Application of Integration in Economics

The concept of integration is widely used in business and economics. In this section, we consider the following applications of integrals in finance and economics:

- Marginal and total revenue, cost, and profit;
- Capital accumulation over a specified period of time;
- Consumer and producer surplus;
- Lorenz curve and Gini coefficient;

1. Marginal and Total Revenue, Cost, and Profit

1.1 Marginal Revenue (MR) is the additional revenue gained by producing one more unit of a good.
It can also be described as the change in total revenue (TR) due to change in quantity produced (X):

$$MR = d(TR)/dX.$$

If a marginal revenue function $MR(X)$ is known, the total revenue can be obtained by integrating the marginal revenue function:

$$TR(X) = \int MR(X)dX,$$

where integration is carried out over a certain interval of X.

1.2 Marginal cost (MC) denotes the additional cost of producing one extra unit of output.
The similar relationship exists between the marginal cost MC and the total cost TC:

$$MC = d(TC)/dX,$$

so

$$TC(X) = \int MC(X)dX.$$

Since profit is defined as

$$TP = TR - TC,$$

we can write the following equation for marginal profit (MP):

$$MP = MR - MC,$$

Or, $dTP/dX = dTR/dX - dTC/dX.$

Example 1.

The marginal revenue of a company is given by $MR=100+20X+3X^2$, where X is number of units sold for a period. the total revenue function if at $X=2$, is equal to 260.

Solution.
We find the total revenue function TR by integrating the marginal revenue function MR:
$TR(X)=\int MR(X)dX=\int(100+20X+3X^2)dX=100X+10X^2+X^3+C.$
The constant of integration C can be determined using the initial condition $TR(X=2)=260$. Hence,
$200+40+8+C=260, \quad C=12.$
So, the total revenue function is given by

$TR(X)=100X+10X^2+X^3+12.$

Example 2- If the Marginal Revenue Function is $MR = 60/(X+3)^2 - 2$
Find total Revenue. Also deduce the demand function.
Solution-
$TR = \int 60/(X+3)^2$

2. Capital Accumulation Over a Period

Let $I(t)$ be the rate of investment. The total capital accumulation K during the time interval $[a,b]$ can be estimated by the formula

Example 2.

The rate of investment is given by $I(t)=6\sqrt{t}$. Calculate the capital growth between the 4th and the 9th years.

Solution.
Using the integration formula

we have

$= \quad = 76.$

Example 3.

Assume the rate of investment is given by the function $I(t)=\ln t$. Compute the total capital accumulation between the 1_{st} and the 5_{th} years.

Solution.
To calculate the capital accumulation, we use the formula

Integrat by parts, we have
$K=(t\ln t-t)|_1^5=(5\ln5-5)-(\ln1-1)=5\ln5-4\approx4.05$

3. Consumer's and Producer's Surplus

The demand function or demand curve shows the relationship between the price of a certain product or service and the quantity demanded over a period of time.
The supply function or supply curve shows the quantity of a product or service that producers will supply over a period of time at any given price.
Both these price-quantity relationships are usually considered as functions of quantity (X). Generally, the demand function P=D(X) is decreasing, because consumers are likely to buy more of a product at lower prices. Unlike the law of demand, the supply function P=S(X) is increasing, because producers are willing to deliver a greater quantity of a product at higher prices. The demand and supply curve intersect each other at point E where the price is P0 and the quantity exchanged is Xo is called the market equilibrium point.

