Application of Differential Calculus

Micro Economics

A.1 Utility

1. Theory of Consumer Behaviour- Maximization of Utility

The problem of a consumer consists of three things.

i. Object -To maximise the satisfaction
ii. Constraints- Unable to satisfy all his/her wants
iii. Decision – In course of maximizing satisfaction, the consumer taken decision about how much of any product will be used

1.1 The Utility Approach

It was developed by Marshall

Assumption

i. Utility can be measured cardinally (in numbers).
ii. Consumer is rational
iii. The consumer has to spend his total income on two products

Maximize U = f(x1, x2)…………………………………………………………………………(1.1)

Where U = consumer's total utility function,
Y =consumer's income
Where x1, x2 are the quantities of commodities consumed, and
P1, p2 are the respective prices of the commodities
Y= (x1p1 + x2p2)……………………………………………………………………………… (1.2)
x1, x2 are endogenous and p1, p2 are exogenous variable

x2 …………………………………………………………………… (1.3)

U = f(x1,)

$dU/dx1 = f_1 + f_2(-p_1/p_2)=0$…………………………………………………….(1.4)

or, $f_1/p_1 = f_2/p_2$ where f_1 and f_2 are marginal utility of x1 and x2

so, MU1/p1 = MU2/p2
………………………………………………………………….(1.5)

extending equation 1.5 to 'n' items

$MU_1/p_1 = MU_2/p_2 = MU_3/p_3 =$…………………….. =MUn/pn………………….. (1.6)

From the second order derivative of (1.4) which will be negative, a sufficient condition to maximise the utility.

Criticisms of the Utility Theory

1. Cardinal measurement (1, 2, 3……) of utility as assumed by A. Marshall is not possible.
2. Marginal utility was considered independent but in case of substitute and complementary goods they are inter-dependent.
3. Marginal utility of money is not constant. As it is known that marginal utility of money decreases as stock of money increases.
4. It is based on assumption of *Ceteris Paribus* (Latin- all other things being unchanged or constant) but is real word it does not happen.

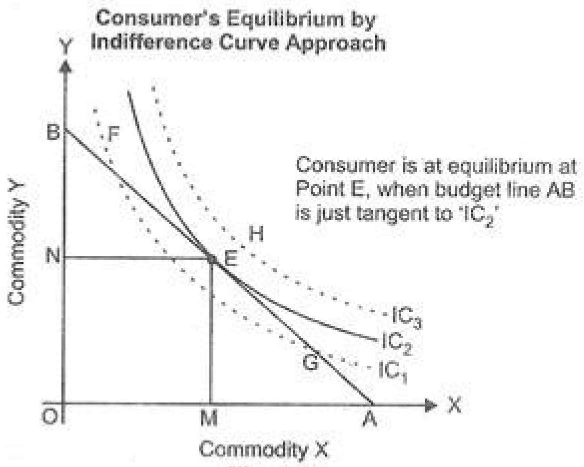
<table>
<tr><td>1.2</td><td>Indifference Curve</td><td>1. Prof. Edgeworth (a British Economist) first introduced the use of indifference curve in 1881.
2. In 1906, Vilfredo Pareto (an Italian Economist) applied the Edgeworth Technique’ in the Welfare Economics.
3. Prof. JR Hicks and Prof. RGD Allen popularised and extended the use of indifference curve analysis in 1930’s</td></tr>
<tr><td colspan="2">Indifference curve can be defined as a locus of points indicating different combinations of Commodity X1 and Commodity x2 of fig 2.12.
ii. $IC_1 < IC_2 < IC_3$ are showing different level of satisfaction.
iii. AB is the budget line where p1 is price of commodity X1 and p2 is the price of commodity x2. All points within ABO is under reach of the consumer.
iv At point E will give the consumer maximum satisfaction where IC2 is tangent to the Budget line.</td><td>

Fig. 2.12
On x axis= x1
Y axis = x2</td></tr>
<tr><td colspan="3">Marginal Rate of Substitution of X_2 for X_1= $MU_1/MU2$
and
Slope of the Budget Line = P_1/P_2
For equilibrium condition,
the marginal rate of substitution (MRS) = the slope of the price line.
All points at an indifference curve is constant we have
U= f (x1, x2)
As you know, for maximiztion,
dU = f1dx1+f2dx2 = 0
or, f1dx1 = -f2dx2………………………………………………(1.7)
or, f1/f2= - dx2/dx1 which is MRS = negative slope of the indifference curve.
Where – dx1 is the amount given away and</td></tr>
</table>

+ $dx2$ is the amount added to the consumption,

Thus, $- dx1 =+ dx2$ ……………………………………………………(1.8)

So, it can be said that f1dx1 is the amount of utility or satisfaction which is consumer gives up to get f2dx2 satisfaction from the commodity X2.

Numerical Example-

Given Total Utility function, U = x1x2

Price of the commodity x1, p1= Rs4

Price of commodity x2, p2 = Rs. 20 and

consumer's income=Y=Rs. 100

Find out the equilibrium level of consumption of commodity x1 and x2

Solution-

Budget constraint as

Y = p1x1+p2x2

Or, 100 = 4x1+20x2

Or, x2 = (100-4x1)/20 = 5- x1/5

Substituting the value of x2 in the utility function,

U = x1x2

= x1(5- x1/5) = $5x_1 - x_1^2/5$

For maximisation first degree differentiation =0 and second degree differentiation should be negative.

Ans= x1= 12.5 and x2 = 5

A.2 Demand

Elasticity of Demand

Demand of a product depends on many factors as price of the product, price of other product, income of the consumer, weather, taste, fashion etc.

