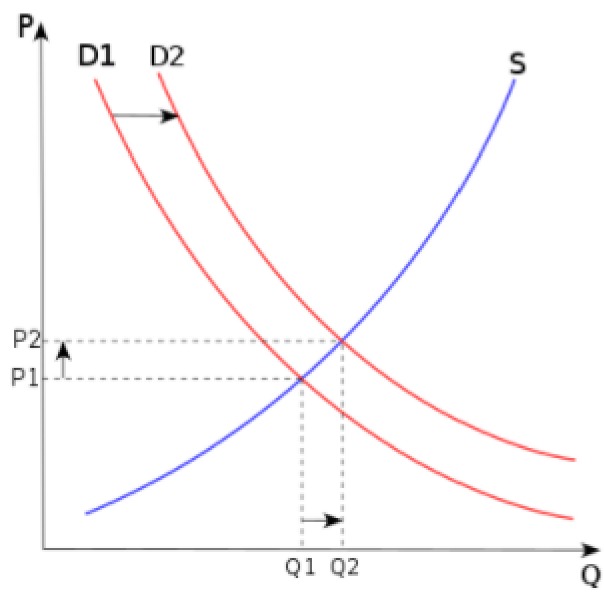
Application of Differential and Difference Equations

1. **Comparative Statics**

In economics, comparative statics is the comparison of two different economic outcomes, before and after a change in some underlying exogenous parameter.

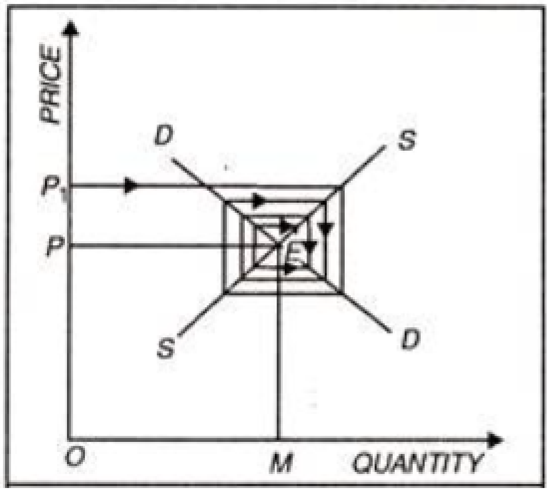
As a type of static analysis it compares two different equilibrium states, after the process of adjustment (if any). It does not study the motion towards equilibrium, nor the process of the change itself. Comparative statics is commonly used in Microeconomics to study changes in supply and demand when analyzing a single market, and in Macroeconomics to study changes in monetary or fiscal policy when analyzing the whole economy.



In this graph, comparative statics shows an increase in demand causing a rise in price and quantity. Comparing two equilibrium states, comparative statics does not describe *how the increases actually occur*.

2. **Dynamic Economic Analysis**

Dynamic economic analysis is a complex approach for the study of economic variables because it is based on time element. In dynamic economics, we study the economic variables like consumption function, income and investment in a dynamic state. Comparative economic statics does not show the path of change of the old to new equilibrium. But in dynamic economics, we also study the path of change or the movement towards equilibrium. This path can be explained with the help of the diagram given below which relates to price determination in a market.



In the diagram drawn above, DD is the demand curve and SS is the supply curve. Suppose the initial price is OP_1. At OP_1 price, supply of the commodity is more than its demand. As a result, price falls. This process of falling price continues till demand becomes equal to the supply of the commodity.

E is the point where demand for and supply of the commodity are equal. This is a point of equilibrium. Here, OP is the equilibrium price. OM is the quantity demanded and supplied. This equilibrium path of the price change is shown through the arrow lines in the figure. Dymanic Economic Analysis may be done through Difference and Differential equation.

6.2 Review of the Concept

	It would be difficult to comprehend the contemporary literature of economics if one does not understand theory of Differential and Difference equations.
	Differential Equation
Definition	An equation that involves a dependent and independent variable and at least one derivative of the dependent variable with respect to the independent variable is called a differential equation.
	A differential equation expresses the rate of change of the current state as a function of the current state. A simple illustration of this type of dependence is changes of the Gross Domestic Product (GDP) over time. Consider state x of the GDP of the economy. The rate of change of the GDP is proportional to the current GDP. $x'(t) = gx(t)$,

