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ALLEN CAREER INSTITUTE KOTA (RAJASTHAN)

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JEE (Main/Advanced) Syllabus

JEE (Main) Syllabus :

Trigonometric Identities.Trigonometric equations.

JEE (Advanced) Syllabus :

Logarithms and their properties. Trigonometric functions, their periodicity and graphs, addition and subtraction formulae, formulae involving multiple and sub-multiple angles. General solution of trigonometric equations.

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FUNDAMENTALS OF MATHEMATICS

1. NUMBER SYSTEM :

Natural Numbers : (N) = $\{1, 2, 3....\infty\}$

Whole Numbers : (W) = $\{0, 1, 2, 3.....\infty\}$

Integers : (I) = $\{-\infty,.....-3, -2, -1, 0, 1, 2, 3.....\infty\}$

Positive Integers : (I^+) = $\{1, 2, 3 ... \infty\}$

Negative Integers : (I^-) = $\{-\infty, ... -3, -2, -1\}$

Non-negative Integers : $\{0, 1, 2, 3.....\}$

Non-positive Integers : $\{-\infty, ... -3, -2, -1, 0\}$

Even Integers = $\{....-6, -4, -2, 0, 2, 4, 6 ...\}$

Odd Integers = $\{-5, -3, -1, 1, 3, 5 .....\}$

Note :

(i) Zero is neither positive nor negative.

(ii) Zero is even number.

(ii) Positive means > 0.

(iv) Non-negative means ≥ 0.

2. FRACTION $\left(\frac{p}{q}\right)$:

(a) Proper Fraction = $\frac{3}{5}$: $N^r < D^r$

(b) Improper Fraction = $\frac{5}{3}$: $N^r > D^r$

(c) Mixed Fraction : $2+\frac{3}{5}$

(d) Compound Fraction : $\frac{\frac{2}{3}}{\frac{5}{6}}$

(e) Complex Fraction : $2\frac{1}{3}$

(f) Continued Fraction : $2+\cfrac{2}{2+\cfrac{2}{+.....}}$

This is usually written in the more compact form $2+\frac{1}{2+}\ \frac{1}{2+}$ .....

3. RATIONAL NUMBERS (Q) :

All the numbers that can be represented in the form p/q, where p and q are integers and $q \neq 0$, are called rational numbers. Integers, Fractions, Terminating decimal numbers, Non-terminating but repeating decimal numbers are all rational numbers. $Q = \left\{\frac{p}{q} : p, q \in I \text{ and } q \neq 0\right\}$

Note :

(i) Integers are rational numbers, but converse need not be true.

(ii) A rational number always exists between two distinct rational numbers, hence infinite rational numbers exist between two rational numbers.

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4. IRRATIONAL NUMBERS (Q^C) :

There are real numbers which can not be expressed in p/q form. Non-Terminating non repeating decimal numbers are irrational number e.g. $\sqrt{2}, \sqrt{5}, \sqrt{3}, \sqrt[3]{10}$; e, π.

$e \approx 2.71$ is called Napier's constant and $\pi \approx 3.14$

Note :

(i) Sum of a rational number and an irrational number is an irrational number e.g. $2 + \sqrt{3}$

(ii) If $a \in Q$ and $b \notin Q$, then ab = rational number, only if a = 0.

(iii) Sum, difference, product and quotient of two irrational numbers need not be an irrational number or we can say, result may be a rational number also.

5. REAL NUMBERS (R) :

The complete set of rational and irrational number is the set of real numbers, $R = Q \cup Q^C$. The real numbers can be represented as a position of a point on the real number line.

6. COMPLEX NUMBERS. (C) :

A number of the form a + ib, where $a, b \in R$ and $i = \sqrt{-1}$ is called a complex number. Complex number is usually denoted by z and the set of all complex numbers is represented by

$C = \{(x + iy) : x, y \in R, i = \sqrt{-1}\}$

$$\boxed{N \subset W \subset I \subset Q \subset R \subset C}$$

7. EVEN NUMBERS :

Numbers divisible by 2, unit's digit 0, 2, 4, 6, 8 & represented by 2n.

8. ODD NUMBERS :

Not divisible by 2, last digit 1, 3, 5, 7, 9 represented by (2n ± 1)

(a) even ± even = even

(b) even ± odd = odd

(c) odd ± odd = even

(d) even × any number = even number

(e) odd × odd = odd

9. PRIME NUMBERS :

Let 'p' be a natural number, 'p' is said to be prime if it has exactly two distinct positive integral factors, namely 1 and itself. e.g. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31 .....

10. COMPOSITE NUMBERS :

A number that has more than two divisors

Note :

(i) '1' is neither prime nor composite.

(ii) '2' is the only even prime number.

(iii) '4' is the smallest composite number.

(iv) Natural numbers which are not prime are composite numbers (except 1)

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11. CO-PRIME NUMBERS/ RELATIVELY PRIME NUMBERS :

Two natural numbers (not necessarily prime) are coprime, if their H.C.F. is one

e.g. (1, 2), (1, 3), (3, 4) (5, 6) etc.

Note :

(i) Two distinct prime number(s) are always co-prime but converse need not be true.

(ii) Consecutive natural numbers are always co-prime numbers.

12. TWIN PRIME NUMBERS :

If the difference between two prime numbers is two, then the numbers are twin prime numbers.

e.g. {3, 5}, {5, 7}, {11, 13} etc.

13. NUMBERS TO REMEMBER :

Number	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Square	4	9	16	25	36	49	64	81	100	121	144	169	196	225	256	289	324	361	400
Cube	8	27	64	125	216	343	512	729	1000	1331	1728	2197	2744	3375	4096	4913	5832	6859	8000
Sq. Root	1.41	1.73	2	2.24	2.45	2.65	2.83	3	3.16										

Note :

(i) Square of a real number is always non negative (i.e. $x^2 \geq 0$)

(ii) Square root of a positive number is always positive e.g. $\sqrt{4} = 2$

(iii) $\sqrt{x^2} = |x|$

14. DIVISIBILITY RULES :

Divisible by	Remark.
2	Last digit of number is 0, 2, 4, 6 or 8
3	Sum of digits of number divisible by 3 (Remainder will be same when number is divided by 3 or sum of digits is divided by 3.)
4	Number formed by last two digits divisible by 4 (Remainder will be same whether we divide the number or its last two digits)
5	Last digit 0 or 5
6	Divisible by 2 and 3 simultaneously.
8	Number formed by last three digits is divisible by 8 (Remainder will be same whether we divide the number or its last three digits)
9	Sum of digits divisible by 9. (Remainder will be same when number is divided by 9 or sum of digit is divided by 9)
10	Last digit 0
11	(Sum of digits at even places) – (sum of digits at odd places) = 0 or divisible by 11

15. LCM AND HCF :

(a) HCF is the highest common factor between any two or more numbers or algebraic expressions. When dealing only with numbers, it is also called "Greatest common divisor" (GCD).

(b) LCM is the lowest common multiple of two or more numbers or algebraic expressions.

(c) The product of HCF and LCM of two numbers (or expressions) is equal to the product of the numbers.

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16. FACTORIZATION :

Formulae :

(a) $(a \pm b)^2 = a^2 \pm 2ab + b^2 = (a \mp b)^2 \pm 4ab$

(b) $a^2 - b^2 = (a+b)(a-b)$

- If $a^2 - b^2 = 1$ then $a + b = \dfrac{1}{a-b}$

For example : $\sec\theta - \tan\theta = \dfrac{1}{\sec\theta + \tan\theta}$ or $\sqrt{3} + \sqrt{2} = \dfrac{1}{\sqrt{3} - \sqrt{2}}$

(c) $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$

(d) $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$

(e) $a^3 + b^3 = (a+b)(a^2 - ab + b^2) = (a+b)^3 - 3ab(a+b)$

(f) $a^3 - b^3 = (a-b)(a^2 + ab + b^2) = (a-b)^3 + 3ab(a-b)$

(g) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

(h) $\mathbf{a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)}$

$$\mathbf{= \frac{1}{2}(a+b+c)\{(a-b)^2 + (b-c)^2 + (c-a)^2\}}$$

(i) $\mathbf{(a+b+c)^3 = a^3 + b^3 + c^3 + 3(a+b)(b+c)(c+a)}$

(j) $\mathbf{a^4 + a^2 + 1 = (a^2+1)^2 - a^2 = (1 + a + a^2)(1 - a + a^2)}$

17. CYCLIC FACTORS :

If an expression remain same after replacing a by b, b by c & c by a, then it is called cyclic expression and its factors are called cyclic factors. e.g. $a(b-c) + b(c-a) + c(a-b)$

18. REMAINDER THEOREM :

If a polynomial $a_1x^n + a_2x^{n-1} + a_3x^{n-2} + \ldots + a_n$ is divided by $x-p$, then the remainder is obtained by putting $x = p$ in the polynomial.

19. FACTOR THEOREM :

A polynomial $a_1x^n + a_2x^{n-1} + a_3x^{n-2} + \ldots + a_n$ is divisible by $x-p$, if the remainder is zero

i.e. if $a_1p^n + a_2p^{n-1} + \ldots + a_n = 0$ then $x - p$ will be a factor of polynomial.

20. RATIO AND PROPORTION :

(a) If $\dfrac{a}{b} = \dfrac{c}{d}$, then : $\dfrac{a+b}{b} = \dfrac{c+d}{d}$ (componendo); $\dfrac{a-b}{b} = \dfrac{c-d}{d}$ (dividendo);

$\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$ (componendo and dividendo); $\dfrac{a}{c} = \dfrac{b}{d}$ (alternendo); $\dfrac{b}{a} = \dfrac{d}{c}$ (invertendo)

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(b) $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=.....=\frac{\lambda_1 a+\lambda_2 c+\lambda_3 e...}{\lambda_1 b+\lambda_2 d+\lambda_3 f...}$, where $\lambda_1,\lambda_2,\lambda_3.....$ are real numbers

(c) If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=.....$, then each ratio = $\left(\frac{a^n+c^n+e^n}{b^n+d^n+f^n}\right)^{\frac{1}{n}}$

Example : $\frac{a}{b}=\frac{c}{d}=\frac{\sqrt{a^2+c^2}}{\sqrt{b^2+d^2}}=\frac{a+c}{b+d}=\frac{a-c}{b-d}$

21. INDICES AND SURDS

Important Results :

1. $a\times a\times a\times....\times a$ (m times) $=a^m$

2. $a^m\times a^n=a^{m+n}$

3. $a^m\div a^n=a^{m-n}$

4. $(a^m)^n=a^{mn}$

5. $a^{-m}=\frac{1}{a^m}$

6. $\left(\frac{x}{y}\right)^m=\frac{x^m}{y^m}$

7. $(xy)^m=x^m.y^m$

8. $\sqrt[n]{x}=x^{1/n}$ **; $n\geq 2$, $n\in N$**

9. $a^0=1$

10. $a^x=a^y\Rightarrow x=y$ or $a=1$ or $a=0$ if $x>0$ & $y>0$

11. $a^x=b^x\Rightarrow a=b$ or $x=0$

12. $a^{p/q}=(a^p)^{1/q}=(a^{1/q})^p$

13. $(x^a)^b\neq x^{a^b}$ but $=x^{ab}$ e.g. $(2^3)^2=2^6=64$ & $2^{3^2}=2^9=512$

22. INTERVALS :

Intervals are basically subsets of R. If there are two numbers a, b $\in$ R such that a < b, we can define four types of intervals as follows :

(a) Open interval : (a, b) = {x : a < x < b} i.e. end points are not included.

(b) Closed interval : [a, b] = {x : a $\leq$ x $\leq$ b} i.e. end points are also included.

This is possible only when both a and b are finite.

(c) Semi open or semi closed interval : (a, b] = {x : a < x $\leq$ b} ; [a, b) = {x : a $\leq$ x < b}

(d) The infinite intervals are defined as follows :

(i) $(a, \infty)=\{x : x>a\}$ (ii) $[a, \infty)=\{x : x\geq a\}$

(iii) $(-\infty, b)=\{x : x<b\}$ (iv) $(-\infty, b]=\{x : x\leq b\}$

(v) $(-\infty, \infty)=R$

Note :

(i) For some particular values of x, we use symbol { } e.g. If x = 1, 2 we can write it as $x\in\{1, 2\}$

(ii) If there is no values of x, then we say $x\in\phi$ (null set)

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23. BASIC CONCEPTS OF GEOMETRY :

(A) BASIC THEOREMS & RESULTS OF TRIANGLES :

(a) Two polygons are similar if (i) their corresponding angles are equal, (ii) the length of their corresponding sides are proportional. (Both conditions are independent & necessary)
In case of a triangle, any one of the conditions is sufficient, other satisfies automatically.

(b) **Thales Theorem (Basic Proportionality Theorem) :** In a triangle, a line drawn parallel to one side, to intersect the other sides in distinct points, divides the two sides in the same ratio.
Converse : If a line divides any two sides of a triangle in the same ratio then the line must be parallel to the third side.

(c) **Similarity Theorem :**

(i) **AAA similarity** : If in two triangles, corresponding angles are equal i.e. two triangles are equiangular, then the triangles are similar.

(ii) **SSS similarity** : If the corresponding sides of two triangles are proportional, then they are similar.

(iii) **SAS similarity :** If in two triangles, one pair of corresponding sides are proportional and the included angles are equal then the two triangles are similar.

(iv) If two triangles are similar then

(1) They are equiangular

(2) The ratio of the corresponding (I) Sides (all), (II) Perimeters, (III) Medians, (IV) Angle bisector segments, (V) Altitudes are same (converse also true)

(3) The ratio of the areas is equal to the ratio of the squares of corresponding (I) Sides (all), (II) Perimeters, (III) Medians, (IV) Angle bisector segments, (V) Altitudes (converse also true)

(d) **Pythagoras theorem :**

(i) In a right triangle the square of hypotenuse is equal to the sum of square of the other two sides.
Converse : In a triangle if square of one side is equal to sum of the squares of the other two sides. then the angle opposite to the first side is a right angle.

(ii) In obtuse Δ

$AC^2 = AB^2 + BC^2 + 2BC \cdot BD$

(iii) In Acute Δ

$AC^2 = AB^2 + BC^2 - 2BC \cdot BD$

(e) The internal/external bisector of an angle of a triangle divides the opposite side internally/externally in the ratio of sides containing the angle (converse is also true) i.e. $\frac{AB}{AC} = \frac{BD}{DC} = \frac{BE}{CE}$

(f) The line joining the mid points of two sides of a triangle is parallel & half of the third side. (It's converse is also true)

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(g) (i) The diagonals of a trapezium divided each other proportionally. (converse is also true) i.e. $\frac{AE}{EC} = \frac{BE}{ED}$

(ii) Any line parallel to the parallel sides of a trapezium divides the non parallel sides proportionally i.e. $\frac{DG}{GA} = \frac{CF}{FB}$

(iii) If three or more parallel lines are intersected by two transversals, then intercepts made by them on transversals are proportional.

(h) In any triangle the sum of squares of any two sides is equal to twice the square of half of the third side together with twice the square of the median which bisects the third side. i.e. $AB^2 + AC^2 = 2\left(\frac{1}{2}BC\right)^2 + 2\,(AD)^2$

$= 2(AD^2 + BD^2)$

(i) In any triangle the three times the sum of squares of the sides of a triangle is equal to four times the sum of the square of the medians of the triangle.

(j) The altitudes, medians and angle bisectors of a triangle are concurrent among themselves.

(B) BASIC THEOREMS & RESULTS OF CIRCLES :

(a) **Concentric circles** : Circles having same centre.

(b) **Congruent circles :** Iff their radii are equal.

(c) **Congruent arcs :** Iff they have same degree measure at the centre.

Theorem 1 :

(i) If two arcs of a circle (or of congruent circles) are congruent, the corresponding chords are equal.
Converse : If two chords of a circle are equal then their corresponding arcs are congruent.

(ii) Equal chords of a circle (or of congruent circles) subtend equal angles at the centre.
Converse : If the angle subtended by two chords of a circle (or of congruent circles) at the centre are equal, the chords are equal.

Theorem 2 :

(i) The perpendicular from the centre of a circle to a chord bisects the chord.
Converse : The line joining the mid point of a chord to the centre of a circle is perpendicular to the chord.

(ii) Perpendicular bisectors of two chords of a circle intersect at its centre.

Theorem 3 :

(i) There is one and only one circle passing through three non collinear points.

(ii) If two circles intersects in two points, then the line joining the centres is perpendicular bisector of common chords.

Theorem 4 :

(i) Equal chords of a circle (or of congruent circles) are equidistant from the centre.
Converse : Chords of a circle (or of congruent circles) which are equidistant from the centre are equal.

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(ii) If two equal chords are drawn from a point on the circle, then the centre of circle will lie on angle bisector of these two chords.

(iii) Of any two chords of a circle larger will be near to centre.

Theorem 5 :

(i) The degree measure of an arc or angle subtended by an arc at the centre is double the angle subtended by it at any point of alternate segment.

(ii) Angle in the same segment of a circle are equal.

(iii) The angle in a semi circle is right angle.

Converse : The arc of a circle subtending a right angle in alternate segment is semi circle.

Theorem 6 :

Any angle subtended by a minor arc in the alternate segment is acute and any angle subtended by a major arc in the alternate segment is obtuse.

Theorem 7 :

If a line segment joining two points subtends equal angles at two other points lying on the same side of the line segment, the four points are concyclic, i.e. lie on the same circle.

(d) Cyclic Quadrilaterals :

A quadrilateral is called a cyclic quadrilateral if its all vertices lie on a circle.

Theorem 1 :

The sum of either pair of opposite angles of a cyclic quadrilateral is 180°

OR

The opposite angles of a cyclic quadrilateral are supplementary.

Converse : If the sum of any pair of opposite angle of a quadrilateral is 180°, then the quadrilateral is cyclic.

Theorem 2 :

If a side of a cyclic quadrilateral is produced, then the exterior angle is equal to the interior opposite angle. i.e. $\angle CBE = \angle ADC$

Theorem 3 :

The quadrilateral PQRS formed by angle bisectors of a cyclic quadrilateral is also cyclic.

Theorem 4 :

If two sides of a cyclic quadrilateral are parallel then the remaining two sides are equal and the diagonals are also equal. i.e. $AB \parallel CD \Leftrightarrow AC = BD \;\&\; AD = BC$

OR

A cyclic trapezium is isosceles and its diagonals are equal.

Converse : If two non-parallel sides of a trapezium are equal, then it is cyclic.

OR

An isosceles trapezium is always cyclic.

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Theorem 5 :

The bisectors of the angles formed by producing the opposite sides of a cyclic quadrilateral (provided that they are not parallel), intersect at right angle.

(C) TANGENTS TO A CIRCLE :

Theorem 1 :

A tangent to a circle is perpendicular to the radius through the point of contact.

Converse : A line drawn through the end point of a radius and perpendicular to it is a tangent to the circle.

Theorem 2 :

If two tangents are drawn to a circle from an external point, then :

(i) they are equal.

(ii) they subtend equal angles at the centre,

(iii) they are equally inclined to the segment, joining the centre to that point.

Theorem 3 :

If two chords of a circle intersect inside or outside the circle when produced, the rectangle formed by the two segments of one chord is equal in area to the rectangle formed by the two segments of the other chord. $PA \times PB = PC \times PD$

Theorem 4 :

If PAB is a secant to a circle intersecting the circle at A and B and PT is tangent segment, then $PA \times PB = PT^2$

OR

Area of the rectangle formed by the two segments of a chord is equal to the area of the square of side equal to the length of the tangent from the point on the circle.

Theorem 5 :

If a chord is drawn through the point of contact of a tangent to a circle, then the angles which this chord makes with the given tangent are equal respectively to the angles formed in the corresponding alternate segments.

$\angle BAQ = \angle ACB$ and $\angle BAP = \angle ADB$

Converse :

If a line is drawn through an end point of a chord of a circle so that the angle formed with the chord is equal to the angle subtended by the chord in the alternate segment, then the line is a tangent to the circle.

(D) COMMON TANGENTS OF TWO CIRCLES :

A common tangent is called direct tangent if both centres of circle lie on same side of it and called transverse tangent if centres lie on opposite side of it.

(a) When $OO' > r + s$ i.e. the distance between the centres is greater than the sum of the radii.

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In this case, the two circles do not intersect with each other and four common tangents can be drawn to two circles. Two of them are called direct (external) common tangents and the other two are known as transverse (internal or indirect) common tangents

(b) When $OO' = r + s$ i.e. the distance between the centres is equal to the sum of the radii.

In this case, the two circles touch each other externally the common point of the two circles is called the point of contact and three common tangents can be drawn to the two circles. Two of them are direct common tangents and one transverse common tangent.

(c) When $|r - s| < OO' < r + s$ i.e. the distance between the centres is less than the sum of the radii and greater than their absolute difference.

In this case, the two circles intersect in two points and there are two direct common tangents only.

(d) When $OO' = r - s$, $r > s$ i.e. the distance between the centres is equal to the difference of the radii.

In this case the two circles touch internally. The common point of the two circles is called their point of contact and there is only one common tangent to the two circles.

(e) When $OO' < r - s$, $r > s$ i.e. the distance between the centres is less than the difference of the radii.

In this case one circle lies inside the other and they do not touch. In such a case there is no common tangent.

Theorem 1 :

If two circles touch each other (internally or externally) the point of contact lies on the line through the centres.

Theorem 2 :

The points of intersection of direct common tangents and transverse common tangents to two circles divide the line segment joining the two centres externally and internally respectively in the ratio of their radii.

(i) P divides OO' externally in the ratio r : s i.e. $\frac{OP}{O'P} = \frac{r}{s}$

(ii) Q divides OO' internally in the ratio r : s i.e. $\frac{OQ}{O'Q} = \frac{r}{s}$

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24. BASIC CONCEPT OF MENSURATION

PLANE

(A) TRIANGLE :

(a) Sum of three angle is 180°

(b) Perimeter = Sum of three sides = a + b + c = 2s

Semi perimeter s = (a + b + c)/2

(c) Area = 1/2 (Base × Height)

$= \frac{1}{2}$ (Any side × Altitude over it) $= \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

Note : Area of triangles formed between two same parallel lines and on the same base is same

Area = $\frac{1}{2}$bh

(d) **Right Angle Triangle** : One angle 90° (Right angle)

& Hypotenuse^2 = Perpendicular^2 + Base^2 (Pythagoras theorem)

Area = $\frac{1}{2}$PB

(e) **Isosceles Triangle :** Two sides equal hence two angle are equal.

Special case : Isosceles Right Triangle : Two sides equal and Base = Perpendicular.

(f) **Equilateral Triangle :** All three sides and angles (60°) are equal; $h = \left(\frac{\sqrt{3}}{2}\right)a$;

$$\text{Area} = \left(\frac{1}{2}\right)\text{base} \times \text{height} = \left(\frac{1}{2}\right)(a) \times \left(\frac{\sqrt{3}}{2}\right)a = \left(\frac{\sqrt{3}}{4}\right)a^2 = \frac{h^2}{\sqrt{3}}$$

(B) QUADRILATERAL :

(a) Sum of all angles is 360°

Area $= \frac{1}{2}(AC)(h_1 + h_2)$ i.e. sum of areas of $\Delta ACD + \Delta ABC = \frac{1}{2}d_1d_2 \sin\theta$

(b) **Parallelogram :**

(i) Opposite sides are parallel and equal.

(ii) Opposite angles are equal. ($\angle B = \angle D$ and $\angle A = \angle C$)

(iii) Diagonals bisects each other. AO = OC & BO = OD

(iv) Perimeter = 2(a + b) ;

(v) Area $= \frac{1}{2}(ah) + \frac{1}{2}(ah) = ah$ i.e. sum of area of

$\Delta ACD + \Delta ABC$ also, Area $= \frac{p_1p_2}{\sin\theta}$

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(c) Special cases of parallelogram :

(i) **Rhombus :** All sides are equal and opposite angles are equal.

$AB = BC = CD = DA = a$

$\angle A = \angle C$ & $\angle B = \angle D$

Diagonals are not equal ($d_1 \neq d_2$) but bisects each other at 90°

$AC \neq BD$ but $AC \perp BD$

$\text{Area} = \frac{1}{2}(d_1 \times d_2)$ i.e. sum of areas of Δ ACD + ΔABC

(ii) **Square :** All sides are equal and all angle are equal (90°)

Diagonals are equal and perpendicular bisectors of each other

$\text{Area} = a^2 = \frac{d^2}{2}$

$AC \perp BD$ & $AO = OC$, $BO = OD$

(iii) **Rectangle :** Opposite sides are equal and parallel, all angles are equal (90°) and diagonal are equal and bisects each other but not at 90°.

Area = a × b; Perimeter = 2(a + b)

(iv) **Trapezium** : Any two opposite sides are parallel but not equal. Diagonals cuts in same proportion. $AD \parallel BC$; $AD \neq BC$; $d_1 \neq d_2$

$\text{Area} = \left(\frac{1}{2}\right)(a + b)\,h$ i.e. sum of area of $\Delta ABC + \Delta ACD$

$\frac{AO}{OC} = \frac{OD}{OB}$ $(\because \sim \Delta BOC \sim \Delta DOA)$

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(C) POLYGON :

A plane figure enclosed by line segments (sides of polygon).

(a) n sides polygon have n sides : Triangle and quadrilaterals are polygon of three and four sides respectively. The polygons having 5 to 10 sides are called, PENTAGON, HEXAGON, HEPTAGON, OCTAGON, NANOGON and DECAGON respectively.

(b) Regular polygon : Polygon which has all equal sides and equal angles and can be inscribed in a circle whose center coincides with the center of polygon. Therefore the center is equidistant from all its vertices.

(i) A regular polygon can also circumscribe a circle.

(ii) **A 'n' sided regular polygon can be divided into 'n' Isosceles Congruent Triangles with a common vertex i.e. centre of polygon.**

(iii) $\text{Area} = n \times \left(\frac{1}{2}\right) \times a \times h$

(iv) Perimeter = na

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(v) Each interior angle of polygon = $\left(\frac{n-2}{n}\right) \times 180°$

(vi) Angle subtended at the centre of inscribed/circumscribed circle by one side = 360°/n

(vii) Each exterior angle = $\left(\frac{360}{n}\right)^{o}$

(viii) Sum of all interior angle = (n – 2) × 180°

(ix) Sum of all exterior angles = 360°

(x) **Convex polygon :** If any two consecutive vertices are joined then remaining all other vertices will lie on same side.

(D) CIRCLE :

Area A = πr^2 ; Circumference (perimeter) = $2\pi r$

(a) Sector of a circle : Bounded by arc of circle (subtending angle 'θ' at center) and two radii. Circle is divided into minor (containing 'θ') and major sectors

(i) Arc length of sector : $\ell = \left(\frac{\theta°}{360°}\right) 2\pi r$

(ii) Area : $A = \left(\frac{\theta°}{360°}\right) \pi r^2 = \left(\frac{1}{2}\right) \ell r$

(iii) Perimeter of sector AOC = $2r + \ell$

(b) Segment of a circle : Bounded by arc of the circle and the chord (determining the segment).

(i) Circle is divided into two segments minor segment and major segment.

(ii) When chord is diameter, sector coincides with segment.

(iii) Area (segment ACB) = Area of sector OACB - Area of ΔAOB

$$= \left(\frac{\theta°}{360°}\right) \times \pi r^2 - \left(\frac{1}{2}\right) \times \left(2r \sin\frac{\theta}{2}\right) \times \left(r \cos\frac{\theta}{2}\right)$$

$$\text{Area} = \left(\frac{\theta°}{360°}\right) \pi r^2 - \left(\frac{1}{2}\right) r^2 \sin\theta$$

SOLIDS

Require three dimension to describe

(a) Surfaces of solids : Plane areas bounding the solid e.g. six rectangle faces bounding a brick. Surface area is measured in square units.

(b) Volume of solids : Space occupied by a solid and is measured in cubic units.

Cuboid Cone Cylinder Sphere

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(A) CUBOID :

Rectangular shaped solid also known as rectangular parallelopiped (e.g. match box, brick)

(a) Have six rectangular faces with opposite faces parallel and congruent.

(b) Have twelve edges (Edge - The line segment where two adjacent faces meets).

(c) Three adjacent faces meet at a point called vertex and cuboid have eight vertices

(d) **Surface area :** $A = 2[\ell \times b + b \times h + h \times \ell]$ square unit.

(e) **Volume :** $V = l \times b \times h$ cubic unit.

(B) CUBE :

Special case of cuboid having all sides equal.

Area = $6\ell^2$; Volume = ℓ^3 Unit cube : Side $\ell = 1$

Volume is 1 cubic unit (From this cubic unit is derived)

(C) CYLINDER :

Having a lateral (curved) surface and two congruent circular cross section.

(e.g. Jar, Circular Pillars, Drums, Pipes etc.)

(a) **Axis :** Line joining the centers of two circular cross section.

(b) **Right circular cylinder :** When axis is perpendicular to circular cross section.

(c) **Generators :** Lines parallel to axis and lying on the lateral surface.

(d) **Base :** With cylinder in vertical position, the lower circular end is base.

(e) **Height (h) :** Distance between two circular faces.

(f) **Radius (r) :** Radius of base or top circle.

(g) **Total surface area :** Base area + curved surface area

$= 2\pi r^2 + 2\pi rh = 2\pi r(h + r)$(including two circular ends).

Without circular ends (Hollow cylinder) $= 2\pi rh$

(h) **Volume :** $V = \pi r^2 h$

(D) CONE :

Have a curved surface with a vertex (V) and circular base radius : r and center O)

(a) **Axis :** Line joining vertex and center of base circle (VO)

(b) **Height of cone (h) :** Length of VO

(c) **Slant height (Q) :** Distance of vertex from any point of base circle

$\ell = \sqrt{r^2 + h^2}$

(d) **Right circular cone :** When axis is perpendicular to base.

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(e) The cross section of a cone parallel to base is a circle and perpendicular to base is an isosceles triangle.

(f) **Volume :** $(1/3)\pi r^2 h$ (volume of a cone is 1/3rd of volume of a cylinder with same height and base radius).

(g) Curved surface Area : $\pi r \ell$

(h) Total surface Area : $\pi r \ell + \pi r^2 = \pi r (\ell + r)$

(i) A right circular cone can be generated by rotating a right angled triangle about its right angle forming side.

(E) SPHERE :

All point on its surface are equidistant from its center, the distance is called radius (r) and any line passing through center with end points on surface is called diameter.

(a) **Volume :** $(4/3)\ \pi r^3$

(b) **Surface area :** $4\pi r^2$

(F) HEMISPHERE :

A sphere is divided into two hemi spheres by a plane passing through center.

(a) **Volume :** $(2/3)\pi r^3$

(b) **Curved surface area :** $= 2\pi r^2$

(c) **Total surface area :** $= 2\pi r^2 + \pi r^2 = 3\pi r^2$

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MODULUS

1. ABSOLUTE VALUE FUNCTION/MODULUS FUNCTION :

The symbol of modulus function is $|x|$

and is defined as : $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Properties of Modulus :

For any a, b $\in$ R

(a) $|a| \geq 0$

(b) $|a| = |-a|$

(c) $|ab| = |a||b|$

(d) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

(e) $|a + b| \leq |a|+|b|$

(f) $|a|-|b| \leq |a - b|$

(g) $|a + b| = |a| + |b| \Rightarrow ab \geq 0$

(h) $\sqrt{x^2} = |x|$

Illustration 1 : If $||x-1| - 2| = 5$, then find x.

Solution : $|x - 1| - 2 = \pm 5$

$|x - 1| = 7, -3$

Case-I : When $|x - 1| = 7 \Rightarrow x - 1 = \pm 7 \Rightarrow x = 8, -6$

Case-II : When $|x - 1| = -3$ (reject)

Illustration 2 : If $|x - 1| + |x + 1| = 2$, then find x.

Solution : **Case-I :** If $x \leq -1$

$-(x - 1) - (x + 1) = 2$

$\Rightarrow -x + 1 - x - 1 = 2$

$\Rightarrow -2x = 2 \Rightarrow x = -1$ .....(i)

Case-II : If $-1 < x < 1$

$-(x - 1) + (x + 1) = 2$

$\Rightarrow -x + 1 + x + 1 = 2$

$\Rightarrow 2 = 2 \Rightarrow -1 < x < 1$ .....(ii)

Case-III : If $x \geq 1$

$x - 1 + x + 1 = 2$

$\Rightarrow x = 1$ .....(iii)

Thus from (i), (ii) and (iii) $-1 \leq x \leq 1$

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Do yourself - 1 :

(i) Solve : $|x + 3| = 2(5 - x)$

(ii) Solve : $x|x| + 7x - 8 = 0$

2. INEQUALITIES INVOLVING MODULUS FUNCTION :

Properties of modulus function :

(i) $|x| \ge a \Rightarrow x \ge a$ or $x \le -a$, where a is positive.

(ii) $|x| \le a \Rightarrow x \in [-a, a]$, where a is positive

(iii) $|x| > |y| \Rightarrow x^2 > y^2$

(iv) $\big||a| - |b|\big| \le |a \pm b| \le |a| + |b|$

(v) $|x + y| = |x| + |y| \Rightarrow xy \ge 0$

(vi) $|x - y| = |x| + |y| \Rightarrow xy \le 0$

Illustration 3 : If x satisfies $|x - 1| + |x - 2| + |x - 3| \ge 6$, then

(A) $0 \le x \le 4$ (B) $x \le -2$ or $x \ge 4$

(C) $x \le 0$ or $x \ge 4$ (D) none of these

Solution : **Case I :** $x \le 1$, then

$1 - x + 2 - x + 3 - x \ge 6 \Rightarrow x \le 0$

Hence $x \le 0$ ...(i)

Case II : $1 < x \le 2$, then

$x - 1 + 2 - x + 3 - x \ge 6 \Rightarrow x \le -2$

But $1 < x \le 2 \Rightarrow$ No solution. ...(ii)

Case III : $2 < x \le 3$, then

$x - 1 + x - 2 + 3 - x \ge 6 \Rightarrow x \ge 6$

But $2 < x \le 3 \Rightarrow$ No solution. ...(iii)

Case IV : $x > 3$, then

$x - 1 + x - 2 + x - 3 \ge 6 \Rightarrow x \ge 4$

Hence $x \ge 4$ ...(iv)

From (i), (ii), (iii) and (iv) the given inequality holds for $x \le 0$ or $x \ge 4$.

Illustration 4 : Solve for x : (a) $||x - 1| + 2| \le 4$. (b) $\dfrac{x-4}{x+2} \le \left|\dfrac{x-2}{x-1}\right|$

Solution : (a) $||x - 1| + 2| \le 4 \Rightarrow -4 \le |x - 1| + 2 \le 4$

$\Rightarrow -6 \le |x - 1| \le 2$

$\Rightarrow |x - 1| \le 2 \Rightarrow -2 \le x - 1 \le 2$

$\Rightarrow -1 \le x \le 3 \Rightarrow x \in [-1, 3]$

(b) **Case 1 :** Given inequation will be statisfied for all x such that

$\dfrac{x-4}{x+2} \le 0 \Rightarrow x \in (-2, 4] - \{1\}$ .....(i)

(Note : $\{1\}$ is not in domain of RHS)

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Case 2 : $\frac{x-4}{x+2} > 0 \quad \Rightarrow \quad x \in (-\infty, -2) \cup (4, \infty)$ .....(ii)

Given inequation becomes

$\frac{x-2}{x-1} \geq \frac{x-4}{x+2}$ or $\frac{x-2}{x-1} \leq -\frac{x-4}{x+2}$

on solving we get $x \in (-2, 4/5) \cup (1, \infty)$

taking intersection with (ii) we get $x \in (4, \infty)$ .....(iii)

on solving we get $x \in (-2, 0] \cup (1, 5/2]$

taking intersection with (ii) we get $x \in \phi$

Hence, solution of the original inequation : $x \in (-2,\infty) - \{1\}$ (taking union of (i) & (iii))

Illustration 5 : The equation $|x| + \left|\frac{x}{x-1}\right| = \frac{x^2}{|x-1|}$ is always true for x belongs to

(A) $\{0\}$ (B) $(1, \infty)$ (C) $(-1, 1)$ (D) $(-\infty, \infty)$

Solution : $\frac{x^2}{|x-1|} = \left|x + \frac{x}{x-1}\right|$

$\because |x| + \left|\frac{x}{x-1}\right| = \left|x + \frac{x}{x-1}\right|$ is true only if $\left(x.\frac{x}{x-1}\right) \geq 0 \Rightarrow x \in \{0\} \cup (1, \infty)$. **Ans (A,B)**

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3. IRRATIONAL INEQUALITIES :

Illustration 6 : Solve for x, if $\sqrt{x^2 - 3x + 2} > x - 2$

Solution :

$$\left[\begin{array}{l} \begin{cases} x^2 - 3x + 2 \geq 0 \\ x - 2 \geq 0 \\ (x^2 - 3x + 2) > (x-2)^2 \end{cases} \\ \text{or} \\ \begin{cases} x^2 - 3x + 2 \geq 0 \\ x - 2 < 0 \end{cases} \end{array}\right. \Rightarrow \left[\begin{array}{l} \begin{cases} (x-1)(x-2) \geq 0 \\ (x-2) \geq 0 \\ x - 2 > 0 \end{cases} \Rightarrow x > 2 \\ \text{or} \\ \begin{cases} (x-1)(x-2) \geq 0 \\ x - 2 < 0 \end{cases} \Rightarrow x \leq 1 \end{array}\right.$$

Hence, solution set of the original inequation is $x \in R - (1,2]$

Do yourself - 2 :

(i) Solve for x if $\frac{|x^2 - 4|}{x^2 + x - 2} > 1$

(ii) Solve for x if $\sqrt{x^2 - x} > (x - 1)$

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LOGARITHM

4. DEFINITION :

Every positive real number N can be expressed in exponential form as $a^x = N$ where 'a' is also a positive real number different than unity and is called the base and 'x' is called an exponent. We can write the relation $a^x = N$ in logarithmic form as $\log_a N = x$. Hence $a^x = N \Leftrightarrow \log_a N = x$ Hence logarithm of a number to some base is the exponent by which the base must be raised in order to get that number.

Limitations of logarithm: $\log_a N$ is defined only when

(i) $N > 0$ (ii) $a > 0$ (iii) $a \neq 1$

Note :

(i) For a given value of N, $\log_a N$ will give us a unique value.

(ii) Logarithm of zero does not exist.

(iii) Logarithm of negative reals are not defined in the system of real numbers.

Illustration 7 : If $\log_4 m = 1.5$, then find the value of m.

Solution : $\log_4 m = 1.5 \Rightarrow m = 4^{3/2} \Rightarrow m = 8$

Illustration 8 : If $\log_5 p = a$ and $\log_2 q = a$, then prove that $\frac{p^4q^4}{100} = 100^{2a-1}$

Solution : $\log_5 p = a \Rightarrow p = 5^a$

$\log_2 q = a \Rightarrow q = 2^a$

$$\Rightarrow \frac{p^4q^4}{100} = \frac{5^{4a}.2^{4a}}{100} = \frac{(10)^{4a}}{100} = \frac{(100)^{2a}}{100} = 100^{2a-1}$$

Illustration 9 : The value of N, satisfying $\log_a[1 + \log_b\{1 + \log_c(1 + \log_p N)\}] = 0$ is -

(A) 4 (B) 3 (C) 2 (D) 1

Solution : $1 + \log_b\{1 + \log_c(1 + \log_p N)\} = a^0 = 1$

$\Rightarrow \log_b\{1 + \log_c(1 + \log_p N)\} = 0 \Rightarrow 1 + \log_c(1 + \log_p N) = 1$

$\Rightarrow \log_c(1 + \log_p N) = 0 \Rightarrow 1 + \log_p N = 1$

$\Rightarrow \log_p N = 0 \Rightarrow N = 1$ **Ans. (D)**

Do yourself - 3 :

(i) Express the following in logarithmic form :

(a) $81 = 3^4$ (b) $0.001 = 10^{-3}$ (c) $2 = 128^{1/7}$

(ii) Express the following in exponential form :

(a) $\log_2 32 = 5$ (b) $\log_{\sqrt{2}} 4 = 4$ (c) $\log_{10} 0.01 = -2$

(iii) If $\log_{2\sqrt{3}} 1728 = x$, then find x.

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5. FUNDAMENTAL IDENTITIES :

Using the basic definition of logarithm we have 3 important deductions :

(a) $\log_a 1 = 0$ i.e. logarithm of unity to any base is zero.

(b) $\log_N N = 1$ i.e. logarithm of a number to the same base is 1.

(c) $\log_{\frac{1}{N}} N = -1 = \log_N \frac{1}{N}$ i.e. logarithm of a number to the base as its reciprocal is –1.

Note : $N = (a)^{\log_a N}$ e.g. $2^{\log_2 7} = 7$

Do yourself - 4 :

(i) Find the value of the following :

(a) $\log_{1.4\overline{3}} \frac{43}{30}$ **(b)** $\left(\frac{1}{2}\right)^{\log_2 5}$

(ii) If $4^{\log_2 2x} = 36$, then find x.

6. THE PRINCIPAL PROPERTIES OF LOGARITHMS :

If m,n are arbitrary positive numbers where $a > 0$, $a \neq 1$ and x is any real number, then-

(a) $\log_a mn = \log_a m + \log_a n$

(b) $\log_a \frac{m}{n} = \log_a m - \log_a n$

(c) $\log_a m^x = x \log_a m$

Illustration 10 : Find the value of $2\log\frac{2}{5} + 3\log\frac{25}{8} - \log\frac{625}{128}$

Solution :

$$2\log\frac{2}{5} + 3\log\frac{25}{8} + \log\frac{128}{625}$$

$$= \log\frac{2^2}{5^2} + \log\left(\frac{5^2}{2^3}\right)^3 + \log\frac{2^7}{5^4}$$

$$= \log\frac{2^2}{5^2}\cdot\frac{5^6}{2^9}\cdot\frac{2^7}{5^4} = \log 1 = 0$$

Illustration 11 : If $\log_e x - \log_e y = a$, $\log_e y - \log_e z = b$ & $\log_e z - \log_e x = c$, then find the value of $\left(\frac{x}{y}\right)^{b-c} \times \left(\frac{y}{z}\right)^{c-a} \times \left(\frac{z}{x}\right)^{a-b}$

Solution :

$$\log_e x - \log_e y = a \Rightarrow \log_e \frac{x}{y} = a \Rightarrow \frac{x}{y} = e^a$$

$$\log_e y - \log_e z = b \Rightarrow \log_e \frac{y}{z} = b \Rightarrow \frac{y}{z} = e^b$$

$$\log_e z - \log_e x = c \Rightarrow \log_e \frac{z}{x} = c \Rightarrow \frac{z}{x} = e^c$$

$$\therefore \left(e^a\right)^{b-c} \times \left(e^b\right)^{c-a} \times \left(e^c\right)^{a-b}$$

$$= e^{a(b-c)+b(c-a)+c(a-b)} = e^0 = 1$$

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Illustration 12 : If $a^2 + b^2 = 23ab$, then prove that $\log\frac{(a+b)}{5} = \frac{1}{2}(\log a + \log b)$.

Solution : $a^2 + b^2 = (a + b)^2 - 2ab = 23ab$

$\Rightarrow (a + b)^2 = 25ab \Rightarrow a+b = 5\sqrt{ab}$ ....(i)

Using (i)

$$\text{L.H.S.} = \log\frac{(a+b)}{5} = \log\frac{5\sqrt{ab}}{5} = \frac{1}{2}\log ab = \frac{1}{2}(\log a + \log b) = \text{R.H.S.}$$

Illustration 13 : If $\log_a x = p$ and $\log_b x^2 = q$, then $\log_x \sqrt{ab}$ is equal to (where a, b, x $\in R^+ - \{1\}$)-

(A) $\frac{1}{p}+\frac{1}{q}$ (B) $\frac{1}{2p}+\frac{1}{q}$ (C) $\frac{1}{p}+\frac{1}{2q}$ (D) $\frac{1}{2p}+\frac{1}{2q}$

Solution : $\log_a x = p \Rightarrow a^p = x \Rightarrow a = x^{1/p}$.

similarly $b^q = x^2 \Rightarrow b = x^{2/q}$

Now, $\log_x \sqrt{ab} = \log_x \sqrt{x^{1/p}x^{2/q}} = \log_x x^{\left(\frac{1}{p}+\frac{2}{q}\right)\cdot\frac{1}{2}} = \frac{1}{2p}+\frac{1}{q}$

Do yourself - 5 :

(i) Show that $\frac{1}{2}\log 9 + 2\log 6 + \frac{1}{4}\log 81 - \log 12 = 3\log 3$

7. BASE CHANGING THEOREM :

Can be stated as "quotient of the logarithm of two numbers is independent of their common base."

Symbolically, $\log_b m = \frac{\log_a m}{\log_a b}$, where $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$

Note :

(i) $\log_b a.\ \log_a b = \frac{\log a}{\log b}.\frac{\log b}{\log a} = 1$; hence $\boxed{\log_b a = \frac{1}{\log_a b}.}$

(ii) $a^{\log_b c} = c^{\log_b a}$

(iii) **Base power formula :** $\log_{a^k} m = \frac{1}{k}\log_a m$

(iv) The base of the logarithm can be any positive number other than 1, but in normal practice, only two bases are popular, these are 10 and e(=2.718 approx). Logarithms of numbers to the base 10 are named as 'common logarithm' and the logarithms of numbers to the base e are called Natural or Napierian logarithm. **We will consider logx as $\log_e x$ or ℓnx.**

(v) Conversion of base e to base 10 & viceversa :

$$\log_e a = \frac{\log_{10} a}{\log_{10} e} = 2.303 \times \log_{10} a\ ; \quad \log_{10} a = \frac{\log_e a}{\log_e 10} = \log_{10} e \times \log_e a = 0.434 \log_e a$$

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Illustration 14 : If a, b, c are distinct positive real numbers different from 1 such that $(\log_b a \,.\, \log_c a - \log_a a) + (\log_a b \,.\, \log_c b - \log_b b) + (\log_a c \,.\, \log_b c - \log_c c) = 0$, then abc is equal to -

(A) 0 (B) e (C) 1 (D) none of these

Solution : $(\log_b a \log_c a - 1) + (\log_a b \,.\, \log_c b - 1) + (\log_a c \log_b c - 1) = 0$

$$\Rightarrow \frac{\log a}{\log b}.\frac{\log a}{\log c} + \frac{\log b}{\log a}.\frac{\log b}{\log c} + \frac{\log c}{\log a}.\frac{\log c}{\log b} = 3$$

$\Rightarrow (\log a)^3 + (\log b)^3 + (\log c)^3 = 3\log a \log b \log c$

$\Rightarrow (\log a + \log b + \log c) = 0$ [$\because$ If $a^3 + b^3 + c^3 - 3abc = 0$, then $a + b + c = 0$ if $a \neq b \neq c$]

$\Rightarrow \log abc = \log 1 \Rightarrow abc = 1$

Illustration 15 : Evaluate : $81^{1/\log_5 3} + 27^{\log_9 36} + 3^{4/\log_7 9}$

Solution : $81^{\log_3 5} + 3^{3\log_9 36} + 3^{4\log_9 7}$

$= 3^{4\log_3 5} + 3^{\log_3 (36)^{3/2}} + 3^{\log_3 7^2}$

$= 625 + 216 + 49 = 890.$

Do yourself - 6 :

(i) Evaluate : $\dfrac{\log_3 135}{\log_{15} 3} - \dfrac{\log_3 5}{\log_{405} 3}$

(ii) Evaluate : $\log_9 27 - \log_{27} 9$

(iii) Evaluate : $2^{\log_3 5} - 5^{\log_3 2}$

(iv) Evaluate : $\log_3 4 \,.\, \log_4 5 \,.\, \log_5 6 \,.\, \log_6 7 \,.\, \log_7 8 \,.\, \log_8 9$

(v) If $\dfrac{1}{\log_3 \pi} + \dfrac{1}{\log_4 \pi} > x$, then x can be -

(A) 2 (B) 3 (C) 3.5 (D) π

(vi) If $\log_a 3 = 2$ and $\log_b 8 = 3$, then $\log_a b$ is -

(A) $\log_3 2$ (B) $\log_2 3$ (C) $\log_3 4$ (D) $\log_4 3$

8. POINTS TO REMEMBER :

(i) If base of logarithm is greater than 1 then logarithm of greater number is greater. i.e. $\log_2 8 = 3$, $\log_2 4 = 2$ etc. and if base of logarithm is between 0 and 1 then logarithm of greater number is smaller. i.e. $\log_{1/2} 8 = -3$, $\log_{1/2} 4 = -2$ etc.

$$\log_a x < \log_a y \Leftrightarrow \begin{cases} x < y & \text{if } a > 1 \\ x > y & \text{if } 0 < a < 1 \end{cases}$$

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(ii) It must be noted that whenever the number and the base are on the same side of unity then logarithm of that number to that base is positive, however if the number and the base are located on different side of unity then logarithm of that number to that base is negative.

e.g. $\log_{10}\sqrt[3]{10}=\frac{1}{3}$; $\log_{\sqrt{7}}49=4$; $\log_{\frac{1}{2}}\left(\frac{1}{8}\right)=3$; $\log_2\left(\frac{1}{32}\right)=-5$; $\log_{10}(0.001) = -3$

(iii) $x+\frac{1}{x}\ge 2$ if x is positive real number and $x+\frac{1}{x}\le -2$ if x is negative real number

(iv) $n \ge 2,\ n \in N$

$\sqrt[n]{a}=a^{1/n} \Rightarrow$ n^{th} root of 'a' ('a' is a non negative number)

Some important values : $\log_{10}2 = 0.3010$; $\log_{10}3 = 0.4771$; $\ell n2 = 0.693$, $\ell n10 = 2.303$

9. CHARACTERISTIC AND MANTISSA :

For any given number N, logarithm can be expressed as $\log_a N$ = Integer + Fraction

The integer part is called characteristic and the fractional part is called mantissa. When the value of log n is given, then to find digits of 'n' we use only the mantissa part. The characteristic is used only in determining the number of digits in the integral part (if $n \ge 1$) or the number of zeros after decimal & before first non-zero digit in the number (if $0 < n < 1$).

Note :

(i) The mantissa part of logarithm of a number is always non-negative $(0 \le m < 1)$

(ii) If the characteristic of $\log_{10}N$ be n, then the number of digits in N is (n + 1)

(iii) If the characteristic of $\log_{10}N$ be (–n), then there exist (n – 1) zeros after decimal in N.

10. ANTILOGARITHM :

The positive real number 'n' is called the antilogarithm of a number 'm' if log n = m

Thus, **log n = m ⇔ n = antilog m**

Do yourself - 7 :

(i) Evaluate : $\log_{10}(0.06)^6$

(ii) Find number of digits in 18^{20}

(iii) Determine number of cyphers (zeros) between decimal & first significant digit in $\left(\frac{1}{6}\right)^{200}$

(iv) Find antilog of $\frac{5}{6}$ to the base 64.

11. LOGARITHMIC INEQUALITIES :

Points to remember :

(i) $\log_a x < \log_a y \Leftrightarrow \begin{cases} x<y & \text{if } a>1 \\ x>y & \text{if } 0<a<1 \end{cases}$

(ii) If a > 1, then (a) $\log_a x < p \Rightarrow 0 < x < a^p$ (b) $\log_a x > p \Rightarrow x > a^p$

(iii) If 0 < a < 1, then (a) $\log_a x < p \Rightarrow x > a^p$ (b) $\log_a x > p \Rightarrow 0 < x < a^p$

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Illustration 16 : Solve for x : (a) $\log_{0.5}(x^2 - 5x + 6) \geq -1$ (b) $\log_{1/3}(\log_4(x^2 - 5)) > 0$

Solution : (a) $\log_{0.5}(x^2 - 5x + 6) \geq -1 \Rightarrow 0 < x^2 - 5x + 6 \leq (0.5)^{-1}$

$\Rightarrow 0 < x^2 - 5x + 6 \leq 2$

$$\begin{cases} x^2 - 5x + 6 > 0 \\ x^2 - 5x + 6 \leq 2 \end{cases} \Rightarrow x \in [1,2) \cup (3,4]$$

Hence, solution set of original inequation : $x \in [1,2) \cup (3,4]$

(b) $\log_{1/3}(\log_4(x^2 - 5)) > 0 \Rightarrow 0 < \log_4(x^2 - 5) < 1$

$$\begin{cases} 0 < \log_4(x^2 - 5) \Rightarrow x^2 - 5 > 1 \\ \log_4(x^2 - 5) < 1 \Rightarrow 0 < x^2 - 5 < 4 \end{cases} \Rightarrow 1 < (x^2 - 5) < 4$$

$\Rightarrow 6 < x^2 < 9 \Rightarrow x \in \left(-3, -\sqrt{6}\right) \cup \left(\sqrt{6}, 3\right)$

Hence, solution set of original inequation : $x \in \left(-3, -\sqrt{6}\right) \cup \left(\sqrt{6}, 3\right)$

Illustration 17 : Solve for x : $\log_2 x \leq \dfrac{2}{\log_2 x - 1}$.

Solution : Let $\log_2 x = t$

$$t \leq \frac{2}{t-1} \Rightarrow t - \frac{2}{t-1} \leq 0$$

$$\Rightarrow \frac{t^2 - t - 2}{t-1} \leq 0 \Rightarrow \frac{(t-2)(t+1)}{(t-1)} \leq 0$$

$\Rightarrow t \in (-\infty, -1] \cup (1,2]$

or $\log_2 x \in (-\infty, -1] \cup (1,2]$

or $x \in \left(0, \dfrac{1}{2}\right] \cup (2,4]$

Illustration 18 : Solve the inequation : $\log_{2x+3} x^2 < \log_{2x+3}(2x+3)$

Solution : This inequation is equivalent to the collection of the systems

$$\left[\begin{array}{l} \begin{cases} 2x+3>1 \\ 0<x^2<2x+3 \end{cases} \\ \text{or} \\ \begin{cases} 0<2x+3<1 \\ x^2>2x+3>0 \end{cases} \end{array}\right. \Rightarrow \left[\begin{array}{l} \begin{cases} x>-1 \\ (x-3)(x+1)<0 \ \& \ x \neq 0 \end{cases} \\ \text{or} \\ \begin{cases} -\frac{3}{2}<x<-1 \\ (x-3)(x+1)>0 \end{cases} \end{array}\right. \Rightarrow \left[\begin{array}{l} \begin{cases} x>-1 \\ -1<x<3 \end{cases} \Rightarrow -1<x<3 \ \& \ x \neq 0 \\ \text{or} \qquad\qquad\qquad\qquad \text{or} \\ \begin{cases} -\frac{3}{2}<x<-1 \\ x<-1 \text{ or } x>3 \end{cases} \Rightarrow -\frac{3}{2}<x<-1 \end{array}\right.$$

Hence, solution of the original inequation is $x \in \left(-\dfrac{3}{2}, -1\right) \cup (-1, 0) \cup (0, 3)$

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12. EXPONENTIAL INEQUATIONS :

If $a^{f(x)} > b \Rightarrow \begin{cases} f(x) > \log_a b \text{ when } a > 1 \\ f(x) < \log_a b \text{ when } 0 < a < 1 \end{cases}$

Illustration 19 : Solve for x : $2^{x+2} > \left(\frac{1}{4}\right)^{\frac{1}{x}}$

Solution : We have $2^{x+2} > 2^{-2/x}$. Since the base 2 > 1, we have $x + 2 > -\frac{2}{x}$

(the sign of the inequality is retained).

Now $x + 2 + \frac{2}{x} > 0 \quad \Rightarrow \quad \frac{x^2 + 2x + 2}{x} > 0$

$\Rightarrow \quad \frac{(x+1)^2 + 1}{x} > 0 \quad \Rightarrow \quad x \in (0, \infty)$

Illustration 20 : Solve for x : $(1.25)^{1-x} < (0.64)^{2(1+\sqrt{x})}$

Solution : We have $\left(\frac{5}{4}\right)^{1-x} < \left(\frac{16}{25}\right)^{2(1+\sqrt{x})}$ or $\left(\frac{4}{5}\right)^{x-1} < \left(\frac{4}{5}\right)^{4(1+\sqrt{x})}$

Since the base $0 < \frac{4}{5} < 1$, the inequality is equivalent to the inequality $x - 1 > 4(1 + \sqrt{x})$

$\Rightarrow \quad \frac{x-5}{4} > \sqrt{x}$

Now, R.H.S. is positive

$\Rightarrow \quad \frac{x-5}{4} > 0 \quad \Rightarrow \quad x > 5 \qquad$ .....(i)

we have $\frac{x-5}{4} > \sqrt{x}$

both sides are positive, so squaring both sides

$\Rightarrow \quad \frac{(x-5)^2}{16} > x \quad$ or $\quad \frac{(x-5)^2}{16} - x > 0$

or $\quad x^2 - 26x + 25 > 0 \quad$ or $\quad (x - 25)(x - 1) > 0$

$\Rightarrow \quad x \in (-\infty, 1) \cup (25, \infty) \qquad$ .....(ii)

intersection (i) & (ii) gives $x \in (25, \infty)$

Do yourself-8 :

(i) Solve for x : (a) $\log_{0.3}(x^2 + 8) > \log_{0.3}(9x)$ (b) $\log_7\left(\frac{2x-6}{2x-1}\right) > 0$

(ii) Solve for x : $\left(\frac{2}{3}\right)^{\frac{|x|-1}{|x|+1}} > 1$

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Miscellaneous Illustrations :

Illustration 21 : Show that $\log_4 18$ is an irrational number.

Solution : $\log_4 18 = \log_4(3^2 \times 2) = 2\log_4 3 + \log_4 2 = 2\dfrac{\log_2 3}{\log_2 4} + \dfrac{1}{\log_2 4} = \log_2 3 + \dfrac{1}{2}$

assume the contrary, that this number $\log_2 3$ is rational number.

$\Rightarrow \quad \log_2 3 = \dfrac{p}{q}$. Since $\log_2 3 > 0$ both numbers p and q may be regarded as natural number

$\Rightarrow \quad 3 = 2^{p/q} \quad \Rightarrow \quad 2^p = 3^q$

But this is not possible for any natural number p and q. The resulting contradiction completes the proof.

Illustration 22 : If in a right angled triangle, a and b are the lengths of sides and c is the length of hypotenuse and $c - b \neq 1$, $c + b \neq 1$, then show that

$\log_{c+b} a + \log_{c-b} a = 2\log_{c+b} a \,.\, \log_{c-b} a$.

Solution : We know that in a right angled triangle

$c^2 = a^2 + b^2$

$c^2 - b^2 = a^2$ ..... (i)

$$\text{LHS} = \frac{1}{\log_a(c+b)} + \frac{1}{\log_a(c-b)} = \frac{\log_a(c-b) + \log_a(c+b)}{\log_a(c+b).\log_a(c-b)}$$

$$= \frac{\log_a(c^2 - b^2)}{\log_a(c+b).\log_a(c-b)} = \frac{\log_a a^2}{\log_a(c+b).\log_a(c-b)} \qquad \text{(using (i))}$$

$$= \frac{2}{\log_a(c+b).\log_a(c-b)} = 2\log_{(c+b)} a \,.\, \log_{(c-b)} a = \text{RHS}$$

ANSWERS FOR DO YOURSELF

1 : **(i)** $\dfrac{7}{3}$ **(ii)** $x = 1$

2 : **(i)** $x \in (-\infty, -2) \cup (1, 3/2)$ **(ii)** $x \in R - (0,1]$

3 : **(i)** **(a)** $\log_3 81 = 4$ **(b)** $\log_{10}(0.001) = -3$ **(c)** $\log_{128} 2 = 1/7$

(ii) **(a)** $32 = 2^5$ **(b)** $4 = (\sqrt{2})^4$ **(c)** $0.01 = 10^{-2}$

(iii) 6

4 : **(i)** **(a)** 1 **(b)** $\dfrac{1}{5}$ **(ii)** 3

6 : **(i)** 3 **(ii)** 5/6 **(iii)** 0 **(iv)** 2 **(v)** (A) **(vi)** (C)

7 : **(i)** $\overline{8}.6686$ **(ii)** 26 **(iii)** 155 **(iv)** 32

8 : **(i)** (a) $x \in (1,8)$ (b) $x \in (-\infty, 1/2)$ **(ii)** $x \in (-1,1)$

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EXERCISE (O-1)

1. Sum of roots of the equation $(x + 3)^2 - 4|x + 3| + 3 = 0$ is -

(A) 4 (B) 12 (C) –12 (D) –4

2. The solution of the inequation $|x^2 - 2x - 3| < |x^2 - x + 5|$ is -

(A) $(-\infty, 5)$ (B) $(-\infty, 2)\cup(3, 8)\cup(8, \infty)$

(C) $(-8, \infty)$ (D) $(3, 8)$

3. The minimum value of $f(x) = |x - 1| + |x - 2| + |x - 3|$ is equal to -

(A) 1 (B) 2 (C) 3 (D) 0

4. The complete solution set of the inequation $\sqrt{x+18} < 2 - x$ is -

(A) $[-18, -2]$ (B) $(-\infty, -2) \cup (7, \infty)$ (C) $(-18, 2) \cup (7, \infty)$ (D) $[-18, -2)$

5. Solution of the inequality, $x - 3 < \sqrt{x^2 + 4x - 5}$ is -

(A) $(-\infty, -5] \cup [1, \infty)$ (B) $(-5, 3]$ (C) $(-\infty, -5] \cup \left(\frac{7}{5}, \infty\right)$ (D) $\left(\frac{7}{5}, \infty\right)$

6. If $\log_y x + \log_x y = 7$, then the value of $(\log_y x)^2 + (\log_x y)^2$, is

(A) 43 (B) 45 (C) 47 (D) 49

7. If $\log_2(\log_3(\log_4(x))) = 0$, $\log_3(\log_4(\log_2(y))) = 0$ and $\log_4(\log_2(\log_3(z))) = 0$, then the sum of x, y and z is-

(A) 89 (B) 58 (C) 105 (D) 50

8. $\log_{10}(\log_2 3) + \log_{10}(\log_3 4) + \log_{10}(\log_4 5) + ... + \log_{10}(\log_{1023} 1024)$ simplifies to

(A) a composite (B) a prime number

(C) rational which is not an integer (D) an integer

9. If $\log_9 x + \log_4 y = \frac{7}{2}$ and $\log_9 x - \log_8 y = -\frac{3}{2}$, then x + y equals

(A) 35 (B) 41 (C) 67 (D) 73

10. If $\log_a b + \log_b c + \log_c a$ vanishes where a, b and c are positive reals different than unity then the value of $(\log_a b)^3 + (\log_b c)^3 + (\log_c a)^3$ is

(A) an odd prime (B) an even prime (C) an odd composite (D) an irrational number

11. Let $x = 2^{\log 3}$ and $y = 3^{\log 2}$ where base of the logarithm is 10, then which one of the following holds good ?

(A) $2x < y$ (B) $2y < x$ (C) $3x = 2y$ (D) $y = x$

12. The sum of all the solutions to the equation $2\log_{10}x - \log_{10}(2x - 75) = 2$

(A) 30 (B) 350 (C) 75 (D) 200

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13. The value of 'a' for which $\frac{\log_a 7}{\log_6 7} = \log_\pi 36$ holds good, is

(A) $1/\pi$ (B) π^2 (C) $\sqrt{\pi}$ (D) 2

14. Let W,X,Y and Z be positive real numbers such that

$\log(W.Z) + \log(W.Y) = 2;\ \log(Y.Z) + \log(Y.X) = 3;\ \log(X.W) + \log(X.Z) = 4.$

The value of the product (WXYZ) equals (base of the log is 10)

(A) 10^2 (B) 10^3 (C) 10^4 (D) 10^9

15. If $\sum_{r=0}^{n-1} \log_2\left(\frac{r+2}{r+1}\right) = \prod_{r=10}^{99} \log_r(r+1)$, then 'n' is equal to

(A) 4 (B) 3 (C) 5 (D) 6

16. The number $N = 6\log_{10}2 + \log_{10}31$, lies between two successive integers whose sum is equal to

(A) 5 (B) 7 (C) 9 (D) 10

17. Number of real solution(s) of the equation $|x-3|^{3x^2-10x+3} = 1$ is -

(A) exactly four (B) exactly three (C) exactly two (D) exactly one

18. The solution set of the inequality $\log_{\frac{5}{8}}\left(2x^2 - x - \frac{3}{8}\right) \geq 1$ is -

(A) $\left[-\frac{1}{2}, -\frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right]$ (B) $\left[-\frac{1}{2}, 1\right]$

(C) $\left[-\frac{1}{2}, \frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right]$ (D) $\left(-\infty, -\frac{1}{4}\right) \cup \left(\frac{3}{4}, \infty\right)$

19. Solution set of the inequality, $2 - \log_2(x^2 + 3x) \geq 0$ is -

(A) $[-4, 1]$ (B) $[-4, -3) \cup (0, 1]$ (C) $(-\infty, -3) \cup (1, \infty)$ (D) $(-\infty, -4) \cup [1, \infty)$

20. If $\log_{1/3}\left(\frac{3x-1}{x+2}\right)$ is less than unity then x must lie in the interval -

(A) $(-\infty, -2) \cup (5/8, \infty)$ (B) $(-2, 5/8)$

(C) $(-\infty, -2) \cup (1/3, 5/8)$ (D) $(-2, 1/3)$

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EXERCISE (S-1)

1. Solve the following equations where $x \in R$.

(a) $(x-1)|x^2-4x+3|+2x^2+3x-5=0$

(b) $|x^2+4x+3|+2x+5=0$

(c) $|x+3|(x+1)+|2x+5|=0$

MATCH THE COLUMN

2. Consider the function $f(x) = |x-1| - 2|x+2| + |x+3|$

	Column-I		Column-II
(A)	If $f(x) = k$ has no solution, then $k \in$	(p)	$(2, 4)$
(B)	If $f(x) = k$ has one solution, then $k \in$	(q)	$(-\infty, -2) \cup (4, \infty)$
(C)	If $f(x) = k$ has two solution, then $k \in$	(r)	$(-2, 2) \cup \{4\}$
(D)	If $f(x) = k$ has more then two solution, then $k \in$	(s)	$\{-2, 2\}$

3. Find the square of the sum of the roots of the equation

$\log_3 x.\log_4 x.\log_5 x = \log_3 x.\log_4 x + \log_4 x.\log_5 x + \log_5 x.\log_3 x$.

4. **(a)** Calculate : $4^{5\log_{4\sqrt{2}}\left(3-\sqrt{6}\right)-6\log_8\left(\sqrt{3}-\sqrt{2}\right)}$

(b) Simplify : $\dfrac{81^{\frac{1}{\log_5 9}}+3^{\frac{3}{\log_{\sqrt{6}}3}}}{409}.\left(\left(\sqrt{7}\right)^{\frac{2}{\log_{25}7}}-\left(125\right)^{\log_{25}6}\right)$

(c) Find the value of $49^{(1-\log_7 2)}+5^{-\log_5 4}$.

(d) Find the value of the expression $\dfrac{2}{\log_4(2000)^6}+\dfrac{3}{\log_5(2000)^6}$

5. If a,b and c are positive real numbers such that $a^{\log_3 7}=27; b^{\log_7 11}=49$ and $c^{\log_{11}25}=\sqrt{11}$. Find the value of $\left(a^{(\log_3 7)^2}+b^{(\log_7 11)^2}+c^{(\log_{11}25)^2}\right)$.

6. Solve for x :

(a) $\dfrac{\log_{10}(x-3)}{\log_{10}(x^2-21)}=\dfrac{1}{2}$

(b) $\log(\log x)+\log(\log x^3-2)=0$; where base of log is 10.

(c) $\log_x 2.\log_{2x}2=\log_{4x}2$

(d) $5^{\log x}+5x^{\log 5}=3$ $(a>0)$; where base of log is a.

(e) If $9^{1+\log x}-3^{1+\log x}-210=0$; where base of log is 3.

7. If $x,y>0$, $\log_y x+\log_x y=\dfrac{10}{3}$ and $xy=144$, then $\dfrac{x+y}{2}=\sqrt{N}$ where N is a natural number, find the value of N.