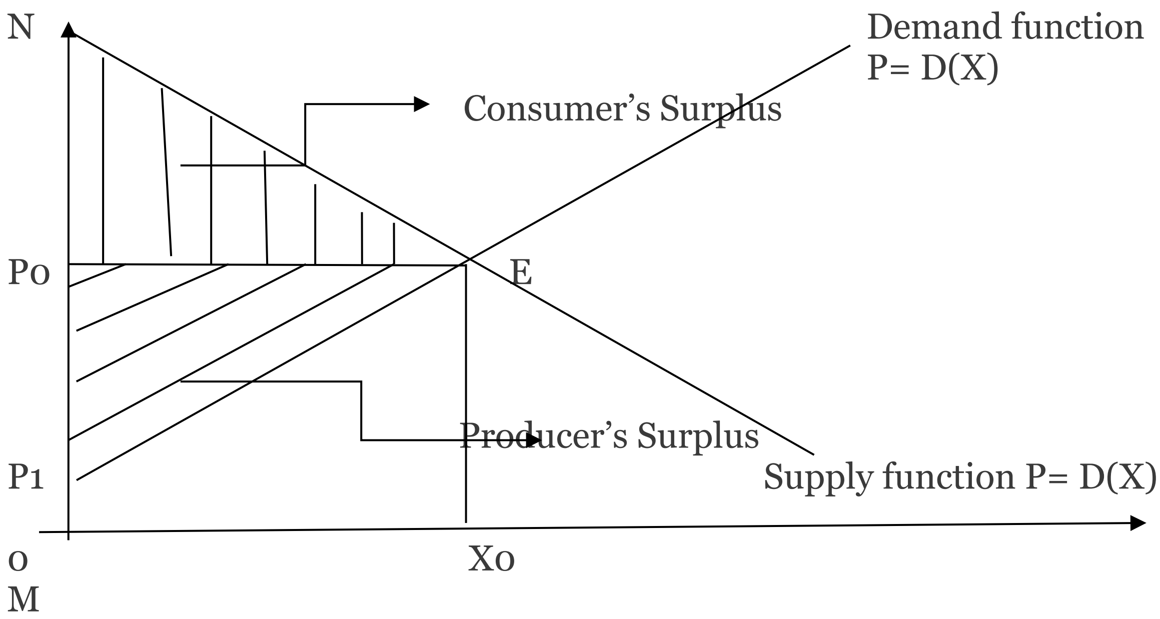


Figure 1: Consumer's and Producer's Surplus

Consumers' Surplus = Amount of money which the consumer is ready to pay – Amount of money which he actually pays. The maximum price a consumer is willing and able to pay is defined by the demand curve P=D(X). For quantities X<X0, it is greater than the equilibrium price P0 in the market. Consumers gain by buying at the equilibrium price rather than at a higher price. This net gain is called consumer surplus.

Consumer surplus is represented by the area under the demand curve P=D(Q) and above the horizontal line P=P0 at the level of the market price.

Consumer surplus (CS) is thus defined by the integration formula

CS

Producers' Surplus = Amount of money which he actually gets - Amount of money at which the producer is ready to sell

A similar analysis shows that producers also gain if they trade their products at the market equilibrium price. Their gain is called producer surplus (PS) and is given by the equation PS=P0X0−X0∫0S(X)dX = X0∫0[P0−S(X)]dX.

Example 4

For a certain product, the demand function is $D(X)=1000-25X$, and the supply function is $S(X)=100+X^2$. Compute the consumer and producer surplus.

Example 5

Assuming the demand function is $D(X)=50-X$ and the supply function is $S(X)=20+\sqrt{X}$, compute the consumer and producer surplus.