Differentiation of quantity of a product with other independent variables shows the magnitude of depends of the dependent variable with respect to change in independent variables when

2.1 Price Elasticity of Demand

1.	Price Elasticity of Demand	$dX/dp.p/X= e_d$ $-\infty<e_d<0$= Normal Goods If, ed=1 -unit price elasticity $e_d>1$; (Luxury commodity) $e_d<1$ (necessary commodity) ed >0 Inferior good

Exercise 1: If the demand is given by $X=25 -4P+P^2$

Find the nature of elasticity of demand. $e_d = - dx/dp.p/x$

or, ed= Marginal function / Average function ……………………………………………………(1)

Exercise- 2: If the demand function X= 20/(p+1) find e_d when price p=4

MR= P (1-1/e)

TR = PX

MR = d(TR)//dX = P+X.dp/dx

= P (1+X/P.dp/dx) = P(1+1/ed) [as ed = dx/dp.p/x]

= AR (1-1 /ledl) if ignore the sign of ed

1.	ed = 1	MR=0
2.	ed > 1	MR>0
3.	ed <1	MR <0

2.2 Cross Elasticity of Demand

$e_d* = dX_1/dP_2.P_2/X_1$

2.	Cross Elasticity of Demand	$dX_1/dp_2.p_2/X_1= e_d$ (i) e_d>0 = Product X1 and X2 are Substitute goods (i) e_d=0 Product X1 and X2 are not at all related to each other (i) e_d<0 = Product X1 and X2 are complimentary good

Exercise 3 - The demand function of the commodity X1 and X2 are

$X1 = P_1^{-1.3} p_2^{0.7}$

$X2 = P_1^{0.6} P_2^{-0.5}$

Find out whether two commodities X1 and X2 are substitute or complementary goods.

Answer- Substitute goods

2.3 Income Elasticity of Demand

3.	Income Elasticity of Demand	e_Y= dX/dY.Y/X where, X – commodity Y – Income i. In case of an inferior good and Geffen goods; $e_Y < 0$ ii. In case of a normal good $e_Y>0$ iii. In case of a luxury good $e_{Y>1}$ iv. In case of a necessary good $e_Y<1$

2.4 Advertisement Elasticity of Demand

Advertisement Elasticity of Demand	e_A = dX/dA.A/X where X= output and A = amount of money spent on advertisement	Though it has become very important ingredient of marketing in the recent time, generally If e_A>1 Luxury commodity e_A<1 Necessary commodity

A2.5. Elasticity of Substitution

Elasticity of substitution may be defined as the extent to which one commodity can be substituted for another as a consequence of a given change in the ratio. Symbolically,

e_s =

Where X/Y = ratio commodities X and Y

And, Px/Py = ratio of the price of x and y.

SN	Elasticity of Substitution	Implications
1.	$e_s = \infty$	Two commodities are perfect substitute. In this case, the rate of substitution between them will be constant or uniform. A fall in the price of commodity X, assuming that price of commodity Y be constant, will lead the consumer to substitute commodity X completely for Y. But in real life we do not find such cases. In case the two commodities are perfect substitutes, then they should be considered as one commodity.
2.	e_s >1 [or, $0<e_s<\infty$]	Two commodities can be substituted for each other
3.	$e_{s=}0$	No possibility of any substitution

A2.6 Mathematical Relationship between Price Elasticity of Demand, Elasticity of Substitution and the income elasticity of demand.

The price elasticity of demand depends on the elasticity of substitution, on the one hand, and the income elasticity of demand on the other. As the price effect is divided into two parts – the substitution effect and the income effect: the price-elasticity of demand can also be divided into two parts: the elasticity of substitution and income elasticity of demand.

Where, $e_p = es \pm \lambda\ e_y$

e_p = price elasticity; e_s= elasticity of substitution; e_y = income elasticity

A3 Elasticity of Supply

The elasticity of supply can be defined as a percentage change in quantity supplied divided by a percentage change in the price.

η =dx/dp.p/x

dx = change in supply

dp= change in price

x- quantity supplied

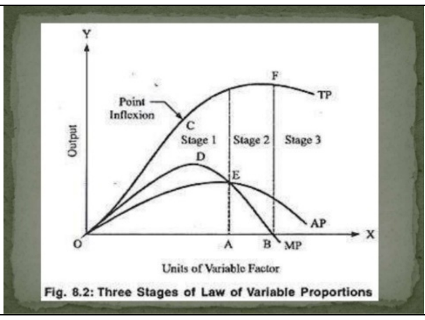
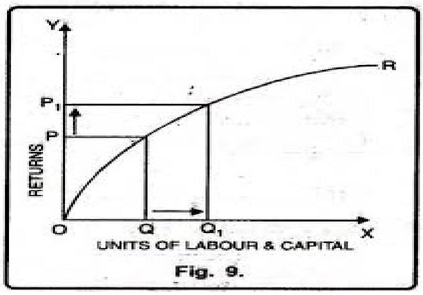
p= price of the product

That is 0< η < ∞

If η = ∞	Perfectly elastic supply curve
η =0	Perfectly inelastic supply curve
η > 1	More than unit elastic supply curve
η < 1	Less than unit elastic supply curve
η =1	Unit elastic supply curve

Elasticity of supply at different points of the supply curve is different as it is different at the Demand curve.

A.3 Production and cost

	Laws of Return	Return to Scale
1.	Only one input is changed while keeping others as constant	All input are changes
2.	Short run	Long run
3.	 Fig. 8.2: Three Stages of Law of Variable Proportions	 Fig. 9.
4.	Increasing Law of Return Constant Law of Return Diminishing Law of Return	Increasing Return to scale Constant Return to Scale Diminishing Return to Scale

Production Function

It gives exact mathematical relationship between input and output.