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8. Solve the system of equations :

$\log_a x \log_a(xyz) = 48$

$\log_a y \log_a(xyz) = 12,\ a > 0,\ a \neq 1$

$\log_a z \log_a(xyz) = 84$

9. Solve the equation for x : $x^{\log x + 4} = 32$, where base of logarithm is 2.

10. Find the product of the positive roots of the equation $\sqrt{(2008)}(x)^{\log_{2008} x} = x^2$.

11. Solve the inequality : $\log_{1/2}(x+1) > \log_2(2-x)$.

12. Solve the inequality : $\log_x 2 \,.\, \log_{2x} 2 \,.\, \log_2 4x > 1$.

EXERCISE (JA)

1. Let (x_0, y_0) be the solution of the following equations **[JEE 2011, 3 (–1)]**

$$(2x)^{\ln 2} = (3y)^{\ln 3}$$

$$3^{\ln x} = 2^{\ln y}$$

Then x_0 is

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) 6

2. The value of $6 + \log_{\frac{3}{2}}\left(\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\ldots\ldots}}}\right)$ is **[JEE 2012, 4M]**

3. If $3^x = 4^{x-1}$, then x = **[JEE-Advanced 2013, 4(–1)]**

(A) $\frac{2\log_3 2}{2\log_3 2 - 1}$ (B) $\frac{2}{2 - \log_2 3}$ (C) $\frac{1}{1 - \log_4 3}$ (D) $\frac{2\log_2 3}{2\log_2 3 - 1}$

4. The value of $\left((\log_2 9)^2\right)^{\frac{1}{\log_2(\log_2 9)}} \times \left(\sqrt{7}\right)^{\frac{1}{\log_4 7}}$ is _____ **[JEE(Advanced)-2018, 3(0)]**

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ANSWER KEY

EXERCISE (O-1)

1. C	**2.** C	**3.** B	**4.** D	**5.** A	**6.** C	**7.** A	**8.** D	**9.** C	**10.** A
11. D	**12.** D	**13.** C	**14.** B	**15.** B	**16.** B	**17.** B	**18.** A	**19.** B	**20.** A

EXERCISE (S-1)

1. (a) 1; (b) $-4, -\sqrt{3}-1$; (c) $-4, -2, -\sqrt{3}-1$ **2.** (A)→(q), (B)→(r), (C)→(p), (D)→(s)

3. 3721 **4.** (a) 9 (b) 1 (c) $\frac{25}{2}$ (d) 1/6 **5.** 469

6. **(a)** x = 5, **(b)** x = 10 **(c)** $x = 2^{\sqrt{2}}$ or $2^{-\sqrt{2}}$ **(d)** $x = 2^{-\log a}$ where base of log is 5 **(e)** x = 5

7. 507 **8.** (a^4, a, a^7) or $\left(\frac{1}{a^4}, \frac{1}{a}, \frac{1}{a^7}\right)$ **9.** x = 2 or 1/32

10. $(2008)^2$ **11.** $-1 < x < \frac{1-\sqrt{5}}{2}$ or $\frac{1+\sqrt{5}}{2} < x < 2$ **12.** $2^{-\sqrt{2}} < x < 2^{-1}$; $1 < x < 2^{\sqrt{2}}$

EXERCISE (JA)

1. C **2.** 4 **3.** A,B,C **4.** 8

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TRIGONOMETRIC RATIOS & IDENTITIES

1. INTRODUCTION TO TRIGONOMETRY :

The word 'trigonometry' is derived from the Greek words 'trigon' and 'metron' and it means 'measuring the sides of a triangle'. The subject was originally developed to solve geometric problems involving triangles. It was studied by sea captains for navigation, surveyor to map out the new lands, by engineers and others. Currently, trigonometry is used in many areas such as the science of seismology, designing electric circuits, describing the state of an atom, predicting the heights of tides in the ocean, analysing a musical tone and in many other areas.

(a) **Measurement of angles :** Commonly two systems of measurement of angles are used.

(i) **Sexagesimal or English System :** Here 1 right angle = 90° (degrees)

1° = 60' (minutes)

1' = 60" (seconds)

(ii) **Circular system :** Here an angle is measured in radians. One radian corresponds to the angle subtended by an arc of length 'r' at the centre of the circle of radius r. It is a constant quantity and does not depend upon the radius of the circle.

(b) Relation between the these systems : $\frac{D}{90} = \frac{R}{\pi/2}$

(c) If θ is the angle subtended at the centre of a circle of radius 'r', by an arc of length 'ℓ' then $\frac{\ell}{r} = \theta$.

Note that here ℓ, r are in the same units and θ is always in radians.

Illustration 1 : If the arcs of same length in two circles subtend angles of 60° and 75° at their centres. Find the ratio of their radii.

Solution : Let r_1 and r_2 be the radii of the given circles and let their arcs of same length 's' subtend angles of 60° and 75° at their centres.

Now, $60° = \left(60 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c$ and $75° = \left(75 \times \frac{\pi}{180}\right)^c = \left(\frac{5\pi}{12}\right)^c$

$\therefore \quad \frac{\pi}{3} = \frac{s}{r_1}$ and $\frac{5\pi}{12} = \frac{s}{r_2}$

$\Rightarrow \quad \frac{\pi}{3} r_1 = s$ and $\frac{5\pi}{12} r_2 = s \Rightarrow \frac{\pi}{3} r_1 = \frac{5\pi}{12} r_2 \Rightarrow 4r_1 = 5r_2 \Rightarrow \quad r_1 : r_2 = 5 : 4$ **Ans.**

Do yourself - 1 :

(i) The radius of a circle is 30 cm. Find the length of an arc of this circle if the length of the chord of the arc is 30 cm.

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2. T-RATIOS (or Trigonometric functions) :

In a right angle triangle

$\sin\theta = \frac{p}{h}$; $\cos\theta = \frac{b}{h}$; $\tan\theta = \frac{p}{b}$; $\operatorname{cosec}\theta = \frac{h}{p}$; $\sec\theta = \frac{h}{b}$ and $\cot\theta = \frac{b}{p}$

'p' is perpendicular ; 'b' is base and 'h' is hypotenuse.

Note : The quantity by which the cosine falls short of unity i.e. $1 - \cos\theta$, is called the versed sine θ of θ and also by which the sine falls short of unity i.e. $1 - \sin\theta$ is called the coversed sine of θ.

3. BASIC TRIGONOMETRIC IDENTITIES :

(1) $\sin\theta . \operatorname{cosec}\theta = 1$

(2) $\cos\theta . \sec\theta = 1$

(3) $\tan\theta . \cot\theta = 1$

(4) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ & $\cot\theta = \frac{\cos\theta}{\sin\theta}$

(5) $\sin^2\theta + \cos^2\theta = 1$ or $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$

(6) $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta = 1 + \tan^2\theta$ or $\tan^2\theta = \sec^2\theta - 1$

(7) $\sec\theta + \tan\theta = \frac{1}{\sec\theta - \tan\theta}$

(8) $\operatorname{cosec}^2\theta - \cot^2\theta = 1$ or $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$ or $\cot^2\theta = \operatorname{cosec}^2\theta - 1$

(9) $\operatorname{cosec}\theta + \cot\theta = \frac{1}{\operatorname{cosec}\theta - \cot\theta}$

(10) Expressing trigonometrical ratio in terms of each other :

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\operatorname{cosec}\theta$
$\sin\theta$	$\sin\theta$	$\sqrt{1-\cos^2\theta}$	$\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\frac{1}{\sqrt{1+\cot^2\theta}}$	$\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\frac{1}{\operatorname{cosec}\theta}$
$\cos\theta$	$\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\frac{1}{\sqrt{1+\tan^2\theta}}$	$\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$\frac{1}{\sec\theta}$	$\frac{\sqrt{\operatorname{cosec}^2\theta-1}}{\operatorname{cosec}\theta}$
$\tan\theta$	$\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\frac{1}{\cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\frac{1}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\cot\theta$	$\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{\tan\theta}$	$\cot\theta$	$\frac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\operatorname{cosec}^2\theta-1}$
$\sec\theta$	$\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\cos\theta}$	$\sqrt{1+\tan^2\theta}$	$\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$\sec\theta$	$\frac{\operatorname{cosec}\theta}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\operatorname{cosec}\theta$	$\frac{1}{\sin\theta}$	$\frac{1}{\sqrt{1-\cos^2\theta}}$	$\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\sqrt{1+\cot^2\theta}$	$\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\operatorname{cosec}\theta$

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Illustration 2 : If $\sin\theta + \sin^2\theta = 1$, then prove that $\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta - 1 = 0$

Solution : Given that $\sin\theta = 1 - \sin^2\theta = \cos^2\theta$

L.H.S. $= \cos^6\theta(\cos^2\theta + 1)^3 - 1 = \sin^3\theta(1 + \sin\theta)^3 - 1 = (\sin\theta + \sin^2\theta)^3 - 1 = 1 - 1 = 0$

Illustration 3 : $4(\sin^6\theta + \cos^6\theta) - 6(\sin^4\theta + \cos^4\theta)$ is equal to

(A) 0 (B) 1 (C) –2 (D) none of these

Solution : $4[(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)] - 6[(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta]$

$= 4[1 - 3\sin^2\theta\cos^2\theta] - 6[1 - 2\sin^2\theta\cos^2\theta]$

$= 4 - 12\sin^2\theta\cos^2\theta - 6 + 12\sin^2\theta\cos^2\theta = -2$ **Ans.(C)**

Do yourself - 2 :

(i) If $\cot\theta = \frac{4}{3}$, then find the value of $\sin\theta$, $\cos\theta$ and $\text{cosec}\theta$ in first quadrant.

(ii) If $\sin\theta + \text{cosec}\theta = 2$, then find the value of $\sin^8\theta + \text{cosec}^8\theta$

4. NEW DEFINITION OF T-RATIOS :

By using rectangular coordinates the definitions of trigonometric functions can be extended to angles of any size in the following way (see diagram). A point P is taken with coordinates (x, y). The radius vector OP has length r and the angle θ is taken as the directed angle measured anticlockwise from the x-axis. The three main trigonometric functions are then defined in terms of r and the coordinates x and y.

$\sin\theta = y/r$,

$\cos\theta = x/r$

$\tan\theta = y/x$,

(The other function are reciprocals of these)

This can give negative values of the trigonometric functions.

5. SIGNS OF TRIGONOMETRIC FUNCTIONS IN DIFFERENT QUADRANTS :

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6. TRIGONOMETRIC FUNCTIONS OF ALLIED ANGLES :

(a) $\sin(2n\pi + \theta) = \sin\theta$, $\cos(2n\pi + \theta) = \cos\theta$, where $n \in I$

(b)

$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$
$\sin(90° - \theta) = \cos\theta$	$\cos(90° - \theta) = \sin\theta$
$\sin(90° + \theta) = \cos\theta$	$\cos(90° + \theta) = -\sin\theta$
$\sin(180° - \theta) = \sin\theta$	$\cos(180° - \theta) = -\cos\theta$
$\sin(180° + \theta) = -\sin\theta$	$\cos(180° + \theta) = -\cos\theta$
$\sin(270° - \theta) = -\cos\theta$	$\cos(270° - \theta) = -\sin\theta$
$\sin(270° + \theta) = -\cos\theta$	$\cos(270° + \theta) = \sin\theta$
$\sin(360° - \theta) = -\sin\theta$	$\cos(360° - \theta) = \cos\theta$
$\sin(360° + \theta) = \sin\theta$	$\cos(360° + \theta) = \cos\theta$

7. VALUES OF T-RATIOS OF SOME STANDARD ANGLES :

Angles / T-ratio	0°	30°	45°	60°	90°	180°	270°
	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$
$\sin\theta$	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0	–1
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0	–1	0
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	N.D.	0	N.D.
$\cot\theta$	N.D.	$\sqrt{3}$	1	$1/\sqrt{3}$	0	N.D.	0
$\sec\theta$	1	$2/\sqrt{3}$	$\sqrt{2}$	2	N.D.	–1	N.D.
$\operatorname{cosec}\theta$	N.D.	2	$\sqrt{2}$	$2/\sqrt{3}$	1	N.D.	–1

N.D. $\rightarrow$ Not Defined

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$ where $n \in I$

(b) $\sin(2n+1)\frac{\pi}{2} = (-1)^n$; $\cos(2n+1)\frac{\pi}{2} = 0$ where $n \in I$

Illustration 4 : If $\sin\theta = -\frac{1}{2}$ and $\tan\theta = \frac{1}{\sqrt{3}}$ then θ is equal to -

(A) 30° (B) 150° (C) 210° (D) none of these

Solution : Let us first find out θ lying between 0 and 360°.

Since $\sin\theta = -\frac{1}{2} \Rightarrow \theta = 210°$ or $330°$ and $\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = 30°$ or $210°$

Hence , $\theta = 210°$ or $\frac{7\pi}{6}$ is the value satisfying both. **Ans. (C)**

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Do yourself - 3 :

(i) If $\cos\theta = -\frac{1}{2}$ and $\pi < \theta < \frac{3\pi}{2}$, then find the value of $4\tan^2\theta - 3\text{cosec}^2\theta$.

(ii) Prove that : (a) $\cos 570° \sin 510° + \sin(-330°)\cos(-390°) = 0$

(b) $\tan\frac{11\pi}{3} - 2\sin\frac{9\pi}{3} - \frac{3}{4}\text{cosec}^2\frac{\pi}{4} + 4\cos^2\frac{17\pi}{6} = \frac{3-2\sqrt{3}}{2}$

8. GRAPH OF TRIGONOMETRIC FUNCTIONS :

(i) y = sinx

(ii) y = cosx

(iii) y = tanx

(iv) y = cotx

(v) y = secx

(vi) y = cosecx

9. DOMAINS, RANGES AND PERIODICITY OF TRIGONOMETRIC FUNCTIONS :

T-Ratio	Domain	Range	Period
sin x	R	[–1,1]	2π
cos x	R	[–1,1]	2π
tan x	$R-\{(2n+1)\pi/2 ; n\in I\}$	R	π
cot x	$R-\{n\pi : n \in I\}$	R	π
sec x	$R-\{(2n+1)\pi/2 : n \in I\}$	$(-\infty,-1]\cup[1,\infty)$	2π
cosec x	$R-\{n\pi : n \in I\}$	$(-\infty,-1]\cup[1,\infty)$	2π

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10. TRIGONOMETRIC RATIOS OF THE SUM & DIFFERENCE OF TWO ANGLES :

(i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$. **(ii)** $\sin(A - B) = \sin A \cos B - \cos A \sin B$.

(iii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$ **(iv)** $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(v) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ **(vi)** $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(vii) $\cot(A + B) = \dfrac{\cot B \cot A - 1}{\cot B + \cot A}$ **(viii)** $\cot(A - B) = \dfrac{\cot B \cot A + 1}{\cot B - \cot A}$

Some more results :

(i) $\sin^2 A - \sin^2 B = \sin(A + B).\sin(A - B) = \cos^2 B - \cos^2 A$.

(ii) $\cos^2 A - \sin^2 B = \cos(A+B).\cos(A - B)$.

Illustration 5 : Prove that $\sqrt{3}\ \text{cosec}20° - \sec 20° = 4$.

Solution : L.H.S. $= \dfrac{\sqrt{3}}{\sin 20°} - \dfrac{1}{\cos 20°} = \dfrac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20°.\cos 20°}$

$$= \frac{4\left(\frac{\sqrt{3}}{2}\cos 20° - \frac{1}{2}\sin 20°\right)}{2\sin 20° \cos 20°} = \frac{4(\sin 60.\cos 20° - \cos 60°.\sin 20°)}{\sin 40°}$$

$$= 4.\frac{\sin(60° - 20°)}{\sin 40°} = 4.\frac{\sin 40°}{\sin 40°} = 4 = \text{R.H.S.}$$

Illustration 6 : Prove that $\tan 70° = \cot 70° + 2\cot 40°$.

Solution : L.H.S. $= \tan 70° = \tan(20° + 50°) = \dfrac{\tan 20° + \tan 50°}{1 - \tan 20° \tan 50°}$

or $\tan 70° - \tan 20° \tan 50° \tan 70° = \tan 20° + \tan 50°$

or $\tan 70° = \tan 70° \tan 50° \tan 20° + \tan 20° + \tan 50° = 2\tan 50° + \tan 20°$

$= \cot 70° + 2\cot 40°$ = R.H.S.

Do yourself - 4 :

(i) If $\sin A = \dfrac{3}{5}$ and $\cos B = \dfrac{9}{41}$, $0 < A \,\&\, B < \dfrac{\pi}{2}$, then find the value of the following :

(a) $\sin(A + B)$ (b) $\sin(A - B)$ (c) $\cos(A + B)$ (d) $\cos(A - B)$

(ii) If $x + y = 45°$, then prove that :

(a) $(1 + \tan x)(1 + \tan y) = 2$ (b) $(\cot x - 1)(\cot y - 1) = 2$

(Remember these results)

11. FORMULAE TO TRANSFORM THE PRODUCT INTO SUM OR DIFFERENCE :

(i) $2\sin A \cos B = \sin(A + B) + \sin(A - B)$. **(ii)** $2\cos A \sin B = \sin(A + B) - \sin(A - B)$.

(iii) $2\cos A \cos B = \cos(A + B) + \cos(A - B)$ **(iv)** $2\sin A \sin B = \cos(A - B) - \cos(A + B)$

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Illustration 7: If sin2A = λ sin2B, then prove that $\frac{\tan(A+B)}{\tan(A-B)}=\frac{\lambda+1}{\lambda-1}$.

Solution : Given sin2A = λ sin2B

$$\Rightarrow \frac{\sin 2A}{\sin 2B}=\frac{\lambda}{1}$$

Applying componendo & dividendo,

$$\frac{\sin 2A+\sin 2B}{\sin 2B-\sin 2A}=\frac{\lambda+1}{1-\lambda}$$

$$\Rightarrow \frac{2\sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right)}{2\cos\left(\frac{2B+2A}{2}\right)\sin\left(\frac{2B-2A}{2}\right)}=\frac{\lambda+1}{1-\lambda}$$

$$\Rightarrow \frac{\sin(A+B)\cos(A-B)}{\cos(A+B)\sin\{-(A-B)\}}=\frac{\lambda+1}{1-\lambda} \quad \Rightarrow \frac{\sin(A+B)\cos(A-B)}{\cos(A+B)\times-\sin(A-B)}=\frac{\lambda+1}{-(\lambda-1)}$$

$$\Rightarrow \frac{\sin(A+B)\cos(A-B)}{\cos(A+B)\sin(A-B)}=\frac{\lambda+1}{\lambda-1} \quad \Rightarrow \tan(A+B)\cot(A-B)=\frac{\lambda+1}{\lambda-1}$$

$$\Rightarrow \frac{\tan(A+B)}{\tan(A-B)}=\frac{\lambda+1}{\lambda-1}$$

12. FORMULAE TO TRANSFORM SUM OR DIFFERENCE INTO PRODUCT :

(i) $\sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(ii) $\sin C-\sin D=2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(iii) $\cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(iv) $\cos C-\cos D=2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

Illustration 8 : $\frac{\sin 5\theta+\sin 2\theta-\sin\theta}{\cos 5\theta+2\cos 3\theta+2\cos^2\theta+\cos\theta}$ is equal to -

(A) tan θ (B) cos θ (C) cot θ (D) none of these

Solution : L.H.S.= $\frac{2\sin 2\theta\cos 3\theta+\sin 2\theta}{2\cos 3\theta.\cos 2\theta+2\cos 3\theta+2\cos^2\theta}=\frac{\sin 2\theta[2\cos 3\theta+1]}{2[\cos 3\theta(\cos 2\theta+1)+(\cos^2\theta)]}$

$$=\frac{\sin 2\theta[2\cos 3\theta+1]}{2[\cos 3\theta(2\cos^2\theta)+\cos^2\theta]}=\frac{\sin 2\theta(2\cos 3\theta+1)}{2\cos^2\theta(2\cos 3\theta+1)}=\tan\theta$$

Ans. (A)

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Illustration 9 : Show that sin12°.sin48°.sin54° = 1/8

Solution : L.H.S. $= \frac{1}{2}[\cos 36° - \cos 60°]\sin 54° = \frac{1}{2}\left[\cos 36° \sin 54° - \frac{1}{2}\sin 54°\right]$

$$= \frac{1}{4}[2\cos 36° \sin 54° - \sin 54°] = \frac{1}{4}[\sin 90° + \sin 18° - \sin 54°]$$

$$= \frac{1}{4}[1 - (\sin 54° - \sin 18°)] = \frac{1}{4}[1 - 2\sin 18° \cos 36°]$$

$$= \frac{1}{4}\left[1 - \frac{2\sin 18°}{\cos 18°}\cos 18° \cos 36°\right] = \frac{1}{4}\left[1 - \frac{\sin 36° \cos 36°}{\cos 18°}\right]$$

$$= \frac{1}{4}\left[1 - \frac{2\sin 36° \cos 36°}{2\cos 18°}\right] = \frac{1}{4}\left[1 - \frac{\sin 72°}{2\sin 72°}\right] = \frac{1}{4}\left[1 - \frac{1}{2}\right] = \frac{1}{8} = \text{R.H.S.}$$

Do yourself - 5 :

(i) Simplify $\frac{\sin 75° - \sin 15°}{\cos 75° + \cos 15°}$

(ii) Prove that

(a) (sin3A + sinA)sinA + (cos3A – cosA)cosA = 0

(b) $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}$

(c) $\frac{\sin 8\theta \cos\theta - \sin 6\theta \cos 3\theta}{\cos 2\theta \cos\theta - \sin 3\theta \sin 4\theta} = \tan 2\theta$

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13. TRIGONOMETRIC RATIOS OF SUM OF MORE THAN TWO ANGLES :

(i) sin (A+B+C) = sinAcosBcosC + sinBcosAcosC + sinCcosAcosB – sinAsinBsinC

= ΣsinA cosB cosC – Πsin A

= cosA cosB cosC [tanA + tanB + tanC – tanA tanB tanC]

(ii) cos (A+B+C) = cosA cosB cosC – sinA sinB cosC – sinA cosB sinC – cosA sinB sinC

= Πcos A – Σsin A sin B cos C

= cos A cos B cos C [1 – tan A tan B – tan B tan C – tan C tan A]

(iii) $\tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A} = \frac{S_1 - S_3}{1 - S_2}$

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14. TRIGONOMETRIC RATIOS OF MULTIPLE ANGLES :

(a) Trigonometrical ratios of an angle 2θ in terms of the angle θ :

(i) $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

(ii) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

(iii) $1 + \cos 2\theta = 2\cos^2\theta$ **(iv)** $1 - \cos 2\theta = 2\sin^2\theta$

(v) $\tan\theta = \dfrac{1-\cos 2\theta}{\sin 2\theta} = \dfrac{\sin 2\theta}{1+\cos 2\theta}$ **(vi)** $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Illustration 10: Prove that : $\dfrac{2\cos 2A + 1}{2\cos 2A - 1} = \tan(60° + A)\tan(60° - A)$.

Solution : R.H.S. = $\tan(60° + A)\tan(60° - A)$

$$= \left(\frac{\tan 60° + \tan A}{1 - \tan 60° \tan A}\right)\left(\frac{\tan 60° - \tan A}{1 + \tan 60° \tan A}\right) = \left(\frac{\sqrt{3} + \tan A}{1 - \sqrt{3}\tan A}\right)\left(\frac{\sqrt{3} - \tan A}{1 + \sqrt{3}\tan A}\right)$$

$$= \frac{3 - \tan^2 A}{1 - 3\tan^2 A} = \frac{3 - \dfrac{\sin^2 A}{\cos^2 A}}{1 - 3\dfrac{\sin^2 A}{\cos^2 A}} = \frac{3\cos^2 A - \sin^2 A}{\cos^2 A - 3\sin^2 A} == \frac{2\cos^2 A + \cos^2 A - 2\sin^2 A + \sin^2 A}{2\cos^2 A - 2\sin^2 A - \sin^2 A - \cos^2 A}$$

$$= \frac{2(\cos^2 A - \sin^2 A) + \cos^2 A + \sin^2 A}{2(\cos^2 A - \sin^2 A) - (\sin^2 A + \cos^2 A)} = \frac{2\cos 2A + 1}{2\cos 2A - 1} = \text{L.H.S.}$$

Do yourself - 6 :

(i) Prove that :

(a) $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$ (b) $\dfrac{1+\sin 2\theta + \cos 2\theta}{1+\sin 2\theta - \cos 2\theta} = \cot\theta$

(b) Trigonometrical ratios of an angle 3θ in terms of the angle θ :

(i) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. **(ii)** $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(iii) $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

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Illustration 11 : Prove that : $\tan A + \tan(60° + A) + \tan(120° + A) = 3\tan 3A$

Solution : L.H.S. $= \tan A + \tan(60° + A) + \tan(120° + A)$

$= \tan A + \tan(60° + A) + \tan\{180° - (60° - A)\}$

$= \tan A + \tan(60° + A) - \tan(60° - A)$ $\quad [\because \tan(180° - \theta) = -\tan\theta]$

$$= \tan A + \frac{\tan 60° + \tan A}{1 - \tan 60° \tan A} - \frac{\tan 60° - \tan A}{1 + \tan 60° \tan A} = \tan A + \frac{\sqrt{3} + \tan A}{1 - \sqrt{3}\tan A} - \frac{\sqrt{3} - \tan A}{1 + \sqrt{3}\tan A}$$

$$= \tan A + \frac{\sqrt{3} + \tan A + 3\tan A + \sqrt{3}\tan^2 A - \sqrt{3} + \tan A + 3\tan A - \sqrt{3}\tan^2 A}{(1 - \sqrt{3}\tan A)(1 + \sqrt{3}\tan A)}$$

$$= \tan A + \frac{8\tan A}{1 - 3\tan^2 A} = \frac{\tan A - 3\tan^3 A + 8\tan A}{1 - 3\tan^2 A}$$

$$= \frac{9\tan A - 3\tan^3 A}{1 - 3\tan^2 A} = 3\left(\frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}\right) = 3\tan 3A = \text{R.H.S.}$$

Do yourself - 7 :

(i) Prove that :

(a) $\cot\theta \cot(60° - \theta) \cot(60° + \theta) = \cot 3\theta$ (b) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(c) $\sin 4\theta = 4\sin\theta\cos^3\theta - 4\cos\theta\sin^3\theta$

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15. TRIGONOMETRIC RATIOS OF SUB MULTIPLE ANGLES :

Since the trigonometric relations are true for all values of angle θ, they will be true if instead of θ be substitute $\frac{\theta}{2}$

(i) $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \dfrac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$

(ii) $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 \quad = 1 - 2\sin^2\frac{\theta}{2} = \dfrac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$

(iii) $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$

(iv) $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$

(v) $\tan\frac{\theta}{2} = \dfrac{1 - \cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1 + \cos\theta}$

(vi) $\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}}$

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(vii) $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$

(viii) $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$

(ix) $\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

(x) $2\sin\frac{\theta}{2} = \pm\sqrt{1+\sin\theta} \pm \sqrt{1-\sin\theta}$

(xi) $2\cos\frac{\theta}{2} = \pm\sqrt{1+\sin\theta} \mp \sqrt{1-\sin\theta}$

(xii) $\tan\frac{\theta}{2} = \frac{\pm\sqrt{1+\tan^2\theta}-1}{\tan\theta}$

for **(vii)** to **(xii)** , we decide the sign of ratio according to value of θ.

Illustration 12: $\sin\ 67\frac{1}{2}^\circ + \cos 67\frac{1}{2}^\circ$ is equal to

(A) $\frac{1}{2}\sqrt{4+2\sqrt{2}}$ (B) $\frac{1}{2}\sqrt{4-2\sqrt{2}}$ (C) $\frac{1}{4}\left(\sqrt{4+2\sqrt{2}}\right)$ (D) $\frac{1}{4}\left(\sqrt{4-2\sqrt{2}}\right)$

Solution : $\sin 67\frac{1}{2}^\circ + \cos 67\frac{1}{2}^\circ = \sqrt{1+\sin 135^\circ} = \sqrt{1+\frac{1}{\sqrt{2}}}$ (using cosA + sinA = $\sqrt{1+\sin 2A}$)

$= \frac{1}{2}\sqrt{4+2\sqrt{2}}$ **Ans.(A)**

Do yourself - 8 :

(i) Find the value of

(a) $\sin\frac{\pi}{8}$ (b) $\cos\frac{\pi}{8}$ (c) $\tan\frac{\pi}{8}$

16. TRIGONOMETRIC RATIOS OF SOME STANDARD ANGLES :

(i) $\sin 18^\circ = \sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4} = \cos 72^\circ = \cos\frac{2\pi}{5}$

(ii) $\cos 36^\circ = \cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4} = \sin 54^\circ = \sin\frac{3\pi}{10}$

(iii) $\sin 72^\circ = \sin\frac{2\pi}{5} = \frac{\sqrt{10+2\sqrt{5}}}{4} = \cos 18^\circ = \cos\frac{\pi}{10}$

(iv) $\sin 36^\circ = \sin\frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54^\circ = \cos\frac{3\pi}{10}$

(v) $\sin 15^\circ = \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75^\circ = \cos\frac{5\pi}{12}$

(vi) $\cos 15^\circ = \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75^\circ = \sin\frac{5\pi}{12}$

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(vii) $\tan 15° = \tan\frac{\pi}{12} = 2-\sqrt{3} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \cot 75° = \cot\frac{5\pi}{12}$

(viii) $\tan 75° = \tan\frac{5\pi}{12} = 2+\sqrt{3} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \cot 15° = \cot\frac{\pi}{12}$

(ix) $\tan(22.5°) = \tan\frac{\pi}{8} = \sqrt{2}-1 = \cot(67.5°) = \cot\frac{3\pi}{8}$

(x) $\tan(67.5°) = \tan\frac{3\pi}{8} = \sqrt{2}+1 = \cot(22.5°) = \cot\frac{\pi}{8}$

Illustration 13: Evaluate sin78° – sin66° – sin42° + sin6°.

Solution : The expression $= (\sin 78° - \sin 42°) - (\sin 66° - \sin 6°) = 2\cos(60°)\sin(18°) - 2\cos 36°.\sin 30°$

$= \sin 18° - \cos 36° = \left(\frac{\sqrt{5}-1}{4}\right) - \left(\frac{\sqrt{5}+1}{4}\right) = -\frac{1}{2}$

Do yourself - 9 :

(i) Find the value of

(a) $\sin\frac{\pi}{10} + \sin\frac{13\pi}{10}$ (b) $\cos^2 48° - \sin^2 12°$

17. CONDITIONAL **TRIGONOMETRIC IDENTITIES :**

If A + B + C = 180°, then

(i) tan A + tan B + tan C = tan A tan B tan C

(ii) cot A cot B + cot B cot C + cot C cot A = 1

(iii) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

(iv) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(v) sin 2A + sin 2B + sin 2C = 4 sinA sinB sinC

(vi) cos 2A + cos 2B + cos 2C =–1–4 cosA cosB cosC

(vii) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(viii) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Illustration 14: In any triangle ABC, sin A – cos B = cos C, then angle B is

(A) π/2 (B) π/3 (C) π/4 (D) π/6

Solution : We have , sin A – cos B = cos C

sin A = cos B + cos C

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$$\Rightarrow \quad 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)$$

$$\Rightarrow \quad 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\cos\left(\frac{\pi-A}{2}\right)\cos\left(\frac{B-C}{2}\right) \qquad \because \quad A+B+C=\pi$$

$$\Rightarrow \quad 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\sin\frac{A}{2}\cos\left(\frac{B-C}{2}\right)$$

$$\Rightarrow \quad \cos\frac{A}{2} = \cos\frac{B-C}{2} \text{ or } A = B - C \;\; ; \text{ But } A+B+C=\pi$$

Therefore $2B = \pi \Rightarrow B = \pi/2$ **Ans.(A)**

Illustration 15: If $A + B + C = \frac{3\pi}{2}$, then cos 2A + cos 2B + cos2C is equal to-

(A) 1 – 4cosA cosB cosC (B) 4 sinA sin B sinC

(C) 1 + 2cosA cosB cosC (D) 1 – 4 sinA sinB sinC

Solution : cos 2A + cos 2B + cos 2C = 2 cos (A + B) cos (A – B) + cos 2C

$$= 2\cos\left(\frac{3\pi}{2} - C\right)\cos(A-B) + \cos 2C \qquad \because A+B+C = \frac{3\pi}{2}$$

$$= -2\sin C\cos(A-B) + 1 - 2\sin^2 C = 1 - 2\sin C[\cos(A-B) + \sin C)$$

$$= 1 - 2\sin C\left[\cos(A-B) + \sin\left(\frac{3\pi}{2} - (A+B)\right)\right]$$

$$= 1 - 2\sin C[\cos(A-B) - \cos(A+B)] = 1 - 4\sin A\sin B\sin C$$

Ans.(D)

Do yourself - 10 :

(i) If ABCD is a cyclic quadrilateral, then find the value of sinA + sinB – sinC – sinD

(ii) If $A + B + C = \frac{\pi}{2}$, then find the value of tanA tanB + tanBtanC + tanC tanA

18. MAXIMUM & MINIMUM VALUES OF TRIGONOMETRIC EXPRESSIONS :

(i) $a\cos\theta + b\sin\theta$ will always lie in the interval $[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$ i.e. the maximum and minimum values are $\sqrt{a^2+b^2}, -\sqrt{a^2+b^2}$ respectively.

(ii) Minimum value of $a^2\tan^2\theta + b^2\cot^2\theta = 2ab$ where a, b > 0

(iii) $-\sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} \le a\cos(\alpha+\theta) + b\cos(\beta+\theta) \le \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)}$ where α and β areknown angles.

(iv) In case a quadratic in sin θ & cos θ is given then the maximum or minimum values can be obtained by making perfect square.

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Illustration 16: Prove that : $-4 \le 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 \le 10$, for all values of θ.

Solution : We have, $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) = 5\cos\theta + 3\cos\theta\cos\frac{\pi}{3} - 3\sin\theta\sin\frac{\pi}{3} = \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta$

Since, $-\sqrt{\left(\frac{13}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2} \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le \sqrt{\left(\frac{13}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2}$

$\Rightarrow \quad -7 \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le 7$

$\Rightarrow \quad -7 \le 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) \le 7$ for all θ.

$\Rightarrow \quad -7 + 3 \le 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 \le 7 + 3$ for all θ.

$\Rightarrow \quad -4 \le 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 \le 10$ for all θ.

Illustration 17: Find the maximum value of $1 + \sin\left(\frac{\pi}{4} + \theta\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$ -

(A) 1 (B) 2 (C) 3 (D) 4

Solution : We have $1 + \sin\left(\frac{\pi}{4} + \theta\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$

$= 1 + \frac{1}{\sqrt{2}}(\cos\theta + \sin\theta) + \sqrt{2}(\cos\theta + \sin\theta) = 1 + \left(\frac{1}{\sqrt{2}} + \sqrt{2}\right)(\cos\theta + \sin\theta)$

$= 1 + \left(\frac{1}{\sqrt{2}} + \sqrt{2}\right) . \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$

$\therefore$ maximum value $= 1 + \left(\frac{1}{\sqrt{2}} + \sqrt{2}\right) . \sqrt{2} = 4$ **Ans. (D)**

Do yourself - 11 :

(i) Find maximum and minimum value of $5\cos\theta + 3\sin\left(\theta + \frac{\pi}{6}\right)$ for all real values of θ.

(ii) Find the minimum value of $\cos\theta + \cos2\theta$ for all real values of θ.

(iii) Find maximum and minimum value of $\cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2$.

19. IMPORTANT RESULTS :

(i) $\sin\theta\sin(60° - \theta)\sin(60° + \theta) = \frac{1}{4}\sin3\theta$

(ii) $\cos\theta.\cos(60° - \theta)\cos(60° + \theta) = \frac{1}{4}\cos3\theta$

(iii) $\tan\theta\tan(60° - \theta)\tan(60° + \theta) = \tan3\theta$

(iv) $\cot\theta\cot(60° - \theta)\cot(60° + \theta) = \cot3\theta$

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(v) (a) $\sin^2\theta + \sin^2(60° + \theta) + \sin^2(60° - \theta) = \frac{3}{2}$

(b) $\cos^2\theta + \cos^2(60° + \theta) + \cos^2(60° - \theta) = \frac{3}{2}$

(c) $\tan\theta + \tan(60° + \theta) + \tan(120° + \theta) = 3\tan 3\theta$

(vi) (a) If $\tan A + \tan B + \tan C = \tan A \tan B \tan C$, then $A + B + C = n\pi, n \in I$

(b) If $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$, then $A + B + C = (2n + 1)\frac{\pi}{2}, n \in I$

(vii) $\cos\theta \cos 2\theta \cos 4\theta \ldots \cos(2^{n-1}\theta) = \frac{\sin(2^n\theta)}{2^n \sin\theta}$

(viii) (a) $\cot A - \tan A = 2\cot 2A$ (b) $\cot A + \tan A = 2\operatorname{cosec} 2A$

(ix) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots \sin(\alpha + \overline{n-1}\,\beta) = \dfrac{\sin\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

(x) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha + 2\beta) + \ldots + \cos(\alpha + \overline{n-1}\,\beta) = \dfrac{\cos\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

Do yourself - 12 :

(i) Evaluate $\sin\frac{\pi}{n} + \sin\frac{3\pi}{n} + \sin\frac{5\pi}{n} + \ldots\ldots\ldots$ to n terms

Miscellaneous Illustration :

Illustration 18: Prove that

$\tan\alpha + 2\tan 2\alpha + 2^2\tan^2\alpha + \ldots\ldots + 2^{n-1}\tan 2^{n-1}\alpha + 2^n \cot 2^n\alpha = \cot\alpha$

Solution : We know $\tan\theta = \cot\theta - 2\cot 2\theta$ .....(i)

Putting $\theta = \alpha, 2\alpha, 2^2\alpha, \ldots\ldots\ldots\ldots$in (i), we get

$\tan\alpha = (\cot\alpha - 2\cot 2\alpha)$

$2(\tan 2\alpha) = 2(\cot 2\alpha - 2\cot 2^2\alpha)$

$2^2(\tan 2^2\alpha) = 2^2(\cot 2^2\alpha - 2\cot 2^3\alpha)$

.....

$2^{n-1}(\tan 2^{n-1}\alpha) = 2^{n-1}(\cot 2^{n-1}\alpha - 2\cot 2^n\alpha)$

Adding,

$\tan\alpha + 2\tan 2\alpha + 2^2\tan^2\alpha + \ldots\ldots + 2^{n-1}\tan 2^{n-1}\alpha = \cot\alpha - 2^n\cot 2^n\alpha$

$\therefore$ $\tan\alpha + 2\tan 2\alpha + 2^2\tan^2\alpha + \ldots\ldots + 2^{n-1}\tan 2^{n-1}\alpha + 2^n\cot 2^n\alpha = \cot\alpha$

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Illustration 19: If A,B,C and D are angles of a quadrilateral and $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}=\frac{1}{4}$, prove that A = B = C = D = π/2.

Solution : $\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right)\left(2\sin\frac{C}{2}\sin\frac{D}{2}\right)=1$

$$\Rightarrow \left\{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right\}\left\{\cos\left(\frac{C-D}{2}\right)-\cos\left(\frac{C+D}{2}\right)\right\}=1$$

Since, A + B = 2π – (C + D), the above equation becomes,

$$\Rightarrow \left\{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right\}\left\{\cos\left(\frac{C-D}{2}\right)+\cos\left(\frac{A+B}{2}\right)\right\}=1$$

$$\Rightarrow \cos^2\left(\frac{A+B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\left\{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{C-D}{2}\right)\right\}+1-\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{C-D}{2}\right)=0$$

This is a quadratic equation in $\cos\left(\frac{A+B}{2}\right)$ which has real roots.

$$\Rightarrow \left\{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{C-D}{2}\right)\right\}^2-4\left\{1-\cos\left(\frac{A-B}{2}\right).\cos\left(\frac{C-D}{2}\right)\right\}\geq 0$$

$$\left(\cos\frac{A-B}{2}+\cos\frac{C-D}{2}\right)^2\geq 4$$

$\Rightarrow \cos\frac{A-B}{2}+\cos\frac{C-D}{2}\geq 2$, Now both $\cos\frac{A-B}{2}$ and $\cos\frac{C-D}{2}\leq 1$

$\Rightarrow \cos\frac{A-B}{2}=1 \;\&\; \cos\frac{C-D}{2}=1$

$\Rightarrow \frac{A-B}{2}=0=\frac{C-D}{2}$

$\Rightarrow$ A = B, C = D.

Similarly A = C, B = D ⇒ A = B = C = D = π/2

ANSWERS FOR DO YOURSELF

1 :	**(i)**	10π cm		**2 :**	**(i)**	$\frac{3}{5},\frac{4}{5},\frac{5}{3}$					**(ii)**	2			
3 :	**(i)**	8		**4 :**	**(i)**	(a) $\frac{187}{205}$	(b) $\frac{-133}{205}$	(c) $\frac{-84}{205}$	(d) $\frac{156}{205}$						
5 :	**(i)**	$\frac{1}{\sqrt{3}}$		**8:**	**(i)**	(a) $\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$		(b) $\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$		(c) $\sqrt{2}-1$					
9:	**(i)**	(a) $-\frac{1}{2}$	(b) $\frac{\sqrt{5}+1}{8}$	**10 :**	**(i)**	0			**(ii)**	1					
11 :	**(i)**	7 & –7	**(ii)** $-\frac{9}{8}$		**(iii)**	$4+\sqrt{10}$ & $4-\sqrt{10}$			**12 :**	**(i)**	0				

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EXERCISE (O-1)

1. If $\sin x + \sin^2 x = 1$, then the value of $\cos^2 x + \cos^4 x$ is -

(A) 0 (B) 2 (C) 1 (D) 3

2. $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1$ is equal to -

(A) 2 (B) 0 (C) 4 (D) 6

3. If $\tan A = -\frac{1}{2}$ and $\tan B = -\frac{1}{3}$, (where A,B > 0), then A + B can be

(A) $\frac{\pi}{4}$ (B) $\frac{3\pi}{4}$ (C) $\frac{5\pi}{4}$ (D) $\frac{7\pi}{4}$

4. $\cos^2 48° - \sin^2 12°$ is equal to -

(A) $\frac{\sqrt{5}-1}{4}$ (B) $\frac{\sqrt{5}+1}{8}$ (C) $\frac{\sqrt{3}-1}{4}$ (D) $\frac{\sqrt{3}+1}{2\sqrt{2}}$

5. The expression $\frac{\sin 8\theta\cos\theta - \sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta - \sin 3\theta\sin 4\theta}$ is equals -

(A) $\tan\theta$ (B) $\tan 2\theta$ (C) $\sin 2\theta$ (D) $\cos 2\theta$

6. If $3\sin\alpha = 5\sin\beta$, then $\frac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}} =$

(A) 1 (B) 2 (C) 3 (D) 4

7. $\frac{\sin(A-C) + 2\sin A + \sin(A+C)}{\sin(B-C) + 2\sin B + \sin(B+C)}$ is equal to -

(A) $\tan A$ (B) $\frac{\sin A}{\sin B}$ (C) $\frac{\cos A}{\cos B}$ (D) $\frac{\sin C}{\cos B}$

8. $\frac{1+\sin 2\theta + \cos 2\theta}{1+\sin 2\theta - \cos 2\theta} =$

(A) $\frac{1}{2}\tan\theta$ (B) $\frac{1}{2}\cot\theta$ (C) $\tan\theta$ (D) $\cot\theta$

9. If $A = \tan 6° \tan 42°$ and $B = \cot 66° \cot 78°$, then -

(A) A = 2B (B) A = 1/3B (C) A = B (D) 3A = 2B

10. If $x = y\cos\frac{2\pi}{3} = z\cos\frac{4\pi}{3}$, then xy + yz + zx =

(A) –1 (B) 0 (C) 1 (D) 2

11. If $\tan\alpha = (1+2^{-x})^{-1}$, $\tan\beta = (1+2^{x+1})^{-1}$, then $\alpha + \beta =$

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $\pi/2$

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12. If tan A + tan B + tan C = tan A. tan B. tan C, then -

(A) A,B,C must be angles of a triangle

(B) the sum of any two of A,B,C is equal to the third

(C) A+B+C must be n integral multiple of π

(D) None of these

13. The value of $\sin 10^\circ + \sin 20^\circ + \sin 30^\circ + ... + \sin 360^\circ$ is equal to -

(A) 0 (B) 1 (C) $\sqrt{3}$ (D) 2

14. The number of real solutions of the equation $\sin(e^x) = 2^x + 2^{-x}$ is -

(A) 1 (B) 0 (C) 2 (D) Infinite

15. If $f(x) = \frac{\sin 3x}{\sin x}$, $x \neq n\pi$, then the range of values of f(x) for real values of x is -

(A) $[-1,3]$ (B) $(-\infty,-1]$ (C) $(3, +\infty)$ (D) $[-1,3)$

16. If $\cos x + \cos y + \cos \alpha = 0$ and $\sin x + \sin y + \sin \alpha = 0$, then $\cot\left(\frac{x+y}{2}\right) =$

(A) $\sin \alpha$ (B) $\cos \alpha$ (C) $\cot \alpha$ (D) $2 \sin \alpha$

17. The value of $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$ is :-

(A) $\frac{1}{16}$ (B) $\frac{1}{8}$ (C) $\frac{1}{2}$ (D) 1

18. Maximum and minimum value of $2\sin^2\theta - 3\sin\theta + 2$ is -

(A) $\frac{1}{4}, -\frac{7}{4}$ (B) $\frac{1}{4}, \frac{21}{4}$ (C) $\frac{21}{4}, -\frac{3}{4}$ (D) $7, \frac{7}{8}$

19. For $\theta \in (0, \pi/2)$, the maximum value of $\sin\left(\theta+\frac{\pi}{6}\right)+\cos\left(\theta+\frac{\pi}{6}\right)$ is attained at $\theta =$

(A) $\frac{\pi}{12}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$

20. Minimum value of the expression $\cos^2\theta - (6 \sin\theta \cos\theta) + 3 \sin^2\theta + 2$, is -

(A) $4+\sqrt{10}$ (B) $4-\sqrt{10}$ (C) 0 (D) 4

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EXERCISE (S-1)

1. Prove that : $\cos^2\alpha + \cos^2(\alpha+\beta) - 2\cos\alpha\cos\beta\cos(\alpha+\beta) = \sin^2\beta$

2. Prove that : $\cos 2\alpha = 2\sin^2\beta + 4\cos(\alpha+\beta)\sin\alpha\sin\beta + \cos 2(\alpha+\beta)$

3. Prove that : $\sin^4\frac{\pi}{16} + \sin^4\frac{3\pi}{16} + \sin^4\frac{5\pi}{16} + \sin^4\frac{7\pi}{16} = \frac{3}{2}$

4. If $X = \sin\left(\theta+\frac{7\pi}{12}\right)+\sin\left(\theta-\frac{\pi}{12}\right)+\sin\left(\theta+\frac{3\pi}{12}\right), Y = \cos\left(\theta+\frac{7\pi}{12}\right)+\cos\left(\theta-\frac{\pi}{12}\right)+\cos\left(\theta+\frac{3\pi}{12}\right)$ then prove that $\frac{X}{Y}-\frac{Y}{X} = 2\tan 2\theta$.

5. Find the positive integers p,q,r,s satisfying $\tan\frac{\pi}{24} = \left(\sqrt{p}-\sqrt{q}\right)\left(\sqrt{r}-s\right)$.

6. If $m\tan(\theta-30°) = n\tan(\theta+120°)$, show that $\cos 2\theta = \frac{m+n}{2(m-n)}$.

7. If $\cos(\alpha+\beta) = \frac{4}{5}$; $\sin(\alpha-\beta) = \frac{5}{13}$ & α, β lie between 0 & $\frac{\pi}{4}$, then find the value of $\tan 2\alpha$

8. If the value of the expression $\sin 25°.\sin 35°.\sin 85°$ can be expressed as $\frac{\sqrt{a}+\sqrt{b}}{c}$ where $a,b,c \in N$ and are in their lowest form, find the value of $(a+b+c)$.

9. Prove that $(4\cos^2 9° - 3)(4\cos^2 27° - 3) = \tan 9°$.

10. Prove that $4\cos\frac{2\pi}{7}.\cos\frac{\pi}{7} - 1 = 2\cos\frac{2\pi}{7}$.

11. Let $P(k) = \left(1+\cos\frac{\pi}{4k}\right)\left(1+\cos\frac{(2k-1)\pi}{4k}\right)\left(1+\cos\frac{(2k+1)\pi}{4k}\right)\left(1+\cos\frac{(4k-1)\pi}{4k}\right)$ then find the value of **(a)** P(5) and **(b)** P(6).

Calculate without using trigonometric tables (Q.12 to Q.15) :

12. $4\cos 20° - \sqrt{3}\cot 20°$

13. $\frac{2\cos 40° - \cos 20°}{\sin 20°}$

14. $\cos^6\frac{\pi}{16} + \cos^6\frac{3\pi}{16} + \cos^6\frac{5\pi}{16} + \cos^6\frac{7\pi}{16}$

15. $\tan 10° - \tan 50° + \tan 70°$

16. Given that $(1+\tan 1°)(1+\tan 2°)....(1+\tan 45°) = 2^n$, find n.

17. Prove that from the equality $\frac{\sin^4\alpha}{a} + \frac{\cos^4\alpha}{b} = \frac{1}{a+b}$ follows the relation; $\frac{\sin^8\alpha}{a^3} + \frac{\cos^8\alpha}{b^3} = \frac{1}{(a+b)^3}$.

18. (a) If $y = 10\cos^2 x - 6\sin x\cos x + 2\sin^2 x$, then find the greatest & least value of y.

(b) If $y = 1 + 2\sin x + 3\cos^2 x$, find the maximum & minimum values of $y\ \forall\ x \in R$.

(c) If $y = 9\sec^2 x + 16\operatorname{cosec}^2 x$, find the minimum value of y for all permissible value of x.

(d) If $a \le 3\cos\left(\theta+\frac{\pi}{3}\right) + 5\cos\theta + 3 \le b$, find a and b, where a is the minimum value & b is the maximum value.

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19. Let k = 1°, then prove that $\sum_{n=0}^{88}\frac{1}{\cos nk.\cos(n+1)k}=\frac{\cos k}{\sin^2 k}$

20. If x and y are real number such that $x^2 + 2xy - y^2 = 6$, find the minimum value of $(x^2 + y^2)^2$.

21. If A + B + C = π; prove that $\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\ge 1$.

EXERCISE (JM)

1. If $5(\tan^2x - \cos^2x) = 2\cos 2x + 9$, then the value of cos4x is :- **[JEE(Main)-2017]**

(1) $-\frac{7}{9}$ (2) $-\frac{3}{5}$ (3) $\frac{1}{3}$ (4) $\frac{2}{9}$

2. For any $\theta \in \left(\frac{\pi}{4},\frac{\pi}{2}\right)$, the expression $3(\sin\theta - \cos\theta)^4 + 6(\sin\theta + \cos\theta)^2 + 4\sin^6\theta$ equals : **[JEE(Main)-Jan 19]**

(1) $13 - 4\cos^6\theta$ (2) $13 - 4\cos^4\theta + 2\sin^2\theta\cos^2\theta$

(3) $13 - 4\cos^2\theta + 6\cos^4\theta$ (4) $13 - 4\cos^2\theta + 6\sin^2\theta\cos^2\theta$

3. The value of $\cos\frac{\pi}{2^2}\cdot\cos\frac{\pi}{2^3}\cdot.....\cdot\cos\frac{\pi}{2^{10}}\cdot\sin\frac{\pi}{2^{10}}$ is : **[JEE(Main)-Jan 19]**

(1) $\frac{1}{256}$ (2) $\frac{1}{2}$ (3) $\frac{1}{512}$ (4) $\frac{1}{1024}$

4. Let $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$ for k = 1, 2, 3, ... Then for all $x \in R$, the value of $f_4(x) - f_6(x)$ is equal to :- **[JEE(Main)-Jan 19]**

(1) $\frac{5}{12}$ (2) $\frac{-1}{12}$ (3) $\frac{1}{4}$ (4) $\frac{1}{12}$

5. The maximum value of $3\cos\theta+5\sin\left(\theta-\frac{\pi}{6}\right)$ for any real value of θ is : **[JEE(Main)-Jan 19]**

(1) $\sqrt{19}$ (2) $\frac{\sqrt{79}}{2}$ (3) $\sqrt{31}$ (4) $\sqrt{34}$

6. If $\cos(\alpha+\beta)=\frac{3}{5},\sin(\alpha-\beta)=\frac{5}{13}$ and $0<\alpha,\beta<\frac{\pi}{4}$, then tan(2α) is equal to : **[JEE(Main)-Apr 19]**

(1) $\frac{21}{16}$ (2) $\frac{63}{52}$ (3) $\frac{33}{52}$ (4) $\frac{63}{16}$

7. The value of $\cos^2 10° - \cos 10°\cos 50° + \cos^2 50°$ is : **[JEE(Main)-Apr 19]**

(1) $\frac{3}{2}(1+\cos 20°)$ (2) $\frac{3}{4}$ (3) $\frac{3}{4}+\cos 20°$ (4) $\frac{3}{2}$

8. The value of sin 10° sin30° sin50° sin70° is :- **[JEE(Main)-Apr 19]**

(1) $\frac{1}{36}$ (2) $\frac{1}{32}$ (3) $\frac{1}{18}$ (4) $\frac{1}{16}$

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EXERCISE (JA)

1. If $\frac{\sin^4 x}{2}+\frac{\cos^4 x}{3}=\frac{1}{5}$, then **[JEE 2009, 4]**

(A) $\tan^2 x = \frac{2}{3}$ (B) $\frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}=\frac{1}{125}$ (C) $\tan^2 x = \frac{1}{3}$ (D) $\frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}=\frac{2}{125}$

2. For $0<\theta<\frac{\pi}{2}$, the solution(s) of $\sum_{m=1}^{6}\operatorname{cosec}\left(\theta+\frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta+\frac{m\pi}{4}\right)=4\sqrt{2}$ is (are) - **[JEE 2009, 4]**

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{12}$ (D) $\frac{5\pi}{12}$

3. The maximum value of the expression $\frac{1}{\sin^2\theta+3\sin\theta\cos\theta+5\cos^2\theta}$ is **[JEE 2010, 3]**

4. Let $P=\{\theta:\sin\theta-\cos\theta=\sqrt{2}\cos\theta\}$ and $Q=\{\theta:\sin\theta+\cos\theta=\sqrt{2}\sin\theta\}$ be two sets. Then

(A) $P\subset Q$ and $Q-P\neq\varnothing$ (B) $Q\not\subset P$

(C) $P\not\subset Q$ (D) $P = Q$ **[JEE 2011,3]**

5. The value of $\sum_{k=1}^{13}\frac{1}{\sin\left(\frac{\pi}{4}+\frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4}+\frac{k\pi}{6}\right)}$ is equal to **[JEE(Advanced)-2016, 3(–1)]**

(A) $3-\sqrt{3}$ (B) $2(3-\sqrt{3})$ (C) $2(\sqrt{3}-1)$ (D) $2(2+\sqrt{3})$

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6. For non-negative integers n, let $f(n)=\frac{\sum_{k=0}^{n}\sin\left(\frac{k+1}{n+2}\pi\right)\sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^{n}\sin^2\left(\frac{k+1}{n+2}\pi\right)}$

Assuming $\cos^{-1}x$ takes values in $[0, \pi]$, which of the following options is/are correct ? **[JEE(Advanced)-2019, 4(–1)]**

(1) $\sin(7\cos^{-1}f(5))=0$ (2) $f(4)=\frac{\sqrt{3}}{2}$

(3) $\lim_{n\to\infty}f(n)=\frac{1}{2}$ (4) If $\alpha=\tan(\cos^{-1}f(6))$, then $\alpha^2+2\alpha-1=0$

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ANSWER KEY

EXERCISE (O-1)

1. C	**2.** B	**3.** D	**4.** B	**5.** B	**6.** D	**7.** B	**8.** D
9. C	**10.** B	**11.** B	**12.** C	**13.** A	**14.** B	**15.** D	**16.** C
17. B	**18.** D	**19.** A	**20.** B				

EXERCISE (S-1)

5. p = 3, q = 2; r = 2; s = 1 **7.** $\frac{56}{33}$ **8.** 24 **11. (a)** $\frac{3-\sqrt{5}}{32}$; **(b)** $\frac{2-\sqrt{3}}{16}$

12. –1 **13.** $\sqrt{3}$ **14.** $\frac{5}{4}$ **15.** $\sqrt{3}$ **16.** n = 23

18. (a) $y_{max}=11$, $y_{min}=1$; **(b)** $y_{max}=\frac{13}{3}$, $y_{min}=-1$; **(c)** 49; **(d)** a = –4 & b = 10 **20.** 18

EXERCISE (JM)

1. 1 **2.** 1 **3.** 3 **4.** 4 **5.** 1 **6.** 4 **7.** 2
8. 4

EXERCISE (JA)

1. A,B **2.** C,D **3.** 2 **4.** D **5.** C **6.** 1,2,4

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TRIGONOMETRIC EQUATION

1. TRIGONOMETRIC EQUATION :

An equation involving one or more trigonometrical ratios of unknown angles is called a trigonometrical equation.

2. SOLUTION OF TRIGONOMETRIC EQUATION :

A value of the unknown angle which satisfies the given equation is called a solution of the trigonometric equation.

(a) **Principal solution :-** The solution of the trigonometric equation lying in the interval $[0, 2\pi)$.

(b) **General solution :-** Since all the trigonometric functions are many one & periodic, hence there are infinite values of θ for which trigonometric functions have the same value. All such possible values of θ for which the given trigonometric function is satisfied is given by a general formula. Such a general formula is called general solution of trigonometric equation.

(c) **Particular solution :-** The solution of the trigonometric equation lying in the given interval.

3. GENERAL SOLUTIONS OF SOME TRIGONOMETRIC EQUATIONS (TO BE REMEMBERED) :

(a) If $\sin\theta = 0$, then $\theta = n\pi$, $n \in I$ (set of integers)

(b) If $\cos\theta = 0$, then $\theta = (2n+1)\frac{\pi}{2}$, $n \in I$

(c) If $\tan\theta = 0$, then $\theta = n\pi$, $n \in I$

(d) If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha$ where $\alpha \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$

(e) If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$, $n \in I$, $\alpha \in [0,\pi]$

(f) If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$, $n \in I$, $\alpha \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

(g) If $\sin\theta = 1$, then $\theta = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}$, $n \in I$

(h) If $\cos\theta = 1$ then $\theta = 2n\pi$, $n \in I$

(i) If $\sin^2\theta = \sin^2\alpha$ or $\cos^2\theta = \cos^2\alpha$ or $\tan^2\theta = \tan^2\alpha$, then $\theta = n\pi \pm \alpha$, $n \in I$

(j) For $n \in I$, $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$, $n \in I$
$\sin(n\pi + \theta) = (-1)^n \sin\theta$ $\cos(n\pi + \theta) = (-1)^n \cos\theta$

(k) $\cos n\pi = (-1)^n$, $n \in I$

If n is an odd integer, then $\sin\frac{n\pi}{2} = (-1)^{\frac{n-1}{2}}$, $\cos\frac{n\pi}{2} = 0$,

$$\sin\left(\frac{n\pi}{2} + \theta\right) = (-1)^{\frac{n-1}{2}}\cos\theta$$

$$\cos\left(\frac{n\pi}{2} + \theta\right) = (-1)^{\frac{n+1}{2}}\sin\theta$$

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Illustration 1 : Find the set of values of x for which $\frac{\tan 3x - \tan 2x}{1+\tan 3x.\tan 2x} = 1$.

Solution : We have, $\frac{\tan 3x - \tan 2x}{1+\tan 3x.\tan 2x} = 1 \Rightarrow \tan(3x - 2x) = 1 \Rightarrow \tan x = 1$

$\Rightarrow \tan x = \tan\frac{\pi}{4} \Rightarrow x = n\pi + \frac{\pi}{4}, n \in I$ {using $\tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha$)

But for this value of x, tan 2x is not defined.

Hence the solution set for x is ϕ.

Do yourself-1 :

(i) Find general solutions of the following equations :

(a) $\sin\theta = \frac{1}{2}$ (b) $\cos\left(\frac{3\theta}{2}\right) = 0$ (c) $\tan\left(\frac{3\theta}{4}\right) = 0$

(d) $\cos^2 2\theta = 1$ (e) $\sqrt{3}\sec 2\theta = 2$ (f) $\operatorname{cosec}\left(\frac{\theta}{2}\right) = -1$

4. IMPORTANT POINTS TO BE REMEMBERED WHILE SOLVING TRIGONOMETRIC EQUATIONS :

(a) For equations of the type $\sin\theta = k$ or $\cos\theta = k$, one must check that $|k| \le 1$.

(b) Avoid squaring the equations, if possible, because it may lead to extraneous solutions. Reject extra solutions if they do not satisfy the given equation.

(c) Do not cancel the common variable factor from the two sides of the equations which are in a product because we may loose some solutions.

(d) The answer should not contain such values of θ, which make any of the terms undefined or infinite.

(i) Check that denominator is not zero at any stage while solving equations.

(ii) If $\tan\theta$ or $\sec\theta$ is involved in the equations, θ should not be odd multiple of $\frac{\pi}{2}$.

(iii) If $\cot\theta$ or $\operatorname{cosec}\theta$ is involved in the equation, θ should not be multiple of π or 0.

5. DIFFERENT STRATEGIES FOR SOLVING TRIGONOMETRIC EQUATIONS :

(a) Solving trigonometric equations by factorisation.

e.g. $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$

$\therefore$ $(2\sin x - \cos x)(1 + \cos x) - (1 - \cos^2 x) = 0$

$\therefore$ $(1 + \cos x)(2\sin x - \cos x - 1 + \cos x) = 0$

$\therefore$ $(1 + \cos x)(2\sin x - 1) = 0$

$\Rightarrow \cos x = -1$ or $\sin x = \frac{1}{2}$

$\Rightarrow \cos x = -1 = \cos\pi \Rightarrow x = 2n\pi + \pi = (2n+1)\pi, n \in I$

or $\sin x = \frac{1}{2} = \sin\frac{\pi}{6} \Rightarrow x = k\pi + (-1)^k\frac{\pi}{6}, k \in I$

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Illustration 2 : If $\frac{1}{6}\sin\theta$, $\cos\theta$ and $\tan\theta$ are in G.P. then the general solution for θ is -

(A) $2n\pi \pm \frac{\pi}{3}$ (B) $2n\pi \pm \frac{\pi}{6}$ (C) $n\pi \pm \frac{\pi}{3}$ (D) none of these

Solution : Since, $\frac{1}{6}\sin\theta$, $\cos\theta$, $\tan\theta$ are in G.P.

$\Rightarrow \cos^2\theta = \frac{1}{6}\sin\theta \,.\, \tan\theta \Rightarrow 6\cos^3\theta + \cos^2\theta - 1 = 0$

$\therefore (2\cos\theta - 1)(3\cos^2\theta + 2\cos\theta + 1) = 0$

$\Rightarrow \cos\theta = \frac{1}{2}$ (other values of $\cos\theta$ are imaginary)

$\Rightarrow \cos\theta = \cos\frac{\pi}{3} \Rightarrow \theta = 2n\pi \pm \frac{\pi}{3}, n \in I.$ **Ans. (A)**

(b) Solving of trigonometric equation by reducing it to a quadratic equation.

e.g. $6 - 10\cos x = 3\sin^2 x$

$\therefore 6 - 10\cos x = 3 - 3\cos^2 x \Rightarrow 3\cos^2 x - 10\cos x + 3 = 0$

$\Rightarrow (3\cos x - 1)(\cos x - 3) = 0 \Rightarrow \cos x = \frac{1}{3}$ or $\cos x = 3$

Since $\cos x = 3$ is not possible as $-1 \le \cos x \le 1$

$\therefore \cos x = \frac{1}{3} = \cos\left(\cos^{-1}\frac{1}{3}\right) \Rightarrow x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right), n \in I$

Illustration 3 : Solve $\sin^2\theta - \cos\theta = \frac{1}{4}$ for θ and write the values of θ in the interval $0 \le \theta \le 2\pi$.

Solution : The given equation can be written as

$1 - \cos^2\theta - \cos\theta = \frac{1}{4} \Rightarrow \cos^2\theta + \cos\theta - 3/4 = 0$

$\Rightarrow 4\cos^2\theta + 4\cos\theta - 3 = 0 \Rightarrow (2\cos\theta - 1)(2\cos\theta + 3) = 0$

$\Rightarrow \cos\theta = \frac{1}{2}, -\frac{3}{2}$

Since, $\cos\theta = -3/2$ is not possible as $-1 \le \cos\theta \le 1$

$\therefore \cos\theta = \frac{1}{2} \Rightarrow \cos\theta = \cos\frac{\pi}{3} \Rightarrow \theta = 2n\pi \pm \frac{\pi}{3}, n \in I$

For the given interval, n = 0 and n = 1.

$\Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$ **Ans.**

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Illustration 4 : Find the number of solutions of tanx + secx = 2cosx in [0, 2π].

Solution : Here, tanx + secx = 2cosx $\Rightarrow$ sinx + 1 = 2 $\cos^2$x

$\Rightarrow$ $2\sin^2x + \sin x - 1 = 0$ $\Rightarrow$ $\sin x = \frac{1}{2}, -1$

But sinx = –1 $\Rightarrow x = \frac{3\pi}{2}$ for which tanx + secx = 2 cosx is not defined.

Thus $\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\Rightarrow$ number of solutions of tanx + secx = 2cos x is 2. **Ans.**

Illustration 5 : Solve the equation $5\sin^2x - 7\sin x \cos x + 16\cos^2 x = 4$

Solution : To solve this equation we use the fundamental formula of trigonometric identities,

$\sin^2x + \cos^2x = 1$

writing the equation in the form,

$5\sin^2x - 7\sin x \,.\, \cos x + 16\cos^2x = 4(\sin^2x + \cos^2x)$

$\Rightarrow$ $\sin^2x - 7\sin x \cos x + 12\cos^2 x = 0$

dividing by $\cos^2x$ on both side we get,

$\tan^2x - 7\tan x + 12 = 0$

Now it can be factorized as :

$(\tan x - 3)(\tan x - 4) = 0$

$\Rightarrow$ tanx = 3, 4

i.e., $\tan x = \tan(\tan^{-1}3)$ or $\tan x = \tan(\tan^{-1}4)$

$\Rightarrow$ $x = n\pi + \tan^{-1} 3$ or $x = n\pi + \tan^{-1} 4$, $n \in I$. **Ans.**

Illustration 6 : If $x \neq \frac{n\pi}{2}$, $n \in I$ and $(\cos x)^{\sin^2 x - 3\sin x + 2} = 1$, then find the general solutions of x.

Solution : As $x \neq \frac{n\pi}{2}$ $\Rightarrow$ $\cos x \neq 0, 1, -1$

So, $(\cos x)^{\sin^2 x - 3\sin x + 2} = 1$ $\Rightarrow$ $\sin^2x - 3\sin x + 2 = 0$

$\therefore$ $(\sin x - 2)(\sin x - 1) = 0 \Rightarrow$ sinx = 1, 2

where sinx = 2 is not possible and sinx = 1 which is also not possible as $x \neq \frac{n\pi}{2}$

$\therefore$ no general solution is possible. **Ans.**

Illustration 7 : Solve the equation $\sin^4x + \cos^4 x = \frac{7}{2}\sin x \,.\, \cos x$.

Solution : $\sin^4x + \cos^4x = \frac{7}{2}\sin x \,.\, \cos x$ $\Rightarrow$ $(\sin^2x + \cos^2x)^2 - 2\sin^2x \cos^2x = \frac{7}{2}\sin x \,.\, \cos x$

$\Rightarrow$ $1 - \frac{1}{2}(\sin 2x)^2 = \frac{7}{4}(\sin 2x)$ $\Rightarrow 2\sin^2 2x + 7\sin 2x - 4 = 0$

$\Rightarrow$ $(2\sin 2x - 1)(\sin 2x + 4) = 0$ $\Rightarrow$ $\sin 2x = \frac{1}{2}$ or $\sin 2x = -4$ (which is not possible)

$\Rightarrow$ $2x = n\pi + (-1)^n \frac{\pi}{6}$, $n \in I$

i.e., $x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}$, $n \in I$ **Ans.**

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Do yourself-2 :

(i) Solve the following equations :

(a) $3\sin x + 2\cos^2 x = 0$ (b) $\sec^2 2\alpha = 1 - \tan 2\alpha$

(c) $7\cos^2\theta + 3\sin^2\theta = 4$ (d) $4\cos\theta - 3\sec\theta = \tan\theta$

(ii) Solve the equation : $2\sin^2\theta + \sin^2 2\theta = 2$ for $\theta \in (-\pi, \pi)$.

(c) Solving trigonometric equations by introducing an auxilliary argument.

Consider, $a \sin\theta + b\cos\theta = c$ ..... (i)

$$\therefore \quad \frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta = \frac{c}{\sqrt{a^2+b^2}}$$

equation (i) has a solution only if $|c| \le \sqrt{a^2+b^2}$

let $\frac{a}{\sqrt{a^2+b^2}} = \cos\phi$, $\frac{b}{\sqrt{a^2+b^2}} = \sin\phi$ & $\phi = \tan^{-1}\frac{b}{a}$

by introducing this auxillary argument ϕ, equation (i) reduces to

$\sin(\theta + \phi) = \frac{c}{\sqrt{a^2+b^2}}$ Now this equation can be solved easily.

Illustration 8 : Find the number of distinct solutions of $\sec x + \tan x = \sqrt{3}$, where $0 \le x \le 3\pi$.

Solution : Here, $\sec x + \tan x = \sqrt{3} \Rightarrow 1 + \sin x = \sqrt{3}\cos x$

or $\sqrt{3}\cos x - \sin x = 1$

dividing both sides by $\sqrt{a^2+b^2}$ i.e. $\sqrt{4} = 2$, we get

$$\Rightarrow \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = \frac{1}{2}$$

$$\Rightarrow \cos\frac{\pi}{6}\cos x - \sin\frac{\pi}{6}\sin x = \frac{1}{2} \Rightarrow \cos\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$$

As $0 \le x \le 3\pi$

$$\frac{\pi}{6} \le x + \frac{\pi}{6} \le 3\pi + \frac{\pi}{6}$$

$$\Rightarrow x + \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3} \quad \Rightarrow \quad x = \frac{\pi}{6}, \frac{3\pi}{2}, \frac{13\pi}{6}$$

But at $x = \frac{3\pi}{2}$, tanx and secx is not defined.

