Mechanical Properties - Stresses & Strains

Types of Deformation : Elasic Plastic Anelastic

Elastic deformation is defined as *instantaneous recoverable deformation*

Hooke's law : For *tensile* loading, $\sigma = E\,\varepsilon$

where σ is *stress* defined as the load per unit area : $\sigma = P/A_o$, N/m^2, Pa

and *strain* is given by the change in length per unit length $\varepsilon = \frac{\Delta l}{l_o}$, %

The proportional constant E is the *Young's modulus or modulus of Elasticity* :

E ~ 10×10^6 psi [68.9 GPa] for metals [varying from 10×10^6 psi for Al, 30×10^6 for Fe and 59×10^6 for W].

Poisson's Ratio [ν] : ratio of lateral contraction to longitudinal elongation $\nu = -\varepsilon_x / \varepsilon_z = -\varepsilon_y / \varepsilon_z$ [for isotropic materials]; in general, $\nu \sim 0.3$

Thus the total contractile strains is *less* than the expansion along the tensile axis thereby resulting in a slight increase in the volume of the material under stress - this is known as *Elastic Dilation* .

Modulus Of Rigidity or Shear Modulus [G] : $G = \tau / \gamma$; G is the shear modulus and is related to E and ν, $G = E / 2(1+\nu)$.

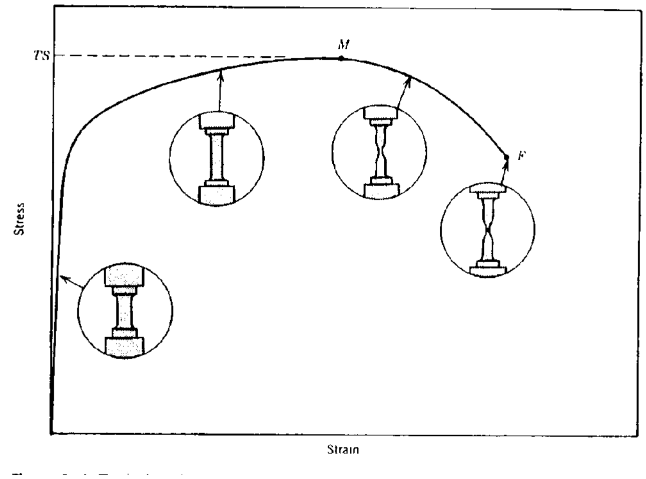
Bulk Modulus [κ] : the change in volume to the original volume is proportional to the hydrostatic pressure [σ_{hyd}] : $\Delta V/V = \beta\, \sigma_{hyd}$, where β is the *compressibility* .

The inverse of the compressibility is the bulk modulus [κ] : $\kappa = 1/\beta$.

κ is also related to E and ν : $\kappa = E / 3(1-2\nu) = 2G(1+\nu) / 3(1-2\nu)$.

• • • Thus one can evaluate the various elastic moduli from one or more experimentally evaluated constants. Note that the elastic moduli are related to the interatomic bonding and thus decrease [slightly] with *increasing* temperature. Any change in the crystal structure, for example following a phase change [polymorphism], one notes a distinct change in the elastic moduli.

Stress - Strain Curve



Definitions

Nominal (engineering) *vs* *True*

$S = \frac{P}{A_o}$, $e = \frac{\Delta l}{l_o}$

$\sigma = \frac{P}{A}$, $\varepsilon = \ln\left(\frac{l}{l_o}\right)$

Nominal (engineering)	True
Proportional limit (PL)	$\sigma = S(1+e)$ & $\varepsilon = \ln(1+e)$
Yield strength (S_y) 0.2% offset; (S_{LY})	*true* Yield stress (σ_y) 0.2% offset
Tensile strength (TS or UTS or S_{UTS})	*true* Tensile strength (TS or UTS or σ_{UTS})
Fracture strength (S_F)	*true* Fracture strength (σ_F)
Uniform elongation (e_u)	*true* Uniform strain (ε_u)
Total elongation *(ductility)* (e_t or e_f *in 2”*)	*true* Total elongation *(ductility)* (ε_t or e_f *in 2”*)
Necking strain ($e_n = e_t - e_u$)	*true* Necking strain ($\varepsilon_n = \varepsilon_t - \varepsilon_u$)
Reduction in area *(ductility)* (RA)	*Volume **is** conserved* $A_o l_o = Al$
Volume increases *(Elastic Dilation)*	

