Mathematical Economics

1. Background

In the recent times, Mathematical Economics and Econometrics have attracted attention of engineering students. Some of the important reasons are:

i. Need to understand the economic forces and develop models at Micro level and
ii. Secondly, to understand the economic environment at Macro level.
iii. Some of them like to explore other disciplines where they can use their basic analytical reasoning and vast knowledge of mathematics, statistics and other similar subjects.
iv. There is so much discussion on Big data.

2. What it Mathematical Economics?

Mathematical economics is not a branch of Economics as Monetary Economics or International Economics but just application of Mathematics to the purely theoretical aspects of economic analysis. It mainly expresses economic theory in mathematical forms.

It may be considered as a theoretical and applied science, whose purpose is to mathematically formalized description of economic objects, processes, and phenomena. Most of the economic theories are presented in terms of economic models. In mathematical economics, the properties of these models are studied based on formalizations of economic concepts and notions. In mathematical economics, theorems on the existence of extreme values of certain parameters are proved, properties of equilibrium states and equilibrium growth trajectories are studied, etc. This creates the impression that the proof of the existence of a solution (optimal or equilibrium) and its calculation is the main aim of mathematical economics. However, the most important purpose is to formulate economic notions and concepts in mathematical form, which will be mathematically adequate and self-consistent, and then, on their basis to construct mathematical models of economic processes and phenomena. Moreover, it is not enough to prove the existence of a solution and find it in an analytic or numerical form, but it is necessary to give an economic interpretation of these to obtain mathematical results (Tarasov, 2019).

3. What is Econometrics?

As the word ‘Metric’ refers to measurement, Econometrics in simple words means measurement of economic data to verify an Economic Theory. It deals with the study of empirical observations for estimation, inference and forecasting.

Econometrics may be defined as the quantitative analysis of actual economic phenomena based on the concurrent development of theory and observation, related by appropriate methods of inferences (P.A. Samuelson T.C. Koopmans and J.R.N. Stone, "Report of the Evaluation Committee for Econometrica,"Econometrica, 22(2)1954 pp. 141-146).

Econometrics may be defined as the social science in which the tools of economic theory, mathematics, and statistical inferences are applied to the analysis of economic phenomena (Arthur S. Golberger, Econometric Theory, John Wiley & Sons, New York, 1964, p.1).

In the beginning, statistics was used for Econometric studies. However, over the years other similar branches as operational research game theory, fuzzy logic etc. are being used for Econometrics studies.

As empirical studies and theoretical analysis are often complementary and mutual reinforcing, knowledge of both Mathematical Economics and Econometrics are important but knowledge of Mathematical Economics may be considered as the more basic of the two to undertake Econometrics study. Hence, knowledge of Mathematical Economics is useful not only for those who are interested in theoretical Economics, but also for those seeking a foundation for the pursuit of Econometric studies.

4. Difference between Mathematical Economics and Econometrics

SN	Mathematical Economics	Econometrics
1.	Mathematical Economics refers to the application of mathematics to the purely theoretical aspects of economic analysis, with little or no concern about such statistical problem as the errors of measurement of the variables under study.	The 'metric' part implies that econometrics is mainly concerned with measurement of economic data.
2.	Deductive reasoning- dealing primarily with theoretical material	Inductive reasoning- dealing with empirical material
3.	On the one hand, theories must be tested against empirical data for validity before they can be applied with confidence.	Statistical work needs economic theory as a guide, in order to determine the most relevant and fruitful direction of research.
4	In Mathematics and Physics, a deterministic system has no randomness, is involved in the development of future state of system. A deterministic model will produce the same	Randomness is involved. Significance of Stochastic Disturbance Term The disturbance term, 'µ' represents all those variables that are omitted from the model but that collectively affect, 'Y'. <u>Assumption of '*Ceritus Paribus*'</u>

	output. In economics, a production function is a deterministic model. Or when you calculate price elasticity of demand that is deterministic relationship. The Ramsay-Cass-Koopmans model is deterministic model. But in stochastic relationship,	*(when other variables remain same)* As we have studied that D inversely related to Price of the product when other things as taste, fashion, price of the substitute goods, complimentary goods remain same. But due to paucity of time or data, we may not study that all other variables to remain same or not. So, we cannot say that
5.	Empirical studies and theoretical analysis are often complimentary and mutually reinforcing.	

5. Development of Mathematical Economics

Economics is a subject of social science. It studies complex economic processes with cause and effect on various factors. As it is affected by many non-economic factors also initially, it was called as Political Economy but it was Neo Classical Economists who discussed Economics as a subject like Chemistry or Physics. A subject which is rational, logical, quantifiable, measurable and predictable. It is generally believed that the use of mathematics as a tool of economics dates from the pioneering work of Cournot (1838). However, there were many others who used mathematics in the analysis of economic ideas before Cournot. They may be discussed as below:

Ecoomists	Area of Study
Sir William Petty 1623-1687	Romsey, Hampshire, England—died December 16, 1687, London), English political economist and statistician. **regarded as the first economic statistician and The inventor of Economics** Medicine, Political Philosophy and Economics. In his Discourses on Political Arithmetic (1690), he declared that he wanted to reduce economic matters in terms of number, weight, and measure.
Had a strong aptitude for mathematics, William spent two years making a land register for the whole of Ireland. After being able to chalk out a model to value land effectively for the register, he went into insurance and became a forerunner of modern insurance business. He wrote at length on issues that would later concern economists, such as taxation, Velocity of money and national income and touched on	

<table>
<tr><td colspan="2">the labour theory of value. However, while his analysis was numerical, he rejected abstract mathematical methodology. He first became prominent serving Oliver Cromwell and the Commonwealth in Ireland. He developed efficient methods to survey the land that was to be confiscated and given to Cromwell's soldiers. He used detailed numerical data (along with John Graunt) which influenced statisticians and economists for some time, even though Petty's works were largely ignored by English scholars (Schumpeter,1954).

