 can see two blue hats (option 1) or two red hats (option 6) then he will know the colour of his own hat and quickly shout out the colour. Therefore the probability that boy 3 correctly deduces the colour of his own hat is 2/6 = 1/3.

If boy 3 sees one blue and one red hat, he will not be able to determine the colour of his own hat, so if there is a long silence then boy 2 can deduce that boy 3 is unable to see two red hats or two blue hats. He can then further deduce that he is wearing the "opposite" colour to boy 1. Therefore the probability that boy 2 correctly deduces the colour of his own hat is 4/6 = 2/3.

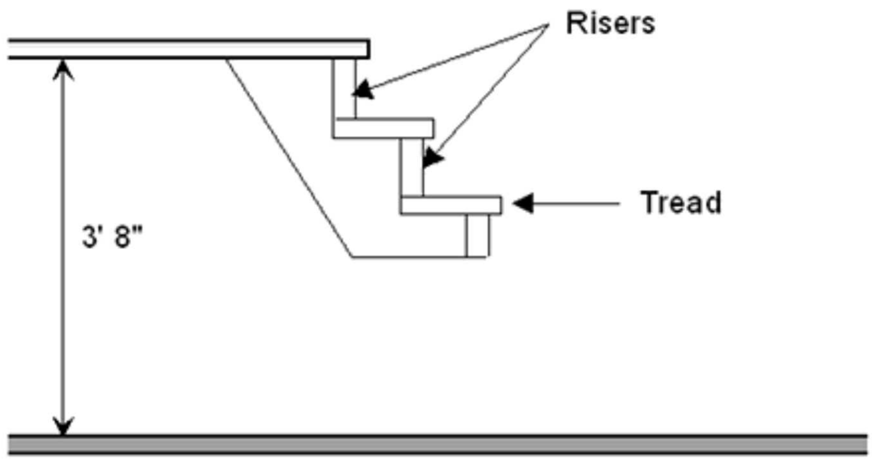
Boy 1 will never be able to guess the colour of his hat, so it is the second boy that is most likely to escape homework this evening.

Of course, if boy 3 could see two red hats or two blue hats and was feeling particularly mean he could deliberately say nothing, knowing that boy 2 would (wrongly) deduce the colour of his own hat.

FRACTIONAL STEPS

Problem

A set of steps leading up to a platform are to be constructed. The platform is 3' 8" above the ground and each tread has a fixed thickness of $1^1/_8$". For safety and comfort each riser should measure between 5 and 7 inches and practically cannot be cut more accurate than $^1/_8$".



How many risers should be used and what size should they be?

Solution

From the bottom to the top each step comprises of a riser and a tread. However, the top step will have no tread as this is where a riser meets with the platform. To make calculations easier, and so as to work with a whole number of complete steps, we shall imagine that the top step includes a tread. Working in inches, 3' 8" = 44", so the total drop shall be considered to be $45^1/_8$".

Given the requirement that each riser must be between 5 and 7 inches, the height of a single step must be between $6^1/_8$ and $8^1/_8$ inches.

To simplify calculations further, and to include the constraint that risers cannot be cut any more accurate than $^1/_8$", we shall use $^1/_8$" as our base unit.

Drop = $45^1/_8$" $\Leftrightarrow 45 \times 8 + 1 = 361$ units
Minimum step height = $6^1/_8$" $\Leftrightarrow$ 49 units
Maximum step height = $8^1/_8$" $\Leftrightarrow$ 65 units

For practical purposes most carpenters attempt to make each step the same height and custom cut the last riser *in situ* to accomodate any errors in previous cuts. The standard method is to divide the drop by the number of steps to determine the height of each step. This value is rounded and the overall gain/loss of using this rounded value is calculated. The carptenter can then distribute this gain/loss among the steps. However, if this difference is no more than $^1/_2$" it is common to allow the top or bottom step to make up the difference.

Working with the minimum height step, $361/49 \approx 7.4$, which is an incomplete number of steps. However, if we used eight steps then each step would be smaller and would fall below the minimum height requirement. So the maximum number of steps we can use is seven.

Similarly, working with the maximum height step, $361/65 \approx 5.6$, which means that if we tried to use 5 steps then each would be taller than the maximum requirement. So the minimum number of steps we can use is six.

6 Steps: $361/6 \approx 60.2$ and $6 \times 60 = 360$, which is 1 unit short, so we could make the top step 61 units tall.

7 Steps: $361/7 \approx 51.6$ and $7 \times 52 = 364$, which is 3 units over, so we could make the top step 49 units tall. But as this is at the limit of code it leaves no room for error. So using $7 \times 51 = 357$, which is 4 units short, and we could then make the top step 55 units tall.

Subtracting 9 units from these measurements (for the $1^1/_8$" tread) and converting back to inches we have two practical options:

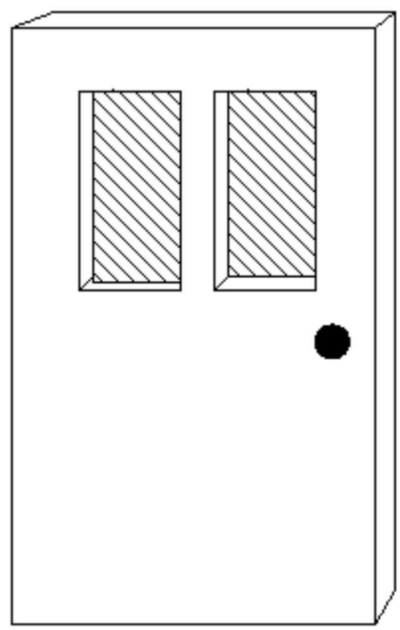
<u>Option 1:</u> 6 steps comprising of 5 risers @ $6^3/_8$" and 1 @ $6^1/_2$".

<u>Option 2:</u> 7 steps comprising of 6 risers @ $5^1/_4$" and 1 @ $5^3/_4$".

GLASS IN THE DOOR

Problem

A 52 kg piece of solid hardword measuring 3" by 6"6" and 3" thick is made into a door by cutting out two rectangular panels measuring 9" by 2"6" and replacing them with glass.



If the door and glass weigh 45 kg together, what is the weight of the glass?

Solution

Volume of solid wood = $3 \times 13/2 \times 1/4 = 39/8$ feet^3.

Assuming uniform density, 39/8 feet^3 weighs 52 kg, so 1 foot^3 weighs $52/(39/8)= 32/3$ kg.

Volume of one panel = $3/4 \times 5/2 \times 1/4 = 15/32$ feet^3, so volume of two panels = 15/16 feet^3.

Therefore, wood removed weighs, $15/16 \times 32/3 = 10$ kg.

The door without the glass will be, $52 - 10 = 42$ kg.

Hence the glass must weigh, $45 - 42 = 3$ kg.

What if the thickness of the door was 6"?

HALF FRACTIONS

Problem

By concatenating all of the digits 1, 2, 3, and 4 to form two numbers, it is impossible to find a pair that divide to make one-half. However, using the digits 1, 2, 2, and 4, it is possible to make one-half in exactly two different ways.

$$\frac{12}{24} = \frac{21}{42} = 0.5$$

By concatenating all of the digits 1, 2, 3, 4, and 5 to form the ratio of two numbers, how many ways can you make one-half?

Solution

The solution must be of the form:

$$\frac{\text{2-digits}}{\text{3-digits}} = 0.5$$

As the denominator must be exactly twice the numerator, the only 2-digit numbers we can form that double to make a 3-digit number are 51, 52, 53 and 54. However, their doubles are 102, 104, 106 and 108, respectively.

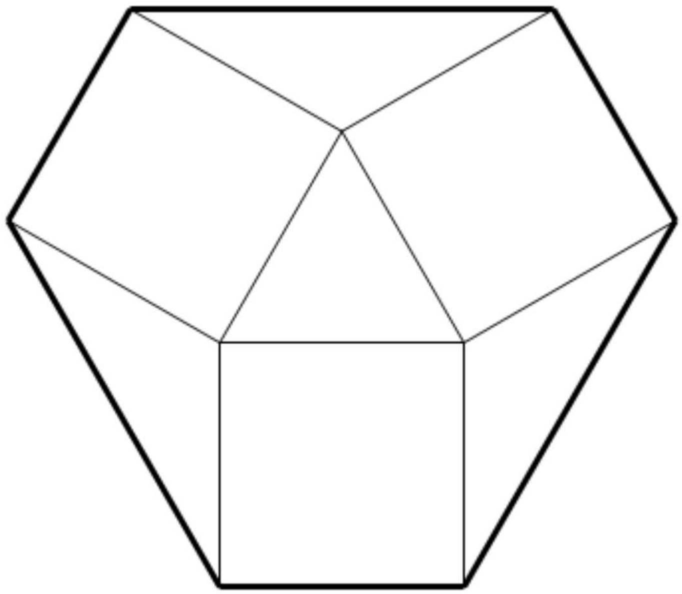
Hence there exist no solutions using the digits 1 to 5.

Investigate using the digits 1 to *n*.

HEXAGON PERIMETER

Problem

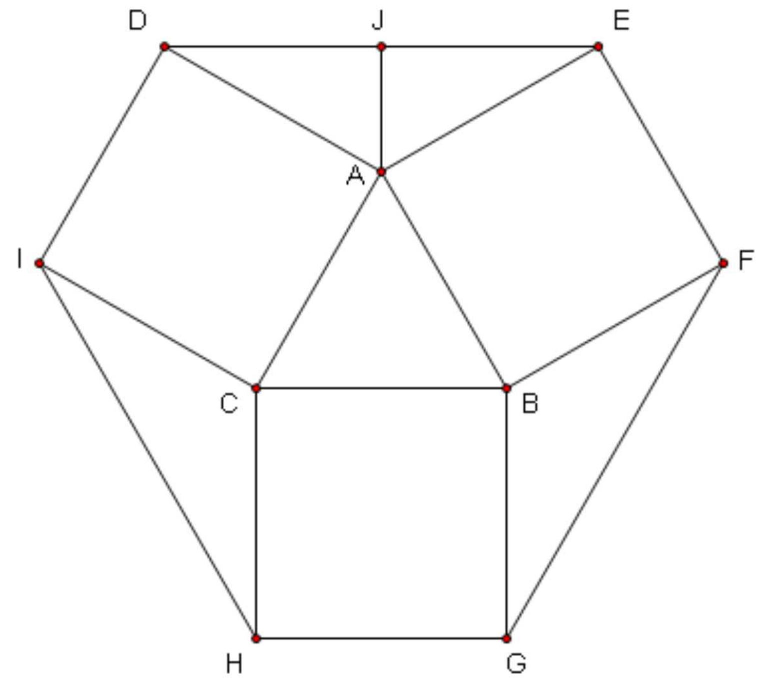
In the following diagram each of the squares have unit length sides.



Find the perimeter of the hexagon.

Solution

Consider the following diagram.



Angle DAE = 360 – (60 + 90 + 90) = 120^o ⇒ angle DAJ = 120/2 = 60^o and angle ADJ = 30^o

∴ DJ = $\cos(30^o) = \sqrt{3}/2 \Rightarrow DE = \sqrt{3}$

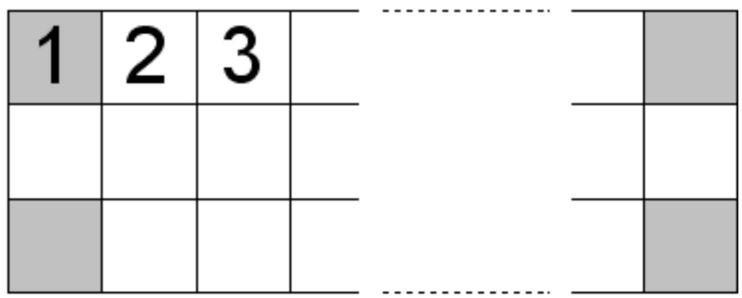
Hence the perimeter of the hexagon is $3(\sqrt{3} + 1)$.

Find the area of the hexagon.

Hockey Lockers

Problem

As the visiting year 10 hockey team were first to arrive at the changing rooms, they had their choice of lockers. Every locker is given a number in order and they are arranged in three rows, numbered 1, 2, 3, ... on the top row. The sequence continuing on the next row. Four friends quickly claimed the corner lockers,



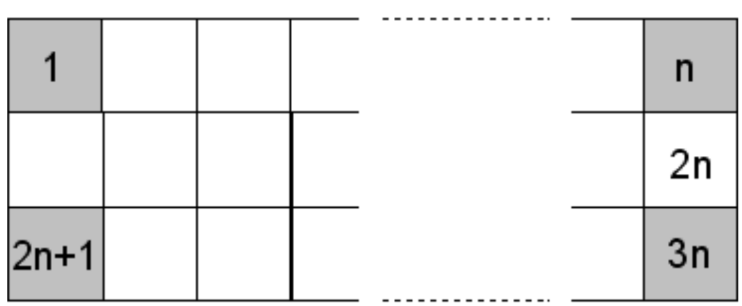
Rather amazingly the four friends found that the numbers on their lockers added up to the sum of their ages.

How many lockers are there?

Note: In England, a year 10 student will be 14 or 15 years old.

Solution

Let there be n lockers on the top row,



Therefore, the sum of four corners is $1 + n + 2n + 1 + 3n = 6n + 2$

As team are year ten they can be 14 or 15 years old.
Minimum sum = $4 \times 14 = 56$ and maximum sum = $4 \times 15 = 60$.

But using formula: 6 x 9 + 2 = 56 and 6 x 10 + 2 = 62, hence n = 9.

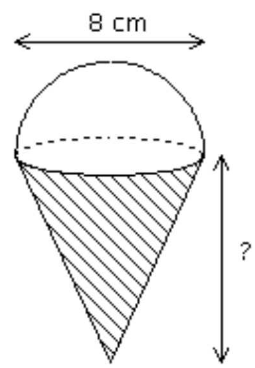
So there are $3n$ = 3 x 9 = 27 lockers.

Find the sum of the four corner lockers with 1, 2, 3, ... , m rows of n lockers.

ICE CREAM CONE

Problem

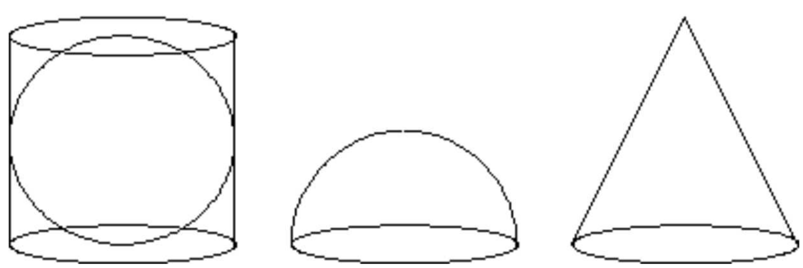
An ice cream cone is packed full of ice cream and a generous hemisphere of ice cream is placed on top.



If the volume of ice cream inside the cone is the same as the volume of ice cream outside the cone, find the height of the cone.

Solution

Using the result that a cone occupies exactly 1/3 of a cylinder with equal base and height and a sphere occupies 2/3 of the cylinder it fits inside, the hemisphere must have the same volume as 1/3 of the cylinder, i.e. the same volume as the cone.



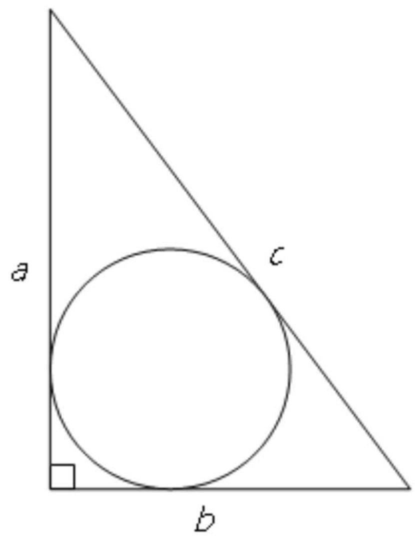
That is the cone is twice the height of the hemisphere.

Hence the height of cone is 8 cm.

INSCRIBED CIRCLE IN RIGHT ANGLED TRIANGLE

Problem

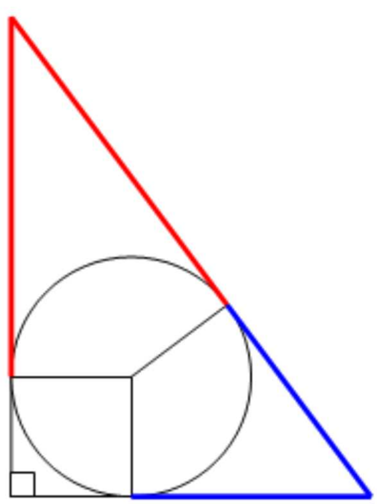
A circle is inscribed in a right angled triangle with the given dimensions.



Find the radius of the circle in terms of *a*, *b*, and *c*.

Solution

Consider the following diagram.



