Sequences and Series: Convergence, Algebraic Properties and Applications

Review

Term	Definition	Examples
Sequence	It is a set of terms (or numbers) arranged in a definite order.	5, 12, 19, 26, . . . In this sequence each term is obtained by adding 7 to the previous term. So the next term would be 32.
		16, 25, 36, 49, . . . This sequence can be rewritten as 4^2 , 5^2 , 6^2 , 7^2 , . . . The next term is 8^2 , or 64.
infinite sequence.	The dots(. . .) indicate that the sequence continues indefinitely	4, 6, 8, 10 ………
finite sequence	stopping after a finite number of terms	4, 6, 8, 10, 12
Recurrence relation	there may be a sequence where next term of a sequence is calculated from the preceding one(s),	un = un−1 + un−2, u1 = 1, u2 = 1 The sequence will be as 1, 1, 2, 3, 5, 8, 13, . . .
Series	A series is formed when the terms of a sequence are added together.	Which means $U_1+U_2+……+U_n$
Arithmetic Series		Example- application of simple interest
Geometric Series		One of the applications of geometric series is the calculation of compound interest. Here the sum on which interest is paid includes the interest that has been earned in previous years. For example, if Rs100 is invested at 5% per annum compound interest, then after 1 year the interest earned is Rs5 (100 x 0.05) and the capital invested for the second year is Rs105. The interest earned in the second year is then Rs5.25 (105 x 0.05) and this capital amount is carried forward to year 3.
		To start, let c be a real number that lies strictly etween −1 and 1. • We often write this as c ∈ (−1 , 1). • Here (−1,1) denotes the collection of all real numbers that are strictly less than 1 and strictly greater than −1.

		• The Symbol $\in$means *in* or *belongs to the set after the symbol*.
Infinite Geometric Series		The first type of geometric that interests us is the infinite series $1+c+c^2+c^3+\cdots$ Where $\cdots\cdots$ means that the series continues without end. The key formula is $$1+c+c^2+c^3+\cdots= 1/(1-c) \quad (1)$$ To prove key formula (1), multiply both sides by $(1-c)$ and verify that if $c\in(-1,1)$, then the outcome is the equation 1=1
Finite Geometric Series		The second series that interests us is the finite geometric series $$1+c+c^2+c^3+\cdots+c^T$$ where T is a positive integer. The key formula here is $$1+c+c^2+c^3+\cdots+c^T =(1-c^{T+1})/(1-c)$$ **Remark:** The above formula works for any value of the scalar c. We don't have to restrict c to be in the set (−1,1).

Definition of Sequence and Series in Maths

Sequence: The sequence is defined as the list of numbers which are arranged in a specific pattern. Each number in the sequence is considered a term. For example, 5, 10, 15, 20, 25, … is a sequence. The three dots at the end of the sequence represents that the pattern will continue further. Here, 5 is the first term, 10 is the second term,15 is the third term and so on. Each term in the sequence can have a common different, and the pattern will continue with the common difference. In the example given above, the common difference is 5. The sequence can be classified into different types, such as:

The order of a sequence matters. Hence, a sequence 5, 6, 7 is different from 7, 6, 5.

However, in case of series 5 + 6 + 7 is same as 7 + 6 + 5.

Use in Mathematical Economnics

Example: The Money Multiplier in Fractional Reserve Banking

Meaning

1. The **Money Multiplier** refers to how an initial deposit can lead to a bigger final increase in the total **money** supply. For example, if the commercial banks gain deposits of ₹1 million and this leads to a final **money** supply of ₹10 million. Than money multiplier is 10. The **money multiplier** is a key element of the fractional banking system.
2. In a **fractional-reserve banking system** that has legal **reserve** requirements, the total amount of loans that commercial **banks** are allowed to extend (the commercial **bank money** that they can legally create) is equal to a multiple of the amount of **reserves**.

In a fractional reserve banking system, banks hold only a fraction $r \in (0,1)$ of cash behind each **deposit receipt** that they issue

- When the UK, France and the US were on either a gold or silver standard (before 1914, for example)
  - cash was a gold or silver coin
  - a *deposit receipt* was a *bank note* that the bank promised to convert into gold or silver on demand; (sometimes it was also a checking or savings account balance)

- In 1893 **India** abandoned the **silver standard**,(The silver standard is a monetary system in which the national currency is backed by physical silver.
- It involves currency holders being able to exchange their national currency in favor of set amounts of silver.
- While the silver standard has a long history throughout the world, there are no longer any countries utilizing it today.)

the Indian **rupee** was a **silver coin** ('raupya' in Sanskrit means **silver**), which made the **rupee to** be **pegged** at a value of 1 shilling 4 pence (i.e., 15 **rupees** = 1 **pound**).

and in 1898 went on the **gold**-exchange **standard through British pound.**

The **gold standard**, which started as a part of the rebuilding process after World War II, ended somewhat ironically with the Vietnam War in 1971.