Nature of production function may take following forms:

i. Homogenous production function

All inputs are changed in the same proportion or in other words, factor proportions are held constant over time. It refers to a situation of constant return to scale.

ii. Non-homogenous production function

Only one input changes while others are kept constant. Therefore, it refers to the situation of Laws of Returns.

iii. Linear- homogenous Production function-

the firm operates under constant return to scale. Example- Cobb Douglas Production function. If both inputs labour and capital increase by a constant term λ resulting the resulting output increases by a λ^1.

iv. Linear-non-homogenous Production function

The firm operates under laws of constant return

v. Non-linear-homogenous production function

The firm operates under either increasing return to scale or diminishing return to scale. If the Production Function is homogenous of degree 2, then following an increase in all inputs by a constant λ, the output will increase to a new level λ^2. In such a situation (when the production function is of degree greater than one), the firm operates under increasing returns to scale. If the production function is homogenous of degree less than one (non-linear homogenous) than a firm operates under diminishing returns to scale, that is output increases less than proportionately with a given increase in the level of all inputs.

vi. Non-linear-non-homogenous production function

The firm operates under either increasing return or diminishing returns.

3.1 Cost

Expenses of production are known as the explicit cost of production and, a producers' effort and sacrifices, incurred in production (rent cost of production)are known as the implicit costs of production. Both explicit and implicit cost need to be calculated. Mathematically,

$$C= f(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.1)$$

Where C= Total cost function

X = output; There is a positive relationship between C and x.

Short-run Cost Function- Short-run is a period in which a firm is unable to change all of its inputs of production. In this time period, they can only change variable factors of production. So shor run cost function:

$$C = f(x) + a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.2)$$

Where, a= fixed cost which is independent of the level of output

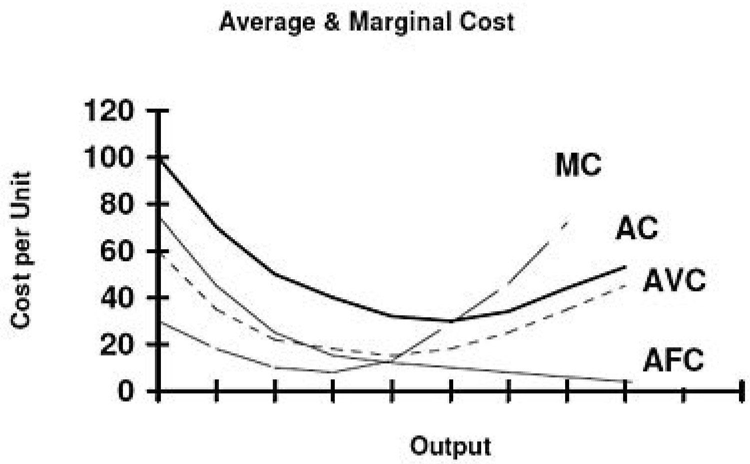
$$TC = f(x) + a = TVC + TFC$$

AC = TC/x = Average cost function

AFC – a/x

AVC = f(x)/x

MC- df(TC)/dx = Marginal Cost function



3.2 The Concept of Cost Elasticity

The cost elasticity measures the responsiveness of cost to changes in output. It can be defined as the ratio of proportionate change in total costs to proportionate change in output.

Mathematically,

Ec= dTC/dX.X/TC ……………………………………….(3.3)

Where TC = Total Cost

	Minimization of Cost	

Situation:

Minimize total cost TC

Where, TC = f (L, K) = $P_L \cdot L + P_K \cdot K$ ………………..(3.4)

P_L and P_k are price of labour and capital

Constraint: produce amount Xo = f(L,K) where X0 = fixed level of output

To obtain the Lagrangean expression for this constrained into the standard form,

Xo - f(L,K) =0…………………………………………………..(3.5)

Multiply 3.5 by λ, and adding it to the expression for TC in (3.4), which the firm is trying to minimize:

Lagrangean expression as

Z = TC + λ[X0-f(L,K)]

$= P_L \cdot L + P_K \cdot K + \lambda[X0\text{-}f(L,K)]$ …………………… (3.6)

To minimize equ. 3.6, each of the partial derivative need to be equal to zero as,

$\delta Z/ \delta L = P_L - \lambda f_{1L} = 0$ ……………………… (3.7)

$\delta Z/ \delta K = P_K - \lambda f_{1K} = 0$ ………………………… (3.8)

$\delta Z/ \delta \lambda = X0 - f(L, K) = 0$ ……………………… (3.9)

from equ. 3.7 and 3.8, we get

$F_{1L}/F_{1K} = P_L/P_K$

The second order condition require that the relevant Bordered Hessian determinant be positive.

Numerical example: Find the firm's expansion path expressed in terms of its total expenditure on its inputs, given the production functions,

$X = 8\log L + 20\log K$

and the price $P_L = 1$ and $P_K = 5$.

Solution:

The objective is to maximize output X, subject to the expenditure (cost outlay) constraint

$TC = 1L + 5K$

To solve the problem, a new function is formed using lagrangean multiplier λ as,

$V = 8\log L + 20\log K - \lambda (TC - 1L - 5K)$

Taking the partial derivatives of V with respect to L, K and λ and equate them to zero, we get

$\delta V/ \delta L = 8/L - \lambda = 0$ ……………………………(3.10)

$\delta V/ \delta K = 20/K - 5\lambda = 0$ ………………………… (3.11)

$\delta Z/ \delta \lambda = TC - 1L - 5K = 0$ …………………………(3.12)

From, equ. (3.10) and (3.11) we get, $8/L = 4/K$

Or, $K/L = 1/2$ or $L = 2K$ ………………………………………(3.13)

Substituting the value in the constraint equation, we get

$TC = L + 5K$;