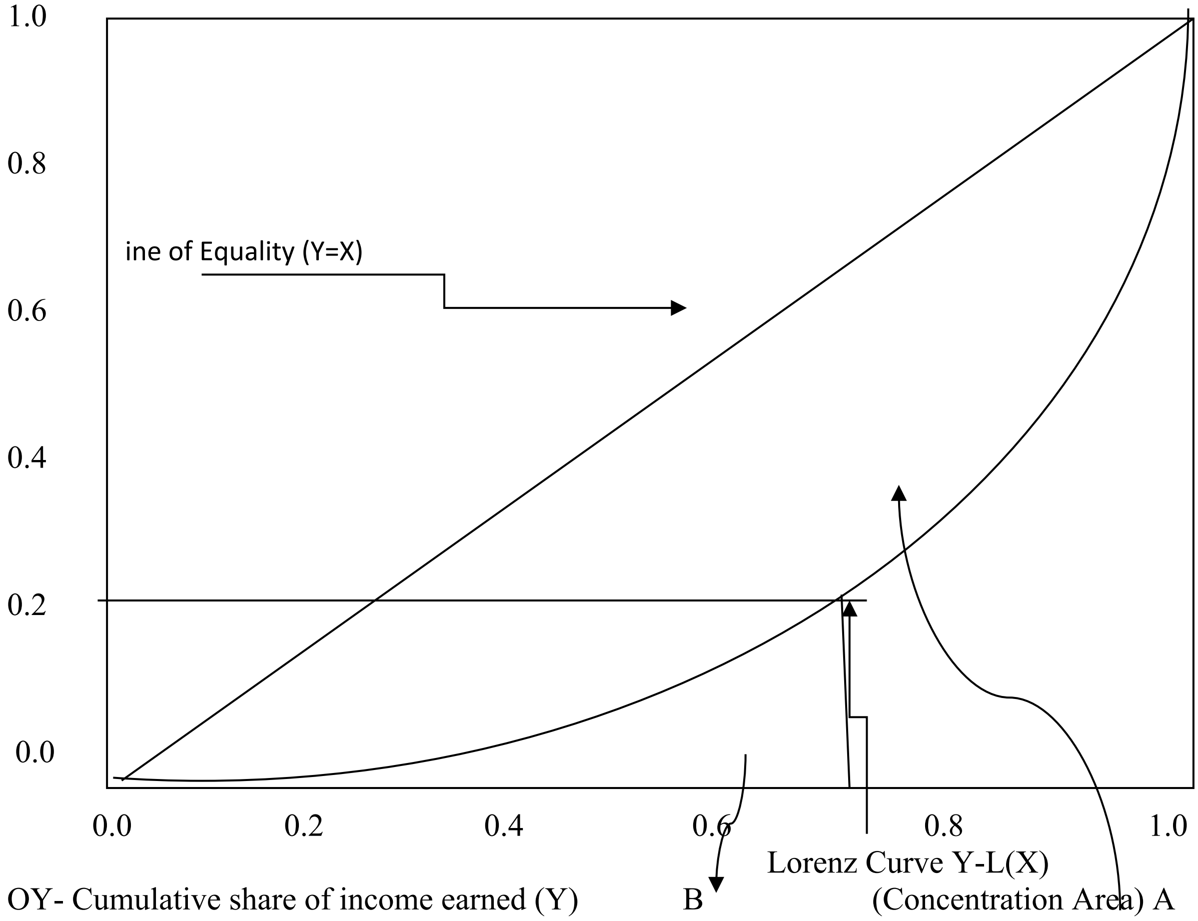
Example 6

Assume the demand and supply functions for a product are $D(X)=(X-2a)^2$ and $S(Q)=Q^2$, where $a>0$ is a parameter. Compute the consumer and producer surplus.

Lorenz Curve and Gini Coefficient

The Lorenz curve is a graphical representation of percentage cumulative frequency.

The horizontal axis on a Lorenz curve typically shows the portion or percentage of total population, and the vertical axis shows the portion of total income or wealth. For instance, if a Lorenz curve has a point with coordinates (0.7,0.2), this means that the first 70% of population (ranked by income in increasing order) earned 20% of total income.



OY- Cumulative share of income earned (Y)
OX – Cumulative share of people from the lower income (X)

Figure 2.

The Lorenz Curve is represented by a convex curve. It is used to show:

1. Income/ wealth distribution among population.
2. It can also be used to show land distribution. A more convex Lorenz curve implies more inequality in income distribution.