	where t stands for time and x'(t) the derivative of the function x with respect to t. The growth rate of the GDP is x'/x. If the growth rate g is given at any time t, the GDP at t is given by solving the differential equation. The solution is x(t) = x(o)e^gt The solution tells that the GDP decays (increases) exponentially in time when g is negative/ positive.
Example	• dy/dx = X + 14 • xdy = ydx
	The Order of a differential equation is the order of the heighest derivatives or differential coefficient occurring in it.
Use	A differential equation expresses the rate of change of the current state as a function of the current state.
	Differential equations can describe exponential growth and decay, the population growth of species or the change in investment return over time.
	Differential Equations are used to figure out what something does over time.
	Type
Ordinary	a single independent variable only
Partial	if there are two or more independent variables.
	An ordinary differential equation containing two or more dependent variables with their differential coefficients with respect to a single independent variable is called a total differential equation.
	Difference Equations
Definition	A Difference Equation may be considered as a function in which a dependent variable is dependent on its own previous value. The system may be initialized at some point t_0 (where $t_0 = 0$). The variable X_t takes values in some metric space (X, d) where d denotes the metric on X. X is sometimes referred to as the state space. Most of the time, we identify this metric space with Rn endowed with the Euclidean metric. If in an equation variable, changes in quantity is discussed is discussed at different point of time and time is measured as discret value, that is called difference equation Difference equation (D.E.) is an equation which involves in it the derivatives (dy/dx) of a function y = f(x). For example, dy/dx + py = q , while a difference equation (d.e.) involves differences of terms in a sequence and it can be expressed in terms of shift operator E or forward difference operator Δ. They are such equations which has dependent and independent variables with successive difference. An equation relating a function to one or more of its derivatives is called a ***DIFFERENTIAL EQUATION***. Difference equation is also called Recurrence relation.

Example	In its most general form, it can be written as $F(Xt , Xt-1, Xt-2, . . . , Xt-p, t)=0$ (6.1) The variable X_t = endogenous or dependent variable and is an n-vector, i.e $Xt \in Rn$, $n \geq 1$. In dynamical system theory, Xt =the state of the system and Rn = state space. n is called the dimension of the system. The difference between the largest and the smallest time index of the dependent variable explicitly involved is called the order of the difference equation. In the formulation (1.1), this is p with $p \geq 1$. In the difference equation above the time index appears explicitly as an argument of the function F.
Use of Difference Equation in Economics	When calculating change in quantity (such as income, consumption) (dependent variable) at a discrete time interval (independent variable. These quantities are dated according to the period which they refer.
Solution of Difference Equation	$X_t = a^t (x_0 - b/1-a) + b/1-a$ [when $a \neq 1$]---------------------------- (1)
	Classification of difference equation depends on following factors
Order of the equation	The order of the equation is the highest order of difference contained in the equation. i.A first-order difference equation only contains the first difference of a variable between two consecutive periods, like $y(t + 1) - y(t)$. ii.A second-order difference equation also contains the second difference in a variable between every two successive time periods, like $y(t + 2) - y(t)$.
Autonomous **an d** Nonautonomous	i. A difference equation is said to be autonomous if it does not depend on time explicitly, and it is this type of equations that will be mainly analyzed here. Autonomous difference equations of the form xn+1 = f(xn) may model populations of species with nonoverlaping generations such as fish, orchard pests, etc. The drawback of such models is that they do not account for environmental fluctuations or seasonal changes. Hence we are led to nonautonomous difference equations of the form xn+1 = fn(xn), $n \in Z +$.
Example	Economic growth and fluctuations

1. Difference between Difference Equation and Differential Equation

S N	Difference Equation	Differential Equation
1	Difference equation involves difference of terms in a sequence of numbers	Differential equation involves derivatives of function.
2.	there is discrete data range. The time index t takes on discrete values and	There is continuous data range.

	typically runs over all integer numbers Z, e.g. t = . . . , −2, −1, 0, 1, 2, . . .	
	If the change happens incrementally rather than continuously then, difference equations are used which are recursively defined sequences.	Differential equations are used in formulation of Economic Models where there is a continuously changing variables.
3.	Ex- If the change happens incrementally rather than continuously Interest, Income, saving over time	Ex- Used in Modelling
	Examples of incrementally changes include interest that is compound monthly, and seasonal businesses such as resorts business at a tourist place. role in the solution of most queueing models	

Important Concept of Economics

Concepts	Discussion
Multiplier	It measures how much an endogenous variable changes in response to a change in some exogenous variable. Keynes was pioneer to discuss the theory of multiplier.
Endogenous variable	In an economic model, an endogenous variable is a variable whose measure is determined by the model.
Exogenous variable	In an economic model, an exogenous variable is one whose measure is determined outside the model and is imposed on the model, and an exogenous change is a change in an exogenous variable.
Marginal Rate of Substitution	The marginal rate of substitution (MRS) is the quantity of one good that a consumer can forego for additional units of another good at the same utility level. MRS is one of the central tenets in the modern theory of consumer behavior as it measures the relative marginal utility.