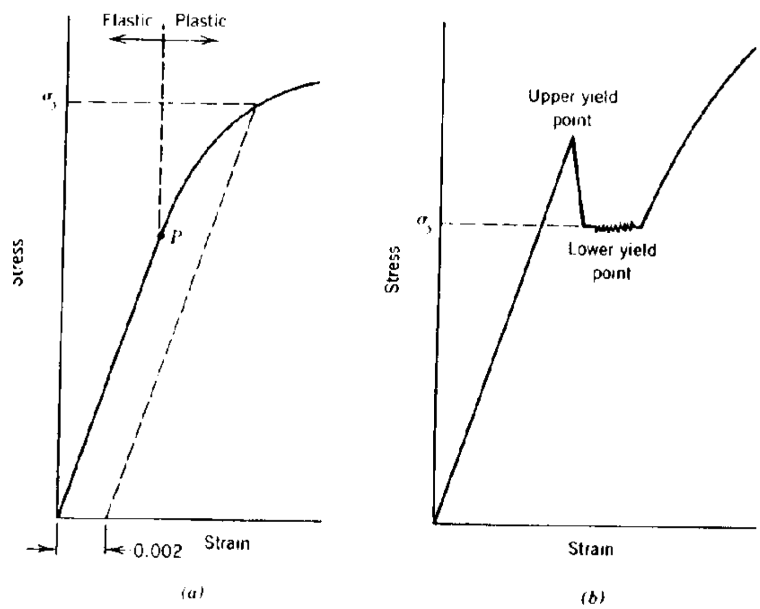
Energy to fracture

Elastic ($\sigma = E\,\varepsilon$) *Plastic* ($\sigma = K\,\varepsilon^n$)

Resilience ($U^{el} = \frac{\sigma^2}{2E}$) units (J/m^3) Toughness ($J = K\,\frac{\varepsilon^{n+1}}{n+1}$)

σ - ε curves

smooth (SSs & fcc)   with yield point (steels & bcc)



Rate Effects

Plastic deformation is rate dependent

(generally at high temperatures) : $\sigma \equiv f(\dot{\varepsilon}) = A\ \dot{\varepsilon}^m$, $m = SRS = \left(\dfrac{d\ln\sigma}{d\ln\dot{\varepsilon}}\right)_{T,\varepsilon}$

$m \sim 0$ at low temperatures $m|_{max} = 1$

$m\uparrow \quad e_t\uparrow \quad$ vs $\quad n\uparrow \quad e_u\uparrow$

Group Work :

left of the instructor : (1) Derive relation between σ and S :

right of the instructor : (2) Derive relation between ε and e :