$\therefore$ Total number of solutions are 2. **Ans.**

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Illustration 9 : Prove that the equation kcosx – 3sinx = k + 1 possess a solution iff $k \in (-\infty, 4]$.

Solution : Here, k cosx – 3sinx = k + 1, could be re-written as :

$$\frac{k}{\sqrt{k^2+9}}\cos x - \frac{3}{\sqrt{k^2+9}}\sin x = \frac{k+1}{\sqrt{k^2+9}}$$

or $\cos(x+\phi) = \frac{k+1}{\sqrt{k^2+9}}$, where $\tan\phi = \frac{3}{k}$

which possess a solution only if $-1 \le \frac{k+1}{\sqrt{k^2+9}} \le 1$

i.e., $\left|\frac{k+1}{\sqrt{k^2+9}}\right| \le 1$

i.e., $(k+1)^2 \le k^2+9$

i.e., $k^2 + 2k + 1 \le k^2 + 9$

or $k \le 4$

$\Rightarrow$ The interval of k for which the equation (kcosx – 3sinx = k + 1) has a solution is $(-\infty, 4]$.

Ans.

Do yourself-3 :

(i) Solve the following equations :

(a) $\sin x + \sqrt{2} = \cos x$.

(b) $\operatorname{cosec}\theta = 1 + \cot\theta$.

(d) Solving trigonometric equations by transforming sum of trigonometric functions into product.

e.g. $\cos 3x + \sin 2x - \sin 4x = 0$

$\cos 3x - 2\sin x \cos 3x = 0$

$\Rightarrow$ $(\cos 3x)(1 - 2\sin x) = 0$

$\Rightarrow$ $\cos 3x = 0$ or $\sin x = \frac{1}{2}$

$\Rightarrow$ $\cos 3x = 0 = \cos\frac{\pi}{2}$ or $\sin x = \frac{1}{2} = \sin\frac{\pi}{6}$

$\Rightarrow$ $3x = 2n\pi \pm \frac{\pi}{2}$ or $x = m\pi + (-1)^m \frac{\pi}{6}$

$\Rightarrow$ $x = \frac{2n\pi}{3} \pm \frac{\pi}{6}$ or $x = m\pi + (-1)^m \frac{\pi}{6}$; $(n, m \in I)$

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Illustration 10 : Solve : $\cos\theta + \cos3\theta + \cos5\theta + \cos7\theta = 0$

Solution : We have $\cos\theta + \cos7\theta + \cos3\theta + \cos5\theta = 0$

$\Rightarrow$ $2\cos4\theta\cos3\theta + 2\cos4\theta\cos\theta = 0 \Rightarrow \cos4\theta(\cos3\theta + \cos\theta) = 0$

$\Rightarrow$ $\cos4\theta(2\cos2\theta\cos\theta) = 0$

$\Rightarrow$ Either $\cos\theta = 0 \Rightarrow \theta = (2n_1 + 1)\,\pi/2,\ n_1 \in I$

or $\cos2\theta = 0 \Rightarrow \theta = (2n_2 + 1)\dfrac{\pi}{4},\ n_2 \in I$

or $\cos4\theta = 0 \Rightarrow \theta = (2n_3 + 1)\dfrac{\pi}{8},\ n_3 \in I$ **Ans.**

(e) Solving trigonometric equations by transforming a product into sum.

e.g. $\sin5x.\ \cos3x = \sin6x.\ \cos2x$

$\sin8x + \sin2x = \sin8x + \sin4x$

$\therefore$ $2\sin2x\ .\ \cos2x - \sin2x = 0$

$\Rightarrow$ $\sin2x(2\cos 2x - 1) = 0$

$\Rightarrow$ $\sin2x = 0$ or $\cos2x = \dfrac{1}{2}$

$\Rightarrow$ $\sin2x = 0 = \sin0$ or $\cos2x = \dfrac{1}{2} = \cos\dfrac{\pi}{3}$

$\Rightarrow$ $2x = n\pi + (-1)^n \times 0,\ n \in I$ or $2x = 2m\pi \pm \dfrac{\pi}{3},\ m \in I$

$\Rightarrow$ $x = \dfrac{n\pi}{2},\ n \in I$ or $x = m\pi \pm \dfrac{\pi}{6},\ m \in I$

Illustration 11 : Solve : $\cos\theta\cos2\theta\cos3\theta = \dfrac{1}{4}$; where $0 \le \theta \le \pi$.

Solution : $\dfrac{1}{2}(2\cos\theta\cos3\theta)\cos2\theta = \dfrac{1}{4} \Rightarrow (\cos2\theta + \cos4\theta)\cos2\theta = \dfrac{1}{2}$

$\Rightarrow$ $\dfrac{1}{2}[2\cos^2 2\theta + 2\cos4\theta\cos2\theta] = \dfrac{1}{2} \Rightarrow 1 + \cos4\theta + 2\cos4\theta\cos2\theta = 1$

$\therefore$ $\cos4\theta\,(1 + 2\cos2\theta) = 0$

$\cos4\theta = 0$ or $(1 + 2\cos2\theta) = 0$

Now from the first equation : $2\cos4\theta = 0 = \cos(\pi/2)$

$\therefore$ $4\theta = \left(n + \dfrac{1}{2}\right)\pi \Rightarrow \theta = (2n+1)\dfrac{\pi}{8},\ n \in I$

for $n = 0,\ \theta = \dfrac{\pi}{8}$; $n = 1,\ \theta = \dfrac{3\pi}{8}$; $n = 2,\ \theta = \dfrac{5\pi}{8}$; $n = 3,\ \theta = \dfrac{7\pi}{8}$ $(\because\ 0 \le \theta \le \pi)$

and from the second equation :

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$$\cos 2\theta = -\frac{1}{2} = -\cos(\pi/3) = \cos(\pi-\pi/3) = \cos(2\pi/3)$$

$\therefore \quad 2\theta = 2k\pi \pm 2\pi/3 \quad \therefore \quad \theta = k\pi \pm \pi/3,\ k \in I$

again for $k = 0,\ \theta = \frac{\pi}{3};\ k = 1,\ \theta = \frac{2\pi}{3} \quad (\because\ 0 \le \theta \le \pi)$

$\therefore \quad \theta = \frac{\pi}{8}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{7\pi}{8}$ **Ans.**

Do yourself-4 :

(i) Solve $4\sin\theta \sin 2\theta \sin 4\theta = \sin 3\theta$.

(ii) Solve for x : $\sin x + \sin 3x + \sin 5x = 0$.

(f) Solving equations by a change of variable :

(i) Equations of the form $P(\sin x \pm \cos x, \sin x.\cos x) = 0$, where $P(y,z)$ is a polynomial, can be solved by the substitution :

$\cos x \pm \sin x = t \quad \Rightarrow \quad 1 \pm 2\sin x.\cos x = t^2$.

e.g. $\sin x + \cos x = 1 + \sin x.\cos x$.

put $\sin x + \cos x = t$

$\Rightarrow \quad \sin^2 x + \cos^2 x + 2\sin x.\cos x = t^2$

$\Rightarrow \quad 2\sin x \cos x = t^2 - 1 \quad (\because\ \sin^2 x + \cos^2 x = 1)$

$\Rightarrow \quad \sin x.\cos x = \left(\frac{t^2-1}{2}\right)$

Substituting above result in given equation, we get :

$t = 1 + \frac{t^2-1}{2}$

$\Rightarrow \quad 2t = t^2 + 1 \Rightarrow t^2 - 2t + 1 = 0$

$\Rightarrow \quad (t-1)^2 = 0 \quad \Rightarrow \quad t = 1$

$\Rightarrow \quad \sin x + \cos x = 1$

Dividing both sides by $\sqrt{1^2+1^2}$ i.e. $\sqrt{2}$, we get

$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{1}{\sqrt{2}} \Rightarrow \quad \cos x \cos\frac{\pi}{4} + \sin x.\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{4} \quad \Rightarrow \quad x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

$\Rightarrow \quad x = 2n\pi$ or $x = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2},\ n \in I$

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(ii) Equations of the form of asinx + bcosx + d = 0, where a, b & d are real numbers can be solved by changing sin x & cos x into their corresponding tangent of half the angle.

e.g. 3 cos x + 4 sin x = 5

$$\Rightarrow \quad 3\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right) + 4\left(\frac{2\tan x/2}{1+\tan^2 x/2}\right) = 5$$

$$\Rightarrow \quad \frac{3-3\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} + \frac{8\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = 5$$

$$\Rightarrow \quad 3 - 3\tan^2\frac{x}{2} + 8\tan\frac{x}{2} = 5 + 5\tan^2\frac{x}{2} \quad \Rightarrow \quad 8\tan^2\frac{x}{2} - 8\tan\frac{x}{2} + 2 = 0$$

$$\Rightarrow \quad 4\tan^2\frac{x}{2} - 4\tan\frac{x}{2} + 1 = 0 \quad \Rightarrow \quad \left(2\tan\frac{x}{2} - 1\right)^2 = 0$$

$$\Rightarrow \quad 2\tan\frac{x}{2} - 1 = 0 \Rightarrow \tan\frac{x}{2} = \frac{1}{2} = \tan\left(\tan^{-1}\frac{1}{2}\right)$$

$$\Rightarrow \quad \frac{x}{2} = n\pi + \tan^{-1}\left(\frac{1}{2}\right), n \in I \quad \Rightarrow \quad x = 2n\pi + 2\tan^{-1}\frac{1}{2}, n \in I$$

(iii) Many equations can be solved by introducing a new variable.

e.g. $\sin^4 2x + \cos^4 2x = \sin 2x.\cos 2x$

substituting $\sin 2x.\cos 2x = y$ $\because$ $(\sin^2 2x + \cos^2 2x)^2 = \sin^4 2x + \cos^4 2x + 2\sin^2 2x.\cos^2 2x$

$\Rightarrow$ $\sin^4 2x + \cos^4 2x = 1 - 2\sin^2 2x.\cos^2 2x$ substituting above result in given equation :

$1 - 2y^2 = y$

$$\Rightarrow \quad 2y^2 + y - 1 = 0 \quad \Rightarrow \quad 2(y+1)\left(y - \frac{1}{2}\right) = 0$$

$$\Rightarrow \quad y = -1 \quad \text{or} \quad y = \frac{1}{2} \Rightarrow \sin 2x.\cos 2x = -1 \quad \text{or} \quad \sin 2x.\cos 2x = \frac{1}{2}$$

$\Rightarrow$ $2\sin 2x.\cos 2x = -2$ or $2\sin 2x.\cos 2x = 1$

$\Rightarrow$ $\sin 4x = -2$ (which is not possible) or $2\sin 2x.\cos 2x = 1$

$$\Rightarrow \quad \sin 4x = 1 = \sin\frac{\pi}{2} \quad \Rightarrow 4x = n\pi + (-1)^n\frac{\pi}{2}, n \in I \Rightarrow x = \frac{n\pi}{4} + (-1)^n\frac{\pi}{8}, n \in I$$

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Illustration 12 : Find the general solution of equation $\sin^4 x + \cos^4 x = \sin x \cos x$.

Solution : Using half-angle formulae, we can represent given equation in the form :

$$\left(\frac{1-\cos 2x}{2}\right)^2 + \left(\frac{1+\cos 2x}{2}\right)^2 = \sin x \cos x$$

$\Rightarrow$ $(1 - \cos 2x)^2 + (1 + \cos 2x)^2 = 4\sin x \cos x$

$\Rightarrow$ $2(1 + \cos^2 2x) = 2\sin 2x \Rightarrow$ $1 + 1 - \sin^2 2x = \sin 2x$

$\Rightarrow$ $\sin^2 2x + \sin 2x = 2$

$\Rightarrow$ $\sin 2x = 1$ or $\sin 2x = -2$ (which is not possible)

$\Rightarrow$ $2x = 2n\pi + \frac{\pi}{2}$, $n \in I$

$\Rightarrow$ $x = n\pi + \frac{\pi}{4}$, $n \in I$ **Ans.**

(g) Solving trigonometric equations with the use of the boundness of the functions involved.

e.g. $\sin x\left(\cos\frac{x}{4} - 2\sin x\right) + \left(1 + \sin\frac{x}{4} - 2\cos x\right).\cos x = 0$

$\therefore$ $\sin x \cos\frac{x}{4} + \cos x \sin\frac{x}{4} + \cos x = 2$

$\therefore$ $\sin\left(\frac{5x}{4}\right) + \cos x = 2$

$\Rightarrow$ $\sin\left(\frac{5x}{4}\right) = 1$ & $\cos x = 1$ (as $\sin\theta \le 1$ & $\cos\theta \le 1$)

Now consider

$\cos x = 1$ $\Rightarrow$ $x = 2\pi, 4\pi, 6\pi, 8\pi$ .....

and $\sin\frac{5x}{4} = 1$ $\Rightarrow$ $x = \frac{2\pi}{5}, \frac{10\pi}{5}, \frac{18\pi}{5}$ .....

Common solution to above APs will be the AP having

First term = 2π

Common difference = LCM of 2π and $\frac{8\pi}{5} = \frac{40\pi}{5} = 8\pi$

$\therefore$ General solution will be general term of this AP i.e. $2\pi + (8\pi)n$, $n \in I$

$\Rightarrow$ $x = 2(4n + 1)\pi$, $n \in I$

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Illustration 13 : Solve the equation $(\sin x + \cos x)^{1+\sin 2x} = 2$, when $0 \le x \le \pi$.

Solution : We know, $-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$ and $-1 \le \sin\theta \le 1$.

$\therefore$ $(\sin x + \cos x)$ admits the maximum value as $\sqrt{2}$

and $(1 + \sin 2x)$ admits the maximum value as 2.

Also $\left(\sqrt{2}\right)^2 = 2$.

$\therefore$ the equation could hold only when, $\sin x + \cos x = \sqrt{2}$ and $1 + \sin 2x = 2$

Now, $\sin x + \cos x = \sqrt{2}$ $\Rightarrow$ $\cos\left(x - \frac{\pi}{4}\right) = 1$

$\Rightarrow$ $x = 2n\pi + \pi/4,\ n \in I$ ..... (i)

and $1 + \sin 2x = 2$ $\Rightarrow$ $\sin 2x = 1 = \sin\frac{\pi}{2}$

$\Rightarrow$ $2x = m\pi + (-1)^m\frac{\pi}{2},\ m \in I$ $\Rightarrow$ $x = \frac{m\pi}{2} + (-1)^m\frac{\pi}{4}$ ..... (ii)

The value of x in $[0, \pi]$ satisfying equations (i) and (ii) is $x = \frac{\pi}{4}$ (when $n = 0$ & $m = 0$)

Ans.

Note : $\sin x + \cos x = -\sqrt{2}$ and $1 + \sin 2x = 2$ also satisfies but as $x \ge 0$, this solution is not in domain.

Illustration 14 : Solve for x and y : $2^{\frac{1}{\cos^2 x}}\sqrt{y^2 - y + 1/2} \le 1$

Solution : $2^{\frac{1}{\cos^2 x}}\sqrt{y^2 - y + 1/2} \le 1$ ..... (i)

$$2^{\frac{1}{\cos^2 x}}\sqrt{\left(y - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \le 1$$

Minimum value of $2^{\frac{1}{\cos^2 x}} = 2$

Minimum value of $\sqrt{\left(y - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}$

$\Rightarrow$ Minimum value of $2^{\frac{1}{\cos^2 x}}\sqrt{y^2 - y + \frac{1}{2}}$ is 1

$\Rightarrow$ (i) is possible when $2^{\frac{1}{\cos^2 x}}\sqrt{\left(y - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1$

$\Rightarrow$ $\cos^2 x = 1$ and $y = 1/2$ $\Rightarrow \cos x = \pm 1 \Rightarrow x = n\pi$, where $n \in I$.

Hence $x = n\pi,\ n \in I$ and $y = 1/2$. **Ans.**

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Illustration 15 : The number of solution(s) of $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2+\frac{1}{x^2}$, $0 \le x \le \pi/2$, is/are -

(A) 0 (B) 1 (C) infinite (D) none of these

Solution : Let $y = 2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2+\frac{1}{x^2}$ $\Rightarrow$ $y = (1 + \cos x)\sin^2 x$ and $y = x^2 + \frac{1}{x^2}$

when $y = (1 + \cos x)\sin^2 x = (\text{a number} < 2)(\text{a number} \le 1)$ $\Rightarrow$ $y < 2$ ..... (i)

and when $y = x^2 + \frac{1}{x^2} = \left(x - \frac{1}{x}\right)^2 + 2 \ge 2$ $\Rightarrow$ $y \ge 2$ ..... (ii)

No value of y can be obtained satisfying (i) and (ii), simultaneously

$\Rightarrow$ No real solution of the equation exists. **Ans. (A)**

Note: If L.H.S. of the given trigonometric equation is always less than or equal to k and RHS is always greater than k, then no solution exists. If both the sides are equal to k for same value of θ, then solution exists and if they are equal for different values of θ, then solution does not exist.

Do yourself-5 :

(i) If $x^2 - 4x + 5 - \sin y = 0$, $y \in [0, 2\pi)$, then -

(A) $x = 1, y = 0$ (B) $x = 1, y = \pi/2$ (C) $x = 2, y = 0$ (D) $x = 2, y = \pi/2$

(ii) If $\sin x + \cos x = \sqrt{y + \frac{1}{y}}$, $y > 0$, $x \in [0, \pi]$, then find the least positive value of x satisfying the given condition.

6. TRIGONOMETRIC INEQUALITIES :

There is no general rule to solve trigonometric inequations and the same rules of algebra are valid provided the domain and range of trigonometric functions should be kept in mind.

Illustration 16 : Find the solution set of inequality $\sin x > 1/2$.

Solution : When $\sin x = \frac{1}{2}$, the two values of x between 0 and 2π are $\pi/6$ and $5\pi/6$.

From the graph of $y = \sin x$, it is obvious that between 0 and 2π,

$\sin x > \frac{1}{2}$ for $\pi/6 < x < 5\pi/6$

Hence, $\sin x > 1/2$

$\Rightarrow$ $2n\pi + \pi/6 < x < 2n\pi + 5\pi/6$, $n \in I$

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Thus, the required solution set is $\bigcup_{n\in I}\left(2n\pi+\frac{\pi}{6}, 2n\pi+\frac{5\pi}{6}\right)$ **Ans.**

Illustration 17 : Find the value of x in the interval $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ for which $\sqrt{2}\sin 2x+1 \le 2\sin x+\sqrt{2}\cos x$

Solution : We have, $\sqrt{2}\sin 2x+1 \le 2\sin x+\sqrt{2}\cos x \Rightarrow 2\sqrt{2}\sin x\cos x-2\sin x-\sqrt{2}\cos x+1\le 0$

$\Rightarrow \quad 2\sin x(\sqrt{2}\cos x-1)-1(\sqrt{2}\cos x-1)\le 0 \Rightarrow (2\sin x-1)(\sqrt{2}\cos x-1)\le 0$

$\Rightarrow \quad \left(\sin x-\frac{1}{2}\right)\left(\cos x-\frac{1}{\sqrt{2}}\right)\le 0$

Above inequality holds when :

Case-I : $\sin x-\frac{1}{2}\le 0$ and $\cos x-\frac{1}{\sqrt{2}}\ge 0 \Rightarrow \sin x\le\frac{1}{2}$ and $\cos x\ge\frac{1}{\sqrt{2}}$

Now considering the given interval of x :

for $\sin x\le\frac{1}{2}$: $x\in\left[-\frac{\pi}{2}, \frac{\pi}{6}\right]\cup\left[\frac{5\pi}{6}, \frac{3\pi}{2}\right]$ and for $\cos x\ge\frac{1}{\sqrt{2}}$: $x\in\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

For both to simultaneously hold true : $x\in\left[-\frac{\pi}{4}, \frac{\pi}{6}\right]$

Case-II : $\sin x-\frac{1}{2}\ge 0$ and $\cos x\le\frac{1}{\sqrt{2}}$

Again, for the given interval of x :

for $\sin x\ge\frac{1}{2}$: $x\in\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ and for $\cos x\le\frac{1}{\sqrt{2}}$: $x\in\left[-\frac{\pi}{2}, -\frac{\pi}{4}\right]\cup\left[\frac{\pi}{4}, \frac{3\pi}{2}\right]$

For both to simultaneously hold true : $x\in\left[\frac{\pi}{4}, \frac{5\pi}{6}\right]$

$\therefore$ Given inequality holds for $x\in\left[-\frac{\pi}{4}, \frac{\pi}{6}\right]\cup\left[\frac{\pi}{4}, \frac{5\pi}{6}\right]$ **Ans.**

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Illustration 18 : Find the values of α lying between 0 and π for which the inequality : $\tan\alpha > \tan^3\alpha$ is valid.

Solution : We have : $\tan\alpha - \tan^3\alpha > 0 \Rightarrow \tan\alpha\,(1 - \tan^2\alpha) > 0$

$\Rightarrow$ $(\tan\alpha)(\tan\alpha + 1)(\tan\alpha - 1) < 0$

So $\tan\alpha < -1,\ 0 < \tan\alpha < 1$

$\therefore$ Given inequality holds for $\alpha \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$ **Ans.**

Do yourself - 6 :

(i) Find the solution set of the inequality : $\cos x \geq -1/2$.

(ii) Find the values of x in the interval $[0, 2\pi]$ for which $4\sin^2 x - 8\sin x + 3 \leq 0$.

Miscellaneous Illustration :

Illustration 19 : Solve the following equation : $\tan^2\theta + \sec^2\theta + 3 = 2(\sqrt{2}\sec\theta + \tan\theta)$

Solution : We have $\tan^2\theta + \sec^2\theta + 3 = 2\sqrt{2}\sec\theta + 2\tan\theta$

$\Rightarrow$ $\tan^2\theta - 2\tan\theta + \sec^2\theta - 2\sqrt{2}\sec\theta + 3 = 0$

$\Rightarrow$ $\tan^2\theta + 1 - 2\tan\theta + \sec^2\theta - 2\sqrt{2}\sec\theta + 2 = 0$

$\Rightarrow$ $(\tan\theta - 1)^2 + (\sec\theta - \sqrt{2})^2 = 0$ $\Rightarrow$ $\tan\theta = 1$ and $\sec\theta = \sqrt{2}$

As the periodicity of $\tan\theta$ and $\sec\theta$ are not same, we get

$\theta = 2n\pi + \frac{\pi}{4},\ n \in I$ **Ans.**

Illustration 20 : Find the solution set of equation $5^{(1 + \log_5 \cos x)} = 5/2$.

Solution : Taking log to base 5 on both sides in given equation :

$(1 + \log_5 \cos x).\ \log_5 5 = \log_5(5/2)$ $\Rightarrow \log_5 5 + \log_5 \cos x = \log_5 5 - \log_5 2$

$\Rightarrow \log_5 \cos x = -\log_5 2 \Rightarrow \cos x = 1/2$ $\Rightarrow x = 2n\pi \pm \pi/3,\ n \in I$ **Ans.**

Illustration 21 : If the set of all values of x in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfying $|4\sin x + \sqrt{2}| < \sqrt{6}$ is $\left(\frac{a\pi}{24}, \frac{b\pi}{24}\right)$ then find the value of $\left|\frac{a-b}{3}\right|$.

Solution : $|4\sin x + \sqrt{2}| < \sqrt{6}$

$\Rightarrow$ $-\sqrt{6} < 4\sin x + \sqrt{2} < \sqrt{6}$ $\Rightarrow$ $-\sqrt{6} - \sqrt{2} < 4\sin x < \sqrt{6} - \sqrt{2}$

$\Rightarrow$ $\frac{-(\sqrt{6} + \sqrt{2})}{4} < \sin x < \frac{\sqrt{6} - \sqrt{2}}{4}$ $\Rightarrow$ $-\frac{5\pi}{12} < x < \frac{\pi}{12}$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Comparing with $\frac{a\pi}{24} < x < \frac{b\pi}{24}$, we get, $a = -10,\ b = 2$

$\therefore \left|\frac{a-b}{3}\right| = \left|\frac{-10-2}{3}\right| = 4$ **Ans.**

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Illustration 22 : Find the values of x in the interval $[0,2\pi]$ which satisfy the inequality : $3|2\sin x - 1| \geq 3 + 4\cos^2 x$.

Solution : The given inequality can be written as :

$3|2\sin x - 1| \geq 3 + 4(1 - \sin^2 x) \Rightarrow 3|2\sin x - 1| \geq 7 - 4\sin^2 x$

Let $\sin x = t \Rightarrow 3|2t - 1| \geq 7 - 4t^2$

Case I : For $2t - 1 \geq 0$ i.e. $t \geq 1/2$ we have , $|2t - 1| = (2t - 1)$

$\Rightarrow 3(2t - 1) \geq 7 - 4t^2 \Rightarrow 6t - 3 \geq 7 - 4t^2$

$\Rightarrow 4t^2 + 6t - 10 \geq 0 \Rightarrow 2t^2 + 3t - 5 \geq 0$

$\Rightarrow (t-1)(2t+5) \geq 0 \Rightarrow t \leq -\frac{5}{2}$ and $t \geq 1$

Now for $t \geq \frac{1}{2}$, we get $t \geq 1$ from above conditions i.e. $\sin x \geq 1$

The inequality holds true only for x satisfying the equation $\sin x = 1 \therefore x = \frac{\pi}{2}$

(for $x \in [0,2\pi]$)

Case II : For $2t - 1 < 0 \Rightarrow t < \frac{1}{2}$

we have, $|2t - 1| = -(2t - 1)$

$\Rightarrow -3(2t - 1) \geq 7 - 4t^2 \Rightarrow -6t + 3 \geq 7 - 4t^2$

$\Rightarrow 4t^2 - 6t - 4 \geq 0 \Rightarrow 2t^2 - 3t - 2 \geq 0$

$\Rightarrow (t - 2)(2t + 1) \geq 0 \Rightarrow t \leq -\frac{1}{2}$ and $t \geq 2$

Again, for $t < \frac{1}{2}$ we get $t \leq -\frac{1}{2}$ from above conditions

i.e. $\sin x \leq -\frac{1}{2} \Rightarrow \frac{7\pi}{6} \leq x \leq \frac{11}{6}\pi$ (for $x \in [0,2\pi]$)

Thus, $x \in \left[\frac{7\pi}{6}, \frac{11\pi}{6}\right] \cup \left\{\frac{\pi}{2}\right\}$ **Ans.**

Illustration 23 : Find the values of θ, for which $\cos 3\theta + \sin 3\theta + (2\sin 2\theta - 3)(\sin\theta - \cos\theta)$ is always positive.

Solution : Given expression can be written as :

$4\cos^3\theta - 3\cos\theta + 3\sin\theta - 4\sin^3\theta + (2\sin 2\theta - 3)(\sin\theta - \cos\theta)$

Applying given condition, we get

$\Rightarrow -4(\sin^3\theta - \cos^3\theta) + 3(\sin\theta - \cos\theta) + (\sin\theta - \cos\theta)(2\sin 2\theta - 3) > 0$

$\Rightarrow -4(\sin\theta - \cos\theta)(\sin^2\theta + \cos^2\theta + \sin\theta\cos\theta) + 3(\sin\theta - \cos\theta) + (\sin\theta - \cos\theta)(2\sin 2\theta - 3) > 0$

$\Rightarrow -4(\sin\theta - \cos\theta)(1 + \sin\theta\cos\theta) + 3(\sin\theta - \cos\theta) + (\sin\theta - \cos\theta)(4\sin\theta\cos\theta - 3) > 0$

$\Rightarrow (\sin\theta - \cos\theta)\{-4 - 4\sin\theta\cos\theta + 3 + 4\sin\theta\cos\theta - 3\} > 0$

$\Rightarrow -4(\sin\theta - \cos\theta) > 0$

$\Rightarrow -4\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right) > 0 \Rightarrow \sin\left(\theta - \frac{\pi}{4}\right) < 0 \Rightarrow 2n\pi - \pi < \theta - \frac{\pi}{4} < 2n\pi,\ n \in I$

$\Rightarrow 2n\pi - \frac{3\pi}{4} < \theta < 2n\pi + \frac{\pi}{4} \Rightarrow \theta \in \left(2n\pi - \frac{3\pi}{4}, 2n\pi + \frac{\pi}{4}\right),\ n \in I$ **Ans.**

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Illustration 24 : The number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3\sin^2 x - 7\sin x + 2 = 0$ is - **[JEE 98]**

(A) 0 (B) 5 (C) 6 (D) 10

Solution : $3\sin^2 x - 7\sin x + 2 = 0$

$\Rightarrow$ $(3\sin x - 1)(\sin x - 2) = 0$

$\because$ $\sin x \neq 2$

$\Rightarrow$ $\sin x = \frac{1}{3} = \sin\alpha$ (say)

where α is the least positive value of x such that $\sin\alpha = \frac{1}{3}$.

Clearly $0 < \alpha < \frac{\pi}{2}$. We get the solution,

$x = \alpha, \pi - \alpha, 2\pi + \alpha, 3\pi - \alpha, 4\pi + \alpha$ and $5\pi - \alpha$.

Hence total six values in $[0, 5\pi]$ **Ans. (C)**

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1 : **(i)** **(a)** $\theta = n\pi + (-1)^n \frac{\pi}{6}, n \in I$ **(b)** $\theta = (2n+1)\frac{\pi}{3}, n \in I$ **(c)** $\theta = \frac{4n\pi}{3}, n \in I$

(d) $\theta = \frac{n\pi}{2}, n \in I$ **(e)** $\theta = n\pi \pm \frac{\pi}{12}, n \in I$ **(f)** $\theta = 2n\pi + (-1)^{n+1}\pi, n \in I$

2 : **(i)** **(a)** $x = n\pi + (-1)^{n+1}\frac{\pi}{6}, n \in I$ **(b)** $\alpha = \frac{n\pi}{2}$ or $\alpha = \frac{k\pi}{2} + \frac{3\pi}{8}, n, k \in I$

(c) $\theta = n\pi \pm \frac{\pi}{3}, n \in I$ **(d)** $\theta = n\pi + (-1)^n\alpha$, where $\alpha = \sin^{-1}\left(\frac{\sqrt{17}-1}{8}\right)$ or $\sin^{-1}\left(\frac{-1-\sqrt{17}}{8}\right), n \in I$

(ii) $\theta = \left\{-\frac{\pi}{4}, -\frac{3\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{2}\right\}$

3 : **(i)** **(a)** $x = 2n\pi - \frac{\pi}{4}, n \in I$ **(b)** $2m\pi + \frac{\pi}{2}, m \in I$

4 : **(i)** $\theta = n\pi$ or $\theta = \frac{m\pi}{3} \pm \frac{\pi}{9}$; $n, m \in I$ **(ii)** $x = \frac{n\pi}{3}, n \in I$ and $k\pi \pm \frac{\pi}{3}, k \in I$

5 : **(i)** D **(ii)** $x = \frac{\pi}{4}$

6 : **(i)** $\bigcup_{n \in I}\left[2n\pi - \frac{2\pi}{3}, 2n\pi + \frac{2\pi}{3}\right]$ **(ii)** $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$

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EXERCISE (O-1)

1. The general solution of tan 3x = 1 is -

(A) $n\pi + \frac{\pi}{4}(n\in I)$ (B) $\frac{n\pi}{3}+\frac{\pi}{12}(n\in I)$ (C) $n\pi\ (n\in I)$ (D) $n\pi \pm \frac{\pi}{4}(n\in I)$

2. If $2\tan^2\theta = \sec^2\theta$, then the general solution of θ -

(A) $n\pi + \frac{\pi}{4}(n\in I)$ (B) $n\pi - \frac{\pi}{4}(n\in I)$ (C) $n\pi \pm \frac{\pi}{4}(n\in I)$ (D) $2n\pi \pm \frac{\pi}{4}(n\in I)$

3. Number of principal solution(s) of the equation $4\cdot 16^{\sin^2 x} = 2^{6\sin x}$ is

(A) 1 (B) 2 (C) 3 (D) 4

4. If $\tan\theta - \sqrt{2}\sec\theta = \sqrt{3}$, then the general solution of θ is -

(A) $n\pi + (-1)^n\frac{\pi}{4}-\frac{\pi}{3}$ (B) $n\pi + (-1)^n\frac{\pi}{3}-\frac{\pi}{4}$ (C) $n\pi + (-1)^n\frac{\pi}{3}+\frac{\pi}{4}$ (D) $n\pi + (-1)^n\frac{\pi}{4}+\frac{\pi}{3}$

where $n \in I$

5. The number of solutions of the equation $\tan^2 x - \sec^{10}x + 1 = 0$ in (0, 10) is -

(A) 3 (B) 6 (C) 10 (D) 11

6. The solution set of $(5 + 4\cos\theta)(2\cos\theta + 1) = 0$ in the interval $[0,2\pi]$ is :

(A) $\left\{\frac{\pi}{3},\frac{2\pi}{3}\right\}$ (B) $\left\{\frac{\pi}{3},\pi\right\}$ (C) $\left\{\frac{2\pi}{3},\frac{4\pi}{3}\right\}$ (D) $\left\{\frac{2\pi}{3},\frac{5\pi}{3}\right\}$

7. The equation $\sin x\cos x = 2$ has :

(A) one solution (B) two solutions (C) infinite solutions (D) no solution

8. If $0 \le x \le 3\pi$, $0 \le y \le 3\pi$ and $\cos x.\sin y = 1$, then the possible number of values of the ordered pair (x, y) is-

(A) 6 (B) 12 (C) 8 (D) 15

9. If $\frac{\tan 2\theta + \tan\theta}{1-\tan\theta\tan 2\theta} = 0$, then the general value of θ is -

(A) $n\pi$; $n \in I$ (B) $\frac{n\pi}{3}$; $n \in I$ (C) $\frac{n\pi}{4}$ (D) $\frac{n\pi}{6}$; $n \in I$

where $n \in I$

10. $\cos(\alpha - \beta) = 1$ and $\cos(\alpha + \beta) = 1/e$, where $\alpha, \beta \in [-\pi, \pi]$, numbers of pairs of α, β which satisfy both the equations is

(A) 0 (B) 1 (C) 2 (D) 4

11. If $0 < \theta < 2\pi$, then the intervals of values of θ for which $2\sin^2\theta - 5\sin\theta + 2 > 0$, is

(A) $\left(0,\frac{\pi}{6}\right)\cup\left(\frac{5\pi}{6},2\pi\right)$ (B) $\left(\frac{\pi}{8},\frac{5\pi}{6}\right)$ (C) $\left(0,\frac{\pi}{8}\right)\cup\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$ (D) $\left(\frac{41\pi}{48},\pi\right)$

12. The number of solutions of the pair of equations

$$2\sin^2\theta - \cos 2\theta = 0$$
$$2\cos^2\theta - 3\sin\theta = 0$$

in the interval $[0, 2\pi]$ is

(A) zero (B) one (C) two (D) four

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PASSAGE :

Whenever the terms on the two sides of the equation are of different nature, then equations are known as Non standard form, some of them are in the form of an ordinary equation but can not be solved by standard procedures.

Non standard problems require high degree of logic, they also require the use of graphs, inverse properties of functions, inequalities.

On the basis of above information, answer the following questions :

13. The equation $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + x^{-2}$, $0 < x \le \frac{\pi}{2}$ has

(A) one real solutions (B) more than one real solutions

(C) no real solution (D) none of the above

14. The number of real solutions of the equation $\sin(e^x) = 5^x + 5^{-x}$ is-

(A) 0 (B) 1 (C) 2 (D) infinitely many

15. The number of solutions of the equation $\sin x = x^2 + x + 1$ is-

(A) 0 (B) 1 (C) 2 (D) None

EXERCISE (S-1)

1. Solve the equation for x, $5^{\frac{1}{2}} + 5^{\frac{1}{2}+\log_5(\sin x)} = 15^{\frac{1}{2}+\log_{15}\cos x}$

2. Find all the values of θ satisfying the equation; $\sin\theta + \sin 5\theta = \sin 3\theta$ such that $0 \le \theta \le \pi$.

3. Solve the equality: $2\sin 11x + \cos 3x + \sqrt{3}\sin 3x = 0$

4. Find all value of θ, between 0 & π, which satisfy the equation; $\cos\theta \,.\, \cos 2\theta \,.\, \cos 3\theta = 1/4$.

5. Find the general solution of the equation, $2 + \tan x \cdot \cot\frac{x}{2} + \cot x \cdot \tan\frac{x}{2} = 0$

6. Determine the smallest positive value of x which satisfy the equation, $\sqrt{1+\sin 2x} - \sqrt{2}\cos 3x = 0$.

7. Find the general solution of the trigonometric equation $3^{\left(\frac{1}{2}+\log_3(\cos x+\sin x)\right)} - 2^{\log_2(\cos x - \sin x)} = \sqrt{2}$.

8. Find the value of θ, which satisfy $3 - 2\cos\theta - 4\sin\theta - \cos 2\theta + \sin 2\theta = 0$.

9. Find the range of y such that the equation, $y + \cos x = \sin x$ has a real solution. For $y = 1$, find x such that $0 < x < 2\pi$.

10. Find the general values of θ for which the quadratic function $(\sin\theta)\,x^2 + (2\cos\theta)x + \frac{\cos\theta + \sin\theta}{2}$ is the square of a linear function.

11. Prove that the equations

(a) $\sin x \cdot \sin 2x \cdot \sin 3x = 1$ (b) $\sin x \cdot \cos 4x \cdot \sin 5x = -1/2$

have no solution.

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12. If α and β are the roots of the equation, $a\cos\theta + b\sin\theta = c$ then match the entries of **column-I** with the entries of **column-II**.

	Column-I		**Column-II**
(A)	$\sin\alpha + \sin\beta$	(P)	$\frac{2b}{a+c}$
(B)	$\sin\alpha \,.\, \sin\beta$	(Q)	$\frac{c-a}{c+a}$
(C)	$\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}$	(R)	$\frac{2bc}{a^2+b^2}$
(D)	$\tan\frac{\alpha}{2} \,.\, \tan\frac{\beta}{2}$	(S)	$\frac{c^2-a^2}{a^2+b^2}$

EXERCISE (JM)

1. If $0 \le x < 2\pi$, then the number of real values of x, which satisfy the equation $\cos x + \cos 2x + \cos 3x + \cos 4x = 0$, is :- **[JEE(Main) 2016]**

(1) 9 (2) 3 (3) 5 (4) 7

2. If sum of all the solutions of the equation $8\cos x\cdot\left(\cos\left(\frac{\pi}{6}+x\right).\cos\left(\frac{\pi}{6}-x\right)-\frac{1}{2}\right) = 1$ in $[0, \pi]$ is $k\pi$, then k is equal to : **[JEE(Main) 2018]**

(1) $\frac{13}{9}$ (2) $\frac{8}{9}$ (3) $\frac{20}{9}$ (4) $\frac{2}{3}$

3. If $0 \le x < \frac{\pi}{2}$, then the number of values of x for which $\sin x - \sin 2x + \sin 3x = 0$, is **[JEE(Main)-Jan 19]**

(1) 2 (2) 1 (3) 3 (4) 4

4. The sum of all values of $\theta \in \left(0, \frac{\pi}{2}\right)$ satisfying $\sin^2 2\theta + \cos^4 2\theta = \frac{3}{4}$ is : **[JEE(Main)-Jan 19]**

(1) $\frac{\pi}{2}$ (2) π (3) $\frac{3\pi}{8}$ (4) $\frac{5\pi}{4}$

5. Let $S = \{\theta \in [-2\pi, 2\pi] : 2\cos^2\theta + 3\sin\theta = 0\}$. Then the sum of the elements of S is **[JEE(Main)-Apr 19]**

(1) $\frac{13\pi}{6}$ (2) π (3) 2π (4) $\frac{5\pi}{3}$

6. All the pairs (x, y) that satisfy the inequality $2\sqrt{\sin^2 x - 2\sin x + 5}\,.\,\frac{1}{4^{\sin^2 y}} \le 1$ also satisfy the eauation. **[JEE(Main)-Apr 19]**

(1) $\sin x = |\sin y|$ (2) $\sin x = 2\sin y$ (3) $2|\sin x| = 3\sin y$ (4) $2\sin x = \sin y$

7. The number of solutions of the equation $1 + \sin^4 x = \cos^2 3x,\ x \in \left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$ is : **[JEE(Main)-Apr 19]**

(1) 5 (2) 4 (3) 7 (4) 3

8. Let S be the set of all $\alpha \in R$ such that the equation, $\cos 2x + \alpha\sin x = 2\alpha - 7$ has a solution. Then S is equal to : **[JEE(Main)-Apr 19]**

(1) [2, 6] (2) [3,7] (3) R (4) [1,4]

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EXERCISE (JA)

1. The number of values of θ in the interval $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ such that $\theta \neq \frac{n\pi}{5}$ for n = 0, ±1,±2 and $\tan\theta = \cot 5\theta$ as well as $\sin 2\theta = \cos 4\theta$, is **[JEE 2010, 3]**

2. The positive integer value of n > 3 satisfying the equation $\frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$ is **[JEE 2011, 4]**

3. The number of distinct solutions of equation $\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$ in the interval [0, 2π] is **[JEE 2015, 4M, –0M]**

4. Let $S = \left\{x \in (-\pi,\pi) : x \neq 0, \pm\frac{\pi}{2}\right\}$. The sum of all distinct solution of the equation $\sqrt{3}\sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$ in the set S is equal to - **[JEE(Advanced)-2016, 3(–1)]**

(A) $-\frac{7\pi}{9}$ (B) $-\frac{2\pi}{9}$ (C) 0 (D) $\frac{5\pi}{9}$

5. Let a, b, c be three non-zero real numbers such that the equation $\sqrt{3}a\cos x + 2b\sin x = c,\ x \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ has two distinct real roots α and β with $\alpha + \beta = \frac{\pi}{3}$. Then the value of $\frac{b}{a}$ is _____ **[JEE(Advanced)-2018, 3(0)]**

Answer the following by appropriately matching the lists based on the information given in the paragraph

Let $f(x) = \sin(\pi\cos x)$ and $g(x) = \cos(2\pi\sin x)$ be two functions defined for x > 0. Define the following sets whose elements are written in the increasing order :

$X = \{x : f(x) = 0\}$, $Y = \{x : f'(x) = 0\}$, $Z = \{x : g(x) = 0\}$, $W = \{x : g'(x) = 0\}$.

List-I contains the sets X,Y,Z and W. List -II contains some information regarding these sets.

List-I		List-II	
(I)	X	(P)	$\supseteq\left\{\frac{\pi}{2},\frac{3\pi}{2},4\pi,7\pi\right\}$
(II)	Y	(Q)	an arithmetic progression
(III)	Z	(R)	NOT an arithmetic progression
(IV)	W	(S)	$\supseteq\left\{\frac{\pi}{6},\frac{7\pi}{6},\frac{13\pi}{6}\right\}$
		(T)	$\supseteq\left\{\frac{\pi}{3},\frac{2\pi}{3},\pi\right\}$
		(U)	$\supseteq\left\{\frac{\pi}{6},\frac{3\pi}{4}\right\}$

6. Which of the following is the only CORRECT combination ? **[JEE(Advanced)-2019, 3(–1)]**

(1) (II), (R), (S) (2) (I), (P), (R) (3) (II), (Q), (T) (4) (I), (Q), (U)

7. Which of the following is the only CORRECT combination ? **[JEE(Advanced)-2019, 3(–1)]**

(1) (IV), (Q), (T) (2) (IV), (P), (R), (S) (3) (III), (R), (U) (4) (III), (P), (Q), (U)

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ANSWER KEY

EXERCISE (O-1)

1. B	**2.** C	**3.** C	**4.** D	**5.** A	**6.** C	**7.** D	**8.** A
9. B	**10.** D	**11.** A	**12.** C	**13.** C	**14.** A	**15.** A	

EXERCISE (S-1)

1. $x = 2n\pi + \frac{\pi}{6},\ n \in I$

2. $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}$ & π

3. $x = \frac{n\pi}{7} - \frac{\pi}{84}$ or $x = \frac{n\pi}{4} + \frac{7\pi}{48}, n \in I$

4. $\frac{\pi}{8}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{7\pi}{8}$

5. $x = 2n\pi \pm \frac{2\pi}{3}, n \in I$

6. $x = \pi/16$

7. $x = 2n\pi + \frac{\pi}{12}$

8. $\theta = 2n\pi$ **or** $2n\pi + \frac{\pi}{2}; n \in I$

9. $-\sqrt{2} \le y \le \sqrt{2}$; $\frac{\pi}{2}, \pi$

10. $2n\pi + \frac{\pi}{4}$ or $(2n+1)\pi - \tan^{-1}2$, $n \in I$

12. (A) R; (B) S; (C) P; (D) Q

EXERCISE (JM)

1. 4	**2.** 1	**3.** 1	**4.** 1	**5.** 3	**6.** 1	**7.** 1
8. 1						

EXERCISE (JA)

1. 3	**2.** 7	**3.** 8	**4.** C	**5.** 0.5	**6.** 3	**7.** 2

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JEE (Main/Advanced) Syllabus

JEE (Main) Syllabus :
Trigonometric Identities
JEE (Advanced) Syllabus :
Trigonometric functions, their periodicity and graphs, addition and subtraction formulae, formulae involving multiple and sub-multiple angles.

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Important Notes

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TRIGONOMETRIC RATIOS & IDENTITIES

1. INTRODUCTION TO TRIGONOMETRY :

The word 'trigonometry' is derived from the Greek words 'trigon' and 'metron' and it means 'measuring the sides of a triangle'. The subject was originally developed to solve geometric problems involving triangles. It was studied by sea captains for navigation, surveyor to map out the new lands, by engineers and others. Currently, trigonometry is used in many areas such as the science of seismology, designing electric circuits, describing the state of an atom, predicting the heights of tides in the ocean, analysing a musical tone and in many other areas.

(a) **Measurement of angles :** Commonly two systems of measurement of angles are used.

(i) **Sexagesimal or English System :** Here 1 right angle = 90° (degrees)

1° = 60' (minutes)

1' = 60" (seconds)

(ii) **Circular system :** Here an angle is measured in radians. One radian corresponds to the angle subtended by an arc of length 'r' at the centre of the circle of radius r. It is a constant quantity and does not depend upon the radius of the circle.

(b) Relation between the these systems : $\frac{D}{90} = \frac{R}{\pi/2}$

(c) If θ is the angle subtended at the centre of a circle of radius 'r', by an arc of length 'ℓ' then $\frac{\ell}{r} = \theta$.

Note that here ℓ, r are in the same units and θ is always in radians.

Illustration 1 : If the arcs of same length in two circles subtend angles of 60° and 75° at their centres. Find the ratio of their radii.

Solution : Let r_1 and r_2 be the radii of the given circles and let their arcs of same length 's' subtend angles of 60° and 75° at their centres.

$$\text{Now, } 60° = \left(60 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c \text{ and } 75° = \left(75 \times \frac{\pi}{180}\right)^c = \left(\frac{5\pi}{12}\right)^c$$

$$\therefore \quad \frac{\pi}{3} = \frac{s}{r_1} \text{ and } \frac{5\pi}{12} = \frac{s}{r_2}$$

$$\Rightarrow \frac{\pi}{3} r_1 = s \text{ and } \frac{5\pi}{12} r_2 = s \Rightarrow \frac{\pi}{3} r_1 = \frac{5\pi}{12} r_2 \Rightarrow 4r_1 = 5r_2 \Rightarrow r_1 : r_2 = 5 : 4 \quad \textbf{Ans.}$$

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Do yourself - 1 :

1. The radius of a circle is 30 cm. Find the length of an arc of this circle if the length of the chord of the arc is 30 cm.

2. A man is running around a regular hexagonal field of side length 6m, so that he always at a distance 3 metre from the nearest boundary. Find the length of path travelled by him in one round.

3. Convert the following measurement into radians :

(a) 25º 30' 30" (b) 10º 42' 30" (c) 9º 18' 42"

4. A belt is tied up across two circular pulleys of radii 5m and 1m respectively whose centres are seperated at a distance 8m. (as shown). Find the length of the belt required.

5. Find the number of degrees, minutes and seconds in the angle at the centre of a circle, whose radius is 5m, which is subtended by an arc of length 6 m. (Consider $\pi = 22/7$)

2. T-RATIOS (or Trigonometric functions) :

In a right angle triangle

$\sin\theta = \frac{p}{h}$; $\cos\theta = \frac{b}{h}$; $\tan\theta = \frac{p}{b}$; $\operatorname{cosec}\theta = \frac{h}{p}$; $\sec\theta = \frac{h}{b}$ and $\cot\theta = \frac{b}{p}$

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'p' is perpendicular ; 'b' is base and 'h' is hypotenuse.

Note : The quantity by which the cosine falls short of unity i.e. $1 - \cos\theta$, is called the versed sine θ of θ and also by which the sine falls short of unity i.e. $1 - \sin\theta$ is called the coversed sine of θ.

3. BASIC TRIGONOMETRIC IDENTITIES :

(1) $\sin\theta . \operatorname{cosec}\theta = 1$

(2) $\cos\theta . \sec\theta = 1$

(3) $\tan\theta . \cot\theta = 1$

(4) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ & $\cot\theta = \frac{\cos\theta}{\sin\theta}$

(5) $\sin^2\theta + \cos^2\theta = 1$ or $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$

(6) $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta = 1 + \tan^2\theta$ or $\tan^2\theta = \sec^2\theta - 1$

(7) $\sec\theta + \tan\theta = \frac{1}{\sec\theta - \tan\theta}$

(8) $\operatorname{cosec}^2\theta - \cot^2\theta = 1$ or $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$ or $\cot^2\theta = \operatorname{cosec}^2\theta - 1$

(9) $\operatorname{cosec}\theta + \cot\theta = \frac{1}{\operatorname{cosec}\theta - \cot\theta}$

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(10) Expressing trigonometrical ratio in terms of each other :

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\operatorname{cosec}\theta$
$\sin\theta$	$\sin\theta$	$\sqrt{1-\cos^2\theta}$	$\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\frac{1}{\sqrt{1+\cot^2\theta}}$	$\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\frac{1}{\operatorname{cosec}\theta}$
$\cos\theta$	$\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\frac{1}{\sqrt{1+\tan^2\theta}}$	$\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$\frac{1}{\sec\theta}$	$\frac{\sqrt{\operatorname{cosec}^2\theta-1}}{\operatorname{cosec}\theta}$
$\tan\theta$	$\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\frac{1}{\cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\frac{1}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\cot\theta$	$\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{\tan\theta}$	$\cot\theta$	$\frac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\operatorname{cosec}^2\theta-1}$
$\sec\theta$	$\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\cos\theta}$	$\sqrt{1+\tan^2\theta}$	$\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$\sec\theta$	$\frac{\operatorname{cosec}\theta}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\operatorname{cosec}\theta$	$\frac{1}{\sin\theta}$	$\frac{1}{\sqrt{1-\cos^2\theta}}$	$\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\sqrt{1+\cot^2\theta}$	$\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\operatorname{cosec}\theta$

Illustration 2 : If $\sin\theta+\sin^2\theta=1$, then prove that $\cos^{12}\theta+3\cos^{10}\theta+3\cos^8\theta+\cos^6\theta-1=0$

Solution : Given that $\sin\theta = 1 - \sin^2\theta = \cos^2\theta$

L.H.S. $= \cos^6\theta(\cos^2\theta + 1)^3 - 1 = \sin^3\theta(1 + \sin\theta)^3 - 1 = (\sin\theta + \sin^2\theta)^3 - 1 = 1 - 1 = 0$

Illustration 3 : $4(\sin^6\theta + \cos^6\theta) - 6(\sin^4\theta + \cos^4\theta)$ is equal to

(A) 0 (B) 1 (C) –2 (D) none of these

Solution : $4[(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)] - 6[(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta]$

$= 4[1 - 3\sin^2\theta\cos^2\theta] - 6[1 - 2\sin^2\theta\cos^2\theta]$

$= 4 - 12\sin^2\theta\cos^2\theta - 6 + 12\sin^2\theta\cos^2\theta = -2$ **Ans.(C)**

Do yourself - 2 :

1. If $\cot\theta = \frac{4}{3}$, then find the value of $\sin\theta$, $\cos\theta$ and $\operatorname{cosec}\theta$ in first quadrant.

2. If $\sin\theta + \operatorname{cosec}\theta = 2$, then find the value of $\sin^8\theta + \operatorname{cosec}^8\theta$

Prove the following statements in their respective valid domains :

3. $\cos^4A - \sin^4A + 1 = 2\cos^2A$

4. $(\sin A + \cos A)(1 - \sin A\cos A) = \sin^3A + \cos^3A$.

5. $\frac{\sin A}{1+\cos A} + \frac{1+\cos A}{\sin A} = 2\operatorname{cosec} A$

6. $\cos^6A + \sin^6A = 1 - 3\sin^2A\cos^2A$.

7. $\sqrt{\frac{1-\sin A}{1+\sin A}} = \sec A - \tan A \quad (-90^\circ < A < 90^\circ)$

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8. $\dfrac{1}{\cot A + \tan A} = \sin A \cos A$

9. $\dfrac{1-\tan A}{1+\tan A} = \dfrac{\cot A - 1}{\cot A + 1}$

10. $\dfrac{\tan A}{1-\cot A} + \dfrac{\cot A}{1-\tan A} = \sec A \operatorname{cosec} A + 1$

11. $\dfrac{\cos A}{1-\tan A} + \dfrac{\sin A}{1-\cot A} = \sin A + \cos A.$

12. $\sec^4 A - \sec^2 A = \tan^4 A + \tan^2 A.$

13. $\cot^4 A + \cot^2 A = \operatorname{cosec}^4 A - \operatorname{cosec}^2 A.$

14. $\tan^2 A - \sin^2 A = \sin^4 A \sec^2 A = \tan^2 A.\sin^2 A$

15. $(1 + \cot A - \operatorname{cosec} A)(1 + \tan A + \sec A) = 2$

16. $\dfrac{1}{\operatorname{cosec} A - \cot A} - \dfrac{1}{\sin A} = \dfrac{1}{\sin A} - \dfrac{1}{\operatorname{cosec} A + \cot A}$

17. $\dfrac{\cot A.\cos A}{\cot A + \cos A} = \dfrac{\cot A - \cos A}{\cot A.\cos A}$

18. $\left(\dfrac{1}{\sec^2\alpha - \cos^2\alpha} + \dfrac{1}{\operatorname{cosec}^2\alpha - \sin^2\alpha}\right)\cos^2\alpha\sin^2\alpha = \dfrac{1-\cos^2\alpha\sin^2\alpha}{2+\cos^2\alpha\sin^2\alpha}$

19. $\sin^8 A - \cos^8 A = (\sin^2 A - \cos^2 A)(1 - 2\sin^2 A\cos^2 A)$

20. $\dfrac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \dfrac{1+\sin A}{\cos A}$

21. $(\tan\alpha + \operatorname{cosec}\beta)^2 - (\cot\beta - \sec\alpha)^2 = 2\tan\alpha\cot\beta(\operatorname{cosec}\alpha + \sec\beta).$

22. $2\sec^2\alpha - \sec^4\alpha - 2\operatorname{cosec}^2\alpha + \operatorname{cosec}^4\alpha = \cot^4\alpha - \tan^4\alpha.$

23. If $\sin\theta$ equals to $\dfrac{x^2-y^2}{x^2+y^2}$, find the value of $\cos\theta$ and $\cot\theta$, where $\theta \in (0,90^\circ)$.

24. If $\sin\theta = \dfrac{m^2+2mn}{m^2+2mn+2n^2}$, prove that $\tan\theta = \dfrac{m^2+2mn}{2mn+2n^2}$, where $\theta \in (0,90^\circ)$.

25. If $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$, prove that $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$.

26. Prove that $\operatorname{cosec}^6\alpha - \cot^6\alpha = 3\operatorname{cosec}^2\alpha\cot^2\alpha + 1.$

27. Express $2\sec^2 A - \sec^4 A - 2\operatorname{cosec}^2 A + \operatorname{cosec}^4 A$ in terms of $\tan A$.

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4. NEW DEFINITION OF T-RATIOS :

By using rectangular coordinates the definitions of trigonometric functions can be extended to angles of any size in the following way (see diagram). A point P is taken with coordinates (x, y). The radius vector OP has length r and the angle θ is taken as the directed angle measured anticlockwise from the x-axis. The three main trigonometric functions are then defined in terms of r and the coordinates x and y.

$\sin\theta = y/r$,

$\cos\theta = x/r$

$\tan\theta = y/x$,

(The other function are reciprocals of these)

This can give negative values of the trigonometric functions.

5. SIGNS OF TRIGONOMETRIC FUNCTIONS IN DIFFERENT QUADRANTS :

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6. TRIGONOMETRIC FUNCTIONS OF ALLIED ANGLES :

(a) $\sin(2n\pi + \theta) = \sin\theta$, $\cos(2n\pi + \theta) = \cos\theta$, where $n \in I$

(b)

$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$
$\sin(90° - \theta) = \cos\theta$	$\cos(90° - \theta) = \sin\theta$
$\sin(90° + \theta) = \cos\theta$	$\cos(90° + \theta) = -\sin\theta$
$\sin(180° - \theta) = \sin\theta$	$\cos(180° - \theta) = -\cos\theta$
$\sin(180° + \theta) = -\sin\theta$	$\cos(180° + \theta) = -\cos\theta$
$\sin(270° - \theta) = -\cos\theta$	$\cos(270° - \theta) = -\sin\theta$
$\sin(270° + \theta) = -\cos\theta$	$\cos(270° + \theta) = \sin\theta$
$\sin(360° - \theta) = -\sin\theta$	$\cos(360° - \theta) = \cos\theta$
$\sin(360° + \theta) = \sin\theta$	$\cos(360° + \theta) = \cos\theta$

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Do yourself - 3 :

1. If $\sin A = \frac{11}{61}$, find tanA, cosA, and sec A.

2. If $\cos\theta = \frac{4}{5}$, find sin θ and cot θ.

3. If $\tan\theta = \frac{1}{\sqrt{7}}$, find the value of $\frac{\operatorname{cosec}^2\theta - \sec^2\theta}{\operatorname{cosec}^2\theta + \sec^2\theta}$

4. If $\cot\theta = \frac{15}{8}$, find sin θ and cosec θ.

5. If 2 sin θ = 2 – cos θ, find sin θ.

6. If 8 sin θ = 4 + cos θ, find sin θ.

7. If tan θ + sec θ = 1.5, find sin θ.

8. If cot θ + cosec θ = 5, find cos θ.

9. If 3 $\sec^4\theta + 8 = 10\sec^2\theta$, find the values of tan θ.

10. If $\tan^2\theta + \sec\theta = 5$, find cos θ.

11. If tan θ + cot θ = 2, find sin θ.

12. If $\sec^2\theta = 2 + 2\tan\theta$, find tan θ.

13. If $\tan\theta = \frac{2x(x+1)}{2x+1}$, find sin θ and cos θ.

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7. VALUES OF T-RATIOS OF SOME STANDARD ANGLES :

Angles	**0°**	**30°**	**45°**	**60°**	**90°**	**180°**	**270°**
T-ratio	**0**	**π/6**	**π/4**	**π/3**	**π/2**	**π**	**3π/2**
sin θ	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0	–1
cos θ	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0	–1	0
tan θ	0	$1/\sqrt{3}$	1	$\sqrt{3}$	N.D.	0	N.D.
cot θ	N.D.	$\sqrt{3}$	1	$1/\sqrt{3}$	0	N.D.	0
sec θ	1	$2/\sqrt{3}$	$\sqrt{2}$	2	N.D.	–1	N.D.
cosecθ	N.D.	2	$\sqrt{2}$	$2/\sqrt{3}$	1	N.D.	–1

N.D. → Not Defined

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$ where $n \in I$

(b) $\sin(2n+1)\frac{\pi}{2} = (-1)^n$; $\cos(2n+1)\frac{\pi}{2} = 0$ where $n \in I$

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Do yourself - 4 :

1. Verify the following identities for A = 30º as well as for A = 45º

(a) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

(b) $\sin 2A = 2\sin A \cos A$ (c) $\cos 3A = 4\cos^3 A - 3\cos A$

(d) $\sin 3A = 3\sin A - 4\sin^3 A$ (e) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

2. Find the value of

(a) $\sin^2 30^\circ + \sin^2 45^\circ + \sin^2 60^\circ$ (b) $\tan^2 30^\circ + \tan^2 45^\circ + \tan^2 60^\circ$

(c) $\sin 30^\circ \cos 60^\circ + \sin 30^\circ \sin 60^\circ$ (d) $\cos 45^\circ \cos 60^\circ - \sin 45^\circ \sin 60^\circ$

Illustration 4 : If $\sin\theta = -\dfrac{1}{2}$ and $\tan\theta = \dfrac{1}{\sqrt{3}}$ then θ is equal to -

(A) 30° (B) 150° (C) 210° (D) none of these

Solution : Let us first find out θ lying between 0 and 360°.

Since $\sin\theta = -\dfrac{1}{2} \Rightarrow \theta = 210^\circ$ or 330° and $\tan\theta = \dfrac{1}{\sqrt{3}} \Rightarrow \theta = 30^\circ$ or 210°

Hence , $\theta = 210^\circ$ or $\dfrac{7\pi}{6}$ is the value satisfying both. **Ans. (C)**

Do yourself - 5 :

1. (i) If $\cos\theta = -\dfrac{1}{2}$ and $\pi < \theta < \dfrac{3\pi}{2}$, then find the value of $4\tan^2\theta - 3\operatorname{cosec}^2\theta$.

(ii) Prove that : (a) $\cos 570^\circ \sin 510^\circ + \sin(-330^\circ)\cos(-390^\circ) = 0$

(b) $\tan\dfrac{11\pi}{3} - 2\sin\dfrac{9\pi}{3} - \dfrac{3}{4}\operatorname{cosec}^2\dfrac{\pi}{4} + 4\cos^2\dfrac{17\pi}{6} = \dfrac{3-2\sqrt{3}}{2}$

2. Evaluate :

(a) $\sin 420^\circ \cos 390^\circ + \cos(-300^\circ)\sin(-330^\circ)$ (b) $\tan 225^\circ \cot 405^\circ + \tan 765^\circ \cot 675^\circ$

3. What are the values of $\cos A - \sin A$ and $\tan A + \cot A$ when A has the values

(a) $\dfrac{2\pi}{3}$ (b) $\dfrac{7\pi}{4}$ (c) $\dfrac{11\pi}{3}$ (d) $\dfrac{5\pi}{4}$

4. Express the following quantities in terms of the ratios of a positive angle, which is less than 45º

(a) $\sin(-65^\circ)$ (b) $\cos(-928^\circ)$ (c) $\tan 1145^\circ$ (d) $\cot(-1054^\circ)$

5. What is the sign of $\sin A + \cos A$ for the following values of A ?

(a) 140º (b) –356º (c) –1125º.

6. What is the sign of $\sin A - \cos A$ for the following values of A ?

(a) 215º (b) –634º (c) – 457º

7. Find the sines and cosines of all angles in the first four quadrant whose tangents are equal to cos 135º.

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8. GRAPH OF TRIGONOMETRIC FUNCTIONS :

(i) y = sinx

(ii) y = cosx

(iii) y = tanx

(iv) y = cotx

(v) y = secx

(vi) y = cosecx

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9. DOMAINS, RANGES AND PERIODICITY OF TRIGONOMETRIC FUNCTIONS :

T-Ratio	Domain	Range	Period
sin x	R	[–1,1]	2π
cos x	R	[–1,1]	2π
tan x	R–{(2n+1)π/2 ; n∈I}	R	π
cot x	R–{nπ : n ∈ I}	R	π
sec x	R– {(2n+1) π/2 : n ∈ I}	(–∞,–1] ∪[1,∞)	2π
cosec x	R– {nπ : n ∈ I}	(–∞,–1] ∪[1,∞)	2π

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Do yourself - 6 :

1. Prove that the equation $\sin\theta = x + \frac{1}{x}$ is impossible if x be real.

2. Show that the equation $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is only possible when x = y.

3. The number of real solutions of the equation $\sin(e^x) = 2^x + 2^{-x}$ is -
   (A) 1 (B) 0 (C) 2 (D) Infinite

4. Draw graphs of
   (a) y = sin2x (b) y = 2cos3x (c) y = 4 tanx

5. Find number of solutions of the equation $\sin\pi x + x^2 + 1 = 2x$

10. TRIGONOMETRIC RATIOS OF THE SUM & DIFFERENCE OF TWO ANGLES :

(i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$.

(ii) $\sin(A - B) = \sin A \cos B - \cos A \sin B$.

(iii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(iv) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(v) $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

(vi) $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

(vii) $\cot(A + B) = \frac{\cot B \cot A - 1}{\cot B + \cot A}$

(viii) $\cot(A - B) = \frac{\cot B \cot A + 1}{\cot B - \cot A}$

Some more results :

(i) $\sin^2 A - \sin^2 B = \sin(A + B).\sin(A - B) = \cos^2 B - \cos^2 A$.

(ii) $\cos^2 A - \sin^2 B = \cos(A+B).\cos(A - B)$.

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Illustration 5 : Prove that $\sqrt{3}\ \text{cosec}20° - \sec 20° = 4$.

Solution : L.H.S. $= \frac{\sqrt{3}}{\sin 20°} - \frac{1}{\cos 20°} = \frac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20°.\cos 20°}$

$$= \frac{4\left(\frac{\sqrt{3}}{2}\cos 20° - \frac{1}{2}\sin 20°\right)}{2\sin 20° \cos 20°} = \frac{4(\sin 60.\cos 20° - \cos 60°.\sin 20°)}{\sin 40°}$$

$$= 4.\frac{\sin(60° - 20°)}{\sin 40°} = 4.\frac{\sin 40°}{\sin 40°} = 4 = \text{R.H.S.}$$

Illustration 6 : Prove that $\tan 70° = \cot 70° + 2\cot 40°$ **.**

Solution : L.H.S. $= \tan 70° = \tan(20° + 50°) = \frac{\tan 20° + \tan 50°}{1 - \tan 20° \tan 50°}$

or $\tan 70° - \tan 20° \tan 50° \tan 70° = \tan 20° + \tan 50°$

or $\tan 70° = \tan 70° \tan 50° \tan 20° + \tan 20° + \tan 50° = 2\tan 50° + \tan 20°$

$= \cot 70° + 2\cot 40°$ = R.H.S.

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Do yourself - 7 :

1. If $\sin A = \frac{3}{5}$ and $\cos B = \frac{9}{41}$, $0 < A, B < \frac{\pi}{2}$, then find the value of the following :

(a) $\sin(A + B)$ (b) $\sin(A - B)$ (c) $\cos(A + B)$ (d) $\cos(A - B)$

2. If $x + y = 45°$, then prove that :

(a) $(1 + \tan x)(1 + \tan y) = 2$ (b) $(\cot x - 1)(\cot y - 1) = 2$

(Remember these results)

3. If $\sin\alpha = \frac{3}{5}$ and $\cos\beta = \frac{9}{41}$, find the value of $\sin(\alpha - \beta)$ and $\cos(\alpha + \beta)$.

4. If $\sin\alpha = \frac{15}{17}$ and $\cos\beta = \frac{12}{13}$, find the values of $\sin(\alpha + \beta)$, $\cos(\alpha - \beta)$, and $\tan(\alpha + \beta)$

Prove that

5. $\cos(45° - A)\cos(45° - B) - \sin(45° - A)\sin(45° - B) = \sin(A + B)$.

6. $\sin(45° + A)\cos(45° - B) + \cos(45° + A)\sin(45° - B) = \cos(A - B)$.

7. $\frac{\sin(A-B)}{\cos A\cos B} + \frac{\sin(B-C)}{\cos B\cos C} + \frac{\sin(C-A)}{\cos C\cos A} = 0$

8. $\sin105° + \cos105° = \cos45°$

9. $\sin75° - \sin15° = \cos105° + \cos15°$

10. $\cos(\alpha + \beta)\cos\gamma - \cos(\beta + \gamma)\cos\alpha = \sin\beta\sin(\gamma - \alpha)$

11. $\sin(n + 1)A\sin(n - 1)A + \cos(n + 1)A\cos(n - 1)A = \cos2A$.

12. $\sin(n + 1)A\sin(n + 2)A + \cos(n + 1)A\cos(n + 2)A = \cos A$.

13. If $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$, find the values of $\tan(2A + B)$ and $\tan(2A - B)$

14. If $\tan\alpha = \frac{5}{6}$ and $\tan\beta = \frac{1}{11}$, prove that $\alpha + \beta = \frac{\pi}{4}$

Prove that

15. $\tan\left(\frac{\pi}{4} + \theta\right) \times \tan\left(\frac{3\pi}{4} + \theta\right) = -1$

16. $\cot\left(\frac{\pi}{4} + \theta\right)\cot\left(\frac{\pi}{4} - \theta\right) = 1$

17. $1 + \tan A\tan\frac{A}{2} = \tan A\cot\frac{A}{2} - 1 = \sec A$

18. $\tan3A\tan2A\tan A = \tan3A - \tan2A - \tan A$.

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11. FORMULAE TO TRANSFORM THE PRODUCT INTO SUM OR DIFFERENCE :

(i) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$. **(ii)** $2\cos A\sin B = \sin(A+B) - \sin(A-B)$.

(iii) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$ **(iv)** $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Illustration 7 : If $\sin 2A = \lambda \sin 2B$, then prove that $\dfrac{\tan(A+B)}{\tan(A-B)} = \dfrac{\lambda+1}{\lambda-1}$.