Example 1(i) If R is the Marginal Rate of Substitution of y for x, show that one form of individual's utility function is u = (x+b)A (y+b)B where a, b, A and B are constants.

(ii) Obtain the demand function for which elasticity demand is one.

Solution- (i) MRS = + =

or,

or, =

Integrating we get,

B log (y+b) = - A log (x+a) + K [K= const]

Or, B log (y+b) + A log (x+a) = K

Or, log $(y+b)^B$ + log $(x+a)^A$ = K

Or, log $(y+b)^B$ log $(x+a)^A$ = K

Or, $(y+b)^B$ + $(x+a)^A$ = e^K =Ki

Or, $(y+b)^B$ + $(x+a)^A$ = u (utility)

(ii) Elasticity of Demand = - = 1 [q= quantity p= price]

Or, -

Or, -

Integrating we get,

-∫

K – log q = log p

Or, logp+logq = K

Or, logpq = K

Or, pq =e^K =Ki

pq = constant ---- demand function

Example 2: The marginal cost of manufacturing a product is given by C' (x) = 2+ 0.15x. Find the total cost function given c(0) = 100

[Ans- c(x) = 2x + 0.075x^2 +100]

3. Market Price Function

Suppose in the demand and supply function, quantity demanded and supplied are D and S respectively and p is the price

D = a - bp (a, b>0)…………………………………… (1)

S = -c+dp (c, d>0)………………………………………..(2)

And dp/dt = α(D-S) (α>0)……………………………(3)

[equation 3 implies that change in price with respect to time(t) is directly proportional to the excess of demand over supply (=D-S)

Substituting (2) and (1) in equation (3), we get

$$dp/dt = \alpha\{(a - bp) - (-c+dp)\}$$

$$\text{or, } dp/dt = \alpha\{(a+c) – (b+d)p\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Equ, (1) and (2) gives the equlabrium price p (bar):

For this price D = S

Or, a – bp(bar) = -c+ dp (bar)

Therefore, p(bar) = (a+c)/(b+d) or, (a+c) = (b+d) p(bar) …………(5)

On substituting equ. (5) in (4), we get

$$dp/dt = \alpha\{(b+d)\ p(\text{bar}) – (b+d)p\}$$

$$\text{or, } dp/dt + \alpha(b+d)p = \alpha(b+d)\ p(\text{bar})$$

$$\text{or, } dp/dt +kp = kp(\text{bar}) \ldots\ldots.[k= \alpha(b+d)]\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

this is the type of differential equation

$$dy/dx+ay =b$$

the solution of equation shows that when time tend to be ∞p tend to be p (bar) which is price of the equilibrium or in other words, in the long run, price will converge to the equilibrium price and in this way, the dynamic stability will be obtained.

1. Dynamic Multiplier

In ***economics***, a ***multiplier*** broadly refers to an ***economic*** factor that, when increased or changed, causes increases or changes in many other related ***economic*** variables. In terms of gross domestic product, the ***multiplier*** effect causes gains in total output to be greater than the change in spending that caused it.

It simply means that the economy is going to move by some multiple of the initiating change in expenditure. According to Keynes, the same **multiplier** effect that could cause a spiral into a Great Depression could also propel us out of that dismal fate.

For more - Solow, R. (1951). A Note on Dynamic Multipliers. *Econometrica, 19*(3), 306-316. doi:10.2307/1906816

I. With constant Autonomous Investment

We know that: $Yt = Ct+It+At$

Where

Ct = Consumption in period t

It = Induced Investment in period t and (The **Induced Investment** is a capital **investment** that is influenced by the shifts in the economy. These **investments** are made with the intention to generate profit out of such **investments**.)

At = Autonomous investment in period t (**Autonomous investment** is the portion of the total **investment** made by a government or other institution independent of economic considerations. These can include government **investments**, funds allocated to public goods or infrastructure, and any other type of **investment** that is not dependent on changes in GDP)

Now, if $It =0$ and $A_t =A$.