all (3) Show that $\varepsilon_u = n$

Concept of Stress

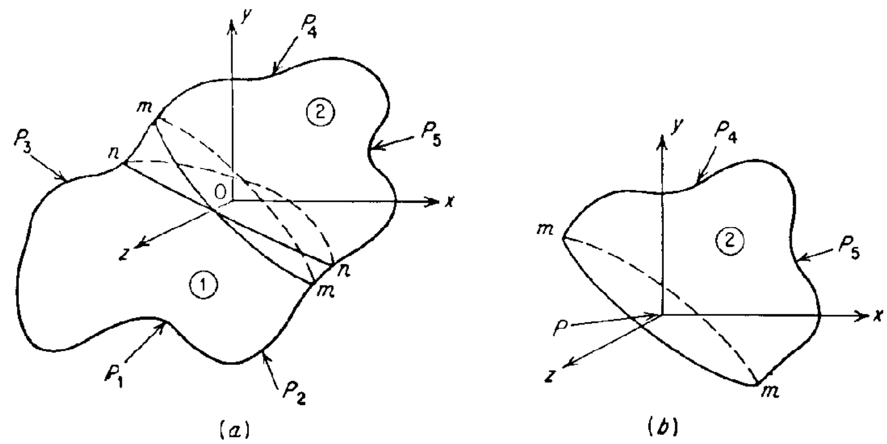


Figure 1-5 (*a*) Body in equilibrium under action of external forces $P_1, \ldots, P_5$; (*b*) forces acting on parts.

$$\sigma = \lim_{A \to 0} \frac{F}{A}$$

Force extended on reference section by <u>remaining sections</u> $\Leftarrow$ body in <u>equilibrium</u>

Normal and Shear Stresses

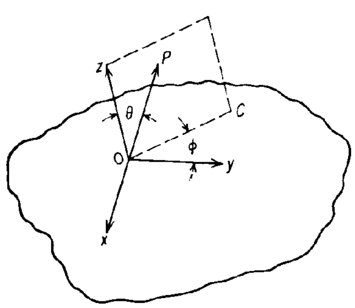


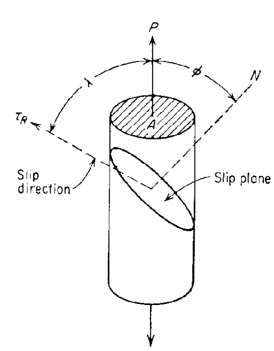
Figure 1-6 Resolution of total stres nents.

$$\sigma_N = \frac{F}{A} \cos\theta$$

$$\sigma_{shear} = \tau = \frac{F}{A} \sin\theta$$

$\tau_x =$

$\tau_y =$



recall RSS **(τ_{RSS}) :**

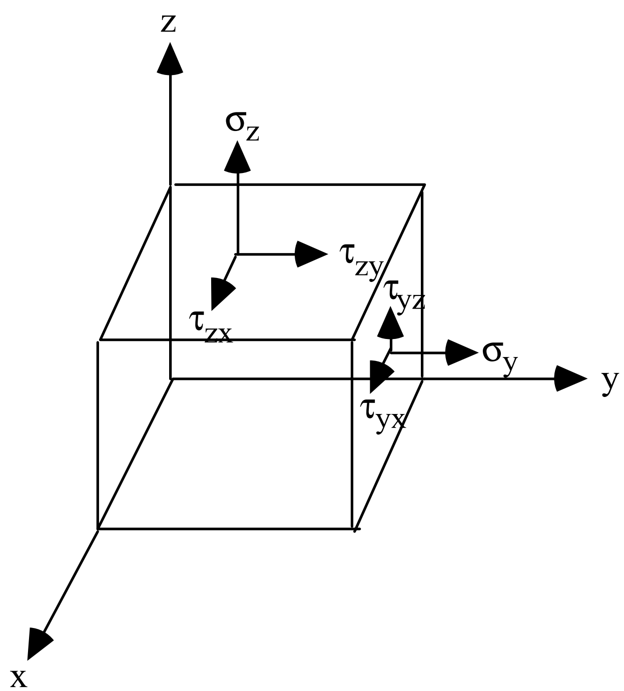
Stress - Strain Relationships

Elastic Behavior

$\overline{F}$, Force is a vector (1st rank tensor)

while σ_{ij} should be specified with 2 directions : plane normal and force direction - acts on plane perpendicular to i along j direction

Sign Convention (Fig. 2.2)



Tension : +ive
Compression : -ve
τ_{ij} (or σ_{ij}) is +ive if both i and j are +ive or –ive
τ_{ij} (or σ_{ij}) is - ive if one of i and j is - ive