If we let the radius be r then the red length on the vertical edge is given by $a - r$; similarly the blue length on the horizontal edge is given by $b - r$.

Using the property that tangential distances are equal we get $a - r + b - r = c$.

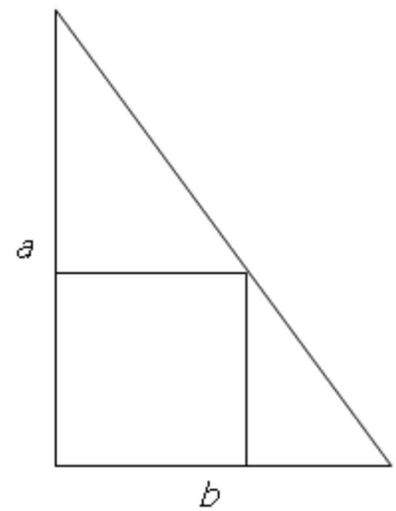
$$\therefore a + b - 2r = c \Rightarrow r = \frac{a + b - c}{2}$$

Why would this result be invalid if the triangle were not right angled?

INSCRIBED SQUARE

Problem

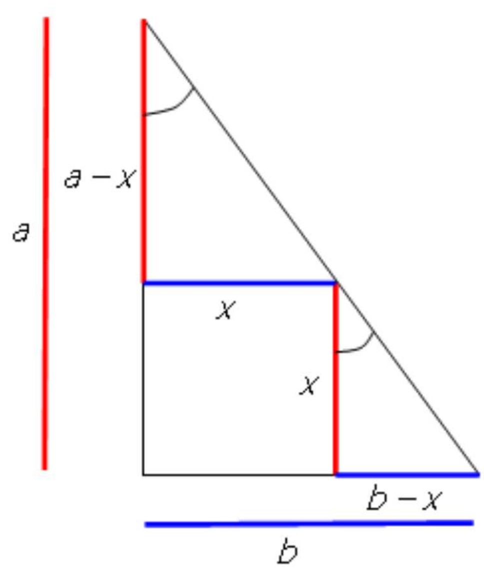
A square is inscribed in a right angled triangle with the given dimensions.



Find the side length of the square in terms of *a* and *b*.

Solution

Consider the following diagram.



As the two marked angles are corresponding we can deduce that the two triangles are similar.

$$\frac{a}{b} = \frac{x}{b-x}$$

$\therefore a(b - x) = bx$
$ab - ax = bx$
$ab = ax + bx = x(a + b)$

$$\therefore x = \frac{ab}{(a+b)}$$

OR

$$\frac{a-x}{x} = \frac{x}{b-x}$$

$\therefore (a - x)(b - x) = x^2$
$ab - (a + b)x + x^2 = x^2$
$(a + b)x = ab$

$$\therefore x = \frac{ab}{(a+b)}$$

Integer Fraction Product

Problem

Prove that n must be odd for $(1 + {}^1/_2)(1 + {}^1/_3)(1 + {}^1/_4)\ldots(1 + {}^1/_n)$ to be integer?

Solution

Begin by using the result:

$$\frac{n+1}{n} = 1 + \frac{1}{n}$$

So that we can write the product in the following way and cancel denominators,

$$\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{4}\right) \times \ldots \times \left(1 + \frac{1}{n}\right)$$

$$= \frac{3}{2} \times \frac{4}{3} \times \frac{5}{4} \times \frac{}{5} \times \ldots \times \frac{n-1}{} \times \frac{n}{n-1} \times \frac{n+1}{n} = \frac{n+1}{2}$$

Hence it will be integer when n is odd.

LARGEST ROOT

Problem

Without the use of a calculator determine which is greater in value, the square root of two or the cube root of three.

Solution

Let $x = \sqrt{2} = 2^{1/2}$ and $y = \sqrt[3]{3} = 3^{1/3}$.

$\therefore x^6 = (2^{1/2})^6 = 2^3 = 8$
$y^6 = (3^{1/3})^6 = 3^2 = 9$

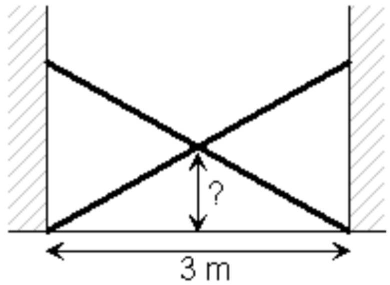
As $y^6 > x^6$ it follows that $\sqrt[3]{3} > \sqrt{2}$.

Although this approach is perfectly valid and leads to the correct solution in this context, caution must be taken when negative numbers are involved. For example, from $(-2)^2 > 1^2$ it does not follow that $-2 > 1$.

LEANING LADDERS

Problem

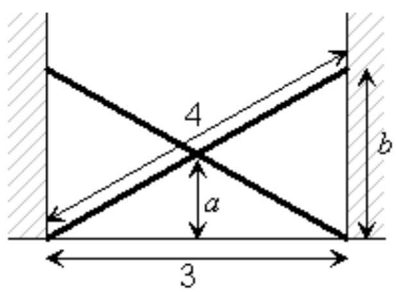
Two ladders, both 4 metres in length, are leaned up against opposite walls in a corridor, 3 metres wide, as shown in the diagram.



How far above the ground do the two ladders cross?

Solution

By symmetry the intersection point is half the height the ladders reach up the opposite wall.



Using the Pythagorean Theorem, $4^2 = b^2 + 3^2 \Rightarrow b = \sqrt{7}$

So, $a = \sqrt{7}/2 \approx 1.32$ metres.

What if the two ladders are different lengths? Try 4 metres and 5 metres.

LETTER PRODUCT

Problem

Let $a = 1$, $b = 2$, $c = 3$, ... , $z = 26$. What is the exact value of $(x - a)(x - b)(x - c)\ldots(x - z)$?

Solution

As one of the factors is $(x - x)$, the value of the product must be zero!

LUCKY GUESS

Problem

Jane shows John four playing cards: Ace of Spades, Two of Clubs, Three of Diamonds, and Four of Hearts. She shuffles the cards and places them face down on a table.

"I would like you to select two cards at random. If you select two cards of the same colour, I'll give you £1, if they're different, you give me £1. As there are two possible outcomes, we both stand an equal chance of winning £1." suggests Jane with a cheeky smile.

John's friend, James, secretly advises John, "Actually, there are three possible outcomes: black and red, black and black, red and red. As two of these outcomes is a win for you, I"d go for it!"

By finding the actual probability of John winning, show that neither Jane nor James are correct.

Solution

The probability of picking two reds, P(RR) = (2/4)(1/3) = 2/12; similarly P(BB) = 2/12.

Therefore, P(same colour) = 4/12 = 1/3.

Alternatively, it doesn't matter whether the first card is red or black,
P(2nd card is the same) = 1/3.

What if there was one red and three black cards?

MAGIC SQUARES

Problem

The numbers 1 to 9 are placed in a 3 by 3 grid in the following way.

1	9	5
8	4	3
6	2	7

The sum of each row and column is 15, however, the diagonal running from top left to bottom right adds to 12.

Show how you can arrange all of the numbers 1 to 9 in a square grid measuring 3 by 3, so that the sum of every row, column, and diagonal is the same.

Solution

We begin by writing representing the value of each square with a letter.

a	b	c
d	e	f
g	h	i

We shall say that the sum of any given row, column, or diagonal is, T.

The sum, $a + b + \ldots + i$, although will not be in the same order, is equivalent to, $1 + 2 + \ldots + 9 = 45$.

If we add all of the rows together, we get,
$(a + b + c) + (d + e + f) + (g + h + i) = 3T$.

Therefore, $3T = 45$ Þ $T = 15$; that is, the "magic" sum of any given row, column, or diagonal will be 15.

Consider the following sum,

$(a + e + i) + (d + e + f) + (g + e + c) + (b + e + h) = 4T = 60$.

We can rearrange this to give, $(a + b + c + d + e + f + g + h + i) + 3e = 60$.

Therefore, $45 + 3e = 60$, $3e = 15 \Rightarrow e = 5$; in other words, the centre square will be 5.

As a result, if a square on one side of the middle square is, $5 + x$, the square on the opposite side will be $5 - x$. Using this idea and the guiding principle that each row, column, and diagonal must add to 15, we can now re-write the grid in the following way.

$5 + x$	$5 - x + y$	$5 - y$
$5 - x - y$	5	$5 + x + y$
$5 + y$	$5 + x - y$	$5 - x$

Clearly, $x \neq y$ (otherwise top left and bottom left would be the same), and as we are dealing with the digits, 1 to 9, $x + 5 \leq 9 \Rightarrow x \leq 4$. However, from the middle right square we must also ensure that $5 + x + y \leq 9$, $x + y \leq 4$. So if $x = 1$, $y = 2, 3$; by symmetry this is equivalent to $y = 1$, $x = 2, 3$.

As top left and top middle squares cannot be equal, $5 + x \neq 5 - x + y$, and so $2x \neq y$.

Hence the only solution is $x = 1$, $y = 3$, producing the following magic square.

6	7	2
1	5	9
8	3	4

Although it is possible to rotate and/or reflect this grid, all variations will all be isomorphic (equivalent).

Extensions

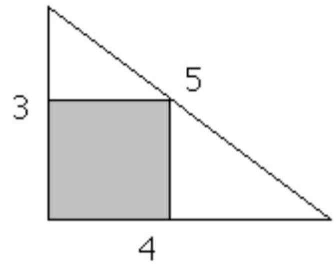
- What about a 4 by 4 magic square, using the numbers 1 to 16?

- If there was no restriction on the numbers you could place in the 3 by 3 grid, what must be special about the magic total, T, so that all of the numbers in the grid are integer?
- What value would the middle square contain?
- For a given total, T, can you always find a magic square so that the grid contains different positive integers?
- For a square grid, measuring n by n, can you find a formula for the magic total, T?
- Can you find a 3 by 3 magic square, such that all the numbers are irrational?

MAXIMUM SQUARE

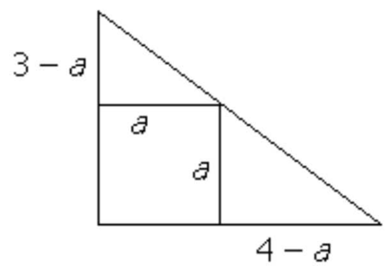
Problem

What are the dimensions of the shaded square inside the 3-4-5 right angle triangle?



Solution

Let length of the side of the square be, *a*.



By similar triangles, $^{(3-a)}/_{a} = {}^{a}/_{(4-a)}$

$\therefore (3-a)(4-a) = a^2$

$a^2 - 7a + 12 = a^2$

$7a = 12 \Rightarrow a = 12/7$

Find the position of the square that has the maximum area.
Can you generalise for an *x-y-z* right angle triangle?
What would be the dimensions of the square in terms of *x, y*, and *z*?

Meaning Of Life

Problem

You were always amazed by your German grandmother's knowledge. Whether it be history, languages, philosophy, science, mathematics or theology, she seemed to know everything. What impressed you most was the sense of contentment and peace that seemed to surround her. Through her lifelong search for knowledge she seemed to have found the answers that most of us never quite find.

Sadly she passed away, but a few days later you receive a letter and computer disc in the post:

> *My dear child,*
> *If you are reading this letter then I am no longer with you. I have sent a copy of this letter to each of my relatives. The accompanying disc contains the IP address of a remote server on the internet, which contains all of the discoveries and great secrets I have uncovered during my lifetime. They have been securely encrypted and can be unlocked with a single password. However, if anyone attempts to access the server they will have one minute to transmit the correct password or all of the data will be destroyed.*
>
> *So as to not cast pearls under the feet of swine you will need to discover the correct password for yourself...*

B	C		A		H	A	H		H		E		I	B	

Solution

Begin by writing the alphabetic position of each character below the cipher.

B	C		A		H	A	H		H		E		I	B	
2	3	0	1	0	8	1	8	0	8	0	5	0	9	2	0

Then read-off pairs: 23 01 08 18 05 09 20.
These represent the alphabetic positions of the message:
W A H R H E I T.
Which is, of course, German for TRUTH.

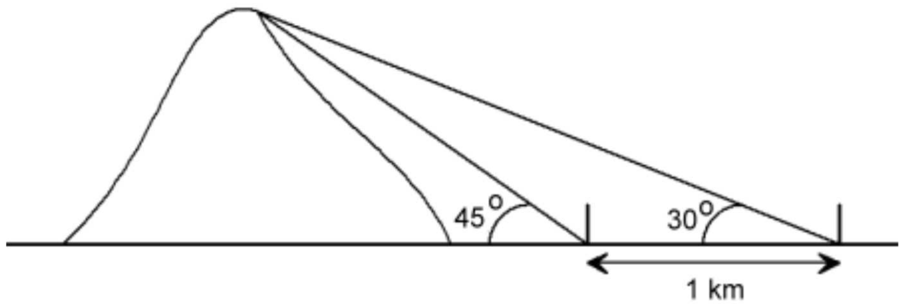
If you keep the message in English, can you see any flaws with this method of encryption? For example, try encoding HELLO.

What could be done to overcome this problem?

MEASURING MOUNTAINS

Problem

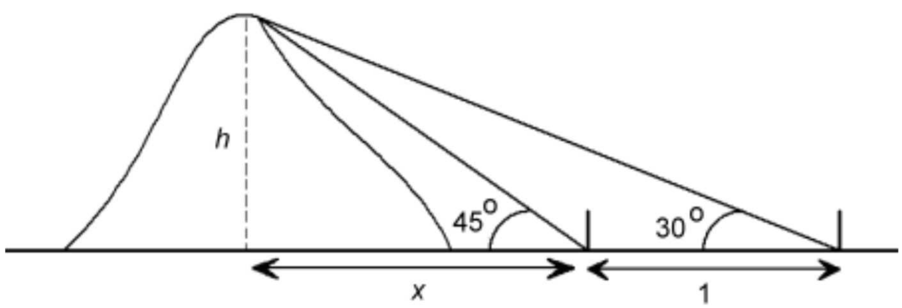
A geographical surveyor places a marker at a position such that the angle between the horizontal and the top of a mountain is 45^o, after driving 1 km away from the mountain, the angle between the horizontal and the top of the mountain is 30^o.



Use the given information to find the height of the mountain.

Solution

Consider the diagram:



In the 45^o triangle, it is evident that $x = h$.

In the 30^o triangle, $\tan 30^o = 1/\sqrt{3} = h/(x + 1)$, leading to $x + 1 = \sqrt{3}h$.

Therefore, $h + 1 = \sqrt{3}h$, $1 = \sqrt{3}h - h$, so $h(\sqrt{3} - 1) = 1$.

Hence, $h = 1/(\sqrt{3} - 1) \approx 1.36$ km.

Missing Weight

Problem

The weights of ten boys are recorded in a stem and leaf diagram.

Weight (kg)

4	7
5	2 4 7 9
6	1 4 6 8
7	2

Key: 5 | 2 means 52 kg

Although the mean of the data is correct, two of the entries had their digits reversed and 75 kg was accidentally written down as 57 kg. Given that the 47 kg is not really the minimum weight, how heavy is the lightest boy?

Solution

The sum of the data in the stem and leaf diagram is 600. If 57 kg is incorrect, 600 – 57 = 543 and 543 + 75 = 618. As the mean is correct, the new total should also be 600, therefore the difference between the other incorrect value and the reverse of its digits must be 18. As we need to lose 18, the incorrect number must be greater than its reverse and the difference between the digits must be 2.

We can see that 64 has this property. That is, 618 – 64 = 554 and 554 + 46 = 600 and so we deduce that the lightest boy weighs 46 kg.

Why did the difference between the digits have to be 2?
Investigate the difference between a 2-digit number and its reverse.

MODEST AGE

Problem

A teacher, attempting to avoid revealing his real age, made the following statement:

> *"I'm only about twenty-three years old if you don't count weekends."*

Can you work out how old the teacher really is?

Solution

If weekends are not counted, the given age only represents $^5/_7$ of his actual age.

Using this information:

$^5/_7$ of 31 = 22.142... ≈ 22
$^5/_7$ of 32 = 22.857... ≈ 23
$^5/_7$ of 33 = 23.571... ≈ 24

So we can deduce that the teacher is about 32 years old.

Be careful, though. You may be thinking, "Why can't I just work out $^{23}/_5 \times 7 = 32.2$?"