By developing and altering different mathematical models, William was the first person to find a method to measure Gross Domestic Product (GDP), find the true importance of banks and money supply, and why and how unemployment affects the economy.

Petty was knighted Sir William Petty for his astounding and remarkable work, yet as an academician he was never truly appreciated. This is because he put data in everything he worked with, and was the only person to use data with economics at that time.
Sir William Petty influenced all the greatest minds of Economics, including Adam Smith, Karl Marx and John Maynard Keynes.</td></tr>
<tr><td>Giovanni Ceva
1647-1734</td><td>An Italian Mathematician, Physicist, and hydraulic engineer best known for the geometric theorem bearing his name concerning straight lines that intersect at a common point when drawn through the vertices of a triangle.</td></tr>
<tr><td colspan="2">in 1711, he wrote a tract in Economics in which mathematical formulas were generously used. He is generally regarded as the first known writer to apply mathematical methods to economic problems.

Ceva wrote De Re Nummeraria in 1711, which was one of the first books in mathematical economics.</td></tr>
<tr><td>Daniel Bernoulli
1700-1782</td><td>Swiss mathematician. English translation in Bernoulli, D. (1954). "Exposition of a New Theory on the Measurement of Risk" (PDF). Econometrica. 22 (1): 23–36. doi:10.2307/1909829. JSTOR 1909829.</td></tr>
<tr><td colspan="2">The in 1738 for the first time used calculus in his analysis of a probability that would result from games of chance rather than from economic problems.

In his 1738 book Specimen theoriae novae de mensura sortis (Exposition of a New Theory on the Measurement of Risk), Bernoulli offered a solution to the St. Petersburg paradox as the basis of the economic theory of risk aversion, risk premium, and utility. Bernoulli often noticed that when making decisions that involved some uncertainty, people did not always try to maximize their possible monetary gain, but rather tried to maximize "utility", an economic term encompassing their personal satisfaction and benefit. Bernoulli realized that for humans, there is a direct relationship between money gained and utility, but that it diminishes as the money gained increases. For example, to a person whose income is $10,000 per year, an additional $100 in income will provide more utility than it would to a person whose income is $50,000 per year.</td></tr>
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Gabriel Cramer (31 July 1704 – 4 January 1752)	He was a Genevan mathematician.

Cramer showed promise in Mathematics from an early age. At 18 he received his doctorate and at 20 he was co-chair of mathematics at the University of Geneva.

In 1728 he proposed a solution to the St. Petersburg Paradox that came very close to the concept of expected utility theory given ten years later by Daniel Bernoulli.

The St. Petersburg paradox is **a situation where a naive decision criterion which takes only the expected value into account predicts a course of action that presumably no actual person would be willing to take**. It is related to probability and decision theory in economics.

François Véron Duverger **de Forbonnais** (1722–1800)	born in Le Mans and educated in Paris. After working for his father's textile business, he settled in Paris and became inspector-general of the French coinage in 1752.

Used mathematical symbols, especially for explaining the rate of exchange between two countries and how an equilibrium is finally established between them. He is best known for his severe attack on Physiocracy.

Johann Heinrich von Thünen (24 June 1783 – 22 September 1850) German	Thünen's work was largely theoretical, but he also mined empirical data in order to attempt to support his generalizations. In comparison to his contemporaries, Thünen built economic models and tools, rather than applying previous tools to new problems (Schumpeter (1954) p. 465-468). His first work, The Isolated State (1826), was an attempt to explain how transportation costs influence the location of agriculture and even the methods of cultivation.

Thünen was a Mecklenburg landowner, who in the first volume of his treatise *The Isolated State* (1826), developed the first serious treatment of spatial economics and economic geography, connecting it with the theory of rent. The importance lies less in the pattern of land use predicted than in its analytical approach.

Thünen developed the basics of the theory of marginal productivity in a mathematically rigorous way, summarizing it in the formula in which

$$R = Y (P-C)- YF_m$$

where R = land rent; Y = yield per unit of land; c = production expenses per unit of commodity; p=market price per unit of commodity; F = freight rate (per agricultural unit, per mile); m=distance to market.

Hermann Heinrich Gossen (7 September 1810 – 13 February 1858)	a Prussian economist who is often regarded as the first to elaborate a general theory of marginal utility. Gossen studied in Bonn, then worked in the Prussian administration until retiring in 1847.

- **Gossen's First Law** is the "law" of diminishing marginal utility: that marginal utilities are diminishing across the ranges relevant to decision-making.
- **Gossen's Second Law**, which presumes that utility is at least weakly quantified, is that in equilibrium an agent will allocate expenditures so that the ratio of marginal utility to price (marginal cost of acquisition) is equal across all goods and services.

$\delta U/\delta X_i/P_i = \delta U/\delta X_j/P_j=$

where

- U = utility
- X_i = quantity of the ith good or service
- P_i = price of the pth good or service

- **Gossen's Third Law** is that scarcity is a precondition for economic value.

Meanwhile, a new cohort of scholars trained in the mathematical methods of the physical sciences started working in economics, advocating and applying those methods to their subject (Philip Mirowski, 1991. "The When, the How and the Why of Mathematical Expression in the History of Economics Analysis", *Journal of Economic Perspectives*, 5(1) pp. 145-157.,[] and described today as moving from geometry to mechanics (Weintraub, E. Roy (2008). "mathematics and economics", *The New Palgrave Dictionary of Economics*, 2nd Edition. Abstract.)	
Neo-Classical Economists	
W.S. Jevons (1 September 1835, United Kingdom- 13 August 1882, United Kingdom	Marginal utility theory; Jevons paradox Jevons studied chemistry and botany at University College, London.