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$ no net moment $\Rightarrow \sigma_{ij} = \sigma_{ji}$ or only 6 components or

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{pmatrix}; \text{ book notation} \Rightarrow \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ & & \sigma_z \end{pmatrix}$$

$\Rightarrow$ Values of σ_{ij} depend on the choice of reference axes (see 2-D example 2.3) $\Leftarrow$

{can determine these components using tensor transformations, Mohr circle, etc.}

First, we look at 3 important examples of Stress States
1. Plane Stress (p.20) 2. Hydrostatic & Deviatoric Stresses (p.46)
3. Principal Stresses (various sections such as 2.14)

Plane Stress (p. 20)

- stresses are zero in one of the primary directions (or 2-D stress state) -

Examples :

1. Thin sheet with loaded in the plane (stresses are zero along the thickness direction)
2. Pressurized thin cylinder (stresses along r or thickness direction are zero for cylinders when wall-thickess is about 1/10th of diameter) :

$$\sigma_\theta = \frac{Pr}{t}, \sigma_z = \frac{Pr}{2t} \text{ with } \sigma_r \approx 0$$

Principal Stresses (p. 22)

Principal Planes are the planes on which maximum normal stresses act with no shear stresses and these stresses are the **Principal Stresses**

Designate σ_1, σ_2, σ_3 $\Leftrightarrow$ implies no shear stresses or $\sigma_{ij} = \sigma_{ij}\,\delta_{ij}$

Proof is clear from Mohr's circle representation (see text Fig.2.6) for 2-D

$$\boxed{\textbf{Note :}\ \tau_{max} = \frac{\sigma_1 - \sigma_3}{2}}$$

Hydrostatic and Deviatoric Stresses (p. 46)

Total stress tensor can be divided into two components (Fig. 2-18)

Hydrostatic or mean stress tensor (σ_m) involving only pure tension or compression

& **Deviatoric** stress tensor (σ'_{ij}) representing pure shear with no normal components

$$\sigma_m = \frac{\sigma_{kk}}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

$$\sigma_{ij} = \sigma'_{ij} + \frac{1}{3}\delta_{ij}\sigma_{kk}$$

Example on Hydrostatic and Deviatoric Stresses

Given the stress state: $\sigma_{ij} = \begin{pmatrix} 80 & 20 & -50 \\ 20 & -40 & 30 \\ -50 & 30 & 50 \end{pmatrix}$, **a**. *Find the hydrostatic part of the stresses*. **b**. *Find the deviatoric part of the stresses*.

Ans. (a) $\sigma_{ij}^{hyd} = \sigma_m \delta_{ij} = \frac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33})\ \delta_{ij}$ where $\sigma_m = \frac{1}{3}(80 - 40 + 50) = 30$ so that

$$\sigma_{ij}^{hyd} = \begin{pmatrix} 30 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 30 \end{pmatrix}.$$

(b) By definition, $\sigma_{ij}^{dev} = \sigma_{ij} - \sigma_{ij}^{hyd} = \begin{pmatrix} 80 & 20 & -50 \\ 20 & -40 & 30 \\ -50 & 30 & 50 \end{pmatrix} - \begin{pmatrix} 30 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 30 \end{pmatrix} = \begin{pmatrix} 50 & 20 & -50 \\ 20 & -70 & 30 \\ -50 & 30 & 20 \end{pmatrix}$

Note that the mean hydrostatic stress for $\sigma_{ij}^{dev} = (\sigma_{11}^{dev} + \sigma_{22}^{dev} + \sigma_{33}^{dev}) = 0$, as expected.