No country follows the **gold standard** anymore. However, it's left certain traces behind. One of its legacies is the **gold** reserves that many countries maintain. ...
The **Indian** government bought 200 tonnes of **gold** from the International Monetary Fund (IMF) in 2009.

- cash consists of pieces of paper issued by the government and INR

- a *deposit* is a balance in a cash or savings account that entitles the owner to ask the bank for immediate payment in cash

Economists and financiers often define the **supply of money** as an economy-wide sum of **cash** plus **deposits**.

In a **fractional reserve banking system** (one in which the reserve ratio r satisfies $0<r<1$, **banks create money** by issuing deposits *backed* by fractional reserves plus loans that they make to their customers.

A geometric series is a key tool for understanding how banks create money (i.e., deposits) in a fractional reserve system.

The geometric series formula is at the heart of the classic model of the money creation process – one that leads us to the celebrated **money multiplier**.

A Simple Model

There is a set of banks named $i=0,1,2,\ldots$

Bank i's loans L_i, deposits D_i, and reserves R_i must satisfy the balance sheet equation (because **balance sheets balance**):

$$L_i+R_i=D_i \qquad (2)$$

The left side of the above equation is the sum of the bank's **assets**, namely, the loans L_i it has outstanding plus its reserves of cash R_i. The right side records bank i's liabilities, namely, the deposits D_i held by its depositors; these are IOU's ("I owe you") from the bank to its depositors in the form of either cash accounts or savings accounts

Each bank i sets its reserves to satisfy the equation

$$R_i=rD_i \qquad (3)$$

where $r\in(0,1)$ is its **reserve-deposit ratio** or **reserve ratio** for short

- the reserve ratio is either set by a RBI or chosen by banks for precautionary reasons

Next we add a theory stating that bank $i+1$'s deposits depend entirely on loans made by bank i, namely

$$D_{i+1}=L_i \text{.................................} (4)$$

Thus, we can think of the banks as being arranged along a line with loans from bank i being immediately deposited in $i+1$

- in this way, the debtors to bank i become creditors of bank $i+1$

Finally, we add an *initial condition* about an exogenous level of bank 0's deposits

$$D_0 \text{ is given exogenously}$$

We can think of D_0 as being the amount of cash that a first depositor put into the first bank in the system, bank number $i=0$.

Combining equations (2) and (3) tells us that

$$L_i=(1-r)D_i \text{..} (5)$$

This states that bank i loans a fraction $(1-r)$ of its deposits and keeps a fraction r as cash reserves.

Combining equation (5) with equation (4) tells us that

$$D_{i+1}=(1-r)D_i \quad \text{for } i \geq 0$$

which implies that

$$D_i=(1-r)^i D_0 \quad \text{for } i \geq 0 \text{.............................} (6)$$

Equation (6) expresses D_i as the ith term in the product of D_0 and the geometric series

$$1,(1-r),(1-r)^2,\cdots$$

Therefore, the sum of all deposits in our banking system $i=0,1,2,\ldots$ is

Money Multiplier

Money multiplier effect

the phenomenon of credit creation.
ıount A, its total reserves are increased.
to hold only an amount equal to r × A in
;, where r is the required reserve ratio.
; reserves i.e. (A − r × A) as loans. When
loan in bank (it is assumed), it increases
A − r × A). Again, the bank is required to
of deposits and it can lend out the rest.
crease in money supply due to an initial
A is equal to m × A, where m is the
in case of a decrease in deposits through

Money Multiplier = Required Reserve Ratio

Required reserve ratio is the fraction of deposits which a bank is required to hold in hand. It can lend out an amount equals to excess reserves which equals (1 − required reserves).

Higher the required reserve ratio, lesser the excess reserves, lesser the banks can lend as loans, and lower the money multiplier. Lower the required reserve ratio, higher the excess reserves, more the banks can lend, and higher is the money multiplier.

In the above relationship it is assumed that there is no currency drainage, i.e. the borrowers keep 100% of the amount received in banks.