Suppose that the teacher claimed to be about 24 years old; $^{24}/_5 \times 7 = 33.6$, and this would suggest that he was really 34 years old. However, consider what happens if we work the other way:

$^5/_7$ of 33 = 23.571... ≈ 24
$^5/_7$ of 34 = 24.285... ≈ 24

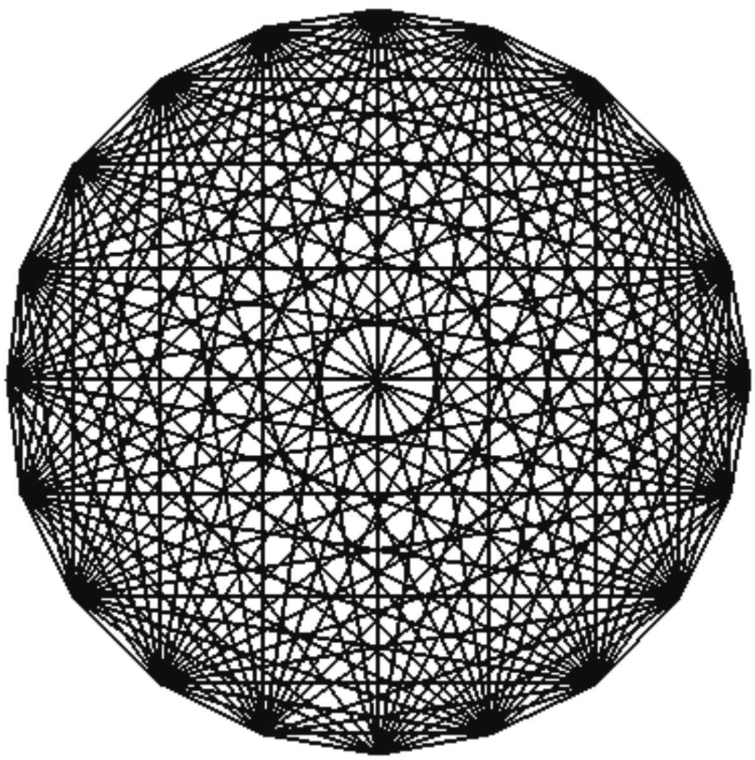
In other words, the ages 33 and 34 both map onto the reduced age of 24, therefore it is impossible to solve the problem in this particular case.

By investigating real ages mapped onto "reduced" ages, can you discover which ages produce unique values and which ages have more than one value?

Mystic Rose

Problem

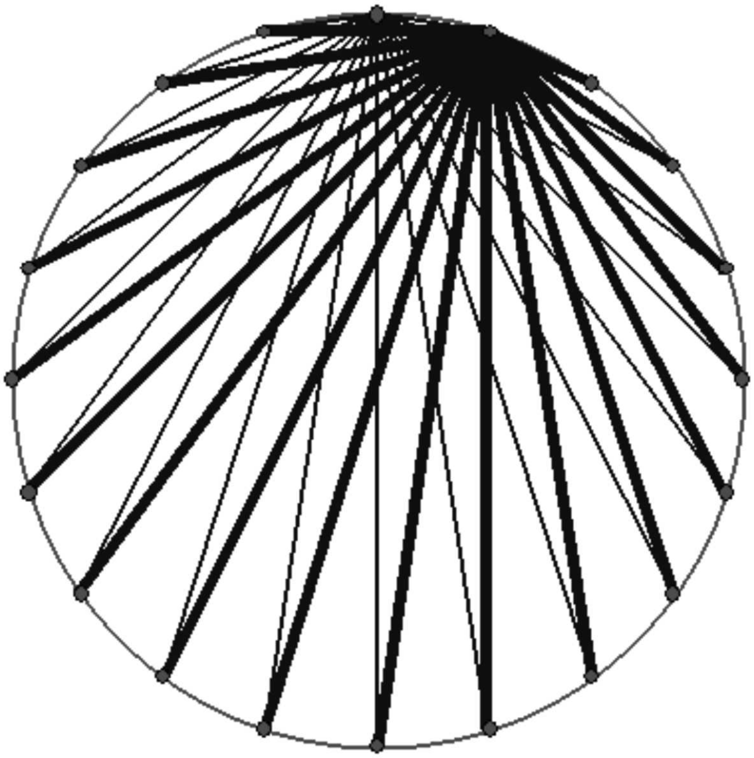
A Mystic Rose consists of twenty points evenly spaced around a circle and every point is joined to every other point.



How many lines are required to construct the Mystic Rose?

Solution

Starting at the first point, it must be joined to nineteen other points. The second point only needs to be joined to eighteen points now. The third point needs to join seventeen points and so on.



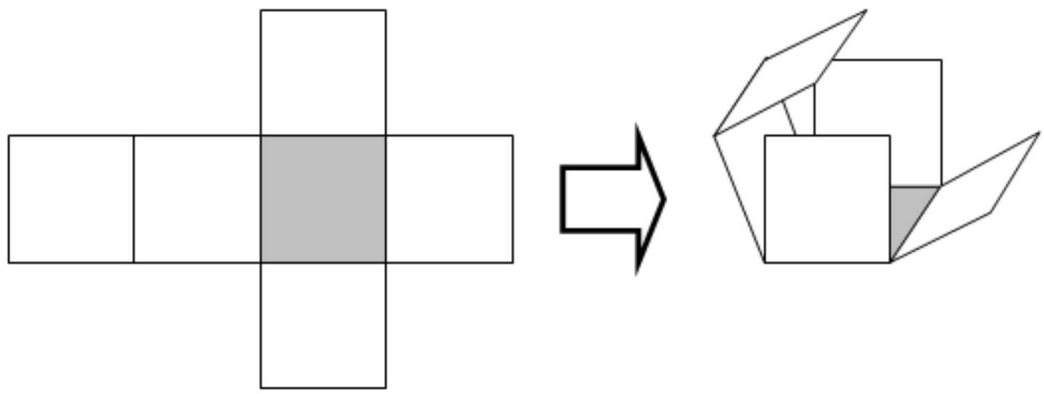
Number of lines, L = 19 + 18 + 17 + ... + 1 = $\frac{1}{2} \times 19 \times (19 + 1) = 190$.

It can be seen that the 1st line is drawn between points 1 and 2. The 2nd line is drawn between points 1 and 3, and so on. Between which two points is the 50th line drawn? What about for the *n* th line?

NET PERIMETER

Problem

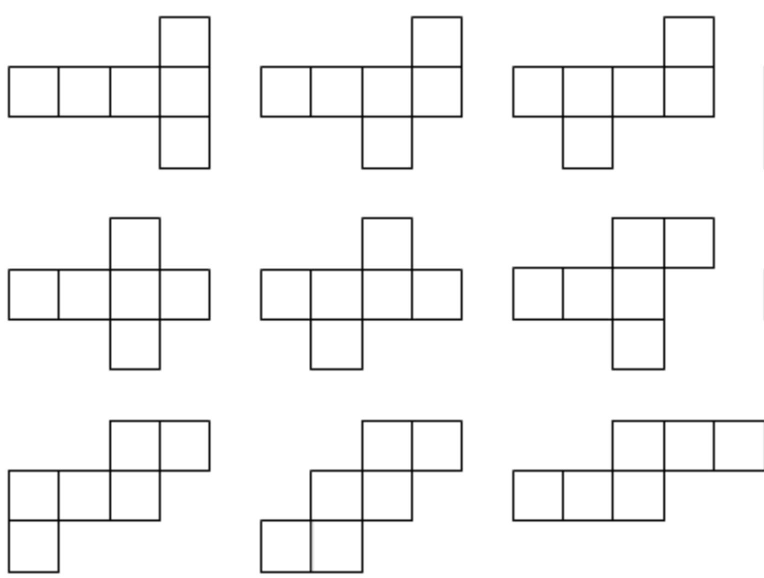
The following net can be folded-up to produce a cube and it can verified that the perimeter of this particular net is 14 units.



Altogether there are eleven distinct nets that can fold to produce a cube. Which of these nets has the maximum perimeter?

Solution

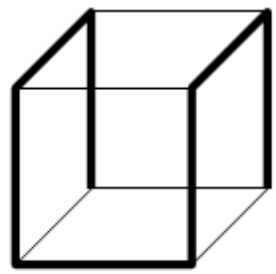
Here are the eleven nets that will fold to produce a cube.



It might seem surprising that all of the nets have exactly the same perimeter: 14 units.

In fact, this result can be confirmed by approaching the problem in a slightly different way.

Imagine starting with a solid cube. In order to flatten the solid it would be necessary to cut along its edges. For example, take a moment to convince yourself that the cube below would unfold to produce the net found in the top left diagram above: the bottom square would remain fixed on the surface, the left and right faces would drop down, and the remaining squares would uncurl out towards the rear.



It should be clear that in producing a properly joined net you would not be allowed to form a "loop" when cutting along the edges. That is, you could not make a cut return to a previously visited corner, otherwise it would cause a square to become separated.

Similarly it would be necessary for each cut to be connected to a previous cut, and each vertex (corner) must be visited by at least one cut, otherwise there would be three edges at a vertex that could not be flattened out.

In graph theory we call the type of path our cutting must follow a spanning tree: a fully connected series of edges (cuts) that visits each vertex once and forms no loops. It can be seen then that each possible spanning tree corresponds with one of the eleven distinct nets above.

As there are eight vertices in a cube, such a tree would comprise of seven edges. And as each cut exposes two edges on the perimeter of the net we confirm that all possible nets would have a perimeter of 14 units.

Never Divides By 5

Problem

Given that x is a positive integer prove that f(x) = $x^2 + x + 1$ will never divide by 5.

Solution

When x is divided by 5 the possible remainders are 0, 1, 2, 3, and 4. The respective remainders of x^2 will be 0, 1, 4, 9 $\equiv$ 4, and 16 $\equiv$ 1. This can be seen more clearly in a table.

mod 5

x	x^2	$x^2 + x + 1$
0	0	$0 + 0 + 1 = 1$
1	1	$1 + 1 + 1 = 3$
2	4	$2 + 4 + 1 = 7 \equiv 2$
3	4	$3 + 4 + 1 = 8 \equiv 3$
4	1	$4 + 1 + 1 = 6 \equiv 1$

Hence we show that f(x) will never divide by 5.

Show that the result holds even if x is a negative integer.
Investigate which other values of n that will never divide into f(x).

NUMBERED DISCS

Problem

Two discs have four different numbers written on them.

As the discs are spun their totals are added together, e.g. 5 + 8 = 13. By spinning the discs the totals 10, 11, 13, and 14 are obtained.

Use this information to work out which numbers are on the other side of the discs.

Solution

Let the numbers on the first disc be 5 and a (on the reverse), and the second disc be 8 and b. As we already know that 5 + 8 = 13, there are three algebraic totals: $a + b$, $a + 8$, and $5 + b$, that need to be matched, in some order, with: 10, 11, and 14.

The remaining three possible totals add to 10 + 11 + 14 = 35.

So $(a + b) + (a + 8) + (5 + b) = 2a + 2b + 13 = 35$,
therefore $2a + 2b = 22$, hence we know that $a + b = 11$.

This leaves two possible totals for $5 + b$: 10 and 14. As all of the numbers are different, $5 + b = 14$, leading to $b = 9$ and $a = 2$.

That is, the numbers 2 and 5 are on the first disc and 8 and 9 are on the second disc.

What if the totals were 11, 12, 13, and 14?

NUMBERED DISCS 2

Problem

Using discs numbered 1 to 6, it is possible to place the discs in two bags, such that no bag contains its double in exactly four different ways:

{1,3,4}	{2,5,6}
{1,3,4,5}	{2,6}
{1,4,6}	{2,3,5}
{1,4,5,6}	{2,3}

If you had discs numbered 1 to 10, how many ways can you separate the discs into the two bags such that no bag contains its double?

Solution

If 1 is placed in one bag, 2 must be placed in the other bag. Now 4 must be placed in the first bag, and 8 must be placed in the second bag. Without loss of generality let us say that the first bag contains {1,4}, and the second bag contains {2,8}.

It is clear that 3 and 6 must be placed in separate bags, as must 5 and 10.

Regardless of how we arrange the other discs, 7 and 9 can be placed in either bag without any conflict. There are four ways we can arrange these two discs: {7} and {9}, {9} and {7}, {7,9} and {}, {} and {7,9}.

In the first bag we can have 3 and 5, 3 and 10, 6 and 5, or 6 and 10.

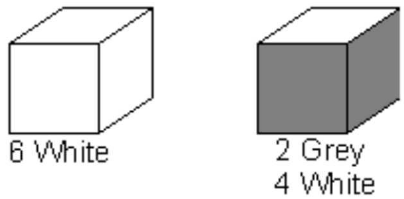
As each of these arrangements has four ways in which we can arrange 7 and 9, there are exactly $4 \times 4 = 16$ different ways in which we can fill the two bags such that no bag contains its double.

Investigate filling the bags with numbers 1 to *n*.

PAINTED CUBES

Problem

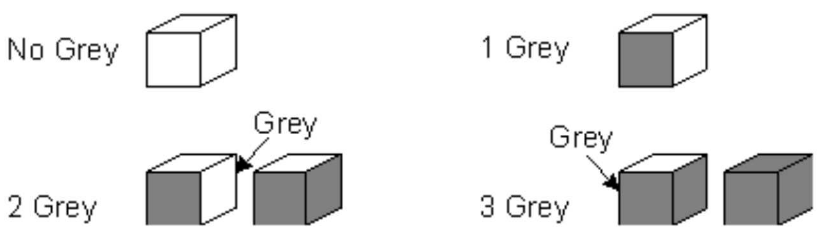
A wooden cube has each of its six faces painted either white or grey.
For example,



How many different cubes can be made?

Solution

Let us consider the number of grey faces painted.



4 grey faces will have 2 white faces and so is equivalent to 2 grey faces, similarly 5 grey faces is equivalent to 1 grey face and 6 grey faces is equivalent to no grey faces.

Number of different way	1	1	2	2	2	1	1
Number of grey/white faces	0/6	1/5	2/4	3/3	4/2	5/1	6/0

So there are 1 + 1 + 2 + 2 + 2 + 1 + 1 = 10 ways to paint a cube with two colours.

How many ways can you paint a cube with three colours?

PASSWORD CRACKER

Problem

Whilst surfing you happen to stumble upon a "hackers" website that is part of a ring of thirteen; each site linking to the next one, until returning to the first. Curiously one of them requires a password and as you cycle through the front pages of the other twelve you notice some strange figures in the bottom right corner. It was not obvious, because each time the web page designer has chosen a colour only slightly different to the background.

Assuming that the password protected site is the last in the ring, you write down the codes from each site sequentially to form a string.

706173733D4861636B65725A

Can you unlock the password?

Solution

The string is made up of 12 hexadecimal bytes (base 16): 70, 61, 73, 73, 3D, 48, 61, 63, 6B, 65, 72, 5A.

Each hexadecimal digit has the following value: 0=0, 1=1, ... , 8=8, 9=9, A=10, B=11, C=12, D=13, E=14, F=15. Similar to decimal (base 10), the first digit in any 2-digit number (reading right to left) is units, but the second digit is 16's (compare with 10's, 100's, 1000's, et cetera).

So the hexadecimal number 5A, the last byte in the message, is worth $5 \times 16 + 10 = 90$.

By converting the hexadecimal bytes to decimal, we form the following string of decimal numbers: 112, 97, 115, 115, 61, 72, 97, 99, 107, 101, 114, 90.

These represent the ASCII (American Standard Code for Information Interchange) codes for the characters in the message. A brief introduction to ASCII can be found in the FAQ section. Alternatively you could do a search for character/ASCII code conversion tables on the internet, ascii table (Google search link). For example, the character for ASCII code 90 is Z (capital). Note that some tables will list the corresponding character values in hexadecimal, so converting to decimal may not be necessary.

By converting the string we read the message: pass=HackerZ.

What is the advantage of using hexadecimal?

Investigate the way in which characters have been mapped.

For example, is it a coincidence that the ASCII codes for 1, 2, 3, ..., are at 31, 32, 33, ... (hex); A, B, C, ..., start at 41 and a, b, c, ... start at 61?

PEN PROBLEM

Problem

A girl bought 15 pens costing £1.84. She paid one pence more for each red pen than each blue pen. How many of each kind did she buy and at what price?

Solution

Let a be the number of blue pens, b be the number of red pens and c the cost of a blue pen, hence a red pen costs $c + 1$.
$\therefore ac + b(c + 1) = 184$
$\therefore ac + bc + b = 184$
$\therefore ac + bc = 184 - b$
$\therefore c(a + b) = 184 - b$

But $a + b = 15$,
$\therefore 15c = 184 - b$

Hence $184 - b$ must be divisible by 15; as $15 \times 12 = 180$, we get $b = 4$,
so $a = 11$ and $c = 12$.

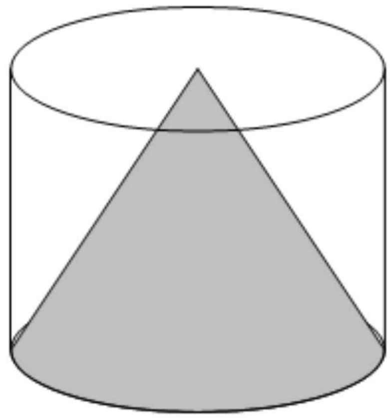
That is, 11 blue pens at 12p each and 4 red pens at 13p each, costing £1.84 in total.