Jevons's book A General Mathematical Theory of Political Economy as the start of the mathematical method in economics. It made the case that economics as a science concerned with quantities

Working in complete independence of one another—Jevons in Manchester, England; LEON WALRAS in Lausanne, Switzerland; and CARL MENGER in Vienna—each scholar developed the theory of marginal utility to understand and explain consumer behaviour. The theory held that the utility (value) of each additional unit of a commodity—the marginal utility—is less and less to the consumer. When you are thirsty, for example, you get great utility from a glass of water. Once your thirst is quenched, the second and third glasses are less and less appealing. He presented paper on a "general mathematical theory of political economy" in 1862, providing an outline for use of the theory of marginal utility in political economy (Jevons, W.S. (1866). "Brief Account of a General Mathematical Theory of Political Economy", *Journal of the Royal Statistical Society*, XXIX (June) pp. 282–87. Read in Section F of the British Association, 1862.). In 1871, he published *The Principles of Political Economy*, declaring that the subject as science "must be mathematical simply because it deals with quantities". Jevons expected that only collection of statistics for price and quantities

would permit the subject as presented to become an exact science (Jevons, W. Stanley (1871). *The Principles of Political Economy, pp. 4, 25*. Macmillan). Others preceded and followed in expanding mathematical representations of economic problems (the preface to Irving Fisher's 1897 work, *A brief introduction to the infinitesimal calculus: designed especially to aid in reading mathematical economics and statistics*). Much of what he said had been said earlier by Hermann Gossen in Germany, Jules Dupuit and Antoine Cournot in France, and Samuel Longfield in Britain. Yet historians of economic thought are sure that Jevons had never read them.	
Antoine **Augustin** **Cournot** (28 August 1801 – 31 March 1877)	was a French philosopher and a professor of mathematics who also contributed to the development of economics.
1. built the tools of the discipline axiomatically around utility, arguing that individuals sought to maximize their utility across choices in a way that could be described mathematically. At the time, it was thought that utility was quantifiable, in units known as utils. Cournot, Walras and Francis Ysidro Edgeworth are considered the precursors to modern mathematical economics. 2. Cournot, a professor of mathematics, developed a mathematical treatment in 1838 for duopoly—a market condition defined by competition between two sellers. This treatment of competition, first published in *Researches into the Mathematical Principles of Wealth*, is referred to as Cournot duopoly. It is assumed that both sellers had equal access to the market and could produce their goods without cost. Further, it assumed that both goods were homogeneous. Each seller would vary his output based on the output of the other and the market price would be determined by the total quantity supplied. The profit for each firm would be determined by multiplying their output and the per unit Market price. Differentiating the profit function with respect to quantity supplied for each firm left a system of linear equations, the simultaneous solution of which gave the equilibrium quantity, price and profits. Cournot's contributions to the mathematization of economics would be neglected for decades, but eventually influenced many of the marginalists. Cournot's models of duopoly and Oligopoly also represent one of the first formulations of non-cooperative games. Today the solution can be given as a Nash equilibrium but Cournot's work preceded modern game theory by over 100 years. Cournot provided a solution for what would later be called partial equilibrium,	
CARL MENGER in Vienna	an Austrian economist and the founder of the Austrian School of Economics. Menger

	contributed to the development of the theories of marginalism and marginal utility, which rejected cost-of-production theory of value, such as developed by the classical economists such as Adam Smith and David Ricardo. As a departure from such, he would go on to call his resultant perspective, the subjective theory of value
Menger used his subjective theory of value to arrive at what he considered one of the most powerful insights in economics: *"both sides gain from exchange"*. Unlike William Jevons, Menger did not believe that goods provide "utils," or units of utility. Rather, he wrote, goods are valuable because they serve various uses whose importance differs. Menger also came up with an explanation of how money develops that is still accepted by some schools of thought today.	
The **subjective theory of value** is an economic theory which advances the idea that the value of a good is not determined by any inherent property, nor by the amount of labour necessary to produce it, but instead is determined by the importance an acting individual places on a good for the achievement of his desired ends	
Léon Walras **Marie-Esprit-Léon Walras** (16 December 1834 – 5 January 1910)	i. was a French mathematical economist. ii. He formulated the marginal theory of value (independently of William Stanley Jevons and Carl Menger) and pioneered the development of general equilibrium theory. iii. Walras is best known for his book *Éléments d'économie politique pure*, a work that has contributed greatly to the mathematization of economics through the concept of general equilibrium. The definition of the role of the entrepreneur found in it was also taken up and amplified by Schumpeter.
1. *Léon Walras attempted to formalize discussion of the economy as a whole through a theory of general competitive equilibrium. The behavior of every economic actor would be considered on both the production and consumption side. Walras originally presented four separate models of exchange, each recursively included in the next. The solution of the resulting system of equations (both linear and non-linear) is the general equilibrium.* *At the time, no general solution could be expressed for a system of arbitrarily many equations, but Walras's attempts produced two famous results in economics. The first*	

is Walras' law and the second is the principle of tâtonnement. Walras' method was considered highly mathematical for the time and Edgeworth commented at length about this fact in his review of Éléments d'économie politique pure (Elements of Pure Economics).