The area between the 45–degree line (the line of equality) and the Lorenz curve can be used as a measure of inequality. It ranges from 0 (or 0%) to 1 (or 100%), with 0 representing perfect equality in a population and 1 representing perfect inequality. The Gini coefficient G is defined as the area between the line of equality and the Lorenz curve, divided by the total area under the line of equality:

$G=A/(A+B)$ = The Gini coefficient equals two concentration areas (2S). Thus,

=

where L(x) is the Lorenz curve defined as follows. Let us assume that a vector of incomes x = (x1, ..., xn) is arranged in non-decreasing order: $x1 \leq x2 \leq ... \leq xn$. The empirical Lorenz function is generated by points, whose first coordinates are numbers i/n, where i = 0, 1, …, n; n is a fixed number, and second coordinates are determined as follows: L(0) = 0 and

where si = x1 + x2 + ... + xi . The Lorenz curve is defined at all points $p \in (0, 1)$ through linear interpolation. One can show that L'(x) > 0 and L"(x) > 0, L(0) = 0 and L(1) = 1.

The Gini Index Around the World

Global Gini

Christoph Lakner of the World Bank and Branko Milanovic of the City University of New York estimate that the global income Gini coefficient was 0.705 in 2008, down from 0.722 in 1988. Figures vary considerably, however. DELTA economists François Bourguignon and Christian Morrisson estimate that the figure was 0.657 in both 1980 and 1992. Bourguignon and Morrisson's work shows a sustained growth in inequality since 1820 when the global Gini coefficient was 0.500. Lakner and Milanovic's shows a decline in inequality around the beginning of the 21st century, as does a 2015 book by Bourguignon.

Shortcomings

Though useful for analyzing economic inequality, the Gini coefficient has some shortcomings. The metric's accuracy is dependent on reliable GDP and income data. Shadow economies and informal economic activity are present in every country. Informal economic activity tends to represent a larger portion of true economic production in developing countries and at the lower end of the income distribution within countries. In both cases this means that the Gini index of measured incomes will overstate true income inequality. Accurate wealth data is even more difficult to come by due to the popularity of tax havens.

Another flaw is that very different income distributions can result in identical Gini coefficients. Because the Gini attempts to distill a two dimensional area (the gap between the Lorenz curve and the equality line) down in to a single number, it obscures information about the "shape" of inequality. In everyday terms, this would be similar to describing the contents of a photo solely by it's length along one edge, or the simple average brightness value of the pixels. While using the Lorenz curve as a supplement can provide more information in this respect, it also does not show demographic variations among subgroups within the distribution, such as a the distribution of incomes across age, race, or social groups. In that vein, understanding demographics can be important for understanding what a given Gini coefficient represents. For example, a large retired population pushes the Gini higher.

Example 7

Calculate the Gini coefficient for the Lorenz function $L(x)=x^3$. Determine the degree of equality of income distribution.

Solution.
Substituting $L(x)=x3$ and evaluating the integral, we find:
$=2_1\int_0(x-x3)dx=2(x^2/2-x_{4/4})^1{}_0=2(1/2-1/4)=0.50$
The Lorenz curve of income distribution within a certain group is given by the formula:

Because G is 50%,.

Example 8

Calculate the Gini coefficient for the Lorenz function $L(x)=x_p$, where $p>1$.
Using the formula

we obtain

$= 2(x^2/2-x^{P+1}/p+1)^1{}_0=2(1/2 -1/p+1)= 1-2/p+1.$
In particular,

$G(p=2)=2-2/2+1=13\approx0.33;$
$G(p=2)=1-2/3+1=12=0.50;$
$G(p=4)=1-2/4+1=35=0.60;$

Example 9

Suppose the Lorenz curve for a society is given by $L(x)=35x^3+15x^2+15x$. Find the Gini coefficient for this income distribution.

Example 10

Suppose the Lorenz curve for a country is given by $L(x)=1-\sqrt{1}-x^2$. Determine the Gini coefficient for this income distribution.