What must be special about the total cost (and number of pens bought) for this problem to have integral solutions?

PERFECT CONE

Problem

A cone is designed so that it fits perfectly into a cylindrical container.



Given the volume of the cone is 100 cm^3 and the curved surface area of the cylinder is 150 cm^2, what is the height of the container?

Solution

Volume of cone = $\frac{1}{3}\pi r^2 h = 100 \Rightarrow \pi r^2 h = 300$.

Curved surface area of cylinder = $2\pi rh = 150$.

$$\therefore \frac{\pi r^2 h}{2\pi rh} = \frac{r}{2} = \frac{300}{150} = 2 \text{ Þ } r = 4 \text{ cm}.$$

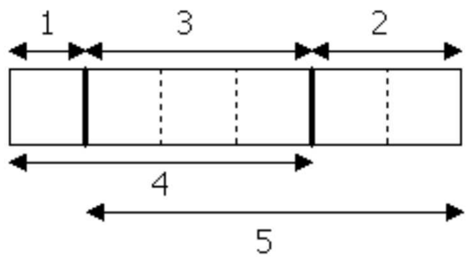
Given that $2\pi rh = 150$, we get $8\pi h = 150$, hence $h = \frac{75}{4\pi}$.

PERFECT RULER

Problem

By marking off six unit lengths on a piece of wood, we can make a ruler to measure all the lengths from 1 to 6 units directly.

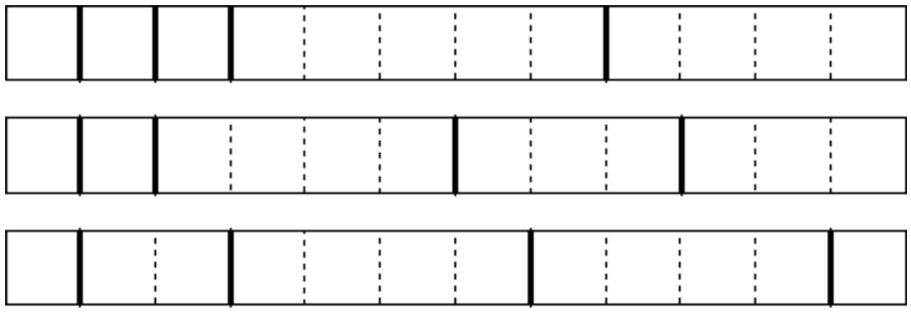
But it is possible to achieve the same thing by using just two marks.



What is the minimum number of marks necessary to measure all the lengths from 1 to 12 units?

Solution

The minimum number of marks is 4 and three different solutions are given.



Some logic can be applied to solve the problem:
Where must you place a mark in order to measure 11?
What do you need to do now to measure 10?
And so on...

How many different solutions for a length 12 ruler exist?
What is the shortest ruler that requires 3 marks?
Investigate different length rulers.

PERMUTED SUMS

Problem

Four digits are selected from the set {1,2,3,4,5} to form a 4-digit number.

Find the sum of all possible permutations.

Solution

There are exactly five sets of four that can be selected ($^5C_4=5$):

1234, 1235, 1245, 1345, 2345

For each combination there are 4! = 24 combinations. Hence there are 5×24 = 120 different 4-digit numbers that can be formed.

Imagine placing each of these 4-digit numbers on top of each other in a long list to be added manually. Each of the digits 1, 2, 3, 4, and 5 will appear equally often in each of the units, tens, hundred, and thousands columns. There are two ways to proceed...

<u>Method 1</u>

> As 120/5 = 24, each digit will contain twenty-four occurrences of each digit and so each column would add to 24(1+2+3+4+5) = 360.
>
> In adding the units column we write 0 and carry 36.
> In the ten colums we get 360+36 = 396: write 6 and carry 39.
> In the hundred and thousands column we get 360+39 = 393: write 9 and carry 39.
>
> Hence the sum is 39 9 9 6 0 = 399,960.

<u>Method 2</u>

> As the mean digit in each column is 3, each number is 3333, on average. Hence the sum is 120×3333 = 399,960.

Find the sum of all possible permutations of *k* digits taken from {1,2,3,...,*n*}.

Prime One Less Than Square

Problem

It can be seen that $2^2 - 1 = 3$ is prime.

Find the next example of a prime which is one less than a perfect square.

Solution

We begin by noting that $n^2 - 1 = (n - 1)(n + 1)$.

As $n^2 - 1$ is the product of $n - 1$ and $n + 1$, it can only be prime when $n - 1 = 1 \Rightarrow n = 2$. That is, 3 is the only example of a prime being one less than a perfect square.

When is $n^2 + 1$ prime?

PRIME SQUARE DIFFERENCES

Problem

Prove that all primes greater than 2 can be written as the difference of two squares.

Solution

By considering the difference of two consecutive square numbers,
$(n + 1)^2 - n^2 = (n^2 + 2n + 1) - n^2 = 2n + 1$

Hence all odd numbers greater than 1, which must include all primes greater than 2, can be written as the difference of two square numbers.

For example, $83 = 2\times41 + 1 = 42^2 - 41^2$

Investigate primes that can be written as the difference of square numbers in general, including non-consecutive squares.

PRIME SQUARE DIVISIBILITY

Problem

Prove that p^2-1 is divisible by 24 for all primes, $p > 3$.

Solution

For all primes greater than 3, $p = 6k\pm1$.

When $p = 6k+1$, $p^2 = 36k^2 + 12k + 1$ and so $p^2-1 = 36k^2 + 12k$.

Similarly, when $p = 6k-1$, $p^2-1 = 36k^2 - 12k$.

Therefore, $p^2-1 = 36k^2 \pm 12k = 12k(3k \pm 1)$.

If k is odd, $3k \pm 1$ will be even and so we prove that p^2-1 will always be divisible by $12 \times 2 = 24$.

What can you say about p^3-1?

What about other powers of p?

PRIME SQUARE SUMS

Problem

The prime number 13 is called a Pythagorean prime because it can be written as the sum of two square numbers.

$$2^2 + 3^2 = 4 + 9 = 13.$$

How many Pythagorean primes are there below 100?

Solution

By working through the primes the following list can be obtained:

2 = 1 + 1	29 = 4 + 25	61 = 25 + 36
5 = 1 + 4	37 = 1 + 36	73 = 9 + 64
13 = 4 + 9	41 = 16 + 25	89 = 25 + 64
17 = 1 + 16	53 = 4 + 49	97 = 16 + 81

That is, there are twelve Pythagorean primes under 100.

Make a complete list of all the primes less than 100 and circle the Pythagorean primes. With the exception of 2, what do you notice?

How many different ways can you write each of the Pythagorean primes as the sum of two square?

PRIME UNIQUENESS

Problem

Prove that seven is the only prime number that is one less than a perfect cube.

Solution

Let p be a prime number one less than a perfect cube, $p = n^3 - 1$

By factoring the right hand side,
$p = (n - 1)(n^2 + n + 1)$

By definition p cannot have any factors, so $n - 1 = 1 \Rightarrow n = 2$.

Hence $p = 2^3 - 1 = 7$.

Investigate this property for other perfect powers.

Proportional Difference

Problem

It can be seen that $12/20 = 3/5 = (12-3)/(20-5) = 9/15$.

If $a/b = c/d$ and $a \neq c$ then prove that $(a-c)/(b-d) = a/b$.

Solution

If $a/b = c/d$ then let $c = ka$ and $d = kb$.

$\therefore (a-c)/(b-d) = (a-ka)/(b-kb) = (a(1-k))/(b(1-k)) = a/b$ **Q. E. D.**

PROPORTION OF ONES

Problem

If we consider 2-digit numbers, there are exactly 18 numbers that contain the digit one:

10, 11, 12, 13, 14, 15, 16, 17, 18, 19,
21, 31, 41, 51, 61, 71, 81, 91

As there are ninety 2-digit numbers, the probability of a 2-digit number containing a one is 18/90 = 1/5.

What proportion of 3-digit numbers contain the digit one?

Solution

There are three mutually exclusive cases for consideration:
1**, (1')1*, and (1')(1')1.

First Digit	Second Digit	Third Digit	Combinations
1	0-9	0-9	$1\times10\times10 = 100$
2-9	1	0-9	$8\times1\times10 = 80$
2-9	2-9 + 0	1	$8\times9\times1 = 72$
		Total	252

There are $9\times10\times10 = 900$ numbers that contain three digits.

Hence the proportion of 3-digit numbers containinig the digit one is 252/900 = 7/25.

Would you expect the proportion of 4-digit numbers containing the digit one to be more or less?
What about *n*-digit numbers?

Quadrant Product Divisibility

Problem

The numbers 1 to 16 have been arranged randomly in the 2 by 2 grid.

6	13	10	1
4	12	7	8
5	16	3	2
11	15	9	14

The product of the numbers in each quadrant are 3744, 560, 13200, and 756, respectively; all but the last quadrant are divisible by 16.

How would you arrange the numbers 1 to 16 in the grid, such that the product of the numbers in each quadrant is divisible by 16?

Solution

Of course there is no restriction on how many numbers we place in each quadrant, but we must ensure that there is at least a factor of 2^4 in each quadrant.

- 2, 6, 10, and 14 contain one factor of 2: $2\times2\times2\times2=2^4$.
- 4 and 12 contain two factors of 2: $2^2\times2^2=2^4$.
- 8 contains three factors of 2: 2^3.
- 16 contains four factors of 2: 2^4.

As we are one factor of 2 short, and no more even numbers are available, we prove that it is impossible to fill the grid with at least one number in each quadrant such that the product of numbers in each quadrant contains a factor of 2^4.

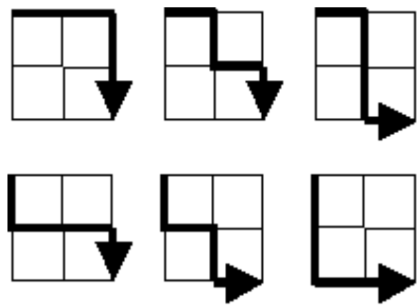
Surprisingly we can solve the problem by bending the "rules" slighty: place 16 in the first quadrant, 8 and 2 in second quadrant, and all the other numbers in the third quadrant. With no numbers in the fourth quadrant, and zero being divisible by 16, we have a solution. However, it could be argued that "the product of numbers" requires at least two numbers in each quadrant; in which case, no solution exists.

Can you arrange the numbers 1 to 20 in the same 2 by 2 grid so that each quadrant contains at least two numbers and the product of numbers in each quadrant is divisible by 20?

RANDOM ROUTES

Problem

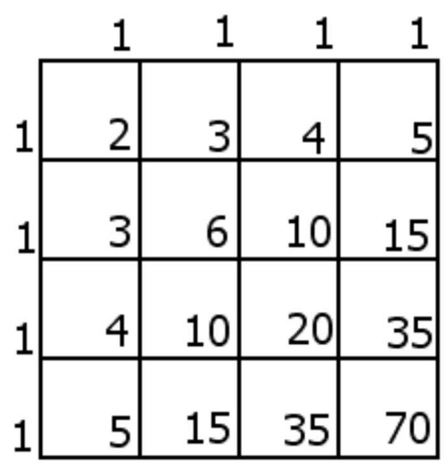
Starting in the top left corner and moving right and down, there are exactly six routes to the bottom right corner of a 2 × 2 grid.



How many different routes can you find through a 4 × 4 grid?

Solution

Consider the diagram.



Each number on the grid represents the number of routes to that node (intersection point on the grid).

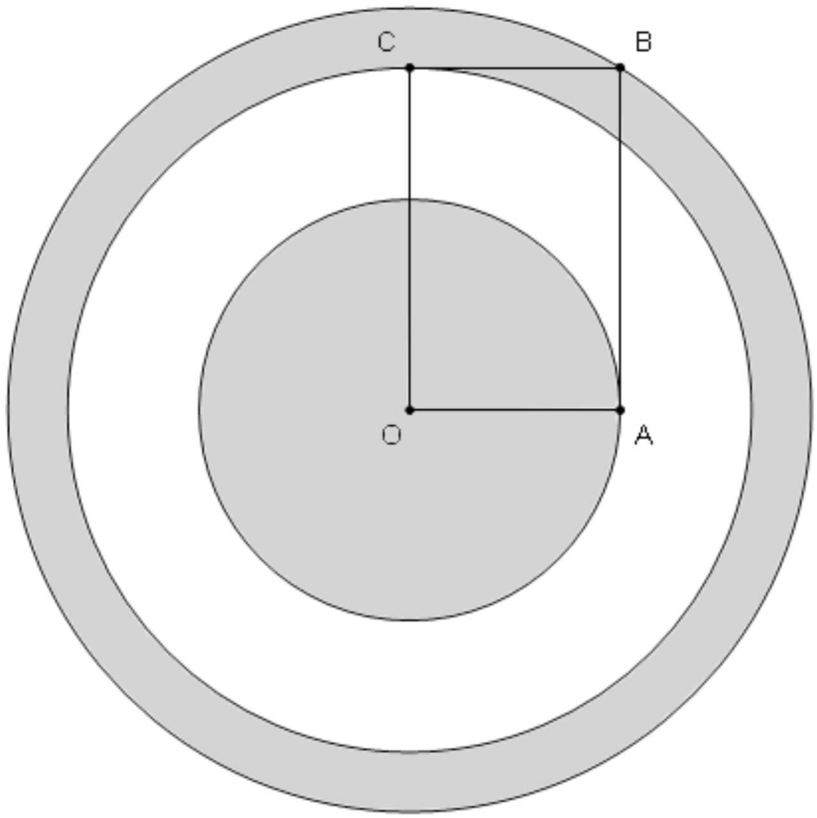
For example, consider the node which has 15 routes to it on the bottom row. As there are 10 routes to the node above it and 5 routes to the node on its the right, there are 10 + 5 = 15 routes to it in total. In the same way, each of the other values have been calculated and it becomes apparent that the total number of different routes through a 4 × 4 grid is 70.

How many routes are there through a 10×10 grid?
What about an $n \times n$ grid?

Rectangular Circles

Problem

Given rectangle OABC, three concentric circles OA, OB, and OC are drawn.



Prove that the area of the annulus generated by the concentric circles OB and OC is always equal to the area of the inner circle OA.

Solution

Using the Pythagorean theorem it is clear that $(OB)^2 = (OA)^2 + (OC)^2$, or $(OA)^2 = (OB)^2 - (OC)^2$.

$\therefore \pi\times(OA)^2 = \pi\times(OB)^2 - \pi\times(OC)^2$

That is, we haven shown that the area of circle OA is equal to the area of the annulus as required.

If the area of the central annulus generated by circles OC and OA is also equal to the area of the inner circle OA, what can be deduced about the ratio of the sides of the rectangle?

Remainder Of One

Problem

Find the smallest number, greater than 1, which has a remainder of 1 when divided by any of the numbers 2, 3, 4, 5, 6, or 7.

Solution

We note that $2 \times 3 = 6$ is the smallest number that divides evenly by 2 or 3. In the same way it should be clear that $2 \times 3 \times 5 = 30$ will be the smallest number that divides by 2, 3, 5, or 6. Although it is obvious that divisibility by 7 requires a factor of 7, to ensure that it divides by $4 = 2 \times 2$, we only need add one extra factor of 2. Hence $2 \times 3 \times 5 \times 2 \times 7 = 420$ is the smallest number that evenly divides by 2, 3, 4, 5, 6, or 7, and so 421 is the smallest number with a remainder of 1.

What is the smallest number which, when divided by any of the number from 2 to 10, has a remainder of 1?
What about the smallest number which when divided by 2 has a remainder of 1, when divided by 3 has a remainder of 2, or when divided by 4 has a remainder of 3?

REVERSE DIGITS

Problem

If the number 41 is added to its reverse, 41 + 14 = 55.

The total, 55, is divisible by 5. But not many numbers have this property. For example, 27 + 72 = 99, which is not divisible by 5.

How many numbers under 100 have this property?

Solution

Let the original number, $n = 10a + b$, so its reverse, $m = 10b + a$.

Therefore, $n + m = 11a + 11b = 11(a + b)$.

Clearly, $n + m$ is divisible by 5 iff $a + b \equiv 0 \bmod 5$.

Listing combinations:

a = 0, b = 5	05,50
a = 1, b = 4 or 9	14, 41, 19, 91
a = 2, b = 3 or 8	23, 32, 28, 82
a = 3, b = 7	37, 73
a = 4, b = 6	46, 64
a = 5, b = 5	55
a = 6, b = 9	69, 96
a = 7, b = 8	78, 87

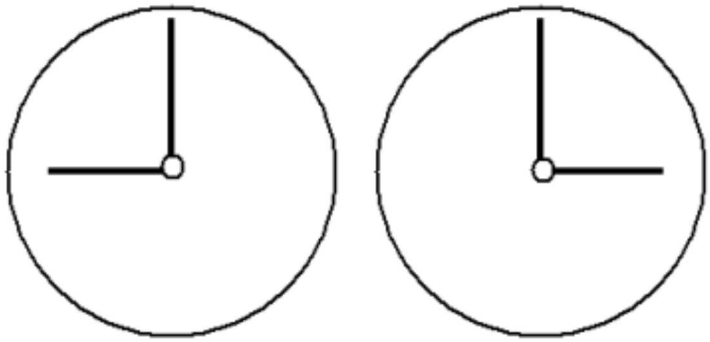
That is, 19 numbers in total.