Principal Stresses (p. 22)

Principal Planes are the planes on which maximum normal stresses act with no shear stresses and these stresses are the **Principal Stresses**

Designate σ_1 , σ_2 , σ_3 ⁄ implies no shear stresses or $\sigma_{ij} = \sigma_{ij}\,\delta_{ij}$

Proof is clear from Mohr's circle representation (see text Fig.2.6) for 2-D

For 3-D such an analogy is not useful and these are determined from the roots of σ of the determinant (cubic in σ) :

$$\begin{Vmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{Vmatrix} = 0 \;;$$

Expand the determinant ┌

$$0 = \sigma^3 - (\sigma_{11} + \sigma_{22} + \sigma_{33})\,\sigma^2 + (\sigma_{11}\sigma_{22}+\sigma_{22}\sigma_{33}+\sigma_{33}\sigma_{11}-\sigma_{12}^2 -\sigma_{23}^2 -\sigma_{31}^2)\,\sigma - (\sigma_{11}\sigma_{22}\sigma_{33}+2\sigma_{12}\sigma_{23}\sigma_{31}-\sigma_{11}\sigma_{23}^2 -\sigma_{22}\sigma_{31}^2 -\sigma_{33}\sigma_{12}^2)$$

$$\boxed{\sigma^3 - I_1\sigma^2 - I_2\sigma - I_3 = 0}$$

where I's are *invariants* of the stress tensor (Eqs. on p.28 of Text) :

$$I_1 = (\sigma_{11} + \sigma_{22} + \sigma_{33})$$

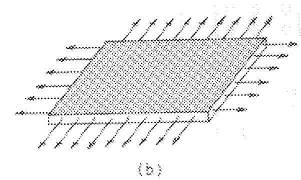
$$I_2 = -(\sigma_{11}\sigma_{22}+\sigma_{22}\sigma_{33}+\sigma_{33}\sigma_{11}-\sigma_{12}^2 -\sigma_{23}^2 -\sigma_{31}^2)$$

$$I_3 = \sigma_{11}\sigma_{22}\sigma_{33}+2\sigma_{12}\sigma_{23}\sigma_{31}-\sigma_{11}\sigma_{23}^2 -\sigma_{22}\sigma_{31}^2 -\sigma_{33}\sigma_{12}^2$$

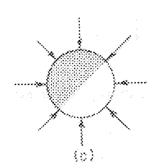
[text - x,y,z ⁄ 1,2,3]

(a) Uniaxial stress : $\begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

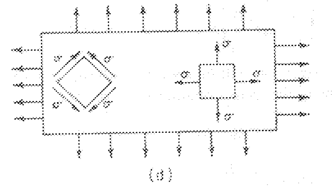
(b) Biaxial stress : $\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$



(c) Hydrostatic pressure : $\begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$



(d) Pure shear : $\begin{pmatrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$



• Special Stress States •

Normal and Shear Stresses on a Given Plane
[Cut-Surface Method]

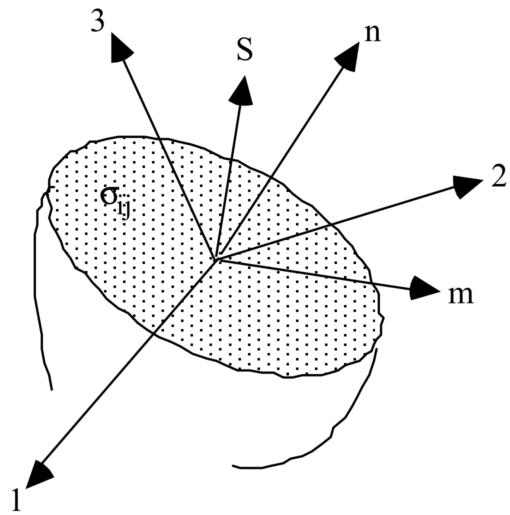
Given σ_{ij} in reference system 1 2 3 ;

$\hat{n}$ is the unit vector normal to the plane =
$n_1 \; n_2 \; n_3$

$\hat{m}$ is the unit vector **in** the plane =
$m_1 \; m_2 \; m_3$

σ_N = normal stress along $\bar{n}$

τ = shear stress along $\bar{m}$



Note : $\hat{n} \cdot \hat{m} = 0;\; n_1^2 + n_2^2 + n_3^2 = 1$ and $m_1^2 + m_2^2 + m_3^2 = 1$

note : if $\bar{n} = 1, 2, 5 \;\Rightarrow\; \hat{n} = \frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}$ $\Leftrightarrow \hat{n}$ is a unit vector
where $\sqrt{1^2 + 2^2 + 5^2} = \sqrt{30}$ so that $n_1^2 + n_2^2 + n_3^2 = 1$.