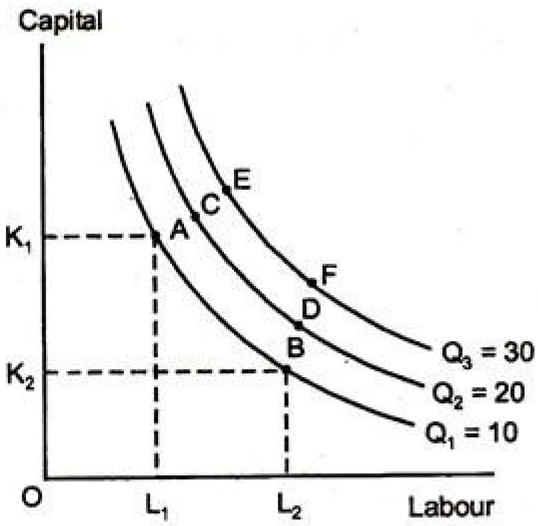
Putting the value of L as 2K from equation (3.13); $TC = 2K + 5K$

or, $K = TC/7$

Also, $L = 2/7C$ and $\lambda = 28/TC$

This constitutes for the required expansion path relationships giving the amount of each input purchased as a function of total cost, TC.

4.	Isoquant Curve The **isoquant curve** is a **graph**, used in the study of microeconomics. Combinations of two inputs which yields same level of production	

		 **Fig. 6.3 : Isoquant Curve/Isoquant Map**
	Example- Which type of production process will be more appropriate for a country like India.	
	Marginal Rate of Technical Substitution (MRTS)	i. It is the amount by which the quantity of one input has to be reduced when one extra unit of another input is used, so that output remains constant. ii. It shows the relations between inputs and the trade-offs amongst them, without changing the level of total output. iii. MRTS = slope of Isoquants. iv. If labour and capital are two inputs $MRTS_{TK} = MP_L/MP_k$ v. Total differential of the Production Function $dP = \delta P/\delta L.dL + \delta P/\delta K.dK$ where, $\delta P/\delta L$ = Marginal Prouctivity of Labour= MP_L $\delta P/\delta K$ = Marginal Productivity of Capital = MP_k vi. Along an Isoquoant, DP=0 vii. So, $\delta P/\delta L.dL + \delta P/\delta K.dK = 0$ viii. $- \delta P/\delta L.dL = \delta P/\delta K.dK$ ix. or, $- MP_L/MP_k = dK/dL$
	Elasticity of Substitution	It is defined as the ratio of the percentage change in the K/L ratio to the percentage change of the $MRTS_{LK}$ • $\sigma = d(K/L)/(K/L).\ d(MRTS)/MRTS$ ……………… (3.14)
	Examples	
	1. Find the MRTS and Elasticity of Substitution for the following Production Function. $P = \{\alpha K^{-\theta} + (1-\alpha)L^{-\theta}\}^{-1/\theta}$ $MP_L = \delta_P/\delta_L = -1/\theta\{\alpha K^{-\theta} + (1-\alpha)L^{-\theta}\}^{-1/\theta-1} .- \theta(1-\alpha)L^{-\theta-1}$ $= (1-\alpha)L^{-\theta-1}\{\alpha K^{-\theta} + (1-\alpha)L^{-\theta}\}^{-1/\theta-1}$ Similarly, MP_K may also be calculated $MP_K = \delta_P/\delta_K = -1/\theta\{\alpha K^{-\theta} + (1-\alpha)L^{-\theta}\}^{-1/\theta-1} .- \theta\alpha K^{-\theta-1}$	

	$= \alpha K^{-\theta-1} \{ \alpha K^{-\theta} + (1-\alpha)L^{-\theta}\}^{-1/\theta-1}$ MRTS = MRL/MPK $=[(1-\alpha)L^{-\theta-1}\cancel{\{\alpha K^{-\theta}+(1-\alpha)L^{-\theta}\}^{-1/\theta-1}}]/[\alpha K^{-\theta-1}\cancel{\{\alpha K^{-\theta}+(1-\alpha)L^{-\theta}\}^{-1/\theta-1}}]$ $= (1-\alpha)L^{-\theta-1} / \alpha K^{-\theta-1} = \{(1-\alpha)/\alpha\} (L/K)^{-1-\theta}$ or, $\{(1-\alpha)/\alpha\} (K/L)^{1+\theta}$ or, $б$= MRTS. L/K (1/ d(MRTS)/d(K/L) (as discussed at No.)----------(3.15) or, $d(MRTS)/d(K/L) = 1+\theta\{(1-\alpha)/\alpha\} (K/L)^{1+\theta-1}$-----------------(3.16) or, Putting the value in = $=\{(1-\alpha)/\alpha\} (K/L)^{1+\theta} [1/1+\theta\{(1-\alpha)/\alpha\} (K/L)^{\theta}]$ Or, $= 1/(1+\theta)$

12.	Labour Productivity	You can measure employee **productivity** with the **labor productivity equation**: total output / total input. Let's say your company generated \$80,000 worth of goods or services (output) utilizing 1,500 **labor** hours (input). To calculate your company's **labor productivity**, you would divide 80,000 by 1,500, which equals 53.
	Example- You can measure employee **productivity** with the **labor productivity equation**: total output / total input. Let's say your company generated \$80,000 worth of goods or services (output) utilizing 1,500 **labor** hours (input). To calculate your company's **labor productivity**, you would divide 80,000 by 1,500, which equals 53.	
13.	Capital Productivity	**Capital productivity** is calculated on the basis of the balance valuation of the fixed production assets (depreciation costs included), using either the average value over the year or the value as of the end of the year. **Capital productivity** is the reciprocal of the **capital**-output ratio.