What about under 1000?

RIGHT TIME

Problem

The angle between the hour hand and the minute hand is 90^o at 9 o"clock and 3 o"clock:



How many times between 9 a.m. and 3 p.m. is the angle between the hour and minute hand 90^o?

Solution

After 9 o"clock, the next time it happens is a little after 9:30, then around 10:05, 10:35, and so on.

It can be seen that it will happen twice during each hour, at roughly the following times:

> 9:00, 9:33, 10:06, 10:37, 11:11, 11:43, 12:17, 12:49, 13:22, 13:54, 14:27, and 15:00.

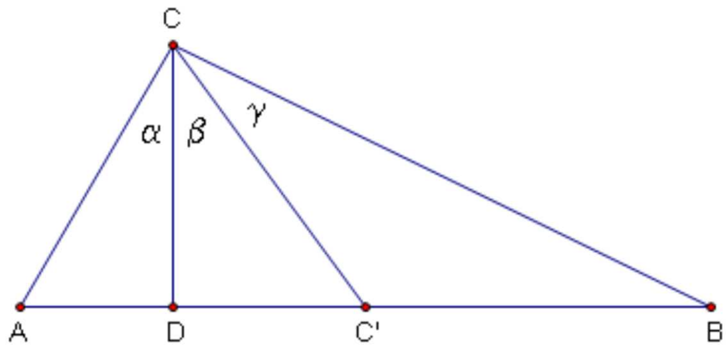
That is, 12 times between 9 a.m. and 3 p.m.

At what time exactly does it first happen after 9 a.m.?
What about the other times?

RIGHT TRIANGLE EQUAL ANGLES

Problem

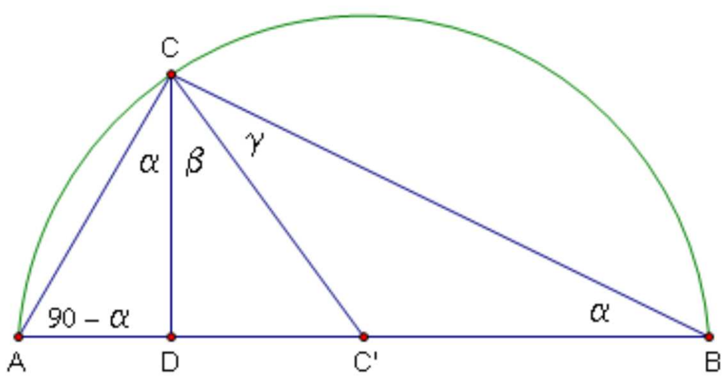
In the triangle ABC, C is a right angle, CD is perpendicular to AB, and C" is the midpoint of AB and C is joined to C".



Prove that $\alpha = \gamma$.

Solution

Consider the following diagram.



As CD is perpendicular to AB, triangle ACD is a right angle triangle. In which case, angle DAC = 90 – α.

We are given that angle ACD is a right angle, so in triangle ABC we can see that angle ABC = α.

As any triangle in a semi-circle is a right angle, C"A = C"B = C"C (radii) and we deduce that triangle BC"C is isosceles. Therefore angle C"BC = angle C"CB; that is, α

$= \gamma$. **QED**

Prove the converse: if $\alpha = \gamma$ then angle ACB must be a right angle.
(see http://www.qbyte.org/puzzles/puzzle13.html#p123)

ROUNDED ROOTS

Problem

A special number machine will accept any whole number, find its square root and then output the result rounded off to the nearest whole number.
For example,

$$\begin{array}{ccccc} \text{Input} & & & & \text{Output} \\ 32 & \rightarrow & \boxed{\sqrt{32} \rightarrow 5.6568542\ldots} & \rightarrow & 6 \end{array}$$

If the output is 6, one possible input is 32, but which other numbers could have gone into the machine to give an output of 6?

Solution

$5.5^2 = 30.25$, so $\sqrt{30} < 5.5$ and $6.5^2 = 42.25$, so $\sqrt{43} > 6.5$.

Hence the solution set is 31, 32, 33, ..., 42

Find the solution sets for each of the integers from 1 to 10.

Is there a rule connecting the output with the solution set?

ROUNDING ERROR

Problem

A number is rounded to two decimal places and this answer is rounded to one decimal place. If the final number is 0.2 then what is the chance that the original number lies between 0.15 and 0.25?

Solution

When rounding to k decimal places (d.p.) we truncate the number at the kth decimal position, but if the next digit is 5 or more then we increase the last digit by 1; this may have a knock-on effect. Consider the following examples of rounding to 2 d.p.:

> 3.61|483 → 3.61
> 7.78|53301 → 7.79
> 0.59|82 → 0.60

Suppose we were told that 0.2 were the result of rounding to one decimal place. The smallest number that would be rounded up to 0.2 is 0.15. However, there is no actual value we can assign to the greatest value that would round down to 0.2. For example, 0.25 would round up to 0.3 and although 0.249 would round down to 0.2, what about 0.2499, or 0.24999, or ... ?

Under these circumstances we define bounds (or limits) in the following way:

- The lower bound is the least value that rounds up to the given number.
- The upper bound is the least value that fails to round down to the given number.

So if 0.2 is the result of rounding to 1 d.p. then the lower and upper bounds are 0.15 and 0.25 respectively.

Working backwards, the lower bound of 0.2 is 0.15 and the lower bound of 0.15 is 0.145. We can easily check this: 0.145 = 0.15 (2 d.p.) and then 0.15 = 0.2 (1 d.p.).

However, we need to tread very carefully when working backwards with the upper bound. Although it is true to say that the upper bound of 0.2 is 0.25, if the original number were rounded to 0.25 (to 2 d.p.), then next process of rounding to 1 d.p. would make it 0.3, not 0.2.

In fact, it is necessary that the result of rounding to 2 d.p. takes it to 0.24; and the upper bound of 0.24 is 0.245.

Hence we are certain that the original number, x, lies somewhere in the interval $0.145 \le x < 0.245$.

If we are considering if the number lies between 0.15 and 0.25 then it would have to lie in the interval $0.15 \leq x < 0.245$.

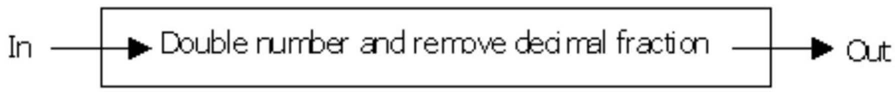
$$\therefore \mathrm{P}(0.15 \leq x < 0.245 \mid 0.145 \leq x < 0.245) = (0.245 - 0.15) / (0.245 - 0.145)$$
$$= 0.095 / 0.1$$
$$= 95\%$$

If instead of rounding to 2 d.p. then to 1 d.p. a number is just rounded to 1 d.p., what is the chance that both methods give the same answer?

ROUNDING MACHINE

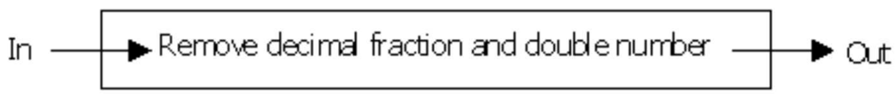
Problem

A particular number machine works as follows.



E.g. 3.4 → 6.8 → 6 or 7.9 → 15.8 → 15

A different number machine does the following.



E.g. 3.4 → 3 → 6 or 7.9 → 7 → 14

Notice that 3.4 came out as 6 from both machines, whereas 7.9 came out differently. What must be special about a number for the same value to come out of each machine?

Solution

If x is the value going into each machine, the machines can be expressed as $[2x]$ and $2[x]$ respectively.

All numbers of the form $n.m$ under the integer part function will become n by definition.

Therefore $2[n.m] = 2n$ (i.e. independent of m)

But, $[2 \times n.m] = [2 \times (n + {}^{m}/_{10})] = [2n + {}^{m}/_{5}]$

If $m < 5$, $0 \le m < 1 \Rightarrow [2n + {}^{m}/_{5}] = 2n$, whereas for $m \ge 5$, $1 \le m < 2 \Rightarrow [2n + {}^{m}/_{5}] = 2n + 1$.

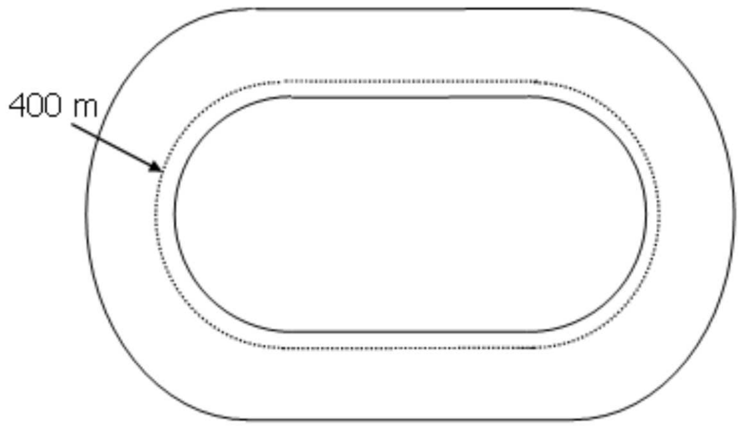
And so the decimal part of x must be less than .5 for $2[x]$ to be equal to $[2x]$.

When is $[x + 0.5] + [x - 0.5]$ equal to $2[x]$ and $[2x]$?

Running Requirements

Problem

An official IAAF (International Amateur Athletics Federation) running track measures 400 m and is made up of a straight section measuring 84.39 m and semi-circular curves with a radius of exactly 36.5 m; the 400 m distance is measured 30 cm from the inside edge of the track.

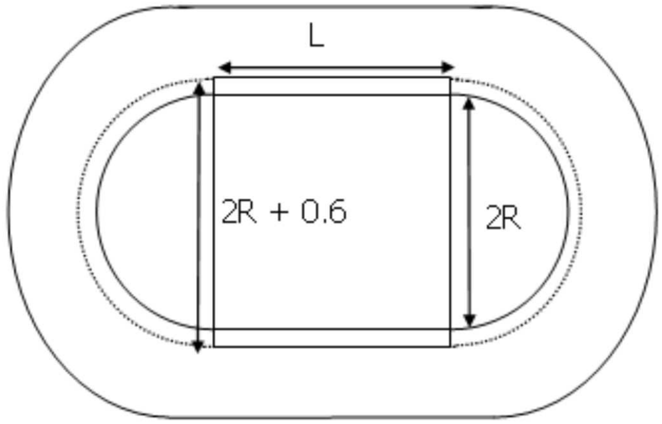


Tracks used to be marked out by using equal quadrant measures, which means that it was made up of 100 m straight sections and 100 m semi-circular curves. However, sport scientists have found that increasing the radius of the curves increases performance of athletes and reduces the chance of injury. In accordance with this, the IAAF has stipulated that the inside radius of the track must lie between 35 m and 38 m.

A school wishes to mark out a running track that satisfies the IAAF regulations. Show that it is necessary to have a non-equal quadrant measure track, and find the bounds of the straight sections to satisfy the requirements.

Solution

Although the inside of the track has radius R, the 400 m is measured 30 cm from this edge. So the radius of the curve that will trace the 400 m distance will have radius R + 0.3, as shown in the diagram below.



It can be seen that the two semi-circles combine to form a circle with a diameter of 2R + 0.6.

If the school built an equal quadrant measure track, $\pi(2R + 0.6) = 200$.

$\therefore \pi(R + 0.3) = 100$
$R + 0.3 = 100/\pi$
$\therefore R = 100/\pi - 0.3 \approx 31.53$ m.

As this falls outside the IAAF requirements, the school must employ a non-equal quadrant measure track.

$\therefore \pi(2R + 0.6) + 2L = 400$
$\pi(R + 0.3) + L = 200$
$\therefore L = 200 - \pi(R + 0.3)$

As $35 \leq R \leq 38$, we get the approximate bounds, $79.68 \leq L \leq 89.10$ m.

The IAAF requires that each lane be 1.22 ± 0.01 m wide. If a 400 m race is to be run, how far must the runner in lane two be ahead of the runner in lane one?

SAME DIGITS

Problem

There are exactly nine 2-digit numbers which have two digits the same: 11, 22, ..., 99.

How many 3-digit numbers have two digits the same?

Solution

In general, the first digit can be 1-9, the second digit can be 0-9, and the final digit can be 0-9; so there are $9\times10\times10 = 900$ 3-digit numbers in total. Of these it is possible to have none, two, or three digits the same.

Let us consider no digits being the same: the first digit can be 1-9, then second digit can be any one of nine digits and the final digit can be any one of eight. That is, there are $9\times9\times8 = 648$ 3-digit numbers for which no digits are the same.

Clearly there are nine 3-digit numbers for which all the digits are the same: 111, 222, ..., 999.

Hence there are $900 - 648 - 9 = 243$ 3-digit numbers for which two digits are the same.

How many 4-digit numbers have two digits the same?
Investigate n-digit numbers.

SAME DIGIT PRIME

Problem

11 is the smallest prime made up of the same digit.

Explain why a number made up of the same digit can only be prime if the repeating digit is one AND the number of digits is itself prime.

Solution

Clearly a number of the form *aaa...* will be divisible by *a*; for example, 77777 is divisible by 7.

Hence such a number can only be prime if *a* = 1.

Consider the number 111111111111111, which is made up of fifteen ones. It should be clear that as 15/5 = 3, it will be divisible by 111.

$$\begin{array}{r|l} & \;\;\;1\;001\;001\;001\;001 \\ \hline 111 & 111\;111\;111\;111\;111 \end{array}$$

That is, 111111111111111/111 = 1001001001001.

In other words, if the number is made up of a string of *k* ones, where *k* = *ab*, then it will certainly be divisible by a string of *a* (or *b*) ones.

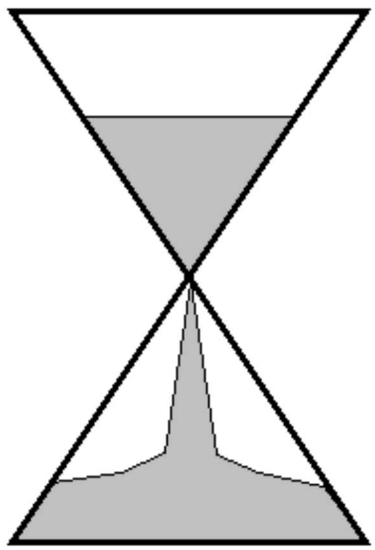
Hence the only numbers made up of the same digit that are prime must be made up of a prime number of ones.

It is interesting to note that there are only five known repunit primes; that is, primes made up of a string of 2, 19, 23, 317, and 1031 ones. However, in 1999 and 2000 respectively strings consisting of 49081 and 86453 ones were found to be *probable* primes, which means that although they are still not proven to be prime, they have passed primailty tests for sufficiently many non-trivial cases that it is highly probable that they are prime.

Sand Glass

Problem

A sand glass is made from two conical glass sections, joined together by a thin glass tube. It is turned upside down and sand is allowed to pour from the top section into the bottom section. A sand glass has a carefully measured volume of sand placed inside it, such that it takes an exact amount of time for the sand to flow completely from one section to another. As the sand cannot escape the device it can be used over and over again, to measure a particular length of time accurately.



Two sand glasses have been designed to measure 9 minutes and 13 minutes respectively.

How would you use them to measure 30 minutes?

Solution

Start the 13 minute sand glass; once it has completed, we know that 13 minutes have elapsed. Turn it over to start it again, and at the same time start the 9 minute timer. When the 9 minute timer has finished, turn it over to start it again; it will be a further 4 minutes before the 13 minute timer finishes.
Once the 13 minutes finishes again, 26 minutes will have elapsed in total and the 9 minute timer will have been been running for 4 minutes.
Now flip the 9 minute back over and when it has completed, 26 + 4 = 30 minutes will have passed.

What if you had a 9 minute and an 11 minute timer?

Investigate different timers to measure 30 minutes.

SAVE RATE

Problem

During a football match in a tournament a goalkeepers save rate was given as 33%. After saving the next shot on target it rose to 40%. How many more shots on target does he need to save to raise his save rate to 50%?