Solution : Given $\sin 2A = \lambda \sin 2B$

$$\Rightarrow \frac{\sin 2A}{\sin 2B} = \frac{\lambda}{1}$$

Applying componendo & dividendo,

$$\frac{\sin 2A + \sin 2B}{\sin 2B - \sin 2A} = \frac{\lambda+1}{1-\lambda}$$

$$\Rightarrow \frac{2\sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right)}{2\cos\left(\frac{2B+2A}{2}\right)\sin\left(\frac{2B-2A}{2}\right)} = \frac{\lambda+1}{1-\lambda}$$

$$\Rightarrow \frac{\sin(A+B)\cos(A-B)}{\cos(A+B)\sin\{-(A-B)\}} = \frac{\lambda+1}{1-\lambda} \quad \Rightarrow \quad \frac{\sin(A+B)\cos(A-B)}{\cos(A+B)\times -\sin(A-B)} = \frac{\lambda+1}{-(\lambda-1)}$$

$$\Rightarrow \frac{\sin(A+B)\cos(A-B)}{\cos(A+B)\sin(A-B)} = \frac{\lambda+1}{\lambda-1} \quad \Rightarrow \quad \tan(A+B)\cot(A-B) = \frac{\lambda+1}{\lambda-1}$$

$$\Rightarrow \frac{\tan(A+B)}{\tan(A-B)} = \frac{\lambda+1}{\lambda-1}$$

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Do yourself - 8 :

1. Simplify $\dfrac{\sin 75° - \sin 15°}{\cos 75° + \cos 15°}$

Express follwing as a sum or difference of angles used in arguments :

2. $2\sin 5\theta \sin 7\theta$.

3. $2\sin 54° \sin 66°$

4. $\cos(36° - A)\cos(36° + A) + \cos(54° + A)\cos(54° - A) = \cos 2A$.

5. $\cos A\sin(B - C) + \cos B\sin(C - A) + \cos C\sin(A - B) = 0$.

6. $\sin(45° + A)\sin(45° - A) = \dfrac{1}{2}\cos 2A$

7. $\sin(\beta - \gamma)\cos(\alpha - \delta) + \sin(\gamma - \alpha)\cos(\beta - \delta) + \sin(\alpha - \beta)\cos(\gamma - \delta) = 0$.

8. $2\cos\dfrac{\pi}{13}\cos\dfrac{9\pi}{13} + \cos\dfrac{3\pi}{13} + \cos\dfrac{5\pi}{13} = 0$

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12. FORMULAE TO TRANSFORM SUM OR DIFFERENCE INTO PRODUCT :

(i) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(ii) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(iii) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(iv) $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

Illustration 8 : $\dfrac{\sin 5\theta + \sin 2\theta - \sin\theta}{\cos 5\theta + 2\cos 3\theta + 2\cos^2\theta + \cos\theta}$ is equal to -

(A) $\tan\theta$ (B) $\cos\theta$ (C) $\cot\theta$ (D) none of these

Solution : L.H.S.= $\dfrac{2\sin 2\theta\cos 3\theta + \sin 2\theta}{2\cos 3\theta.\cos 2\theta + 2\cos 3\theta + 2\cos^2\theta} = \dfrac{\sin 2\theta[2\cos 3\theta + 1]}{2[\cos 3\theta(\cos 2\theta + 1) + (\cos^2\theta)]}$

$= \dfrac{\sin 2\theta[2\cos 3\theta + 1]}{2[\cos 3\theta(2\cos^2\theta) + \cos^2\theta]} = \dfrac{\sin 2\theta(2\cos 3\theta + 1)}{2\cos^2\theta(2\cos 3\theta + 1)} = \tan\theta$ **Ans. (A)**

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Do yourself - 9 :

1. Prove that

(a) $(\sin 3A + \sin A)\sin A + (\cos 3A - \cos A)\cos A = 0$ **(b)** $\dfrac{\sin 8\theta\cos\theta - \sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta - \sin 3\theta\sin 4\theta} = \tan 2\theta$

Prove that

2. $\dfrac{\sin 7\theta - \sin 5\theta}{\cos 7\theta + \cos 5\theta} = \tan\theta$

3. $\dfrac{\cos 2B + \cos 2A}{\cos 2B - \cos 2A} = \cot(A+B)\cot(A-B)$

4. $\dfrac{\sin 2A + \sin 2B}{\sin 2A - \sin 2B} = \dfrac{\tan(A+B)}{\tan(A-B)}$

5. $\cos(A + B) + \sin(A - B) = 2\sin(45^\circ + A)\cos(45^\circ + B)$

6. $\dfrac{\cos 3A - \cos A}{\sin 3A - \sin A} + \dfrac{\cos 2A - \cos 4A}{\sin 4A - \sin 2A} = \dfrac{\sin A}{\cos 2A\cos 3A}$

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7. $\dfrac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = 4\cos 2\theta \cos 4\theta$

8. $\dfrac{\cos 3\theta + 2\cos 5\theta + \cos 7\theta}{\cos\theta + 2\cos 3\theta + \cos 5\theta} = \cos 2\theta - \sin 2\theta \tan 3\theta$

9. $\dfrac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A$

10. $\dfrac{\sin(A-C) + 2\sin A + \sin(A+C)}{\sin(B-C) + 2\sin B + \sin(B+C)} = \dfrac{\sin A}{\sin B}$

11. $\dfrac{\sin A - \sin 5A + \sin 9A - \sin 13A}{\cos A - \cos 5A - \cos 9A + \cos 13A} = \cot 4A$

12. $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \tan\dfrac{A+B}{2}\cot\dfrac{A-B}{2}$

13. $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan\dfrac{A+B}{2}$

14. $\dfrac{\sin A - \sin B}{\cos B - \cos A} = \cot\dfrac{A+B}{2}$

15. $\dfrac{\cos(A+B+C) + \cos(-A+B+C) + \cos(A-B+C) + \cos(A+B-C)}{\sin(A+B+C) + \sin(-A+B+C) - \sin(A-B+C) + \sin(A+B-C)} = \cot B$

16. $\cos\left\{\theta + \left(n - \dfrac{3}{2}\right)\phi\right\} - \cos\left\{\theta + \left(n + \dfrac{3}{2}\right)\phi\right\} = 2\sin\dfrac{3\phi}{2}.\sin(\theta + n\phi)$

17. $\sin\dfrac{\theta}{2}\sin\dfrac{7\theta}{2} + \sin\dfrac{3\theta}{2}\sin\dfrac{11\theta}{2} = \sin 2\theta \sin 5\theta$

18. $\cos 2\theta \cos\dfrac{\theta}{2} - \cos 3\theta \cos\dfrac{9\theta}{2} = \sin 5\theta \sin\dfrac{5\theta}{2}$

19. $\dfrac{2\sin(A-C)\cos C - \sin(A-2C)}{2\sin(B-C)\cos C - \sin(B-2C)} = \dfrac{\sin A}{\sin B}$

20. $\dfrac{\sin A \sin 2A + \sin 3A \sin 6A + \sin 4A \sin 13A}{\sin A \cos 2A + \sin 3A \cos 6A + \sin 4A \cos 13A} = \tan 9A$

21. $\dfrac{\cos 2A \cos 3A - \cos 2A \cos 7A + \cos A \cos 10A}{\sin 4A \sin 3A - \sin 2A \sin 5A + \sin 4A \sin 7A} = \cot 6A \cot 5A$

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13. TRIGONOMETRIC RATIOS OF SUM OF MORE THAN TWO ANGLES :

(i) $\sin(A+B+C) = \sin A\cos B\cos C + \sin B\cos A\cos C + \sin C\cos A\cos B - \sin A\sin B\sin C$

$= \Sigma\sin A\cos B\cos C - \Pi\sin A$

$= \cos A\cos B\cos C\,[\tan A + \tan B + \tan C - \tan A\tan B\tan C]$

(ii) $\cos(A+B+C) = \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C$

$= \Pi\cos A - \Sigma\sin A\sin B\cos C$

$= \cos A\cos B\cos C\,[1 - \tan A\tan B - \tan B\tan C - \tan C\tan A]$

(iii) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A} = \dfrac{S_1 - S_3}{1 - S_2}$

14. TRIGONOMETRIC RATIOS OF MULTIPLE ANGLES :

(a) Trigonometrical ratios of an angle 2θ in terms of the angle θ :

(i) $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

(ii) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

(iii) $1 + \cos 2\theta = 2\cos^2\theta$ **(iv)** $1 - \cos 2\theta = 2\sin^2\theta$

(v) $\tan\theta = \dfrac{1-\cos 2\theta}{\sin 2\theta} = \dfrac{\sin 2\theta}{1+\cos 2\theta}$ **(vi)** $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Illustration 9 : Prove that : $\dfrac{2\cos 2A + 1}{2\cos 2A - 1} = \tan(60° + A)\tan(60° - A)$.

Solution : R.H.S. $= \tan(60° + A)\tan(60° - A)$

$$= \left(\frac{\tan 60° + \tan A}{1 - \tan 60°\tan A}\right)\left(\frac{\tan 60° - \tan A}{1 + \tan 60°\tan A}\right) = \left(\frac{\sqrt{3} + \tan A}{1 - \sqrt{3}\tan A}\right)\left(\frac{\sqrt{3} - \tan A}{1 + \sqrt{3}\tan A}\right)$$

$$= \frac{3 - \tan^2 A}{1 - 3\tan^2 A} = \frac{3 - \dfrac{\sin^2 A}{\cos^2 A}}{1 - 3\dfrac{\sin^2 A}{\cos^2 A}} = \frac{3\cos^2 A - \sin^2 A}{\cos^2 A - 3\sin^2 A} == \frac{2\cos^2 A + \cos^2 A - 2\sin^2 A + \sin^2 A}{2\cos^2 A - 2\sin^2 A - \sin^2 A - \cos^2 A}$$

$$= \frac{2(\cos^2 A - \sin^2 A) + \cos^2 A + \sin^2 A}{2(\cos^2 A - \sin^2 A) - (\sin^2 A + \cos^2 A)} = \frac{2\cos 2A + 1}{2\cos 2A - 1} = \text{L.H.S.}$$

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Do yourself - 10 :

Prove that :

1. $\dfrac{\sin 2A}{1+\cos 2A} = \tan A$ (Remember)

2. $\dfrac{\sin 2A}{1-\cos 2A} = \cot A$ (Remember)

3. $\tan A - \cot A = -2\cot 2A$. (Remember)

4. $\dfrac{1+\sin 2\theta+\cos 2\theta}{1+\sin 2\theta-\cos 2\theta} = \cot\theta$

5. $\tan A + \cot A = 2\,\text{cosec}\,2A$

6. $\dfrac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)} = \tan\dfrac{A}{2}\cot\dfrac{B}{2}$

7. $\dfrac{\sec 8A-1}{\sec 4A-1} = \dfrac{\tan 8A}{\tan 2A}$

8. $\dfrac{1+\tan^2(45°-A)}{1-\tan^2(45°-A)} = \text{cosec}\,2A$

9. $\dfrac{\sin^2 A-\sin^2 B}{\sin A\cos A-\sin B\cos B} = \tan(A+B)$

10. $\tan\left(\dfrac{\pi}{4}+\theta\right)-\tan\left(\dfrac{\pi}{4}-\theta\right) = 2\tan 2\theta$

11. $\cot(A + 15°) - \tan(A - 15°) = \dfrac{4\cos 2A}{1+2\sin 2A}$

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12. $\dfrac{\sin(n+1)A+2\sin nA+\sin(n-1)A}{\cos(n-1)A-\cos(n+1)A} = \cot\dfrac{A}{2}$

13. $\sin 3A + \sin 2A - \sin A = 4\sin A\cos\dfrac{A}{2}\cos\dfrac{3A}{2}$

14. $\tan 2A = (\sec 2A+1)\sqrt{\sec^2 A-1}$

15. $\cos^3 2\theta + 3\cos 2\theta = 4(\cos^6\theta - \sin^6\theta)$

16. $1+\cos^2 2\theta = 2(\cos^4\theta + \sin^4\theta)$

17. $\sec^2 A(1 + \sec 2A) = 2\sec 2A$

18. $\text{cosec}A - 2\cot 2A\cos A = 2\sin A$

19. $\cot A = \dfrac{1}{2}\left(\cot\dfrac{A}{2}-\tan\dfrac{A}{2}\right)$

20. $\dfrac{2\cos 2^n\theta+1}{2\cos\theta+1} = (2\cos\theta-1)(2\cos 2\theta-1)(2\cos 2^2\theta-1)....(2\cos 2^{n-1}\theta-1)$

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(b) Trigonometrical ratios of an angle 3θ in terms of the angle θ :

(i) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. **(ii)** $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(iii) $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

Illustration 10 : Show that sin12°.sin48°.sin54° = 1/8

Solution : L.H.S. $= \frac{1}{2}[\cos 36° - \cos 60°]\sin 54° = \frac{1}{2}\left[\cos 36° \sin 54° - \frac{1}{2}\sin 54°\right]$

$$= \frac{1}{4}[2\cos 36° \sin 54° - \sin 54°] = \frac{1}{4}[\sin 90° + \sin 18° - \sin 54°]$$

$$= \frac{1}{4}[1 - (\sin 54° - \sin 18°)] = \frac{1}{4}[1 - 2\sin 18° \cos 36°]$$

$$= \frac{1}{4}\left[1 - \frac{2\sin 18°}{\cos 18°}\cos 18° \cos 36°\right] = \frac{1}{4}\left[1 - \frac{\sin 36° \cos 36°}{\cos 18°}\right]$$

$$= \frac{1}{4}\left[1 - \frac{2\sin 36° \cos 36°}{2\cos 18°}\right] = \frac{1}{4}\left[1 - \frac{\sin 72°}{2\sin 72°}\right] = \frac{1}{4}\left[1 - \frac{1}{2}\right] = \frac{1}{8} = \text{R.H.S.}$$

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Illustration 11 : Prove that : tanA + tan(60° + A) + tan(120° + A) = 3tan3A

Solution : L.H.S. = tanA + tan(60° + A) + tan(120° + A)

= tanA + tan(60° + A) + tan{180° –(60° – A)}

= tanA + tan(60° + A) – tan(60° – A) [∵ tan(180° – θ) = –tanθ]

$$= \tan A + \frac{\tan 60° + \tan A}{1 - \tan 60° \tan A} - \frac{\tan 60° - \tan A}{1 + \tan 60° \tan A} = \tan A + \frac{\sqrt{3} + \tan A}{1 - \sqrt{3}\tan A} - \frac{\sqrt{3} - \tan A}{1 + \sqrt{3}\tan A}$$

$$= \tan A + \frac{\sqrt{3} + \tan A + 3\tan A + \sqrt{3}\tan^2 A - \sqrt{3} + \tan A + 3\tan A - \sqrt{3}\tan^2 A}{(1 - \sqrt{3}\tan A)(1 + \sqrt{3}\tan A)}$$

$$= \tan A + \frac{8\tan A}{1 - 3\tan^2 A} = \frac{\tan A - 3\tan^3 A + 8\tan A}{1 - 3\tan^2 A}$$

$$= \frac{9\tan A - 3\tan^3 A}{1 - 3\tan^2 A} = 3\left(\frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}\right) = 3\tan 3A = \text{R.H.S.}$$

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Do yourself - 11 :

1 Prove that :

(a) $\cot\theta \cot(60° - \theta) \cot(60° + \theta) = \cot 3\theta$ (b) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(c) $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}$

Prove that

2. $\sin\alpha \sin(60° - \alpha) \sin(60° + \alpha) = \frac{1}{4}\sin 3\alpha$

3. $\cos\alpha \cos(60° - \alpha) \cos(60° + \alpha) = \frac{1}{4}\cos 3\alpha$

4. $\cot\alpha + \cot(60° + \alpha) - \cot(60° - \alpha) = 3\cot 3\alpha$.

5. $\cos 4\alpha = 1 - 8\cos^2\alpha + 8\cos^4\alpha$

6. $\sin 4A = 4\sin A\cos^3 A - 4\cos A\sin^3 A$.

7. $\cos 6\alpha = 32\cos^6\alpha - 48\cos^4\alpha + 18\cos^2\alpha - 1$.

8. If $\cos x + \sin x = a, \left(-\frac{\pi}{2} < x < -\frac{\pi}{4}\right)$, then $\cos 2x$ is equal to

(A) a^2 (B) $a\sqrt{(2-a)}$ (C) $a\sqrt{(2+a)}$ (D) $a\sqrt{(2-a^2)}$

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9. If $\cos A = \frac{3}{4}$, then the value of expression $32\sin\frac{A}{2}\sin\frac{5A}{2}$ is equal to

(A) 11 (B) –11 (C) 12 (D) 4

15. TRIGONOMETRIC RATIOS OF SUB MULTIPLE ANGLES :

Since the trigonometric relations are true for all values of angle θ, they will be true if instead of θ be substitute $\frac{\theta}{2}$

(i) $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

(ii) $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2} = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

(iii) $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$ **(iv)** $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$

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(v) $\tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}=\frac{\sin\theta}{1+\cos\theta}$

(vi) $\tan\theta=\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

(vii) $\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}$

(viii) $\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos\theta}{2}}$

(ix) $\tan\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

(x) $2\sin\frac{\theta}{2}=\pm\sqrt{1+\sin\theta}\pm\sqrt{1-\sin\theta}$

(xi) $2\cos\frac{\theta}{2}=\pm\sqrt{1+\sin\theta}\mp\sqrt{1-\sin\theta}$

(xii) $\tan\frac{\theta}{2}=\frac{\pm\sqrt{1+\tan^2\theta}-1}{\tan\theta}$

for **(vii)** to **(xii)** , we decide the sign of ratio according to value of θ.

Illustration 12: $\sin 67\frac{1}{2}^\circ+\cos 67\frac{1}{2}^\circ$ is equal to

(A) $\frac{1}{2}\sqrt{4+2\sqrt{2}}$ (B) $\frac{1}{2}\sqrt{4-2\sqrt{2}}$ (C) $\frac{1}{4}\left(\sqrt{4+2\sqrt{2}}\right)$ (D) $\frac{1}{4}\left(\sqrt{4-2\sqrt{2}}\right)$

Solution : $\sin 67\frac{1}{2}^\circ+\cos 67\frac{1}{2}^\circ=\sqrt{1+\sin 135^\circ}=\sqrt{1+\frac{1}{\sqrt{2}}}$ (using $\cos A+\sin A=\sqrt{1+\sin 2A}$)

$=\frac{1}{2}\sqrt{4+2\sqrt{2}}$ **Ans.(A)**

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Do yourself - 12 :

1 Find the value of

(a) $\sin\frac{\pi}{8}$ (b) $\cos\frac{\pi}{8}$ (c) $\tan\frac{\pi}{8}$

2. $\frac{\cos A}{1\mp\sin A}=\tan\left(45^\circ\pm\frac{A}{2}\right)$

3. $\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}=\tan\frac{\theta}{2}$

4. If $\sin\theta=\frac{1}{2}$ and $\sin\phi=\frac{1}{3}$, find the values of $\sin(\theta+\phi)$ and $\sin(2\theta+2\phi)$.

5. If $\cos\alpha=\frac{11}{61}$ and $\sin\beta=\frac{4}{5}$, find the values of $\sin^2\frac{\alpha-\beta}{2}$ and $\cos^2\frac{\alpha+\beta}{2}$, the angles α and β being positive acute angles.

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6. If $\cos\alpha = \frac{3}{5}$ and $\cos\beta = \frac{4}{5}$, find the value of $\cos\frac{\alpha-\beta}{2}$, the angles α and β being positive acute angles.

7. Given $\sec\theta = 1\frac{1}{4}$, find $\tan\frac{\theta}{2}$ and $\tan\theta$.

8. Find the values of (a) $\sin 7\frac{1}{2}^\circ$ (b) $\cos 7\frac{1}{2}^\circ$ (c) $\tan 11\frac{1}{4}^\circ$

9. If $\sin\theta + \sin\phi = a$ and $\cos\theta + \cos\phi = b$, find the value of $\tan\frac{\theta-\phi}{2}$.

Prove that

10. $(\cos\alpha+\cos\beta)^2+(\sin\alpha-\sin\beta)^2 = 4\cos^2\frac{\alpha+\beta}{2}$.

11. $(\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2 = 4\cos^2\frac{\alpha-\beta}{2}$.

12. $(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2 = 4\sin^2\frac{\alpha-\beta}{2}$

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13. $\sec\left(\frac{\pi}{4}+\theta\right)\sec\left(\frac{\pi}{4}-\theta\right) = 2\sec 2\theta$

14. $\tan\left(45^\circ+\frac{A}{2}\right) = \sqrt{\frac{1+\sin A}{1-\sin A}} = \sec A + \tan A$

15. $\sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right)-\sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right) = \frac{1}{\sqrt{2}}\sin A$

16. $\cos^2\alpha + \cos^2(\alpha + 120^\circ) + \cos^2(\alpha - 120^\circ) = \frac{3}{2}$

17. $\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8} = \frac{3}{2}$

18. $\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8} = \frac{3}{2}$

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16. TRIGONOMETRIC RATIOS OF SOME STANDARD ANGLES :

(i) $\sin 18° = \sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4} = \cos 72° = \cos\frac{2\pi}{5}$

(ii) $\cos 36° = \cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4} = \sin 54° = \sin\frac{3\pi}{10}$

(iii) $\sin 72° = \sin\frac{2\pi}{5} = \frac{\sqrt{10+2\sqrt{5}}}{4} = \cos 18° = \cos\frac{\pi}{10}$

(iv) $\sin 36° = \sin\frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54° = \cos\frac{3\pi}{10}$

(v) $\sin 15° = \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75° = \cos\frac{5\pi}{12}$

(vi) $\cos 15° = \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75° = \sin\frac{5\pi}{12}$

(vii) $\tan 15° = \tan\frac{\pi}{12} = 2-\sqrt{3} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \cot 75° = \cot\frac{5\pi}{12}$

(viii) $\tan 75° = \tan\frac{5\pi}{12} = 2+\sqrt{3} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \cot 15° = \cot\frac{\pi}{12}$

(ix) $\tan(22.5°) = \tan\frac{\pi}{8} = \sqrt{2}-1 = \cot(67.5°) = \cot\frac{3\pi}{8}$

(x) $\tan(67.5°) = \tan\frac{3\pi}{8} = \sqrt{2}+1 = \cot(22.5°) = \cot\frac{\pi}{8}$

Illustration 13 : Evaluate sin78° – sin66° – sin42° + sin6°.

Solution : The expression $= (\sin 78° - \sin 42°) - (\sin 66° - \sin 6°) = 2\cos(60°)\sin(18°) - 2\cos 36°.\sin 30°$

$$= \sin 18° - \cos 36° = \left(\frac{\sqrt{5}-1}{4}\right) - \left(\frac{\sqrt{5}+1}{4}\right) = -\frac{1}{2}$$

Do yourself - 13 :

1 Find the value of

(a) $\sin\frac{\pi}{10} + \sin\frac{13\pi}{10}$ (b) $\cos^2 48° - \sin^2 12°$

Evaluate :

2. $\sin^2 72° - \sin^2 60°$

3. $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5}$

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4. $\tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ$

5. $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}$

6. $16\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{14\pi}{15}$

7. Two parallel chords of a circle, which are on the same side of the centre, subtend angles of 72° and 144° respectively at the centre. Prove that the perpendicular distance between the chords is half the radius of the circle.

8. In any circle prove that the chord which subtends 108° at the centre is equal to the sum of the two chords which subtend angles of 36° and 60°.

9. If $A = \cos\frac{\pi}{9}\cos\frac{4\pi}{9}\cos\frac{16\pi}{9}, B = \cos\frac{2\pi}{9}\cos\frac{8\pi}{9}\cos\frac{32\pi}{9}$, then value of $\left|\frac{1}{16A.B}\right|$ is

10. The value of $\cos\frac{\pi}{10}\cos\frac{2\pi}{10}\cos\frac{4\pi}{10}\cos\frac{8\pi}{10}\cos\frac{16\pi}{10}$ is

(A) $\frac{\sqrt{10+2\sqrt{5}}}{64}$ (B) $-\frac{\cos(\pi/10)}{16}$ (C) $\frac{\cos(\pi/10)}{16}$ (D) $-\frac{\sqrt{10+2\sqrt{5}}}{16}$

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17. CONDITIONAL TRIGONOMETRIC IDENTITIES :

If A + B + C = 180°, then

(i) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(ii) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(iii) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

(iv) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(v) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(vi) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(vii) $\sin A + \sin B + \sin C = 4 \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(viii) $\cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

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Illustration 14 : In any triangle ABC, sin A – cos B = cos C, then angle B is (Where ∠B > ∠C)

(A) π/2 (B) π/3 (C) π/4 (D) π/6

Solution : We have , sin A – cos B = cos C

sin A = cos B + cos C

$$\Rightarrow \quad 2\sin\frac{A}{2}\cos\frac{A}{2}=2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)$$

$$\Rightarrow \quad 2\sin\frac{A}{2}\cos\frac{A}{2}=2\cos\left(\frac{\pi-A}{2}\right)\cos\left(\frac{B-C}{2}\right) \qquad \because \quad A+B+C=\pi$$

$$\Rightarrow \quad 2\sin\frac{A}{2}\cos\frac{A}{2}=2\sin\frac{A}{2}\cos\left(\frac{B-C}{2}\right)$$

$$\Rightarrow \quad \cos\frac{A}{2}=\cos\frac{B-C}{2} \quad \text{or} \quad A=B-C \quad ; \text{ But } A+B+C=\pi$$

Therefore 2B = π ⇒ B = π/2 **Ans.(A)**

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Illustration 15 : If $A + B + C = \frac{3\pi}{2}$, then cos 2A + cos 2B + cos2C is equal to-

(A) 1 – 4cosA cosB cosC (B) 4 sinA sin B sinC

(C) 1 + 2cosA cosB cosC (D) 1 – 4 sinA sinB sinC

Solution : cos 2A + cos 2B + cos 2C = 2 cos (A + B) cos (A – B) + cos 2C

$$= 2\cos\left(\frac{3\pi}{2}-C\right)\cos(A-B)+\cos 2C \qquad \because A+B+C=\frac{3\pi}{2}$$

$$= -2\sin C\cos(A-B)+1-2\sin^2 C = 1-2\sin C\,[\cos(A-B)+\sin C]$$

$$= 1-2\sin C\left[\cos(A-B)+\sin\left(\frac{3\pi}{2}-(A+B)\right)\right]$$

$$= 1-2\sin C\,[\cos(A-B)-\cos(A+B)] = 1-4\sin A\sin B\sin C$$

Ans.(D)

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Do yourself - 14 :

1. If ABCD is a cyclic quadrilateral, then find the value of sinA + sinB – sinC – sinD

2. If $A + B + C = \frac{\pi}{2}$, then find the value of tanA tanB + tanBtanC + tanC tanA

If A + B + C = 180°, then prove that

3. sin2A + sin2B – sin2C = 4cosA cosB sinC

4. cos2A + cos2B – cos2C = 1 – 4sinA sinB cosC

5. $\sin A + \sin B - \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

6. $\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A\sin B\cos C$

7. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C$.

8. $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2\sin A\sin B\cos C$

9. $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = 1 - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

10. $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2} = 1 - 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

11. $\sin(B + 2C) + \sin(C + 2A) + \sin(A + 2B) = 4\sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}$.

12. $\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} - 1 = 4\sin\frac{\pi - A}{4}\sin\frac{\pi - B}{4}\sin\frac{\pi - C}{4}$

13. $\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

14. sin(B + C – A) + sin(C + A – B) + sin(A + B – C) = 4sinA sinB sinC

15. If x + y + z = xyz prove that

(a) $\frac{3x - x^3}{1 - 3x^2} + \frac{3y - y^3}{1 - 3y^2} + \frac{3z - z^3}{1 - 3z^2} = \frac{3x - x^3}{1 - 3x^2}\cdot\frac{3y - y^3}{1 - 3y^2}\cdot\frac{3z - z^3}{1 - 3z^2}$

(b) $x(1 - y^2)(1 - z^2) + y(1 - z^2)(1 - x^2) + z(1 - x^2)(1 - y^2) = 4xyz$

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18. MAXIMUM & MINIMUM VALUES OF TRIGONOMETRIC EXPRESSIONS :

(i) $a\cos\theta + b\sin\theta$ will always lie in the interval $[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$ i.e. the maximum and minimum values are $\sqrt{a^2+b^2}, -\sqrt{a^2+b^2}$ respectively.

(ii) Minimum value of $a^2\tan^2\theta + b^2\cot^2\theta = 2ab$ where a, b > 0

(iii) $-\sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} \le a\cos(\alpha+\theta) + b\cos(\beta+\theta) \le \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)}$ where α and β areknown angles.

(iv) In case a quadratic in $\sin\theta$ & $\cos\theta$ is given then the maximum or minimum values can be obtained by making perfect square.

Illustration 16: Prove that : $-4 \le 5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right) + 3 \le 10$, for all values of θ.

Solution : We have, $5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right) = 5\cos\theta + 3\cos\theta\cos\frac{\pi}{3} - 3\sin\theta\sin\frac{\pi}{3} = \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta$

Since, $-\sqrt{\left(\frac{13}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2} \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le \sqrt{\left(\frac{13}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2}$

$\Rightarrow \quad -7 \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le 7$

$\Rightarrow \quad -7 \le 5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right) \le 7$ for all θ.

$\Rightarrow \quad -7+3 \le 5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right) + 3 \le 7+3$ for all θ.

$\Rightarrow \quad -4 \le 5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right) + 3 \le 10$ for all θ.

Illustration 17: Find the maximum value of $1 + \sin\left(\frac{\pi}{4}+\theta\right) + 2\cos\left(\frac{\pi}{4}-\theta\right)$ -

(A) 1 (B) 2 (C) 3 (D) 4

Solution : We have $1 + \sin\left(\frac{\pi}{4}+\theta\right) + 2\cos\left(\frac{\pi}{4}-\theta\right)$

$= 1 + \frac{1}{\sqrt{2}}(\cos\theta + \sin\theta) + \sqrt{2}(\cos\theta + \sin\theta) = 1 + \left(\frac{1}{\sqrt{2}} + \sqrt{2}\right)(\cos\theta + \sin\theta)$

$= 1 + \left(\frac{1}{\sqrt{2}} + \sqrt{2}\right) . \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$

$\therefore$ maximum value $= 1 + \left(\frac{1}{\sqrt{2}} + \sqrt{2}\right) . \sqrt{2} = 4$ **Ans. (D)**

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Do yourself - 15 :

1. Find maximum and minimum value of $5\cos\theta + 3\sin\left(\theta + \frac{\pi}{6}\right)$ for all real values of θ.

2. Find the minimum value of $\cos\theta + \cos2\theta$ for all real values of θ.

3. Find maximum and minimum value of $\cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2$.

4. Find the maximum and minimum values of

(i) $\cos2x + \cos^2x$ (ii) $\cos^2\left(\frac{\pi}{4} + x\right) + (\sin x - \cos x)^2$

5. If $\alpha + \beta = 90^\circ$, then find the maximum value of $\sin\alpha.\sin\beta$.

6. Find the maximum and minimum value of $1 + 2\sin x + 3\cos^2x$

7. Find the minimum value of $4\sec^2x + 9\operatorname{cosec}^2x$

8. Find the maximum and minimum value of $9\cos^2x + 48\sin x.\cos x - 5\sin^2x - 2$

9. Find the maximum and minimum value of $2\sin\left(\theta + \frac{\pi}{6}\right) + \sqrt{3}\cos\left(\theta - \frac{\pi}{6}\right)$

10. Find minimum value of (i) $3\sin^2x + 27\operatorname{cosec}^2x$ (ii) $27\sin^2x + 3\operatorname{cosec}^2x$

19. IMPORTANT RESULTS :

(i) $\sin\theta \sin(60^\circ - \theta)\sin(60^\circ + \theta) = \frac{1}{4}\sin3\theta$

(ii) $\cos\theta.\cos(60^\circ - \theta)\cos(60^\circ + \theta) = \frac{1}{4}\cos3\theta$

(iii) $\tan\theta\tan(60^\circ - \theta)\tan(60^\circ + \theta) = \tan3\theta$

(iv) $\cot\theta\cot(60^\circ - \theta)\cot(60^\circ + \theta) = \cot3\theta$

(v) (a) $\sin^2\theta + \sin^2(60^\circ + \theta) + \sin^2(60^\circ - \theta) = \frac{3}{2}$

(b) $\cos^2\theta + \cos^2(60^\circ + \theta) + \cos^2(60^\circ - \theta) = \frac{3}{2}$

(c) $\tan\theta + \tan(60^\circ + \theta) + \tan(120^\circ + \theta) = 3\tan3\theta$

(vi) (a) If $\tan A + \tan B + \tan C = \tan A\tan B\tan C$, then $A + B + C = n\pi, n \in I$

(b) If $\tan A\tan B + \tan B\tan C + \tan C\tan A = 1$, then $A + B + C = (2n + 1)\frac{\pi}{2}, n \in I$

(vii) $\cos\theta\cos2\theta\cos4\theta \ldots \cos(2^{n-1}\theta) = \frac{\sin(2^n\theta)}{2^n\sin\theta}$

(viii) (a) $\cot A - \tan A = 2\cot2A$ (b) $\cot A + \tan A = 2\operatorname{cosec}2A$

(ix) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots \sin(\alpha + \overline{n-1}\,\beta) = \dfrac{\sin\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

(x) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha + 2\beta) + \ldots + \cos(\alpha + \overline{n-1}\,\beta) = \dfrac{\cos\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

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Do yourself - 16 :

1 Evaluate $\sin\frac{\pi}{n}+\sin\frac{3\pi}{n}+\sin\frac{5\pi}{n}+.....$ to n terms

2. Prove that : $\sin\theta + \sin3\theta + \sin5\theta +...+ \sin(2n-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}$

3. Find the average of $\sin2^\circ, \sin4^\circ, \sin6^\circ, ..., \sin180^\circ$

4. Prove that : $\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}=\frac{1}{2}$

5. Find sum of the following series :

(a) $\cos\frac{\pi}{2n+1}+\cos\frac{3\pi}{2n+1}+\cos\frac{5\pi}{2n+1}+...$ up to n terms.

(b) $\sin2\alpha + \sin3\alpha + \sin4\alpha +...+ \sin n\alpha$, where $(n+2)\alpha = 2\pi$

6. If $S=\cos^2\frac{\pi}{n}+\cos^2\frac{2\pi}{n}+...+\cos^2\frac{(n-1)\pi}{n}$, then S equals (where $n \ge 2$, $n \in N$)

(A) $\frac{n}{2(n+1)}$ (B) $\frac{1}{2(n-1)}$ (C) $\frac{n-2}{2}$ (D) $\frac{n}{2}$

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Miscellaneous Illustration :

Illustration 18 : Prove that

$\tan\alpha + 2\tan2\alpha + 2^2\tan^2\alpha + ... + 2^{n-1}\tan 2^{n-1}\alpha + 2^n\cot 2^n\alpha = \cot\alpha$

Solution : We know $\tan\theta = \cot\theta - 2\cot 2\theta$ .....(i)

Putting $\theta = \alpha, 2\alpha, 2^2\alpha$, ...in (i), we get

$\tan\alpha = (\cot\alpha - 2\cot 2\alpha)$

$2(\tan 2\alpha) = 2(\cot 2\alpha - 2\cot 2^2\alpha)$

$2^2(\tan 2^2\alpha) = 2^2(\cot 2^2\alpha - 2\cot 2^3\alpha)$

.....

$2^{n-1}(\tan 2^{n-1}\alpha) = 2^{n-1}(\cot 2^{n-1}\alpha - 2\cot 2^n\alpha)$

Adding,

$\tan\alpha + 2\tan2\alpha + 2^2\tan^2\alpha + ... + 2^{n-1}\tan 2^{n-1}\alpha = \cot\alpha - 2^n\cot 2^n\alpha$

$\therefore$ $\tan\alpha + 2\tan2\alpha + 2^2\tan^2\alpha + ... + 2^{n-1}\tan 2^{n-1}\alpha + 2^n\cot 2^n\alpha = \cot\alpha$

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Illustration 19 : If A,B,C and D are angles of a quadrilateral and $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}=\frac{1}{4}$, prove that A = B = C = D = π/2.

Solution : $\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right)\left(2\sin\frac{C}{2}\sin\frac{D}{2}\right)=1$

$$\Rightarrow \left\{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right\}\left\{\cos\left(\frac{C-D}{2}\right)-\cos\left(\frac{C+D}{2}\right)\right\}=1$$

Since, A + B = 2π – (C + D), the above equation becomes,

$$\Rightarrow \left\{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right\}\left\{\cos\left(\frac{C-D}{2}\right)+\cos\left(\frac{A+B}{2}\right)\right\}=1$$

$$\Rightarrow \cos^2\left(\frac{A+B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\left\{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{C-D}{2}\right)\right\}+1-\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{C-D}{2}\right)=0$$

This is a quadratic equation in $\cos\left(\frac{A+B}{2}\right)$ which has real roots.

$$\Rightarrow \left\{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{C-D}{2}\right)\right\}^2-4\left\{1-\cos\left(\frac{A-B}{2}\right).\cos\left(\frac{C-D}{2}\right)\right\}\geq 0$$

$$\left(\cos\frac{A-B}{2}+\cos\frac{C-D}{2}\right)^2\geq 4$$

$\Rightarrow \cos\frac{A-B}{2}+\cos\frac{C-D}{2}\geq 2$, Now both $\cos\frac{A-B}{2}$ and $\cos\frac{C-D}{2}\leq 1$

$$\Rightarrow \cos\frac{A-B}{2}=1 \;\&\; \cos\frac{C-D}{2}=1$$

$$\Rightarrow \frac{A-B}{2}=0=\frac{C-D}{2}$$

$\Rightarrow$ A = B, C = D.

Similarly A = C, B = D ⇒ A = B = C = D = π/2

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EXERCISE (O-1)

1. If $\sin x + \sin^2 x = 1$, then the value of $\cos^2 x + \cos^4 x$ is -

(A) 0 (B) 2 (C) 1 (D) 3

TR0001

2. $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1$ is equal to -

(A) 2 (B) 0 (C) 4 (D) 6

TR0002

3. If $x = y\cos\frac{2\pi}{3} = z\cos\frac{4\pi}{3}$, then $xy + yz + zx =$

(A) –1 (B) 0 (C) 1 (D) 2

TR0003

4. The value of $\sin 10^\circ + \sin 20^\circ + \sin 30^\circ + ... + \sin 360^\circ$ is equal to -

(A) 0 (B) 1 (C) $\sqrt{3}$ (D) 2

TR0004

5. If $\cos(\alpha + \beta) + \sin(\alpha - \beta) = 0$ and $2010\tan\beta + 1 = 0$, then $\tan\alpha$ is equal to

(A) 1 (B) –1 (C) 2010 (D) $\frac{1}{2010}$

TR0005

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6. If $\cos x + \cos y + \cos\alpha = 0$ and $\sin x + \sin y + \sin\alpha = 0$, then $\cot\left(\frac{x+y}{2}\right) =$

(A) $\sin\alpha$ (B) $\cos\alpha$ (C) $\cot\alpha$ (D) $2\sin\alpha$

TR0006

7. $\frac{\sin(A-C) + 2\sin A + \sin(A+C)}{\sin(B-C) + 2\sin B + \sin(B+C)}$ is equal to -

(A) $\tan A$ (B) $\frac{\sin A}{\sin B}$ (C) $\frac{\cos A}{\cos B}$ (D) $\frac{\sin C}{\cos B}$

TR0007

8. The expression $\frac{\sin 8\theta\cos\theta - \sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta - \sin 3\theta\sin 4\theta}$ is equals -

(A) $\tan\theta$ (B) $\tan 2\theta$ (C) $\sin 2\theta$ (D) $\cos 2\theta$

TR0008

9. $\frac{1 + \sin 2\theta + \cos 2\theta}{1 + \sin 2\theta - \cos 2\theta} =$

(A) $\frac{1}{2}\tan\theta$ (B) $\frac{1}{2}\cot\theta$ (C) $\tan\theta$ (D) $\cot\theta$

TR0009

10. If $\tan x + \tan y = 25$ and $\cot x + \cot y = 30$, then the value of $\tan(x + y)$ is

(A) 150 (B) 200 (C) 250 (D) 100

TR0010

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11. If A = tan 6° tan 42° and B = cot 66° cot 78°, then -

(A) A = 2B (B) A = 1/3B (C) A = B (D) 3A = 2B

TR0011

12. In a right angled triangle the hypotenuse is $2\sqrt{2}$ times the perpendicular drawn from the opposite vertex. Then the other acute angles of the triangle are

(A) $\frac{\pi}{3}$ and $\frac{\pi}{6}$ (B) $\frac{\pi}{8}$ and $\frac{3\pi}{8}$ (C) $\frac{\pi}{4}$ and $\frac{\pi}{4}$ (D) $\frac{\pi}{5}$ and $\frac{3\pi}{10}$

TR0012

13. If $\tan\alpha = (1+2^{-x})^{-1}$, $\tan\beta = (1+2^{x+1})^{-1}$, then $\alpha + \beta =$

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $\pi/2$

TR0013

14. If 3 sin α = 5 sin β, then $\dfrac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}} =$

(A) 1 (B) 2 (C) 3 (D) 4

TR0014

15. If $\cos\alpha = \dfrac{2\cos\beta - 1}{2-\cos\beta}$ then $\tan\frac{\alpha}{2}\cdot\cot\frac{\beta}{2}$ has the value equal to {where $\alpha, \beta \in (0, \pi)$}

(A) 2 (B) $\sqrt{2}$ (C) 3 (D) $\sqrt{3}$

TR0015

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16. If $\tan B = \dfrac{n\sin A\cos A}{1-n\cos^2 A}$ then $\tan(A+B)$ equals

(A) $\dfrac{\sin A}{(1-n)\cos A}$ (B) $\dfrac{(n-1)\cos A}{\sin A}$ (C) $\dfrac{\sin A}{(n-1)\cos A}$ (D) $\dfrac{\sin A}{(n+1)\cos A}$

TR0016

17. The value of $\operatorname{cosec}\frac{\pi}{18} - \sqrt{3}\sec\frac{\pi}{18}$ is a

(A) surd (B) rational which is not integral
(C) negative integer (D) natural number

TR0017

18. The value of $\cot x + \cot(60° + x) + \cot(120° + x)$ is equal to :

(A) $\cot 3x$ (B) $\tan 3x$ (C) $3\tan 3x$ (D) $\dfrac{3 - 9\tan^2 x}{3\tan x - \tan^3 x}$

TR0018

19. If $\frac{5\pi}{2} < x < 3\pi$, then the value of the expression $\dfrac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}$ is

(A) $-\cot\frac{x}{2}$ (B) $\cot\frac{x}{2}$ (C) $\tan\frac{x}{2}$ (D) $-\tan\frac{x}{2}$

TR0019

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20. The value of $\cot 7\frac{1}{2}^{\circ} + \tan 67\frac{1}{2}^{\circ} - \cot 67\frac{1}{2}^{\circ} - \tan 7\frac{1}{2}^{\circ}$ is :

(A) a rational number
(B) irrational number
(C) $2(3 + 2\sqrt{3})$
(D) $2(3 - \sqrt{3})$

TR0020

21. If $x + y = 3 - \cos 4\theta$ and $x - y = 4\sin 2\theta$ then

(A) $x^4 + y^4 = 9$
(B) $\sqrt{x} + \sqrt{y} = 16$
(C) $x^3 + y^3 = 2(x^2 + y^2)$
(D) $\sqrt{x} + \sqrt{y} = 2$

TR0021

22. If $A = \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}$ and $B = \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7}$ then $\sqrt{A^2 + B^2}$ is equal to

(A) 1
(B) $\sqrt{2}$
(C) 2
(D) $\sqrt{3}$

TR0022

23. If tan A + tan B + tan C = tan A. tan B. tan C, then -

(A) A,B,C must be angles of a triangle
(B) the sum of any two of A,B,C is equal to the third
(C) A+B+C must be n integral multiple of π
(D) None of these

TR0023

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24. If $f(x) = \frac{\sin 3x}{\sin x}$, $x \neq n\pi$, then the range of values of f(x) for real values of x is -

(A) $[-1,3]$
(B) $(-\infty,-1]$
(C) $(3, +\infty)$
(D) $[-1,3)$

TR0024

25. Maximum and minimum value of $2\sin^2\theta - 3\sin\theta + 2$ is -

(A) $\frac{1}{4}, -\frac{7}{4}$
(B) $\frac{1}{4}, \frac{21}{4}$
(C) $\frac{21}{4}, -\frac{3}{4}$
(D) $7, \frac{7}{8}$

TR0025

26. For $\theta \in (0, \pi/2)$, the maximum value of $\sin\left(\theta + \frac{\pi}{6}\right) + \cos\left(\theta + \frac{\pi}{6}\right)$ is attained at $\theta =$

(A) $\frac{\pi}{12}$
(B) $\frac{\pi}{6}$
(C) $\frac{\pi}{3}$
(D) $\frac{\pi}{4}$

TR0026

27. Minimum value of the expression $\cos^2\theta - (6\sin\theta\cos\theta) + 3\sin^2\theta + 2$, is -

(A) $4 + \sqrt{10}$
(B) $4 - \sqrt{10}$
(C) 0
(D) 4

TR0027

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28. The value of $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$ is :-

(A) $\frac{1}{16}$ (B) $\frac{1}{8}$ (C) $\frac{1}{2}$ (D) 1

TR0028

29. The exact value of $\frac{96\sin 80^\circ \sin 65^\circ \sin 35^\circ}{\sin 20^\circ + \sin 50^\circ + \sin 110^\circ}$ is equal to

(A) 12 (B) 24 (C) –12 (D) 48

TR0029

30. If m and n are positive integers satisfying

$1 + \cos 2\theta + \cos 4\theta + \cos 6\theta + \cos 8\theta + \cos 10\theta = \frac{\cos m\theta \cdot \sin n\theta}{\sin\theta}$ then $(m + n)$ is equal to

(A) 9 (B) 10 (C) 11 (D) 12

TR0030

EXERCISE (O-2)

Multiple Objective Type :

1. If cosx = tanx, then which of the following is/are true ?

(A) $\frac{1}{\sin x} + \cos^4 x = 1$ (B) $\frac{1}{\sin x} + \cos^4 x = 2$ (C) $\cos^4 x + \cos^2 x = 1$ (D) $\cos^4 x + \cos^2 x = 2$

TR0031

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2. If $\tan A = -\frac{1}{2}$ and $\tan B = -\frac{1}{3}$, then A + B =

(A) $\frac{\pi}{4}$ (B) $\frac{3\pi}{4}$ (C) $\frac{5\pi}{4}$ (D) $\frac{7\pi}{4}$

TR0032

3. If $\cos(A - B) = \frac{3}{5}$ and tanA tanB = 2, then which of the following is/are correct ?

(A) cosA cosB = $-\frac{1}{5}$ (B) $\sin A \sin B = \frac{2}{5}$ (C) $\cos(A + B) = -\frac{1}{5}$ (D) $\sin A \cos B = \frac{4}{5}$

TR0033

4. If $\sin t + \cos t = \frac{1}{5}$, then $\tan\frac{t}{2}$ is equal to

(A) –1 (B) $-\frac{1}{3}$ (C) 2 (D) $-\frac{1}{6}$

TR0034

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5. If $\tan^2\alpha + 2\tan\alpha.\tan2\beta = \tan^2\beta + 2\tan\beta.\tan2\alpha$, then

(A) $\tan^2\alpha + 2\tan\alpha.\tan2\beta = 0$
(B) $\tan\alpha + \tan\beta = 0$
(C) $\tan^2\beta + 2\tan\beta.\tan2\alpha = 1$
(D) $\tan\alpha = \tan\beta$

TR0035

6. If $3\sin\beta = \sin(2\alpha + \beta)$, then $\tan(\alpha + \beta) - 2\tan\alpha$ is

(A) independent of α
(B) independent of β
(C) dependent of both α and β
(D) independent of α but dependent of β

TR0036

7. If $L = \cos^2 84^\circ + \cos^2 36^\circ + \cos 36^\circ \cos 84^\circ$

$M = \cot 73^\circ \cot 47^\circ \cot 13^\circ$

$N = 4 \sin 156^\circ \sin 84^\circ \sin 36^\circ$, then which of the following option(s) is(are) correct ?

(A) $L < 1$
(B) $M > \tan 2$
(C) $N > \sin\frac{\pi}{4}$
(D) $0 < LMN$

TR0037

8. If $\frac{\cos 3x}{\cos x} = \frac{1}{3}$ for some angle x, $0 \le x \le \frac{\pi}{2}$, which of the following is/are true ?

(A) $\frac{\sin 3x}{\sin x} = \frac{7}{3}$
(B) $\cos 2x = \frac{2}{3}$
(C) $\tan x = \frac{1}{\sqrt{5}}$
(D) $\sin 2x = \frac{2\sqrt{5}}{6}$

TR0038

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9. If $\alpha + \beta + \gamma = 2\pi$, then

(A) $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

(B) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2} = 1$

(C) $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = -\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

(D) $\tan\frac{\alpha}{4}\tan\frac{\beta}{4} + \tan\frac{\beta}{4}\tan\frac{\gamma}{4} + \tan\frac{\gamma}{4}\tan\frac{\alpha}{4} = 1$

TR0039

10. If $x + y = z$, then $\cos^2x + \cos^2y + \cos^2z - 2\cos x \cos y \cos z$ is equal to

(A) $\cos^2z$
(B) $\sin^2z$
(C) $\cos(x + y - z)$
(D) 1

TR0040

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11. If A,B,C are angles of a triangle ABC and tanA tanC = 3; tan B tanC = 6 then, which is (are) correct?

(A) $A = \frac{\pi}{4}$ (B) tanA tanB = 2 (C) $\frac{\tan A}{\tan C} = 3$ (D) tanB = 2 tanA

TR0041

12. Which of the following is/are true ?

(A) $\sin\left(\frac{\pi}{4}+\theta\right)\sin\left(\frac{\pi}{4}-\theta\right) = \frac{\cos 2\theta}{2}$

(B) In a ΔABC, if tanA = 2, tanB = 3, then tanC = 1

(C) Minimum value of $4\tan^2\theta + \cot^2\theta$ is 4 (wherever defined)

(D) Range of $3\sin^2\theta + 4\sin\theta\cos\theta + 5\cos^2\theta$ is $\left[4-\sqrt{5},\ 4+\sqrt{5}\right]$

TR0042

13. Let $f_n(\theta) = \sum_{n=0}^{n} \frac{1}{4^n}\sin^4\left(2^n\theta\right)$. Then which of the following alternative(s) is/are correct ?

(A) $f_2\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ (B) $f_3\left(\frac{\pi}{8}\right) = \frac{2+\sqrt{2}}{4}$ (C) $f_4\left(\frac{3\pi}{2}\right) = 1$ (D) $f_5(\pi) = 0$

TR0043

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14. Let $y = \frac{\cos x + \cos 2x + \cos 3x + \cos 4x + \cos 5x + \cos 6x + \cos 7x}{\sin x + \sin 2x + \sin 3x + \sin 4x + \sin 5x + \sin 6x + \sin 7x}$, then which of the following hold good?

(A) The value of y when $x = \pi/8$ is not defined. (B) The value of y when $x = \pi/16$ is 1.

(C) The value of y when $x = \pi/32$ is $\sqrt{2}-1$. (D) The value of y when $x = \pi/48$ is $2+\sqrt{3}$.

TR0044

Paragraph for Question Nos. 15 to 17

Consider the polynomial $P(x) = (x - \cos 36°)(x - \cos 84°)(x - \cos 156°)$

15. The coefficient of x^2 is

(A) 0 (B) 1 (C) $-\frac{1}{2}$ (D) $\frac{\sqrt{5}-1}{2}$

TR0045

16. The coefficient of x is

(A) $\frac{3}{2}$ (B) $-\frac{3}{2}$ (C) $-\frac{3}{4}$ (D) zero

TR0045

17. The absolute term in P(x) has the value equal to

(A) $\frac{\sqrt{5}-1}{4}$ (B) $\frac{\sqrt{5}-1}{16}$ (C) $\frac{\sqrt{5}+1}{16}$ (D) $\frac{1}{16}$

TR0045

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Paragraph for Question 18 to 19

Let a,b,c are respectively the sines and p,q,r are respectively the cosines of $\alpha, \alpha+\frac{2\pi}{3}$ and $\alpha+\frac{4\pi}{3}$, then

18. The value of (a + b + c) + (ab + bc + ca) is

(A) 0 (B) $\frac{3}{4}$ (C) 1 (D) $-\frac{3}{4}$

TR0046

19. The value of (qc – rb) is-

(A) 0 (B) $-\frac{\sqrt{3}}{2}$ (C) $\frac{\sqrt{3}}{2}$ (D) depends on α

TR0046

INTEGER TYPE

20. If acute angle A = 3B and $\sin A=\frac{4}{5}$ then $\left|\frac{3\sec B-4\operatorname{cosec}B}{2}\right|$ is

TR0047

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21. If the value of $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}+\cos\frac{7\pi}{7}=-\frac{\ell}{2}$. Find the value of ℓ.

TR0048

22. If cot(θ – α), 3cotθ, cot(θ + α) are in AP (where, $\theta\neq\frac{n\pi}{2}, \alpha\neq k\pi, n, k\in I$), then $\frac{2\sin^2\theta}{\sin^2\alpha}$ is equal to

TR0049

23. If k_1 = tan27θ – tanθ and $k_2=\frac{\sin\theta}{\cos3\theta}+\frac{\sin3\theta}{\cos9\theta}+\frac{\sin9\theta}{\cos27\theta}$ then $\left(\frac{k_1}{k_2}\right)$ is equal to

TR0050

24. The value of the expression $\frac{1-4\sin10^\circ\sin70^\circ}{2\sin10^\circ}$ is

TR0051

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MATRIX MATCH TYPE

25. In the following matrix match **Column-I** has some quantities and **Column-II** has some comments or other quantities

Match the each element in **Column-I** with corresponding element(s) in **Column-II**

Column-I	**Column-II**
(A) The value of $\left\|4\left(2\cos^3\frac{\pi}{7}-\cos^2\frac{\pi}{7}-\cos\frac{\pi}{7}\right)\right\|$ is	(P) 4
(B) If A + B + C = π and cosA = cosB. cosC then tanB. tanC has the value equal to	(Q) 8
(C) $4\left(\frac{\cos 20^\circ+8\sin 70^\circ\sin 50^\circ\sin 10^\circ}{\sin^2 80^\circ}\right)$ is equal to	(R) 2
(D) The maximum value of $12\sin\theta - 9\sin^2\theta$ is	(S) 1
	(T) 6

TR0052

EXERCISE (S-1)

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1. If $X=\sin\left(\theta+\frac{7\pi}{12}\right)+\sin\left(\theta-\frac{\pi}{12}\right)+\sin\left(\theta+\frac{3\pi}{12}\right), Y=\cos\left(\theta+\frac{7\pi}{12}\right)+\cos\left(\theta-\frac{\pi}{12}\right)+\cos\left(\theta+\frac{3\pi}{12}\right)$, then prove that $\frac{X}{Y}-\frac{Y}{X}=2\tan 2\theta$.

TR0053

2. Prove that :

(a) tan 20° . tan 40° . tan 60° . tan 80° = 3

TR0054

(b) $\sin^4\frac{\pi}{16}+\sin^4\frac{3\pi}{16}+\sin^4\frac{5\pi}{16}+\sin^4\frac{7\pi}{16}=\frac{3}{2}$

TR0055

(c) $\cos^2\alpha + \cos^2(\alpha+\beta) - 2\cos\alpha\ \cos\beta\ \cos(\alpha+\beta) = \sin^2\beta$

TR0056

(d) $(4\cos^2 9^\circ - 3)(4\cos^2 27^\circ - 3) = \tan 9^\circ$.

TR0057

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3. If $m \tan(\theta - 30°) = n \tan(\theta + 120°)$, show that $\cos 2\theta = \dfrac{m+n}{2(m-n)}$.

TR0058

4. If $\cos(\alpha + \beta) = \dfrac{4}{5}$; $\sin(\alpha - \beta) = \dfrac{5}{13}$ & α, β lie between 0 & $\dfrac{\pi}{4}$, then find the value of $\tan 2\alpha$.

TR0059

5. If the value of the expression $\sin 25°.\sin 35°.\sin 85°$ can be expressed as $\dfrac{\sqrt{a}+\sqrt{b}}{c}$, where $a,b,c \in N$ and are in their lowest form, find the value of $(a + b + c)$.

TR0060

6. If $\alpha + \beta = \gamma$, prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 + 2\cos\alpha\cos\beta\cos\gamma$.

TR0061

7. Let $P(k) = \left(1+\cos\dfrac{\pi}{4k}\right)\left(1+\cos\dfrac{(2k-1)\pi}{4k}\right)\left(1+\cos\dfrac{(2k+1)\pi}{4k}\right)\left(1+\cos\dfrac{(4k-1)\pi}{4k}\right)$, then find the value of **(a)** P(5) and **(b)** P(6).

TR0062

8. Calculate without using trigonometric tables :

(a) $4\cos 20° - \sqrt{3}\cot 20°$

TR0063

(b) $\dfrac{2\cos 40° - \cos 20°}{\sin 20°}$

TR0064

(c) $\cos^6\dfrac{\pi}{16} + \cos^6\dfrac{3\pi}{16} + \cos^6\dfrac{5\pi}{16} + \cos^6\dfrac{7\pi}{16}$

TR0065

(d) $\tan 10° - \tan 50° + \tan 70°$

TR0066

9. Given that $(1 + \tan 1°)(1 + \tan 2°)....(1 + \tan 45°) = 2^n$, find n.

TR0067

10. (a) If $y = 10\cos^2 x - 6\sin x\cos x + 2\sin^2 x$, then find the greatest & least value of y.

TR0068

(b) If $y = 1 + 2\sin x + 3\cos^2 x$, find the maximum & minimum values of y $\forall\ x \in R$.

TR0069

(c) If $y = 9\sec^2 x + 16\,\text{cosec}^2 x$, find the minimum value of y for all permissible value of x.

TR0070

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(d) If $a \le 3\cos\left(\theta+\frac{\pi}{3}\right)+5\cos\theta+3 \le b$, find a and b, where a is the minimum value & b is the maximum value.

TR0071

EXERCISE (S-2)

1. Prove that :

(a) $\cos 2\alpha = 2\sin^2\beta + 4\cos(\alpha+\beta)\sin\alpha\ \sin\beta + \cos 2(\alpha+\beta)$

TR0072

(b) $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = \cot\alpha$.

TR0073

2. Prove that : $\tan 9° - \tan 27° - \tan 63° + \tan 81° = 4$. (c)

TR0074

3. Let k = 1°, then prove that $\sum_{n=0}^{88}\frac{1}{\cos nk.\cos(n+1)k} = \frac{\cos k}{\sin^2 k}$

TR0075

4. If A + B + C = π; prove that $\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\ge 1$.

TR0076

5. Find the positive integers p,q,r,s satisfying $\tan\frac{\pi}{24}=\left(\sqrt{p}-\sqrt{q}\right)\left(\sqrt{r}-s\right)$.

TR0077

6. If $f(\theta)=\sum_{n=1}^{6}\operatorname{cosec}\left(\theta+\frac{(n-1)\pi}{4}\right)\operatorname{cosec}\left(\theta+\frac{n\pi}{4}\right)$, where $0<\theta<\frac{\pi}{2}$, then find the minimum value of f(θ).

TR0078

7. Find the exact value of $\tan^2\frac{\pi}{16}+\tan^2\frac{3\pi}{16}+\tan^2\frac{5\pi}{16}+\tan^2\frac{7\pi}{16}$

TR0079

8. If 'θ' is eliminated from the equations cos θ – sin θ = b and cos3θ + sin3θ = a, find the eliminant.

TR0080

9. In a right angled triangle, acute angles A and B satisfy

$\tan A + \tan B + \tan^2 A + \tan^2 B + \tan^3 A + \tan^3 B = 70$

find the angle A and B in radians.

TR0081

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10. If A + B + C = π, prove that $\sum\left(\frac{\tan A}{\tan B.\tan C}\right)=\sum(\tan A)-2\sum(\cot A)$.

TR0082

EXERCISE (JM)

1. In a ΔPQR, if 3 sinP + 4 cosQ = 6 and 4 sinQ + 3 cos P = 1, then the angle R is equal to : **[AIEEE-2012]**

(1) $\frac{3\pi}{4}$ (2) $\frac{5\pi}{6}$ (3) $\frac{\pi}{6}$ (4) $\frac{\pi}{4}$

TR0083

2. The expression $\frac{\tan A}{1-\cot A}+\frac{\cot A}{1-\tan A}$ can be written as **[JEE-MAIN 2013]**

(1) sinA cosA + 1 (2) secA cosecA + 1 (3) tanA + cotA (4) secA + cosecA

TR0084

3. Let $f_K(x) = \frac{1}{k}\left(\sin^k x+\cos^k x\right)$ where x ∈ R and k ≥ 1. Then $f_4(x) - f_6(x)$ equals : **[JEE-MAIN 2014]**

(1) $\frac{1}{6}$ (2) $\frac{1}{3}$ (3) $\frac{1}{4}$ (4) $\frac{1}{12}$

TR0085

4. If $5(\tan^2x - \cos^2x) = 2\cos 2x + 9$, then the value of cos4x is :- **[JEE-MAIN 2017]**

(1) $-\frac{7}{9}$ (2) $-\frac{3}{5}$ (3) $\frac{1}{3}$ (4) $\frac{2}{9}$

TR0086

5. For any $\theta \in \left(\frac{\pi}{4},\frac{\pi}{2}\right)$, the expression $3(\sin\theta - \cos\theta)^4 + 6(\sin\theta + \cos\theta)^2 + 4\sin^6\theta$ equals : **[JEE-MAIN 2019]**

(1) $13 - 4\cos^6\theta$ (2) $13 - 4\cos^4\theta + 2\sin^2\theta\cos^2\theta$

(3) $13 - 4\cos^2\theta + 6\cos^4\theta$ (4) $13 - 4\cos^2\theta + 6\sin^2\theta\cos^2\theta$

TR0087

6. The value of $\cos\frac{\pi}{2^2}\cdot\cos\frac{\pi}{2^3}\cdot.....\cdot\cos\frac{\pi}{2^{10}}\cdot\sin\frac{\pi}{2^{10}}$ is : **[JEE-MAIN 2019]**

(1) $\frac{1}{256}$ (2) $\frac{1}{2}$ (3) $\frac{1}{512}$ (4) $\frac{1}{1024}$

TR0088

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7. Let $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$ for k = 1, 2, 3, ... Then for all $x \in R$, the value of $f_4(x) - f_6(x)$ is equal to :- **[JEE-MAIN 2019]**

(1) $\frac{5}{12}$ (2) $\frac{-1}{12}$ (3) $\frac{1}{4}$ (4) $\frac{1}{12}$

TR0089

8. The maximum value of $3\cos\theta+5\sin\left(\theta-\frac{\pi}{6}\right)$ for any real value of θ is : **[JEE-MAIN 2019]**

(1) $\sqrt{19}$ (2) $\frac{\sqrt{79}}{2}$ (3) $\sqrt{31}$ (4) $\sqrt{34}$

TR0090

9. Let α and β be two real roots of the equation $(k + 1)\tan^2 x - \sqrt{2}\,.\,\lambda \tan x = (1 - k)$, where $k(\neq -1)$ and λ are real numbers. If $\tan^2(\alpha + \beta) = 50$, then a value of λ is ; **[JEE-MAIN 2020]**

(1) 5 (2) 10 (3) $5\sqrt{2}$ (4) $10\sqrt{2}$

TR0091

10. The value of $\cos^3\left(\frac{\pi}{8}\right)\cdot\cos\left(\frac{3\pi}{8}\right)+\sin^3\left(\frac{\pi}{8}\right)\cdot\sin\left(\frac{3\pi}{8}\right)$ is : **[JEE-MAIN 2020]**

(1) $\frac{1}{4}$ (2) $\frac{1}{\sqrt{2}}$ (3) $\frac{1}{2\sqrt{2}}$ (4) $\frac{1}{2}$

TR0092

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EXERCISE (JA)

1. Let $\theta \in \left(0, \frac{\pi}{4}\right)$ and $t_1 = (\tan\theta)^{\tan\theta}$, $t_2 = (\tan\theta)^{\cot\theta}$, $t_3 = (\cot\theta)^{\tan\theta}$, $t_4 = (\cot\theta)^{\cot\theta}$, then - **[JEE 06, 3M,–1M]**

(A) $t_1 > t_2 > t_3 > t_4$ (B) $t_4 > t_3 > t_1 > t_2$ (C) $t_3 > t_1 > t_2 > t_4$ (D) $t_2 > t_3 > t_1 > t_4$

TR0093

2. **(a)** If $\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$, then **[JEE 2009, 4 + 4]**

(A) $\tan^2 x = \frac{2}{3}$ (B) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$

(C) $\tan^2 x = \frac{1}{3}$ (D) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{2}{125}$

TR0094

(b) For $0 < \theta < \frac{\pi}{2}$, the solution(s) of $\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$ is (are) -

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{12}$ (D) $\frac{5\pi}{12}$

TR0095

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3. **(a)** The maximum value of the expression $\frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$ is

TR0096

(b) Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3}+1$ apart. If the chords subtend at the center, angles of $\frac{\pi}{k}$ and $\frac{2\pi}{k}$ where k > 0, then the value of [k] is -

[**Note :** [k] denotes the largest integer less than or equal to k] **[JEE 2010, 3+3]**

TR0097

4. Let $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and $Q = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets. Then

(A) $P \subset Q$ and $Q - P \neq \varnothing$ (B) $Q \not\subset P$

(C) $P \not\subset Q$ (D) $P = Q$ **[JEE 2011,3]**

TR0098

5. The value of $\sum_{k=1}^{13} \frac{1}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$ is equal to **[JEE(Advanced)-2016, 3(–1)]**

(A) $3-\sqrt{3}$ (B) $2(3-\sqrt{3})$ (C) $2(\sqrt{3}-1)$ (D) $2(2+\sqrt{3})$

TR0099

6. Let α and β be nonzero real numbers such that 2(cosβ – cosα) + cosα cosβ = 1. Then which of the following is/are true ? **[JEE(Advanced)-2017, 4]**

(A) $\tan\left(\frac{\alpha}{2}\right) - \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$ (B) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\beta}{2}\right) = 0$

(C) $\tan\left(\frac{\alpha}{2}\right) + \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$ (D) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right) = 0$

TR0100

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ANSWERS

Do yourself-1

1. 10π cm **2.** $36 + 6\pi$ **3.** (a) $\left(\frac{3061}{21600}\pi\right)$ (b) $\left(\frac{257}{4320}\pi\right)$ (c) $\frac{5587}{108000}\pi$

4. $\frac{22\pi}{3}+8\sqrt{3}$ **5.** 68º 43' 38"

Do yourself-2

1. $\frac{3}{5},\frac{4}{5},\frac{5}{3}$ **2.** 2 **23.** $\frac{2xy}{x^2+y^2},\frac{2xy}{x^2-y^2}$ **27.** $\left(\frac{1}{\tan^4 A}-\tan^4 A\right)$

Do yourself-3

1. $\left\{\frac{11}{60},\frac{60}{61},\frac{61}{60}\right\},\left\{\frac{-11}{60},\frac{-60}{61},\frac{-61}{60}\right\}$ **2.** $\left\{\frac{3}{5},\frac{4}{3}\right\},\left\{\frac{-3}{5},\frac{-4}{3}\right\}$ **3.** $\left\{\frac{3}{4}\right\}$

4. $\left\{\frac{8}{17},\frac{17}{8}\right\},\left\{\frac{-8}{17},\frac{-17}{8}\right\}$ **5.** $1,\frac{3}{5}$ **6.** $\left\{\frac{3}{5},\frac{5}{13}\right\}$ **7.** $\frac{5}{13}$ **8.** $\left\{\frac{12}{13}\right\}$

9. $\left\{\pm 1,\pm\frac{1}{\sqrt{3}}\right\}$ **10.** $\left\{\frac{1}{2},-\frac{1}{3}\right\}$ **11.** $\pm\frac{1}{\sqrt{2}}$

12. $\left\{1\pm\sqrt{2}\right\}$ **13.** $\pm\frac{2x(x+1)}{2x^2+2x+1};\pm\frac{2x+1}{2x^2+2x+1}$

Do yourself-4

2. (a) $\frac{3}{2}$ (b) $4\frac{1}{3}$ (c) $\frac{1+\sqrt{3}}{4}$ (d) $-\frac{\sqrt{3}-1}{2\sqrt{2}}$

Do yourself-5

1. **(i)** 8 **2.** (a) 1 (b) 0

3. (a) $-\left(\frac{\sqrt{3}+1}{2}\right),-\frac{4}{\sqrt{3}}$ (b) $\left(\sqrt{2},-2\right)$ (c) $\frac{\sqrt{3}+1}{2},-\frac{4}{\sqrt{3}}$ (d) 0,2

4. (a) $-\cos(25^\circ)$ (b) $-\cos(28^\circ)$ (c) $\cot 25^\circ$ (d) $\cot(26^\circ)$

5. (a) negative (b) positive (c) zero

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6. (a) positive (b) positive (c) negative **7.** $-\frac{1}{\sqrt{3}}$ and $\frac{\sqrt{2}}{\sqrt{3}}$; $\frac{1}{\sqrt{3}}$ and $-\frac{\sqrt{2}}{\sqrt{3}}$

Do yourself-6

3. B

4. (a)

(b)

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(c)

5. 2

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Do yourself-7

1. (a) $\frac{187}{205}$ (b) $\frac{-133}{205}$ (c) $\frac{-84}{205}$ (d) $\frac{156}{205}$

3. $\sin(\alpha-\beta)=-\frac{133}{205}$ or $\frac{187}{205}$; $\cos(\alpha+\beta)=-\frac{84}{205}$ or $\frac{156}{205}$

4. $\sin(\alpha+\beta)=\frac{220}{221}$ or $\frac{140}{221}$; $\cos(\alpha-\beta)=\pm\frac{171}{221}$ or $\pm\frac{21}{221}$; $\sin(\alpha+\beta)=\frac{220}{221}$ or $\frac{140}{221}$; $\tan(\alpha+\beta)=\pm\frac{220}{21}$ or $\pm\frac{140}{171}$

13. 3 and $\frac{9}{13}$

Do yourself-8

1. $\frac{1}{\sqrt{3}}$ **2.** $\cos 2\theta-\cos 12\theta$ **3.** $\cos 12^\circ-\cos 120^\circ$

Do yourself-11

8. D **9.** A

Do yourself-12

1 (a) $\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$ (b) $\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$ (c) $\sqrt{2}-1$ **4.** $\frac{\pm 2\sqrt{2}\pm\sqrt{3}}{6}$; $\frac{\pm 7\sqrt{3}\pm 4\sqrt{2}}{18}$

5. $\frac{16}{305}$; $\frac{49}{305}$ **6.** $\frac{7}{5\sqrt{2}}$ **7.** $\pm\frac{1}{3}$; $\pm\frac{3}{4}$

8. (a) $\frac{\sqrt{4-\sqrt{2}-\sqrt{6}}}{2\sqrt{2}}$; (b) $\frac{\sqrt{4+\sqrt{2}+\sqrt{6}}}{2\sqrt{2}}$; (c) $\sqrt{4+2\sqrt{2}}-(\sqrt{2}+1)$ **9.** $\pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$

Do yourself-13

1. (a) $-\frac{1}{2}$ (b) $\frac{\sqrt{5}+1}{8}$ **2.** $\frac{\sqrt{5}-1}{8}$ **3.** $\frac{5}{16}$ **4.** 1 **5.** $\frac{1}{2^7}$

6. 1 **9.** 4 **10.** B

Do yourself-14

1. 0 **2.** 1

Do yourself-15

1. 7 & –7 **2.** $-\frac{9}{8}$ **3.** $4+\sqrt{10}$ & $4-\sqrt{10}$ **4.** (i) 2, – 1 (ii) 3,0 **5.** $\frac{1}{2}$

6. $\left\{\frac{13}{3},-1\right\}$ **7.** 25 **8.** {25,–25} **9.** $\{\sqrt{13},-\sqrt{13}\}$ **10.** (i) 30 (ii) 18

Do yourself-16

1. 0 **3.** $\frac{\cot 1^\circ}{90}$ **5.** (a) $\frac{1}{2}$ (b) 0 **6.** C

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EXERCISE (O-1)

1. C **2.** B **3.** B **4.** A **5.** B **6.** C **7.** B **8.** B **9.** D **10.** A **11.** C **12.** B
13. B **14.** D **15.** D **16.** A **17.** D **18.** D **19.** D **20.** B **21.** D **22.** B **23.** C
24. D **25.** D **26.** A **27.** B **28.** B **29.** B **30.** C

EXERCISE (O-2)

1. B,C **2.** B,D **3.** B,C **4.** B,C **5.** B,C,D **6.** A,B **7.** A,B,C,D
8. A,B,C,D **9.** A,D **10.** C,D **11.** A,B,D **12.** A,B,C,D **13.** C,D **14.** B,D
15. A **16.** C **17.** B **18.** D **19.** C **20.** **5** **21.** **3**
22. **3** **23.** **2** **24.** **1** **25.** (A)→(S); (B)→(R); (C)→(Q); (D)→(P)

EXERCISE (S-1)

4. $\frac{56}{33}$ **5.** 24 **7.** (a) $\frac{3-\sqrt{5}}{32}$; (b) $\frac{2-\sqrt{3}}{16}$ **8.** (a) –1, (b) $\sqrt{3}$, (c) $\frac{5}{4}$, (d) $\sqrt{3}$

9. n = 23 **10.** (a) y_{max} =11, y_{min} = 1; (b) $y_{max} = \frac{13}{3}$, y_{min} = –1; (c) 49; (d) a =–4 & b = 10

EXERCISE (S-2)

5. p = 3, q = 2; r = 2; s = 1 **6.** $2\sqrt{2}$ **7.** 28 **8.** $a = 3b - 2b^3$

9. $\frac{\pi}{12}$ and $\frac{5\pi}{12}$

EXERCISE (JM)

1. 3 **2.** 2 **3.** 4 **4.** 1 **5.** 1 **6.** 3 **7.** 4 **8.** 1
9. 2 **10.** 3

EXERCISE (JA)

1. B **2.** **(a)** A, B; **(b)** C,D **3.** (a) 2; (b) k = 3 **4.** D **5.** C **6.** **Bonus**

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Chapter 02

LOGARITHM

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JEE (Main/Advanced) Syllabus

JEE (Advanced) Syllabus :
Logarithms and their properties.