1. Find the stress vector ($\bar{S}$)

{the *stress vector* : the vector force per unit area acting on the *cut* }: $\bar{S} = \sigma \cdot \hat{n} \;\Rightarrow$

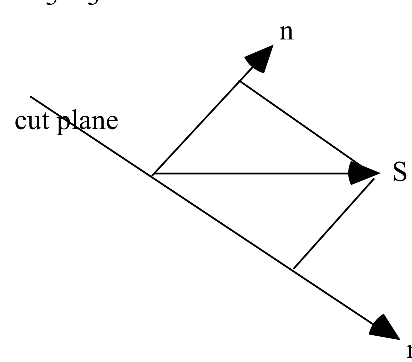
$$\begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \Rightarrow S_i = \sum_{k=1}^{3} \sigma_{ik}\, n_k \text{ ; i.e. } S_1 = \sigma_{11}n_1 + \sigma_{12}n_2 + \sigma_{13}n_3 \text{ ; etc.}$$

2. σ_N and τ follow as : $\sigma_N = \bar{S} \cdot \hat{n} = S_1\, n_1 + S_2\, n_2 + S_3\, n_3$

$\tau = \bar{S} \cdot \hat{m} = S_1\, m_1 + S_2\, m_2 + S_3\, m_3$

and τ_{max} occurs when n, S and m are in the same plane, or from Fig.

$|\bar{S}|^2 = \sigma_N^2 + \tau_{max}^2$



Normal and Shear Stresses on a Given Plane
[Cut-Surface Method]

• EXAMPLE •

A stress state in a given reference frame is (MPa): $\sigma_{ij} = \begin{pmatrix} 8 & 2 & -5 \\ 2 & -4 & 3 \\ -5 & 3 & 6 \end{pmatrix}$

Assume that the stresses are independent of position (uniform stress state).
A plane "cut" is made through the body such that the normal to the cut is $\bar{n} = \sqrt{2}, 2, -2$.

a. What is the normal stress σ_N on the plane ?

b. What is the shear stress τ along the direction $\bar{m}$ = 0, 1, 1 in the plane ?

c. What is the *maximum* shear stress in the plane (consider all directions in the plane)?

Answer :

$\hat{n} = \frac{\sqrt{2}}{\sqrt{10}}, \frac{2}{\sqrt{10}}, \frac{-2}{\sqrt{10}}$ and $\hat{m} = 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$.

$$S_1 = \sigma_{11}n_1 + \sigma_{12}n_2 + \sigma_{13}n_3 = \frac{1}{\sqrt{10}}\left[8(\sqrt{2}) + 2(2) + (-5)(-2)\right] = 8 \text{ MPa},$$

Similarly, $S_2 = -3.53$ MPa and $S_3 = -4.13$ MPa.

(a) $\sigma_N = S_1n_1 + S_2n_2 + S_3n_3 = \frac{1}{\sqrt{10}}\left[8(\sqrt{2}) + (-3.53)(2) + (-4.13)(-2)\right] = 3.96$ MPa

(b) $\tau = \bar{S}\cdot\hat{m} = S_1 m_1 + S_2 m_2 + S_3 m_3 = \frac{1}{\sqrt{2}}\left[8(0) + (-3.53)(1) + (-4.13)(1)\right] = -5.42$ MPa.

note: $\boxed{\text{“-” sign means the shear stress acts in the } -\hat{m} \text{ direction}}$