Then the above equation turns into $Yt = C_t+A$. In case consumption is lagged function of income, i.e. todays consumption depends upon yesterday's income then:

$C_t = cY_{t-1}$

Where c is MPC (In economics, the marginal propensity to consume (MPC) is defined as the proportion of an aggregate raise in pay that a consumer spends on the consumption of goods and services, as opposed to saving it. Marginal propensity to consume is a component of Keynesian macroeconomic theory and is calculated as the change in consumption divided by the change in income. MPC is depicted by a consumption line, which is a sloped line created by plotting the change in consumption on the vertical "y" axis and the change in income on the horizontal "x" axis.)

Therefore, $Yt -cYt-1 =A$---(3)

Solution of (3) will be:

$$Yt = Y^* +(Y0 -Y^*)c^t,$$

Where the equilibrium income Y^* is obtained by putting

$Yt =Y^*$ and $Yt-1 =Y^*$ in equ (3)

$Y^* = A/(1-c)$

Since MPC is usually less than 1(but greater than zero), c^t will tend to zero as t tend to be infinite, therefore, in the long run Yt converges steadily to Y. there is stable equilibrium. The speed of response depends upon (i) time-lag, and (ii) MPC. Shorter the lag, the more will be the speed of response and smaller the MPC, smaller will be the change in Y_t.

II. Dynamic Multipler (with change in Autonomous Investment)

Now suppose autonomous investment grows according to the geometric law:

$A_t = A_0(1+r)^t$ where A_0 is the initial autonomous investment and 'r' is the rate of growth of autonomous investment.

Hence, $Y_t = cY_{t-1} + A_0(1+r)^t$

$$Y_t - cY_{t-1} = A_0(1+r)^t$$

which still has the homogenous equation of the form:

$$Y_t - cY_{t-1} = 0$$

Of which the complimentary function (Y_c) will be:

$Y_c = Ec^t$ where E is an arbitrary constant**Dynamic Multiplier**

The concept of employment multiplier was first introduced in Economic Theory by Prof. R.F. Kahn, a Cambridge economist, in his article entitled, "The Relation of Home Investment to Unemployment" . The theory was further developed by Keynes when he discussed investment multiplier. The investment multiplier explains the cumulative effects of changes in investment on income via their effects on consumption expenditure.

$\triangle Y = \triangle I.k$

Where , $\triangle Y$ = Change of National Income

$\triangle I$=Change in Investment

K = Value of Multipler

Relation between Consumption and Income

C= $\alpha + \beta$ Y

Where C= consumption

α = intercept

β = slope of the curve

As income increases, consumption also increases but the slop of consumption curve than the Income curve.

E.3 Marginal Propensity to Consume (MPC)-

MPC is the proportion of increase in consumption with additional unit of increase in income. For example, if a household earns one extra dollar of disposable income, and the marginal propensity to consume is 0.65, then of that dollar, the household will spend 65 cents and save 35 cents.

MPC= dc/dY

Where dc= increase in consumption

dY = increase in additional unit of Household income (in Economics, Household is smallest unit. All those who eat from same kitchen are member of same household)

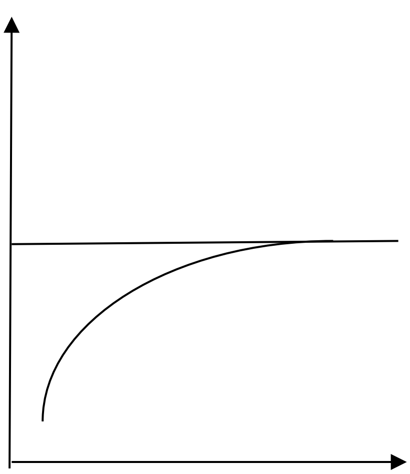
MPS = 1- MPC

Where MPS = Marginal Propensity to save

So, MRS+MPC = 1

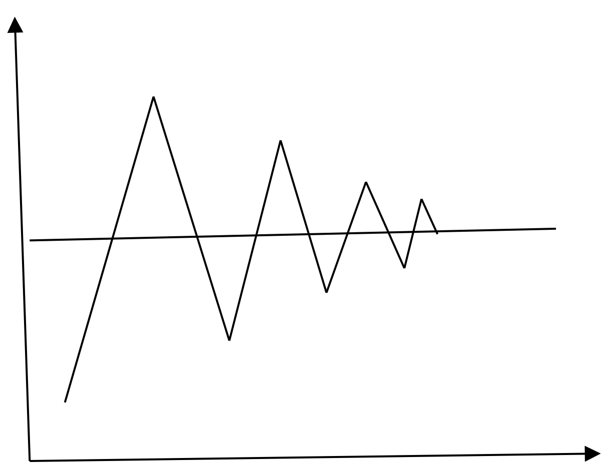
BUSINESS CYCLE

All major theories of business cycle consider fluctuation in investment as primary reason for income fluctuations. Initial causes of investment fluctuation may be both external factors as technological innovation, population growth, territorial growth, etc and internal factors as propensities to consume, inducement to invest, availability of bank credits etc.