Currency drainage

In reality, borrowers do keep a fraction of loans received in cash. This reduces the money multiplier. When there is some currency drainage, money multiplier is calculated as per following formula:

Money multiplier when there is currency drainage=1 + drainage ratiorequired reserve ratio + drainage ratio

The **money multiplier** is a number that tells the multiplicative factor by which an exogenous injection of cash into bank 0 leads to an increase in the total deposits in the banking system.

Equation (7) asserts that the **money multiplier** is 1r

- An initial deposit of cash of D_0 in bank 0 leads the banking system to create total deposits of D_0r.

- The initial deposit D_0 is held as reserves, distributed throughout the banking system according to

  D_0=

Example 1

Ishkebar is an alien country that has seen little financial innovation. Its central bank requires commercial banks to keep 100% of their deposits as reserves. Calculate money multiplier for the economy.

Money multiplier = 1/required reserve ratio = 1/100% = 1

The country has a money multiplier of 1. No money creation is possible because in response to an increase in bank deposits of say 100 million Ishkebar dollars (I$), the money supply will increase by 1 × I$100 million = I$100 million.

Example 2

North Sarrawak is run by a dictator who knows no economics and is not willing to listen to any advice. He thinks he can always print money whenever a depositor wants to withdraw so he does not think having any required reserve ratio for the sole bank of the country is necessary. What could be the consequences?

Zero required reserve ratio means infinite money multiplier and infinite money creation. Infinite money creation means no scarcity of money which means money would no longer be money since it would no longer be a store of value.

Example 3

Palmolive has required reserve ratio of 30% and a currency drainage of 15%. Calculate the money multiplier and compare it with Parazuela, a country where drainage is zero and required reserve ratio is 30%.

Money multiplier in Palmolive = (1 + 15%) ÷ (30% + 15%) = 2.56

Money multiplier in Parazuela = 1/30% = 3.33

Parazuel has higher money multiplier which makes sense because it has zero drainage. Zero drainage means all of the excess reserves loaned out in round 1 form part of total reserves in round 2.

https://www.economicsdiscussion.net/multiplier/investment-multiplier/investment-multiplier-definition-logic-and-assumptions/20789

The Keynesian Multiplier

T ın Maynard Keynes and his followers created a simple
m ine national income y in circumstances in which

ıl unemployed resources, in particular **excess**
d capital
ates fail to adjust to make aggregate **supply equal**
s and interest rates are frozen)
ntirely determined by aggregate demand

S

A model of national income determination consists of
th ibe aggregate demand for y and its components.

T onal income identity asserting that
c ment i equals national income y:

.... (1)

T eynesian consumption function asserting that people
c 1) of their income:

c=by

T led the **marginal propensity to consume**.

ity to Consume is the proportion of an increase in
spent on consumption.
come level. MPC is typically lower at higher

eterminant of the Keynesian multiplier, which
ct of increased investment or government
conomic stimulus.

T called the **marginal propensity to save**.

 save equals the ratio of a change in **saving** to a change in

pensity to save

ds = Change in saving

dY = Change in Income

The sum of the **propensity** to consume and the **propensity to save** always equals one (see **propensity** to consume)

MPC + MPS = 1

The third equation simply states that investment is exogenous at level i.

- *exogenous* means *determined outside this model*.

Substituting the second equation into the first gives $(1-b)y=i$.

Solving this equation for Y gives

$$Y=1-bi$$

The quantity $1-b$ is called the **investment multiplier** or simply the **multiplier**.

The term **investment multiplier** refers to the concept that any increase in public or private **investment** spending has a more than proportionate positive impact on aggregate income and the general economy. **It is** rooted in the economic theories of John Maynard Keynes. Applying the formula for the sum of an infinite geometric series, we can write the above equation as

y =i

$$y=i$$

where t is a non-negative integer.

So, we arrive at the following equivalent expressions for the multiplier:

$$1-b= i$$

The expression motivates an interpretation of the multiplier as the outcome of a dynamic process that we describe next.

DYNAMIC VERSION

We arrive at a dynamic version by interpreting the non-negative integer t as indexing time and changing our specification of the consumption function to take time into account

- we add a one-period lag in how income affects consumption

We let c_t be consumption at time t and i_t be investment at time t.

We modify our consumption function to assume the form

$$c_t = b y_{t-1}$$

so that b is the marginal propensity to consume (now) out of last period's income.

We begin with an initial condition stating that

$$y_{-1} = 0$$

We also assume that

$$i_t = i \quad \text{for all } t \geq 0$$

so that investment is constant over time.