(c) To find the maximum shear stress in the plane : Since $\tau = \bar{S}\cdot\hat{m}$, the maximum *projection* of $\bar{S}$ along $\hat{m}$ will occur when these 2 vectors are coplanar (containing $\hat{n}$ also) - see Fig. below:

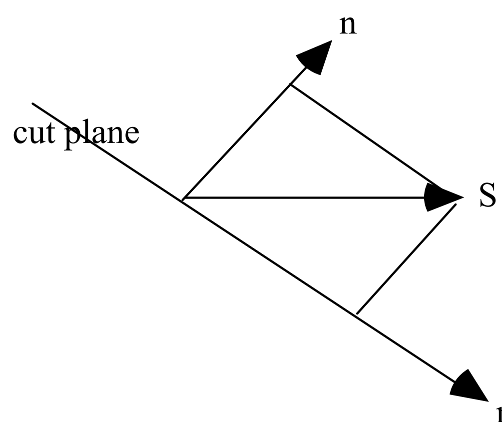
$$|\bar{S}|^2 = S_1^2 + S_2^2 + S_3^2 = (8)^2 + (-4.13)^2 + (-5.42)^2 = 93.5.$$

From the figure, note that

$$|\bar{S}|^2 = \sigma_N^2 + \tau_{max}^2 \quad \text{or}$$

$$\tau_{max} = \sqrt{|\bar{S}|^2 - \sigma_N^2} = \sqrt{93.5 - (3.96)^2} =$$

8.82 MPa.

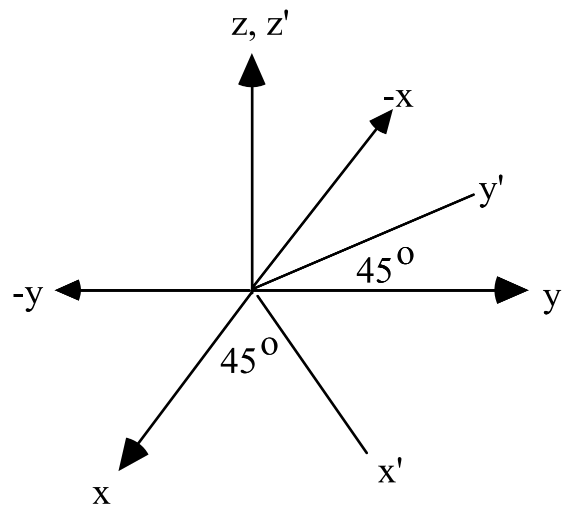


Example : Stress Tensor Transformations

Given $\sigma_{ij} = \begin{pmatrix} 10 & 5 & 0 \\ 5 & 20 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ wrt x,y,z axes.

New axes (x',y',z') are rotated 45˚ around z-axis.

Need to find σ'_{ij}. $\boxed{\sigma'_{ij} = a_{ik}\, a_{jn}\, \sigma_{kn}}$



a. Calculate τ'_{xy} .

$$\theta_{ij} : \begin{array}{c|ccc} & x & y & z \\ x' & 45^o & 45^o & 90^o \\ y' & 135^o & 45^o & 90^o \\ z' & 90^o & 90^o & 0^o \end{array}$$

$$\begin{aligned} \tau'_{xy} = {} & a_{xx}\, a_{yx}\sigma_{xx} + a_{xx}\, a_{yy}\sigma_{xy} + a_{xx}\, a_{yz}\sigma_{xz} && = \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right)(10) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)(5) + 0 \\ & + a_{xy}\, a_{yx}\sigma_{yx} + a_{xy}\, a_{yy}\sigma_{yy} + a_{xy}\, a_{yz}\sigma_{yz} && + \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right)(5) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)(20) + 0 \\ & + a_{xz}\, a_{yx}\sigma_{zx} + a_{xz}\, a_{yy}\sigma_{zy} + a_{xZ}\, a_{yz}\sigma_{zz} && + \quad 0 \quad + \quad 0 \quad + 0 \end{aligned}$$