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Important Notes

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LOGARITHM

1. DEFINITION :

The logarithm of a number N to a base 'a' is an exponent indicating the power to which the base 'a' must be raised to obtain the number N. This number is designated as $\log_a N$. (Read it "Log N on base a"). Here N is usually called argument of the Logarithm and 'a' is called base of the Logarithm.

Hence : $\log_a N = x \Leftrightarrow a^x = N$, $a > 0$, $a \neq 1$ and $N > 0$

By the definition of logarithm, $\log_2 16$ is the exponent indicating the power to which 2 must be raised in order to obtain 16.

As $2^4 = 16$, hence $\log_2 16 = 4$.

Similarly

Exponential Form		**Logarithmic Form**
$3^5 = 243$	$\Leftrightarrow$	$\log_3 243 = 5$
$5^4 = 625$	$\Leftrightarrow$	$\log_5 625 = 4$
$2^{-3} = \frac{1}{8}$	$\Leftrightarrow$	$\log_2 \frac{1}{8} = -3$
$7^0 = 1$	$\Leftrightarrow$	$\log_7 1 = 0$

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Note that the expressions $\log_3(-27)$, $\log_1 16$, $\log_0 5$ and $\log_2 0$ has no sense in real numbers since the equations $3^x = -27$, $1^x = 16$, $0^x = 5$, $2^x = 0$ are absurd for any real x, the reason being obvious that no such exponent x in real number could be found.

In general, the expression $\log_a N$ is meaningful if and only if, $a > 0$, $a \neq 1$ and $N > 0$.

The existence and uniqueness of the number $\log_a N$ follows from the properties of exponential functions.

Illustration 1 : If $\log_4 m = 1.5$, then find the value of m.

Solution : $\log_4 m = 1.5 \Rightarrow m = 4^{3/2} \Rightarrow m = 8$

Illustration 2 : If $\log_5 p = a$ and $\log_2 q = a$, then prove that $\frac{p^4 q^4}{100} = 100^{2a-1}$

Solution : $\log_5 p = a \Rightarrow p = 5^a$

$\log_2 q = a \Rightarrow q = 2^a$

$$\Rightarrow \frac{p^4 q^4}{100} = \frac{5^{4a}.2^{4a}}{100} = \frac{(10)^{4a}}{100} = \frac{(100)^{2a}}{100} = 100^{2a-1}$$

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Illustration 3 : The value of N, satisfying $\log_a[1+\log_b\{1+\log_c(1+\log_p N)\}]=0$ is -

(A) 4 (B) 3 (C) 2 (D) 1

Solution : $1+\log_b\{1+\log_c(1+\log_p N)\}=a^0=1$

$\Rightarrow \log_b\{1+\log_c(1+\log_p N)\}=0 \quad \Rightarrow \quad 1+\log_c(1+\log_p N)=1$

$\Rightarrow \log_c(1+\log_p N)=0 \quad \Rightarrow \quad 1+\log_p N=1$

$\Rightarrow \log_p N=0 \quad \Rightarrow \quad N=1$ **Ans. (D)**

Do yourself - 1 :

1 Express the following in logarithmic form :

(a) $81=3^4$ (b) $0.001=10^{-3}$ (c) $2=128^{1/7}$

2. Express the following in exponential form :

(a) $\log_2 32=5$ (b) $\log_{\sqrt{2}}4=4$ (c) $\log_{10}0.01=-2$

3. If $\log_{2\sqrt{3}}1728=x$, then find x.

4. Find the logarithms of the following numbers to the base 2 :

(i) 1 (ii) 2 (iii) 4 (iv) 8 (v) $\frac{1}{2}$

(vi) $\frac{1}{32}$ (vii) $\frac{1}{16}$ (viii) $\sqrt{2}$ (ix) $\sqrt[3]{8}$ (x) $2\sqrt{2}$

(xi) $\frac{1}{\sqrt[5]{2}}$ (xii) $\frac{1}{\sqrt[7]{8}}$

5. Find the logarithms of the following numbers to the base $\frac{1}{2}$:

(i) 1 (ii) $\frac{1}{2}$ (iii) $\frac{1}{8}$ (iv) 16

(v) $\sqrt{2}$ (vi) $\frac{1}{\sqrt{2}}$ (vii) $2\sqrt{2}$ (viii) $\frac{1}{4\sqrt[4]{2}}$

6. Find the logarithms of the following numbers to the base 3 :

(i) 1 (ii) 3 (iii) 9 (iv) 81 (v) $\frac{1}{3}$

(vi) $\sqrt{3}$ (vii) $\frac{1}{3\sqrt{3}}$ (viii) $27\sqrt{3}$ (ix) $\sqrt[7]{9}$

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7. Find the logarithms of the following numbers to the base $\frac{1}{3}$:

(i) 1 (ii) $\frac{1}{3}$ (iii) $\frac{1}{9}$ (iv) 3 (v) 9

(vi) 81 (vii) $\sqrt[3]{3}$ (viii) $\frac{1}{\sqrt[7]{3}}$ (ix) $9\sqrt{3}$ (x) $\frac{1}{9\sqrt[4]{3}}$

8. Find the logarithms of the following numbers to the base 5.

(i) 1 (ii) 5 (iii) 25 (iv) 625 (v) $\frac{1}{5}$

(vi) $\frac{1}{25}$ (vii) $\frac{1}{\sqrt{5}}$ (viii) $\sqrt{\sqrt{5}}$ (ix) $5^{\frac{1}{2}}$ (x) $5^{\frac{1}{3}}$

(xi) $\sqrt[4]{5\sqrt[3]{5}}$

9. Find all values of 'a' for which each of the following equalities hold true :

(i) $\log_2 a = 2$ (ii) $\log_a 2 = 1$ (iii) $\log_a 1 = 0$

(iv) $\log_{10}(a(a+3)) = 1$ (v) $\log_{1/3}(a^2 - 1) = -1$ (vi) $\log_2(a^2 - 5) = 2$

(vii) $\log_3 a = 2$ (viii) $\log_{1/3}(a) = 4$ (ix) $\log_{1/3}(a) = 0$

(x) $\log_a(a+2) = 2$ (xi) $\log_3(a^2 + 1) = 1$

10. Evaluate the following :

(i) $\log_5\left(\frac{1}{\sqrt{5}}\right)$ (ii) $\log_{\sqrt{5}+1}(6+2\sqrt{5})$

(iii) $\log_3\left(4\sin^2(x) + 4\cos^2(x) - 1\right)$ (iv) $\log_{\sqrt{3}-\sqrt{2}}\left(\sqrt{5-2\sqrt{6}}\right)$

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2. FUNDAMENTAL LOGARITHMIC IDENTITY :

From the definition of the logarithm of the number N to the base 'a' , we have an identity :

$$a^{\log_a N} = N \ , \ a > 0 \ , \ a \neq 1 \text{ and } N > 0$$

This is known as the **Fundamental Logarithmic Identity .**

Note :

Using the basic definition of logarithm we have 3 important deductions :

(a) $\log_a 1 = 0$ i.e. logarithm of unity to any base is zero ($a > 0$; $a \neq 1$).

(b) $\log_N N = 1$ i.e. logarithm of a number to the same base is 1. ($N > 0$; $N \neq 1$)

(c) $\log_{\frac{1}{N}} N = -1 = \log_N \frac{1}{N}$ i.e. logarithm of a number to the base as its reciprocal is –1. ($N > 0$; $N \neq 1$)

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Do yourself - 2 :

1 Find the value of the following :

(a) $\log_{1.4\overline{3}} \frac{43}{30}$ **(b)** $\left(\frac{1}{2}\right)^{\log_2 5}$

2. If $4^{\log_2 2x} = 36$, then find x.

3. THE PRINCIPAL PROPERTIES OF LOGARITHMS :

If m,n are arbitrary positive numbers where $a > 0$, $a \neq 1$ and x is any real number, then

(a) $\log_a mn = \log_a m + \log_a n$ (b) $\log_a \frac{m}{n} = \log_a m - \log_a n$ (c) $\log_a m^x = x \log_a m$

Illustration 4 : Find the value of $2\log\frac{2}{5} + 3\log\frac{25}{8} - \log\frac{625}{128}$

Solution : $2\log\frac{2}{5} + 3\log\frac{25}{8} + \log\frac{128}{625}$

$= \log\frac{2^2}{5^2} + \log\left(\frac{5^2}{2^3}\right)^3 + \log\frac{2^7}{5^4}$

$= \log\frac{2^2}{5^2}\cdot\frac{5^6}{2^9}\cdot\frac{2^7}{5^4} = \log 1 = 0$

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Illustration 5 : If $\log_e x - \log_e y = a$, $\log_e y - \log_e z = b$ & $\log_e z - \log_e x = c$, then find the value of

$\left(\frac{x}{y}\right)^{b-c} \times \left(\frac{y}{z}\right)^{c-a} \times \left(\frac{z}{x}\right)^{a-b}$.

Solution : $\log_e x - \log_e y = a \Rightarrow \log_e \frac{x}{y} = a \Rightarrow \frac{x}{y} = e^a$

$\log_e y - \log_e z = b \Rightarrow \log_e \frac{y}{z} = b \Rightarrow \frac{y}{z} = e^b$

$\log_e z - \log_e x = c \Rightarrow \log_e \frac{z}{x} = c \Rightarrow \frac{z}{x} = e^c$

$\therefore \left(e^a\right)^{b-c} \times \left(e^b\right)^{c-a} \times \left(e^c\right)^{a-b}$

$= e^{a(b-c)+b(c-a)+c(a-b)} = e^0 = 1$

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Illustration 6 : If $a^2 + b^2 = 23ab$, then prove that $\log\frac{(a+b)}{5} = \frac{1}{2}(\log a + \log b)$.

Solution : $a^2 + b^2 = (a + b)^2 - 2ab = 23ab$

$\Rightarrow (a + b)^2 = 25ab \Rightarrow a+b = 5\sqrt{ab}$ ....(i)

Using (i)

$$\text{L.H.S.} = \log\frac{(a+b)}{5} = \log\frac{5\sqrt{ab}}{5} = \frac{1}{2}\log ab = \frac{1}{2}(\log a + \log b) = \text{R.H.S.}$$

Illustration 7 : If $\log_a x = p$ and $\log_b x^2 = q$, then $\log_x \sqrt{ab}$ is equal to (where a, b, x $\in R^+ - \{1\}$)-

(A) $\frac{1}{p}+\frac{1}{q}$ (B) $\frac{1}{2p}+\frac{1}{q}$ (C) $\frac{1}{p}+\frac{1}{2q}$ (D) $\frac{1}{2p}+\frac{1}{2q}$

Solution : $\log_a x = p \Rightarrow a^p = x \Rightarrow a = x^{1/p}$.

similarly $b^q = x^2 \Rightarrow b = x^{2/q}$

Now, $\log_x \sqrt{ab} = \log_x \sqrt{x^{1/p}x^{2/q}} = \log_x x^{\left(\frac{1}{p}+\frac{2}{q}\right)\cdot\frac{1}{2}} = \frac{1}{2p}+\frac{1}{q}$

Do yourself - 3 :

1. Show that $\frac{1}{2}\log 9 + 2\log 6 + \frac{1}{4}\log 81 - \log 12 = 3\log 3$

2. Find the value of following

(i) $\log_{12} 8 + \log_{12} 3 + \log_{12} 6$ (ii) $\log_5 \frac{500}{3} - \log_5 \frac{4}{3}$

(iii) $\log_{39}\frac{15}{7} + \log_{39}\frac{13}{3} - \log_{39}\frac{5}{21}$ (iv) $2\log_6 2 + 3\log_6 3 + \log_6 12$

4. BASE CHANGING THEOREM :

Can be stated as "quotient of the logarithm of two numbers is independent of their common base."

Symbolically, $\log_b m = \frac{\log_a m}{\log_a b}$, where $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$

Note :

(i) $\log_b a.\ \log_a b = \frac{\log a}{\log b}.\frac{\log b}{\log a} = 1$; hence $\boxed{\log_b a = \frac{1}{\log_a b}.}$

(ii) $a^{\log_b c} = c^{\log_b a}$

(iii) Base power formula : $\log_{a^k} m = \frac{1}{k}\log_a m$

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(iv) The base of the logarithm can be any positive number other than 1, but in normal practice, only two bases are popular, these are 10 and e(=2.718 approx). Logarithms of numbers to the base 10 are named as 'common logarithm' and the logarithms of numbers to the base e are named as Natural or Napierian logarithm. **We will consider log x as $\log_e x$ or ℓn x.**

(v) Conversion of base e to base 10 & viceversa :

$$\log_e a = \frac{\log_{10} a}{\log_{10} e} = 2.303 \times \log_{10} a \; ; \quad \log_{10} a = \frac{\log_e a}{\log_e 10} = \log_{10} e \times \log_e a = 0.434 \log_e a$$

(vi) Some important values : $\log_{10}2 \approx 0.3010$; $\log_{10}3 \approx 0.4771$; $\ell n2 \approx 0.693$, $\ell n10 \approx 2.303$

(vii) The positive real number 'n' is called the antilogarithm of a number 'm' to base 'a' if $\log_a n = m$

Thus, $\log_a n = m \Leftrightarrow n = \text{antilog}_a m$

Illustration 8 : If a, b, c are distinct positive real numbers different from 1 such that $(\log_b a \,.\, \log_c a - \log_a a) + (\log_a b \,.\, \log_c b - \log_b b) + (\log_a c \,.\, \log_b c - \log_c c) = 0$, then abc is equal to -

(A) 0 (B) e (C) 1 (D) none of these

Solution : $(\log_b a \log_c a - 1) + (\log_a b \,.\, \log_c b - 1) + (\log_a c \log_b c - 1) = 0$

$$\Rightarrow \frac{\log a}{\log b}.\frac{\log a}{\log c} + \frac{\log b}{\log a}.\frac{\log b}{\log c} + \frac{\log c}{\log a}.\frac{\log c}{\log b} = 3$$

$\Rightarrow (\log a)^3 + (\log b)^3 + (\log c)^3 = 3\log a \log b \log c$

$\Rightarrow (\log a + \log b + \log c) = 0$ [$\because$ If $a^3 + b^3 + c^3 - 3abc = 0$, then $a + b + c = 0$ if $a \neq b \neq c$]

$\Rightarrow \log abc = \log 1 \Rightarrow abc = 1$

Illustration 9 : Evaluate : $81^{1/\log_5 3} + 27^{\log_9 36} + 3^{4/\log_7 9}$

Solution : $81^{\log_3 5} + 3^{3\log_9 36} + 3^{4\log_9 7}$

$= 3^{4\log_3 5} + 3^{\log_3 (36)^{3/2}} + 3^{\log_3 7^2} = 625 + 216 + 49 = 890.$

Illustration 10 : Show that $\log_4 18$ is an irrational number.

Solution : $\log_4 18 = \log_4(3^2 \times 2) = 2\log_4 3 + \log_4 2 = 2\dfrac{\log_2 3}{\log_2 4} + \dfrac{1}{\log_2 4} = \log_2 3 + \dfrac{1}{2}$

assume the contrary, that this number $\log_2 3$ is rational number.

$\Rightarrow \log_2 3 = \dfrac{p}{q}$. Since $\log_2 3 > 0$ both numbers p and q may be regarded as natural number

$\Rightarrow 3 = 2^{p/q} \Rightarrow 2^p = 3^q$

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But this is not possible for any natural number p and q. The resulting contradiction completes the proof.

Illustration 11 : If in a right angled triangle, a and b are the lengths of sides and c is the length of hypotenuse and $c - b \neq 1$, $c + b \neq 1$, then show that

$\log_{c+b}a + \log_{c-b}a = 2\log_{c+b}a \,.\, \log_{c-b}a.$

Solution : We know that in a right angled triangle

$c^2 = a^2 + b^2$

$c^2 - b^2 = a^2$ ..... (i)

$$\text{LHS} = \frac{1}{\log_a(c+b)} + \frac{1}{\log_a(c-b)} = \frac{\log_a(c-b) + \log_a(c+b)}{\log_a(c+b).\log_a(c-b)}$$

$$= \frac{\log_a(c^2 - b^2)}{\log_a(c+b).\log_a(c-b)} = \frac{\log_a a^2}{\log_a(c+b).\log_a(c-b)} \qquad \text{(using (i))}$$

$$= \frac{2}{\log_a(c+b).\log_a(c-b)} = 2\log_{(c+b)}a \,.\, \log_{(c-b)} a = \text{RHS}$$

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Do yourself - 4 :

1. **(i)** Evaluate : $\dfrac{\log_3 135}{\log_{15} 3} - \dfrac{\log_3 5}{\log_{405} 3}$ **(ii)** Evaluate : $\log_9 27 - \log_{27} 9$

(iii) Evaluate : $2^{\log_3 5} - 5^{\log_3 2}$

(iv) Evaluate : $\log_3 4 \,.\, \log_4 5 \,.\, \log_5 6 \,.\, \log_6 7 \,.\, \log_7 8 \,.\, \log_8 9$

(v) If $\dfrac{1}{\log_3 \pi} + \dfrac{1}{\log_4 \pi} > x$, then x can be -

(A) 2 (B) 3 (C) 3.5 (D) π

(vi) If $\log_a 3 = 2$ and $\log_b 8 = 3$, then $\log_a b$ is -

(A) $\log_3 2$ (B) $\log_2 3$ (C) $\log_3 4$ (D) $\log_4 3$

2. Find value of following

(i) $\log_7 3.\log_5 2.\log_3 7.\log_2(125)$ (ii) $25^{\log_5 3}$

(iii) $6^{\log_6 5} + 3^{\log_9 16}$ (vi) $\log_6 4 + \dfrac{1}{\log_9 6}$

3. Find antilog of $\dfrac{5}{6}$ to the base 64.

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5. LOGARITHMIC EQUATIONS

Illustration 12 : $\log_{\frac{1}{2}}\left(\log_2 \sqrt{2}x\right)=1$, then find x ?

Solution : $\log_{\frac{1}{2}}\left(\log_2 \sqrt{2}x\right)=1$

$\Rightarrow \log_2\left(\sqrt{2}x\right)=\frac{1}{2}$

$\Rightarrow \sqrt{2}x=2^{\frac{1}{2}}$

$\Rightarrow x = 1$

Illustration 13 : Solve the equation $2\log_2\left(\log_2 x\right)+\log_{1/2}\left(\frac{3}{2}+\log_2 x\right)=1$.

Solution : Let $\log_2 x = t$

$\Rightarrow 2\log_2(t)+\log_{\frac{1}{2}}\left(\frac{3}{2}+t\right)=1$

$\Rightarrow 2\log_2 t-\log_2\left(\frac{3}{2}+t\right)=1$

$\Rightarrow \log_2\left(\frac{t^2}{\frac{3}{2}+t}\right)=1$

$\Rightarrow \frac{2t^2}{3+2t}=2$

$\Rightarrow t^2 - 2t - 3 = 0$

$\Rightarrow (t + 1)(t - 3) = 0$

$\Rightarrow t = 3 \because t > 0$

$\Rightarrow x = 8$

Illustration 14 : Solve the equation $\log x^2 - \log(2x) = 3\log 3 - \log 6$.

Solution : $\log x^2 - \log 2x = 3\log 3 - \log 6$

$x > 0 \Rightarrow 2\log x - \log 2 - \log x = 3\log 3 - \log 2 - \log 3 \Rightarrow \log x = 2\log 3 \Rightarrow \log x = \log 9$

$\Rightarrow x = 9$

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Illustration 15 : Solve the equation $\left(\log_5 x\right)^2 + \log_5 x + 1 = \dfrac{7}{\log_5 x - 1}$

Solution : Put $\log_5 x = t$, we get $t^2 + t + 1 = \dfrac{7}{t-1}$

$(t-1)(t^2+t+1) = 7 \Rightarrow t^3 + t^2 + t - t^2 - t - 1 = 7 \Rightarrow t^3 - 8 = 0$

$\Rightarrow (t-2)(t^2+2t+4) = 0 \Rightarrow t - 2 = 0;\ t^2 + 2t + 4 \neq 0 \Rightarrow t = 2$

Now, $t = \log_5 x$, so $\log_5 x = 2$

$x = 5^2 \Rightarrow x = 25$

Illustration 16 : Solve the equation $\left|x-1\right|^{\log_2 x^2 - 2\log_x 4} = \left(x-1\right)^7$

Solution : Obviously $x = 2$ is a solution. Since, left side is positive, $x - 1 > 0$.

The equation reduces $\log_2 x^2 - 2\log_x 4 = 7$

$\Rightarrow 2t - \dfrac{4}{t} = 7, t = \log_2 x$

$\Rightarrow 2t^2 - 7t - 4 = 0 \Rightarrow t = 4, -\dfrac{1}{2}$

But $t > 0$ since $x > 1$. $\therefore t = 4$

$\Rightarrow x = 2^4 = 16$

$\therefore x = 2, 16$

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Illustration 17 : Solve the equation $x^{\log_3 x^2 + (\log_3 x)^2 - 10} = \dfrac{1}{x^2}$

Solution : Taking $\log_3$ on both sides, we get

$(2t + t^2 - 10)t = -2t, \quad t = \log_3 x$

$\Rightarrow t(t^2 + 2t - 8) = 0 \Rightarrow t = 0, 2, -4$

$\Rightarrow x = 1, 9, \dfrac{1}{81}$

Illustration 18 : Solve the equation $4^{\log_2 \ell nx} = \ell nx - \left(\ell nx\right)^2 + 1$

Solution : $4^{\log_2 \ell nx} = 2^{2\log_2(\ell nx)} = 2^{\log_2(\ell nx)^2} = \left(\ell nx\right)^2$

$\Rightarrow \left(\ell n\, x\right)^2 = \ell n\, x - \left(\ell n\, x\right)^2 + 1 \Rightarrow 2\left(\ell n\, x\right)^2 - \ell n\, x - 1 = 0 \Rightarrow \ell n\, x = 1, -\dfrac{1}{2}$

But $\ell nx > 0$

$\therefore \ell nx = 1 \Rightarrow x = e.$

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Illustration 19 : Solve the equation x : $\log_{x+1}(x^2+x-6)^2 = 4$

Solution : We have,

$\log_{x+1}(x^2+x-6)^2 = 4 \Rightarrow (x^2+x-6)^2 = (x+1)^4 = (x^2+2x+1)^2$

$\Rightarrow (x^2+x-6-x^2-2x-1)(x^2+x-6+x^2+2x+1) = 0$

$\Rightarrow (-x-7)(2x^2+3x-5) = 0 \Rightarrow (x+7)(x-1)(2x+5) = 0 \Rightarrow x = -7, -5/2, 1$

The values x = –7 and x = –5/2 are rejected because they make the base x + 1 negative

Hence, x = 1 is the only solution of the given equation.

Do yourself - 5 :

1. Find all values of x for which the following equalities hold true.

(i) $\log_2 x^2 = 1$ (ii) $\log_3 x = \log_3(2-x)$ (iii) $\log_4 x^2 = \log_4 x$

(iv) $\log_{1/2}(2x+1) = \log_{1/2}(x+1)$ (v) $\log_{1/3}(x^2+8) = -2$

2. Find all the values of x for which the following equalities hold true.

(i) $\log_2 x^2 = 2$ (ii) $\log_{1/4} x^2 = 1$

(iii) $\log_{1/2} x - \log_{1/2}(3-x) = 0$ (iv) $\log_2(x+1) - \log_2(2x-3) = 0$

3. $\log_3(4^x + 15.2^x + 27) - 2\log_3(4.2^x - 3) = 0.$

4. $\dfrac{\log_8\left(8/x^2\right)}{\left(\log_8 x\right)^2} = 3$

5. $\log_x 3.\log_{x/3} 3 + \log_{x/81} 3 = 0.$

6. $1 + 2\log_x 2.\log_4(10-x) = 2/\log_4 x.$

7. $x^{x+1} = x.$

8. $x^{\log_x (x+3)^2} = 16$

9. $\sqrt{x^{\log_{10}\sqrt{x}}} = 10$

10. $3^{\log_a x} + 3.x^{\log_a 3} = 2$

11. $\log_a\left(1-\sqrt{1+x}\right) = \log_{a^2}\left(3-\sqrt{1+x}\right)$

12. $2\log_8(2x) + \log_8(x^2+1-2x) = 4/3$

13. $|\log_2 x| = 3$

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6. GRAPHS OF LOGARITHMIC FUNCTION AND ITS INEQUALITIES

Graph of $y = \log_a x$:

when a > 1

When 0 < a < 1

(i) $\log_a x < \log_a y \Leftrightarrow \begin{bmatrix} x < y & \text{if} & a > 1 \\ x > y & \text{if} & 0 < a < 1 \end{bmatrix}$

(ii) If a > 1, then $\log_a x < p \Rightarrow 0 < x < a^p$ and $\log_a x > p \Rightarrow x > a^p$

(iii) If 0 < a < 1, then $\log_a x < p \Rightarrow x > a^p$ and $\log_a x > p \Rightarrow 0 < x < a^p$

Note :

(i) If base of logarithm is greater than on that base 1 then logarithm of greater number is greater. i.e. $\log_2 8 = 3$, $\log_2 4 = 2$ etc. and if base of logarithm is between 0 and 1 then on that base logarithm of greater number is smaller. i.e. $\log_{1/2} 8 = -3$, $\log_{1/2} 4 = -2$ etc.

(ii) It must be noted that whenever the number and the base are on the same side of unity then logarithm of that number to that base is positive, however if the number and the base are located on different side of unity then logarithm of that number to that base is negative.

e.g. $\log_{10} \sqrt[3]{10} = \frac{1}{3}$; $\log_{\sqrt{7}} 49 = 4$; $\log_{\frac{1}{2}}\left(\frac{1}{8}\right) = 3$; $\log_2\left(\frac{1}{32}\right) = -5$; $\log_{10}(0.001) = -3$

Illustration 20 : Solve for x : $x^{\log_5 x} > 5$

Solution : as x > 0 (for existance)

now solving inequality

$x^{\log_5 x} > 5$. Taking 'log' with base '5' we have $\log_5 x . \log_5 x > 1$

$\Rightarrow (\log_5 x - 1)(\log_5 x + 1) > 0 \Rightarrow \log_5 x > 1$ or $\log_5 < -1$

$\Rightarrow x > 5$ or $x < 1/5$. Also we must have $x > 0$

Thus, $x \in (0,1/5)$ or $x \in (5,\infty)$

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Illustration 21 : Solve for x : $\log_3(2x+1) < \log_3 5$.

Solution : Checking existance

$2x + 1 > 0 \Rightarrow x > -\frac{1}{2}$

Now solving inequality we have $2x + 1 < 5$

$\Rightarrow \quad 2x > -1$ and $2x < 4$

$\Rightarrow \quad x > -1/2$ and $x < 2$,

$\Rightarrow \quad x \in (-1/2, 2)$

Illustration 22: Solve for x : $(\log_{10}100x)^2 + (\log_{10}10x)^2 + \log_{10}x \le 14$.

Solution : Checking existance

$x > 0$

Now solving inequality,

Let $u = \log_{10}x$

$(2+u)^2 + (1+u)^2 + u \le 14 \Rightarrow u^2 + 4u + 4 + u^2 + 2u + 1 + u \le 14$

$\Rightarrow 2u^2 + 7u - 9 \le 0 \Rightarrow 2u^2 + 9u - 2u - 9 \le 0$

$\Rightarrow u(2u+9) - 1(2u+9) \le 0 \Rightarrow (2u+9)(u-1) \le 0$

$\Rightarrow \frac{-9}{2} \le u \le 1 \Rightarrow \frac{-9}{2} \le \log_{10}x \le 1 \Rightarrow 10^{-\frac{9}{2}} \le x \le 10$

Illustration 23: Solve for x : $\log_3((x+2)(x+4)) + \log_{1/3}(x+2) < \frac{1}{2}\log_{\sqrt{3}}7$.

Solution : Checking existance,

$(x+2)(x+4) > 0$ and $(x+2) > 0$

$x < -4$ or $x > -2$ and $x > -2$

Now solving inequality.

$\log_3((x+2)(x+4)) + \log_{1/3}(x+2) < \frac{1}{2}\log_{\sqrt{3}}7$.

$\Rightarrow \log_3(x+2)(x+4) - \log_3(x+2) < \log_3 7$

$\Rightarrow \log_3(x+4) < \log_3 7$

$\Rightarrow x + 4 < 7 \Rightarrow x < 3$

$\Rightarrow -2 < x < 3 \Rightarrow \quad x \in (-2, 3)$

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Illustration 24: Solve for x : $\log_{1/3}\log_4(x^2-5)>0$

Solution : Checking existence,

(i) $\log_4(x^2-5)>0 \Rightarrow x^2-5>1$

$$\Rightarrow \left(x-\sqrt{6}\right)\left(x+\sqrt{6}\right)>0$$

$$\Rightarrow x\in\left(-\infty,-\sqrt{6}\right)\cup\left(\sqrt{6},\infty\right)$$

(ii) $x^2-5>0 \Rightarrow x\in\left(-\infty,-\sqrt{5}\right)\cup\left(\sqrt{5},\infty\right)$

solving inequality,

$$\log_{\frac{1}{3}}\log_4\left(x^2-5\right)>0$$

$$\Rightarrow \log_4\left(x^2-5\right)<1$$

$\Rightarrow x^2-5<4 \Rightarrow x^2-9<0$

$\Rightarrow (x-3)(x+3)<0$

$\Rightarrow x\in(-3,3)$ ...(iii)

$\therefore$ Answer : (i) $\cap$ (ii) $\cap$ (iii)

$$\Rightarrow x\in\left(-3,-\sqrt{6}\right)\cup\left(\sqrt{6},3\right)$$

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Illustration 25 : Solve for x : $\dfrac{\log_2\left(4x^2-x-1\right)}{\log_2\left(x^2+1\right)}>1$

Solution : Checking existance

(i) $x^2+1\neq 1 \Rightarrow x\neq 0$

(ii) $4x^2-x-1>0$

Now solving inequality,

$$\frac{\log_2\left(4x^2-x-1\right)}{\log_2\left(x^2+1\right)}>1$$

$\Rightarrow \log_2(4x^2-x-1)>\log_2(x^2+1) \Rightarrow \log_2(4x^2-x-1)-\log_2(x^2+1)>0$

$$\Rightarrow \log_2\frac{4x^2-x-1}{x^2+1}>0$$

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$\Rightarrow 4x^2 - x - 1 - x^2 - 1 > 0 \Rightarrow 3x^2 - x - 2 > 0 \Rightarrow 3x^2 - 3x + 2x - 2 > 0$

$\Rightarrow 3x(x-1) + 2(x-1) > 0 \Rightarrow (x-1)(3x+2) > 0 \Rightarrow x < -2/3$ or $x > 1$

$\Rightarrow x \in (-\infty, -2/3)$ or $x \in (1, \infty)$

Note : $x < -\frac{2}{3}$ and $x > 1$; $4x^2 - x - 1 > 0$

Illustration 26: Solve for x : $\log_4(3^x - 1)\log_{1/4}\left(\frac{3^x - 1}{16}\right) \le \frac{3}{4}$

Solution : Checking existance $3^x - 1 > 0 \Rightarrow x > 0$,

Now solving inequality

$$\log_4(3^x - 1)\log_{1/4}\left(\frac{3^x - 1}{16}\right) \le \frac{3}{4}$$

$$\Rightarrow \log_4(3^x - 1).\left[-\log_4(3^x - 1) + \log_4 16\right] \le \frac{3}{4}$$

$$\Rightarrow \log_4(3^x - 1)\left[-\log_4(3^x - 1) + 2\right] \le \frac{3}{4} \Rightarrow -\left[\log_4(3^x - 1)\right]^2 + 2\left[\log_4(3^x - 1)\right] \le \frac{3}{4}$$

Put $\log_4(3^x - 1) = t \Rightarrow -t^2 + 2t \le \frac{3}{4}$

$\Rightarrow -4t^2 + 8t - 3 \le 0 \Rightarrow 4t^2 - 8t + 3 \ge 0 \Rightarrow 4t^2 - 6t - 2t + 3 \ge 0$

$\Rightarrow 2t(2t-3) - 1(2t-3) \ge 0 \Rightarrow (2t-3)(2t-1) \ge 0$

$$\Rightarrow \log_4(3^x - 1) \le \frac{1}{2} \text{ or } \log_4(3^x - 1) \ge \frac{3}{2}$$

$\Rightarrow 0 < 3^x - 1 \le 4^{1/2}$ or $3^x - 1 \ge 4^{3/2}$

$\Rightarrow 1 < 3^x \le 3$ or $3^x \ge 9$

$\Rightarrow 0 < x \le 1$ or $x \ge 2$

$\Rightarrow x \in (0, 1] \cup [2, \infty)$

Illustration 27: Solve for x : $\log_{1/3}(x^2 - 6x + 18) - 2\log_{1/3}(x - 4) < 0$

Solution : Checking existance

(1) $x^2 - 6x + 18 > 0 \Rightarrow x \in \mathbb{R}$

(2) $x - 4 > 0 \Rightarrow x > 4$

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Now solving inequality

$\log_{1/3}(x^2 - 6x + 18) - \log_{1/3}(x - 4)^2 < 0$

$\Rightarrow \log_{1/3}\dfrac{(x^2-6x+18)}{(x-4)^2} < 0$ and $2\log_{1/3}(x-4) = \log_{1/3}(x-4)^2$

only when $x - 4 > 0$, so we get

$\log_{1/3}\left(\dfrac{x^2-6x+18}{(x-4)^2}\right) < 0$ and $x - 4 > 0 \Rightarrow x > 4$ ...(1)

$\Rightarrow x^2 - 6x + 18 > (x-4)^2 \Rightarrow x^2 - 6x + 18 > x^2 - 8x + 16$

$2x + 2 > 0 \Rightarrow x > -1 \Rightarrow x \in (-1,\infty)$ ...(2)

from equation (1) and (2), we get $x \in (4,\infty)$

Illustration 28: Solve for $x : \log_e(x^2 - 2x - 2) \le 0$

Solution : The values of x satisfying the inequality $\log_e(x^2 - 2x - 2) \le 0$ must be such that

$0 < x^2 - 2x - 2 \le 1$

we have, $x^2 - 2x - 2 > 0 \Rightarrow (x-1)^2 > 3$

$\Rightarrow |x-1|^2 > 3 \Rightarrow |x-1| > \sqrt{3} \Rightarrow x - 1 > \sqrt{3}$ or $x - 1 < -\sqrt{3}$

$\Rightarrow x > 1 + \sqrt{3}$ or $x < 1 - \sqrt{3}$ ...(1)

Again $x^2 - 2x - 2 \le 1 \Rightarrow x^2 - 2x \le 3 \Rightarrow (x-1)^2 \le 4 \Rightarrow |x-1|^2 \le 4$

$\Rightarrow |x-1| \le 2 \Rightarrow -2 \le x - 1 \le 2 \Rightarrow -1 \le x \le 3$ ...(2)

The value of x satisfying both the inequalities equation (1) and (2) are given by;

Hence, $x \in \left[-1, 1-\sqrt{3}\right) \cup \left(1+\sqrt{3}, 3\right]$

Illustration 29: Solve for x : $\log_x\left(2x - \dfrac{3}{4}\right) > 2$

Solution : For existanc of logarithm

$2x - \dfrac{3}{4} > 0$ and $x > 0$ and $x \ne 1$

so, $x \in \left(\dfrac{3}{8}, \infty\right) - \{1\}$

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To find the value of x satisfying the inequality $\log_x[2x-(3/4)] > 2$

Case I. Let $0 < x < 1$

Then, $\log_x[2x-(3/4)] > 2 \Rightarrow [2x-(3/4)] < x^2$

$\Rightarrow x^2 - 2x + (3/4) > 0 \Rightarrow 4x^2 - 8x + 3 > 0 \Rightarrow (2x-1)(2x-3) > 0$

$$\Rightarrow \left(x-\frac{1}{2}\right)\left[x-(3/2)\right] > 0 \Rightarrow x > 3/2 \text{ or } x < 1/2$$

$\Rightarrow x < 1/2$ because we have $0 < x < 1$.

$\therefore$ But for $\log[2x-(3/4)]$ to be meaningful, we must have

$2x - (3/4) > 0 \Rightarrow x > 3/8$

Therefore, if $0 < x < 1$, the values of x satisfying the given inequality are given by :

$3/8 < x < 1/2$

Case II. Let $x > 1$

Then, $\log_x[2x-(3/4)] > 2 \Rightarrow [2x-(3/4)] > x^2$

$\Rightarrow x^2 - 2x + (3/4) < 0 \Rightarrow 4x^2 - 8x + 3 < 0 \Rightarrow (2x-1)(2x-3) < 0$

$$\Rightarrow \left(x-\frac{1}{2}\right)[x-(3/2)] < 0 \Rightarrow 1/2 < x < 3/2$$

But we have $x > 1$

$\therefore$ We must have $1 < x < 3/2$ and obviously these values of x make $2x - (3/4) > 0$

Thereofre, if $x > 1$, the values of x satisfying the given inequality are given by, $1 < x < 3/2$

$$x \in \left(\frac{3}{8}, \frac{1}{2}\right) \cup \left(1, \frac{3}{2}\right)$$

Illustration 30 : Solve for x : $\log_{0.5}(x^2 - 5x + 6) \geq -1$

Solution : Checking existance $x^2 - 5x + 6 > 0$,

$\Rightarrow x \in (-\infty, 2) \cup (3, \infty)$

Now solving inequality

$\log_{0.5}(x^2 - 5x + 6) \geq -1 \Rightarrow 0 < x^2 - 5x + 6 \leq (0.5)^{-1}$

$\Rightarrow x^2 - 5x + 6 \leq 2$

$$\begin{cases} x^2 - 5x + 6 > 0 \\ x^2 - 5x + 6 \leq 2 \end{cases} \Rightarrow x \in [1,2) \cup (3,4]$$

Hence, solution set of original inequation : $x \in [1,2) \cup (3,4]$

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Illustration 31 : Solve for x : $\log_2 x \le \dfrac{2}{\log_2 x - 1}$.

Solution : Let $\log_2 x = t$

$$t \le \frac{2}{t-1} \Rightarrow t - \frac{2}{t-1} \le 0$$

$$\Rightarrow \quad \frac{t^2 - t - 2}{t-1} \le 0 \Rightarrow \frac{(t-2)(t+1)}{(t-1)} \le 0$$

$$\Rightarrow \quad t \in (-\infty, -1] \cup (1,2]$$

or $\log_2 x \in (-\infty, -1] \cup (1,2]$

or $x \in \left(0, \dfrac{1}{2}\right] \cup (2,4]$

Illustration 32 : Find all x such that $\log_{1/2} x > \log_{1/3} x$.

Solution : We have $\log_{1/2} x > \log_{1/3} x$.

$\Rightarrow \quad -\log_2 x > -\log_3 x \quad \Rightarrow \quad \log_2 x < \log_3 x$

$$\Rightarrow \quad \log_2 x < \frac{\log_2 x}{\log_2 3}$$

$\Rightarrow \quad \log_2 3 \log_2 x < \log_2 x \qquad$ (as $\log_2 3 > 0$)

$\Rightarrow \quad \log_2 x (\log_2 3 - 1) < 0$

Since $\log_2 3 - 1 > 0$, from the latter inequality we obtain $\log_2 x < 0$, hence $x < 1$. But the original inequality is meaningful only when $x > 0$. Therefore all x that satisfy the original inequality lie in the interval $0 < x < 1$.

Answer : $x \in (0, 1)$

Illustration 33 : Solve the inequation : $\log_{2x+3} x^2 < \log_{2x+3}(2x+3)$

Solution : For existance of logarithm

(i) $x^2 > 0$ (ii) $2x + 3 > 0$ (iii) $2x + 3 \ne 1 \Rightarrow x \in \left(-\dfrac{3}{2}, \infty\right) - \{-1, 0\}$ ...(i)

Now solving inequality

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Case I : $0 < 2x + 3 < 1 \Rightarrow -\frac{3}{2} < x < -1$

$\because \log_{2x+3} x^2 < \log_{2x+3} 2x+3$

$\Rightarrow x^2 > 2x+3$

$\Rightarrow (x-3)(x+1) > 0$

$\Rightarrow x \in (-\infty,-1) \cup (3,\infty)$; but $-\frac{3}{2} < x < -1$

$\Rightarrow x \in \left(-\frac{3}{2},-1\right)$ intersection with (i) $\Rightarrow x \in \left(-\frac{3}{2},-1\right)$

Case II : 2x + 3 > 1 ⇒ x > –1

$\because \log_{2x+3} x^2 < \log_{2x+3} 2x+3$

$\Rightarrow x^2 < 2x+3$

$\Rightarrow (x-3)(x+1) < 0$

$\Rightarrow x \in (-1,3)$; but $x > -1$

$\Rightarrow x \in (-1,3)$ intersection with (i) $\Rightarrow x \in (-1,3) - \{0\}$

$\therefore x \in$ case I $\cup$ case II

$\Rightarrow x \in \left(-\frac{3}{2},-1\right) \cup (-1,0) \cup (0,3)$

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Do yourself-6 :

Solve for x :

1. $\log_{0.3}(x^2 + 8) > \log_{0.3}(9x)$ **2.** $\log_7\left(\frac{2x-6}{2x-1}\right) > 0$

3. $\log_{x^2}(2+x) < 1$. **4.** $\log_x \frac{4x+5}{6-5x} < -1$. **5.** $\log_{1/3}\sqrt{x^2-2x} > -\frac{1}{2}$

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7. GRAPH OF EXPONENTIAL FUNCTION, ITS EQUATION AND INEQUALITIES

$$\text{If } a^{f(x)} > b \Rightarrow \begin{cases} f(x) > \log_a b \text{ when } a > 1 \\ f(x) < \log_a b \text{ when } 0 < a < 1 \end{cases}$$

Illustration 34 : **Solve the equation** $3^x.8^{\frac{x}{x+2}} = 6$.

Solution : Some students solved this equation thus : rewriting it as

$$3^x.2^{\frac{3x}{x+2}} = 3^1.2^1$$

They chose a root x so that the exponents of the respective bases were the same :

$$x = 1, \frac{3x}{x+2} = 1$$

hence the "answer" x = 1.

But this "answer" is incorrect in the sense that only one root of the equation is found and nothing has been said about any other roots. Actually, if the exponents on the appropriate bases are equal, then the products of these powers are equal, however the converse is not in any way implied and is simply incorrect. For instance, the equation

$$3^1.\ 2^1 = 3^2.\ 2^{\log_2(2/3)}$$

is valid, but $1 \neq 2$ and $1 \neq \log_2(2/3)$. Therefore, the foregoing reasoning may lead to a loss of roots, and this is exactly what occurred in the equation at hand.

Taking logarithms of both members of the original to the base 10, we get

$$x\log_{10}3 + \frac{3x}{x+2}\log_{10}2 = \log_{10}6$$

or $$x^2\log_{10} 3 + x\,(3\log_{10} 2 + 2\log_{10} 3 - \log_{10}6) - 2\log_{10}6 = 0$$

We now have to solve this quadratic equation. This can be done using a familiar formula, but we will try to simplify the solution by an ingenious device, since we have already seen, by trial and error, that $x_1 = 1$ is a root of the original equation and, consequently, satisfies the equivalent quadratic equation. For this reason, by Viete's theorem the second root of the quadratic equation is $x_2 = (-2\log_{10}6)/\log_{10} 3 = -2\log_3 6$ and so the original equation has two roots;

$$x_1 = 1,\ x_2 = -2\log_3 6.$$

Thus, it is useful to be able to guess a root, but never consider the guessing as the whole solution.

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Illustration 35 : $\begin{cases} 2^{y-x}(x+y)=1, \\ (x+y)^{x-y}=2. \end{cases}$

Solution : The domain of definition is $x + y > 0$.

$$\begin{cases} x+y=\dfrac{1}{2^{y-x}}=2^{x-y}, \\ (x+y)^{x-y}=2. \end{cases}$$

From the first equation we find $x + y = 2^{x-y}$ and substitute it into the second equation.

Then $(2^{x-y})^{x-y} = 2 \Rightarrow 2^{(x-y)^2} = 2 \Rightarrow (x-y)^2 = 1$.

The solution of the given system is the solution of the collection of systems

$$\begin{cases} x-y=1, \\ x+y=2, \end{cases} \text{ or } \begin{cases} x-y=-1, \\ x+y=\dfrac{1}{2}. \end{cases}$$

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Answer : $\left(\dfrac{3}{2},\dfrac{1}{2}\right), \left(-\dfrac{1}{4},\dfrac{3}{4}\right)$.

Illustration 36 : Solve for x : $2^{x+2} > \left(\dfrac{1}{4}\right)^{\frac{1}{x}}$

Solution : We have $2^{x+2} > 2^{-2/x}$. Since the base $2 > 1$, we have $x + 2 > -\dfrac{2}{x}$

(the sign of the inequality is retained).

Now $x+2+\dfrac{2}{x}>0 \quad \Rightarrow \quad \dfrac{x^2+2x+2}{x}>0$

$\Rightarrow \quad \dfrac{(x+1)^2+1}{x}>0 \quad \Rightarrow \quad x \in (0, \infty)$

Illustration 37 : Solve for x : $(1.25)^{1-x} < (0.64)^{2(1+\sqrt{x})}$

Solution : We have $\left(\dfrac{5}{4}\right)^{1-x} < \left(\dfrac{16}{25}\right)^{2(1+\sqrt{x})}$ or $\left(\dfrac{4}{5}\right)^{x-1} < \left(\dfrac{4}{5}\right)^{4(1+\sqrt{x})}$

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Since the base $0 < \frac{4}{5} < 1$, the inequality is equivalent to the inequality $x - 1 > 4(1+\sqrt{x})$

$$\Rightarrow \quad \frac{x-5}{4} > \sqrt{x}$$

Now, R.H.S. is positive

$$\Rightarrow \quad \frac{x-5}{4} > 0 \quad \Rightarrow \quad x > 5 \qquad .....(i)$$

we have $\frac{x-5}{4} > \sqrt{x}$

both sides are positive, so squaring both sides

$$\Rightarrow \quad \frac{(x-5)^2}{16} > x \qquad \text{or} \qquad \frac{(x-5)^2}{16} - x > 0$$

or $\quad x^2 - 26x + 25 > 0 \quad$ or $\quad (x-25)(x-1) > 0$

$\Rightarrow \quad x \in (-\infty, 1) \cup (25, \infty) \qquad .....(ii)$

intersection (i) & (ii) gives $x \in (25, \infty)$

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Do yourself - 7 :

Solve for $x \in \mathbb{R}$

1. $4^x - 10.2^{x-1} = 24$

2. $4.2^{2x} - 6^x = 18.3^{2x}$

3. $3^{2x-3} - 9^{x-1} + 27^{2x/3} = 675.$

4. $7^{x+2} - \frac{1}{7}.7^{x+1} - 14.7^{x-1} + 2.7^x = 48$

5. $\left(\frac{5}{3}\right)^{x+1} . \left(\frac{9}{25}\right)^{x^2+2x-11} = \left(\frac{5}{3}\right)^9$

6. $\left(3^{x^2-7.2x+3.9} - 9\sqrt{3}\right)\log(7-x) = 0$

7. $5^{2x} = 3^{2x} + 2.5^x + 2.3^x$

8. $\left(\frac{2}{3}\right)^{\frac{|x|-1}{|x|+1}} > 1$

9. $(1.25)^{1-(\log_2 x)^2} < (0.64)^{2+\log_{\sqrt{2}} x}$

10. $\left(\frac{1}{2}\right)^{\log_3 \log_{1/5}\left(x^2 - \frac{4}{5}\right)} < 1$

11. $4^x < 2^{x+1} + 3$

12. $e^{\frac{x^2+2}{x^2-1}} < \frac{1}{e^2}$

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8. CHARACTERISTIC AND MANTISSA :

For any given number N, logarithm can be expressed as $\log_a N$ = Integer + Fraction

The integer part is called characteristic and the fractional part (always taken non negative) is called mantissa. When the value of $\log_{10} N$ is given, then to find digits of 'N' we use only the mantissa part. The characteristic is used only in determining the number of digits in the integral part (if $N \geq 1$) or the number of zeros after decimal & before first non-zero digit in the number (if $0 < N < 1$).

Note :

(i) The mantissa part of logarithm of a number is always non-negative $(0 \leq m < 1)$

(ii) If the characteristic of $\log_{10}N$ be 'C' and $C \geq 0$, then the number of digits in N is $(C + 1)$

(iii) If the characteristic of $\log_{10}N$ be '– C' and $C > 0$, then there exist $(C - 1)$ zeros after decimal in N.

In summary, if characteristic of $\log_{10}N$ is 'C' then number of digits ($N \geq 1, N \in \mathbb{N}$) or number of zeros after decimal in N ($0 < N < 1$) = $|C + 1|$.

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Do yourself - 8 :

1. Evaluate : $\log_{10}(0.06)^6$

2. Find number of digits in 18^{20}

3. Determine number of cyphers (zeros) between decimal & first significant digit in $\left(\frac{1}{6}\right)^{200}$

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EXERCISE (O-1)

1. If $2^a = 3$ and $9^b = 4$ then value of (ab) is-

(A) 1 (B) 2 (C) 3 (D) 4

LG0001

2. If $\log_2\left(4+\log_3(x)\right)=3$, then sum of digits of x is -

(A) 3 (B) 6 (C) 9 (D) 18

LG0002

3. Sum of all the solution(s) of the equation $\log_{10}(x)+\log_{10}(x+2)-\log_{10}(5x+4)=0$ is-

(A) –1 (B) 3 (C) 4 (D) 5

LG0003

4. The product of all the solutions of the equation $x^{1+\log_{10}x}=100000x$ is-

(A) 10 (B) 10^5 (C) 10^{-5} (D) 1

LG0004

5. If x_1 and x_2 are the roots of equation $e^{3/2}.\,x^{2\ell nx}=x^4$, then the product of the roots of the equation is -

(A) e^2 (B) e (C) $e^{3/2}$ (D) e^{-2}

LG0005

6. If $\log_2(x^2+1)+\log_{13}(x^2+1)=\log_2(x^2+1)\log_{13}(x^2+1)$, $(x\neq 0)$, then $\log_7(x^2+24)$ is equal to

(A) 1 (B) 2 (C) 3 (D) 4

LG0006

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7. Given $\log_3 a = p = \log_b c$ and $\log_b 9 = \dfrac{2}{p^2}$. If $\log_9\left(\dfrac{a^4b^3}{c}\right)=\alpha p^3+\beta p^2+\gamma p+\delta$ $(\forall\ p\in R-\{0\})$, then $(\alpha+\beta+\gamma+\delta)$ equals

(A) 1 (B) 2 (C) 3 (D) 4

LG0007

8. If $\log_a\left(1-\sqrt{1+x}\right)=\log_{a^2}\left(3-\sqrt{1+x}\right)$, then number of solutions of the equation is-

(A) 0 (B) 1 (C) 2 (D) infinitely many

LG0008

9. The number of solution(s) of $\sqrt{\log_3(3x^2).\log_9(81x)}=\log_9 x^3$ is-

(A) 0 (B) 1 (C) 2 (D) 3

LG0009

10. If x_1 & x_2 are the two values of x satisfying the equation $7^{2x^2}-2\left(7^{x^2+x+12}\right)+7^{2x+24}=0$, then (x_1+x_2) equals-

(A) 0 (B) 1 (C) –1 (D) 7

LG0010

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11. If $x, y \in 2^n$ when $n \in I$ and $1 + \log_x y = \log_2 y$, then the value of $(x + y)$ is

(A) 2 (B) 4 (C) 6 (D) 8

LG0011

12. If $n \in N$ such that characteristic of n^2 to the base 8 is 2, then number of possible values of n is-

(A) 14 (B) 15 (C) 448 (D) infinite

LG0012

13. If $x = \log_2\left(\sqrt{56+\sqrt{56+\sqrt{56+\sqrt{56+.....\infty}}}}\right)$, then which of the following statements holds good ?

(A) $x < 0$ (B) $0 < x < 2$ (C) $2 < x < 4$ (D) $3 < x < 4$

LG0013

14. The greatest value of $(4\log_{10}x - \log_x(.0001))$ for $0 < x < 1$ is-

(A) 4 (B) –4 (C) 8 (D) –8

LG0014

15. The number $\log_2 7$ is

(A) an integer (B) a rational number

(C) an irrational number (D) a prime number

LG0015

16. The number of integral solutions of $|\log_5 x^2 - 4| = 2 + |\log_5 x - 3|$ is-

(A) 1 (B) 2 (C) 3 (D) 0

LG0016

17. If $60^a = 3$ and $60^b = 5$ then the value of $12^{\frac{1-a-b}{2(1-b)}}$ equals

(A) 2 (B) 3 (C) $\sqrt{3}$ (D) $\sqrt{12}$

LG0017

18. Let ABC be a triangle right angled at C. The value of $\dfrac{\log_{b+c} a + \log_{c-b} a}{\log_{b+c} a \cdot \log_{c-b} a}$ $(b + c \neq 1, c - b \neq 1)$ equals

(A) 1 (B) 2 (C) 3 (D) 1/2

LG0018

19. If α and β are the roots of the equation $(\log_2 x)^2 + 4(\log_2 x) - 1 = 0$ then the value of $\log_\beta \alpha + \log_\alpha \beta$ equals

(A) 18 (B) – 16 (C) 14 (D) – 18

LG0019

20. If $\log_{0.3}(x-1) < \log_{0.09}(x-1)$, then x lies in the interval

(A) $(2, \infty)$ (B) $(1, 2)$ (C) $(1, \infty)$ (D) none of these

LG0020

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21. If $\log_{10}\left(\frac{1}{2^x + x - 1}\right) = x\left(\log_{10} 5 - 1\right)$, then x =

(A) 4 (B) 3 (C) 2 (D) 1

LG0021

22. The number of solutions of the equation $\log_{x-3}(x^3 - 3x^2 - 4x + 8) = 3$ is equal to

(A) 4 (B) 3 (C) 2 (D) 1

LG0022

23. Sum of the roots of the equation $9^{\log_3(\log_2 x)} = \log_2 x - (\log_2 x)^2 + 1$ is equal to

(A) 2 (B) 4 (C) 6 (D) 8

LG0023

24. If x satisfies the inequality $\log_{25} x^2 + (\log_5 x)^2 < 2$, then $x \in$

(A) $\left(\frac{1}{25}, 5\right)$ (B) (1, 2) (C) (4, 5) (D) (0, 1)

LG0024

25. If 1, $\log_9(3^{1-x} + 2)$ and $\log_3(4.3^x - 1)$ are in A.P. then x can be

(A) $\log_4 3$ (B) $\log_3 4$ (C) $1 - \log_3 4$ (D) $\log_3 0.25$

LG0025

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EXERCISE (O-2)

Multiple Type From 1 to 8

1. If $\left(\log_\beta \alpha\right)^2 + \left(\log_\alpha \beta\right)^2 = 79$, $(\alpha > 0, \beta > 0, \alpha \neq 1, \beta \neq 1)$ then value of $(\log_\beta \alpha) + (\log_\alpha \beta)$ can be-

(A) 7 (B) –9 (C) 9 (D) –7

LG0026

2. Which of the following statements is(are) correct ?

(A) $7^{1/7} > (42)^{1/14} > 1$

(B) $\log_3(5)\log_7(9)\log_{11}(13) > -2$

(C) $\sqrt{99 + 70\sqrt{2}} + \sqrt{99 - 70\sqrt{2}}$ is rational

(D) $\frac{1}{\log_4 3} + \frac{1}{\log_7 3} > 3$

LG0027

3. For $a > 0, \neq 1$ the roots of the equation $\log_{ax} a + \log_x a^2 + \log_{a^2x} a^3 = 0$ can be

(A) $a^{-3/4}$ (B) $a^{-4/3}$ (C) $a^{-1/2}$ (D) none of these

LG0028

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4. The solution of the equation $5^{\log_a x} + 5x^{\log_a 5} = 3$, $(a > 0, a \neq 1)$ is

(A) $a^{-\log_5 2}$ (B) $a^{\log_5 2}$ (C) $2^{-\log_5 a}$ (D) $2^{\log_5 a}$

LG0029

5. If $2^{x+y} = 6^y$ and $3^{x-1} = 2^{y+1}$, then the value of $(\log 3 - \log 2)/(x - y)$ is

(A) 1 (B) $\log_2 3 - \log_3 2$ (C) $\log(3/2)$ (D) $\log 3 - \log 2$

LG0030

6. The equation $x^{(3/4)(\log_2 x)^2 + \log_2 x - 5/4} = \sqrt{2}$ has

(A) at least one real solution (B) exactly three real solutions

(C) exactly one irrational solution (D) complex roots

LG0031

7. The inequality $-1 \le \left(\frac{1}{3}\right)^x < 2$ is satisfied by

(A) $x \in [0, 1]$ (B) $x > -\log_3 2$ (C) $x < -\log_3 2$ (D) $x < -1$

LG0032

Linked Comprehension Type

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Paragraph for Question 8 to 10

Let $\log_2 5 = \mathbf{a}$; $\log_5 3 = \mathbf{b}$

8. The value of $\log_{50} 30$ is equal to

(A) $\frac{\mathbf{b} + \mathbf{ab} + 1}{\mathbf{b} + 2}$ (B) $\frac{\mathbf{a} + \mathbf{ab} + 1}{2\mathbf{a} + 1}$ (C) $\frac{\mathbf{a} + \mathbf{ab} + 1}{\mathbf{a} + 2}$ (D) $\frac{\mathbf{a} + \mathbf{ab} + \mathbf{b}}{2\mathbf{a} + 1}$

LG0033

9. $\log_2 9 + 3\mathbf{b}.\log_5 9 + \log_2 25 + 3\log_5 9$ lies in the interval given as :

(A) (12,24) (B) (6,12] (C) (24,28) (D) [24,28)

LG0033

10. Roots of the equation $\log_{10} 2.\log_{10} 5x^2 - (\log_{10} 5 + \log_{10} 15.\log_{10} 2)x + \log_{10} 15 = 0$ are

(A) **a,b** (B) 1 + **a**, 1 + **b** (C) **a** – 1, 1 – **b** (D) **a** + 2; **b** + 2

LG0033

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Matrix Match Type

11. In the following matrix match **Column-I** has some quantities and **Column-II** has some comments or other quantities

Match the each element in **Column-I** with corresponding element(s) in **Column-II**

	Column-I		Column-II
(A)	$\log_{\sin \pi/6} e$	(P)	$x^{\log_x 2}$; $x > 0$; $x \neq 1$
(B)	2	(Q)	$2^{\log_5 7}$
(C)	$\log_{\tan \pi/3} \pi$	(R)	negative
(D)	$7^{\log_5 2}$	(S)	positive

LG0034

Reasoning Type

(A) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1.

(B) Statement-1 is True, Statement-2 is True ; Statement-2 is NOT a correct explanation for Statement-1.

(C) Statement-1 is True, Statement-2 is False.

(D) Statement-1 is False, Statement-2 is True.

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12. **Statement-1 :** $\log_{1/5} \frac{4x+6}{x} \geq 0$ has no solution for $x \in \left(-\frac{3}{2}, -1\right)$

and

Statement-2 : $2^{(y-x)}.(x+y) = 1$, $(x+y)^{x-y} = 2$ has only one pair of solution.

LG0035

13. **Statement - 1 :** For $3^x + 4^x < 5^x \Rightarrow x > 2$.

and

Statement - 2 : $6^x + 6^{x+1} = 2^x + 2^{x+1} + 2^{x+2}$ has no non-zero solutions.

LG0036

INTEGER TYPE

14. The value of $\log_5 9 . \log_2(\sqrt{5}+2) . \log_{(\sqrt{5}-2)} 5 . \log_3\left(\frac{1}{4}\right)$ is

LG0037

15. Number of solutions of $\log_{(1+x)}\left(\frac{1-x}{1+x}\right) - 2\log_{(1-x)}(1+x) = 0$ is

LG0038

16. Number of solutions of $\log_{x^2+1}(x^2-2)^2 = 2$ is

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LG0039

17. If x_1, x_2 are the value(s) of x satisfying the equation $\log_2^2(x-2)+\log_2(x-2)\log_2\left(\frac{3}{x}\right)-2\log_2^2\left(\frac{3}{x}\right)=0$, then x_1+x_2 is equal to

LG0040

18. If characteristic of $\log_{10}(0.00006)$ is A & characteristic $\log_3 750$ is B, then A + B is

LG0041

19. If $\log_{10}2 = .3010$, $\log_{10}3 = .4771$, then number of digits in $4^8.3^7.5^3$ is 'P', then $\frac{P-1}{2}$ is

LG0042

20. There exist positive integers A, B and C with no common factor greater than 1 such that $A\log_{200}5 + B\log_{200}2 = C$, then the value of A + B + C is

LG0043

21. The minimum possible real x which satisfy the equation, $2\log_2\log_2 x+\log_{1/2}\log_2\left(2\sqrt{2}x\right)=1$.

LG0044

EXERCISE (S-1)

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1. Let A denotes the value of $\log_{10}\left(\frac{ab+\sqrt{(ab)^2-4(a+b)}}{2}\right)+\log_{10}\left(\frac{ab-\sqrt{(ab)^2-4(a+b)}}{2}\right)$

when a = 43 and b = 57 and B denotes the value of the expression $\left(2^{\log_6 18}\right).\left(3^{\log_6 3}\right)$.

Find the value of (A.B).

LG0045

2. Compute the following :

(a) $\log_{1/3}\sqrt[4]{729.\sqrt[3]{9^{-1}.27^{-4/3}}}$

LG0046

(b) $a^{\frac{\log_b(\log_b N)}{\log_b a}}$

LG0047

3. Find the square of the sum of the solution of the equation $\log_3x.\log_4x.\log_5x=\log_3x.\log_4x+\log_4x.\log_5x+\log_5x.\log_3x$.

LG0048

4. Calculate : $4^{5\log_{4\sqrt{2}}\left(3-\sqrt{6}\right)-6\log_8\left(\sqrt{3}-\sqrt{2}\right)}$

LG0049

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5. Simplify : $\dfrac{81^{\frac{1}{\log_5 9}}+3^{\frac{3}{\log_{\sqrt{6}} 3}}}{409}.\left(\left(\sqrt{7}\right)^{\frac{2}{\log_{25} 7}}-(125)^{\log_{25} 6}\right)$

LG0050

6. Simplify : $5^{\log_{1/5}\left(\frac{1}{2}\right)}+\log_{\sqrt{2}}\dfrac{4}{\sqrt{7}+\sqrt{3}}+\log_{1/2}\dfrac{1}{10+2\sqrt{21}}$

LG0051

7. Given that $\log_2 a = s$, $\log_4 b = s^2$ and $\log_{c^2}(8) = \dfrac{2}{s^3+1}$. Write $\log_2 \dfrac{a^2b^5}{c^4}$ as a function of 's'

(a,b,c>0, c≠1).

LG0052

8. Find the value of $49^{(1-\log_7 2)}+5^{-\log_5 4}$.

LG0053

9. Prove that $\dfrac{\log_2 24}{\log_{96} 2}-\dfrac{\log_2 192}{\log_{12} 2}=3$.

LG0054

10. Prove that $a^x - b^y = 0$ where $x=\sqrt{\log_a b}$ & $y=\sqrt{\log_b a}$, $a>0$, $b>0$ & a , $b \neq 1$.

LG0055

11. Solve the following equations :

i. $\log_{x-1} 3 = 2$

LG0056

ii. $\log_4(2\log_3(1+\log_2(1+3\log_3 x)))=\dfrac{1}{2}$

LG0057

iii. $\log_3(1+\log_3(2^x-7))=1$

LG0058

iv. $\log_3(3^x-8)=2-x$

LG0059

v. $\dfrac{\log_2(9-2^x)}{3-x}=1$

LG0060

vi. $\log_{5-x}(x^2-2x+65)=2$

LG0061

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vii. $\log_{10}5+\log_{10}(x+10)-1=\log_{10}(21x-20)-\log_{10}(2x-1)$

LG0062

viii. $x^{1+\log_{10}x}=10x$

LG0063

ix. $2(\log_x\sqrt{5})^2-3\log_x\sqrt{5}+1=0$

LG0064

x. $3+2\log_{x+1}3=2\log_3(x+1)$

LG0065

12. Solve the inequality. Where ever base is not given take it as 10.

(i) $(\log 100x)^2+(\log 10x)^2+\log x\le 14$

LG0066

(ii) $\log_{1/2}(x+1)>\log_2(2-x)$.