It follows that

$$y_0 = i + c_0 = i + b y_{-1} = i$$

and

$$y_1 = c_1 + i = b y_0 + i = (1+b) i$$

and

$$y_2 = c_2 + i = b y_1 + i = (1 + b + b^2) i$$

and more generally

$$y_t = b y_{t-1} + i = (1 + b + b^2 + \cdots + b^t) i$$

or

$$y_t = \frac{1 - b^{t+1}}{1 - b} i$$

yt=1−bt+11−bi

Evidently, as $t \to +\infty$,

$$y_t \to 1 - bi$$

Remark 1: The above formula is often applied to assert that an exogenous increase in investment of Δi at time 0 ignites a dynamic process of increases in national income by successive amounts

$$\Delta i, (1+b)\Delta i, (1+b+b_2)\Delta i, \cdots$$

at times $0, 1, 2, \ldots$

Remark 2 Let g_t be an exogenous sequence of government expenditures.

If we generalize the model so that the national income identity becomes

$$c_t + i_t + g_t = y_t$$

then a version of the preceding argument shows that the **government expenditures multiplier** is also $1-b$, so that a permanent increase in government expenditures ultimately leads to an increase in national income equal to the multiplier times the increase in government expenditures.

Interest Rates and Present Values

We can apply our formula for geometric series to study how interest rates affect values of streams of dollar payments that extend over time.

We work in discrete time and assume that $t = 0, 1, 2, \ldots$ indexes time.

We let $r \in (0,1)$ be a one-period **net nominal interest rate**

- if the nominal interest rate is 5 percent, then $r = .05$r=.05

A one-period **gross nominal interest rate** R is defined as

$$R = 1 + r \in (1, 2)$$

- if $r = .05$, then $R = 1.05$

Remark: The gross nominal interest rate R is an **exchange rate** or **relative price** of dollars at between times t and $t+1$. The units of R are dollars at time $t+1$.

- A real interest rate is adjusted to remove the effects of inflation and gives the real rate of a bond or loan.
- A nominal interest rate refers to the interest rate before taking inflation into account.
- To calculate the real interest rate, you need to subtract the actual or expected rate of inflation from the nominal interest rate.

When people borrow and lend, they trade dollars now for dollars later or dollars later for dollars now.

The price at which these exchanges occur is the gross nominal interest rate.

- If I sell x dollars to you today, you pay me Rx dollars tomorrow.
- This means that you borrowed x dollars for me at a gross interest rate R and a net interest rate r.

We assume that the net nominal interest rate rr is fixed over time, so that R is the gross nominal interest rate at times $t=0,1,2,\ldots$

Two important geometric sequences are

$$1,R,R^2,\cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),\text{and}$$
$$1,R^{-1},R^{-2},\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Sequence (8) tells us how dollar values of an investment **accumulate** through time.

Sequence (9) tells us how to **discount** future dollars to get their values in terms of today's dollars.

Accumulation

Geometric sequence (8) tells us how one dollar invested and re-invested in a project with gross one period nominal rate of return accumulates

- here we assume that net interest payments are reinvested in the project
- thus, 1 dollar invested at time 0 pays interest r dollars after one period, so we have r+1=R dollars at time1
- at time 1, we reinvest 1+r=R dollars and receive interest of rR dollars at time 2 plus the *principal* R dollars, so we receive rR+R=(1+r)R=R2 dollars at the end of period 2
- and so on

Evidently, if we invest x dollars at time 0 and reinvest the proceeds, then the sequence

$$x, xR, xR^2, \cdots$$

tells how our account accumulates at dates $t=0,1,2,\ldots$

DISCOUNTING

Geometric sequence (9) tells us how much future dollars are worth in terms of today's dollars.

Remember that the units of R are dollars at $t+1$ per dollar at t.

It follows that

- the units of R^{-1} are dollars at t per dollar at $t+1$
- the units of R^{-2} are dollars at t per dollar at $t+2$
- and so on; the units of R^{-j} are dollars at t per dollar at $t+j$

So if someone has a claim on x dollars at time $t+j$, it is worth xR^{-j} dollars at time t (e.g., today).

Application to Asset Pricing

A **lease** requires a payments stream of x_t dollars at times $t=0,1,2,\ldots$ where

$$x_t = G^t x_0$$

where $G=(1+g)$ and $g\in(0,1)$.

Thus, lease payments increase at g percent per period.

For a reason soon to be revealed, we assume that $G<R$.

The **present value** of the lease is

$$p_0 = x_0 + x_1/R + x_2/(R^2) + \ddots = x_0(1+GR^{-1}+G^2R^{-2}+\cdots) = x_0\frac{1}{1-GR^{-1}} \quad p_0 = x_0 + x_1/R + x_2/(R^2) + \ddots = x_0(1+GR^{-1}+G^2R^{-2}+\cdots) = x_0\frac{1}{1-GR^{-1}}$$

where the last line uses the formula for an infinite geometric series.