Step no.	General Step	Example
	Cobb-Douglas Production Function	$X = 10 \cdot L^{2/3} K^{1/3}$ Xo = 40 P_L = Rs.50, P_K = Rs 200
1.	take partial derivatives of X to get the tangency condition (TC): $MP_L/MP_K = P_L/P_K$	$MRTS = MP_L/MP_K$ So here, MRTS = $((2/3)/(1/3)) \cdot K/L = 2 \cdot K/L$ PL/PK = \$50/\$200 = 1/4 So TC $2 \cdot K/L$ = ¼ Or,
2.	rearrange the tangency condition to express K as the dependent variable.	(2) $2 \cdot K/L = 1/4 \Rightarrow K=L/8$
3.	plug the expression for K into the output constraint to solve for L.	(3) $Qo = 10 \cdot L^{2/3} K^{1/3}$ $40 = 10 \cdot L^{2/3} (L/8)^{1/3}$ $40 = 5 \cdot L$ L = 8
		(4) K = L/8 = 8/8 = 1

4.	plug the solution for L into the formula for K derived in Step 2 to solve for K.		
5.	Plug your solutions for L and K into the cost equation (TC = PL·L + PK·K) to find out the minimum cost of producing Qo.	(5) TC = \$50·8 + \$200·1 = \$600.	
	To check your answers: Is the tangency condition met? Are you producing your targeted level of output (Qo)?	Checking results: tc: MRTS: 2·K/L = 2·(1/8) = 1/4 PL/PK = \$50/\$200 = 1/4. Qo: Q = 10·(8)$^{2/3}$ (1)$^{1/3}$ = 10·4·1 = 40.	

Try the following example:

Given: $Q = L^{1/2}K^{1/2}$

PL = \$4, PK = \$1

Goal: Produce Qo = 16 units as cheaply as possible.

(1) Solve for the cost-minimizing input combination:

L = _______ K = _______ TC = _______

(2) Depict the optimum in the diagram to the right. Use actual numerical values to label (a) your isocost line endpoints, (b) your isoquant, and (c) the values of L and K at your optimum.

A.4 Market

	Maximization of Profit	

4.1 Monopoly

Only one producer

Industry = Firm

Exercise 4 - Consider a monopolist who faces a linear demand function= p= 100-4x and a total cost function TC= 50+20x. Determine the optimum level of output.

Solution:

Total Revenue (TR) = x.p = (100- 4x)x = 100x $-4x^2$ ---(4.1)

total cost function TC= 50+20x

Π (profit) = TR-TC -- (4.2)

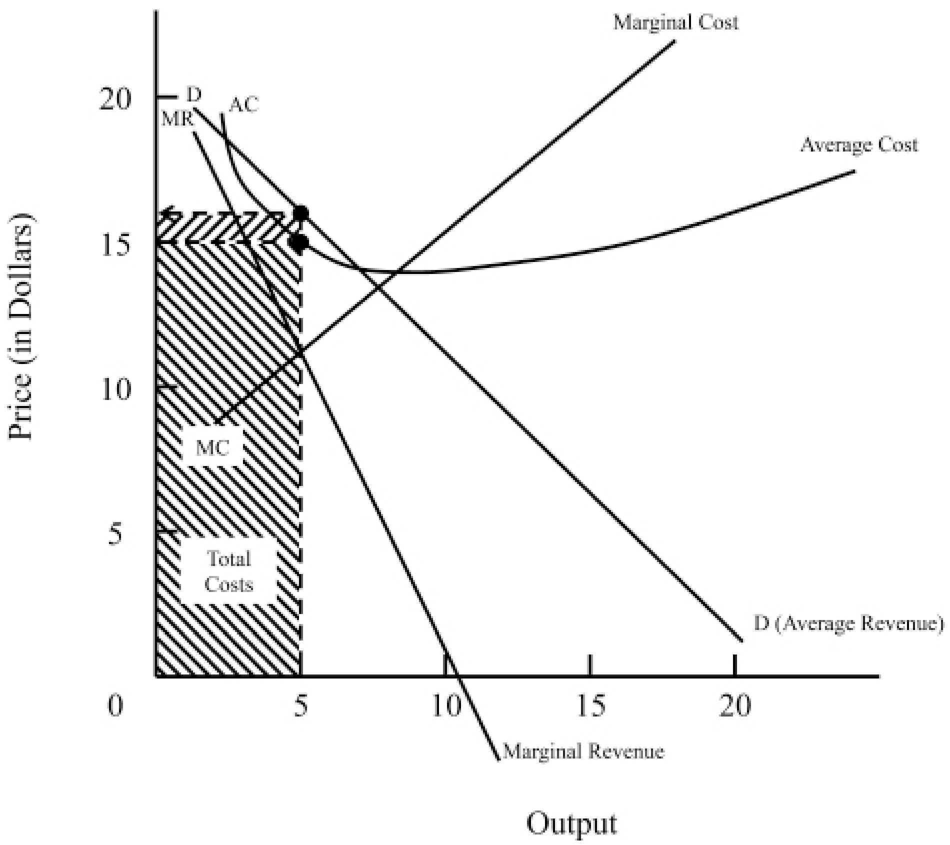
Or, $100x-4x^2 - 50-20x = 0$

or, $80x - 4x^2 -50 =0$;

$d\pi/dX = 80-8X = 0$ (necessary condition)

Or, $x= 10$

or, $d^2x/dp^2 = -8<0$ (sufficient condition hence proved



4.2 Discriminating Monopoly

A monopolist can sell its product at two different market at two different price given

i. The two markets are not connected to each other.
ii. The price elasticity of demand is different in two markets. Price will be higher in the market where the price elasticity is lower in the market where price elasticity is more.

Assumption

i. There are only two markets i.e. market 1 and market 2 where the monopoly firm sells its product.

Ii The monopoly firm sells its product in such a way that revenue of the firm is maximised. So, $MR1= MR2$..(4.3)

$MR_1= MR_2= MC$ .... (4.4)

Unless the Discriminating Monopoly achieves condition of equ 4.3 and equ. 4.4, it is not in equilibrium.