O Time(t)

Figure1: Damped non-oscillatory



O Time (t)

Figure: Damped oscillatory income

Income movement   movement

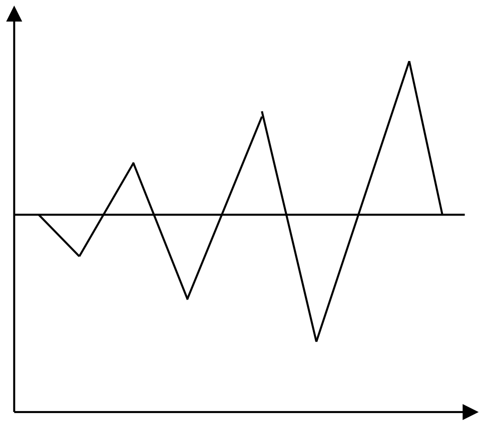


Figure 3- Explosive oscillatory Income movement

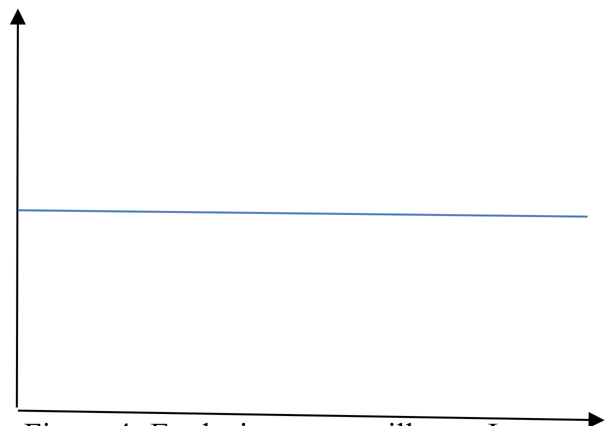


Figure 4- Explosive non-oscillatory Income movement (Could not draw the curve)

The Cobweb Model

Introduction

Nicholas kaldor analysed the model in 1934, coining the term "cobweb theorem", citing previous analyses in German by Henry Schultz and Umberto Ricci. Cobweb theory is the idea that price fluctuation can lead to fluctuations in supply which cause a cycle of raising and falling prices. In a simple cobweb model, we assume there is an agricultural market where supply can vary due to variable factors, such as the weather.

Assumptions:

● In an agricultural market, farmers have to decide how much to produce a year in advance - before they know what the market price will be (supply is price inelastic in short run).

● A key determinant of supply will be the price from the previous year.

● A low price will mean some farmers go out of business. Also,a low price will discourage Farmers from growing that crop in the next year.

● Demand for agricultural goods is usually price inelastic (a fall in price only causes a smaller percentage increase in demand).

2 1. If there is a very good harvest, then supply will be greater than expected and this will cause a fall in price.

2. However, this fall in price may cause some farmers to go out of business. Next year farmers may be put off by the low price and produce something else. The consequence is that if we have one year of low prices, next year farmers reduce the supply.

3. if supply is reduced, then this will cause the price to rise. 4. If farmers see high prices (and high profits), the next year they are inclined to increase supply because that product is more profitable. In this theory, the market could fluctuate between high price and low price as suppliers respond to past prices.

3 Price divergence: If the slope of the supply curve is less than the demand curve, then the price changes could become magnified and the market more unstable. Price convergence: At the equilibrium point, if the demand curve is more elastic than supply curve, we get the price volatility falling, and the price will convergence on the equilibrium .

4 Limitations:

Rational expectations: The model assumes farmers base next years supply purely on the previous price and assume that next year's price will be the same as last year (adaptive expectations). However, that really applies in the real world. Farmers are more likely to see it as a "good” year or "bad" year and learn price volatility.

Price divergence is unrealistic and not empirically seen: The idea that farmers only base supply on last year's price means, in theory, prices could increasingly diverge, but farmers would learn from this and pre-empty changes in price.