Recall that $R=1+r$ and $G=1+g$ and that $R>G$ and $r>g$ and that r and g are typically small numbers, e.g., .05 or .03.

Use the Taylor series of $\frac{1}{1+r}$ about $r=0$, namely,

$$\frac{1}{1+r} = 1 - r + r^2 - r^3 + \cdots$$

and the fact that r is small to approximate 11+r≈1−r.

Use this approximation to write p0p0 as

$$p0=x011−GR−1=x011−(1+g)(1−r)=x011−(1+g−r−rg)≈x01r−g$$

where the last step uses the approximation rg≈0.

The approximation

$$p0=x0r−g$$

is known as the **Gordon formula** for the present value or current price of an infinite payment stream x0Gtx0Gt when the nominal one-period interest rate is r and when r>g.

We can also extend the asset pricing formula so that it applies to finite leases.

Let the payment stream on the lease now be xt for t=1,2,…,T, where again

$$xt=Gtx0$$

The present value of this lease is:

$$p0=x0+x1/R+⋯+xT/RT=x0(1+GR−1+⋯+GTR−T)=x0(1−GT+1R−(T+1))1−GR−1$$

Applying the Taylor series to R−(T+1) about r=0 we get:

$$1(1+r)T+1=1−r(T+1)+12r2(T+1)(T+2)+⋯≈1−r(T+1)$$

Similarly, applying the Taylor series to GT+1 about g=0:

$$(1+g)T+1=1+(T+1)g(1+g)T+(T+1)Tg2(1+g)T−1+⋯≈1+(T+1)g$$

Thus, we get the following approximation:

$$p0=x0(1−(1+(T+1)g)(1−r(T+1)))1−(1−r)(1+g)$$

Expanding:

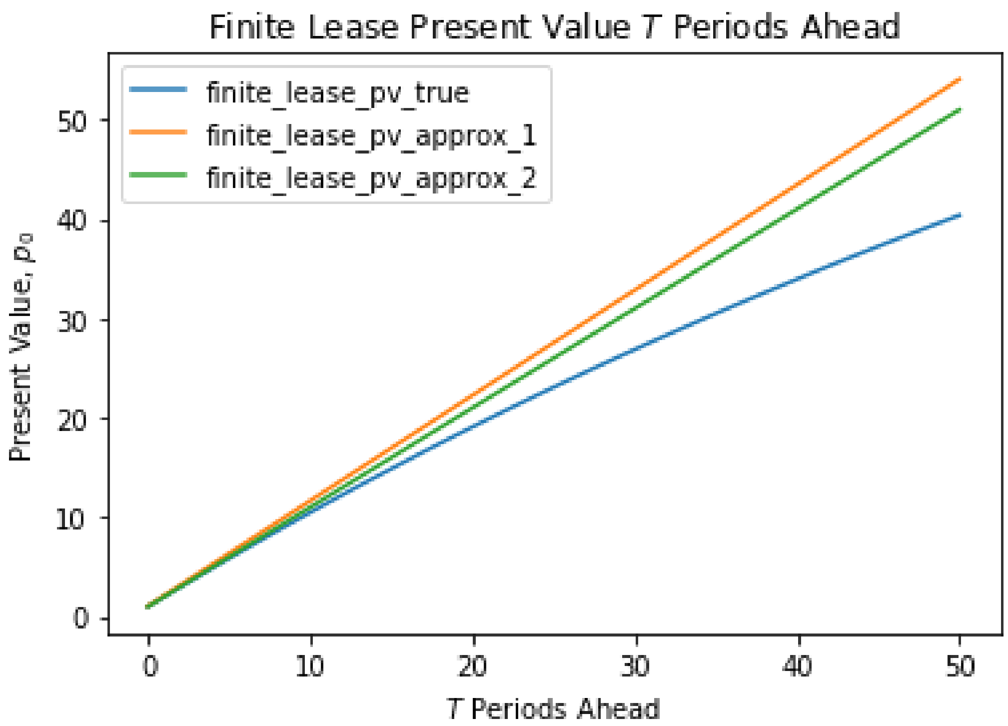
$$p0=x0(1−1+(T+1)2rg−r(T+1)+g(T+1))1−1+r−g+rg=x0(T+1)((T+1)rg+r−g)r−g+rg≈x0(T+1)(r−g)r−g+x0rg(T+1)r−g=x0(T+1)+x0rg(T+1)r−g$$

We could have also approximated by removing the second term rgx0(T+1) when Tis relatively small compared to 1/(rg) to get x0(T+1) as in the finite stream approximation.