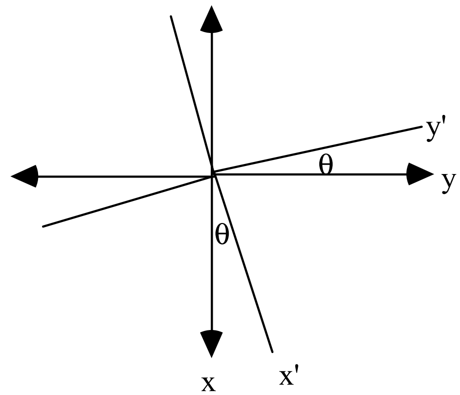
$$= 5 \text{ MPa}$$

b. Show that $\sigma'_x = 20$ MPa and $\sigma'_y = 10$ MPa.

Thus $\sigma'_{ij} = \begin{pmatrix} 20 & 5 & 0 \\ 5 & 10 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Note that $\sigma_x + \sigma_y = \sigma'_x + \sigma'_y$ (= 30 MPa)

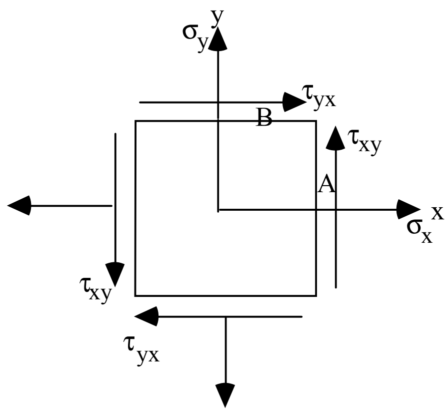
Example $\boxed{2}$: Same as above but in 2-D for a general case

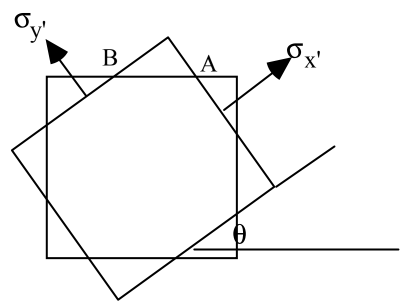
x', y' rotation by θ



$$\theta_{ij}: \begin{array}{c|cc} & x & y \\ \hline x' & \theta & 90-\theta \\ y' & 90+\theta & \theta \end{array}$$

$$a_{ij} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$





$$\tau'_{xy} \text{ (or } \tau'_{xy}) = a_{xx}\, a_{yx}\, \sigma_{xx} + a_{xx}\, a_{yy}\, \tau_{xy} + a_{xy}\, a_{yx}\, \tau_{yx} + a_{xy}\, a_{yy}\, \sigma_{yy}$$

$$= -\sin\theta\cos\theta\,\sigma_{xx} + \cos^{y}\theta\,\tau_{xy} - \sin^{y}\theta\,\tau_{xy} + \sin\theta\cos\theta\,\sigma_{yy}$$

$$= \frac{\sigma_{yy} - \sigma_{xx}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \Rightarrow \text{Eq. 2.7}$$

Similarly find $\sigma'_x = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$ (Eq 2.5)

and $\sigma'_y = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$ (Eq 2.6)

& Fig. 2.4 shows variation of these 3 stresses with θ.

Note : $\sigma_x + \sigma_y = \sigma'_x + \sigma'_y$ as should be since I_1 is invariant ⇔

i.e., sum of normal stresses on mutually perpendicular planes is invariant.

same thing can be done using Mohr's circle representation (easier for 2-D case)

If $\tau'_{xy} = 0$, principal planes and stresses: $\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$ (Eq. 2.8)

whereas maximum shear stresses when $\tan 2\theta = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$ (Eq.2.10) & τ_{max} given by Eq. 2.11.