LG0067

(iii) $\log_x 2\,.\,\log_{2x}2\,.\,\log_2 4x>1$.

LG0068

(iv) $\log_{1/5}(2x^2+5x+1)<0$.

LG0069

(v) $\log_x\dfrac{4x+5}{6-5x}<-1$

LG0070

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EXERCISE (S-2)

1. Let a and b be real numbers greater than 1 for which there exists a positive real number c, different from 1, such that $2(\log_a c+\log_b c)=9\log_{ab}c$. Find the largest possible value of $\log_a b$.

LG0071

2. Find the value of the expression $\dfrac{2}{\log_4(2000)^6}+\dfrac{3}{\log_5(2000)^6}$

LG0072

3. Given that $\log_2 3=a$, $\log_3 5=b$ and $\log_7 2=c$, express the logarithm of the number 63 to the base 140 in terms of a,b & c.

LG0073

4. If a,b and c are positive real numbers such that $a^{\log_3 7}=27; b^{\log_7 11}=49$ and $c^{\log_{11}25}=\sqrt{11}$. Find the value of $\left(a^{(\log_3 7)^2}+b^{(\log_7 11)^2}+c^{(\log_{11}25)^2}\right)$.

LG0074

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5. If 'x' and 'y' are real numbers such that $2\log(2y - 3x) = \log x + \log y$, find $\frac{x}{y}$.

LG0075

6. The real x and y satisfy $\log_8 x + \log_4 y^2 = 5$ and $\log_8 y + \log_4 x^2 = 7$, find xy.

LG0076

7. If $a = \log_{12} 18$ & $b = \log_{24} 54$ then find the value of $ab + 5(a - b)$.

LG0077

8. Solve the following equations :

i. $\log_{1/2}^2(4x) + \log_2\left(\frac{x^2}{8}\right) = 8$

LG0078

ii. $\log_{0.5x} x^2 - 14\log_{16x} x^3 + 40\log_{4x}\sqrt{x} = 0$

LG0079

iii. $\log_3(4.3^x - 1) = 2x + 1$

LG0080

iv. $\log_5\left(\frac{2+x}{10}\right) = \log_5\left(\frac{2}{x+1}\right)$

LG0081

v. $1 + 2\log_{(x+2)} 5 = \log_5(x + 2)$

LG0082

vi. $\log_4 2^{4x} = 2^{\log_2 4}$

LG0083

vii. $\log_2(4.3^x - 6) - \log_2(9^x - 6) = 1$

LG0084

viii. $2\log_8(2x) + \log_8(x^2 + 1 - 2x) = \frac{4}{3}$

LG0085

ix. $\log_3^2 6 - \log_3^2 2 = (\log_{10}^2 x - 2)\log_3 12$

LG0086

x. $\log_6 2^{x+3} - \log_6(3^x - 2) = x$

LG0087

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9. Solve the system of equations :

$\log_a x \log_a(xyz) = 48$

$\log_a y \log_a(xyz) = 12, a > 0, a \neq 1$

$\log_a z \log_a(xyz) = 84$

LG0088

10. Let $y = \sqrt{\log_2 3.\log_2 12.\log_2 48.\log_2 192 + 16} - \log_2 12.\log_2 48 + 10$. Find $y \in N$.

LG0089

11. Let **'L'** denotes the antilog of 0.4 to the base 1024.

and **'M'** denotes the number of digits in 6^{10} (Given $\log_{10}2 = 0.3010$, $\log_{10}3 = 0.4771$)

and **'N'** denotes the number of positive integers which have the characteristic 2, when base of the logarithm is 6.

Find the value of LMN.

LG0090

12. Solve the inequality. Where ever base is not given take it as 10.

(i) $(\log_2 x)^4 - \left(\log_{\frac{1}{2}} \frac{x^5}{4}\right)^2 - 20\log_2 x + 148 < 0$.

LG0091

(ii) $\log_{1/2} x + \log_3 x > 1$.

LG0092

(iii) $\log_{x^2}(2+x) < 1$

LG0093

(iv) $(\log_{|x+6|} 2) . \log_2(x^2 - x - 2) \geq 1$

LG0094

(v) $\log_3 \frac{|x^2 - 4x| + 3}{x^2 + |x-5|} \geq 0$

LG0095

13. Find the product of the positive roots of the equation $\sqrt{(2017)}(x)^{\log_{2017} x} = x^2$.

LG0096

14. If (x_1, y_1) and (x_2, y_2) are the solution of the system of equation.

$\log_{225}(x) + \log_{64}(y) = 4$

$\log_x(225) - \log_y(64) = 1$,

then show that the value of $\log_{30}(x_1 y_1 x_2 y_2) = 12$.

LG0097

15. (a) Given : $\log_{10}34.56 = 1.5386$, find $\log_{10}3.456$; $\log_{10}0.3456$ & $\log_{10}0.003456$.

LG0098

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(b) Find the number of positive integers which have the characteristic 3, when the base of the logarithm is 7.

LG0099

(c) If $\log_{10}2 = 0.3010$ & $\log_{10}3 = 0.4771$, find the value of $\log_{10}(2.25)$

LG0100

(d) Find the antilogarithm of 0.75, if the base of the logarithm is 2401.

LG0101

16. (a) If x,y > 0, $\log_y x + \log_x y = \frac{10}{3}$ and xy = 144, then $\frac{x+y}{2} = \sqrt{N}$ where N is a natural number, find the value of N.

LG0102

(b) If $x = 1 + \log_a bc$, $y = 1 + \log_b ca$ and $z = 1 + \log_c ab$, then prove that $xyz = xy + yz + zx$.

LG0103

(c) If $\frac{\log_a N}{\log_c N} = \frac{\log_a N - \log_b N}{\log_b N - \log_c N}$ where $N > 0$ & $N \neq 1$, a,b,c > 0 & not equal to 1, then prove that $b^2 = ac$.

LG0104

17. $\log_{3/4}\log_8(x^2+7) + \log_{1/2}\log_{1/4}(x^2+7)^{-1} = -2$.

LG0105

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EXERCISE (JA)

1. Number of solutions of $\log_4(x-1) = \log_2(x-3)$ is **[JEE 2001 (Screening)]**

(A) 3 (B) 1 (C) 2 (D) 0

LG0106

2. Let (x_0, y_0) be the solution of the following equations **[JEE 2011, 3 (–1)]**

$$(2x)^{\ln 2} = (3y)^{\ln 3}$$

$$3^{\ln x} = 2^{\ln y}$$

Then x_0 is

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) 6

LG0107

3. The value of $6 + \log_{\frac{3}{2}}\left(\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\ldots\ldots}}}\right)$ is **[JEE 2012, 4M]**

LG0108

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4. If $3^x = 4^{x-1}$, then x = **[JEE-Advanced 2013, 4, (–1)]**

(A) $\frac{2\log_3 2}{2\log_3 2-1}$ (B) $\frac{2}{2-\log_2 3}$ (C) $\frac{1}{1-\log_4 3}$ (D) $\frac{2\log_2 3}{2\log_2 3-1}$

LG0109

5. The value of $\left((\log_2 9)^2\right)^{\frac{1}{\log_2(\log_2 9)}} \times \left(\sqrt{7}\right)^{\frac{1}{\log_4 7}}$ is _____ **[JEE(Advanced)-2018]**

LG0110

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ANSWERS

Do yourself - 1

1 **(a)** $\log_3 81 = 4$ **(b)** $\log_{10}(0.001) = -3$ **(c)** $\log_{128} 2 = 1/7$

2. **(a)** $32 = 2^5$ **(b)** $4 = (\sqrt{2})^4$ **(c)** $0.01 = 10^{-2}$

3. 6

4. (i) 0 (ii) 1 (iii) 2 (iv) 3 (v) –1
(vi) –5 (vii) –4 (viii) 1/2 (ix) 1 (x) 3/2
(xi) –1/5 (xii) –3/7

5. (i) 0 (ii) 1 (iii) 3 (iv) –4 (v) –1/2
(vi) 1/2 (vii) –3/2 (viii) 9/4

6. (i) 0 (ii) 1 (iii) 2 (iv) 4 (v) –1
(vi) 1/2 (vii) –3/2 (viii) 7/2 (ix) 2/7

7. (i) 0 (ii) 1 (iii) 2 (iv) –1 (v) –2
(vi) –4 (vii) –1/3 (viii) 1/7 (ix) –5/2 (x) 9/4

8. (i) 0 (ii) 1 (iii) 2 (iv) 4 (v) –1
(vi) –2 (vii) –1/2 (viii) 1/4 (ix) 1/2 (x) 1/3
(xi) 1/3

9. (i) 4 (ii) 2 (iii) $a > 0, a \neq 1$ (iv) –5,2 (v) –2,2
(vi) –3,3 (vii) 9 (viii) 1/81 (ix) 1 (x) 2
(xi) $\sqrt{2}, -\sqrt{2}$

10. (i) $\frac{-1}{2}$ (ii) 2 (iii) 1 (iv) 1

Do yourself - 2

1 **(a)** 1 **(b)** $\frac{1}{5}$ **2.** 3

Do yourself - 3

2. (i) 2 (ii) 3 (iii) 1 (iv) 4

Do yourself - 4

1. **(i)** 3 **(ii)** 5/6 **(iii)** 0 **(iv)** 2 **(v)** (A)
(vi) (C)

2. (i) 3 (ii) 9 (iii) 9 (vi) 2

3. 32

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Do yourself - 5

1. (i) $\sqrt{2}, -\sqrt{2}$ (ii) 1 (iii) 1 (iv) 0 (v) 1,–1

2. (i) $x = \pm 2$ (ii) $x = \pm\frac{1}{2}$ (iii) $x = \frac{3}{2}$ (iv) $x = 4$

3. $\log_2 3$ **4.** $\frac{1}{8}, 2$ **5.** $9, \frac{1}{9}$ **6.** 2,8 **7.** 1,0

8. ϕ **9.** $100, \frac{1}{100}$ **10.** $2^{-\log_3 a}$ **11.** ϕ **12.** 2 13. $x = 8, \frac{1}{8}$

Do yourself - 6

1. $x \in (1,8)$ **2.** $x \in (-\infty, 1/2)$

3. $(-2,-1)\cup(-1,0)\cup(0,1)\cup(2,\infty)$ **4.** $\left(\frac{1}{2},1\right)$ **5.** $(-1,0)\cup(2,3)$

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Do yourself - 7

1. $x = 3$ **2.** $x = -2$ **3.** $x = 3$ **4.** $x = 0$ **5.** $x = \frac{-7}{2}, 2$

6. $x = \frac{1}{5}, 6$ **7.** $x = 1$ **8.** $x \in (-1,1)$ **9.** $\left(0,\frac{1}{2}\right)\cup(32,\infty)$

10. $\left(-1,-\frac{2}{\sqrt{5}}\right)\cup\left(\frac{2}{\sqrt{5}},1\right)$ **11.** $(-\infty, \log_2 3)$ **12.** $(-1,0) \cup (0,1)$

Do yourself - 8

1. $\overline{8}.6686$ **2.** 26 **3.** 155

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EXERCISE (O-1)

1. A	**2.** C	**3.** C	**4.** D	**5.** A	**6.** B	**7.** C	**8.** A
9. B	**10.** B	**11.** D	**12.** B	**13.** C	**14.** D	**15.** C	**16.** A
17. A	**18.** B	**19.** D	**20.** A	**21.** D	**22.** D	**23.** A	**24.** A
25. C							

EXERCISE (O-2)

1. B,C **2.** A,B,D **3.** B,C **4.** A,C **5.** C,D **6.** A,B,C,D **7.** A,B

8. B **9.** A **10.** B **11.** **(A)→(R); (B)→(P,S); (C)→(S); (D)→(Q,S)**

12. C **13.** B **14.** 4 **15.** 0 **16.** 2 **17.** 9 **18.** 1 **19.** 5

20. 6 **21.** $x = 8$

EXERCISE (S-1)

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1. 12 **2.** **(a)** -1 **(b)** $\log_b N$ **3.** 3721 **4.** 9 **5.** 1 **6.** 6

7. $2s + 10s^2 - 3(s^3 + 1)$ **8.** $\frac{25}{2}$

11. **i.** $\{1+\sqrt{3}\}$ **ii.** $\{3\}$ **iii.** $\{4\}$ **iv.** $\{2\}$ **v.** $\{0\}$ **vi.** $\{-5\}$

vii. $\{3/2, 10\}$ **viii.** $\{10^{-1}, 10\}$ **ix.** $\{\sqrt{5}, 5\}$ **x.** $\{-(3-\sqrt{3})/3, 8\}$

12 **(i)** $\frac{1}{\sqrt{10^9}} \le x \le 10$ **(ii)** $-1 < x < \frac{1-\sqrt{5}}{2}$ or $\frac{1+\sqrt{5}}{2} < x < 2$

(iii) $2^{-\sqrt{2}} < x < 2^{-1}$; $1 < x < 2^{\sqrt{2}}$ **(iv)** $(-\infty, -2.5) \cup (0, \infty)$ **(v)** $\frac{1}{2} < x < 1$

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EXERCISE (S-2)

1. 2 **2.** 1/6 **3.** $\frac{1+2ac}{2c+abc+1}$ **4.** 469 **5.** 4/9 **6.** $xy = 2^9$ **7.** 1

8. **i.** $\{2^{-7}, 2\}$ **ii.** $\{1/\sqrt{2}, 1, 4\}$ **iii.** $\{-1, 0\}$ **iv.** $\{3\}$ **v.** $\{-9/5, 23\}$

vi. $\{2\}$ **vii.** $\{1\}$ **viii.** $\{2\}$ **ix.** $\{10^{-\sqrt{3}}, 10^{\sqrt{3}}\}$ **x.** $\{\log_3 4\}$

9. (a^4, a, a^7) or $\left(\frac{1}{a^4}, \frac{1}{a}, \frac{1}{a^7}\right)$ **10.** y = 6 **11.** 23040

12. **(i)** $x \in \left(\frac{1}{16}, \frac{1}{8}\right) \cup (8, 16)$ **(ii)** $0 < x < 3^{1/1-\log 3}$ (where base of log is 2)

(iii) $-2<x<-1,\ -1<x<0,\ 0<x<1,\ x>2$ **(iv)** $x < -7,\ -5 < x \le -2,\ x \ge 4$

(v) $x \le -\frac{2}{3};\ \frac{1}{2} \le x \le 2$

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13. $(2017)^2$

15. **(a)** 0.5386; $\bar{1}.5386; \bar{3}.5386$ **(b)** 2058 **(c)** 0.3522 **(d)** 343

16. (a) 507 **17.** x = 3 or –3

EXERCISE (JA)

1. B **2.** C **3.** 4 **4.** A,B,C **5.** 8

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Chapter 03 QUADRATIC EQUATIONS

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JEE (Main/Advanced) Syllabus

JEE (Main) Syllabus :

Sets and their representation ; Union, intersection and complement of sets and their algebraic properties; power sets ; polynomials.

JEE (Advanced) Syllabus :

Absolute value, polynomial.

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Important Notes

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QUADRATIC EQUATION

1. INTRODUCTION :

The algebraic expression of the form $ax^2 + bx + c$, $a \neq 0$ is called a quadratic expression, because the highest order term in it is of second degree. Quadratic equation means, $ax^2 + bx + c = 0$. In general whenever one says zeroes of the expression $ax^2 + bx + c$, it implies roots of the equation $ax^2 + bx + c = 0$, unless specified otherwise.

A quadratic equation has exactly two roots which may be real (equal or unequal) or imaginary.

2. SOLUTION OF QUADRATIC EQUATION & RELATION BETWEEN ROOTS & CO-EFFICIENTS :

(a) The general form of quadratic equation is $ax^2 + bx + c = 0$, $a \neq 0$.

The roots can be found in following manner :

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0 \quad \Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \quad \Rightarrow \quad \boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

This expression can be directly used to find the two roots of a quadratic equation.

(b) The expression $b^2 - 4ac \equiv D$ is called the discriminant of the quadratic equation.

(c) If α & β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then :

(i) $\alpha + \beta = -b/a$ (ii) $\alpha\beta = c/a$ (iii) $|\alpha - \beta| = \frac{\sqrt{D}}{|a|}$

(d) A quadratic equation whose roots are α & β is $(x - \alpha)(x - \beta) = 0$ i.e.

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$ i.e. $x^2 -$ (sum of roots) x + product of roots = 0.

Illustration 1 : If α, β are the roots of a quadratic equation $x^2 - 3x + 5 = 0$, then the equation whose roots are $(\alpha^2 - 3\alpha + 7)$ and $(\beta^2 - 3\beta + 7)$ is -

(A) $x^2 + 4x + 1 = 0$ (B) $x^2 - 4x + 4 = 0$ (C) $x^2 - 4x - 1 = 0$ (D) $x^2 + 2x + 3 = 0$

Solution : Since α, β are the roots of equation $x^2 - 3x + 5 = 0$

So $\alpha^2 - 3\alpha + 5 = 0$

$\beta^2 - 3\beta + 5 = 0$

$\therefore$ $\alpha^2 - 3\alpha = -5$

$\beta^2 - 3\beta = -5$

Putting in $(\alpha^2 - 3\alpha + 7)$ & $(\beta^2 - 3\beta + 7)$ .....(i)

$-5 + 7, -5 + 7$

$\therefore$ 2 and 2 are the roots.

$\therefore$ The required equation is $x^2 - 4x + 4 = 0$. **Ans. (B)**

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Illustration 2 : If α and β are the roots of $ax^2 + bx + c = 0$, find the value of $(a\alpha + b)^{-2} + (a\beta + b)^{-2}$.

Solution : We know that $\alpha + \beta = -\frac{b}{a}$ & $\alpha\beta = \frac{c}{a}$

$$(a\alpha + b)^{-2} + (a\beta + b)^{-2} = \frac{1}{(a\alpha+b)^2} + \frac{1}{(a\beta+b)^2}$$

$$= \frac{a^2\beta^2 + b^2 + 2ab\beta + a^2\alpha^2 + b^2 + 2ab\alpha}{(a^2\alpha\beta + ba\beta + ba\alpha + b^2)^2} = \frac{a^2(\alpha^2+\beta^2) + 2ab(\alpha+\beta) + 2b^2}{(a^2\alpha\beta + ab(\alpha+\beta) + b^2)^2}$$

($\alpha^2 + \beta^2$ can always be written as $(\alpha + \beta)^2 - 2\alpha\beta$)

$$= \frac{a^2\left[(\alpha+\beta)^2 - 2\alpha\beta\right] + 2ab(\alpha+\beta) + 2b^2}{(a^2\alpha\beta + ab(\alpha+\beta) + b^2)^2} = \frac{a^2\left[\frac{b^2-2ac}{a^2}\right] + 2ab\left(-\frac{b}{a}\right) + 2b^2}{\left(a^2\frac{c}{a} + ab\left(-\frac{b}{a}\right) + b^2\right)^2} = \frac{b^2-2ac}{a^2c^2}$$

Alternatively :

If α & β are roots of $ax^2 + bx + c = 0$

then, $a\alpha^2 + b\alpha + c = 0$

$$\Rightarrow \quad a\alpha + b = -\frac{c}{\alpha}$$

same as $a\beta + b = -\frac{c}{\beta}$

$$\therefore \quad (a\alpha+b)^{-2} = (\alpha\beta+b)^{-2} = \frac{\alpha^2}{c^2} + \frac{\beta^2}{c^2}$$

$$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{c^2}$$

$$= \frac{(-b/a)^2 - 2(c/a)}{c^2}$$

$$= \frac{b^2-2ac}{a^2c^2}$$

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Do yourself - 1 :

1. Find the roots of following equations :
   (a) $x^2 + 3x + 2 = 0$ (b) $x^2 - 8x + 16 = 0$ (c) $x^2 - 2x - 1 = 0$
2. Find the roots of the equation $a(x^2 + 1) - (a^2 + 1)x = 0$, where $a \neq 0$.
3. Solve : $\frac{6-x}{x^2-4} = 2 + \frac{x}{x+2}$
4. If the roots of $4x^2 + 5k = (5k + 1)x$ differ by unity, then find the values of k.

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5. Form a quadratic equation whose roots are the numbers $\frac{1}{10-\sqrt{72}}$ and $\frac{1}{10+6\sqrt{2}}$.

6. For what values of a is the sum of the roots of the equation $x^2 + (2 - a - a^2)x - a^2 = 0$ equal to zero ?

7. For what values of a is the ratio of the roots of the equation $ax^2 - (a + 3)x + 3 = 0$ equal to 1.5?

8. The roots x_1 and x_2 of the equation $x^2 + px + 12 = 0$ are such that $x_2 - x_1 = 1$. Find p.

9. Find k in the equation $5x^2 - kx + 1 = 0$ such that the difference between the roots of the equation is unity.

10. Find p in the equation $x^2 - 4x + p = 0$ if it is know that the sum of the squares of its roots is equal to 16.

11. For what values of a is the difference between the roots of the equation $2x^2 - (a + 1)x + (a - 1) = 0$ equal to their product ?

12. Express $x_1^3 + x_2^3$ in terms of the coefficients of the equation $x^2 + px + q = 0$, where x_1 and x_2 are the roots of the equation.

13. Assume that x_1 and x_2 are roots of the equation $3x^2 - ax + 2a - 1 = 0$. Calculate $x_1^3 + x_2^3$.

14. Without solving the equation $3x^2 - 5x - 2 = 0$, find the sum of the cubes of its roots.

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3. NATURE OF ROOTS :

(a) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in \mathbb{R}$ & $a \neq 0$ then ;

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

(i) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal).

(ii) $D = 0 \Leftrightarrow$ roots are real & coincident (equal)

(iii) $D < 0 \Leftrightarrow$ roots are imaginary.

(iv) If $p + i\,q$ is one root of a quadratic equation, then the other root must be the conjugate $p - i\,q$ & vice versa. $(p, q \in \mathbb{R}$ & $i = \sqrt{-1})$.

(b) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in \mathbb{Q}$ & $a \neq 0$ then ;

(i) If D is a perfect square, then roots are rational.

(ii) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is an irrational) then other root will be $p - \sqrt{q}$.

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Illustration 3 : If roots of the equation $(a-b)x^2+(c-a)x+(b-c)=0$ are equal, then

(A) $2b=a+c$ (B) $b=\frac{2ac}{a+c}$ (C) $b^2=ac$ (D) none of these

Solution : $(a-b)x^2+(c-a)x+(b-c)=0$

As roots are equal so

$B^2-4AC=0 \Rightarrow (c-a)^2-4(a-b)(b-c)=0 \Rightarrow (c-a)^2-4ab+4b^2+4ac-4bc=0$

$\Rightarrow (c-a)^2+4ac-4b(c+a)+4b^2=0 \Rightarrow (c+a)^2-2\,.\,(2b)(c+a)+(2b)^2=0$

$\Rightarrow [c+a-2b]^2=0 \Rightarrow c+a-2b=0 \Rightarrow c+a=2b$

Alternative method :

$\because$ Sum of the coefficients = 0

Hence one root is 1 and other root is $\frac{b-c}{a-b}$.

Given that both roots are equal, so

$1=\frac{b-c}{a-b} \Rightarrow a-b=b-c \Rightarrow 2b=a+c$

Illustration 4 : If equation $\frac{x^2-bx}{ax-c}=\frac{k-1}{k+1}$ has roots equal in magnitude & opposite in sign, then the value of k is -

(A) $\frac{a+b}{a-b}$ (B) $\frac{a-b}{a+b}$ (C) $\frac{a}{b}+1$ (D) $\frac{a}{b}-1$

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Solution : Let the roots are α & $-\alpha$.

given equation is

$(x^2-bx)(k+1)=(k-1)(ax-c)$ {Considering, $x \neq c/a$ & $k \neq -1$}

$\Rightarrow x^2(k+1)-bx(k+1)=ax(k-1)-c(k-1)$

$\Rightarrow x^2(k+1)-bx(k+1)-ax(k-1)+c(k-1)=0$

Now sum of roots = 0 ($\because \alpha-\alpha=0$)

$\therefore b(k+1)+a(k-1)=0 \Rightarrow k=\frac{a-b}{a+b}$ **Ans. (B)**

Illustration 5 : If the coefficient of the quadratic equation are rational & the coefficient of x^2 is 1, then find the equation one of whose roots is $\sqrt{2}-1$.

Solution : Irrational roots always occur in conjugational pairs.

Hence if one root is $(-1+\sqrt{2})$, the other root will be $(-1-\sqrt{2})$. Equation is

$(x-(-1+\sqrt{2}))(x-(-1-\sqrt{2}))=0 \Rightarrow x^2+2x-1=0$

Illustration 6 : Find all the integral values of a for which the quadratic equation $(x-a)(x-10)+1=0$ has integral roots.

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Solution : Here the equation is $x^2 - (a + 10)x + 10a + 1 = 0$. Since integral roots will always be rational it means D should be a perfect square.

From (i) $D = a^2 - 20a + 96$.

$\Rightarrow \quad D = (a - 10)^2 - 4 \Rightarrow \quad 4 = (a - 10)^2 - D$

If D is a perfect square it means we want difference of two perfect square as 4 which is possible only when $(a - 10)^2 = 4$ and $D = 0$.

$\Rightarrow \quad (a - 10) = \pm 2 \quad \Rightarrow \quad a = 12, 8$ **Ans.**

Illustration 7 : If D_1 and D_2 are the disrcriminant of two Quadratic Equations with real coefficients respectively then comment upon the nature of the roots of the Quadratic Equations under the following conditions

(i) $D_1 + D_2 \geq 0$ (ii) $D_1 + D_2 < 0$ (iii) $D_1 D_2 < 0$ (iv) $D_1 D_2 > 0$ (v) $D_1 D_2 = 0$

Solution :

(i) If D_1 and D_2 are discriminant of two quadratic equations and $D_1 + D_2 \geq 0$, then at least one of D_1 and $D_2 \geq 0$

$\Rightarrow$ At least one of the equation has real roots.

(ii) $D_1 + D_2 < 0 \Rightarrow$ At least one of D_1 and $D_2 < 0$

$\Rightarrow$ At least one of the equation has imaginary roots.

(iii) If $D_1 D_2 < 0$. $\Rightarrow (D_1 > 0$ and $D_2 < 0)$ or $(D_1 < 0$ and $D_2 > 0)$.

Then one of the equations has real root and other equation has imaginary roots.

(iv) $D_1 D_2 > 0$

Case I : If $D_1 > 0$ and $D_2 > 0$, then both the equations has real roots

Case II : If $D_1 < 0$ and $D_2 < 0$, then both the equations has non real roots

(v) If $D_1 D_2 = 0$, then $D_1 = 0$ or $D_2 = 0$

$\Rightarrow$ at least of equation has equal roots.

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Illustration 8 : Let $a > 0$, $b > 0$ and $c > 0$. Then both the roots of the equation $ax^2 + bx + c = 0$ can

(a) be real and negative (b) have negative real parts

(c) be rational numbers (d) none of these

Solution : (abc)

We have $D = b^2 - 4ac$.

If $D \geq 0$, then the roots of the equation are given by $x = \dfrac{-b \pm \sqrt{D}}{2a}$

As $D = b^2 - 4ac < b^2$ ($\because a > 0, c > 0$), it follows that the roots of the quadratic equation are negative.

In case $D < 0$, then the roots of the equation are given by $x = \dfrac{-b \pm i\sqrt{-D}}{2a}$

which have negative real parts. ($\because$ Both a and b are positive)

Roots will be rational numbers when $\dfrac{b}{a}, \dfrac{c}{a} \in \mathbb{Q}$ and $\dfrac{D}{4a^2}$ is perfect square of some rational number.

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4. SYMMETRIC EXPRESSIONS OF α AND β.

Let a and b be the roots of the equation $ax^2 + bx + c = 0$ such that

$f(\alpha, \beta) = f(\beta, \alpha)$ then $f(\alpha, \beta)$ denotes symmetric expression of the roots.

e.g. $f(\alpha,\beta) = \alpha^2\beta + \alpha\beta^2$

It is to be noted that every symmetric expression in α,β can be expressed in terms of two symmetric expression $\alpha + \beta$ and $\alpha\beta$

Do yourself - 2 :

1. For the following equations, find the nature of the roots (real & distinct, real & coincident or imaginary).

(a) $x^2 - 6x + 10 = 0$

(b) $x^2 - (7+\sqrt{3})x + 6(1+\sqrt{3}) = 0$

(c) $4x^2 + 28x + 49 = 0$

2. Prove that roots of $(x - a)(x - b) = h^2$ are always real.

3. For what values of a does the equation $9x^2 - 2x + a = 6 - ax$ posses equal roots ?

4. Find the value of k for which the equation $(k - 1)x^2 + (k + 4)x + k + 7 = 0$ has equal roots.

5. Find the values of a for which the roots of the equation $(2a - 5)x^2 - 2(a - 1)x + 3 = 0$ are equal.

6. For what values of 'a' does the quadratic equation $x^2 + (2a\sqrt{a^2 - 3})x + 4 = 0$ possess equal roots ?

7. Find the least integral value of k for which the equation $x^2 - 2(k + 2)x + 12 + k^2 = 0$ has two different real roots.

8. If ℓ, m are real and $\ell \neq m$, then show that the roots of $(\ell - m)x^2 - 5(\ell + m)x - 2(\ell - m) = 0$ are real and unequal.

9. For what values of m does the equation $x^2 - x + m = 0$ possess no real roots ?

10. For what values of c does the equation $(c - 2)x^2 + 2(c - 2)x + 2 = 0$ possess no real roots ?

11. Find integral values of k for which the quadratic equation $(k - 12)x^2 + 2(k - 12)x + 2 = 0$ possess no real roots ?

12. For what values of m does the equation $x^2 - x + m^2 = 0$ possess no real roots ?

13. For what values of m does the equation $mx^2 - (m + 1)x + 2m - 1 = 0$ possess no real roots ?

14. If $2+\sqrt{3}$ is a root of the equation $x^2 + bx + c = 0$, where $b, c \in \mathbb{Q}$, find b, c.

15. Find the value of $x^3 + x^2 - x + 22$ when $x = 1 + 2i$.

16. Find the value of $x^3 - 3x^2 - 8x + 15$ when $x = 3 + i$.

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17. Consider $f(x) = x^2 + bx + c$.

(a) Find c if $x = 0$ is a root of $f(x) = 0$.

(b) Find c if $\alpha, \frac{1}{\alpha}$ are roots of $f(x) = 0$.

(c) Comment on sign of b & c, if $\alpha < 0 < \beta$ & $|\beta| > |\alpha|$, where α, β are roots of $f(x) = 0$.

18. Which of the following expressions in α, β will denote the symmetric functions in α,β. Give proper reasoning.

(i) $f(\alpha, \beta) = \alpha^2 - \beta$ (ii) $f(\alpha, \beta) = \alpha^2\beta + \alpha\beta^2$

(iii) $f(\alpha, \beta) = \alpha^3 + \beta^3$ (iv) $f(\alpha, \beta) = \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

19. If α,β are the roots of the equation $ax^2 + bx + c = 0$, find the value of

(A) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ (B) $\alpha^4\beta^7 + \alpha^7\beta^4$ (C) $\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)^2$

20. Calculate $\frac{1}{x_1^3} + \frac{1}{x_2^3}$, where x_1 and x_2 are roots of the equation $2x^2 - 3ax - 2 = 0$.

21. For what value of a is the difference between the roots of the equation $(a - 2)x^2 - (a - 4)x - 2 = 0$ equal to 3 ?

22. Find the value of a for which one root of the equation $x^2 + (2a - 1)x + a^2 + 2 = 0$ is twice as large as the other.

23. For what values of a is the ratio of the roots of the equation $x^2 + ax + a + 2 = 0$ equal to 2 ?

24. For what values of a do the roots x_1 and x_2 of the equation $x^2 - (3a + 2)x + a^2 = 0$ satisfy the relation $x_1 = 9x_2$? Find the roots.

25. Find a such that one of the roots of the equation $x^2 - \frac{15}{4}x + a = 0$ is the square of the other.

26. Find all the values of a for which the sum of the roots of the equation $x^2 - 2a(x - 1) - 1 = 0$ is equal to the sum of the squares of its roots.

27. Find the coefficients of the equation $x^2 + px + q = 0$ such that its roots are equal to p and q.

5. TRANSFORMATION OF THE EQUATION :

Let $ax^2 + bx + c = 0$ be a quadratic equation with two roots α and β. If we have to find an equation whose roots are $f(\alpha)$ and $f(\beta)$, i.e. some expression in α & β, then this equation can be found by finding α in terms of y. Now as α satisfies given equation, put this α in terms of y directly in the equation.

$$y = f(\alpha)$$

By transformation, $\alpha = g(y)$

$$a(g(y))^2 + b(g(y)) + c = 0$$

This is the required equation in y.

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Illustration 9 : If the roots of $ax^2 + bx + c = 0$ are α and β, then find the equation whose roots are :

(a) $\frac{-2}{\alpha}, \frac{-2}{\beta}$ (b) $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$ (c) α^2, β^2

Solution : (a) $\frac{-2}{\alpha}, \frac{-2}{\beta}$

put, $y = \frac{-2}{\alpha} \Rightarrow \alpha = \frac{-2}{y}$

$a\left(-\frac{2}{y}\right)^2 + b\left(\frac{-2}{y}\right) + c = 0 \Rightarrow cy^2 - 2by + 4a = 0$

Required equation is $cx^2 - 2bx + 4a = 0$

(b) $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$

put, $y = \frac{\alpha}{\alpha+1} \Rightarrow \alpha = \frac{y}{1-y}$

$\Rightarrow a\left(\frac{y}{1-y}\right)^2 + b\left(\frac{y}{1-y}\right) + c = 0 \Rightarrow (a + c - b)y^2 + (-2c + b)y + c = 0$

Required equation is $(a + c - b)x^2 + (b - 2c)x + c = 0$

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(c) α^2, β^2

put $y = \alpha^2 \Rightarrow \alpha = \sqrt{y}$

$ay + b\sqrt{y} + c = 0$

$b^2y = a^2y^2 + c^2 + 2acy$

$\Rightarrow a^2y^2 + (2ac - b^2)y + c^2 = 0$

Required equation is $a^2x^2 + (2ac - b^2)x + c^2 = 0$

Illustration 10 : If the roots of $ax^3 + bx^2 + cx + d = 0$ are α, β, γ then find equation whose roots are $\frac{1}{\alpha\beta}, \frac{1}{\beta\gamma}, \frac{1}{\gamma\alpha}$.

Solution : Put $y = \frac{1}{\alpha\beta} = \frac{\gamma}{\alpha\beta\gamma} = -\frac{a\gamma}{d}$ $(\because \alpha\beta\gamma = -\frac{d}{a})$

Put $x = -\frac{dy}{a}$

$\Rightarrow a\left(-\frac{dy}{a}\right)^3 + b\left(-\frac{dy}{a}\right)^2 + c\left(-\frac{dy}{a}\right) + d = 0$

Required equation is $d^2x^3 - bdx^2 + acx - a^2 = 0$

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Do yourself - 3 :

1. If α, β are the roots of $ax^2 + bx + c = 0$, then find the equation whose roots are

(a) $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$ (b) $\frac{1}{a\alpha+b}, \frac{1}{a\beta+b}$ (c) $\alpha+\frac{1}{\beta}, \beta+\frac{1}{\alpha}$

2. If α and β are the roots of $x^2 + px + q = 0$, form the equation whose roots are $(\alpha - \beta)^2$ and $(\alpha + \beta)^2$.

3. If α, β are roots of $x^2 - px + q = 0$, then find the quadratic equation whose root are $(\alpha^2-\beta^2)(\alpha^3-\beta^3)$ and $\alpha^2\beta^3+\alpha^3\beta^2$.

6. EQUATION VS IDENTITY :

An equation which is true for every value of the variable within the domain is called an identity , for example : $5(a-3) = 5a - 15$, $(a+b)^2 = a^2 + b^2 + 2ab$ for all $a, b \in R$.

Note : An equation of degree ≤ 2 cannot have three or more roots & if it has , it becomes an identity.

If $ax^2 + bx + c = 0$ is an identity $\Leftrightarrow a = b = c = 0$

Illustration 11 : If the equation $(\lambda^2 - 5\lambda + 6)x^2 + (\lambda^2 - 3\lambda + 2)x + (\lambda^2 - 4) = 0$ has more than two roots, then find the value of λ ?

Solution : As the equation has more than two roots so it becomes an identity. Hence

$\lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda = 2, 3$

and $\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda = 1, 2$

and $\lambda^2 - 4 = 0 \Rightarrow \lambda = 2, -2$

So $\lambda = 2$ **Ans. $\lambda = 2$**

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7. COMMON ROOTS OF TWO QUADRATIC EQUATIONS :

(a) Atleast one common root.

Let α be the common root of $ax^2 + bx + c = 0$ & $a'x^2 + b'x + c' = 0$ then

$a\alpha^2 + b\alpha + c = 0$ & $a'\alpha^2 + b'\alpha + c' = 0$. By Cramer's Rule $\frac{\alpha^2}{bc'-b'c} = \frac{\alpha}{a'c-ac'} = \frac{1}{ab'-a'b}$

Therefore, $\alpha = \frac{ca'-c'a}{ab'-a'b} = \frac{bc'-b'c}{a'c-ac'}$

So the condition for a common root is $(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$.

(b) If both roots are same then $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

Illustration 12 : Find p and q such that $px^2 + 5x + 2 = 0$ and $3x^2 + 10x + q = 0$ have both roots in common.

Solution : $a_1 = p, b_1 = 5, c_1 = 2$ and $a_2 = 3, b_2 = 10, c_2 = q$

We know that :

$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow \frac{p}{3} = \frac{5}{10} = \frac{2}{q} \Rightarrow p = \frac{3}{2}; q = 4$

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Illustration 13 : Find the possible value(s) of a for which the equations $x^2 + ax + 1 = 0$ and $x^2 + x + a = 0$ have atleast one common root.

Solution : Let α is a common root

then $\alpha^2 + a\alpha + 1 = 0$

& $\alpha^2 + \alpha + a = 0$

by cramer's rule

$$\frac{\alpha^2}{a^2-1} = \frac{\alpha}{1-a} = \frac{1}{1-a}$$

$\Rightarrow \quad (1-a)^2 = (a^2-1)(1-a)$

$\Rightarrow \quad a = 1, -2$

Illustration 14 : If the equations $x^2 - ax + b = 0$ and $x^2 - cx + d = 0$ have one root in common and second equation has equal roots, prove that $ac = 2(b + d)$

Solution : The equation $x^2 - cx + d = 0$ has equal roots.

$\Rightarrow D = 0 \Rightarrow D = c^2 - 4d = 0$ ...(i)

Also, the equal roots are $x = -\frac{b}{2a}$ (for $ax^2 + bx + c = 0$ having equal roots)

$\Rightarrow x = \frac{c}{2}$ is the equal root of this equation.

Now this should be the common root.

$\therefore x = \frac{c}{2}$ will satisfy the first equation

$\Rightarrow \frac{c^2}{4} - a\left(\frac{c}{2}\right) + b = 0 \quad \Rightarrow c^2 + 4b = 2ac$

$\Rightarrow 4d + 4b = 2ac$ [Using (i)]

$\Rightarrow 2(d + b) = ac$

Hence $ac = 2(b + d)$

Illsutration 15 : If $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a root in common, find the relation between a, b and c.

Solution : Solve the two equations :

$ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$

$$\frac{x^2}{ba-c^2} = \frac{-x}{a^2-bc} = \frac{1}{ac-b^2}$$

$\Rightarrow (a^2 - bc)^2 = (ba - c^2)(ac - b^2)$

Simplify to get : $a(a^3 + b^3 + c^3 - 3abc) = 0$

$\Rightarrow a = 0$ or $a^3 + b^3 + c^3 = 3abc$

This is the relation between a, b and c.

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Illsutration 16 : Find the conditions on a, b, c and d such that equations

$2ax^3 + bx^2 + cx + d = 0$ and $2ax^2 + 3bx + 4c = 0$ have a common root.

Solution : Let 'α' be a common root of the given equations,

then $2a\alpha^3 + b\alpha^2 + c\alpha + d = 0$ ...(1)

and $2a\alpha^2 + 3b\alpha + 4c = 0$ ...(2)

Multiply (2) by α and then subtract (1) from it, to get

$2b\alpha^2 + 3c\alpha - d = 0$ ...(3)

Now (2) and (3) are quadratic having a common root α, so

$$\frac{\alpha^2}{3bd-12c^2} = \frac{\alpha}{8bc+2ad} = \frac{1}{6ac-6b^2}, \quad \alpha^2 = \frac{bd+4c^2}{2b^2-2ac}, \quad \alpha = \frac{4bc+ad}{3ac-3b^2}$$

Eliminating α from these two equations, we get

$(4bc + ad)^2 = \frac{9}{2}(bd + 4c^2)(b^2 - ac)$, which is the required condition.

Illustration 17 : The values of 'a' for which the equations $x^3 - 6x^2 + (6 + a)x - 6 = 0$ and $x^2 - ax + 4 = 0$ have a common root.

Solution : Let 'α' is the common root.

$\alpha^3 - 6\alpha^2 + (6 + a)\alpha - 6 = 0$

$\alpha^2 - a\alpha + 4 = 0$

Adding

$\alpha^3 - 5\alpha^2 + 6\alpha - 2 = 0$

$\Rightarrow \alpha = 1$ is one solution of above equation.

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$$\begin{array}{r|l} & \alpha^2 - 4\alpha + 2 \\ \hline \alpha - 1 & \alpha^3 - 5\alpha^2 + 6\alpha - 2 \\ & \alpha^3 - \alpha^2 \\ \hline & -4\alpha^2 + 6\alpha - 2 \\ & -4\alpha^2 + 4\alpha \\ \hline & 2\alpha - 2 \\ & 2\alpha - 2 \\ \hline & 0 \end{array}$$

$\Rightarrow \quad (\alpha - 1)(\alpha^2 - 4\alpha + 2) = 0$

$\Rightarrow \quad (\alpha-1)\{\alpha-(2-\sqrt{2})\}\{\alpha-(2+\sqrt{2})\} = 0$

$\alpha = 1,\ \alpha = 2-\sqrt{2}$ or $\alpha = 2+\sqrt{2}$

This common root can be $1,\ 2-\sqrt{2},\ 2+\sqrt{2}$.

We can obtain values of 'a'.

For $\alpha = 1$, $1 - a + 4 = 0 \Rightarrow a = 5$

For $\alpha = 2-\sqrt{2}$, $(6-4\sqrt{2}) - a(2-\sqrt{2}) + 4 = 0$

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$\Rightarrow \quad a(2-\sqrt{2}) = 10 - 4\sqrt{2}$

$\Rightarrow \quad a = \dfrac{10-4\sqrt{2}}{2-\sqrt{2}} = \dfrac{(10-4\sqrt{2})(2+\sqrt{2})}{(4-2)}$

$\Rightarrow \quad a = (5-2\sqrt{2})(2+\sqrt{2})$

$\Rightarrow \quad a = 10 + 5\sqrt{2} - 4\sqrt{2} - 4$

$\Rightarrow \quad a = 6 + \sqrt{2}$

For $\alpha = 2+\sqrt{2} \Rightarrow (6+4\sqrt{2}) - a(2+\sqrt{2}) + 4 = 0$

$\Rightarrow \quad a = \dfrac{10+4\sqrt{2}}{2+\sqrt{2}} = \dfrac{(10+4\sqrt{2})(2-\sqrt{2})}{2}$

$\Rightarrow \quad a = (5+2\sqrt{2})(2-\sqrt{2})$

$\Rightarrow \quad a = 10 - 5\sqrt{2} + 4\sqrt{2} - 4$

$\Rightarrow \quad a = 6 - \sqrt{2}$

Thus possible values of 'a' are $5, 6 \pm \sqrt{2}$.

Illustration 18 : If each pair of the following three equations $x^2 + ax + b = 0$, $x^2 + cx + d = 0$, $x^2 + ex + f = 0$ has exactly one root in common, then show that $(a + c + e)^2 = 4(ac + ce + ea - b - d - f)$.

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Solution : Given equations are

$x^2 + ax + b = 0$ ...(1)

$x^2 + cx + d = 0$ ...(2)

$x^2 + ex + f = 0$ ...(3)

Let α, β be the roots of (1), β, γ be the roots of (2) and γ, α be the roots of (3).

$\therefore\ \alpha + \beta = -a,\ \alpha\beta = b$ ...(4)

$\beta + \gamma = -c,\ \beta\gamma = d$ ...(5)

$\gamma + \alpha = -e,\ \gamma\alpha = f$ ...(6)

LHS $= (a + c + e)^2$

$= (-\alpha - \beta - \beta - \gamma - \gamma - \alpha)^2$ {from (4), (5), (6)}

$= 4(\alpha + \beta + \gamma)^2$ ...(7)

RHS $= 4(ac + ce + ea - b - d - f)$

$= 4\{(\alpha + \beta)(\beta + \gamma) + (\beta + \gamma)(\gamma + \alpha) + (\alpha + \beta)(\gamma + \alpha) - \alpha\beta - \beta\gamma - \gamma\alpha\}$ {from (4), (5), (6)}

$= 4(\alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha)$

$= 4(\alpha + \beta + \gamma)^2$ ...(8)

From (7) and (8),

$(a + c + e)^2 = 4(ac + ce + ea - b - d - f)$

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Do yourself - 4 :

1. For what values of a, the equation

$(a + 3)(a - 1)x^2 + (a - 1)(4a + 3)x + (a^2 - 3a + 2) = 0$ has more than two roots.

2. If $a.\frac{(x-b)(x-c)}{(a-b)(a-c)} + b.\frac{(x-c)(x-a)}{(b-c)(b-a)} + c.\frac{(x-a)(x-b)}{(c-a)(c-b)} = x$ then prove that it is an identity.

3. If $x^2 + bx + c = 0$ & $2x^2 + 9x + 10 = 0$ have both roots in common, find b & c.

4. If $x^2 - 7x + 10 = 0$ & $x^2 - 5x + c = 0$ have a common root, find c.

5. Show that $x^2 + (a^2 - 2)x - 2a^2 = 0$ and $x^2 - 3x + 2 = 0$ have exactly one common root for all $a \in R$.

6. For what values of a do the equations $x^2 + ax + 1 = 0$ and $x^2 + x + a = 0$ have exactly one root in common ?

7. Given two quadratic equations $x^2 - x + m = 0$ and $x^2 - x + 3m = 0$, $m \neq 0$. Find the value of m for which one of the roots of the second equation is equal to double the root of the first equation.

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8. QUADRATIC EXPRESSION AND IT'S GRAPHS :

Consider the quadratic expression, $y = ax^2 + bx + c$, $a \neq 0$ & $a, b, c \in R$ then ;

(a) The graph between x, y is always a parabola. If $a > 0$ then the shape of the parabola is concave upwards & if $a < 0$ then the shape of the parabola is concave downwards.

(b) The graph of $y = ax^2 + bx + c$ can be divided in 6 broad categories which are as follows : (Let the real roots of quadratic equation $ax^2 + bx + c = 0$ be α & β where $\alpha \leq \beta$).

Fig. 1

Roots are real & distinct

$ax^2 + bx + c > 0 \; \forall x \in (-\infty, \alpha) \cup (\beta, \infty)$

$ax^2 + bx + c < 0 \; \forall x \in (\alpha, \beta)$

Fig. 4

Fig. 2

Roots are coincident

$ax^2 + bx + c > 0 \; \forall x \in R - \{\alpha\}$

$ax^2 + bx + c = 0$ for $x = \alpha = \beta$

Fig. 5

Fig.3

Roots are complex conjugate

$ax^2 + bx + c > 0 \; \forall x \in R$

Fig.6

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Roots are real & distinct

$ax^2 + bx + c > 0 \ \forall \ x \in (\alpha, \beta)$

$ax^2 + bx + c < 0 \ \forall \ x \in (-\infty, \alpha) \cup (\beta, \infty)$

Roots are coincident

$ax^2 + bx + c < 0 \ \forall \ x \in R - \{\alpha\}$

$ax^2 + bx + c = 0$ for $x = \alpha = \beta$

Roots are complex conjugate

$ax^2 + bx + c < 0 \ \forall \ x \in R$

Important Note :

(i) The quadratic expression $ax^2 + bx + c > 0$ for each $x \in R \Rightarrow a > 0, D < 0$ & vice-versa (Fig. 3)

(ii) The quadratic expression $ax^2 + bx + c < 0$ for each $x \in R \Rightarrow a < 0, D < 0$ & vice-versa (Fig. 6)

9. INEQUALITIES (rational, irrational and modulus)

9.1 Rational inequalities (When numerator or denomerator contains irreducible quadratic factor)

Illustration 19: Find x such that $3x^2 - 7x + 6 < 0$

Solution : $f(x) = 3x^2 - 7x + 6$

$D = 49 - 72 < 0$ and $a > 0$

$\Rightarrow 3x^2 - 7x + 6 > 0 \ \forall \ x \in R \Rightarrow x \in \phi$

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Illustration 20 : Find x such that $2x^2 + 4x + 9 > 0$

Solution : $f(x) = 2x^2 + 4x + 9$

$a > 0$ and $D < 0$

$\Rightarrow 2x^2 + 4x + 9 > 0 \ \forall \ x \in R$

$\Rightarrow x \in (-\infty, \infty)$

Illustration-21 : Solve for x : $\dfrac{(x^2-1)^2+1}{x^2-1} \geq 2$.

Solution : Recall :

The sum of any real number except zero and its reciprocal should not lie in (–2, 2).

Case I :

When $t + \dfrac{1}{t} \geq 2$. This holds $\forall \ t > 0$ and equality holds only when t = 1.

Case II :

When $t + \dfrac{1}{t} \leq -2$. This holds $\forall \ t < 0$ and equality holds if t = –1.

Note : $x^2 - 1 \neq 0, \ x^2 \neq 1 \Rightarrow x \neq \pm 1$

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Rewrite the given inequality $\dfrac{(x^2-1)^2+1}{x^2-1} \geq 2$

as $(x^2-1)+\dfrac{1}{(x^2-1)} \geq 2$

$\Rightarrow x^2 - 1 > 0 \Rightarrow (x + 1)(x - 1) > 0$

$\Rightarrow x \in (-\infty, -1) \cup (1, \infty)$

Do yourself- 5 :
Solve for x ∈ ℝ

1. $x^2 - 3x + 4 > 0$

2. $\dfrac{x}{x^2+2} < 1$

3. $\dfrac{x^2-5x+12}{x^2-4x+5} < 3$

4. $\dfrac{(2x-1)(x+3)(2-x)(1-x)^2}{x^4(x+6)(x-9)(2x^2+4x+9)} < 0$

5. $\dfrac{7x-17}{x^2-3x+4} \geq 1$

6. $\dfrac{x^2+6x-7}{x^2+1} \leq 2$

7. $\dfrac{2x}{x^2-9} \leq \dfrac{1}{x+2}$

8. $\dfrac{x^2-5x+12}{x^2-4x+5} > 3$

9. $\dfrac{x^2-5x+6}{x^2+x+1} < 0$

10 $\dfrac{x^2+4x+4}{2x^2-x-1} > 0$

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9.2. IRRATIONAL INEQUALITIES :

While solving Irrational Inequality, first we have to find largest interval of x for which left hand side and right hand side both make sense. This interval is called domain of the Inequality.

Note :

(i) For $\sqrt{f(x)}$ to be well defined $f(x)$ must be greater than or equal to 0.

(ii) $\sqrt{f(x)} \geq 0$ whenever defined.

Type 1 : $\sqrt{f(x)} \geq g(x)$

Case-I : $f(x) \geq 0$ and $g(x) > 0$ ... (1)

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Above inequality become

$f(x) \geq g^2(x)$ ... (2)

Answer in this case is intersection of solutions of (1) and (2)

Case-II : $f(x) \geq 0$ and $g(x) \leq 0$.

In this type final answer of the given Inequality is union of case I and case II.

Type 2 : $g(x) \geq \sqrt{f(x)}$

Since $\sqrt{f(x)} \geq 0$ for $\forall\ f(x) \geq 0$

So g(x) must be non-negative

$g(x) \geq 0$ ... (1)

$f(x) \geq 0$ ... (2)

$g^2(x) \geq f(x)$ ... (3)

Answer of the inequality is intersection of (1), (2) and (3).

Illustration-22 : Solve : $x + 1 \geq \sqrt{5 - x}$

Solution : $5 - x \geq 0$ and $x + 1 \geq 0$

$\Rightarrow x \in [-1, 5]$ ... (1)

For $x \in [-1, 5]$

$x + 1 \geq \sqrt{5 - x}$

$\Rightarrow (x + 1)^2 \geq 5 - x \Rightarrow x^2 + 3x - 4 \geq 0$

$\Rightarrow (x - 1)(x + 4) \geq 0 \Rightarrow x \in (-\infty, -4] \cup [1, \infty)$

$\Rightarrow x \in [1, 5]$

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Illustration-23 : Solve : $\sqrt{6 + x} \geq 6 - x$

Solution :

Case-I $6 + x \geq 0$ and $6 - x > 0$

$x \in [-6, 6)$

$\sqrt{6 + x} \geq 6 - x \Rightarrow 6 + x \geq 36 - 12x + x^2$

$\Rightarrow 0 \geq x^2 - 13x + 30 \Rightarrow (x - 10)(x - 3) \leq 0 \Rightarrow x \in [3, 10]$

$x \in [3, 10] \cap [-6, 6) = [3, 6)$

Case-II $6 + x \geq 0$ and $6 - x \leq 0 \Rightarrow x \in [6, \infty)$

In this case given Inequality is always true.

From case-I and case-II,

$x \in [3, \infty)$.

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Illustration 24 : Solve for x, if $\sqrt{x^2-3x+2} > x-2$

Solution :

$$\left[\begin{array}{l}\begin{cases} x^2-3x+2 \ge 0 \\ x-2 \ge 0 \\ (x^2-3x+2) > (x-2)^2 \end{cases} \\ \text{or} \\ \begin{cases} x^2-3x+2 \ge 0 \\ x-2 < 0 \end{cases}\end{array}\right. \Rightarrow \left[\begin{array}{l}\begin{cases} (x-1)(x-2) \ge 0 \\ (x-2) \ge 0 \\ x-2 > 0 \end{cases} \Rightarrow x > 2 \\ \text{or} \\ \begin{cases} (x-1)(x-2) \ge 0 \\ x-2 < 0 \end{cases} \Rightarrow x \le 1\end{array}\right.$$

Hence, solution set of the original inequation is $x \in R - (1,2]$

Do yourself- 6 :

Solve for x if

1 $\sqrt{x^2-x} > (x-1)$ **2.** $\sqrt{\dfrac{x-2}{1-2x}} > -1$ **3.** $\dfrac{\sqrt{x-3}}{x-2} > 0$

4. $2\sqrt{x-1} < x$ **5.** $\sqrt{9x-20} < x$ **6.** $\sqrt{5-2x} < 6x-1$

7. $\sqrt{2x-x^2} < 5-x$ **8.** $\sqrt{x+18} < 2-x$

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9.3 Modulus inequalities

- $|x| \le a$, if $a \ge 0$
  $\Rightarrow -a \le x \le a$
  if $a < 0$, then $x \in \phi$
- $|x| \ge a$
  if $a \ge 0$, then $x \in (-\infty, -a] \cup [a, \infty)$
  if $a < 0$, then $x \in (-\infty, \infty)$
- $|xy| = |x|\,|y|$
- $\big||x|-|y|\big| \le |x+y| \le |x|+|y|$
- $|x| + |y| = |x + y| \Leftrightarrow xy \ge 0$
- $|x| + |y| = |x - y| \Leftrightarrow xy \le 0$

Illustration -25 : Represent x on the number line when

(i) $|x| \ge 2$ (ii) $|x - 3| \le 1$ (iii) $|5 - x| < 2$

(iv) $1 \le |x + 2| < 5$ (v) $|2x - 5| < 1$

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Solution : (i) $|x| \geq 2 \Rightarrow$ distance of x from O on the number line is greater than or equal to 2.

(ii) $|x - 3| \leq 1 \Rightarrow$ distance of x from 3 is less than or equal to 1.

(iii) $|5 - x| < 2 \Rightarrow$ distance of x from 2 is less than 2.

(iv) $1 \leq |x + 2| < 5$

$\Rightarrow 1 \leq |x - (2)| < 5$

$\Rightarrow$ distance of x from –2 is greater than or equal to 1 and less than 5.

(v) $|2x - 5| < 1$

$\Rightarrow$ distance of 2x from 5 on the number line is less than or equal to 1.

$\Rightarrow 4 < 2x < 6 \quad \Rightarrow 2 < x < 3$

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Illustration -26 : Solve following

(i) $|x^2 - 4x| = x + 1$

(ii) $|x^2 + 1| = |x - 2|$

(iii) $|x^2 - 16| - 8|x - 2| = x(8 - x)$

(iv) $a^2|x + a| + |a^2x + 1| = 1 - a^3$

where a is a real constant.

Solution : (i) Case-1

$x^2 - 4x \geq 0$

$\Rightarrow x \in (-\infty, 0] \cup [4, \infty) \qquad ...(1)$

$|x^2 - 4x| = x + 1$

$\Rightarrow x^2 - 4x = x + 1 \Rightarrow x^2 - 5x - 1 = 0$

$$\Rightarrow x = \frac{5 \pm \sqrt{29}}{2}$$

from (1) and (2)

$$x = \frac{5 \pm \sqrt{29}}{2}$$

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Case -2

where $x^2 - 4x < 0$

$\Rightarrow x \in (0,4)$ ...(3)

$|x^2 - 4x| = x + 1$

$\Rightarrow 4x - x^2 = x + 1 \Rightarrow x^2 - 3x + 1 = 0$

$\Rightarrow x = \frac{3 \pm \sqrt{5}}{2}$ ...(4)

From (3) and (4)

Set of all solutions of the given equation is

$$\left\{\frac{5-\sqrt{29}}{2}, \frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}, \frac{5+\sqrt{29}}{2}\right\}$$

(ii) $|x^2 + 1| = |x - 2|$

As $x^2 + 1 = 0 \ \forall \ x \in \mathbb{R}$

$\Rightarrow |x^2 + 1| = x^2 + 1, \ \forall \ x \in \mathbb{R}$

Case-1

$x - 2 \geq 0 \quad \Rightarrow x \in [2,\infty)$

$|x^2 + 1| = |x - 2|$

$\Rightarrow x^2 + 1 = x - 2 \Rightarrow x^2 - x + 3 = 0$

$\Rightarrow x = \frac{1 \pm \sqrt{11}i}{2}$

Case-2

$x - 2 < 0 \Rightarrow x \in (-\infty,2)$

$|x^2 + 1| = |x - 2|$

$\Rightarrow x^2 + 1 = -x + 2 \Rightarrow x^2 + x - 1 = 0$

$\Rightarrow x = \frac{-1 \pm \sqrt{5}}{2}$

Both x belongs to $(-\infty,2)$

Hence set of solutions is $\left\{\frac{-1 \pm \sqrt{5}}{2}\right\}$

(iii) $|x^2 - 16| - 8|x - 2| = x(8 - x)$

Case 1 : $x \leq -4$

$x^2 - 16 + 8x - 16 = 8x - x^2$

$2x^2 = 32$

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Case 2 : $x \in [-4,2]$

$-x^2 + 16 + 8x - 16 = 8x - x^2$

$x = x$ always true

$x \in [-4,2]$

Case 3 : $x \in [2,4]$

$-x^2 + 76 - 8x + 16 = 8x - x^2$

$16x = 32$

$x = 2$

Case 4 : $x \geq 4$

$x^2 - 16 - 8x + 16 = 8x - x^2$

$2x^2 = 16x$

$x = 8, 0$ (rejected)

$x \in [-4,2] \cup \{8\}$

(iv) $a^2|x + a| + |a^2x + 1| = 1 - a^3 \Rightarrow |a^2x + a^3| + |a^2x + 1| = 1 - a^3$

Let $A = a^2x + a^3$, $B = a^2x + 1$ and $C = 1 - a^3$

Given equation can be written as

$|A| + |B| = C = B - A$

$\Rightarrow B \geq 0$ and $A \leq 0$

$a^2x + 1 \geq 0$ and $a^2x + a^3 \leq 0$

when $a = 0$, $x \in R$

when $a \neq 0$, $x \geq \frac{-1}{a^2}$ and $x \leq -a$ $\Rightarrow x \in \left[-\frac{1}{a^2}, \infty\right) \cap (-\infty, -a]$

Case-1

when $-\frac{1}{a^2} > -a$ $\Rightarrow \frac{1}{a^2} < a \Rightarrow 1 < a^3$ $\Rightarrow a \in (1,\infty)$

$x \in \left[-\frac{1}{a^2}, \infty\right) \cap (-\infty, -a] = \phi$

Case-2

when $-a \geq -\frac{1}{a^2} \Rightarrow a \leq \frac{1}{a^2}$ and $a \neq 0$

$\Rightarrow a^3 \leq 1$ and $a \neq 0$ $\Rightarrow a < 1$ and $a \neq 0 \Rightarrow a \in (-\infty, 1] - \{0\}$

$x \in \left[-\frac{1}{a^2}, \infty\right) \cap (-\infty, -a]$

$x \in \left[-\frac{1}{a^2}, -a\right]$

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Solutions of the given equation is

$x \in \mathbb{R}$ when $a = 0$

$x \in \phi$ when $a > 1$

$x \in \left[-\frac{1}{a^2}, -a\right]$ when $a \in (-\infty, 1] - \{0\}$

Illustrations-27 : Let $A = \{x : x \in R, |x| < 1]$; $B = [x : x \in R, |x - 1| \geq 1]$ and $A \cup B = R - D$, then the set D is-

(1) $[x : 1 < x \leq 2]$ (2) $[x : 1 \leq x < 2]$ (3) $[x : 1 \leq x \leq 2]$ (4) none of these

Solution : $A = [x : x \in R, -1 < x < 1]$

$B = [\, x : x \in R : x - 1 \leq -1$ or $x - 1 \geq 1]$

$= [x : x \in R : x \leq 0$ or $x \geq 2]$

$\therefore A \cup B = R - D$

where $D = [x : x \in R, 1 \leq x < 2]$

Thus (2) is the correct answer.

Illustration-28 : If $|x - 1||x - 2| = -(x^2 - 3x + 2)$, then find the interval in which x lies ?

Solution : $|x - 1||x - 2| = -(x - 2)(x - 1)$

$\Rightarrow (x - 1)(x - 2) \leq 0$

$\Rightarrow 1 \leq x \leq 2$

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Illustration-29 : Solve : (i) $||x - 2| - 5| \geq 1$

Solution : $||x - 2| - 5| \geq 1$

$\Rightarrow |x - 2| - 5 \geq 1$ or $|x - 2| - 5 \leq -1$

$\Rightarrow |x - 2| \geq 6$ or $|x - 2| \leq 4$

Case-1

$|x - 2| \geq 6$

$\Rightarrow x - 2 \geq 6$ or $x - 2 \leq -6$ $\Rightarrow x \geq 8$ or $x \leq -4$

$\Rightarrow x \in (-\infty, -4] \cup [8, \infty)$ ...(1)

Case-2

$|x - 2| \geq 4$

$\Rightarrow -4 \leq x - 2 \leq 4$ $\Rightarrow -2 \leq x \leq 6$

$\Rightarrow x \in [-2, 6]$ ...(2)

From (1) and (2) $x \in (-\infty, -4] \cup [-2, -6] \cup [8, \infty)$

Illustration 30 : If x satisfies $|x - 1| + |x - 2| + |x - 3| \geq 6$, then

(A) $0 \leq x \leq 4$ (B) $x \leq -2$ or $x \geq 4$

(C) $x \leq 0$ or $x \geq 4$ (D) none of these

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Solution : **Case I :** $x \le 1$, then

$1 - x + 2 - x + 3 - x \ge 6 \Rightarrow x \le 0$

Hence $x \le 0$ ...(i)

Case II : $1 < x \le 2$, then

$x - 1 + 2 - x + 3 - x \ge 6 \Rightarrow x \le -2$

But $1 < x \le 2 \Rightarrow$ No solution. ...(ii)

Case III : $2 < x \le 3$, then

$x - 1 + x - 2 + 3 - x \ge 6 \Rightarrow x \ge 6$

But $2 < x \le 3 \Rightarrow$ No solution. ...(iii)

Case IV : $x > 3$, then

$x - 1 + x - 2 + x - 3 \ge 6 \Rightarrow x \ge 4$

Hence $x \ge 4$ ...(iv)

From (i), (ii), (iii) and (iv) the given inequality holds for $x \le 0$ or $x \ge 4$.

Illustration 31 : Solve for x : (a) $||x - 1| + 2| \le 4$. (b) $\dfrac{x-4}{x+2} \le \left|\dfrac{x-2}{x-1}\right|$

Solution : (a) $||x - 1| + 2| \le 4 \Rightarrow -4 \le |x - 1| + 2 \le 4$

$\Rightarrow -6 \le |x - 1| \le 2$

$\Rightarrow |x - 1| \le 2 \Rightarrow -2 \le x - 1 \le 2$

$\Rightarrow -1 \le x \le 3 \Rightarrow x \in [-1, 3]$

(b) **Case 1 :** Given inequation will be statisfied for all x such that

$\dfrac{x-4}{x+2} \le 0 \Rightarrow x \in (-2, 4] - \{1\}$ .....(i)

(Note : $\{1\}$ is not in domain of RHS)

Case 2 : $\dfrac{x-4}{x+2} > 0 \Rightarrow x \in (-\infty, -2) \cup (4, \infty)$ .....(ii)

Given inequation becomes

$\dfrac{x-2}{x-1} \ge \dfrac{x-4}{x+2}$ or $\dfrac{x-2}{x-1} \le -\dfrac{x-4}{x+2}$

on solving we get on solving we get

$x \in (-2, 4/5) \cup (1, \infty)$ $x \in (-2, 0] \cup (1, 5/2]$

taking intersection with (ii) we get taking intersection with (ii) we get

$x \in (4, \infty)$ .....(iii) $x \in \phi$

Hence, solution of the original inequation : $x \in (-2,\infty) - \{1\}$ (taking union of (i) & (iii))

Illustration 32 : The equation $|x| + \left|\dfrac{x}{x-1}\right| = \dfrac{x^2}{|x-1|}$ is always true for x belongs to

(A) $\{0\}$ (B) $(1, \infty)$ (C) $(-1, 1)$ (D) $(-\infty, \infty)$

Solution : $\dfrac{x^2}{|x-1|} = \left|x + \dfrac{x}{x-1}\right|$

$\because |x| + \left|\dfrac{x}{x-1}\right| = \left|x + \dfrac{x}{x-1}\right|$ is true only if $\left(x.\dfrac{x}{x-1}\right) \ge 0 \Rightarrow x \in \{0\} \cup (1, \infty)$. **Ans (A,B)**

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Do yourself-7

Solve for x $\in \mathbb{R}$

1.	$\lvert x\rvert \geq 10$	**2.**	$\lvert x-5\rvert \leq 5$	**3.**	$\lvert 2-x\rvert \leq 4$
4.	$1 \leq \lvert 2x-3\rvert \leq 10$	**5.**	$\lvert x\rvert < \sqrt{2}$	**6.**	$\lvert x\rvert < 0$
7.	$\lvert 2x\rvert < 3$	**8.**	$\lvert 3x\rvert + 5 < 0$	**9.**	$\lvert 3x-5\rvert < 2$
10.	$\lvert 1-x\rvert \leq 2$	**11.**	$\lvert x\rvert > 2$	**12.**	$\lvert 2-7x\rvert < 8$
13.	$\lvert 2x\rvert + 4 > 0$	**14.**	$\lvert 2x\rvert - 7 > 0$		

10. MAXIMUM & MINIMUM VALUES OF QUADRATIC EXPRESSIONS : $y = ax^2 + bx + c$:

We know that $y = ax^2 + bx + c$ takes following form : $y = a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{(b^2-4ac)}{4a^2}\right]$,

which is a parabola. $\therefore$ vertex $= \left(\frac{-b}{2a}, \frac{-D}{4a}\right)$

When $a > 0$, y will take a minimum value at vertex ; $x = \frac{-b}{2a}$; $y_{min} = \frac{-D}{4a}$

When $a < 0$, y will take a maximum value at vertex; $x = \frac{-b}{2a}$; $y_{max} = \frac{-D}{4a}$.