We will plot the true finite stream present-value and the two approximations, under different values of T, and g and r in Python.

First we plot the true finite stream present-value after computing it below

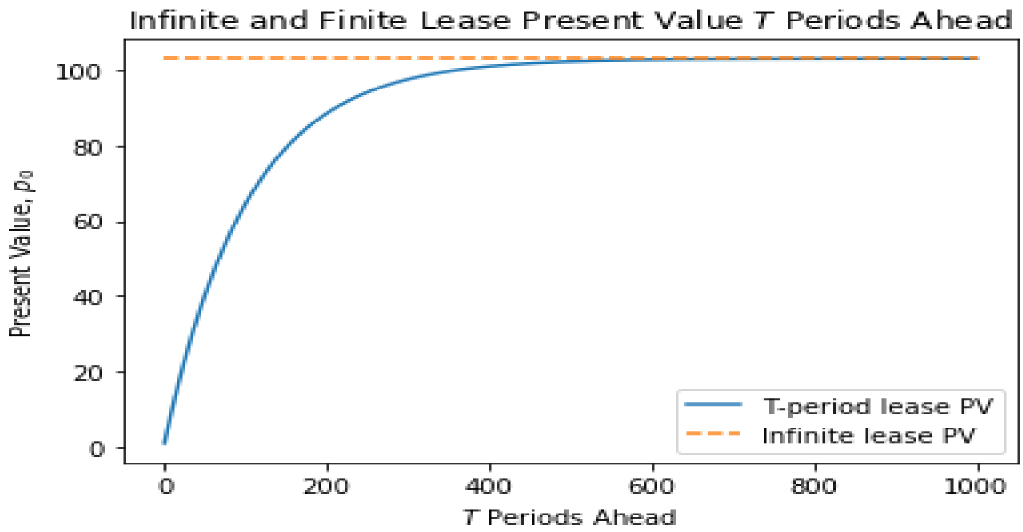
Now that we have defined our functions, we can plot some outcomes.



Evidently our approximations perform well for small values of T.

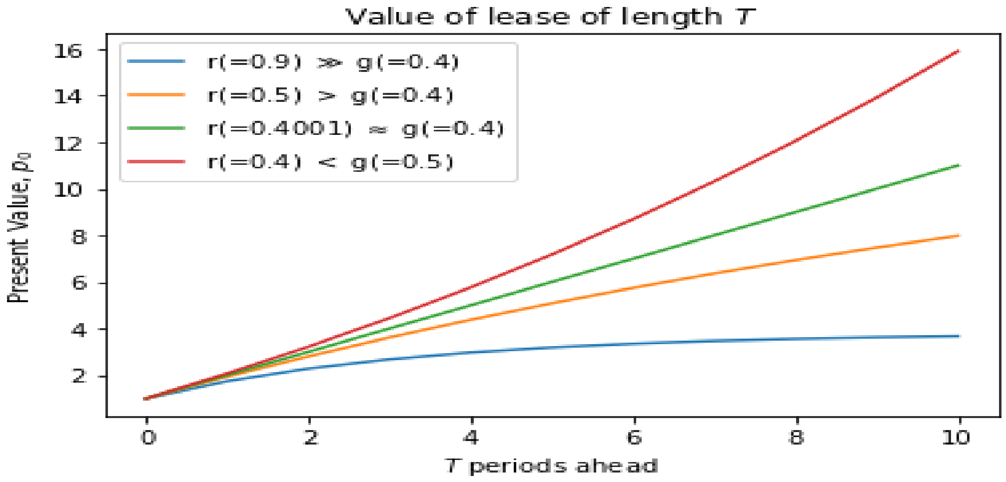
However, holding gg and r fixed, our approximations deteriorate as T increases.

Next we compare the infinite and finite duration lease present values over different lease lengths T.



The graph above shows how as duration $T \to +\infty$, the value of a lease of duration T approaches the value of a perpetual lease.