Walras' law was introduced as a theoretical answer to the problem of determining the solutions in general equilibrium. His notation is different from modern notation but can be constructed using more modern summation notation. Walras assumed that in equilibrium, all money would be spent on all goods: every good would be sold at the market price for that good and every buyer would expend their last dollar on a basket of goods. Starting from this assumption, Walras could then show that if there were n markets and n-1 markets cleared (reached equilibrium conditions) that the nth market would clear as well. This is easiest to visualize with two markets (considered in most texts as a market for goods and a market for money). If one of two markets has reached an equilibrium state, no additional goods (or conversely, money) can enter or exit the second market, so it must be in a state of equilibrium as well. Walras used this statement to move toward a proof of existence of solutions to general equilibrium but it is commonly used today to illustrate market clearing in money markets at the undergraduate level.[27]

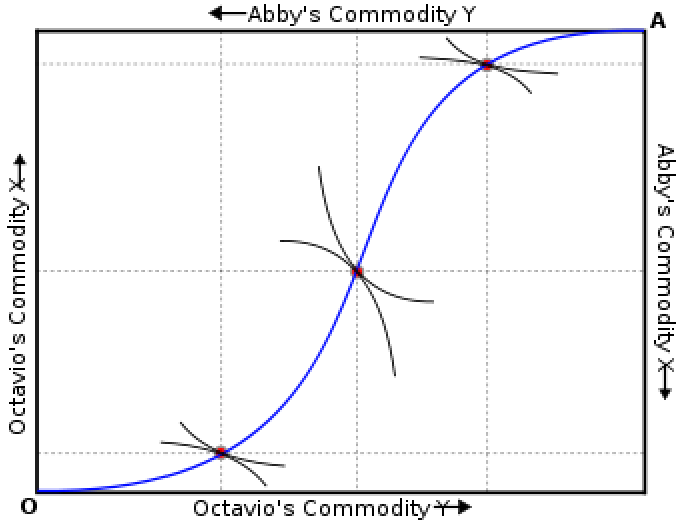
Tâtonnement (roughly, French for *groping toward*) was meant to serve as the practical expression of Walrasian general equilibrium. Walras abstracted the marketplace as an auction of goods where the auctioneer would call out prices and market participants would wait until they could each satisfy their personal reservation prices for the quantity desired (remembering here that this is an auction on *all* goods, so everyone has a reservation price for their desired basket of goods).[28]

Only when all buyers are satisfied with the given market price would transactions occur. The market would "clear" at that price—no surplus or shortage would exist. The word *tâtonnement* is used to describe the directions the market takes in *groping toward* equilibrium, settling high or low prices on different goods until a price is agreed upon for all goods. While the process appears dynamic, Walras only presented a static model, as no transactions would occur until all markets were in equilibrium. In practice, very few markets operate in this manner.

Francis Ysidro Edgeworth

Edgeworth introduced mathematical elements to Economics explicitly in *Mathematical Psychics: An Essay*

. He adopted Jeremy Bentham's felicific calculus to economic behavior, allowing the outcome of each decision to be converted into a change in utility. Using this assumption, Edgeworth built a model of exchange on three assumptions: individuals are self-interested,

on the Application of Mathematics to the Moral Sciences, published in 1881	individuals act to maximize utility, and individuals are "free to recontract with another independently of...any third party". 

An Edgeworth box displaying the contract curve on an economy with two participants. Referred to as the "core" of the economy in modern parlance, there are infinitely many solutions along the curve for economies with two participants. The felicific calculus is an algorithm formulated by utilitarian philosopher Jeremy Bentham for calculating the degree or amount of pleasure that a specific action is likely to induce

2. n economics, an Edgeworth box, sometimes referred to as an Edgeworth-Bowley box, is a graphical representation of a market with just two commodities, X and Y, and two consumers. The dimensions of the box are the total quantities Ω_x and Ω_y of the two goods. Let the consumers be Octavio and Abby.Given two individuals, the set of solutions where the both individuals can maximize utility is described by the *contract curve* on what is now known as an Edgeworth Box. Technically, the construction of the two-person solution to Edgeworth's problem was not developed graphically until 1924 by Arthur Lyon Bowley. The contract curve of the Edgeworth box (or more generally on any set of solutions to Edgeworth's problem for more actors) is referred to as the core of an economy.

Edgeworth devoted considerable effort to insisting that mathematical proofs were appropriate for all schools of thought in economics. While at the helm of *The Economic Journal*, he published several articles criticizing the mathematical rigor of rival researchers, including Edwin Robert Anderson Seligman, a noted skeptic of mathematical economics. The articles focused on a back and forth over tax incidence and responses by producers. Edgeworth noticed that a monopoly producing a good that had jointness of supply but not jointness of demand (such as first class and economy on an airplane, if the

plane flies, both sets of seats fly with it) might actually lower the price seen by the consumer for one of the two commodities if a tax were applied. Common sense and more traditional, numerical analysis seemed to indicate that this was preposterous. Seligman insisted that the results Edgeworth achieved were a quirk of his mathematical formulation. He suggested that the assumption of a continuous demand function and an infinitesimal change in the tax resulted in the paradoxical predictions. Harold Hotelling later showed that Edgeworth was correct and that the same result (a "diminution of price as a result of the tax") could occur with a discontinuous demand function and large changes in the tax rate.
Modern Mathematical Economists From the later-1930s, an array of new mathematical tools from the differential calculus and differential equations, convex sets, and graph theory were deployed to advance economic theory in a way similar to new mathematical methods earlier applied to physics. The process was later described as moving from mechanics to axiomatics

Modern Economics	

1. In the landmark treatise *Foundations of Economic Analysis* (1947), Paul Samuelson identified a common paradigm and mathematical structure across multiple fields in the subject, building on previous work by Alfred Marshall. *Foundations* took mathematical concepts from physics and applied them to economic problems. This broad view (for example, comparing Le Chatelier's principle to tâtonnement) drives the fundamental premise of mathematical economics: systems of economic actors may be modeled and their behavior described much like any other system. This extension followed on the work of the marginalists in the previous century and extended it significantly. Samuelson approached the problems of applying individual utility maximization over aggregate groups with comparative statics, which compares two different equilibrium states after an exogenous change in a variable. This and other methods in the book provided the foundation for mathematical economics in the 20th century.
Differential calculus