If quadratic expression ax^2+bx+c is a perfect square, then $a > 0$ and $D = 0$

Illustration 33 : The value of the expression $x^2 + 2bx + c$ will be positive for all real x if -

(A) $b^2 - 4c > 0$ (B) $b^2 - 4c < 0$ (C) $c^2 < b$ (D) $b^2 < c$

Solution : As $a > 0$, so this expression will be positive if $D < 0$

so $4b^2 - 4c < 0$

$b^2 < c$ **Ans. (D)**

Illustration 34 : The minimum value of the expression $4x^2 + 2x + 1$ is -

(A) 1/4 (B) 1/2 (C) 3/4 (D) 1

Solution : Since $a = 4 > 0$

therefore its minimum value $= -\frac{D}{4a} = \frac{4(4)(1)-(2)^2}{4(4)} = \frac{16-4}{16} = \frac{12}{16} = \frac{3}{4}$ **Ans. (C)**

Illustration 35 : If $y = x^2 - 2x - 3$, then find the range of y when :

(i) $x \in R$ (ii) $x \in [0,3]$ (iii) $x \in [-2,0]$

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Solution : We know that minimum value of y will occur at

$$x = -\frac{b}{2a} = -\frac{(-2)}{2\times 1} = 1$$

$$y_{min} = -\frac{D}{4a} = \frac{-(4+3\times 4)}{4} = -4$$

(i) $x \in R$;

$y \in [-4,\infty)$ **Ans.**

(ii) $x \in [0, 3]$

$f(0) = -3,\ f(1) = -4,\quad f(3) = 0$

$\because\quad f(3) > f(0)$

$\therefore\quad$ y will take all the values from minimum to f(3).

$y \in [-4, 0]$ **Ans.**

(iii) $x \in [-2, 0]$

This interval does not contain the minimum value of y for $x \in R$.

y will take values from f(0) to f(–2)

$f(0) = -3$

$f(-2) = 5$

$y \in [-3, 5]$ **Ans.**

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Illustration 36 : If $ax^2 + bx + 10 = 0$ does not have real & distinct roots, find the minimum value of 5a–b.

Solution : Either $f(x) \ge 0\ \forall\ x \in R$ or $f(x) \le 0\ \forall\ x \in R$

$\because\quad f(0) = 10 > 0 \Rightarrow f(x) \ge 0\ \forall\ x \in R$

$\Rightarrow\quad f(-5) = 25a - 5b + 10 \ge 0$

$\Rightarrow\quad 5a - b \ge -2$ **Ans.**

Illustration-37 Find possible values (range) of x^2 if

(i) $x \in [1,3]$ (ii) $x \in [-2,0]$ (iii) $x \in [-1,2]$ (iv) $x \in [-2,1]$ (v) $x \in [-2,2]$

Solution : (i) $1 \le x \le 3 \Rightarrow 1 \le x^2 \le 9 \Rightarrow x^2 \in [1,9]$

(ii) $-2 \le x \le 0 \Rightarrow 4 \ge x^2 \ge 0 \Rightarrow x^2 \in [0,4]$

(iii) $-1 \le x < 0$ or $0 \le x \le 2$

$\Rightarrow 1 \ge x^2 > 0$ or $0 \le x^2 \le 4\ \Rightarrow 0 \le x^2 \le 4 \Rightarrow x^2 \in [0,4]$

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(iv) $-2 \le x < 0$ or $0 \le x \le 1$

$\Rightarrow 4 \ge x^2 > 0$ or $0 \le x^2 \le 1 \quad \Rightarrow x^2 \in [0,4]$

(v) $-2 \le x < 0$ or $0 \le x \le 2$

$\Rightarrow 4 \ge x^2 > 0$ or $0 \le x^2 \le 4 \quad \Rightarrow 0 \le x^2 \le 4 \Rightarrow x^2 \in [0,4]$

Illustration-38 : If $x \in \left[-\frac{3}{2}, 1\right]$ then find range of (i) $(2x-1)^2 - 3$ (ii) $4x^3 + 1$

Solution : (i) $[-3,13]$ (ii) $\left[-\frac{25}{2}, 5\right]$

Do yourself - 8

1. Find the minimum value of :

(a) $y = x^2 + 2x + 2$ (b) $y = 4x^2 - 16x + 15$

2. For following graphs of $y = ax^2 + bx + c$ with $a,b,c \in R$, comment on the sign of :

(i) a (ii) b (iii) c (iv) D (v) $\alpha + \beta$ (vi) $\alpha\beta$

(1)

(2)

(3)

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3. Given the roots of equation $ax^2 + bx + c = 0$ are real & distinct, where $a,b,c \in R^+$, then the vertex of the graph will lie in which quadrant.

4. Find the range of 'a' for which :

(a) $ax^2 + 3x + 4 > 0 \quad \forall x \in R$ (b) $ax^2 + 4x - 2 < 0 \quad \forall x \in R$

5. The trinomial $ax^2 + bx + c$ has no real roots, $a + b + c < 0$. Find the sign of the number c.

6. For what values of a do the graphs of the functions $y = 2ax + 1$ and $y = (a-6)x^2 - 2$ not intersect?

7. For what values of k is the inequality $x^2 - (k-3)x - k + 6 > 0$ valid for all real x?

8. For what integral k is the inequality $x^2 - 2(4k-1)x + 15k^2 - 2k - 7 > 0$ valid for any real x ?

9. For what values of a is the inequality $ax^2 + 2ax + 0.5 > 0$ valid for all real x ?

10. For what least integral k is the expression $(k-2)x^2 + 8x + k + 4 > 0$ for all values of x ?

11. Find all values of a for which the inequality $(a-3)x^2 - 2ax + 3a - 6 > 0$ is true for all $x \in R$.

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11. MAXIMUM & MINIMUM VALUES OF RATIONAL ALGEBRAIC EXPRESSIONS :

$$y = \frac{a_1x^2+b_1x+c_1}{a_2x^2+b_2x+c_2}, \frac{1}{ax^2+bx+c}, \frac{a_1x+b_1}{a_2x^2+b_2x+c_2}, \frac{a_1x^2+b_1x+c_1}{a_2x+b_2}:$$

Sometime we have to find range of expression of form $\frac{a_1x^2+b_1x+c_1}{a_2x^2+b_2x+c_2}$.

The following procedure is used :

Step 1 : Equate the given expression to y i.e. $y = \frac{a_1x^2+b_1x+c_1}{a_2x^2+b_2x+c_2}$

Step 2 : By cross multiplying and simplifying, obtain a quadratic equation in x.

$(a_1 - a_2y)x^2 + (b_1 - b_2y)x + (c_1 - c_2y) = 0$

Step 3 : Put Discriminant ≥ 0 and solve the inequality for possible set of values of y.

Illustration : 39 If $x \in [-2, 3]$, find all possible values (Range) of

(i) $\frac{1}{x+1}$ (ii) $\frac{x+2}{x+4}$ (iii) $\frac{2x}{x-1}$

Solution : (i) $-2 \leq x \leq 3 \Rightarrow -1 \leq x+1 \leq 4$

(For $x + 1 = 0$, $\frac{1}{x+1}$ is undefined)

so, $-1 \leq x+1 < 0$ or $0 < x+1 \leq 4$

$\Rightarrow -1 \geq \frac{1}{x+1} > -\infty$ or $\infty > x+1 \geq \frac{1}{4}$

So range of $\frac{1}{x+1}$ is $(-\infty, -1] \cup \left[\frac{1}{4}, \infty\right)$

(ii) $\frac{x+2}{x+4} = 1 - \frac{2}{x+4}$

Given $-2 \leq x \leq 3$

$\Rightarrow 2 \leq x+4 \leq 7 \Rightarrow \frac{1}{2} \geq \frac{1}{x+4} \geq \frac{1}{7}$

$\Rightarrow \frac{2}{2} \geq \frac{2}{x+4} \geq \frac{2}{7} \Rightarrow -1 \leq \frac{-2}{x+4} \leq \frac{-2}{7}$

$\Rightarrow 0 \leq 1 - \frac{2}{x+4} \leq \frac{5}{7} \Rightarrow$ Range $= \left[0, \frac{5}{7}\right]$

(iii) $\frac{2x}{x-1} = 2 + \frac{2}{x-1}$

for $-2 \leq x \leq 3 \Rightarrow -3 \leq x - 1 \leq 2$

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$\Rightarrow -3 \le x - 1 < 0$ or $0 < x - 1 \le 2$

$\Rightarrow \frac{-2}{3} \ge \frac{2}{x-1} > -\infty$ or $\infty > \frac{2}{x-1} \ge \frac{2}{2}$

$\Rightarrow \frac{4}{3} \ge 2 + \frac{2}{x-1}$ or $2 + \frac{2}{x-1} \ge 3$

so range $= \left(-\infty, \frac{4}{3}\right] \cup [3, \infty)$

Illustration 40 : For $x \in R$, find the set of values attainable by $\frac{x^2 - 3x + 4}{x^2 + 3x + 4}$.

Solution : Let $y = \frac{x^2 - 3x + 4}{x^2 + 3x + 4}$

$x^2(y - 1) + 3x(y + 1) + 4(y - 1) = 0$

Case- I : $y \ne 1$

For $y \ne 1$ above equation is a quadratic equation.

So for $x \in R$, $D \ge 0$

$\Rightarrow \quad 9(y + 1)^2 - 16(y - 1)^2 \ge 0 \quad \Rightarrow \quad 7y^2 - 50y + 7 \le 0$

$\Rightarrow \quad (7y - 1)(y - 7) \le 0 \quad \Rightarrow \quad y \in \left[\frac{1}{7}, 7\right] - \{1\}$

Case II : when $y = 1$

$\Rightarrow \quad 1 = \frac{x^2 - 3x + 4}{x^2 + 3x + 4}$

$\Rightarrow \quad x^2 + 3x + 4 = x^2 - 3x + 4$

$\Rightarrow \quad x = 0$

Hence $y = 1$ for real value of x.

so range of y is $\left[\frac{1}{7}, 7\right]$

Illustration 41 : Find the values of a for which the expression $\frac{ax^2 + 3x - 4}{3x - 4x^2 + a}$ assumes all real values for real values of x.

Solution : Let $y = \frac{ax^2 + 3x - 4}{3x - 4x^2 + a}$

$x^2(a + 4y) + 3(1 - y)x - (4 + ay) = 0$

If $x \in R$, $D \ge 0$

$\Rightarrow \quad 9(1 - y)^2 + 4(a + 4y)(4 + ay) \ge 0 \quad \Rightarrow \quad (9 + 16a)y^2 + (4a^2 + 46)y + (9 + 16a) \ge 0$

for all $y \in R$, $(9 + 16a) > 0$ & $D \le 0$

$\Rightarrow \quad (4a^2 + 46)^2 - 4(9 + 16a)(9 + 16a) \le 0 \quad \Rightarrow \quad 4(a^2 - 8a + 7)(a^2 + 8a + 16) \le 0$

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$\Rightarrow \quad a^2 - 8a + 7 \le 0 \quad \Rightarrow \quad 1 \le a \le 7$

$9 + 16a > 0$ & $1 \le a \le 7$

Taking intersection, $a \in [1, 7]$

Now, checking the boundary values of a

For $a = 1$

$$y = \frac{x^2 + 3x - 4}{3x - 4x^2 + 1} = -\frac{(x-1)(x+4)}{(x-1)(4x+1)}$$

$\because \quad x \ne 1 \Rightarrow y \ne -1$

$\Rightarrow \quad a = 1$ is not possible.

if $\quad a = 7$

$$y = \frac{7x^2 + 3x - 4}{3x - 4x^2 + 7} = \frac{(7x-4)(x+1)}{(7-4x)(x+1)} \qquad \because x \ne -1 \Rightarrow y \ne -1$$

So y will assume all real values for some real values of x.

So $\quad a \in (1,7)$

Illustration 42 : Find the values of m so that the inequality : $\left|\frac{x^2 + mx + 1}{x^2 + x + 1}\right| < 3$ holds for all $x \in R$.

Solution : We know that

$|a| < b \Rightarrow -b < a < b \qquad$ (for $b > 0$)

Hence $\left|\frac{x^2 + mx + 1}{x^2 + x + 1}\right| < 3$. $\quad \Rightarrow -3 < \frac{x^2 + mx + 1}{x^2 + x + 1} < 3$

Case I :

$$\frac{x^2 + mx + 1}{x^2 + x + 1} < 3$$

$$\Rightarrow \frac{(x^2 + mx + 1) - 3(x^2 + x + 1)}{x^2 + x + 1} < 0 \Rightarrow \frac{-2x^2 + (m-3)x - 2}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} < 0$$

Multiplying both sides by denominator, we get :

$\Rightarrow -2x^2 + (m - 3)x - 2 < 0$ (because denominator is always positive)

$\Rightarrow 2x^2 - (m - 3)x + 2 > 0$

A quadratic expression in x is always positive if coefficient of $x^2 > 0$ and $D < 0$.

$\Rightarrow (m - 3)^2 - 4(2)(2) < 0 \Rightarrow (m - 3)^2 - 4^2 < 0 \Rightarrow (m - 7)(m + 1) < 0 \Rightarrow m \in (-1, 7)$ ...(i)

Case II :

$$-3 < \frac{x^2 + mx + 1}{x^2 + x + 1}$$

$$\Rightarrow \quad \frac{(x^2 + mx + 1) + 3(x^2 + x + 1)}{x^2 + x + 1} > 0 \Rightarrow 4x^2 + (m + 3)x + 4 > 0$$

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For this to be true for all $x \in \mathbb{R}$, coefficient of $x^2 > 0$ and $D < 0$

$\Rightarrow$ $(m + 3)^2 - 4(4)(4) < 0$

$\Rightarrow$ $(m + 3 - 8)(m + 3 + 8) < 0$

$\Rightarrow$ $(m - 5)(m + 11) < 0$

$\Rightarrow$ $m \in (-11, 5)$ ...(ii)

We will combine (i) and (ii) because both must be satisfied.

$\Rightarrow$ The common solution is $m \in (-1, 5)$.

Illustration 43 : Find the values of m for which the expression : $\dfrac{2x^2 - 5x + 3}{4x - m}$ can take all real values for $x \in R - \left\{\dfrac{m}{4}\right\}$

Solution : Let $\dfrac{2x^2 - 5x + 3}{4x - m} = y \Rightarrow 2x^2 - (4y + 5)x + 3 + my = 0$

$\Rightarrow$ As $x \in R$, discriminant ≥ 0

$\Rightarrow$ $(4y + 5)^2 - 8(3 + my) \geq 0$

$\Rightarrow$ $16y^2 + (40 - 8m)y + 1 \geq 0$ ...(i)

A quadratric in y is non-negative for all values of y if coefficient of y^2 is positive and discriminant ≤ 0.

$\Rightarrow$ $(40 - 8m)^2 - 4(16)(1) \leq 0 \Rightarrow (5 - m)^2 - 1 \leq 0$

$\Rightarrow$ $(m - 5 - 1)(m - 5 + 1) \leq 0 \Rightarrow (m - 6)(m - 4) \leq 0$

$\Rightarrow$ $m \in [4, 6]$ but $m = 4,6$ will be rejected as shown in Illustration 41

So for the given expression to take all real values, m should take values : $m \in (4, 6)$.

Doyourself - 9:

1. If x is real prove that the expression $\dfrac{8x - 4}{x^2 + 2x - 1}$ where x is real cannot have values between 2 and 4, in its range.

2. Find the range of $\dfrac{x^2 + 2x + 1}{x^2 + 2x + 7}$, where x is real

3. If x is real, then prove that $\dfrac{x^2 - x + 1}{x^2 + x + 1}$ lies from $\dfrac{1}{3}$ to 3.

4. If x be real, prove that $\dfrac{x}{x^2 - 5x + 9}$ must lie between 1 and $-\dfrac{1}{11}$ (both inclusive).

5. If x be real, prove that $\dfrac{x^2 + 34x - 71}{x^2 + 2x - 7}$ can have no value between 5 and 9.

6. Find the greatest value of $\dfrac{x + 2}{2x^2 + 3x + 6}$ for real values of x.

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12. LOCATION OF ROOTS :

This article deals with an elegant approach of solving problems on quadratic equations when the roots are located / specified on the number line with variety of constraints :

Consider the quadratic equation **$ax^2 + bx + c = 0$ with $a > 0$ and let $f(x) = ax^2 + bx + c$**

Type-1 :

Both roots of the quadratic equation are greater than a specific number (say d).

The necessary and sufficient condition for this are :

(i) $D \geq 0$; (ii) $f(d) > 0$; (iii) $-\frac{b}{2a} > d$

Note : When both roots of the quadratic equation are less than a specific number d than the necessary and sufficient condition will be :

(i) $D \geq 0$; (ii) $f(d) > 0$; (iii) $-\frac{b}{2a} < d$

Type-2 :

Both roots lie on either side of a fixed number say (d). Alternatively one root is greater than 'd' and other root less than 'd' or 'd' lies between the roots of the given equation.

The necessary and sufficient condition for this are : $f(d) < 0$

Note : Consideration of discriminant is not needed.

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Type-3 :

Exactly one root lies in the interval (d, e).

The necessary and sufficient condition for this are :

$f(d) \cdot f(e) < 0$

Note : The extremes of the intervals found by given conditions give 'd' or 'e' as the root of the equation.

Hence in this case also check for end points.

Type-4 :

When both roots are confined between the number d and e (d < e).

The necessary and sufficient condition for this are :

(i) $D \geq 0$; (ii) $f(d) > 0$; (iii) $f(e) > 0$

(iv) $d < -\frac{b}{2a} < e$

Type-5 :

One root is greater than e and the other roots is less than d (d < e).

The necessary and sufficient condition for this are : $f(d) < 0$ and $f(e) < 0$

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Note : If a < 0 in the quadratic equation $ax^2 + bx + c = 0$ then we divide the whole equation by 'a'. Now assume $x^2 + \frac{b}{a}x + \frac{c}{a}$ as f(x). This makes the coefficient of x^2 positive and hence above cases are applicable.

Illustration 44 : Let the quadratic equation $ax^2 + bx + c = 0$, $a,b,c \in R$ has real roots then find the conditions such that

(a) roots are equal in magnitude but opposite in sign

(b) one root is zero other is – b/a

(c) roots are reciprocal to each other

(d) roots are of opposite signs

(e) both roots are negative.

(f) both roots are positive.

(g) Greater root in magnitude is negative.

(h) Greater root in magnitude is positive.

(i) one root is 1 and second root is c/a or (–b–a)/a.

Solution

(a) $b = 0, D > 0$ **(b)** $c = 0$ **(c)** $a = c, D \geq 0$

(d) $a.c < 0$ **(e)** $D \geq 0, \frac{b}{a} > 0, \frac{c}{a} > 0$ **(f)** $D \geq 0, \frac{b}{a} < 0, \frac{c}{a} > 0$

(g) $\frac{b}{a} > 0, D \geq 0$ **(h)** $\frac{b}{a} < 0, D \geq 0$ **(i)** $a + b + c = 0$ and $a \neq 0$

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Illustration 45 : Find the values of the parameter 'a' for which the roots of the quadratic equation $x^2 + 2(a - 1)x + a + 5 = 0$ are

(i) real and distinct (ii) equal

(iii) opposite in sign (iv) equal in magnitude but opposite in sign

(v) positive (vi) negative

(vii) greater than 3 (viii) smaller than 3

(ix) such that both the roots lie in the interval (1, 3)

Solution : Let $f(x) = x^2 + 2(a - 1)x + a + 5 = Ax^2 + Bx + C$ (say)

$\Rightarrow$ $A = 1, B = 2(a - 1), C = a + 5.$

Also $D = B^2 - 4AC = 4(a - 1)^2 - 4(a + 5) = 4(a + 1)(a - 4)$

(i) $D > 0$

$\Rightarrow$ $(a + 1)(a - 4) > 0 \Rightarrow$ $a \in (-\infty, -1) \cup (4, \infty)$.

(ii) $D = 0$

$\Rightarrow$ $(a + 1)(a - 4) = 0 \Rightarrow$ $a = -1, 4$.

(iii) This means that 0 lies between the roots of the given equation.

$\Rightarrow$ $f(0) < 0$ and $D > 0$ i.e. $a \in (-\infty, -1) \cup (4, \infty)$

$\Rightarrow$ $a + 5 < 0 \Rightarrow a < -5 \Rightarrow a \in (-\infty, -5)$.

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(iv) This means that the sum of the roots is zero

$\Rightarrow$ $-2(a-1)=0$ and $D>0$ i.e. $a \in (-\infty,-1)\cup(4,\infty)$ $\Rightarrow$ $a=1$

which does not belong to $(-\infty,-1)\cup(4,\infty)$

$\Rightarrow$ $a \in \phi$.

(v) This implies that both the roots are greater than zero

$\Rightarrow$ $-\frac{B}{A}>0,\ \frac{C}{A}>0, D\ge 0$ $\Rightarrow$ $-(a-1)>0,\ a+5>0,\ a\in(-\infty,-1]\cup[4,\infty)$

$\Rightarrow$ $a<1,\ -5<a,\ a\in(-\infty,-1]\cup[4,\infty)$ $\Rightarrow$ $a\in(-5,-1]$.

(vi) This implies that both the roots are less than zero

$\Rightarrow$ $-\frac{B}{A}<0,\ \frac{C}{A}>0, D\ge 0$ $\Rightarrow$ $-(a-1)<0,\ a+5>0,\ a\in(-\infty,-1]\cup[4,\infty)$

$\Rightarrow$ $a>1,\ a>-5,\ a\in(-\infty,-1]\cup[4,\infty)$ $\Rightarrow$ $a\in[4,\infty)$.

(vii) In this case

$-\frac{B}{2a}>3$, $A.f(3)>0$ and $D\ge 0$.

$\Rightarrow$ $-(a-1)>3,\ 7a+8>0$ and $a\in(-\infty,-1]\cup[4,\infty)$

$\Rightarrow$ $a<-2,\ a>-8/7$ and $a\in(-\infty,-1]\cup[4,\infty)$

Since no value of 'a' can satisfy these conditions simultaneously, there can be no value of a for which both the roots will be greater than 3.

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(viii) In this case

$-\frac{B}{2a}<3$, $A.f(3)>0$ and $D\ge 0$.

$\Rightarrow$ $a>-2,\ a>-8/7$ and $a\in(-\infty,-1]\cup[4,\infty)$ $\Rightarrow$ $a\in(-8/7,-1]\cup[4,\infty)$

(ix) In this case

$1<-\frac{B}{2A}<3$, $A.f(1)>0$, $A.f(3)>0$, $D\ge 0$.

$\Rightarrow$ $1<-1(a-1)<3,\ 3a+4>0,\ 7a+8>0,\ a\in(-\infty,-1]\cup[4,\infty)$

$\Rightarrow$ $-2<a<0,\ a>-4/3,\ a>-8/7,\ a\in(-\infty,-1]\cup[4,\infty)$ $\Rightarrow a\in\left(-\frac{8}{7},-1\right]$

Illustration 46 : Find value of k for which one root of equation $x^2-(k+1)x+k^2+k-8=0$ exceeds 2 & other is less than 2.

Solution : $4-2(k+1)+k^2+k-8<0$ $\Rightarrow$ $k^2-k-6<0$

$(k-3)(k+2)<0$ $\Rightarrow$ $-2<k<3$

Taking intersection, $k\in(-2,3)$.

Illustration 47 : Find all possible values of a for which exactly one root of $x^2-(a+1)x+2a=0$ lies in interval (0,3).

Solution : $f(0)\,.\,f(3)<0$

$\Rightarrow$ $2a(9-3(a+1)+2a)<0$ $\Rightarrow$ $2a(-a+6)<0$

$\Rightarrow$ $a(a-6)>0$ $\Rightarrow$ $a<0$ or $a>6$

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Checking the extremes.

If a = 0, $x^2 - x = 0$

$x = 0, 1$

$1 \in (0, 3)$

If a = 6, $x^2 - 7x + 12 = 0$

$x = 3, 4$ But $4 \notin (0, 3)$

Hence solution set is

$a \in (-\infty, 0] \cup (6, \infty)$

Illustration 48 : Let $x^2 - (m-3)x + m = 0$ $(m \in R)$ be a quadratic equation. Find the value of m for which the roots of the equation are

(i) real and distinct
(ii) equal
(iii) not real
(iv) opposite in sign
(v) equal in magnitude but opposite in sign
(vi) positive
(vii) negative
(viii) such that at least one is positive
(ix) one root is smaller than 2 and the other root is greater than 2
(x) both the roots are greater than 2
(xi) both the roots are smallers than 2
(xii) exactly one root lies in the interval (1, 2)
(xiii) both the roots lie in the interval (1, 2)
(xiv) such that at least one root lie in the interval (1, 2)
(xv) one root is greater than 2 and the other root is smaller than 1

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Solution :

Let $f(x) = x^2 - (m-3)x + m$

(i) Both the roots are real and distinct $\Rightarrow D > 0$

$\Rightarrow (m-3)^2 - 4m > 0 \Rightarrow m^2 - 10m + 9 > 0$

$\Rightarrow (m-1)(m-9) > 0 \Rightarrow m \in (-\infty, 1) \cup (9, \infty)$

(ii) Both the roots are equal $\Rightarrow D = 0 \Rightarrow m = 9$ or $m = 1$

(iii) Both the roots are imaginary $\Rightarrow D < 0$

$(m-1)(m-9) < 0 \Rightarrow m \in (1, 9)$

(iv) Both the roots are opposite in sign

$\Rightarrow f(0) < 0 \Rightarrow m < 0 \Rightarrow m \in (-\infty, 0)$

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(v) Roots are equal in magnitude but opposite in sign
$\Rightarrow$ sum of roots is zero as well as $D \geq 0$
$\Rightarrow m \in (-\infty, 1] \cup [9, \infty)$ and $m - 3 = 0$ i.e., $m = 3$
$\Rightarrow$ no such m exists

(vi) Both the roots are positive $\Rightarrow D \geq 0$,
both sum and product of roots are positive
$\Rightarrow m - 3 > 0$, $m > 0$ and $m \in (-\infty, 1] \cup [9, \infty)$
$\Rightarrow m \in [9, \infty)$

(vii) Both the roots are negative

$\Rightarrow D \geq 0$, sum of roots is negative but product of roots is positive.
$\Rightarrow m - 3 < 0$, $m > 0$, $m \in (-\infty, 1] \cup [9, \infty) \Rightarrow m \in (0, 1]$

(viii) at least one root is positive

$\Rightarrow$ either one root is positive or both the roots are positive
$\Rightarrow$ union of (iv) and (vi) with $m = 0$
(i.e., one root is zero and in this case other root becomes negative) $\Rightarrow m \in (-\infty, 0) \cup [9, \infty)$

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(ix) one root is smaller than 2 and

other root is greater than 2 $\Rightarrow$ 2 lies between the roots
$\Rightarrow f(2) < 0 \Rightarrow 4 - 2(m - 3) + m < 0 \Rightarrow m > 10$

(x) Both roots are greater than 2

$\Rightarrow f(2) > 0$, $D \geq 0$, $\dfrac{-b}{2a} > 2$

$\Rightarrow m < 10$ and $m \in (-\infty, 1] \cup [9, \infty)$ and $m - 3 > 4$
$\Rightarrow m \in [9, 10)$

(xi) Both the roots are smaller than 2

$\Rightarrow f(2) > 0$, $D \geq 0$, $\dfrac{-b}{2a} < 2 \Rightarrow m \in (-\infty, 1]$

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(xii) Exactly one root lies between (1, 2)

$\Rightarrow f(1).f(2) < 0 \Rightarrow 4(10 - m) < 0 \Rightarrow m \in (10, \infty)$

(xiii) Both roots lie in the interval (1, 2). Then

$f(1) > 0,\ 1 - m + 3 + m > 0 \Rightarrow 4 > 0$

$f(2) > 0,\ 4 - 2m + 6 + m > 0 \Rightarrow m < 10$

$1 < \frac{m-3}{2} < 2 \Rightarrow 5 < m < 7$

$D \geq 0,\ m \in (-\infty, 1] \cup [9, \infty) \Rightarrow$ no solution

(xiv) **Case I :** Exactly one root lies in (1, 2)

$f(1).f(2) < 0 \Rightarrow m > 10$

Case II : Both roots lie in (1, 2)

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From (xiii) $m \in \phi$

At least one root lies in (1, 2) $\Rightarrow m \in (10, \infty)$

(xv) For one root greater than 2 and other is smaller than 1, conditions are

$f(1) < 0$ ...(1)

and $f(2) < 0$ ...(2)

From (1), $f(1) < 0$, but $f(1) = 4$ which is not possible

Thus no such 'm' exists.

Illustration 49 : Find the value(s) of 'a' for which $ax^2 + (a - 3)x + 1 < 0$ for at least one positive x.

Solution : Let $f(x) = ax^2 + (a - 3)x + 1$

Case I :

If $a > 0$, then $f(x)$ will be negative only for those values of x, which lie between the roots. From the graph we can see that, $f(x)$ will be less than zero for at least one positive real x, when $f(x) = 0$ has distinct roots and at least one of these roots is positive real root.

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For this D > 0, i.e., $(a-3)^2 - 4a > 0$

$\Rightarrow$ $a < 1$ or $a > 9$ ...(1)

It is easy to see that for figure 1, a < 3 and figure 2 is not possible as $f(0) = 1$. Hence for case-I $0 < a < 1$.

Case II :

If a < 0, then $f(x)$ will certainly be negative for infintely many positive x. Thus $a \in (-\infty, 0)$.

Case III :

if a = 0, $f(x) = -3x + 1 \Rightarrow f(x) < 0\ \forall\ x > 1/3$

Hence the required set of values of 'a' is $(-\infty, 1)$

Illustration 50 : Find the values of 'a' for which $4^t - (a-4)2^t + \frac{9}{4}a < 0\ \forall\ t \in (1, 2)$.

Solution : Let $2^t = x$ and $f(x) = x^2 - (a-4)x + \frac{9}{4}a$

We want

$f(x) < 0\ \forall\ x \in (2^1, 2^2)$ i.e., $\forall\ x \in (2, 4)$

Since we want $f(x) < 0\ \forall\ x \in (2, 4)$, one of the roots of $f(x)$ should be less than or equal to 2 and the other must be greater than or equal to 4

i.e., $f(2) \le 0$ and $f(4) \le 0 \Rightarrow a \le -48$ and $a \ge 128/7$, which is not possible.

Hence no such 'a' exists.

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Illustration 51 : Find the value(s) of 'a' for which the inequality $\tan^2 x + (a+1)\tan x - (a-3) < 0$, is true for at least one $x \in \left(0, \frac{\pi}{2}\right)$.

Solution : The required condition will be satisfied if

(i) The quadratic expression (quadratic in tanx)

$f(x) = \tan^2 x + (a+1)\tan x - (a-3)$ has positive discriminant, and

(ii) At least one root of $f(x) = 0$ is positive, as $\tan x > 0$, $\forall\ x \in (0, \pi/2)$

For (i) Discriminant > 0 $\Rightarrow (a+1)^2 + 4(a-3) > 0$

$\Rightarrow$ $a > 2\sqrt{5} - 3$ or $a < -\left(2\sqrt{5} + 3\right)$

For (ii), we first find the condition, that both the roots of $t^2 + (a+1)t - (a-3) = 0$

(t = tanx) are non-positive for which

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Sum of roots < 0 product of roots ≥ 0

$\Rightarrow -(a+1) < 0$ and $-(a-3) \geq 0 \Rightarrow -1 < a \leq 3$

Condition (ii) will be fulfilled if $a \leq -1$ or $a > 3$ ...(2)

Required values of a is given by intersection of (1) and (2)

Hence $a \in \left(-\infty, -\left(2\sqrt{5}+3\right)\right) \cup (3, \infty)$

Do yourself - 10 :

1. If α, β are roots of $7x^2 + 9x - 2 = 0$, find their position with respect to following $(\alpha < \beta)$:
(a) –3 (b) 0 (c) 1

2. If $a > 1$, roots of the equation $(1-a)x^2 + 3ax - 1 = 0$ are -
(a) one positive one negative (b) both negative
(c) both positive (d) both non-real

3. Find the set of value of a for which the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are less than 3.

4. If α, β are the roots of $x^2 - 3x + a = 0$, $a \in R$ and $\alpha < 1 < \beta$, then find the values of a.

5. If α, β are roots of $4x^2 - 16x + \lambda = 0$, $\lambda \in R$ such that $1 < \alpha < 2$ and $2 < \beta < 3$, then find the range of λ.

6. For what values of a does the equation $(2-x)(x+1) = a$ posses real and positive roots ?

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13. GENERAL QUADRATIC EXPRESSION IN TWO VARIABLES :

$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factors if ;

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \text{OR} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Illustration 52 : If $x^2 + 2xy + 2x + my - 3$ have two linear factor then m is equal to -
(A) 6, 2 (B) –6, 2 (C) 6, –2 (D) –6, –2

Solution : Here a =1, h =1, b = 0, g = 1, f = m/2, c = –3

$$\text{So} \quad \Delta = 0 \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & m/2 \\ 1 & m/2 & -3 \end{vmatrix} = 0$$

$$\Rightarrow -\frac{m^2}{4} - (-3 - m/2) + m/2 = 0 \qquad \Rightarrow -\frac{m^2}{4} + m + 3 = 0$$

$$\Rightarrow m^2 - 4m - 12 = 0 \qquad \Rightarrow m = -2, 6$$

Ans. (C)

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Do yourself - 11 :

1. Find the value of k for which the expression $x^2 + 2xy + ky^2 + 2x + k = 0$ can be resolved into two linear factors.
2. For what values of m will the expression $y^2 + 2xy + 2x + my - 3$ be capable of resolution into two rational factors ?
3. Find the value of m which will make $2x^2 + mxy + 3y^2 - 5y - 2$ equivalent to the product of two linear factors.
4. Find the condition that the expression $\ell x^2 + mxy + ny^2$, $\ell' x^2 + m'xy + n'y^2$ may have a common linear factor.
5. Find the condition that the expression $ax^2 + 2hxy + by^2$, $a'x^2 + 2h'xy + b'y^2$ may be respectively divisible by factors of the form $y - mx, my + x$.
6. If x and y are two real quantities connected by the equation $9x^2 + 2xy + y^2 - 92x - 20y + 244 = 0$, then show that x will lie between 3 and 6. and y between 1 and 10.

14. THEORY OF EQUATIONS :

Let $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ are roots of the equation, $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$,

where $a_0, a_1, \ldots a_n$ are constants and $a_0 \neq 0$.

$f(x) = a_0(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots (x - \alpha_n)$

$\therefore \quad a_0x^n + a_1x^{n-1} + \ldots a_{n-1}x + a_n = a_0(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$

Comparing the coefficients of like powers of x, we get

$$\sum \alpha_i = -\frac{a_1}{a_0} = S_1 \text{ (say)}$$

or $$S_1 = -\frac{\text{coefficient of } x^{n-1}}{\text{coefficient of } x^n}$$

$$S_2 = \sum_{i \neq j} \alpha_i\alpha_j = (-1)^2 \frac{a_2}{a_0}$$

$$S_3 = \sum_{i \neq j \neq k} \alpha_i\alpha_j\alpha_k = (-1)^3 \frac{a_3}{a_0}$$

$\vdots$

$$S_n = \alpha_1\alpha_2 \ldots \alpha_n = (-1)^n \frac{a_n}{a_0} = (-1)^n \frac{\text{constant term}}{\text{coefficient of } x^n}$$

where S_k denotes the sum of the product of root taken k at a time.

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Quadratic equation : If α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$,

then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

Cubic equation : If α, β, γ are roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$, then

$\alpha + \beta + \gamma = -\frac{b}{a}$, $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$ and $\alpha\beta\gamma = -\frac{d}{a}$

Note :

(i) If α is a root of the equation $f(x) = 0$, then the polynomial $f(x)$ is exactly divisible by $(x-\alpha)$ or $(x-\alpha)$ is a factor of $f(x)$ and conversely.

(ii) Every equation of nth degree ($n \geq 1$) has exactly n root & if the equation has more than n roots, it is an identity.

(iii) If the coefficients of the equation $f(x) = 0$ are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root. i.e. **imaginary roots occur in conjugate pairs.**

(iv) If the coefficients in the equation are all rational & $\alpha + \sqrt{\beta}$ is one of its roots, then $\alpha - \sqrt{\beta}$ is also a root where $\alpha, \beta \in Q$ & β is not a perfect square.

(v) If there be any two real numbers 'a' & 'b' such that f(a) & f(b) are of opposite signs, then $f(x) = 0$ must have atleast one real root between 'a' and 'b'.

(vi) Every equation $f(x) = 0$ of degree odd and leading coefficient positive has atleast one real root of a sign opposite to that of its non zero constant term.

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Illustration 53 : If two roots are equal, find the roots of $4x^3 + 20x^2 - 23x + 6 = 0$.

Solution : Let roots be α, α and β

$\therefore \quad \alpha + \alpha + \beta = -\frac{20}{4} \quad \Rightarrow \quad 2\alpha + \beta = -5$ ..... (i)

$\therefore \quad \alpha \,.\, \alpha + \alpha\beta + \alpha\beta = -\frac{23}{4} \quad \Rightarrow \quad \alpha^2 + 2\alpha\beta = -\frac{23}{4}$ & $\alpha^2\beta = -\frac{6}{4}$

from equation (i)

$\alpha^2 + 2\alpha(-5 - 2\alpha) = -\frac{23}{4} \Rightarrow \alpha^2 - 10\alpha - 4\alpha^2 = -\frac{23}{4} \Rightarrow 12\alpha^2 + 40\alpha - 23 = 0$

$\therefore \quad \alpha = 1/2, -\frac{23}{6}$

when $\alpha = \frac{1}{2}$

$\alpha^2\beta = \frac{1}{4}(-5 - 1) = -\frac{3}{2}$

when $\alpha = -\frac{23}{6} \Rightarrow \alpha^2\beta = \frac{23 \times 23}{36}\left(-5 - 2x\left(-\frac{23}{6}\right)\right) \neq -\frac{3}{2} \quad \Rightarrow \quad \alpha = \frac{1}{2} \qquad \beta = -6$

Hence roots of equation $= \frac{1}{2}, \frac{1}{2}, -6$ **Ans.**

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Illustration 54 : If α, β, γ are the roots of $x^3 - px^2 + qx - r = 0$, find :

(i) $\sum \alpha^3$ (ii) $\alpha^2(\beta+\gamma)+\beta^2(\gamma+\alpha)+\gamma^2(\alpha+\beta)$

Solution : We know that $\alpha + \beta + \gamma = p$

$\alpha\beta + \beta\gamma + \gamma\alpha = q$

$\alpha\beta\gamma = r$

(i) $\alpha^3+\beta^3+\gamma^3 = 3\alpha\beta\gamma+(\alpha+\beta+\gamma)\{(\alpha+\beta+\gamma)^2-3(\alpha\beta+\beta\gamma+\gamma\alpha)\}$

$= 3r + p\{p^2 - 3q\} = 3r + p^3 - 3pq$

(ii) $\alpha^2(\beta+\gamma)+\beta^2(\alpha+\gamma)+\gamma^2(\alpha+\beta) = \alpha^2(p-\alpha)+\beta^2(p-\beta)+\gamma^2(p-\gamma)$

$= p(\alpha^2+\beta^2+\gamma^2) - 3r - p^3 + 3pq = p(p^2 - 2q) - 3r - p^3 + 3pq = pq - 3r$

Illustration 55 : If $b^2 < 2ac$ and $a, b, c, d \in \mathbb{R}$, then prove that $ax^3 + bx^2 + cx + d = 0$ has exactly one real root.

Solution : Let α, β, γ be the roots of $ax^3 + bx^2 + cx + d = 0$

Then $\alpha + \beta + \gamma = -\dfrac{b}{a}$

$\alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{c}{a}$

$\alpha\beta\gamma = \dfrac{-d}{a}$

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = \dfrac{b^2}{a^2} - \dfrac{2c}{a} = \dfrac{b^2 - 2ac}{a^2}$

$\Rightarrow$ $\alpha^2+\beta^2+\gamma^2 < 0$, which is not possible if all α, β, γ are real. So atleast one root is non-real, but complex roots occurs in pair. Hence given cubic equation has two non-real and one real roots.

Illustration 56 : If the sum of two roots of the equation $x^3 - px^2 + qx - r = 0$ is zero, then prove that $pq =$ r.

Solution : Let the roots of the given equation by α, β, γ such that $\alpha + \beta = 0$, Then,

$\alpha+\beta+\gamma = -\dfrac{(-p)}{1} \Rightarrow \alpha + \beta + \gamma = p \Rightarrow \gamma = p$ $[\because \alpha + \beta = 0]$

But γ is a root of the given equation. Therefore,

$\gamma^3 - p\gamma^2 + q\gamma - r = 0 \Rightarrow p^3 - p^3 + qp - r = 0 \Rightarrow pq = r$

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Illustration 57 : Find the condition that the roots of the equation $x^3 - px^2 + qx - r = 0$ may be in A.P.

Solution : Let the roots of the given equation be $a - d, a, a + d$.

Then,

$$(a - d) + a + (a + d) = \frac{-(-p)}{1} \Rightarrow a = p/3$$

Since a is a root of the given equation. Therefore,

$$a^3 - pa^2 + qa - r = 0 \Rightarrow \frac{p^3}{27} - \frac{p^3}{9} + \frac{qp}{3} - r = 0 \Rightarrow 2p^3 - 9pq + 27r = 0$$

This is the required condition.

Illustration 58 : If $x^2 + x + 1$ is a factor of $ax^3 + bx^2 + cx + d$, then find the real root of

$ax^3 + bx^2 + cx + \ d = 0$.

Solution : $x^2 + x + 1$ is a factor of $ax^3 + bx^2 + cx + d$. So roots of $x^2 + x + 1 = 0$ are also the roots of the equation $ax^3 + bx^2\ cx + d = 0$.

Let third roots of $ax^3 + bx^2 + cx + d$ is α.

Now we can write,

$ax^3 + bx^2 + cx + d = a(x^2 + x + 1)(x - \alpha)$

Comparing constant term on both sides, we get : $d = -a\alpha$

Hence, the real root of $ax^3 + bx^2 + cx + d = 0$ is $-\frac{d}{a}$.

Note : Roots of $x^2 + x + 1 = 0$ are complex

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Do yourself - 12 :

1. Let α, β be two of the roots of the equation $x^3 - px^2 + qx - r = 0$. If $\alpha + \beta = 0$, then show that $pq = r$
2. If two roots of $x^3 + 3x^2 - 9x + c = 0$ are equal, then find the value of c.
3. If α, β, γ be the roots of $ax^3 + bx^2 + cx + d = 0$, then find the value of

(a) $\sum \alpha^2$ (b) $\sum \frac{1}{\alpha}$ (c) $\sum \alpha^2(\beta + \gamma)$

Miscellaneous Illustrations :

Illustrations 59: If α, β are the roots of $x^2 + px + q = 0$, and γ, δ are the roots of $x^2 + rx + s = 0$, evaluate $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$ in terms of p, q, r and s. Deduce the condition that the equations have a common root.

Solution : α, β are the roots of $x^2 + px + q = 0$

$\therefore \quad \alpha + \beta = -p, \alpha\beta = q$ .....(1)

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and γ, δ are the roots of $x^2 + rx + s = 0$

$\therefore \quad \gamma + \delta = -r, \quad \gamma\delta = s$ .....(2)

Now, $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$

$= [\alpha^2 - \alpha(\gamma + \delta) + \gamma\delta][\beta^2 - \beta(\gamma + \delta) + \gamma\delta]$

$= (\alpha^2 + r\alpha + s)(\beta^2 + r\beta + s)$

$= \alpha^2\beta^2 + r\alpha\beta(\alpha + \beta) + r^2\alpha\beta + s(\alpha^2 + \beta^2) + sr(\alpha + \beta) + s^2$

$= \alpha^2\beta^2 + r\alpha\beta(\alpha + \beta) + r^2\alpha\beta + s((\alpha + \beta)^2 - 2\alpha\beta)) + sr(\alpha + \beta) + s^2$

$= q^2 - pqr + r^2q + s(p^2 - 2q) + sr(-p) + s^2$

$= (q - s)^2 - rpq + r^2q + sp^2 - prs$

$= (q - s)^2 - rq(p - r) + sp(p - r)$

$= (q - s)^2 + (p - r)(sp - rq)$

For a common root (Let $\alpha = \gamma$ or $\beta = \delta$) .....(3)

then $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta) = 0$ .....(4)

from (3) and (4), we get

$(q - s)^2 + (p - r)(sp - rq) = 0$

$\Rightarrow \quad (q - s)^2 = (p - r)(rq - sp)$, which is the required condition.

Illustrations 60 : If $(y^2 - 5y + 3)(x^2 + x + 1) < 2x$ for all $x \in R$, then find the interval in which y lies.

Solution : $(y^2 - 5y + 3)(x^2 + x + 1) < 2x, \ \forall\ x \in R$

$\Rightarrow \quad y^2 - 5y + 3 < \dfrac{2x}{x^2 + x + 1}$

Let $\dfrac{2x}{x^2 + x + 1} = P$

$\Rightarrow \quad px^2 + (p - 2)x + p = 0$

(1) Since x is real, $(p - 2)^2 - 4p^2 \geq 0$

$\Rightarrow \quad -2 \leq p \leq \dfrac{2}{3}$

(2) The minimum value of $2x/(x^2 + x + 1)$ is -2.

So, $y^2 - 5y + 3 < -2 \quad \Rightarrow \quad y^2 - 5y + 5 < 0$

$\Rightarrow \quad y \in \left(\dfrac{5 - \sqrt{5}}{2}, \dfrac{5 + \sqrt{5}}{2}\right)$

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EXERCISE (O-1)

1. If the roots of the equation $x^2 - 5x + 16 = 0$ are α, β and the roots of the equation $x^2 + px + q = 0$ are $(\alpha^2 + \beta^2)$ and $\frac{\alpha\beta}{2}$, then -

(A) p = 1 and q = 56 (B) p = 1 and q = –56

(C) p = –1 and q = 56 (D) p = –1 and q = –56

QE0001

2. If the roots of the equation $x^2 + px + q = 0$ are 8 and 2 and the roots of $x^2 + rx + s = 0$ are 3 and 3, then roots of $x^2 + px + s = 0$ are

(A) –1, –9 (B) 1, 9 (C) 8, 3 (D) None

QE0002

3. If α and β be the roots of the equation $(x - a)(x - b) = c$ and $c \neq 0$, then roots of the equation $(x - \alpha)(x - \beta) + c = 0$ are-

(A) a and c (B) b and c (C) a and b (D) a + b and b + c

QE0003

4. If $\alpha^2 = 5\alpha - 3$, $\beta^2 = 5\beta - 3$ then the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ is-

(A) 19/3 (B) 25/3 (C) –19/3 (D) none of these

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QE0004

5. The value of a for which one roots of the quadratic equation $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice as large as the other is

(A) – 2/3 (B) 1/3 (C) – 1/3 (D) 2/3

QE0005

6. The number of real solutions of the equation $x^2 - 3|x| + 2 = 0$, is-

(A) 4 (B) 1 (C) 3 (D) 2

QE0006

7. If (1 – p) is a root of quadratic equation $x^2 + px + (1 - p) = 0$ then its roots are-

(A) 0, – 1 (B) – 1, 1 (C) 0, 1 (D) –1, 2

QE0007

8. If the roots of the equation $x^2 - bx + c = 0$ be two consecutive integers, then $b^2 - 4c$ equals-

(A) 1 (B) 2 (C) 3 (D) –2

QE0008

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9. If $x^2 - A(x + 1) + C = 0$ has roots x_1 & x_2, then the value of $x_1^2 + x_2^2 + (2 + A)\, x_1x_2$, is-

(A) AC (B) $A^2 + AC$ (C) $A^2 - AC$ (D) $-AC$

QE0009

10. The quadratic $x^2 + ax + b + 1 = 0$ has roots which are positive integers, then $(a^2 + b^2)$ can be equal to-

(A) 50 (B) 17 (C) 29 (D) 53

QE0010

11. **Statement-1 :** If sum of the roots of quadratic equation $2x^2 + bx + c = 0$ is equal to sum of the squares of the roots, then $b^2 + 2b = 4c$.

Statement-2 : If one root of quadratic equation $x^2 + px + q = 0$, is $4+\sqrt{3}$, then other root is $4-\sqrt{3}$.

(A) Statement-1 is true and Statement-2 is false.
(B) Statement-1 and Statement-2 both are true.
(C) Statement-1 is false and Statement-2 is true.
(D) Statement-1 and Statement-2 both are false.

QE0011

12. If one root of the equation $x^2 + px + 12 = 0$ is 4, while the equation $x^2 + px + q = 0$ has equal roots, then the value of 'q' is-

(A) 3 (B) 12 (C) 49/4 (D) 4

QE0012

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13. The sum of the values of m for which the quadratic polynomial $P(x) = x^2 + (m + 5)x + (5m + 1)$ is a perfect square $(m \in R)$ is

(A) 3 (B) 7 (C) 8 (D) 10

QE0013

14. If equations $x^2 - 5x + 5 = 0$ and $x^3 + ax^2 + bx + 5 = 0$ have common root, then value of a + b $(a, b \in Q)$ is -

(A) 4 (B) –4 (C) 0 (D) can't find

QE0014

15. If the equation $ax^2 + bx + c = 0$ has distinct real roots, both negative, then-

(A) a,b,c must be of same sign
(B) a,b must be of opposite sign
(C) a,c must be of opposite sign
(D) a,b must be of same sign and opposite to sign of c

QE0015

16. If $P(x) = x^2 - (2 - p)x + p - 2$ assumes both positive and negative value, then the complete set of values of 'p' is-

(A) $(-\infty,2)$ (B) $(6,\infty)$ (C) $(2,6)$ (D) $(-\infty,2) \cup (6, \infty)$

QE0016

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17. If value of a for which the sum of the squares of the roots of the equation $x^2 - (a-2)x - a - 1 = 0$ assume the least value is-

(A) 2 (B) 3 (C) 0 (D) 1

QE0017

18. If $x^2 + 2ax + 10 - 3a > 0$ for all $x \in R$, then

(A) $-5 < a < 2$ (B) $a < -5$

(C) $a \geq 5$ (D) $2 < a < 5$

QE0018

19. If the expression $y = 8x - x^2 - 15$ is negative, then x lies in the interval-

(A) (3,5) (B) (5,50)

(C) $(3,\infty)$ (D) $(-\infty,3) \cup (5,\infty)$

QE0019

20. Let $f(x) = ax^2 + bx + 8$ $(a,b \in R)$ be a quadratic polynomial whose graph is symmetric about the line $x = 2$. If minimum value of $f(x)$ is 6, then the value of $2a - b$ is-

(A) 0 (B) 1 (C) 2 (D) 3

QE0020

21. Let $g(x) = x^2 - (b+1)x + (b-1)$, where b is a real parameter. The largest natural number b satisfying $g(x) > -2\ \forall\ x \in R$, is -

(A) 1 (B) 2 (C) 3 (D) 4

QE0021

22. $y = x^2 - 6x + 5$, $x \in [2,4]$, then-

(A) least value of y is -3 (B) least value of y is 3

(C) greatest value of y is 4 (D) greatest value of y is -3

QE0022

23. Range of the expression $\frac{16x^2 - 12x + 9}{16x^2 + 12x + 9} : (x \in R)$ is-

(A) $\left[\frac{1}{3}, 3\right]$ (B) $\left(-\infty, \frac{1}{3}\right]$ (C) $[3,\infty)$ (D) R

QE0023

24. If the roots of equation
$(4p - p^2 - 5)x^2 - (2p-1)x + 3p = 0$ lie on either side of unity then the number of integral values of p is-

(A) 4 (B) 2 (C) 3 (D) 1

QE0024

25. If exactly one root of the equation $2^k x^2 - 4^k x + 2^k - 1 = 0$ lies in $[0,1)$, then complete range of k is-

(A) $(-\infty,0]$ (B) $(-\infty,0)$ (C) $(0,\infty)$ (D) $[0,\infty)$

QE0025

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26. If $\forall$ p $\in$ R one root of the equation $x^2 + 2px + q^2 - p^2 - 6 = 0$ is less than 1 and other root is greater than 1, then range of q is -

(A) $(-\infty, -2)$ (B) $(-2, 2)$ (C) $(-\sqrt{5}, \sqrt{5})$ (D) $(2, \infty)$

QE0026

27. Let $f(x) = 2x^2 + px + 1$ is given. If $f(x)$ is negative integer for only one real value of x, then product of all possible values of p is -

(A) –3 (B) –16 (C) 5 (D) –7

QE0027

28. Let r_1, r_2, r_3 be roots of equation $x^3 - 2x^2 + 4x + 5074 = 0$, then the value of $(r_1 + 2)(r_2 + 2)(r_3 + 2)$ is

(A) 5050 (B) –5050 (C) –5066 (D) –5068

QE0028

29. Let a, b, c are roots of equation $x^3 + 8x + 1 = 0$, then the value of

$$\frac{bc}{(8b+1)(8c+1)} + \frac{ac}{(8a+1)(8c+1)} + \frac{ab}{(8a+1)(8b+1)}$$ is equal to

(A) 0 (B) –8 (C) –16 (D) 16

QE0029

30. Let $f(x) = x^3 + x + 1$ and P(x) be a cubic polynomial such that P(0) = –1 and the roots of P(x) = 0 are the squares of the roots of $f(x) = 0$, then value of P(9) is -

(A) 98 (B) 899 (C) 80 (D) 898

QE0030

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EXERCISE (O-2)

1. If α and β are the roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{2009} + \beta^{2009} =$

(A) –2 (B) –1 (C) 1 (D) 2

QE0031

2. The sum of all the real values of x satisfying the equation $2^{(x-1)(x^2+5x-50)} = 1$ is :

(A) 16 (B) –5 (C) –4 (D) 14

QE0032

3. The number of integral values of m for which the equation $(1 + m^2)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$ has no real root is :

(A) infinitely many (B) 2 (C) 3 (D) 1

QE0033

4. Sum of all distinct integral value(s) of α such that equation $x^2 - \alpha x + \alpha + 1 = 0$ has integral roots, is equal to-

(A) 2 (B) 4 (C) 3 (D) None of these

QE0034

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5. For the equation, $3x^2 + px + 3 = 0$, $p > 0$ if one of the roots is square of the other, then p is equal to -
(A) 1/3 (B) 1 (C) 3 (D) 2/3

QE0035

6. If one root of the equation $x^2 + px + q = 0$ is the square of the other, then
(A) $p^3 + q^2 - q(3p + 1) = 0$ (B) $p^3 + q^2 + q(1 + 3p) = 0$
(C) $p^3 + q^2 + q(3p - 1) = 0$ (D) $p^3 + q^2 + q(1 - 3p) = 0$

QE0036

7. Let α, β, be the roots of the equation $x^2 - px + r = 0$ and $\frac{\alpha}{2}$, 2β be the roots of the equation $x^2 - qx + r = 0$. Then the value of r is -
(A) $\frac{2}{9}(p-q)(2q-p)$ (B) $\frac{2}{9}(q-p)(2p-q)$ (C) $\frac{2}{9}(q-2p)(2q-p)$ (D) $\frac{2}{9}(2p-q)(2q-p)$

QE0037

8. Let $p, q \in \mathbb{Q}$. If $2-\sqrt{3}$ is a root of the quadratic equation, $x^2 + px + q = 0$, then :
(A) $q^2 + 4p + 14 = 0$ (B) $p^2 - 4q - 12 = 0$ (C) $q^2 - 4p - 16 = 0$ (D) $p^2 - 4q + 12 = 0$

QE0038

9. If α and β are the roots of the quadratic equation, $x^2 + x\sin\theta - 2\sin\theta = 0$, $\theta \in \left(0, \frac{\pi}{2}\right)$, then $\frac{\alpha^{12}+\beta^{12}}{\left(\alpha^{-12}+\beta^{-12}\right)(\alpha-\beta)^{24}}$ is equal to :

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(A) $\frac{2^6}{(\sin\theta+8)^{12}}$ (B) $\frac{2^{12}}{(\sin\theta-8)^6}$ (C) $\frac{2^{12}}{(\sin\theta-4)^{12}}$ (D) $\frac{2^{12}}{(\sin\theta+8)^{12}}$

QE0039

10. If f(x) is a quadratic expression such that $f(1) + f(2) = 0$, and -1 is a root of $f(x) = 0$, then the other root of $f(x) = 0$ is :-
(A) $-\frac{5}{8}$ (B) $\frac{8}{5}$ (C) $-\frac{8}{5}$ (D) $\frac{5}{8}$

QE0040

11. If the equations $x^2 + bx - 1 = 0$ and $x^2 + x + b = 0$ have a common root different from -1, then 'b' is equal to :-
(A) $\sqrt{2}$ (B) $\sqrt{3}\,i$ (C) 3i (D) 2

QE0041

12. Let p(x) be a quadratic polynomial such that $p(0) = 1$. If p(x) leaves remainder 4 when divided by $x - 1$ and it leaves remainder 6 when divided by $x + 1$; then :
(A) $p(2) = 19$ (B) $p(-2) = 19$ (C) $p(-2) = 11$ (D) $p(2) = 11$

QE0042

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13. If α,β are the roots of the equation $ax^2 + bx + c = 0$, then the roots of the equation $a(2x + 1)^2 + b(2x + 1)(x - 1) + c(x - 1)^2 = 0$ are

(A) $\frac{2\alpha+1}{\alpha-1}, \frac{2\beta+1}{\beta-1}$ (B) $\frac{2\alpha-1}{\alpha+1}, \frac{2\beta-1}{\beta+1}$ (C) $\frac{\alpha+1}{\alpha-2}, \frac{\beta+1}{\beta-2}$ (D) $\frac{2\alpha+3}{\alpha-1}, \frac{2\beta+3}{\beta-1}$

QE0043

14. Let a, b, c be the sides of a triangle. No two of them are equal and $\lambda \in R$. If the roots of the equation $x^2 + 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$ are real then -

(A) $\lambda < \frac{4}{3}$ (B) $\lambda > \frac{5}{3}$ (C) $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$ (D) $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$

QE0044

15. The equations $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$, where $b^2 - 4ac \neq 0$ have a common root, then $a^3 + b^3 + c^3$ is equal to $(a \neq 0)$

(A) 3abc (B) abc (C) 0 (D) 1

QE0045

16. If α & $\beta(\alpha < \beta)$, are the roots of the equation, $x^2 + bx + c = 0$, where $c < 0 < b$, then -

(A) $0 < \alpha < \beta$ (B) $\alpha < 0 < \beta < |\alpha|$ (C) $\alpha < \beta < 0$ (D) $\alpha < 0 < |\alpha| < \beta$

QE0046

17. If $b > a$, then the equation, $(x - a)(x - b) - 1 = 0$, has -

(A) both roots in $[a, b]$ (B) both roots in $(-\infty, a)$

(C) both roots in $[b, \infty)$ (D) one root in $(-\infty, a)$ & the other in $(b, +\infty)$

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QE0047

18. The graph of $y = ax^2 + bx + c$ is shown. Which of the following does **NOT** hold good?

(A) $ab^2c^3 > 0$

(B) $ab^3c^2 < 0$

(C) $ab^3c^5 > 0$

(D) $b^2 > 4ac$

QE0048

19. The sum of the solutions of the equation $\left|\sqrt{x} - 2\right| + \sqrt{x}\left(\sqrt{x} - 4\right) + 2 = 0, (x > 0)$ is equal to :

(A) 4 (B) 9 (C) 10 (D) 12

QE0049

20. The set of all real numbers x for which $x^2 - |x + 2| + x > 0$, is

(A) $(-\infty, -2) \cup (2, \infty)$ (B) $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

(C) $(-\infty, -1) \cup (1, \infty)$ (D) $(\sqrt{2}, \infty)$

QE0050

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21. If α,β are roots of equation $3x^2 - 9x - \lambda = 0$, $\lambda \in R$ such that $1 < \alpha < 3$ & $3 < \beta < 5$ then λ lies in
(A) $(-6,30)$ (B) $(-6,0)$ (C) $(0,30)$ (D) Null set

QE0051

22. The set of values of 'a' for which $f(x) = ax^2 + 2x(1-a) - 4$ is negative for exactly three integral values of x, is-
(A) $(0,2)$ (B) $(0,1]$ (C) $[1,2)$ (D) $[2,\infty)$

QE0052

23. If $\alpha,\beta,\gamma,\delta$ are the roots of equation $x^4 - \beta x + 3 = 0$, then the equation whose solutions are $\frac{\alpha+\beta+\gamma}{\delta^2}, \frac{\alpha+\beta+\delta}{\gamma^2}, \frac{\alpha+\gamma+\delta}{\beta^2}, \frac{\beta+\gamma+\delta}{\alpha^2}$ is-

(A) $3x^4 - \beta x^3 - 1 = 0$ (B) $3x^4 - \beta x^3 + 1 = 0$
(C) $3x^4 + \beta x^3 - 1 = 0$ (D) $3x^4 + \beta x^3 + 1 = 0$

QE0053

[MATCHING COLUMN TYPE]

24. Let $f(x) = \frac{x^2-6x+5}{x^2-5x+6}$

Match the expressions/statements in Column I with expressions/statements in Column II.

	Column-I		**Column-II**
(A)	If $-1 < x < 1$, then f(x) satisfies	(p)	$0 < f(x) < 1$
(B)	If $1 < x < 2$, then f(x) satisfies	(q)	$f(x) < 0$
(C)	If $3 < x < 5$, then f(x) satisfies	(r)	$f(x) > 0$
(D)	If $x > 5$, then f(x) satisfies	(s)	$f(x) < 1$

QE0054

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EXERCISE (S-1)

1. α, β are the roots of the equation $K(x^2 - x) + x + 5 = 0$. If K_1 & K_2 are the two values of K for which the roots α, β are connected by the relation $(\alpha/\beta) + (\beta/\alpha) = 4/5$. Find the value of $(K_1/K_2) + (K_2/K_1)$.

QE0055

2. Let the quadratic equation $x^2 + 3x - k = 0$ has roots a, b and $x^2 + 3x - 10 = 0$ has roots c, d such that modulus of difference of the roots of the first equation is equal to twice the modulus of the difference of the roots of the second equation. If the value of 'k' can be expressed as rational number in the lowest form as m/n then find the value of $(m + n)$.

QE0056

3. If α, β are the roots of $ax^2 + bx + c = 0$, $(a \neq 0)$ and $\alpha + \delta, \beta + \delta$ are the roots of $Ax^2 + Bx + C = 0$, $(A \neq 0)$ for some constant δ, then prove that, $\frac{b^2 - 4ac}{a^2} = \frac{B^2 - 4AC}{A^2}$.

QE0057

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4. If the quadratic equations, $x^2 + bx + c = 0$ and $bx^2 + cx + 1 = 0$ have a common root then prove that either $b + c + 1 = 0$ or $b^2 + c^2 + 1 = bc + b + c$.

QE0058

5. Find the value of *m* for which the quadratic equations $x^2 - 11x + m = 0$ and $x^2 - 14x + 2m = 0$ may have common root.

QE0059

6. Let a, b be arbitrary real numbers. Find the smallest natural number 'b' for which the equation $x^2 + 2(a + b)x + (a - b + 8) = 0$ has unequal real roots for all $a \in R$.

QE0060

7. Find the range of values of a, such that $f(x) = \dfrac{ax^2 + 2(a+1)x + 9a + 4}{x^2 - 8x + 32}$ is always negative.

QE0061

8. Consider the quadratic polynomial $f(x) = x^2 - 4ax + 5a^2 - 6a$

(a) Find the smallest positive integral value of 'a' for which f(x) positive for every real x.

(b) Find the largest distance between the roots of the equation $f(x) = 0$

(c) Find the set of values of 'a' for which range of f(x) is $[-8, \infty)$

QE0062

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9. We call 'p' a good number if the inequality $\dfrac{2x^2 + 2x + 3}{x^2 + x + 1} \le p$ is satisfied for any real x. Find the smallest integral good number.

QE0063

10. Number of integral values of 'a' for which $2x^2 - 2ax + a^2 - a - 6 = 0$ has roots of opposite sign is

QE0064

11. Find all values of p for which the roots of the equation $(p - 3)x^2 - 2px + 5p = 0$ are real and positive.

QE0065

12. Find all the values of the parameter 'a' for which both roots of the quadratic equation $x^2 - ax + 2 = 0$ belong to the interval (0, 3).

QE0066

13. At what values of 'a' do all the zeroes of the function $f(x) = (a - 2)x^2 + 2ax + a + 3$ lie on the interval $(-2, 1)$?

QE0067

14. If both the roots of the quadratic equation $x^2 - 2kx + k^2 + k - 5 = 0$ are less than 5, then find k.

QE0068

15. When $y^2 + my + 2$ is divided by $(y - 1)$ then the quotient is f (y) and the remainder is R_1. When $y^2 + my + 2$ is divided by $(y + 1)$ then quotient is g (y) and the remainder is R_2. If $R_1 = R_2$ then find the value of *m*.

QE0069

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EXERCISE (S-2)

1. Let P (x) = $x^2 + bx + c$, where b and c are integer. If P(x) is a factor of both $x^4 + 6x^2 + 25$ and $3x^4 + 4x^2 + 28x + 5$, find the value of P(1).

QE0070

2. Find the complete set of real values of 'a' for which both roots of the quadratic equation

$(a^2 - 6a + 5)x^2 - \sqrt{a^2 + 2a}\ x + (6a - a^2 - 8) = 0$ lie on either side of the origin.

QE0071

3. Let α, β and γ are the roots of the cubic $x^3 - 3x^2 + 1 = 0$. Find a cubic whose roots are $\frac{\alpha}{\alpha - 2}$, $\frac{\beta}{\beta - 2}$ and $\frac{\gamma}{\gamma - 2}$. Hence or otherwise find the value of $(\alpha - 2)(\beta - 2)(\gamma - 2)$. **QE0072**

4. Find the product of the real roots of the equation, $x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}$

QE0073

5. If the roots of $x^2 - ax + b = 0$ are real & differ by a quantity which is less than c (c > 0), prove that b lies between $(1/4)(a^2 - c^2)$ & $(1/4)a^2$.

QE0074

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6. Suppose a, b, c $\in \mathbb{Z}$ such that greatest common divisor of $x^2 + ax + b$ and $x^2 + bx + c$ is $(x + 1)$ and the least common multiple of $x^2 + ax + b$ and $x^2 + bx + c$ is $(x^3 - 4x^2 + x + 6)$. Find the value of $(a + b + c)$.

QE0075

7. If roots of the equation $(x - \alpha)(x - 4 + \beta) + (x - 2 + \alpha)(x + 2 - \beta) = 0$ are p and q then find the absolute value of the sum of the roots of the equations $2(x - p)(x - q) - (x - \alpha)(x - 4 + \beta) = 0$ and $2(x - p)(x - q) - (x - 2 + \alpha)(x + 2 - \beta) = 0$.

QE0076

8. Let $P(x) = 4x^2 + 6x + 4$ and $Q(y) = 4y^2 - 12y + 25$. Find the unique pair of real numbers (x, y) that satisfy $P(x) \cdot Q(y) = 28$.

QE0077

9. If largest real root of the equation $x^4 - 4x^3 + 5x^2 - 4x + 1 = 0$ can be expressed as $\frac{a + \sqrt{b}}{c}$ (where a, b, c are relatively prime natural numbers in lowest form), then the value of $a + b - c$ is

QE0078

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10. Find the values of 'a' for which $-3 < [(x^2+ax-2)/(x^2+x+1)] < 2$ is valid for all real x.