$\Pi = TR_1 + TR_2 – MC (X_1+X_2)$.. (4.5)

Maximixe the equation 4.5 by first degree partial derivative = 0 and second order partial derivative is negative.

Important Theorem

$MR_1= MR_2$ does not necessarily imply that

$P1 =P2$. It will be lower in the market with price elasticity of demand >1

and vice –versa.

Proof:

We know, $MR_1= MR_2$

and $MR_1= P_1 (1-1/e_1)$

$MR_2 = P_2 (1-1/e_2)$

Where P_1 = Price or Average Revenue of Market 1

P_2 = Price or Average Revenue of Market 2

e_1 = Price Elasticity of Market 1

e_2 = Price Elasticity of Market 2

therefore, $P_1 (1-1/e_1) = P_2 (1-1/e_2)$

or, $P_1 / P_2 = (1-1/e_2)/ (1-1/e_1)$

if, $e_1 = 2$; $e_2 = 4$ (market 2 is more elastic than the market 1)

or, $P1 =3/2 P$

or, $P1> P2$

So, Price is more in the less elastic market than in the more elastic market, even if $MR1=MR2$.

Hence Proved

Example.1: Given the following demand for two separate markets and the total cost function of the monopoly firm,

$P_1 = 17-2X_1$
$P_2 = 25-3X_2$
$TC = 2+X_1+X_2$

What will be the prices, output and marginal reveunues in the two markets and monopolist's total profits under Price Discrimination?

Solution: $TR_1 = P_1X_1 = 17X_1 - 2x_1^2$; $MR_1 = 17 - 4X_1$ ....(6)

$TR2 = P_2X_2 = 25X_2 - 3X_2^2$ or, $MR_2 = 25 - 6X_2$ ....(7)

$\Pi = TR_1 + TR_2 - MC\,(X_1+X_2)$

After putting the value of TR1, TR2 and TC

$\Pi = 17X_1 - 2x_1^2 + 25X_2 - 3X_2^2 - 2 - X_1 - X_2 = 16X_1 - 2x_1^2 + 24X_2 - 3X_2^2 - 2$...............................(8)

First degree partial derivative =0 second degree ,0

(Marginal Revenue)$d\,\Pi_1 = 16-4X_1$ or, $X_1 = 4$; P1 = 9 putting value MR1 = 1 Answer

(Marginal Revenue) $d\,\Pi_2 = 24 - 6X2$ or, X2 = 4; P2 =13; MR2 = 1 Answer

$\Pi = 78$ Answer

Example.2: Find the Value of Example.1; Find the corresponding value of Π, MR_1, MR_2; P_1; P_2, if the monopoly cannot discriminate.

Solution – If a monopolist cannot discrimination then,

P1 = P2

In such a green, profits of the monopolist must be maximised subject to the constraint, P1=P2

or, $17-2X_1 = 25-3X_2$; or, $17-2X_1 - 25+3X_2 = 0$
or, $8 - 3X_2 + 2X_1 = 0$ .... (9)(equation of Constraint)

Using the Langrangian expression
$\Pi^* = \pi + \lambda\,(8 - 3X_2 + 2X_1)$..(10)

Where, λ = Langrangian multiplier to the constraint
Put the value of π function from the (8) in the equation (10),

$\Pi^* = 16X_1 - 2X_1^2 + 24X_2 - 3X_2^2 - 2 + \lambda\,(8 - 3X_2 + 2X_1)$

Taking the partial derivative of Π^* with respect to X_1, X_2 and λ, and equate them equal to zero, we get

$\delta\Pi^*/\delta X1 = 16 - 4X_1^2 + 2\lambda$

$\lambda = (-16 + 4X_1^2)/2 = 2X_1^2 - 8$ ....(11)

$\delta\Pi^*/\delta X2 = 24 - 6X_2 - 3\,\lambda = 0$

or, $\lambda = 8 - 2X_2$ ....(12)

$\delta\Pi^*/\delta\lambda = 8 - 3X_2 + 2X_1 = 0$ ....(13)

From equation (11) and (12)

$2X_1 + 2X_2 - 16 = 0$..(14)

Solving equation, (13) and (14)

X1 = 3.2; X2= 4.8; and λ = -1.6

And substituting these values in the relevant functions, we have,

$\delta\Pi^* = 16X_1 - 2X_1^2 + 24X_2 - 3X_2^2 - 2 + \lambda(8 - 3X_2 + 2X_1) = 74.80$

	Monopoly	Discriminating Monopoly
Profit	74.80	78

Example- A company manufactures two types of typewriters – electrical (E) and manual (M). The revenue function of the company, in thousands, is: TR=8E+ 5M + 2EM – E 2 - 2M2 +20, Determine the quantity of electrical and manual typewriters which lead to maximum revenue. Also calculate the maximum revenue.

Solution- $TR = 8E + 5M + 2EM - E^2 - 2M^2 + 20$

$\delta(TR)/\delta M = 5 + 2E - 4M = 0$

or, 4M-2E =5……………………………………………………………………()

Again, $\delta(TR)/\delta E = 8 + 2M - 2E = 0$

Or, 2E-2M = 8…………………………………..()

From equation () and equation (), we get 4M-2E =5

-2M+2E = 8

M= 6.5

Putting the value of M in the MR equation, 2E-2M = 8

Or, 2E= 8+13 = 21

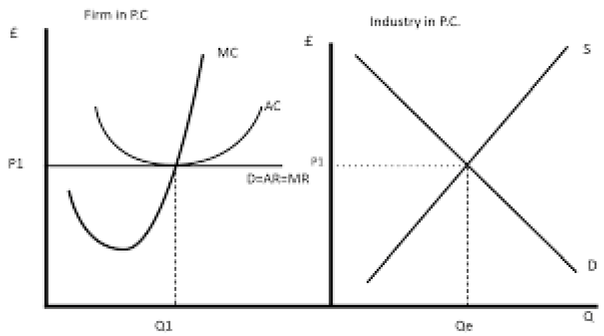
Or, E =10.5

$A = \delta^2(TR)/\delta E^2 = -2$; $B = \delta^2(TR)/\delta EM = 2$; $C = \delta^2(TR)/\delta M^2 = -4$

A&C < 0 and $B^2 > 0$

TR =8X10..5+ 5X6.5+2X6.5X10.5- $(10.5)^2$ -2$(6.5)^2$ +20 = Rs. 78.25 Ans.