Vilfredo Federico Damaso Pareto	An Italian civil engineer, sociologist, economist, political scientist, and philosopher. He made several important contributions to economics, particularly in the study of income distribution and in the analysis of individuals'

Analysed Microeconomics by treating decisions by economic actors as attempts to change a given allotment of goods to another, more preferred allotment. Sets of allocations could then be treated as Pareto efficient (Pareto optimal is an equivalent term) when no exchanges could occur between actors that could make at least one individual better off without making any

other individual worse off. Pareto's proof is commonly conflated with Walrassian equilibrium or informally ascribed to Adam Smith's Invisible hand hypothesis.[Rather, Pareto's statement was the first formal assertion of what would be known as the first fundamental theorem of welfare economics.[42] These models lacked the inequalities of the next generation of mathematical economics.

Input-output analysis

In 1936, the Russian–born economist Wassily Leontief built his model of input-output analysis from the 'material balance' tables constructed by Soviet economists, which themselves followed earlier work by the physiocrats. With his model, which described a system of production and demand processes, Leontief described how changes in demand in one economic sector would influence production in another.[52] In practice, Leontief estimated the coefficients of his simple models, to address economically interesting questions. In production economics, "Leontief technologies" produce outputs using constant proportions of inputs, regardless of the price of inputs, reducing the value of Leontief models for understanding economies but allowing their parameters to be estimated relatively easily. In contrast, the von Neumann model of an expanding economy allows for choice of techniques, but the coefficients must be estimated for each technology.

In allowing inequality constraints, the Kuhn–Tucker approach generalized the classic method of Lagrange multipliers, which (until then) had allowed only equality constraints. The Kuhn–Tucker approach inspired further research on Lagrangian duality, including the treatment of inequality constraints. The duality theory of nonlinear programming is particularly satisfactory when applied to convex minimization problems, which enjoy the convex-analytic duality theory of Fenchel and Rockafellar; this convex duality is particularly strong for polyhedral convex functions, such as those arising in linear programming. Lagrangian duality and convex analysis are used daily in operations research, in the scheduling of power plants, the planning of production schedules for factories, and the routing of airlines (routes, flights, planes, crews).

Nash, John Harsanyi

Game theory

In 1994, , and Reinhard Selten received the Nobel Memorial Prize in Economic Sciences their work on non–cooperative games. Harsanyi and Selten were awarded for their work on repeated games. Later work extended their results

	to computational methods of modeling.

John von Neumann, working with Oskar Morgenstern on the theory of games, broke new mathematical ground in 1944 by extending functional analytic methods related to convex sets and topological fixed-point theory to economic analysis.[8][85] Their work thereby avoided the traditional differential calculus, for which the maximum–operator did not apply to non-differentiable functions. Continuing von Neumann's work in cooperative game theory, game theorists Lloyd S. Shapley, Martin Shubik, Hervé Moulin, Nimrod Megiddo, Bezalel Peleg influenced economic research in politics and economics. For example, research on the fair prices in cooperative games and fair values for voting games led to changed rules for voting in legislatures and for accounting for the costs in public–works projects. For example, cooperative game theory was used in designing the water distribution system of Southern Sweden and for setting rates for dedicated telephone lines in the USA.

Earlier neoclassical theory had bounded only the *range* of bargaining outcomes and in special cases, for example bilateral monopoly or along the contract curve of the Edgeworth box.[88] Von Neumann and Morgenstern's results were similarly weak. Following von Neumann's program, however, John Nash used fixed–point theory to prove conditions under which the bargaining problem and noncooperative games can generate a unique equilibrium solution.[89] Noncooperative game theory has been adopted as a fundamental aspect of experimental economics, behavioral economics, information economics, industrial organization, and political economy. It has also given rise to the subject of mechanism design (sometimes called reverse game theory), which has private and public-policy applications as to ways of improving economic efficiency through incentives for information sharing.

Choosing among Competing Model

Advice given by Clive Granger is worth keeping in mind:

i. What purpose does it have? What economic decision does it help with?
ii. Is there any evidence being presented that allows me to evaluate its quality compared to alternative theories or models?

Differential decline and rise

John von Neumann's work on functional analysis and topology broke new ground in mathematics and economic theory.[44][85] It also left advanced mathematical economics with fewer applications of differential calculus. In particular, general equilibrium theorists used general topology, convex geometry, and optimization theory more than differential calculus, because the approach of differential calculus had failed to establish the existence of an equilibrium.

However, the decline of differential calculus should not be exaggerated, because differential calculus has always been used in graduate training and in applications. Moreover, differential calculus has returned to the highest levels of mathematical economics, general equilibrium theory (GET), as practiced by the "GET-set" (the humorous designation due to Jacques H. Drèze). In the 1960s and 1970s, however, Gérard Debreu and Stephen Smale led a revival of the use of differential calculus in mathematical economics. In particular, they were able to prove the existence of a general equilibrium, where earlier writers had failed, because of their novel mathematics: Baire category from general topology and Sard's lemma from differential topology. Other economists associated with the use of differential analysis include Egbert Dierker, Andreu Mas-Colell, and Yves Balasko.[86][87] These advances have changed the traditional narrative of the history of mathematical economics, following von Neumann, which celebrated the abandonment of differential calculus.