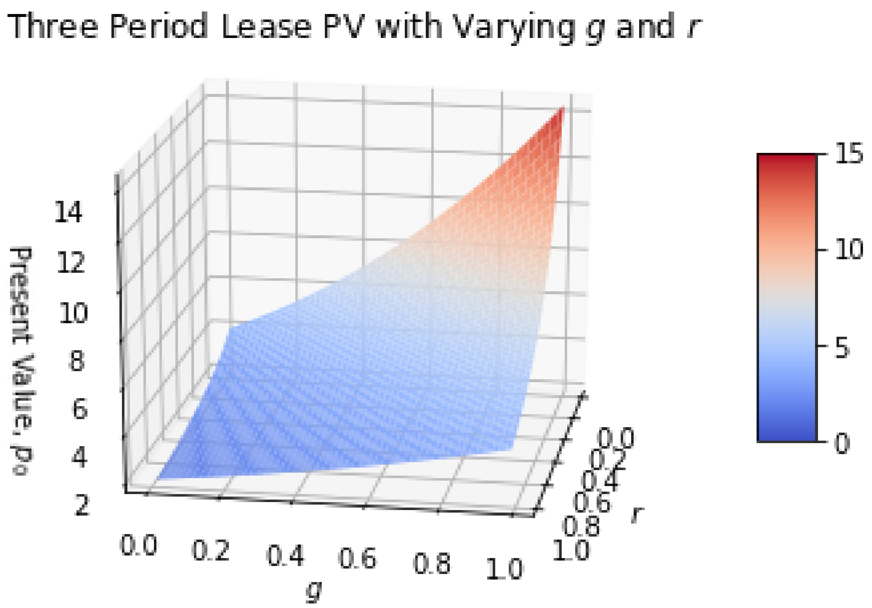
Now we consider two different views of what happens as r and g covary



This graph gives a big hint for why the condition $r>g$r>g is necessary if a lease of length $T=+\infty$ is to have finite value.

For fans of 3-d graphs the same point comes through in the following graph.

If you aren't enamored of 3-d graphs, feel free to skip the next visualization!



We can use a little calculus to study how the present value p0 of a lease varies with r and g.

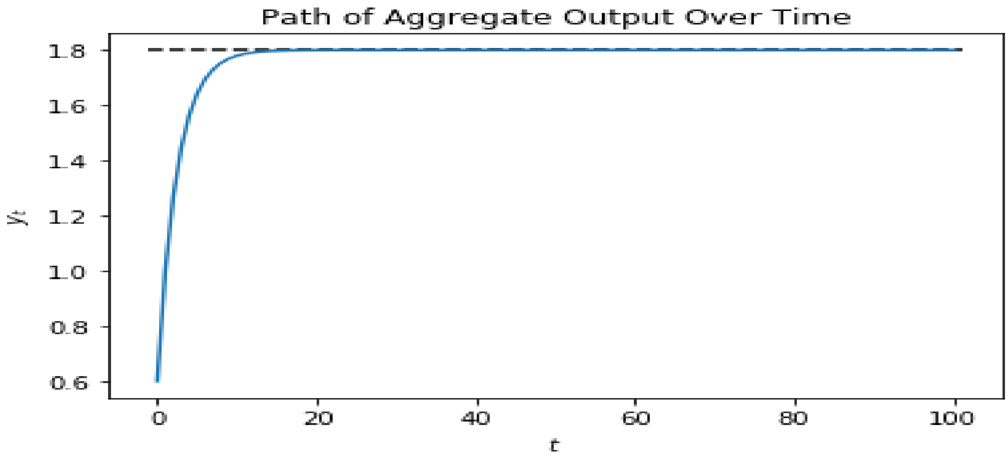
We will use a library called SymPy.

SymPy enables us to do symbolic math calculations including computing derivatives of algebraic equations.

We will illustrate how it works by creating a symbolic expression that represents our present value formula for an infinite lease.