4.3 Perfect Competition



Equilibrium of the Industry

Example- Given the demand Function and Supply Function as

$D= 500-100P$
$S = 50 + 50P$

Find out the price and output level of the industry. P= 3 and D=S = 200 Ans

For perfect competition, for firms to be in equilibrium

MR=MC

Example- Given the demand function and total cost function of the perfect competition firm as

$P=32-X$; and $C =X^2+8X +4$,

what level of output will maximise total profit and what are the corresponding value of price (P) profit (π) and Total Revenue (R).

Solution

For maximization, following two conditions must be satisfied

(i) $d\pi/dX = 0$; MR=MC
(ii) $d^2\pi/dX^2 <0$; X=6; P= 26; TR =156; π= 68 Ans.

4.4 Monopolistic Competititon

Example A firm under monopolistic competition faces a linear demand function and a linear cost function as

$P =16-2X$; $TC =2+8x$

Find the total profit and its monopoly power

Solution

$TR = PX = 16X -2X^2$

Π = TR-TC

$16X - 2X^2 - 2 - 8X$;

$\delta\Pi/\delta X = 16 - 4X - 8 = 0$

or X=2

MR = 16- 4X = 2 putting the value of X in total profit and MR function we get

Π = 6; P= 12; MR=8

In perfect Competition = MR= Price = 8; In Monopolistic Competition P> MR

The extent to which the price of the monopolistic firm is higher than the MR is called as Monopolic power. Here the price is 50% more than the MR. Therefore, the firm has a monopoly power to increase price by 50% more than the price in Perfect competition.

4.5 Oligopoly

The Cournot Solution

Cournot gave a solution of a duopoly problem where two firms are assumed to produce a homogenous product. He defined the demand function as:

$$P = f(X1+X2) \quad \ldots\ldots(4.5.1)$$

Where, X1= output of duopolist I
X2 = output of the duopolist II
P = price (which is equal to both the duopolists)
Also, TRI =x1P = X1f(X1+X2) = TR1(X1,X2)……(4.5.2)
Where, TR1= Total revenue function of the duopolist I
TRII =x2P = X1f(X1+X2) = TR2(X1,X2)……(4.5.3)
Where, TR1= Total revenue function of the duopolist II
It indicates that total revenue of each duopolist is a function of his own output and that of his competitor.
The profit function of each duopolist are:
$\Pi 1$ = TR1(X1, X2)- TC1(X1)……(4.5.4)
$\Pi 2$= TR2(X1, X2) -TC2(X2)……(4.5.5)
Where, TC1(X1)- Total Cost of the duopolist1
TC2(X2)= Total Cost of the duopolist II
This means that revenue function of each duopolist depends on X1 and X2 while cost function of each duopolist depends upon his own output level.
The basic requirement of Counot Solution is that each duopolist maximises profit with respect to his own level assuming that his rival's output Is unchanging or constant. In other words, duopolist I maximises $\pi 1$ with respect to X1 assuming X2 as a parameter and the duopolist II maximises π 2 with respect to X2 assuming X1 as a parameter.
For maximisesation,
For duopolist I: $\delta\pi 1/\delta x1 = \delta TR1/\delta x1 - \delta TC1/\delta x1 = 0$;
or, $\delta TR1/\delta x1 = \delta TC1/\delta x1$ …… (4.5.6)
MR1=MCI
And, $\delta^2 TR1/\delta x1^2 - \delta TC1/\delta x1 < 0$

or, $\delta^2TR1/\delta x1^2 < \delta^2TC1/\delta x1^2$ …………………………… (4.5.7)
(change in MR1) (Change in MC1)

For duopolist II $\delta\pi 2/\delta x2 = \delta TR2/\delta x2 - \delta TC2/\delta x2 = 0$

Or, $\delta TR2/\delta x2 = \delta TC2/\delta x2$ ……………………….(4.5.8)

or, MR2=MC2

And $\delta^2TR2/\delta x2^2 - \delta^2TC2/\delta x2^2 < 0$

Or, $\delta^2TR2/\delta x2^2 < \delta^2TC2/\delta x2^2$……………………..(4.5.9)
(Change in MR2) (Change in MC2)

The equilibrium solution can be obtained if equations (4.5.6), (4.5.7), (4.5.8), and (4.5.9) are satisfied. And neither of the duopolists desires to alter his output while the other maximises his output.

Important Theorem – The Marginal Revenues of the Duopolists (MR1 and MR2 respectively) are not necessarily equal, but the duopolist with the greatest output will have smaller MR.

Proof:

Total Revenue of both the duopolists is = TR =TR1+TR2 = PX1 +PX2 = P(X1+X2) = PX

Where X =X1+X2

(General Marginal Revenue) MR = $\delta TR/\delta X = P+X.dP/dX = P+(X1+X2)dP/dX$

Where, $dP/dX<0$

Let X1> X2 and X2 is a parameter for X1. In such a situation MR1<MR2.