Linear models

1. Restricted models of general equilibrium were formulated by John von Neumann in 1937. Unlike earlier versions, the models of von Neumann had inequality constraints. For his model of an expanding economy, von Neumann proved the existence and uniqueness of an equilibrium using his generalization of Brouwer's fixed point theorem. Von Neumann's model of an expanding economy considered the matrix pencil $\boldsymbol{A} - \lambda \boldsymbol{B}$ with nonnegative matrices **A** and **B**; von Neumann sought probability vectors p and q and a positive number λ that would solve the complementarity equation

2. $p^T (\boldsymbol{A} - \lambda \boldsymbol{B})\, q = 0,$

3. along with two inequality systems expressing economic efficiency. In this model, the (transposed) probability vector p represents the prices of the goods while the probability vector q represents the "intensity" at which the production process would run. The unique solution λ represents the rate of growth of the economy, which equals the interest rate. Proving the existence of a positive growth rate and proving that the growth rate equals the interest rate were remarkable achievements, even for von Neumann. Von Neumann's results have been viewed as a special case of linear programming, where von Neumann's model uses only nonnegative matrices.[48] The study of von Neumann's model of an expanding economy continues to interest mathematical economists with interests in computational economics.[49][50][51]

1. It was in the course of proving of the existence of an optimal equilibrium in his 1937 model of economic growth that John von Neumann introduced functional analytic methods to include topology in economic theory, in particular, fixed-point theory through his

generalization of Brouwer's fixed-point theorem.[8][44][76] Following von Neumann's program, Kenneth Arrow and Gérard Debreu formulated abstract models of economic equilibria using convex sets and fixed–point theory. In introducing the Arrow–Debreu model in 1954, they proved the existence (but not the uniqueness) of an equilibrium and also proved that every Walras equilibrium is Pareto efficient; in general, equilibria need not be unique.[77] In their models, the ("primal") vector space represented *quantities* while the "dual" vector space represented *prices*.[78]

2. In Russia, the mathematician Leonid Kantorovich developed economic models in partially ordered vector spaces, that emphasized the duality between quantities and prices.[79] Kantorovich renamed *prices* as "objectively determined valuations" which were abbreviated in Russian as "o. o. o.", alluding to the difficulty of discussing prices in the Soviet Union.[78][80][81]

3. Even in finite dimensions, the concepts of functional analysis have illuminated economic theory, particularly in clarifying the role of prices as normal vectors to a hyperplane supporting a convex set, representing production or consumption possibilities. However, problems of describing optimization over time or under uncertainty require the use of infinite–dimensional function spaces, because agents are choosing among functions or stochastic processes.[78][82][83][84]

Following are two post-Keynesian models of Income. So, they can be said as an improvement over Keynesian model.

i. PERMANENT INCOME HYPOTHESIS OF MILTON FRIEDMAN

The permanent income hypothesis was formulated by the Nobel Prize winning economist Milton Friedman in 1957. The hypothesis implies that changes in consumption behavior are not predictable, because they are based on individual expectations. This has broad implications concerning economic policy. Under this theory, even if economic policies are successful in increasing income in the economy, the policies may not kick off a multiplier effect from increased consumer spending. Rather, the theory predicts there will not be an uptick in consumer spending until workers reform expectations about their future incomes.

(source: http://www.investopedia.com/terms/p/permanent-income-hypothesis.asp dated 22/2/2016)

Milton Friedman was an American economist who received the 1976 Nobel Memorial Prize in Economic Sciences for his research on consumption analysis, monetary history and theory and the complexity of stabilization policy. Wikipedia

Born: July 31, 1912, Brooklyn, New York City, New York, United States

Died: November 16, 2006, San Francisco, California, United States

Influenced by: John Maynard Keynes, Friedrich Hayek, more

Influenced: Margaret Thatcher, Gary Becker, Mart Laar, more

ii. LIFE CYCLE PERMANENT INCOME HYPOTHESIS of ROBERT HALL

According to the hypothesis, consumers form estimates of their ability to consume in the long run and then set current consumption to the appropriate fraction of that estimate. The estimate may be stated in the form of wealth, following Modigliani, in which case the fraction is the annuity value of wealth, or as permanent income, following Friedman, in which case the fraction should be very close to one. The major problem in empirical research based on the hypothesis has arisen in fitting the part of the model that relates current and past observed income to expected future income.

(Source_ Hall Robert E. (1978), "Stochastic Implications of the Life Cycle – Permanent Income Hypothesis: Theory and evidence" Journal of Political Economy, vol86, no. 6 pp 971-987)

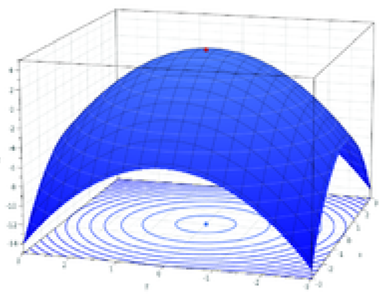
Robert Ernest "Bob" Hall *(born August 13, 1943)*	is an American economist and a Robert and Carole McNeil Senior Fellow at Stanford University's Hoover Institution. He is generally considered a macroeconomist, but he describes himself as an "applied economist".[1]

Bob Hall received a BA in Economics at the University of California, Berkeley and a PhD in Economics from MIT for thesis titled Essays on the Theory of Wealth under the supervision of Robert Solow. He is a member of the Hoover Institution, the National Academy of Sciences, a fellow at both American Academy of Arts and Sciences and the Econometric Society, and a member of the NBER, where he is the program director of the business cycle dating committee. Hall served as President of the American Economic Association in 2010

4. *Economic dynamics* allows for changes in economic variables over time, including in dynamic systems. The problem of finding optimal functions for such changes is studied in variational calculus and in optimal control theory. Before the Second World War, Frank Ramsey and Harold Hotelling used the calculus of variations to that end.

5. Following Richard Bellman's work on dynamic programming and the 1962 English translation of L. Pontryagin *et al.*'s earlier work, optimal control theory was used more extensively in economics in addressing dynamic problems, especially as to economic growth equilibrium and stability of economic systems, of which a textbook example is optimal consumption and saving. A crucial distinction is between deterministic and stochastic control models. Other applications of optimal control theory include those in finance, inventories, and production for example.