Solution

A save rate of 33% means 1 in 3, 2 in 6, 3 in 9, etc. (assuming 33% has been rounded down from 33.333...%).
i.e. $^1/_3 = {}^2/_6 = {}^3/_9 = {}^4/_{12} = \ldots$

If the next shot on target was a save, both the numerator and denominator will have increased by 1.
$40\% = {}^2/_5 = {}^4/_{10} = {}^6/_{15} = \ldots$

We can see that $^{3+1}/_{9+1} = {}^4/_{10}$, so the goalkeeper must currently have a save rate of 4 out of 10 shots on target.

If all subsequent shots on target are saved,
$^4/_{10} \rightarrow {}^5/_{11} \rightarrow {}^6/_{12} = {}^1/_2 = 50\%$.

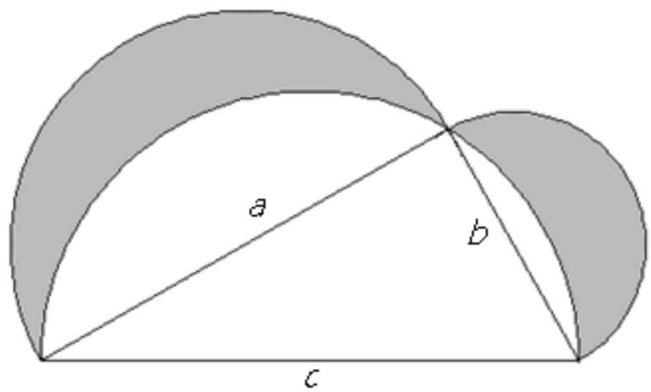
So it will necessary to save the next two shots on target.

What happens if one of the next two shots on target are not saved?

SEMI-CIRCLE LUNES

Problem

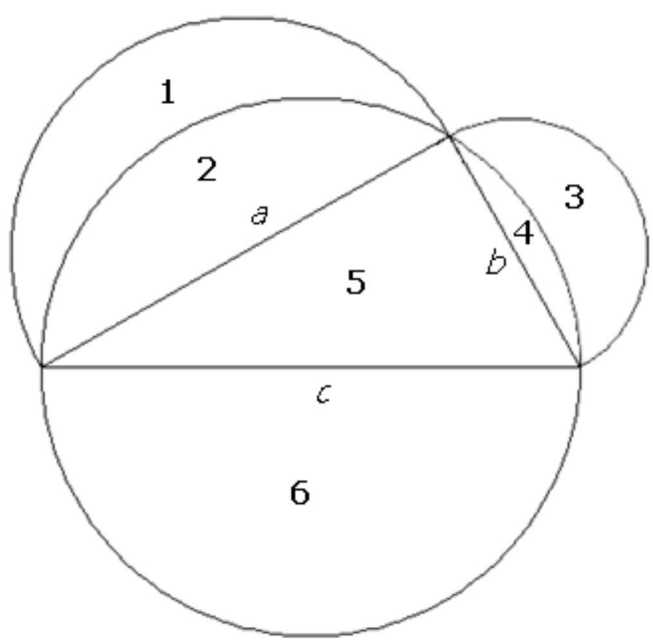
A triangle is formed by connecting the two ends of the diameter of a semi-circle, length *c*, to a point on the circumference. Circles are constructed on the two shorter sides with diameters *a* and *b* respectively, so as to form two lunes (the shaded part).



Find the total area of the two lunes in terms of *a*, *b*, and *c*.

Solution

Consider the diagram, with regions identified by the numbers 1 to 6.



In a semi-circle the triangle will be right-angled, so using the Pythagorean Theorem, $a^2 + b^2 = c^2$.

Multiplying through by $\pi/8$ we get, $\frac{1}{2}\pi(a/2)^2 + \frac{1}{2}\pi(b/2)^2 = \frac{1}{2}\pi(c/2)^2$. In other words, if semi-circles are drawn on the sides of a right-angle triangle, the area of the semi-circles on the shorter sides will be equal to area of the semi-circle on the hypotenuse.

That is, $(A_1 + A_2) + (A_3 + A_4) = A_6$; in addition, $A_6 = A_2 + A_4 + A_5$.

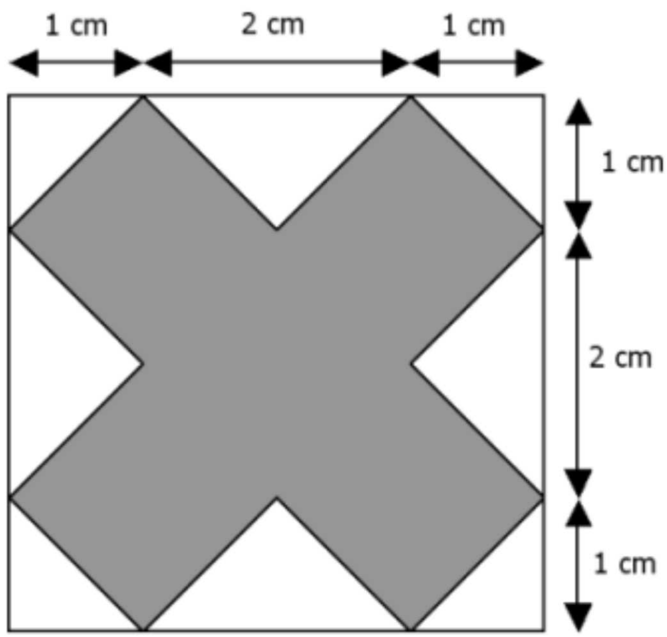
$\therefore A_1 + A_2 + A_3 + A_4 = A_2 + A_4 + A_5 \Rightarrow A_1 + A_2 = A_5$

Hence the area of the two lunes are equal to area of triangle, $\frac{1}{2}ab$.

SHADED CROSS

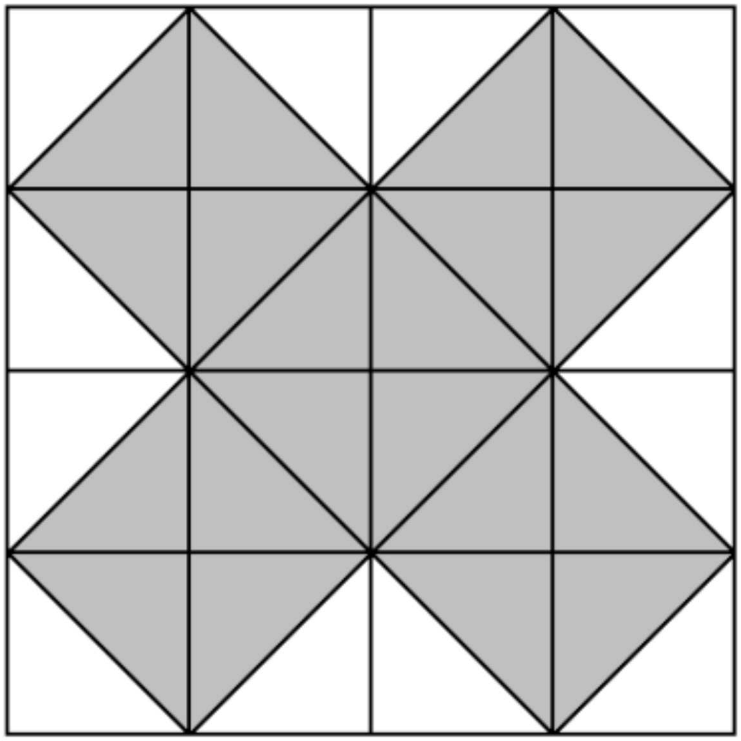
Problem

Find the area of the diagram below that is shaded.



Solution

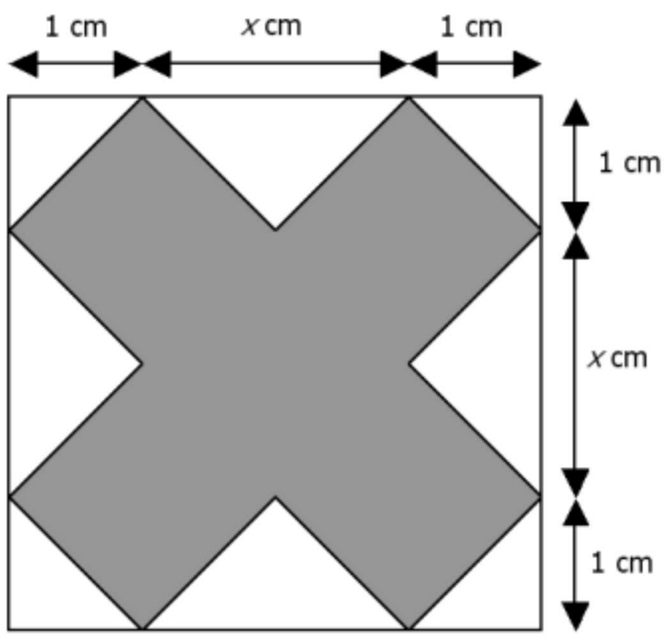
Surprisingly this can be tackled relatively easily by splitting the diagram up in the following way.



It should be clear that $^{20}/_{32} = {}^{5}/_{8}$ of the diagram is shaded.

As the area of the square is 16 cm^2,
Area shaded = $^{5}/_{8}$ of 16 = 10 cm^2

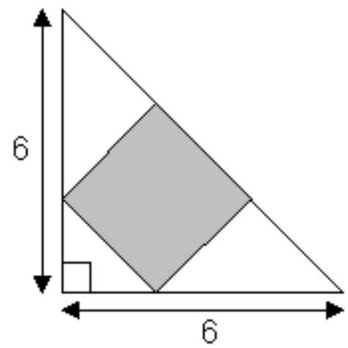
What about the following diagram?



SHADED TRIANGLE

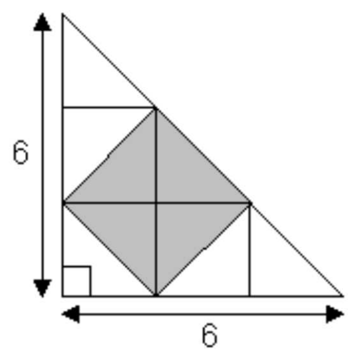
Problem

Find the area of the shaded square.



Solution

By splitting the diagram into triangles,



It can be seen that $^4/_9$ of the diagram is shaded. That is, $^4/_9 \times 18 = 8$ units^2.

What is the perimeter of the shaded square?

Simple Fractions Symmetry

Problem

Given that k and n are positive integers, and $k<n$, we shall call k/n a simple fraction if it cannot be cancelled.

For example, when $n=9$, there are exactly six simple fractions:

$$1/9 \quad 2/9 \quad 4/9 \quad 5/9 \quad 7/9 \quad 8/9$$

For a given denominator, $n>2$, prove there will always be an even number of simple fractions.

Solution

For any given denominator, n, there will be $n-1$ proper fractions:
$1/n$, $2/n$, ... , $(n-1)/n$.

If HCF$(k,n)\neq 1$, it follows that $n-k$ will also share the same common factor. For example, HCF(8,28)=4 and as both 8 and 20 divide by 4, 20–8=12 will also divide by 4. That is, if k/n cancels, $(n-k)/n$ will also cancel.

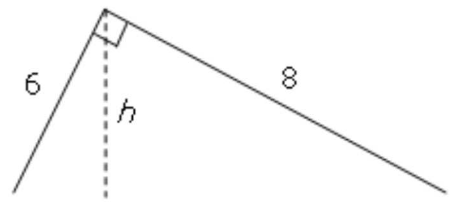
Using this idea, we can see how fractions will cancel in pairs. The exception is n being even, in which case there will be an odd number of proper fractions. However, the numerator of the middle fraction will be $n/2$, which will cancel.

For a given denominator, n, how many simple fractions exist?

SLIDE HEIGHT

Problem

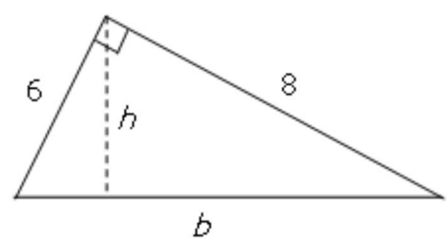
A slide is made by joining a 6 foot ladder to an 8 foot chute at right angles.



Find the vertical height of the slide.

Solution

We shall begin by using the Pythagorean Theorem to calculate the base length of the slide.



$\therefore b^2 = 6^2 + 8^2 = 36 + 64 = 100 \Rightarrow b = 10$

By considering 6 to be the "base" and 8 to be the "height" of the triangle we can see that the area of the triangle is given by $6\times 8/2 = 24$.

However, by considering the triangle in the current orientation,
area of triangle = $bh/2 = 10h/2 = 24 \Rightarrow 10h = 48 \Rightarrow h = 4.8$".

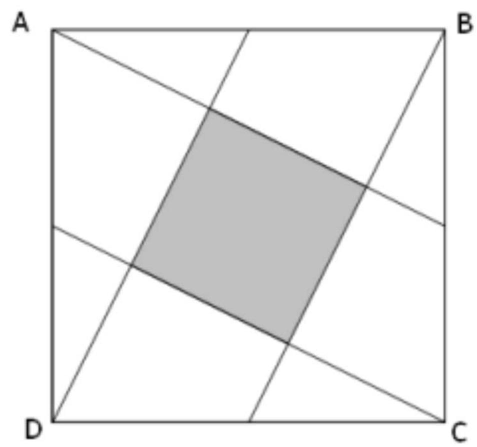
(Difficult) Extension:

> Given that the pair of lengths (6,8) produces a slide with a vertical height of 4.8, find the smallest pair of integer lengths that produce a slide with a vertical height that is also integer.

SLOPING SQUARE

Problem

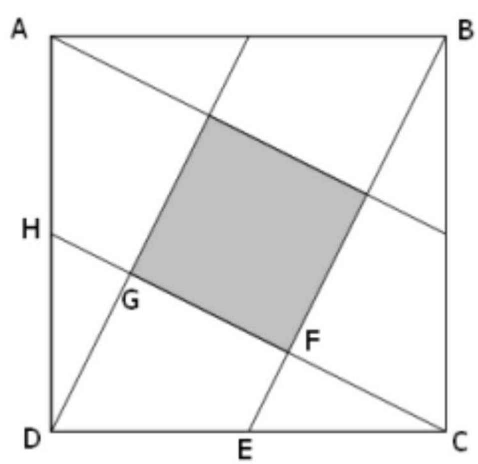
In unit square, ABCD, A is joined to the midpoint of BC, B is joined to the midpoint of CD, C is joined to the midpoint of DA, and D is joined to the midpoint of AB.



Find the area of the shaded square formed by this construction.

Solution

We shall solve this in two different ways. First the hard way... (c;



Using the Pythagorean Theorem, $BE^2 = 1^2 + (1/2)^2 \Rightarrow BE = \sqrt{5}/2$.

As ΔBEC is similar to ΔFEC,

$BE/BC = CE/FC$, $\sqrt{5}/2 = (1/2)/FC \Rightarrow FC = 1/\sqrt{5}$.

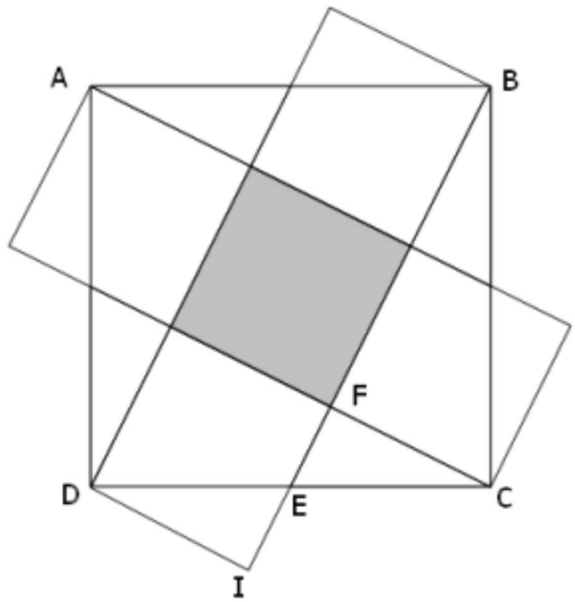
FE/CE = CE/BE, FE/(1/2) = (1/2)/($\sqrt{5}$/2) ⇒ FE = 1/(2$\sqrt{5}$).

HC = HG + GF + FC, and as HG = FE, $\sqrt{5}$/2 = 1/(2$\sqrt{5}$) + GF + 1/$\sqrt{5}$.

∴ 5/(2$\sqrt{5}$) = 3/(2$\sqrt{5}$) + GF ⇒ GF = 1/$\sqrt{5}$.

Hence the area of the shaded square = 1/5.

Now the easy way...

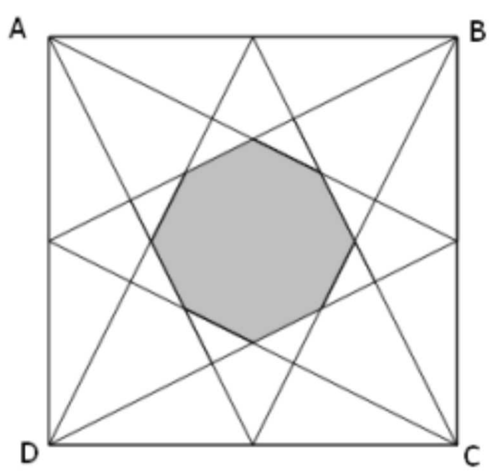


It can be seen that ΔEFC is congruent with ΔEID.