The Cournot Solution includes only two- firms case, but it can be extended to more than two firms. Here also the basic requirement of profit maximization will be the same.

Criticised to assume that each duopolist to maximise his profits treating other's output as a parameter. However, if Duopolist I and Duopolist II have a collusion to act in union or under a uniform policy in order to maximise the total profit of the industry (here comprising of two firms only). This situation becomes very similar to that of monopoly and is known as Collusion solution of the oligopoly problem (Willim J. Baumol p. 326)

Example: If the demand and cost functions are given as

$$P = 80 - 0.4\,(X_1+X_2)$$

$$TC_1 = 4X_1;\ TC_2 = 0.4X_2^2$$

Find out X1, X2, P and Π1, Π2. Also prove the second order condition for profit maximisation. Discuss whether a rise of either duopolist's output level will cause a reduction of other's output level?

Solution: We establish

$TR1 = PX1= [80 - 0.4\,(X_1+X_2)]X1 = 80X_1 - 0.4X_1^2 - 0.4X1X2$

$TR2 = PX2 = [80 - 0.4\,(X_1+X_2)]X2 = 80X_2 - 0.4X1X2 - 0.4X_2^2$

Now, their profit equations are

$\Pi_1 = TR_1 - TC_1 = 80X_1 - 0.4X_1^2 - 0.4X1X2 - 4X_1 = 76X1 - 0.4X_1^2 - 0.4X1X2$ …………………… (4.5.10)

$\Pi_2 = TR_2 - TC_2 = 80X_2 - 0.4X_1X_2 - 0.4X_2^2 - 0.4X_2^2 = 80X_2 - 0.4X_1X_2 - 0.8X_2^2$...............................(4.5.11)

First order condition for profit maximisation

$\delta\pi/\delta X_1 = 76 - 0.8X1 - 0.4X2 = 0$..(4.5.12)

$\delta\pi/\delta X_2 = 80 - 0.4X1 - 1.6X2 = 0$..(4.5.13)

Solving equation (4.5.12) and (4.5.13), we get X1= 80 and X2 =30

Substituting the value of X1 and X2 we get, P= 36; π1 = 4480 and π 2=720 Ans.

Second order condition for profit maximization

$\delta\pi/\delta X_1$ - 0.8 <0

$\delta^2\pi/\delta X_2^2 = -1.6 < 0$ Hence proved

We may now get the corresponding reaction functions of the duopoilist from the equ. (4.5.12) and (4.5.13) as:

$X_1 = (76 - 0.4X_2)/0.8$ ....(4.5.14)

$X2 = (80 - 0.4X1)/1.6$ ....(4..5.15)

The above equation (4.5.14) and (4.5.15) states hat a rise of either duopolist's output level will cause a reduction of the other's optimum output level. Hence each duopolist acts as if his rival's output were fixed.

SN	Models of Duopoly	Discussion of the Model
1.	Cournot's Model	Duopolist I maximises π1 with respect to X1, treating x2 as a parameter and the Duopolist maximises π2 with respect to X2, treating 1 as a parameter.
2.	Collusion/ Cartel Model	Duopolist maximises their joint profits (π=π1+π2).
3.	Stackelberg Model	
4.		

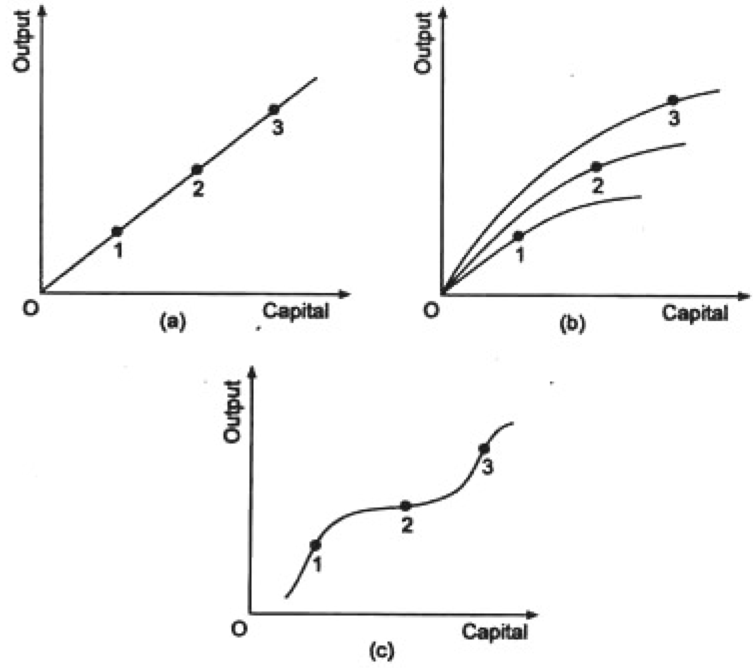
The Stackelberg Solution of the Duopolist (Leadership Solution or Followership Solution)

Developed by German economist Heinrichvon Stackelberg.
Assumption:
Profit of each dupolist is function of the output level of both i.e.

$$\pi 1 = f(X_1, X_2)$$
$$\pi 2 = f(X_1, X_2)$$

But one of the them is leader and the other is follower.

B Macro Economics

B.1.	Marginal Propensity to consume	
	Example	
	Example	
	Example-	
B.2	Incremental Capital-Output ratio	The **incremental capital output ratio** (**ICOR**) explains the relationship between the level of investment made in the economy and the consequent increase in GDP. **ICOR** is a metric that assesses the marginal amount of investment **capital** necessary for a country or other entity to generate the next unit of production.
	Diagram	 Fig. 13 : Estimating the capital-output ratio from historical data
	Example	
	How efficiency of capital can be achieved?	It is possible mainly through technological progress. When there is superior technology, capital will be efficient to produce more output and capital output ratio will be lower.
	Example-	