6. Mathematical optimization



7.

8. Red dot in z direction as maximum for paraboloid function of (x, y) inputs

	9. Mathematical optimization and Dual problem *See also: Convexity in economics and Non-convexity (economics)*

In mathematics, mathematical optimization (or optimization or mathematical programming) refers to the selection of a best element from some set of available alternatives. In the simplest case, an optimization problem involves maximizing or minimizing a real function by selecting input values of the function and computing the corresponding values of the function. The solution process includes satisfying general necessary and sufficient conditions for optimality. For optimization problems, specialized notation may be used as to the function and its input(s). More generally, optimization includes finding the best available element of some function given a defined domain and may use a variety of different computational optimization techniques.

Economics is closely enough linked to optimization by agents in an economy that an influential definition relatedly describes economics *qua* science as the "study of human behavior as a relationship between ends and scarce means" with alternative uses. Optimization problems run through modern economics, many with explicit economic or technical constraints. In microeconomics, the utility maximization problem and its dual problem, the expenditure minimization problem for a given level of utility, are economic optimization problems. Theory posits that consumers maximize their utility, subject to their budget constraints and that firms maximize their profits, subject to their production functions, input costs, and market demand.

Economic equilibrium is studied in optimization theory as a key ingredient of economic theorems that in principle could be tested against empirical data. Newer developments have occurred in dynamic programming and modeling optimization with risk and uncertainty, including applications to portfolio theory, the economics of information, and search theory.

Optimality properties for an entire market system may be stated in mathematical terms, as in formulation of the two fundamental theorems of welfare economics and in the Arrow–Debreu model of general equilibrium (also discussed below). More concretely, many problems are

amenable to analytical (formulaic) solution. Many others may be sufficiently complex to require numerical methods of solution, aided by software.[56] Still others are complex but tractable enough to allow computable methods of solution, in particular computable general equilibrium models for the entire economy.

Linear optimization	*Linear programming and Simplex algorithm*

1. Linear programming was developed to aid the allocation of resources in firms and in industries during the 1930s in Russia and during the 1940s in the United States. During the Berlin airlift (1948), linear programming was used to plan the shipment of supplies to prevent Berlin from starving after the Soviet blockade.

2. Linear and nonlinear programming have profoundly affected microeconomics, which had earlier considered only equality constraints.[64] Many of the mathematical economists who received Nobel Prizes in Economics had conducted notable research using linear programming: Leonid Kantorovich, Leonid Hurwicz, Tjalling Koopmans, Kenneth J. Arrow, Robert Dorfman, Paul Samuelson and Robert Solow.[65] Both Kantorovich and Koopmans acknowledged that George B. Dantzig deserved to share their Nobel Prize for linear programming. Economists who conducted research in nonlinear programming also have won the Nobel prize, notably Ragnar Frisch in addition to Kantorovich, Hurwicz, Koopmans, Arrow, and Samuelson.

Agent-based computational economics

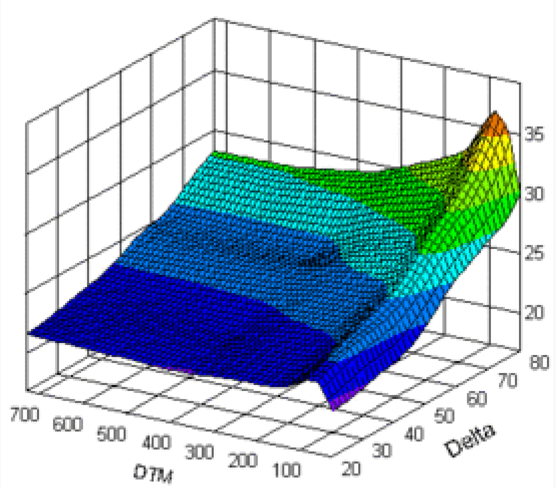
Agent-based computational economics (ACE) as a named field is relatively recent, dating from about the 1990s as to published work. It studies economic processes, including whole economies, as dynamic systems of interacting agents over time. As such, it falls in the paradigm of complex adaptive systems. In corresponding agent-based models, agents are not real people but "computational objects modeled as interacting according to rules" ... "whose micro-level interactions create emergent patterns" in space and time. The rules are formulated to predict behavior and social interactions based on incentives and information. The theoretical assumption of mathematical *optimization* by agents markets is replaced by the less restrictive postulate of agents with *bounded* rationality *adapting* to market forces.

3. ACE models apply numerical methods of analysis to computer-based simulations of complex dynamic problems for which more conventional methods, such as theorem formulation, may not find ready use. Starting from specified initial conditions, the computational economic system is modeled as evolving over time as its constituent agents repeatedly interact with each other. In these respects, ACE has been characterized as a bottom-up culture-dish approach to the study of the economy. In contrast to other standard

modeling methods, ACE events are driven solely by initial conditions, whether or not equilibria exist or are computationally tractable. ACE modeling, however, includes agent adaptation, autonomy, and learning.[102] It has a similarity to, and overlap with, game theory as an agent-based method for modeling social interactions. Other dimensions of the approach include such standard economic subjects as competition and collaboration, market structure and industrial organization, transaction costs, welfare economics and mechanism design, information and uncertainty, and macroeconomics.

Mathematicization of economics

Volatility Surface



The surface of the Volatility smile is a 3-D surface whereby the current market implied volatility (Z-axis) for all options on the underlier is plotted against strike price and time to maturity (X & Y-axes).