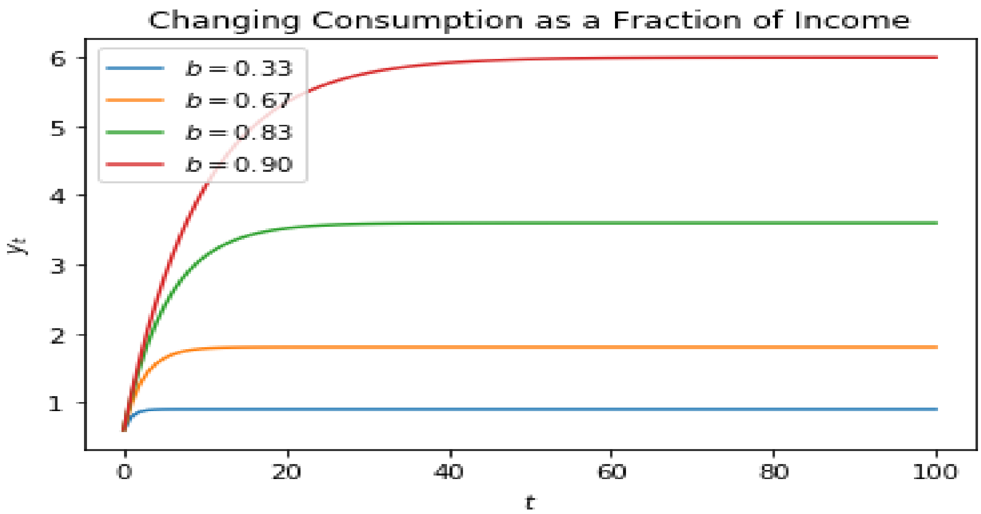
Back to the Keynesian Multiplier

We will now go back to the case of the Keynesian multiplier and plot the time path of yt, given that consumption is a constant fraction of national income, and investment is fixed.



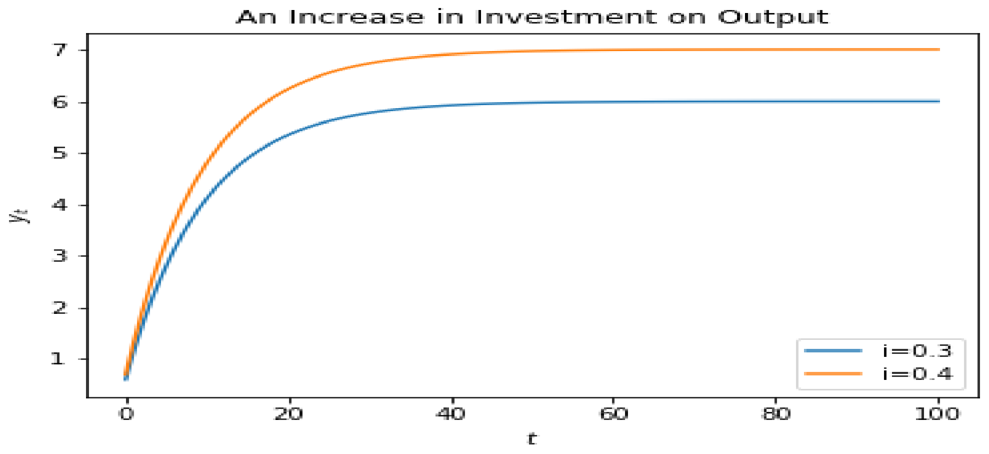
In this model, income grows over time, until it gradually converges to the infinite geometric series sum of income.

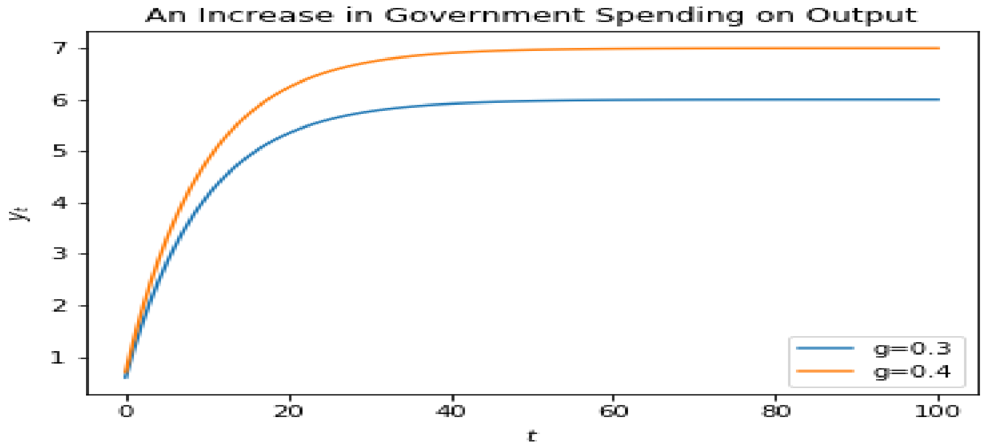
We now examine what will happen if we vary the so-called **marginal propensity to consume**, i.e., the fraction of income that is consumed



Increasing the marginal propensity to consume b increases the path of output over time.

Now we will compare the effects on output of increases in investment and government spending.





Notice here, whether government spending increases from 0.3 to 0.4 or investment increases from 0.3 to 0.4, the shifts in the graphs are identical.

Source

https://python.quantecon.org/geom_series.html

https://www.varsitytutors.com/hotmath/hotmath_help/topics/geometric-series

https://math.stackexchange.com/questions/1321389/solving-the-exponent-function-for-x