It may not be easy or desirable to switch supply: A potato grower may concentrate on potatoes because that is his speciality. It is not easy to give up potatoes and take to aubergines.

Other factors affecting price: There are many other factors affecting price than a farmers decision to supply. In global markets, supply fluctuation will be minimised by the role of importing from abroad. Also, demand may vary. Also, supply can vary due to weather factors.

Buffer stock schemes: Governments aur producers could band together to limit price volatility by buying surplus. Conclusion: In spite of its shortcomings the cobweb model is important besides its application as an explanation for the cyclical behaviour of wheat and other agricultural product's markets. It concentrates attention on the important fact that the present events depend upon the past happenings it furnishes us with technique to demonstrate the process of change over time.

6.4 Application of Difference Equation in Economics

In Economic dynamic Concerned with time element- According to Prof. J.R. Hicks, dynamic economics includes only those parts of economic theory where every quantity must be dated. There are economists who define dynamic economics in relation to equilibrium. According to these economists’ dynamic economics studies the process through which the final position of equilibrium is reached through a series of adjustments over a series of time. Prof. Baumol states that dynamic economics has the power of forecasting future. Following topics are studied through dynamic analysis (difference equation):

Example 3: $2x_t + 3x_{t-1} + 2 = 0$ where $x_0 = -1$

$X_t = -3/2\ X_{t-1} - 1$; a= -3/2 and b= -1

Solution = Put the values in equation (1) and you will get answer

$(-3/5)(3/2)^t - 2/5$

EQUILIBRIUM OF A LINEAR DIFFERENCE EQUATION = X*

Then, $X^* = X_t = X_{t-1}$ [equilibrium is achieved at X* which means value of X will remain same]--(2)

Which may be calculated as $X_t = aX_{t-1} + b$

As equ (2), X* = aX* +b or, X* = b/(1-a)

Time Path

Value of a	Type	Stable/ Unstable
0< a < 1	Convergent	Equilibrium is stable
-1 < a < 0	Damped ossilation (Decreasing fluctuation around X*)	stable equilibrium
a > 1	Xt diverges to +Infinity > a>- infinity (X0 >X*; Xt, diverges to + infinity; X0< X*; Xt diverges to - infinity	Unstable equilibrium
a< -1	Explosive oscillation	Unstable equilibrium
a =-1	Finite oscillation	Unstable equlibrium

Example 4: Suppose the supply and demand curve are as follow:

$Q_{dt} = 6-2P_t$

$Q_{st} = -4 + 2P_{t-1}$

Where Qdt and Qst = demand and supply of onion at time t

Pt = Price at t

When Qdt =Qst,

i. Write down the equation that represents the time path of price.
ii. Find out the steady state price (where the demand and supply curve cut each other
iii. $P_0 = 4$; the time path of price is convergent or divergent

Solution –i. At Qdt = Qst

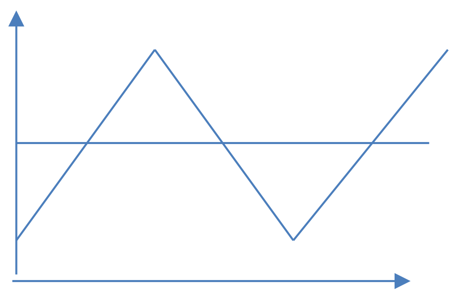
Or, $6\text{-}2P_{t\,=}\ \text{-}4 + 2P_{t-1}$

or, P_t - $-P_{t-1} + 5$

so, a =-1; b = 5

ii. At steady state P* = b / (1- a) = 2.5

iii. Time path



Example 5: if $Y_t = C_t$+It; C_t= $10+0.5Y_{t-1}$ and $I_t = 20+2(Y_t\text{-}Y_{t-1})$. Find the time path of Y forming and solving a difference equation.

[Yt = Income at time t; Ct = Consumption at time t It= Investment at time t]

Hint- $Y_t = C_t$+It;

$Y_{t\,=}\ 10+0.5Y_{t-1} + 20+2Y_t\text{-}2Y_{t-1}$

or, $Y_t\text{-}\ 2Y_{t\,=}\ 10+0.5Y_{t-1} + 20\text{-}2Y_{t-1}$

or, $Y_{t\,=}\ \text{-}20+1.5Y_{t-1}$

Ans- $C(3/2)^t + 60[1\text{-}(3/2)^t]$