The volatility surface is a three-dimensional plot of the implied volatility of a stock option. Implied volatility exists due to discrepancies with how the market prices stock options and what stock option pricing models say the correct prices should be. To gain a full understanding of this phenomenon, it is important to know the basics of stock options, stock option pricing, and the volatility surface.

https://www.investopedia.com/articles/stock-analysis/081916/volatility-surface-explained.asp

Over the course of the 20th century, articles in "core journals" in economics have been almost exclusively written by economists in academia. As a result, much of the material transmitted in those journals relates to economic theory, and "economic theory itself has been continuously more abstract and mathematical." A subjective assessment of mathematical techniques[117] employed in these core journals showed a decrease in articles that use neither geometric representations nor mathematical notation from 95% in 1892 to 5.3% in 1990. A 2007 survey of ten of the top economic journals finds that only 5.8% of the articles published in 2003 and 2004 both lacked statistical analysis of data and lacked displayed mathematical expressions that were indexed with numbers at the margin of the page.

The French engineer, A. J. E. Dupuit, used mathematical symbols to express his concepts of supply and demand. Even though he had no systematic theory he did develop the concepts of utility and diminishing utility, which were clearly stated and presented in graphical form. He viewed the price of a good as dependent on the price of other goods. Another Lausanne economist, Vilfredo Pareto, ranks with the best in mathematical virtuosity. The Pareto optimum and the indifference curve analysis (along with Edgeworth in England) were clearly conceived in the mathematical frame work. A much more profound understanding of the analytical power of mathematics was shown by Leon Walras. He is generally regarded as the founder of the mathematical school of economics. He set out to translate pure theory into pure mathematics. After the pioneering work of Jevons (1871) and Walras (1874), the use of mathematics in economics progressed at very slow pace for a number of years. The first of the latter group is F. Y. Edgeworth. His contribution to mathematical economics is found mainly in the 1881 publication in which he dealt with the theory of probability and statistical theory and the law of error. The indifference analysis was first propounded by Edgeworth in 1881 and restated in 1906 by Pareto and in 1915 by the Russian economist Slutsky, who used elaborate mathematical treatment of the topic. Alfred Marshall made extensive use of mathematics in his Principles of Economics (1890). His interest in mathematics dates from his early schooldays when his first love was for mathematics, not the classics." In the Principles, he used mathematical techniques very effectively. Another economist whose influence spans many years is the American, Irving Fisher. Fisher belongs to other groups as well as to the mathematical moderns. His life's work reveals him as a statistician, econometrician, mathematician, pure theorist, teacher, social crusader, inventor, businessman, and scientist. His contribution to statistical method, The Making of Index Numbers, was great. The well-known Fisher formula, $M V + M' V' = P T$, is an evidence of his contribution to quantitative economics. A strong inducement to formulate economic models in mathematical terms has been the post-World War II development of the electronic computer. In the broad area of economics, there has been a remarkable use of mathematical techniques. Economics, like several other disciplines, has always used quantification to some degree. Common terms such as wealth, income, margins, factor returns, diminishing returns, trade balances, balance of payments, and the many other familiar concepts have a quantitative connotation. All economic data have in some fashion been reduced to numbers which became generally known as economic statistics. The most noteworthy developments, however, have come in the decades since about 1930. It was approximately this time that marked the ebb of neoclassicism, the rise of institutionalism, and the introduction of aggregate economics. Over a period of years scholars have developed new techniques designed to help in the explanation of economic behavior under different market situations using mathematics. Now we discuss some of the economic theories and the techniques of modern analysis. They are given largely on a chronological basis, and their significance is developed in the discussion. The Theory of Games: The pioneering work was done by John von Neumann in 1928. The theory became popular with the publication of Theory of Games and Economic Behaviour in 1944. Basically, The Game Theory holds that the actions of players in gambling games are similar to situations that prevail in economic, political, and social life. The theory of games has many elements in common with real-life situations. Decisions must be made on the basis of available facts, and chances must be taken to win. Strategic moves must be concealed (or anticipated) by the contestants based on past knowledge and future estimates. Success or failure rests, in large measure, on the accuracy of the analysis of the elements. The theory of games introduced an interesting and challenging concept. Economists have made some use of it, and it has also been fitted into other social sciences, notably sociology and political science.

Linear Programming: Linear programming is a specific class of mathematical problems in which a linear function is maximized (or minimized) subject to given linear constraints.

The founders of the subject are generally regarded as George B. Dantzig, who devised the simplex method in 1947, and John von Neumann, who established the theory of duality that same year. The scope of linear programming is very broad. It brings together both theoretical and practical problems in which some quantity is to be maximized or minimized. The data could be almost any fact such as profit, costs, output, distance to or from given points, time, and so on. It also makes allowance for given technology and restraints that may occur in factor markets or in finance. Linear programming has been proven very useful in many areas. It is in common use in agriculture, where chemical combinations of proper foods for plants and animals have been worked out, and in the manufacture of many processed agricultural products. It is necessary in modem materials scheduling, in shipping, and in final production. The Nobel prize in economics was awarded in 1975 to the mathematician Leonid Kantorovich (USSR) and the economist Tjalling Koopmans (USA) for their contributions to the theory of optimal allocation of resources, in which linear programming played a key role.

Input-Output Analysis: In terms of techniques, input-output analysis is a rather special case of linear programming. It was devised originally by Leontief and, in a sense, was a World War II-inspired analysis. Basically it was designed for presenting a general equilibrium theory suited for empirical study. The problem is to determine the interrelationship of sector inputs and outputs on other sectors or on all sectors which use the product. The rational for the term IOA can be explained like this. There is a close interdependence between different sectors of a modern economy. This interdependence arises out of the fact that the output of any given industry is utilized as an input by the other industries and often by the same industry itself. Thus, the IOA analyses the interdependence between different sectors of an economy. The basis of IOA is the input - output table which can be expressed in the form of matrices.