∴ Area five squares = Area of unit square, ABCD = 1

∴ Area of one square = Area of the shaded square = 1/5

What about the area of the shaded octagon in the diagram below?

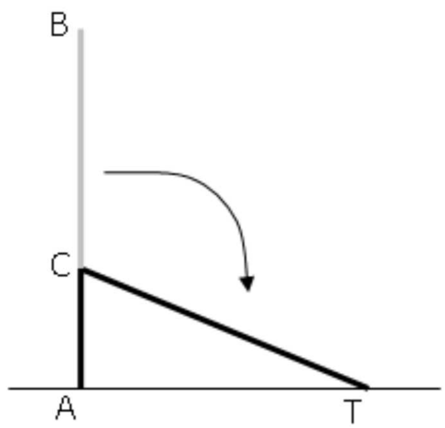


Is it a regular octagon?

SNAPPED POLE

Problem

A vertical pole AB measuring 5 metres snaps at point C. The pole remains in contact at C and the top of the pole touches the ground at point T, a distance of 3 metres from A.



Find the length AC, the point where the pole snapped.

Solution

Let AC = x, so BC = CT = 5 – x.

Using the Pythagorean Theorem,

$$(5 - x)^2 = x^2 + 3^2$$
$$\therefore 25 - 10x + x^2 = x^2 + 9$$
$$\therefore 10x = 16$$

Hence AC = 8/5 metres.

SOLID ENCRYPTION

Problem

Although your mathematics teacher tells excellent jokes, he can be particularly cruel. At the end of a lesson he hands out a small piece of paper to each student.

A cement mixer collided with a prison van...

```
31 51 02 51 81 90 91 02 91 10
81 50 10 91 11 50 40 02 51 20
50 51 41 02 80 50 21 51 51 11
51 12 02 60 51 81 91 90 42 02
50 50 41 80 10 81 40 50 41 50
40 30 81 90 31 90 41 10 21 91
```

Can you discover the punchline?

Solution

The method of encryption is to convert each letter to a number according to its alphabetical position, then reverse the digits:

```
A = 01 = 10
B = 02 = 20
C = 03 = 30
D = 04 = 40
E = 05 = 50
F = 06 = 60
G = 07 = 70
H = 08 = 80
I = 09 = 90
J = 10 = 01
K = 11 = 11
L = 12 = 21
M = 13 = 31
N = 14 = 41
O = 15 = 51
P = 16 = 61
Q = 17 = 71
R = 18 = 81
S = 19 = 91
T = 20 = 02
U = 21 = 12
```

```
V = 22 = 22
W = 23 = 32
X = 24 = 42
Y = 25 = 52
Z = 26 = 62
```

Using this method in reverse to decode the punchline:

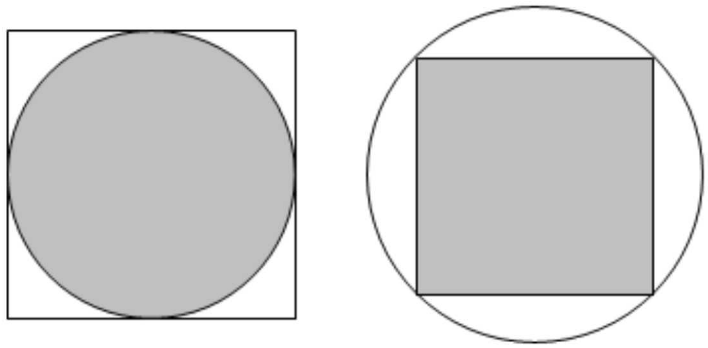
A cement mixer collided with a prison van...

Motorists are asked to be on the lookout for sixteen hardened criminals.

Square And Round Plugs

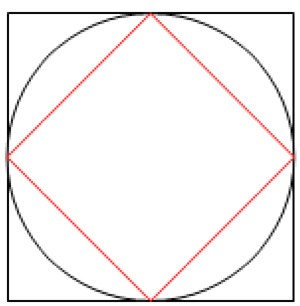
Problem

Without the aid of a calculator determine which fits better: a round plug in a square hole or a square plug in a round hole?



Solution

Consider the following diagram.



If we let the radius of the circle be r then the area of the circle is given by πr^2 and the area of the outside square will be $(2r)^2 = 4r^2$. It should also be clear that the red square is exactly half the area of the outside square, so its area will be $2r^2$.

$\therefore$ (Area of Circle) / (Area of Large Square) = $\pi r^2 / 4r^2 = \pi / 4 \approx 78.5\%$
(Area of Small Square) / (Area of Circle) = $2r^2 / \pi r^2 = 2 / \pi \approx 63.7\%$

But to compare these ratios without the aid of a calculator we write both over the same denominator:

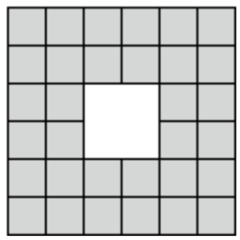
$$\pi / 4 = \pi^2 / 4\pi$$
$$2 / \pi = 8 / 4\pi$$

As $\pi > 3$, $\pi^2 > 9$, so it follows that $\pi / 4 > 2 / \pi$.

SQUARE LAMINAS

Problem

A square lamina can be made from 1, 4, 9, 16, 25, ... unit square tiles. But by placing a square hole in the centre of the lamina it is possible to make a "hollow" square lamina from a non-square number such as 32,

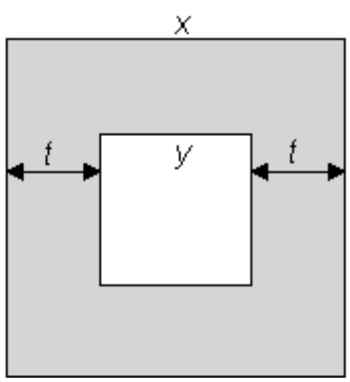


$6^2 - 2^2 = 36 - 4 = 32$

How many different square laminas can be made from 240 unit square tiles?

Solution

If we let x be the dimensions of the lamina and y be the dimensions of the hollow, we are trying to solve $x^2 - y^2 = 240$.



(where t is the thickness)

Writing $x^2 - y^2 = (x + y)(x - y) = 240$.

We are now looking for two integers that multiply to make 240, but some further analysis is required...

If x is odd and the hollow is centrally placed, y must also be odd (equal thickness either side, that is, $y = x - 2t$). Similarly if x is even then y will be even.

In both cases $(x + y)$ and $(x - y)$ will be even, so we are looking for pairs of even factors of 240: They are (2,120), (4,60), (6,40), (8,30), (10,24) and (12,20).

Now $(x + y) + (x - y) = 2x$, that is the sum of the two factors is $2x$. Then y can be found by knowing that $(x - y)$ is the smaller of the two factors.

2, 120	$x = 61$, $y = 59$
4, 60	$x = 32$, $y = 28$
6, 40	$x = 23$, $y = 17$
8, 30	$x = 19$, $y = 11$
10, 24	$x = 17$, $y = 7$
12, 20	$x = 16$, $y = 4$

Can a square lamina be made from any number of starting tiles?

Is there a connection between the number of tiles and the number of solutions?

SQUARE PRODUCT

Problem

Given that $[n(n+1)(n+2)]^2 = 3039162537*6$, find the value of *.

Solution

In any three consecutive integers, n, $n+1$, $n+2$, at least one of the numbers will be even, and one of them will be a multiple of 3. Hence the product, $n(n+1)(n+2)$, will be even and divisible by 3. Furthermore, the square of an even number will be divisible by 4, and the square of a multiple of 3 will be divisible by 9.

If a number is divisible by 9, the sum of the digits will also be divisible by 9: 3 + 3 + 9 + 1 + 6 + 2 + 5 + 3 + 7 + 6 = 45, so the value of * must be 0 or 9.

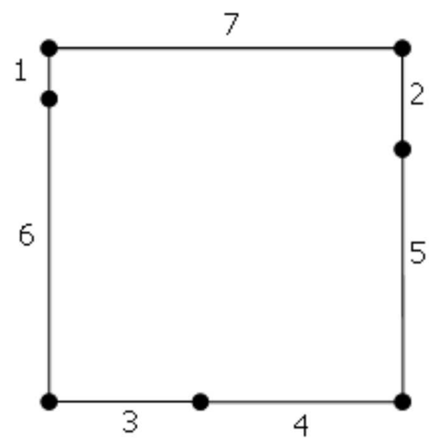
However, if the number is divisible by 4, the last two digits (either 06 or 96) must be divisible by 4. Hence the value of * is 9.

Find the value of n.
How would you solve the equation, $[n(n+1)(n+2)]^2 = k$, in general?

SQUARE RODS

Problem

Using rods of fixed length 1, 2, 3, ..., *n*, the smallest value of *n* for which a square can be formed is *n* = 7.

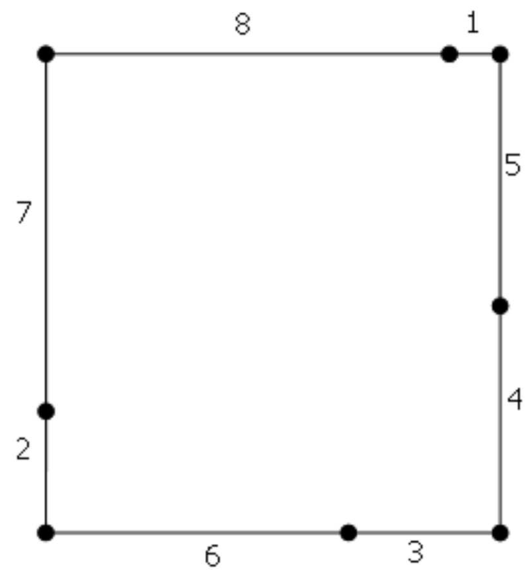


Find the values of *n* for the next two solutions.

Solution

The next solution occurs when *n* = 8: the perimeter of the square will be, 1 + 2 + 3 + ... + 8 = 36. Therefore one side will be 36/4=9 units long.

It can be seen that 1 must be paired with 8, then 2 must be paired with 7, and so on. Hence we must use the following pairings: (1,8), (2,7), (3,6), (4,5).



In general, the perimeter, P = 1 + 2 + 3 + ... + n, must be divisible by 4.

We note that P, the sum of the first n positive integers, is given by $n(n+1)/2$. And as the product of two consecutive numbers, n and $n+1$, will contain one odd factor and one even factor, the even factor must be divisible by 8 in order to be divisible by 4 after being halfed. This explains why $n=8$ is a solution, and $n=7$ is a solution because $n+1=8$. In other words, it is necessary for n or $n+1$ to be divisible by 8.

Hence the next solution should occur when $n=15$, as $n+1=16$; that is, $1 + 2 + ... + 15 = 15\times16/2 = 15\times8 = 120$.

As the side of the square will be 120/4 = 30, we can form a square using the following sets: (15,14,1), (13,12,5), (11,10,9), and (8,7,6,4,3,2).

If n or $n+1$ is divisible by 8, will you always be able to form a square?

STRING OF ONES

Problem

A number consisting entirely of the digit one is called a repunit; for example, 11111.

Find the smallest repunit that is divisible by 63.

Solution

As $63 = 7 \times 9$, the repunit we are seeking must be divisible by both 7 and 9. A number is divisible by 9 if and only if the sum of the digits is a multiple of 9, therefore 111111111 is the smallest repunit that divides by 9.

By systematically working through increasing length repunits we can see that the 6-digit number, 111111, is the smallest repunit that divides by 7; that is, 111111/7 = 15873.

As 7 divides into a block of six ones, it must divide into any repunit that contains a multiple of six ones. Simiarly 9 will divide into any repunit containing a multiple of nine ones.

The lowest common multiple of six and nine is eighteen. Hence the smallest repunit that divides by 63 contains exactly eighteen digits.

Find the smallest repunit that divides by 1353.

TAMING THE SUM

Problem

Show that the sum, $1 \pm 2 \pm 3 \pm ... \pm 99 = 100$, where $\pm$ between each term can be independently set to + or –, has at least one solution.

Solution

Consider the sum, $n - (n+1) - (n+2) + (n+3) = 0$.

In other words, we can set four consecutive integers to zero:

$1 - 2 - 3 + 4 = 0$
$5 - 6 - 7 + 8 = 0$
$9 - 10 - 11 + 12 = 0$
...
$93 - 94 - 95 + 96 = 0$

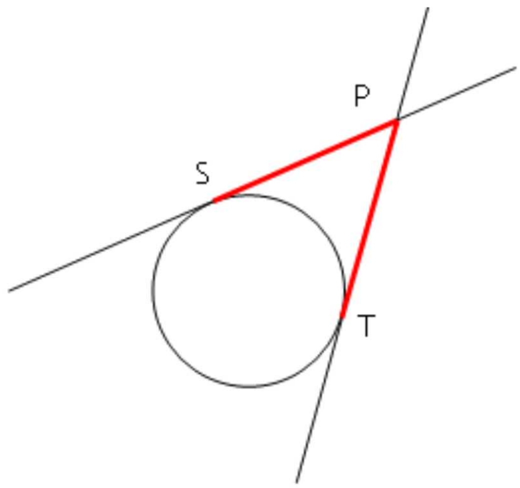
Then $-97 + 98 + 99 = 100$

What about $1^2 \pm 2^2 \pm 3^2 \pm ... \pm 99^2 = 100$?
Is there a solution for cubes?

TANGENTIAL DISTANCES

Problem

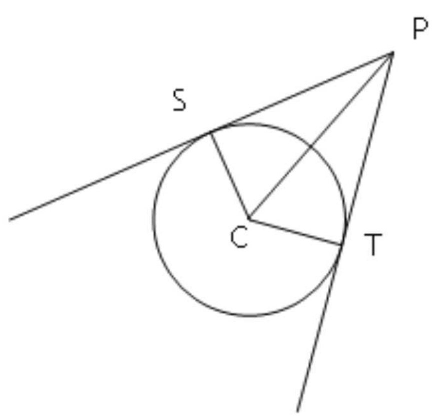
P represents a remote point from a circle, and the lines through PS and PT are tangential to the circle.



Prove that the tangential distances PS and PT are always equal.

Solution

Consider the following diagram where C represents the centre of the circle.



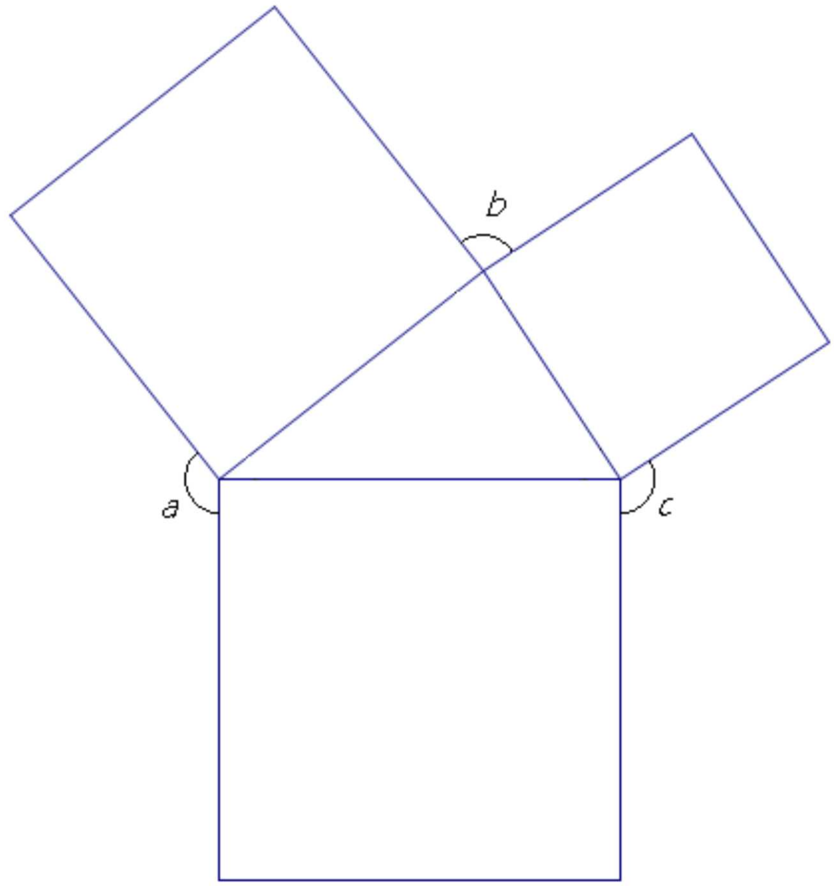
As a radius meets a tangent at a right angle, PCS and PCT are right angled triangles.

By the Pythagorean Theorem, $PS^2 = PC^2 - CS^2$ and $PT^2 = PC^2 - CT^2$. But as CS = CT (they are both radii), it follows that $PS^2 = PT^2 \Rightarrow PS = PT$. **Q.E.D.**

THREE SQUARES

Problem

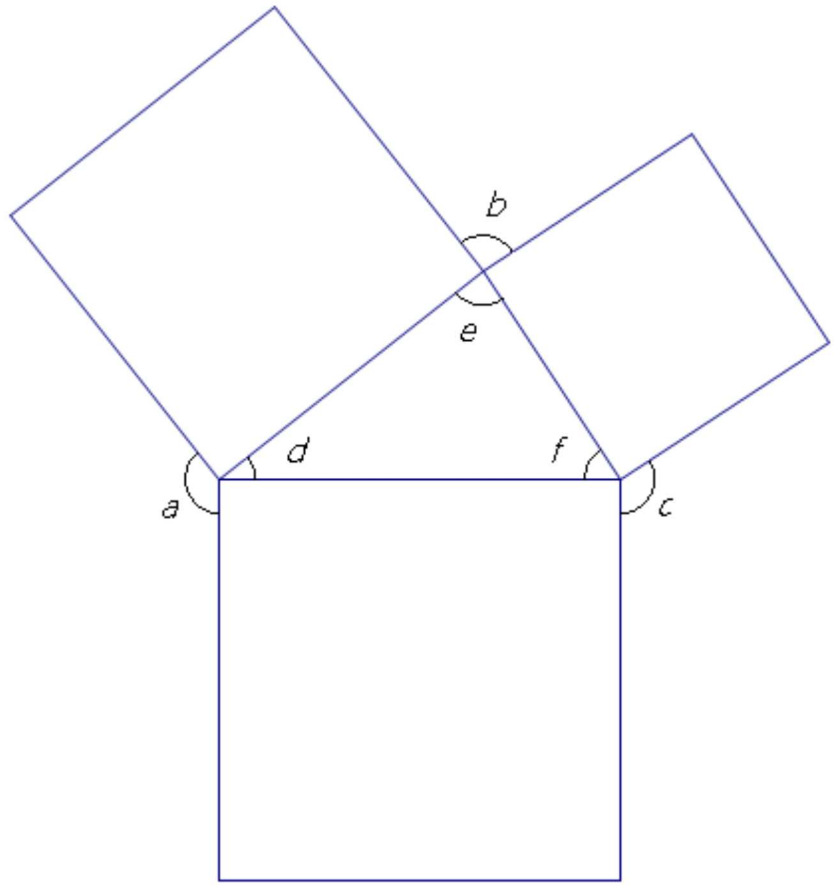
Three squares are joined together to create a triangle.



Prove that $a + b + c = 360^\circ$.

Solution

Consider the following diagram.



The sum of angles around each vertex of the triangle will be 360^o and the sum of angles in the triangle $d + e + f = 180^o$.

$$\begin{aligned} \therefore 3\times360 &= a + b + c + d + e + f + 6\times90 \\ 1080 &= a + b + c + 180 + 540 \\ \therefore 360 &= a + b + c \end{aligned}$$

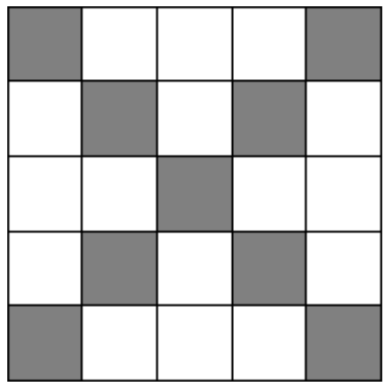
Q. E. D.

What are the conditions for *a*, *b*, and *c* for this result to be true?
What if four squares are joined together to form a quadrilateral?

TILED FLOOR

Problem

The floor of a square room is tiled with square tiles. Along the two diagonals of room measuring 5x5 tiles there are nine tiles.



If there is 81 tiles along both diagonals, how many tiles are there on the floor?

Solution

The diagonal running from top left to bottom right will have one square in every column. So an n x n square will have n tiles in the diagonal.

The same will be true for the other diagonal, but if the dimensions of the square are odd, the diagonals will have a common tile at the centre.

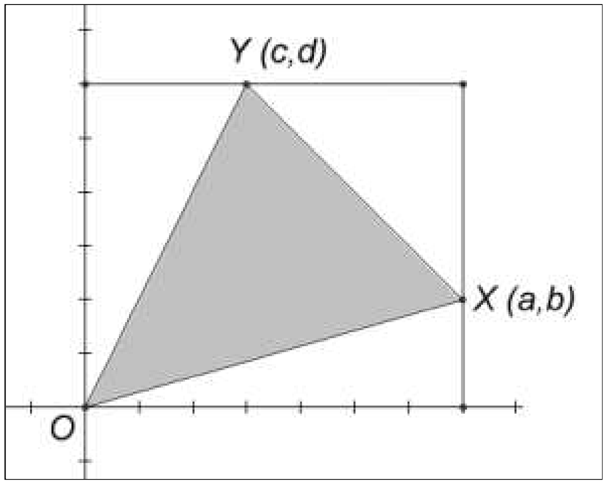
n being even: Tiles in diagonal = 2*n*
n being odd: Tiles in diagonal = 2*n* - 1 (which itself is odd)

So if there are 81 tiles in the diagonal, *n* must be odd.
Solving $2n - 1 = 81 \Rightarrow 2n = 82 \Rightarrow n = 41$.
Hence, there are $41 \times 41 = 1681$ tiles in the room.

TRIANGLE AREA

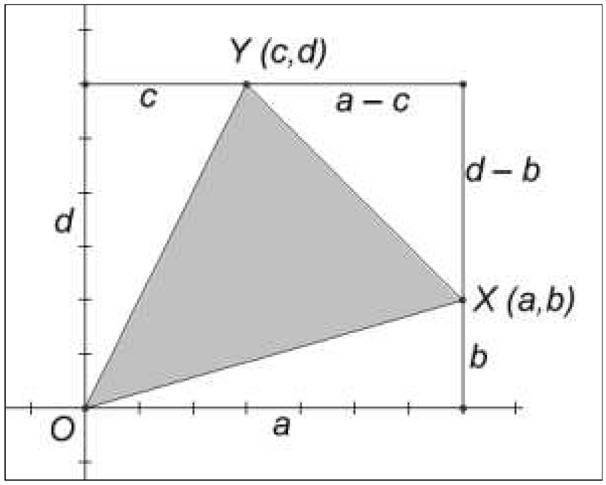
Problem

In the diagram below, find the area of triangle OXY in terms of *a*, *b*, *c*, and *d*.



Solution

Consider the following diagram.



The area of the surrounding rectangle is ad.
The area of each of the surrounding triangles are given by $cd/2$, $(a-c)(d-b)/2 = (ad-ab-cd+bc)/2$, and $ab/2$. Therefore the total area of these triangles is $(ad+bc)/2$.

Hence the area of triangle OXY is $ad - (ad+bc)/2 = (ad-bc)/2$.

Given two general points X (a, b) and Y (c, d), prove that the area of triangle OXY is given by $|ad-bc|/2$, where $|n|$ represents the absolute value of n.

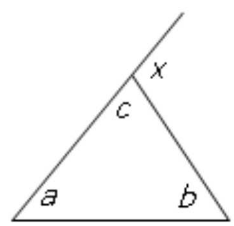
TRI ANGLES

Problem

Prove that exterior angle of a triangle is equal to the sum of two opposite interior angles.

Solution

Consider the following triangle.



Clearly $a + b + c = 180^o$ (angles in a triangle).

Also $x + c = 180^o$ (angles on a straight line).

Therefore $a + b = x$ (**Q.E.D.**).

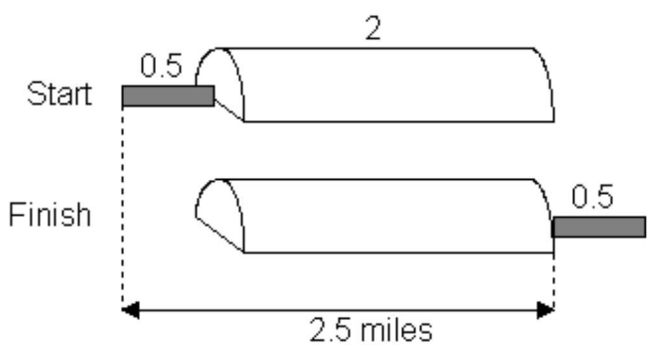
TUNNEL TRAIN

Problem

A train, length ½ mile, travelling at a constant speed of 30 miles per hour enters a tunnel which is 2 miles long. How long will it take for the train to completely pass through the tunnel?

Solution

If the train is travelling at a speed of 30 miles per hour (60 minutes), this is equivalent to 1 mile in 2 minutes.



It can be seen that the train will need to travel a distance of 2.5 miles to pass completely through the tunnel. If the train travels 1 mile in 2 minutes, it will travel 2.5 miles in $2.5 \times 2 = 5$ minutes .

If the train, travelling at a different constant speed, takes 4 minutes to pass completely through the same tunnel, how fast is it travelling?

UNSORTED SOCKS

Problem

A draw contains an unsorted collection of black, navy, white, and brown coloured socks. If socks are taken at random, one at a time, what is the minimum number which must be taken to be certain of finding five matching pairs?

Solution

Obviously we could be lucky and take ten socks of which we have five matching pairs, but equally we might, for example, have taken eight black socks, one white sock, and one brown sock.

We shall approach this problem by a process called induction, which, simply put, means one thing leading to the next. We begin by considering the base case: how many socks are required to be certain that we have one matching pair?

The worst case scenario is that the first four socks could all be different colours, but the next sock must match one of the four. So we can be certain of picking at least one matching pair with five socks.

Suppose we took six socks. By our previous reasoning we can be certain that there is at least one matching pair, so let us remove this pair. However, now we cannot be certain that a matching pair will be found among the four remaining socks. Instead if we took seven socks we know that at least one pair matches, and by removing that pair we have five socks remaining of which we can be certain that a further matching pair could be found. In other words, we can be certain of finding two matching pairs among seven socks.

By continuing with this reasoning we can see that by increasing the number of socks by two each time we can be certain of selecting an additional matching pair. Therefore, the number of socks required to be certain of finding p matching pairs is given by $2p + 3$.

Hence to be certain of selecting five matching pairs we must take $2\times 5 + 1 = 13$ socks.

Can you obtain a formula to find the minimum number of socks which must be taken to be certain of finding p matching pairs when the draw contains an unsorted collection of c different coloured socks?

UP DOWN LEFT RIGHT

Problem

Your sister and your best friend have been acting rather suspiciously recently. One evening you notice your sister secretly pass a piece of paper into your friend's pocket. You wait for the right opportunity and remove the note to discover the following:

```
R1U0R0L4L6D5L7D1 L4U6R0R2U4 U6L0D8 R1D1R2.
R1U0L7L4D1U8R2R1 L2L3L8D4D5L2D3!
```

Confused, you record the secret message and return the piece of paper. The next day, while your sister is out, you look around the house and discover a scrap of paper in one of the waste paper bins. It has the following similar message written on it, but it seems to contain a clue:

```
R2D1R1L8 = TEST
```

Can you decode the original message and work out what your sister and best friend are up to?

Solution

The method is, indeed, very misleading and sneaky...

Using the letters D, L, R and U as key markers in the alphabet, the digits that follow represent the numbers of letter after the key marker that the coded letter lies. For example, D3 (3 letters after D) = G.

ABC**D**EFGHIJK**L**MNOPQ**R**ST**U**VWXYZ

Checking with the message on the scrap of paper, R2D1L7L8: R2 (2 letters after R) = T, D1 (1 letter after D) = E, R1 (1 letter after R) = S and L8 (8 letters after L) = T; spelling TEST. Notice how the last letter, T, was disguised by using L rather than R (the nearer marker) to avoid duplication with the first letter, T.

Using this system,

R1U0R0L4L6D5L7D1 L4U6R0R2U4 U6L0D8 R1D1R2.
R1U0L7L4D1U8R2R1 L2L3L8D4D5L2D3!

becomes

SURPRISE PARTY ALL SET.
SUSPECTS NOTHING!

WEIGHTY LOGIC

Problem

Daddy Bear is 60 kg heavier than Mummy Bear, and Baby Bear is 20 kg heavier than Goldilocks. Given that the lightest and the heaviest weigh 20 kg more than the combined weights of the other two, and the combined weight of all four is 300 kg, how heavy is Goldilocks?

Solution

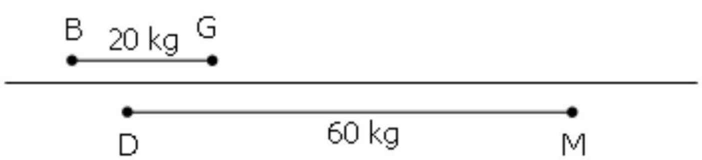
Let the weights of Daddy Bear, Mummy Bear, Baby Bear, and Goldilocks be D, M, B, and G respectively.

We know that $D = M + 60$ and $B = G + 20$.

As $D + M + B + G = (M + 60) + M + (G + 20) + G = 2M + 80 + 2G = 300 \Rightarrow M + G = 110$ (equation 1).

However, we must be careful to make any false assumptions about the order of their weights. There are a number of possibilities for which the difference between the heaviest/lightest pair and the other two need to be considered separately.

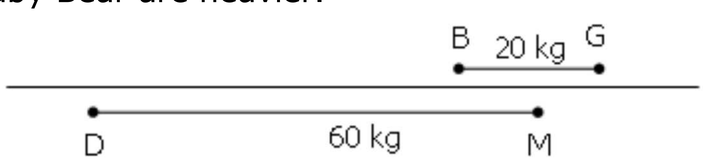
If Baby Bear is the same weight or heavier than Daddy Bear, we can see that Mummy Bear must be the lightest. Note that is doesn't matter which of Daddy Bear or Goldilocks is the heaviest; they're both the "middle" weights:



In which case, $(B + M) - (D + G) = (G + 20 + M) - (M + 60 + G) = -40$.

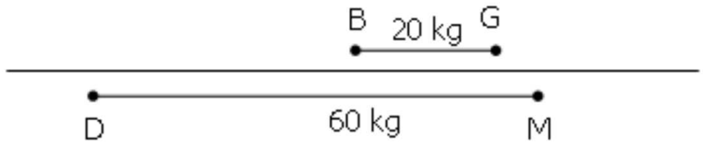
This is not consistent with the claim that the difference is 20 kg. So we know that Daddy Bear is the heaviest, which leaves two possibilities:

i. Goldilocks is the lightest, and note, once more, that it doesn't matter which of Mummy Bear or Baby Bear are heavier.



$(D + G) - (B + M) = (M + 60 + G) - (G + 20 + M) = 40$

ii. Mummy Bear is the lightest.



$$(D + M) - (B + G) = (M + 60 + M) - (G + 20 + G) = 2M + 40 - 2G$$

Only the latter is consistent with the claim that the difference is 20 kg; that is, $2M + 40 - 2G = 20 \Rightarrow M - G = -10$ (equation 2).

(equation 1) – (equation 2):$\rightarrow$ $2G = 120$, which means that Goldilocks weighs 60 kg; for completeness it can be shown that M = 50 kg, D = 110 kg, and B = 80 kg.

X HITS THE SPOT

Problem

Sitting in your comfy chair one evening and reading a good book, you notice something white behind the ventilation grill next to the fireplace. Carefully removing the grill you discover a white envelope containing a diskette and a piece of paper. Excitedly you check-out the disk, which contains a program. When you run the program you are greeted with the cryptic message: X hits the spot, but what does X represent? A small text box on the screen seems to be awaiting an answer. A little disappointed and confused you unfold the piece of paper and read the following:

```
T R O O C H X X I A O O
O P X O N F O O C T X O
H O O X L E X O I A O X
E H O O W T O X E U X O
```

What does X represent?

Solution

A subtle clue was provided in the location of the white envelope; as it was behind a grill, the code makes us of a "grille" principle.

The first stage is to realise that every 3rd and 4th character is either an O or an X. Begin by writing the encrypted message in blocks of four:

```
T R O O
C H X X
I A O O
O P X O
N F O O
C T X O
H O O X
L E X O
I A O X
E H O O
W T O X
E U X O
```

Then to decrypt we use the 3rd and 4th characters on each line as a template/grille. By placing the 3rd and 4th characters on top of the 1st and 2nd characters, we ignore any characters that are not marked by an X:

```
    O O
C H X X
```

```
    O O
O   X O
    O O
C   X O
  O O X
L   X O
  A O X
    O O
  T O X
E   X O
```

Hence the X represents CHOCOLATE – which certainly hits the spot! ;o)