QE0079

11. Find values of k for which the equation $x^2 + (1-2k)|x| + (k^2-1) = 0$ has :

(a) No solution (b) One real solution

(c) Two real solutions (d) Three real solutions

(e) Four real solutions

QE0080

EXERCISE (JM)

1. Let for $a \neq a_1 \neq 0$, $f(x) = ax^2 + bx + c$, $g(x) = a_1x^2 + b_1x + c_1$ and $p(x) = f(x) - g(x)$. If $p(x) = 0$ only for $x = -1$ and $p(-2) = 2$, then the value of $p(2)$ is: **[AIEEE-2011]**

(1) 18 (2) 3 (3) 9 (4) 6

QE0081

2. Sachin and Rahul attempted to solve a quadratic equation. Sachin made a mistake in writing down the constant term and ended up in roots (4, 3). Rahul made a mistake in writing down coefficient of x to get roots (3, 2). The correct roots of equation are: **[AIEEE-2011]**

(1) –4, –3 (2) 6, 1 (3) 4, 3 (4) –6, –1

QE0082

3. If the equations $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$, a, b, c $\in$ R, have a common root, then a : b : c is : **[JEE-MAIN-2013]**

(1) 1 : 2 : 3 (2) 3 : 2 : 1 (3) 1 : 3 : 2 (4) 3 : 1 : 2

QE0083

4. Let α and β be the roots of equation $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$, for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is equal to : **[JEE-MAIN-2015]**

(1) 3 (2) – 3 (3) 6 (4) – 6

QE0084

5. The sum of all real values of x satisfying the equation $\left(x^2-5x+5\right)^{x^2+4x-60} = 1$ is :- **[JEE-MAIN-2016]**

(1) 5 (2) 3 (3) –4 (4) 6

QE0085

6. Let α and β be two roots of the equation $x^2 + 2x + 2 = 0$, then $\alpha^{15} + \beta^{15}$ is equal to : **[JEE(Main)-2019]**

(1) 512 (2) –512 (3) –256 (4) 256

QE0086

7. The number of all possible positive integral values of α for which the roots of the quadratic equation, $6x^2-11x+\alpha = 0$ are rational numbers is : **[JEE(Main)-2019]**

(1) 2 (2) 5 (3) 3 (4) 4

QE0087

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8. Consider the quadratic equation $(c-5)x^2-2cx+(c-4)=0$, $c\neq5$. Let S be the set of all integral values of c for which one root of the equation lies in the interval (0,2) and its other root lies in the interval (2,3). Then the number of elements in S is : **[JEE(Main)-2019]**

(1) 11 (2) 18 (3) 10 (4) 12

QE0088

9. If λ be the ratio of the roots of the quadratic equation in x, $3m^2x^2+m(m-4)x+2=0$, then the least value of m for which $\lambda+\frac{1}{\lambda}=1$, is : **[JEE(Main)-2019]**

(1) $2-\sqrt{3}$ (2) $4-3\sqrt{2}$ (3) $-2+\sqrt{2}$ (4) $4-2\sqrt{3}$

QE0089

10. Let α and β be the roots of the equation $x^2-x-1=0$. If $p_k=(\alpha)^k+(\beta)^k$, $k\geq1$, then which one of the following statements is not true ? **[JEE (Main)-2020]**

(1) $(p_1+p_2+p_3+p_4+p_5)=26$ (2) $p_5=11$

(3) $p_3=p_5-p_4$ (4) $p_5=p_2\cdot p_3$

QE0090

11. The least positive value of 'a' for which the equation $2x^2+(a-10)x+\frac{33}{2}=2a$ has real roots is

[JEE(Main)-2020]

QE0091

12. If $A=\{x\in\mathbf{R}:|x|<2\}$ and $B=\{x\in\mathbf{R}:|x-2|\geq3\}$; then :

(1) $A\cup B=\mathbf{R}-(2,5)$ (2) $A\cap B=(-2,-1)$

(3) $B-A=\mathbf{R}-(-2,5)$ (4) $A-B=[-1,2)$ **[JEE(Main)-2020]**

QE0092

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EXERCISE (JA)

1. The smallest value of k, for which both the roots of the equation, $x^2-8kx+16(k^2-k+1)=0$ are real, distinct and have values at least 4, is **[JEE 2009, 4 (–1)]**

QE0093

2. Let p and q be real numbers such that $p\neq0$, $p^3\neq q$ and $p^3\neq-q$. If α and β are nonzero complex numbers satisfying $\alpha+\beta=-p$ and $\alpha^3+\beta^3=q$, then a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots is **[JEE 2010, 3]**

(A) $(p^3+q)x^2-(p^3+2q)x+(p^3+q)=0$ (B) $(p^3+q)x^2-(p^3-2q)x+(p^3+q)=0$

(C) $(p^3-q)x^2-(5p^3-2q)x+(p^3-q)=0$ (D) $(p^3-q)x^2-(5p^3+2q)x+(p^3-q)=0$

QE0094

3. Let α and β be the roots of $x^2-6x-2=0$, with $\alpha>\beta$. If $a_n=\alpha^n-\beta^n$ for $n\geq1$, then the value of $\frac{a_{10}-2a_8}{2a_9}$ is **[JEE 2011]**

(A) 1 (B) 2 (C) 3 (D) 4

QE0095

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4. A value of b for which the equations

$$x^2 + bx - 1 = 0$$
$$x^2 + x + b = 0,$$

have one root in common is - **[JEE 2011]**

(A) $-\sqrt{2}$ (B) $-i\sqrt{3}$ (C) $i\sqrt{5}$ (D) $\sqrt{2}$

QE0096

5. Let S be the set of all non-zero numbers α such that the quadratic equation $\alpha x^2 - x + \alpha = 0$ has two distinct real roots x_1 and x_2 satisfying the inequality $|x_1 - x_2| < 1$. Which of the following intervals is(are) a subset(s) of S ? **[JEE 2015, 4M, –0M]**

(A) $\left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right)$ (B) $\left(-\frac{1}{\sqrt{5}}, 0\right)$ (C) $\left(0, \frac{1}{\sqrt{5}}\right)$ (D) $\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

QE0097

PARAGRAPH

Let p,q be integers and let α,β be the roots of the equation, $x^2 - x - 1 = 0$, where $\alpha \neq \beta$. For $n = 0,1,2,....$, let $a_n = p\alpha^n + q\beta^n$.

FACT : If a and b are rational numbers and $a + b\sqrt{5} = 0$, then a = 0 = b.

6. If $a_4 = 28$, then $p + 2q =$

(A) 14 (B) 7

(C) 12 (D) 21 **[JEE(Advanced)-2017, 3(–1)]**

QE0098

7. $a_{12} =$

(A) $2a_{11} + a_{10}$ (B) $a_{11} - a_{10}$

(C) $a_{11} + a_{10}$ (D) $a_{11} + 2a_{10}$ **[JEE(Advanced)-2017, 3(–1)]**

QE0098

8. Suppose a, b denote the distinct real roots of the quadratic polynomial $x^2 + 20x - 2020$ and suppose c, d denote the distinct complex roots of the quadratic polynomial $x^2 - 20x + 2020$. Then the value of $ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d)$ is **[JEE(Advanced)-2020]**

(A) 0 (B) 8000 (C) 8080 (D) 16000

QE0169

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ANSWERS

Do yourself-1

1 **(a)** $-1, -2$; **(b)** 4; **(c)** $1 \pm \sqrt{2}$;

2 $a, \frac{1}{a}$; **3.** $\frac{7}{3}$ **4.** $3, -\frac{1}{5}$

5. $28x^2 - 20x + 1 = 0$ **6.** $a_1 = -2, a_2 = 1$

7. $a_1 = 2, a_2 = 9/2$ **8.** $p = \pm 7$

9. $k = \pm 3\sqrt{5}$ **10.** $p = 0$

11. $a = 2$ **12.** $x_1^3 + x_2^3 = 3pq - p^3$

13. $x_1^3 + x_2^3 = \frac{a(a^2 - 18a + 9)}{27}$ **14.** $\frac{215}{27}$

Do yourself-2

1. **(a)** imaginary; **(b)** real & distinct; **(c)** real & coincident

3. $a = 20 \pm 6\sqrt{5}$ **4.** $k_1 = \frac{-22}{3}, k_2 = 2$ **5.** $a = 4$

6. $a = \pm 2$ **7.** $k = 3$ **9.** For all $m \in \left(\frac{1}{4}, +\infty\right)$

10. For all $c \in [2, 4)$ **11.** $k = 13$ **12.** For all $m \in \left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

13. For all $m \in \left(-\infty, -\frac{1}{7}\right) \cup (1, +\infty) \cup \{0\}$ **14.** $b = -4, c = 1$;

15. 7 **16.** -15

17. **(a)** $c = 0$; **(b)** $c = 1$; **(c)** $b \to$ negative, $c \to$ negative **18.** (ii), (iii) and (iv)

19. (A) $\frac{b^2 - 2ac}{c^2}$ (B) $\frac{bc^4(3ac - b^2)}{a^7}$ (C) $\frac{b^2(b^2 - 4ac)}{a^2c^2}$

20. $\frac{1}{x_1^3} + \frac{1}{x_2^3} = -\frac{27a^3 + 36a}{8}$ **21.** $a_1 = 3/2, a_2 = 3$ **22.** $a = -4$

23. $a_1 = -\frac{3}{2}, a_2 = 6.$ **24.** $\{2, 18\}$ for $a = 6$, $\left\{\frac{2}{19}, \frac{18}{19}\right\}$ for $a = -\frac{6}{19}$

25. $a_1 = -\frac{125}{8}, a_2 = \frac{27}{8}$ **26.** $a_1 = 1/2, a_2 = 1$ **27.** $p_1 = 0, q_1 = 0, p_2 = 1, q_2 = -2$

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Do yourself-3

1. **(a)** $c^2y^2 + y(2ac - b^2) + a^2 = 0$; **(b)** $acx^2 - bx + 1 = 0$;
(c) $acx^2 + (a + c)bx + (a + c)^2 = 0$

2. $x^2 - 2(p^2 - 2q)x + p^2(p^2 - 4q) = 0$

3. $x^2 - p(p^4 - 5p^2q + 5q^2)x + p^2q^2(p^2 - 4q)(p^2 - q) = 0$

Do yourself-4

1. 1 **3.** $b = \frac{9}{2}, c = 5$ **4.** $c = 0, c = 6$

6. $a = -2$ **7.** $m = -2$

Do yourself-5

1. $x \in R$ **2.** $x \in R$ **3.** $x \in \left(-\infty, \frac{1}{2}\right) \cup (3, \infty)$

4. $x \in (-6, -3) \cup \left(\frac{1}{2}, 2\right) - \{1\} \cup (9, \infty)$ **5.** $[3,7]$ **6.** $x \in \mathbb{R}$

7. $x \in (-\infty, -3) \cup (-2,3)$ **8.** $\left(\frac{1}{2}, 3\right)$ **9.** $(2, 3)$

10 $(-\infty, -2) \cup \left(-2, -\frac{1}{2}\right) \cup (1, \infty)$

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Do yourself-6

1. $x \in R - (0,1]$ **2.** $(1/2, 2]$ **3.** $(3, +\infty)$

4. $[1, 2) \cup (2, +\infty)$ **5.** $[20/9, 4) \cup (5, +\infty)$ **6.** $\left(\frac{1}{2}, \frac{5}{2}\right]$

7. $[0, 2]$ **8.** $x \in [-18,-2)$

Do yourself-7

1. $(-\infty,-10] \cup [10,\infty)$ **2.** $x \in [0,10]$ **3.** $x \in [-2,6]$

4. $\left[-\frac{7}{2}, 1\right] \cup \left[2, \frac{13}{2}\right]$ **5.** $\left(-\sqrt{2}, \sqrt{2}\right)$ **6.** ϕ

7. $\left(-\frac{3}{2}, \frac{3}{2}\right)$ **8.** ϕ **9.** $\left(1, \frac{7}{3}\right)$

10. $[-1,3]$ **11.** $(-\infty,-2) \cup (2,\infty)$ **12.** $\left(-\frac{6}{7}, \frac{10}{7}\right)$

13. $x \in R$ **14.** $\left(-\infty, -\frac{7}{2}\right) \cup \left(\frac{7}{2}, \infty\right)$

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Do yourself-8

1. **(a)** 1 **(b)** –1

2. (1) (i) $a < 0$ (ii) $b < 0$ (iii) $c < 0$ (iv) $D > 0$ (v) $\alpha + \beta < 0$ (vi) $\alpha\beta > 0$
(2) (i) $a < 0$ (ii) $b > 0$ (iii) $c = 0$ (iv) $D > 0$ (v) $\alpha + \beta > 0$ (vi) $\alpha\beta = 0$
(3) (i) $a < 0$ (ii) $b = 0$ (iii) $c = 0$ (iv) $D = 0$ (v) $\alpha + \beta = 0$ (vi) $\alpha\beta = 0$

3. Third quadrant **4.** (a) $a > 9/16$ (b) $a < -2$ **5.** $c < 0$

6. For all $a \in (-6, 3)$ **7.** $(-3, 5)$ **8.** $k = 3$

9. $\left[0, \frac{1}{2}\right)$ **10.** $k = 5$ **11.** For all $a \in (6, +\infty)$

Do yourself-9

1. $(-\infty, 2] \cup [4, \infty)$ **2.** $[0,1)$ **3.** $\left[\frac{1}{3}, 3\right]$ **4.** $\left[-\frac{1}{11}, 1\right]$

5. $(-\infty, 5] \cup [9, \infty)$ **6.** $\frac{1}{3}$

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Do yourself-10

1. (a) α and β are greater than –3

(b) '0' lies in between the roots α and β.

(c) α and β are less than 1.

2. C **3.** $a < 2$ **4.** $a < 2$ **5.** $12 < \lambda < 16$

6. For all $a \in \left(2, \frac{9}{4}\right]$

Do yourself-11

1. 0, 2 **2.** –2 **3.** ±7

4. $(n\ell' - n'\ell)^2 = (mn' - m'n)(\ell m' - \ell' m)$

5. $(aa' - bb')^2 = -4(ah' + b'h)(a'h + bh')$

Do yourself-12

2. –27, 5

3. **(a)** $\frac{1}{a^2}(b^2 - 2ac)$ **(b)** $-\frac{c}{d}$, **(c)** $\frac{1}{a^2}(3ad - bc)$

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EXERCISE (O-1)

1. D	**2.** B	**3.** C	**4.** A	**5.** D	**6.** A	**7.** A	**8.** A
9. A	**10.** A	**11.** A	**12.** C	**13.** D	**14.** B	**15.** A	**16.** D
7. D	**18.** A	**19.** D	**20.** D	**21.** B	**22.** D	**23.** A	**24.** B
25. D	**26.** B	**27.** B	**28.** B	**29.** C	**30.** B		

EXERCISE (O-2)

1. C	**2.** C	**3.** A	**4.** B	**5.** C	**6.** D	**7.** D	**8.** B
9. D	**10.** B	**11.** B	**12.** B	**13.** C	**14.** A	**15.** A	**16.** B
17. D	**18.** C	**19.** C	**20.** B	**21.** D	**22.** C	**23.** D	

24. (A) $\rightarrow$ (p, r, s); (B) $\rightarrow$ (q, s); (C) $\rightarrow$ (q, s); (D) $\rightarrow$ (p, r, s)

EXERCISE (S-1)

1. 254 **2.** 191 **5.** 0 or 24 **6.** 5 **7.** $a \in \left(-\infty, -\frac{1}{2}\right)$

8. **(a)** 7, **(b)** 6, **(c)** 2 or 4 **9.** 4 **10.** 4 **11.** For all $p \in \left[3, \frac{15}{4}\right]$

12. $2\sqrt{2} \le a < \frac{11}{3}$ **13.** $\left(-\infty, -\frac{1}{4}\right) \cup \{2\} \cup (5, 6]$ **14.** $(-\infty, 4)$ **15.** 0

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EXERCISE (S-2)

1. $P(1) = 4$ **2.** $(-\infty, -2] \cup [0, 1) \cup (2, 4) \cup (5, \infty)$

3. $3y^3 - 9y^2 - 3y + 1 = 0$; $(\alpha - 2)(\beta - 2)(\gamma - 2) = 3$

4. 20 **6.** -6 **7.** 4 **8.** $\left(-\frac{3}{4}, \frac{3}{2}\right)$ **9.** 6

11. (a) $k \in (-\infty, -1) \cup \left(\frac{5}{4}, \infty\right)$ (b) $k = -1$

(c) $k \in (-1, 1) \cup \{5/4\}$ (d) $k = 1$ (e) $k \in (1, 5/4)$

EXERCISE (JM)

1. 1	**2.** 2	**3.** 1	**4.** 1	**5.** 2	**6.** 3	**7.** 3	**8.** 1
9. 2	**10.** 4	**11.** 8.00	**12.** 3				

EXERCISE (JA)

1. 2	**2.** B	**3.** C	**4.** B	**5.** A,D	**6.** C	**7.** C	**8.** D

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Chapter 04

SEQUENCE & SERIES

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JEE (Main/Advanced) Syllabus

JEE (Main) Syllabus :

Arithmetic, geometric and harmonic progressions, arithmetic, geometric and harmonic means, sums of finite arithmetic and geometric progressions, infinite geometric series, sums of squares and cubes of the first n natural numbers.

JEE (Advanced) Syllabus :

Arithmetic and Geometric progressions, insertion of arithmetic, geometric means between two given numbers. Relation between A.M. and G.M. Sum upto n terms of special series: Sn, Sn2, Sn3. Arithmetico – Geometric progression.

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Important Notes

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SEQUENCE & SERIES

1. DEFINITION :

Sequence :

A succession of terms a_1, a_2, a_3, a_4..... formed according to some rule or law.

Examples are : 1, 4, 9, 16, 25

$-1, 1, -1, 1,$.....

$\frac{x}{1!}, \frac{x^2}{2!}, \frac{x^3}{3!}, \frac{x^4}{4!}, ...$

A finite sequence has a finite (i.e. limited) number of terms, as in the first example above. An infinite sequence has an unlimited number of terms, i.e. there is no last term, as in the second and third examples.

Series :

The indicated sum of the terms of a sequence. In the case of a finite sequence $a_1, a_2, a_3,$....., a_n the corresponding series is $a_1 + a_2 + a_3 + ... + a_n = \sum_{k=1}^{n} a_k$. This series has a finite or limited number of terms and is called a finite series.

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2. ARITHMETIC PROGRESSION (A.P.) :

A.P. is a sequence whose terms differ by a fixed number. This fixed number is called the common difference. If a is the first term & d the common difference, then A.P. can be written as

$a, a + d, a + 2d, ..., a + (n - 1) d, ...$

(a) n^{th} term of AP $T_n = a + (n-1)d$, where $d = t_n - t_{n-1}$

(b) The sum of the first n terms : $S_n = \frac{n}{2}[a + \ell] = \frac{n}{2}[2a + (n-1)d]$

where ℓ is n^{th} term.

Note :

(i) n^{th} term of an A.P. is of the form An + B i.e. a linear expression in 'n', in such a case the coefficient of n is the common difference of the A.P. i.e. A.

(ii) Sum of first 'n' terms of an A.P. is of the form $An^2 + Bn$ i.e. a quadratic expression in 'n', in such case the common difference is twice the coefficient of n^2. i.e. 2A

(iii) Also n^{th} term $T_n = S_n - S_{n-1}$

Illustration 1 : If (x + 1), 3x and (4x + 2) are first three terms of an A.P. then its 5th term is -

(A) 14 (B) 19 (C) 24 (D) 28

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Solution : (x + 1), 3x, (4x + 2) are in AP

$\Rightarrow 3x - (x + 1) = (4x + 2) - 3x \qquad \Rightarrow x = 3$

$\therefore a = 4, d = 9 - 4 = 5 \qquad \Rightarrow T_5 = 4 + (4)5 = 24$ **Ans. (C)**

Illustration 2 : The sum of first four terms of an A.P. is 56 and the sum of it's last four terms is 112. If its first term is 11 then find the number of terms in the A.P.

Solution : $a + a + d + a + 2d + a + 3d = 56$

$4a + 6d = 56$

$44 + 6d = 56 \qquad$ (as a = 11)

$6d = 12 \qquad$ hence d = 2

Let total number of terms = n

Now sum of last four terms.

$a + (n - 1)d + a + (n - 2)d + a + (n - 3)d + a + (n - 4)d = 112$

$\Rightarrow 4a + (4n - 10)d = 112 \qquad \Rightarrow 44 + (4n - 10)2 = 112$

$\Rightarrow 4n - 10 = 34$

$\Rightarrow n = 11$ **Ans.**

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Illustration 3 : The sum of first n terms of two A.Ps. are in ratio $\frac{7n+1}{4n+27}$. Find the ratio of their 11th terms.

Solution : Let a_1 and a_2 be the first terms and d_1 and d_2 be the common differences of two A.P.s respectively then

$$\frac{\frac{n}{2}[2a_1 + (n-1)d_1]}{\frac{n}{2}[2a_2 + (n-1)d_2]} = \frac{7n+1}{4n+27} \Rightarrow \frac{a_1 + \left(\frac{n-1}{2}\right)d_1}{a_2 + \left(\frac{n-1}{2}\right)d_2} = \frac{7n+1}{4n+27}$$

For ratio of 11th terms

$$\frac{n-1}{2} = 10 \quad \Rightarrow \quad n = 21$$

so ratio of 11th terms is $\frac{7(21)+1}{4(21)+27} = \frac{148}{111} = \frac{4}{3}$ **Ans.**

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Do yourself - 1 :

1. Write down the sequence whose n^{th} terms is : (a) $\frac{2^n}{n}$ (b) $\frac{3+(-1)^n}{3^n}$

2. For an A.P, show that $t_m + t_{2n+m} = 2t_{m+n}$

3. If the sum of p terms of an A.P. is q and the sum of its q terms is p, then find the sum of its (p + q) term.

Find the

4. sum 49, 44, 39.... to 17 terms.

5. sum $\frac{3}{4}, \frac{2}{3}, \frac{7}{12}$,..... to 19 terms.

6. sum 1.3,–3.1, –7.5,.... to 10 terms

7. sum a – 3b, 2a – 5b, 3a – 7b,.... to 40 terms.

3. PROPERTIES OF A.P. :

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(a) If each term of an A.P. is increased, decreased, multiplied or divided by the some nonzero number, then the resulting sequence is also an A.P.

(b) Three numbers in A.P. : a – d, a, a + d

Four numbers in A.P. : a – 3d, a – d, a + d, a + 3d

Five numbers in A.P. : a – 2d, a – d, a, a + d, a + 2d

Six numbers in A.P. : a – 5d, a – 3d, a – d, a + d, a + 3d, a + 5d etc.

(c) The common difference can be zero, positive or negative.

(d) k^{th} term from the last = $(n - k + 1)^{th}$ term from the beginning (If total number of terms = n).

(e) The sum of the two terms of an AP equidistant from the beginning & end is constant and equal to the sum of first & last terms. $\Rightarrow T_k + T_{n-k+1}$ = constant = $a + \ell$.

(f) Any term of an AP (except the first) is equal to half the sum of terms which are equidistant from it. $a_n = (1/2)(a_{n-k} + a_{n+k}),\ k < n$

For $k = 1,\ a_n = (1/2)(a_{n-1} + a_{n+1})$; For $k = 2,\ a_n = (1/2)(a_{n-2} + a_{n+2})$ and so on.

(g) If a, b, c are in AP, then 2b = a + c.

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Illustration 4 : Four numbers are in A.P. If their sum is 20 and the sum of their squares is 120, then the middle terms are -

(A) 2, 4 (B) 4, 6 (C) 6, 8 (D) 8, 10

Solution : Let the numbers are a – 3d, a – d, a + d, a + 3d

given, $a - 3d + a - d + a + d + a + 3d = 20$ $\Rightarrow 4a = 20 \Rightarrow a = 5$

and $(a - 3d)^2 + (a - d)^2 + (a + d)^2 + (a + 3d)^2 = 120$ $\Rightarrow 4a^2 + 20d^2 = 120$

$\Rightarrow 4 \times 5^2 + 20d^2 = 120$ $\Rightarrow d^2 = 1 \Rightarrow d = \pm 1$

Hence numbers are 2, 4, 6, 8 or 8, 6, 4, 2 **Ans. (B)**

Illustration 5 : If $a_1, a_2, a_3, \ldots\ldots\ldots, a_n$ are in A.P. where $a_i > 0$ for all i, show that :

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots\ldots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{(n-1)}{\sqrt{a_1}+\sqrt{a_n}}$$

Solution : $$\text{L.H.S.} = \frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots\ldots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$$

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$$= \frac{1}{\sqrt{a_2}+\sqrt{a_1}}+\frac{1}{\sqrt{a_3}+\sqrt{a_2}}+\ldots\ldots+\frac{1}{\sqrt{a_n}+\sqrt{a_{n-1}}}$$

$$= \frac{\sqrt{a_2}-\sqrt{a_1}}{(a_2-a_1)}+\frac{\sqrt{a_3}-\sqrt{a_2}}{(a_3-a_2)}+\ldots\ldots+\frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{a_n-a_{n-1}}$$

Let 'd' is the common difference of this A.P.

then $a_2 - a_1 = a_3 - a_2 = \ldots\ldots = a_n - a_{n-1} = d$

Now L.H.S.

$$= \frac{1}{d}\left\{\sqrt{a_2}-\sqrt{a_1}+\sqrt{a_3}-\sqrt{a_2}+\ldots\ldots+\sqrt{a_{n-1}}-\sqrt{a_{n-2}}+\sqrt{a_n}-\sqrt{a_{n-1}}\right\} = \frac{1}{d}\left\{\sqrt{a_n}-\sqrt{a_1}\right\}$$

$$= \frac{a_n-a_1}{d\left(\sqrt{a_n}+\sqrt{a_1}\right)} = \frac{a_1+(n-1)d-a_1}{d\left(\sqrt{a_n}+\sqrt{a_1}\right)} = \frac{1}{d}\frac{(n-1)d}{(\sqrt{a_n}+\sqrt{a_1})} = \frac{n-1}{\sqrt{a_n}+\sqrt{a_1}} = \text{R.H.S.}$$

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Do yourself - 2 :

1. Find the sum of first 24 terms of the A.P. a_1, a_2, a_3....., if it is know that $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$.
2. Find the number of terms common to the two A.P.'s 3, 7, 11, ... 407 and 2, 9, 16, ..., 709
3. In an A.P. the first term is 2, the last term is 29, the sum is 155; find the common difference.
4. The sum of 15 terms of an A.P. is 600, and the common difference is 5; find the first term.
5. The third term of an A.P. is 18, and the seventh term is 30; find the sum of 17 terms.
6. The sum of three numbers in A.P. is 27, and their product is 504; find them.
7. The sum of three numbers in A.P. is 12, and the sum of their cubes is 408; find them.
8. Find the sum of 15 terms of the series whose n^{th} term is $4n + 1$.
9. In an AP if the sum of 7 terms is 49, and the sum of 17 terms is 289, find the sum of n terms.
10. The sum of four integers in A.P. is 24, and their product is 945; find them.
11. If the sum of n terms of an A.P. is $2n + 3n^2$, find the rth term.
12. If the sum of m terms of an A.P. is to the sum of n terms as m^2 to n^2, show that the mth term is to the nth term as $2m - 1$ is to $2n - 1$.

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4. GEOMETRIC PROGRESSION (G.P.) :

G.P. is a sequence of non zero numbers each of the succeeding term is equal to the preceeding term multiplied by a constant. Thus in a GP the ratio of successive terms is constant. This constant factor is called the COMMON RATIO of the sequence & is obtained by dividing any term by the immediately previous term. Therefore a, ar, ar^2, ar^3, ar^4 , ... is a GP with 'a' as the first term & 'r' as common ratio.

(a) n^{th} term ; $T_n = a\,r^{n-1}$

(b) Sum of the first n terms; $S_n = \dfrac{a(r^n - 1)}{r - 1}$, if $r \neq 1$

(c) Sum of infinite G.P. , $S_\infty = \dfrac{a}{1 - r}$; $0 < |r| < 1$

5. PROPERTIES OF GP :

(a) If each term of a G.P. be multiplied or divided by the some non-zero quantity, then the resulting sequence is also a G.P.

(b) Three consecutive terms of a GP : a/r, a, ar ;

Four consecutive terms of a GP : a/r^3, a/r, ar, ar^3 & so on.

(c) If a, b, c are in G.P. then $b^2 = ac$.

(d) If in a G.P, the product of two terms which are equidistant from the first and the last term, is constant and is equal to the product of first and last term. $\Rightarrow T_k . T_{n-k+1}$ = constant = $a.\ell$

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(e) If each term of a G.P. be raised to the same power, then resulting sequence is also a G.P.

(f) In a G.P., $T_r^2 = T_{r-k}.\ T_{r+k}$, $k < r$, $r \neq 1$

(g) If the terms of a given G.P. are chosen at regular intervals, then the new sequence is also a G.P.

(h) If $a_1, a_2, a_3.....a_n$ is a G.P. of positive terms, then $\log a_1, \log a_2,.....\log a_n$ is an A.P. and vice-versa.

(i) If $a_1, a_2, a_3.....$ and $b_1, b_2, b_3.....$ are two G.P.'s then $a_1b_1, a_2b_2, a_3b_3.....$ & $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}.....$ is also in G.P.

Illustration 6 : If a, b, c, d and p are distinct real numbers such that

$\left(a^2+b^2+c^2\right)p^2-2p\left(ab+bc+cd\right)+\left(b^2+c^2+d^2\right)\le 0$ then a, b, c, d are in

(A) A.P. (B) G.P. (C) H.P. (D) None of these

Solution : Here, the given condition $\left(a^2+b^2+c^2\right)p^2-2p\left(ab+bc+ca\right)+b^2+c^2+d^2\le 0$

$\Rightarrow (ap-b)^2+(bp-c)^2+(cp-d)^2\le 0$

$\because$ a square can not be negative

$\therefore\ ap-b=0, bp-c=0, cp-d=0 \Rightarrow p=\frac{b}{a}=\frac{c}{b}=\frac{d}{c} \Rightarrow$ a, b, c, d are in G.P.

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Ans. (B)

Illustration 7 : If a, b, c are in G.P., then the equations $ax^2+2bx+c=0$ and $dx^2+2ex+f=0$ have a common root if $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in -

(A) A.P. (B) G.P. (C) H.P. (D) None of these

Solution : a, b, c are in G.P $\Rightarrow b^2=ac$

Now the equation $ax^2+2bx+c=0$ can be rewritten as $ax^2+2\sqrt{ac}x+c=0$

$\Rightarrow \left(\sqrt{a}x+\sqrt{c}\right)^2=0 \Rightarrow x=-\sqrt{\frac{c}{a}}, -\sqrt{\frac{c}{a}}$

If the two given equations have a common root, then this root must be $-\sqrt{\frac{c}{a}}$.

Thus $d\frac{c}{a}-2e\sqrt{\frac{c}{a}}+f=0 \Rightarrow \frac{d}{a}+\frac{f}{c}=\frac{2e}{c}\sqrt{\frac{c}{a}}=\frac{2e}{\sqrt{ac}}=\frac{2e}{b} \Rightarrow \frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P.

Ans. (A)

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Illustration 8 : A number consists of three digits which are in G.P. the sum of the right hand and left hand digits exceeds twice the middle digit by 1 and the sum of the left hand and middle digits is two third of the sum of the middle and right hand digits. Find the numbers.

Solution : Let the three digits be a, ar and ar^2 then number is

$100a + 10ar + ar^2$ ....(i)

Given, $a + ar^2 = 2ar + 1$

or $a(r^2 - 2r + 1) = 1$

or $a(r - 1)^2 = 1$ ....(ii)

Also given $a + ar = \frac{2}{3}(ar + ar^2)$

$\Rightarrow$ $3 + 3r = 2r + 2r^2$ $\Rightarrow$ $2r^2 - r - 3 = 0$ $\Rightarrow$ $(r + 1)(2r - 3) = 0$

$\therefore$ $r = -1, 3/2$

for $r = -1$, $a = \frac{1}{(r-1)^2} = \frac{1}{4} \notin I$ $\therefore r \neq -1$

for $r = 3/2$, $a = \frac{1}{\left(\frac{3}{2}-1\right)^2} = 4$ {from (ii)}

From (i), number is $400 + 10.4.\frac{3}{2} + 4.\frac{9}{4} = 469$ **Ans.**

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Illustration 9 : Find the value of $0.32\overline{58}$

Solution : Let $R = 0.32\overline{58}$ $\Rightarrow$ $R = 0.32585858....$ ..... (i)

Here number of figures which are not recurring is 2 and number of figures which are recurring is also 2.

then $100\,R = 32.585858.....$ .....(ii)

and $10000\,R = 3258.5858.....$ .....(iii)

Subtracting (ii) from (iii), we get

$9900\,R = 3226$ $\Rightarrow$ $R = \frac{1613}{4950}$

Aliter Method : $R = .32 + .0058 + .0058 + .000058 + ...$

$$= .32 + \frac{58}{10^4}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + ...\infty\right) = .32 + \frac{58}{10^4}\left(\frac{1}{1 - \frac{1}{100}}\right)$$

$$= \frac{32}{100} + \frac{58}{9900} = \frac{3168 + 58}{9900} = \frac{3226}{9900} = \frac{1613}{4950}$$

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Do yourself - 3 :

1. Find a three digit number whose consecutive digits form a G.P. If we subtract 792 from this number, we get a number consisting of the same digits written in the reverse order. Now, if we increase the second digit of the required number by 2, then the resulting digits will form an A.P.

2. If the third term of G.P. is 4, then find the product of first five terms.

3. If a, b, c are respectively the pth, q^{th} and r^{th} terms of the given G.P., then show that $(q-r)\log a+(r-p)\log b+(p-q)\log c=0$, where a, b, c > 0.

4. Find three numbers in G.P., whose sum is 52 and the sum of whose products in pairs is 624.

5. The rational number which equals the number $2.\overline{357}$ with recurring decimal is -

(A) $\frac{2357}{999}$ (B) $\frac{2379}{997}$ (C) $\frac{785}{333}$ (D) $\frac{2355}{1001}$

Find the

6. sum $\frac{3}{4},1\frac{1}{2},3,....$ to 8 terms.

7. sum 1,5,25,... to p terms.

8. sum $1,\sqrt{3},3,...$ to 12 terms.

9. sum $\frac{8}{5},-1,\frac{5}{8},...$ upto infinity

10. sum .45,.015,.0005,...upto infinity

11. sum $3,\sqrt{3},1,...$ upto infinity

12. The sum of the first 6 terms of a G.P. is 9 times the sum of the first 3 terms; find the common ratio.

13. The fifth term of a G.P. is 81, and the second term is 24; find the series.

14. The sum of three numbers in G.P. is 38, and their product is 1728; find them.

15. The continued product of three numbers in G.P. is 216, and the sum of the product of them in pairs is 156; find the numbers.

16. The sum of an infinite number of terms of a G.P. is 4, and the sum of their cubes is 192; find the series.

17. The first two terms of an infinite G.P. are together equal to 5, and every term is 3 times the sum of all the terms that follow it; find the seires.

18. Sum of following series : x + a, x^2 + 2a, x^3 + 3a... to n terms.

6. HARMONIC PROGRESSION (H.P.) :

A sequence is said to be in H.P. if the reciprocal of its terms are in AP.

If the sequence $a_1, a_2, a_3, ..., a_n$ is an HP then $1/a_1, 1/a_2, ..., 1/a_n$ is an AP . Here we do not have the formula for the sum of the n terms of an HP. The general form of a harmonic progression is

$$\frac{1}{a},\frac{1}{a+d},\frac{1}{a+2d},.....\frac{1}{a+(n-1)d}$$

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Note : No term of any H.P. can be zero.

(i) If a, b, c are in HP, then $b = \frac{2ac}{a+c}$ or $\frac{a}{c} = \frac{a-b}{b-c}$

Illustration 10 : The sum of three numbers are in H.P. is 37 and the sum of their reciprocals is 1/4. Find the numbers.

Solution : Three numbers are in H.P. can be taken as

$$\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$$

then $\frac{1}{a-d} + \frac{1}{a} + \frac{1}{a+d} = 37$ .....(i)

and $a - d + a + a + d = \frac{1}{4}$ $\Rightarrow$ $a = \frac{1}{12}$

from (i), $\frac{12}{1-12d} + 12 + \frac{12}{1+12d} = 37$ $\Rightarrow$ $\frac{12}{1-12d} + \frac{12}{1+12d} = 25$

$\Rightarrow$ $\frac{24}{1-144d^2} = 25$ $\Rightarrow$ $1 - 144d^2 = \frac{24}{25}$ $\Rightarrow$ $d^2 = \frac{1}{25 \times 144}$

$\therefore$ $d = \pm\frac{1}{60}$

$\therefore$ a – d, a, a + d are $\frac{1}{15}, \frac{1}{12}, \frac{1}{10}$ or $\frac{1}{10}, \frac{1}{12}, \frac{1}{15}$

Hence, three numbers in H.P. are 15, 12, 10 or 10, 12, 15 **Ans.**

Illustration 11 : Suppose a is a fixed real number such that $\frac{a-x}{px} = \frac{a-y}{qy} = \frac{a-z}{rz}$

If p, q, r are in A.P., then prove that x, y, z are in H.P.

Solution : $\because$ p, q, r are in A.P.

$\therefore$ $q - p = r - q$ ..... (i)

$\Rightarrow$ $p - q = q - r = k$ (let)

given $\frac{a-x}{px} = \frac{a-y}{qy} = \frac{a-z}{rz}$ $\Rightarrow$ $\frac{\frac{a}{x}-1}{p} = \frac{\frac{a}{y}-1}{q} = \frac{\frac{a}{z}-1}{r}$

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$$\Rightarrow \quad \frac{\left(\frac{a}{x}-1\right)-\left(\frac{a}{y}-1\right)}{p-q}=\frac{\left(\frac{a}{y}-1\right)-\left(\frac{a}{z}-1\right)}{q-r} \quad \text{(by law of proportion)}$$

$$\Rightarrow \quad \frac{\frac{a}{x}-\frac{a}{y}}{k}=\frac{\frac{a}{y}-\frac{a}{z}}{k} \quad \{\text{from (i)}\}$$

$$\Rightarrow \quad a\left(\frac{1}{x}-\frac{1}{y}\right)=a\left(\frac{1}{y}-\frac{1}{z}\right) \qquad \Rightarrow \quad \frac{1}{x}-\frac{1}{y}=\frac{1}{y}-\frac{1}{z}$$

$$\therefore \quad \frac{2}{y}=\frac{1}{x}+\frac{1}{z}$$

$$\therefore \quad \frac{1}{x},\frac{1}{y},\frac{1}{z} \text{ are in A.P.}$$

Hence x, y, z are in H.P.

Do yourself - 4 :

1. If the 7th term of a H.P. is 8 and the 8th term is 7. Then find the 28th term.

2. In a H.P., if 5th term is 6 and 3rd term is 10. Find the 2nd term.

3. If the pth, q^{th} and r^{th} terms of a H.P. are a,b,c respectively, then prove that $\frac{q-r}{a}+\frac{r-p}{b}+\frac{p-q}{c}=0$

4. Find the fourth term in each of the following series :

(a) $2, 2\frac{1}{2}, 3\frac{1}{3}, \ldots$ (b) $1\frac{1}{2}, 2, 3\ldots$ (c) $3\frac{3}{5}, 6, 18, \ldots$

5. If a,b,c be in H.P., show that a : a – b = a + c : a – c.

6. If the mth term of a H.P. be equal to n, and the nth term be equal to m, prove that the $(m + n)^{th}$ term is equal to $\frac{mn}{m+n}$.

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7. MEANS

(a) ARITHMETIC MEAN :

If three terms are in A.P. then the middle term is called the A.M. between the other two, so if a, b, c are in A.P., b is A.M. of a & c. So A.M. of a and c = $\frac{a+c}{2}=b$.

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n-ARITHMETIC MEANS BETWEEN TWO NUMBERS :

If a,b be any two given numbers & a, A_1, A_2, ..., A_n, b are in AP, then A_1, A_2,.....A_n are the 'n' A.M's between a & b then. $A_1 = a + d$, $A_2 = a + 2d$,....., $A_n = a + nd$ or $b - d$, where

$$d = \frac{b-a}{n+1}$$

$$\Rightarrow A_1 = a + \frac{b-a}{n+1}, \quad A_2 = a + \frac{2(b-a)}{n+1}, ...$$

Note: Sum of n A.M's inserted between a & b is equal to n times the single A.M. between a & b

i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single A.M. between a & b.

(b) GEOMETRIC MEAN :

If a, b, c are in G.P., then b is the G.M. between a & c, $b^2 = ac$. So G.M. of a and c $= \sqrt{ac} = b$

n-GEOMETRIC MEANS BETWEEN TWO NUMBERS :

If a, b are two given positive numbers & a, G_1, G_2, ..., G_n, b are in G.P. Then G_1, G_2, G_3 ,.....G_n are 'n' G.Ms between a & b. where $b = ar^{n+1} \Rightarrow r = (b/a)^{1/n+1}$

$G_1 = a(b/a)^{1/n+1}$, $G_2 = a(b/a)^{2/n+1}$ ....., $G_n = a(b/a)^{n/n+1}$

$= ar$, $= ar^2$, ... $= ar^n = b/r$

Note : The product of n G.Ms between a & b is equal to n^{th} power of the single G.M. between a & b i.e. $\prod_{r=1}^{n} G_r = (G)^n$ where G is the single G.M. between a & b

(c) HARMONIC MEAN :

If a, b, c are in H.P., then b is H.M. between a & c. So H.M. of a and c $= \frac{2ac}{a+c} = b$.

Insertion of 'n' HM's between a and b :

a, H_1, H_2, H_3 ,....., H_n, b $\rightarrow$ H.P

$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, ... \frac{1}{H_n}, \frac{1}{b} \rightarrow$ A.P.

$$\frac{1}{b} = \frac{1}{a} + (n+1)D \Rightarrow \quad D = \frac{\frac{1}{b} - \frac{1}{a}}{n+1}$$

$$\frac{1}{H_n} = \frac{1}{a} + n\left(\frac{\frac{1}{b} - \frac{1}{a}}{n+1}\right)$$

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Important note :

(i) If A, G, H, are respectively A.M., G.M., H.M. between two positive number a & b then
(a) $G^2 = AH$ (A, G, H constitute a GP) (b) $A \geq G \geq H$ (c) $A = G = H \Leftrightarrow a = b$

(ii) Let $a_1, a_2, \ldots\ldots, a_n$ be n positive real numbers, then we define their arithmetic mean (A), geometric mean (G) and harmonic mean (H) as $A = \dfrac{a_1 + a_2 + \ldots + a_n}{n}$

$G = (a_1 a_2 \ldots\ldots a_n)^{1/n}$ and $H = \dfrac{n}{\left(\dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3} + \ldots \dfrac{1}{a_n}\right)}$

It can be shown that $A \geq G \geq H$. Moreover equality holds at either place if and only if $a_1 = a_2 = \ldots\ldots = a_n$

Illustration 12 : If $2x^3 + ax^2 + bx + 4 = 0$ (a and b are positive real numbers) has 3 real roots, then prove that $a + b \geq 6(2^{1/3} + 4^{1/3})$.

Solution : Let α, β, γ be the roots of $2x^3 + ax^2 + bx + 4 = 0$. Given that all the coefficients are positive, so all the roots will be negative.

Let $\alpha_1 = -\alpha, \alpha_2 = -\beta, \alpha_3 = -\gamma \quad \Rightarrow \quad \alpha_1 + \alpha_2 + \alpha_3 = \dfrac{a}{2}$

$\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = \dfrac{b}{2}$

$\alpha_1\alpha_2\alpha_3 = 2$

Applying AM $\geq$ GM, we have

$\dfrac{\alpha_1 + \alpha_2 + \alpha_3}{3} \geq (\alpha_1\alpha_2\alpha_3)^{1/3} \quad \Rightarrow \quad a \geq 6 \times 2^{1/3}$

Also $\dfrac{\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3}{3} > (\alpha_1\alpha_2\alpha_3)^{2/3} \Rightarrow \quad b \geq 6 \times 4^{1/3}$

Therefore $a + b \geq 6(2^{1/3} + 4^{1/3})$.

Illustration 13 : If $a_i > 0 \ \forall \ i \in N$ such that $\prod_{i=1}^{n} a_i = 1$,

then prove that $(1 + a_1)(1 + a_2)(1 + a_3)\ldots\ldots(1 + a_n) \geq 2^n$

Solution : Using A.M. $\geq$ G.M.

$1 + a_1 \geq 2\sqrt{a_1}$

$1 + a_2 \geq 2\sqrt{a_2}$

$\vdots$

$1 + a_n \geq 2\sqrt{a_n} \quad \Rightarrow \quad (1 + a_1)(1 + a_2)\ldots\ldots\ldots(1 + a_n) \geq 2^n(a_1a_2a_3\ldots\ldots a_n)^{1/2}$

As $a_1a_2a_3\ldots\ldots a_n = 1$

Hence $(1 + a_1)(1 + a_2)\ldots\ldots\ldots\ldots(1 + a_n) \geq 2^n$.

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Illustration 14 : If a, b, x, y are positive natural numbers such that $\frac{1}{x}+\frac{1}{y}=1$ then prove that $\frac{a^x}{x}+\frac{b^y}{y}\ge ab$.

Solution : Consider the positive numbers a^x, a^x,.....y times and b^y, b^y,.....x times

For all these numbers,

$$AM = \frac{\{a^x+a^x+.....y\text{ time}\}+\{b^y+b^y+.....x\text{ times}\}}{x+y} = \frac{ya^x+xa^y}{(x+y)}$$

$$GM = \left\{\left(a^x.a^x.....y\text{ times}\right)\left(b^y.b^y.....x\text{ times}\right)\right\}^{\frac{1}{(x+y)}} = \left[\left(a^{xy}\right).\left(b^{xy}\right)\right]^{\frac{1}{(x+y)}} = (ab)^{\frac{xy}{(x+y)}}$$

As $\frac{1}{x}+\frac{1}{y}=1$, $\frac{x+y}{xy}=1$, i.e, $x+y=xy$

So using AM $\ge$ GM $\frac{ya^x+xa^y}{x+y}\ge(ab)^{\frac{xy}{x+y}}$

$$\therefore \quad \frac{ya^x+xa^y}{xy}\ge ab \quad \text{or} \quad \frac{a^x}{x}+\frac{a^y}{y}\ge ab.$$

Do yourself - 5 :

1. If $\frac{a^n+b^n}{a^{n-1}+b^{n-1}}$ is the G.M. between a & b then find the value of 'n'.

2. If b is the harmonic mean between a and c, then prove that $\frac{1}{b-a}+\frac{1}{b-c}=\frac{1}{a}+\frac{1}{c}$.

3. Insert 19 arithmetic means between $\frac{1}{4}$ and $-9\frac{3}{4}$.

4. Insert 17 arithmetic means between $3\frac{1}{2}$ and $-41\frac{1}{2}$

5. Insert 3 geometric means between $2\frac{1}{4}$ and $\frac{4}{9}$.

6. Insert 5 geometric means between $3\frac{5}{9}$ and $40\frac{1}{2}$.

7. Insert two harmonic means between 5 and 11.

8. Insert four harmonic means between $\frac{2}{3}$ and $\frac{2}{13}$.

9. If between any two quantities there be inserted two arithmetic means A_1, A_2; two geometric means G_1, G_2 ; and two harmonic means H_1, H_2;
show that $G_1G_2 : H_1H_2 = A_1 + A_2 : H_1 + H_2$.

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8. ARITHMETICO - GEOMETRIC SERIES :

A series, each term of which is formed by multiplying the corresponding term of an A.P. & G.P. is called the Arithmetico-Geometric Series , e.g. $1+3x+5x^2+7x^3+.....$

Here 1, 3, 5, ... are in A.P. & 1, x, x^2, x^3 ..... are in G.P.

(a) SUM OF N TERMS OF AN ARITHMETICO-GEOMETRIC SERIES :

Let $S_n = a+(a+d)r+(a+2d)r^2+.....+[a+(n-1)d]r^{n-1}$

then $S_n = \frac{a}{1-r}+\frac{dr(1-r^{n-1})}{(1-r)^2}-\frac{[a+(n-1)d]\ r^n}{1-r}$, $r \neq 1$

(b) SUM TO INFINITY :

If $0<|r|<1$ & $n\to\infty$, then $\lim_{n\to\infty} r^n = 0, S_\infty = \frac{a}{1-r}+\frac{dr}{(1-r)^2}$

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Illustration 15 : Find the sum of series $4-9x+16x^2-25x^3+36x^4-49x^5+.....\infty$.

Solution : Let $S = 4-9x+16x^2-25x^3+36x^4-49x^5+.....\infty$

$-Sx = -4x+9x^2-16x^3+25x^4-36x^5+.....\infty$

On subtraction, we get

$S(1+x) = 4-5x+7x^2-9x^3+11x^4-13x^5+.....\infty$

$-S(1+x)x = -4x+5x^2-7x^3+9x^4-11x^5+.....\infty$

On subtraction, we get

$S(1+x)^2 = 4-x+2x^2-2x^3+2x^4-2x^5+.....\infty$

$= 4-x+2x^2(1-x+x^2-.....\infty) = 4-x+\frac{2x^2}{1+x} = \frac{4+3x+x^2}{1+x}$

$S = \frac{4+3x+x^2}{(1+x)^3}$ **Ans.**

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Illustration 16 : Find the sum of series upto n terms $\left(\frac{2n+1}{2n-1}\right)+3\left(\frac{2n+1}{2n-1}\right)^2+5\left(\frac{2n+1}{2n-1}\right)^3+\ldots\ldots\ldots$.

Solution : For $x \neq 1$, let

$S = x + 3x^2 + 5x^3 + \ldots\ldots + (2n-3)x^{n-1} + (2n-1)x^n$ ..... (i)

$\Rightarrow \quad xS = x^2 + 3x^3 + \ldots\ldots + (2n-5)x^{n-1} + (2n-3)x^n + (2n-1)x^{n+1}$ ..... (ii)

Subtracting (ii) from (i), we get

$(1-x)S = x + 2x^2 + 2x^3 + \ldots\ldots\ldots + 2x^{n-1} + 2x^n - (2n-1)x^{n+1}$

$$= x + \frac{2x^2(1-x^{n-1})}{1-x} - (2n-1)x^{n+1}$$

$$= \frac{x}{1-x}[1 - x + 2x - 2x^n - (2n-1)x^n + (2n-1)x^{n+1}]$$

$$\Rightarrow \quad S = \frac{x}{(1-x)^2}[(2n-1)x^{n+1} - (2n+1)x^n + 1 + x]$$

Thus $\left(\frac{2n+1}{2n-1}\right)+3\left(\frac{2n+1}{2n-1}\right)^2+\ldots\ldots\ldots+(2n-1)\left(\frac{2n+1}{2n-1}\right)^n$

$$=\left(\frac{2n+1}{2n-1}\right)\left(\frac{2n-1}{2}\right)^2\left[(2n-1)\left(\frac{2n+1}{2n-1}\right)^{n+1}-(2n+1)\left(\frac{2n+1}{2n-1}\right)^n+1+\frac{2n+1}{2n-1}\right]$$

$$=\frac{4n^2-1}{4}\cdot\frac{4n}{2n-1}=n(2n+1)$$ **Ans.**

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Do yourself - 6 :

1. Find sum to n terms of the series $3+5\times\frac{1}{4}+7\times\frac{1}{4^2}+\ldots\ldots\ldots$

Find

2. sum $1 + 2a + 3a^2 + 4a^3 + \ldots$ to n terms.

3. sum $1+\frac{3}{4}+\frac{7}{16}+\frac{15}{64}+\frac{31}{256}+\ldots$ to infinity.

4. sum $\frac{2}{3}+\frac{3}{3^2}+\frac{2}{3^3}+\frac{3}{3^4}+\frac{2}{3^5}+\frac{3}{3^6}+\ldots$ to infinity.

5. Find the sum of n terms of the series the rth term of which is $(2r + 1)\,2^r$.

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9. SIGMA NOTATIONS (Σ)

THEOREMS :

(a) $\sum_{r=1}^{n}(a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$ **(b)** $\sum_{r=1}^{n} k\,a_r = k\sum_{r=1}^{n} a_r$ **(c)** $\sum_{r=1}^{n} k = nk$; where k is a constant.

10. RESULTS

(a) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ (sum of the first n natural numbers)

(b) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ (sum of the squares of the first n natural numbers)

(c) $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4} = \left[\sum_{r=1}^{n} r\right]^2$ (sum of the cubes of the first n natural numbers)

(d) $\sum_{r=1}^{n} r^4 = \frac{n}{30}(n+1)(2n+1)(3n^2+3n-1)$

(e) $\sum_{r=1}^{n}(2r-1) = n^2$ (sum of first n odd natural numbers)

(f) $\sum_{r=1}^{n} 2r = n(n+1)$ (sum of first n even natural numbers)

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Note :

If n^{th} term of a sequence is given by $T_n = an^3 + bn^2 + cn + d$ where a, b, c, d are constants,

then sum of n terms $S_n = \Sigma T_n = a\Sigma n^3 + b\Sigma n^2 + c\Sigma n + \Sigma d$

This can be evaluated using the above results.

Illustration 17 : Find the sum of the series to n terms whose n^{th} terms is 3n + 2.

Solution : $S_n = \Sigma T_n = \Sigma(3n + 2) = 3\Sigma n + \Sigma 2 = \frac{3(n+1)n}{2} + 2n = \frac{n}{2}(3n+7)$

Illustration 18 : If $T_k = k^3 + 3^k$, then find $\sum_{k=1}^{n} T_k$.

Solution : $\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 3^k = \left(\frac{n(n+1)}{2}\right)^2 + \frac{3(3^n-1)}{3-1} = \left(\frac{n(n+1)}{2}\right)^2 + \frac{3}{2}(3^n-1)$

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Illustration 19 : Find the value of the expression $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}1$

Solution : $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}1 = \sum_{i=1}^{n}\sum_{j=1}^{i}j = \sum_{i=1}^{n}\frac{i(i+1)}{2} = \frac{1}{2}\left[\sum_{i=1}^{n}i^2+\sum_{i=1}^{n}i\right] = \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}\right]$

$$= \frac{n(n+1)}{12}[2n+1+3] = \frac{n(n+1)(n+2)}{6}.$$

Illustration 20 : Sum up to 16 terms of the series $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+.....$ is

(A) 450 (B) 456 (C) 446 (D) none of these

Solution : $t_n = \frac{1^3+2^3+3^3+....+n^3}{1+3+5+....(2n-1)} = \frac{\left\{\frac{n(n+1)}{2}\right\}^2}{\frac{n}{2}\{2+2(n-1)\}} = \frac{\frac{n^2(n+1)^2}{4}}{n^2} = \frac{(n+1)^2}{4} == \frac{n^2}{4}+\frac{n}{2}+\frac{1}{4}$

$$\therefore \quad S_n = \Sigma t_n = \frac{1}{4}\Sigma n^2+\frac{1}{2}\Sigma n+\frac{1}{4}\Sigma 1 = \frac{1}{4}.\frac{n(n+1)(2n+1)}{6}+\frac{1}{2}.\frac{n(n+1)}{2}+\frac{1}{4}.n$$

$$\therefore \quad S_{16} = \frac{16.17.33}{24}+\frac{16.17}{4}+\frac{16}{4} = 446$$

Ans. (C)

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11. METHOD OF DIFFERENCE :

Some times the nth term of a sequence or a series can not be determined by the method, we have discussed earlier. So we compute the difference between the successive terms of given sequence for obtained the nth terms.

If $T_1, T_2, T_3,.....,T_n$ are the terms of a sequence then some times the terms $T_2 - T_1, T_3 - T_2,.....$ constitute an AP/GP. n^{th} term of the series is determined & the sum to n terms of the sequence can easily be obtained.

Case 1 :

(a) If difference series are in A.P., then

Let $T_n = an^2 + bn + c$, where a, b, c are constant

(b) If difference of difference series are in A.P.

Let $T_n = an^3 + bn^2 + cn + d$, where a, b, c, d are constant

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Case 2 :

(a) If difference are in G.P., then

Let $T_n = ar^n + b$, where r is common ratio & a, b are constant

(b) If difference of difference are in G.P., then

Let $T_n = ar^n + bn + c$, where r is common ratio & a, b, c are constant

Determine constant by putting n = 1, 2, 3 ..... n and putting the value of T_1, T_2, T_3 .....

and sum of series $(S_n) = \sum T_n$

Illustration 21 : Find the sum of n terms of the series 3 + 7 + 14 + 24 + 37 + ...

Solution : Clearly here the differences between the successive terms are

7 – 3, 14 – 7, 24 – 14, ... i.e. 4, 7, 10, 13,....., which are in A.P.

Let $S = 3 + 7 + 14 + 24 + ... + T_n$

$S = \quad 3 + 7 + 14 + ... + T_{n-1} + T_n$

Subtracting, we get

$0 = 3 + [4 + 7 + 10 + 13 + ... (n-1) \text{ terms}] - T_n$

$\therefore \quad T_n = 3 + S_{n-1}$ of an A.P. whose a = 4 and d = 3.

$$\therefore \quad T_n = 3 + \left(\frac{n-1}{2}\right)(2.4 + (n-2)3) = \frac{6 + (n-1)(3n+2)}{4} \quad \text{or,} \quad T_n = \frac{1}{2}(3n^2 - n + 4)$$

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Now putting n = 1, 2, 3,....., n and adding

$$\therefore \quad S_n = \frac{1}{2}[3\sum n^2 - \sum n + 4n] = \frac{1}{2}\left[3\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n\right] = \frac{n}{2}(n^2 + n + 4) \textbf{ Ans.}$$

Aliter Method :

Let $T_n = an^2 + bn + c$

Now, $T_1 = 3 = a + b + c$ .....(i)

$T_2 = 7 = 4a + 2b + c$ .....(ii)

$T_3 = 14 = 8a + 3b + c$ .....(iii)

Solving (i), (ii) & (iii) we get

$$a = \frac{3}{2}, b = -\frac{1}{2} \ \& \ c = 2 \qquad \therefore \quad T_n = \frac{1}{2}(3n^2 - n + 4)$$

$$\Rightarrow s_n = \Sigma T_n = \frac{1}{2}[3\sum n^2 - \sum n + 4n] = \frac{1}{2}\left[3\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n\right] = \frac{n}{2}(n^2 + n + 4)$$

Ans.

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Illustration 22 : Find the sum of n-terms of the series 1 + 4 + 10 + 22 + ...

Solution : Let $S = 1 + 4 + 10 + 22 + ... + T_n$ ..... (i)

$S = 1 + 4 + 10 + ... + T_{n-1} + T_n$ ..... (ii)

(i) – (ii) $\Rightarrow T_n = 1 + (3 + 6 + 12 + ... + T_n - T_{n-1})$

$$T_n = 1 + 3\left(\frac{2^{n-1} - 1}{2 - 1}\right)$$

$T_n = 3 \,.\, 2^{n-1} - 2$

So $S_n = \Sigma T_n = 3\Sigma 2^{n-1} - \Sigma 2$

$$= 3\left(\frac{2^n - 1}{2 - 1}\right) - 2n = 3.2^n - 2n - 3$$ **Ans.**

Aliter Method :

Let $T_n = ar^n + b$, where r = 2

Now $T_1 = 1 = ar + b$ ....(i)

$T_2 = 4 = ar^2 + b$ ....(ii)

Solving (i) & (ii), we get

$a = \frac{3}{2}, b = -2$

$\therefore$ $T_n = 3.2^{n-1} - 2$

$\Rightarrow$ $S_n = \Sigma T_n = 3\Sigma 2^{n-1} - \Sigma 2$

$$= 3\left(\frac{2^n - 1}{2 - 1}\right) - 2n = 3.2^n - 2n - 3$$ **Ans.**

Illustration 23 : Find the general term and sum of n terms of the series 1 + 5 + 19 + 49 + 101 + 181 + 295 + ...

Solution : The sequence of difference between successive term 4, 14, 30, 52, 80 .....

The sequence of the second order difference is 10, 16, 22, 28, ... clearly it is an A.P.

So, let n^{th} term

$T_n = an^3 + bn^2 + cn + d$

$a + b + c + d = 1$ .... (i)

$8a + 4b + 2c + d = 5$ .... (ii)

$27a + 9b + 3c + d = 19$ .... (iii)

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$64a + 16b + 4c + d = 49$ .... (iv)

from (i), (ii), (iii) & (iv)

$a = 1, b = -1, c = 0, d = 1 \Rightarrow T_n = n^3 - n^2 + 1$

$$\therefore S_n = \sum(n^3 - n^2 + 1) = \left(\frac{n(n+1)}{2}\right)^2 - \frac{n(n+1)(2n+1)}{6} + n = \frac{n(n^2-1)(3n+2)}{12} + n$$

Do yourself - 7 :

1. Find the sum of the series upto n terms $1 + \frac{1+2}{2} + \frac{1+2+3}{3} + \frac{1+2+3+4}{4} + ...$

2. Find the sum of 'n' terms of the series whose n^{th} term is

(a) $3n^2 + 2n$. (b) $3n^2 - n$.

(c) $n^3 + \frac{3}{2}n$ (d) $n(n + 2)$

(e) $n^2(2n + 3)$ (f) $3^n - 2^n$

(g) $3(4^n + 2n^2) - 4n^3$

3. Sum the following series to n terms :

(a) $1 . 4 . 7 + 2 . 5 . 8 + 3 . 6 . 9 + ...$

(b) $1 . 5 . 9 + 2 . 6 . 10 + 3 . 7 . 11 + ...$

4. Find the nth term and the sum of n terms of the series :

(a) 4, 14, 30, 52, 80, 114, ...

(b) 9, 16, 29, 54, 103, ...

Miscellaneous Illustration :

Illustration 24 : If $\sum_{r=1}^{n} T_r = \frac{n}{8}(n+1)(n+2)(n+3)$, then find $\sum_{r=1}^{n} \frac{1}{T_r}$.

Solution : $\because \quad T_n = S_n - S_{n-1}$

$$= \sum_{r=1}^{n} T_r - \sum_{r=1}^{n-1} T_r = \frac{n(n+1)(n+2)(n+3)}{8} - \frac{(n-1)n(n+1)(n+2)}{8} = \frac{n(n+1)(n+2)}{8}[(n+3)-(n-1)]$$

$$T_n = \frac{n(n+1)(n+2)}{8}(4) = \frac{n(n+1)(n+2)}{2}$$

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$$\Rightarrow \quad \frac{1}{T_n} = \frac{2}{n(n+1)(n+2)} = \frac{(n+2)-n}{n(n+1)(n+2)} = \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \quad \text{..... (i)}$$

Let $V_n = \dfrac{1}{n(n+1)}$

$$\therefore \quad \frac{1}{T_n} = V_n - V_{n+1}$$

Putting $n = 1, 2, 3, ... n$

$$\Rightarrow \quad \frac{1}{T_1} + \frac{1}{T_2} + \frac{1}{T_3} + ... + \frac{1}{T_n} = (V_1 - V_{n+1}) \Rightarrow \sum_{r=1}^{n} \frac{1}{T_r} = \frac{n^2+3n}{2(n+1)(n+2)}$$

Illustration 25 : Find the sum of n terms of the series $1 . 3 . 5 + 3 . 5 . 7 + 5 . 7 . 9 + ...$

Solution : The nth term is $(2n-1)(2n+1)(2n+3)$

$T_n = (2n-1)(2n+1)(2n+3)$

$T_n = \frac{1}{8}(2n-1)(2n+1)(2n+3)\{(2n+5)-(2n-3)\}$

$= \frac{1}{8}(V_n - V_{n-1})$ [Let $V_n = (2n-1)(2n+1)(2n+3)(2n+5)$]

$S_n = \sum T_n = \frac{1}{8}[V_n - V_0]$

$\therefore \quad S_n = \dfrac{(2n-1)(2n+1)(2n+3)(2n+5)}{8} + \dfrac{15}{8} = n(2n^3 + 8n^2 + 7n - 2)$ **Ans.**

Illustration 26 : The series of natural numbers is divided into groups (1), (2, 3, 4), (5, 6, 7, 8, 9) ..... and so on. Show that the sum of the numbers in n^{th} group is $n^3 + (n-1)^3$

Solution : The groups are (1), (2, 3, 4), (5, 6, 7, 8, 9) .....

The number of terms in the groups are 1, 3, 5.....

$\therefore$ The number of terms in the nth group $= (2n-1)$

the last term of the nth group is n^2

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If we count from last term common difference should be –1

So the sum of numbers in the nth group $= \left(\frac{2n-1}{2}\right)\left\{2n^2 + (2n-2)(-1)\right\}$

$= (2n-1)(n^2 - n + 1) = 2n^3 - 3n^2 + 3n - 1 = n^3 + (n-1)^3$

Illustration 27 : Find the natural number 'a' for which $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$, where the function f satisfied $f(x+y) = f(x).\ f(y)$ for all natural number x,y and further $f(1) = 2$.

Solution : It is given that

$f(x+y) = f(x)\ f(y)$ and $f(1) = 2$

$f(1+1) = f(1)\ f(1) \Rightarrow f(2) = 2^2$, $f(1+2) = f(1)\ f(2) \Rightarrow f(3) = 2^3$, $f(2+2) = f(2)\ f(2) \Rightarrow f(4) = 2^4$

Similarly $f(k) = 2^k$ and $f(a) = 2^a$

Hence , $\sum_{k=1}^{n} f(a+k) = \sum_{k=1}^{n} f(a)f(k) = f(a)\sum_{k=1}^{n} f(k) = 2^a \sum_{k=1}^{n} 2^k = 2^a\{2^1 + 2^2 + ...+2^n\}$

$= 2^a\left\{\frac{2(2^n-1)}{2-1}\right\} = 2^{a+1}(2^n-1)$

But $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$

$2^{a+1}(2^n-1) = 16\,(2^n-1)$

$\therefore \quad 2^{a+1} = 2^4$

$\therefore \quad a+1 = 4 \quad \Rightarrow \quad a = 3$ **Ans.**

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EXERCISE (O-1)

SINGLE CORRECT ONLY

1. If $a_1, a_2, a_3, \ldots, a_n, \ldots$ are in A.P. such that $a_4 - a_7 + a_{10} = m$, then the sum of first 13 terms of this A.P., is :

(A) 15 m (B) 10 m (C) 12 m (D) 13 m

SS0001

2. If a, b, c are in AP, then $(a - c)^2$ equals

(A) $4(b^2 - ac)$ (B) $4(b^2 + ac)$ (C) $4b^2 - ac$ (D) $b^2 - 4ac$

SS0002

3. Let $a_1, a_2, a_3, \ldots$ be an A.P. such that $\frac{a_1 + a_2 + \ldots + a_p}{a_1 + a_2 + a_3 + \ldots + a_q} = \frac{p^2}{q^2}$; $p \neq q$. Then $\frac{a_6}{a_{21}}$ is equal to :

(A) $\frac{121}{1861}$ (B) $\frac{11}{41}$ (C) $\frac{121}{1681}$ (D) $\frac{41}{11}$

SS0003

4. Given sum of the first n terms of an A. P. is $2n + 3n^2$. Another A. P. is formed with the same first term and double of the common difference, the sum of n terms of the new A. P. is :-

(A) $n + 4n^2$ (B) $n^2 + 4n$ (C) $3n + 2n^2$ (D) $6n^2 - n$

SS0004

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5. If the sum of n terms of an AP is $Pn + Qn^2$, where P, Q are constants, then its common difference is

(A) 2Q (B) P + Q (C) 2P (D) P – Q

SS0005

6. The first term of an infinite G.P. is 1 and every term is equals to the sum of the successive terms, then its fourth term will be-

(A) $\frac{1}{2}$ (B) $\frac{1}{8}$ (C) $\frac{1}{4}$ (D) $\frac{1}{16}$

SS0006

7. If $a = \sum_{n=0}^{\infty} x^n, b = \sum_{n=0}^{\infty} y^n, c = \sum_{n=0}^{\infty} (xy)^n$ where $|x|, |y| < 1$; then-

(A) abc = a + b + c (B) ab + bc = ac + b (C) ac + bc = ab + c (D) ab + ac = bc + a

SS0007

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8. If r > 1 and $x = a + \frac{a}{r} + \frac{a}{r^2} + ...\text{to } \infty$, $y = b - \frac{b}{r} + \frac{b}{r^2} - ...\text{to } \infty$ and $z = c + \frac{c}{r^2} + \frac{c}{r^4} + ...\text{to } \infty$, then $\frac{xy}{z} =$

(A) $\frac{ab}{c}$ (B) $\frac{ac}{b}$ (C) $\frac{bc}{a}$ (D) None of these

SS0008

9. In a GP, first term is 1. If $4T_2 + 5T_3$ is minimum, then its common ratio is

(A) $\frac{2}{5}$ (B) $-\frac{2}{5}$ (C) $\frac{3}{5}$ (D) $-\frac{3}{5}$

SS0009

10. Given a sequence of 4 numbers, first three of which are in G.P. and the last three are in A.P. with common difference six. If first and last terms of this sequence are equal, then the last term is :

(A) 8 (B) 16 (C) 2 (D) 4

SS0010

11. If G be the GM between x and y, then the value of $\frac{1}{G^2 - x^2} + \frac{1}{G^2 - y^2}$ is equal to

(A) G^2 (B) $\frac{2}{G^2}$ (C) $\frac{1}{G^2}$ (D) $3G^2$

SS0011

12. If a, b, c are in HP, then $\frac{a-b}{b-c}$ is equal to

(A) $\frac{a}{b}$ (B) $\frac{b}{a}$ (C) $\frac{a}{c}$ (D) $\frac{c}{b}$

SS0012

13. If a, b and c are positive real numbers then $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ is greater than or equal to

(A) 3 (B) 6 (C) 27 (D) 5

SS0013

14. If $a_1, a_2, a_3 .... a_n \in R^+$ and $a_1.a_2.a_3....a_n = 1$, then minimum value of $(1+a_1 + a_1^2)(1 + a_2 + a_2^2)(1 + a_3 + a_3^2) .... (1 + a_n + a_n^2)$ is equal to :-

(A) 3^{n+1} (B) 3^n (C) 3^{n-1} (D) none of these

SS0014

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15. If a, b, c are positive real numbers such that $ab^2c^3 = 64$ then minimum value of $\left(\frac{1}{a}+\frac{2}{b}+\frac{3}{c}\right)$ is equal to:-

(A) 6 (B) 2 (C) 3 (D) None of these

SS0015

16. The value of $1^2 + 3^2 + 5^2 + ...+ 25^2$ is :

(A) 1728 (B) 1456 (C) 2925 (D) 1469

SS0016

17. The sum of the series : $(2)^2 + 2(4)^2 + 3(6)^2 + ...$ upto 10 terms is :

(A) 11300 (B) 12100 (C) 12300 (D) 11200

SS0017

18. $2 + 4 + 7 + 11 + 16 + ...$ to n terms =

(A) $\frac{1}{6}(n^2 + 3n + 8)$ (B) $\frac{n}{6}(n^2 + 3n + 8)$ (C) $\frac{1}{6}(n^2 - 3n + 8)$ (D) $\frac{n}{6}(n^2 - 3n + 8)$

SS0018

19. The sum $\frac{3}{1^2}+\frac{5}{1^2+2^2}+\frac{7}{1^2+2^2+3^2}+.....$ upto 11-terms is :-

(A) $\frac{11}{4}$ (B) $\frac{60}{11}$ (C) $\frac{7}{2}$ (D) $\frac{11}{2}$

SS0019

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20. The sum of the series : $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + ...$ up to 10 terms, is:

(A) $\frac{22}{13}$ (B) $\frac{18}{11}$ (C) $\frac{20}{11}$ (D) $\frac{16}{9}$

SS0020

EXERCISE (O-2)

SINGLE CORRECT ONLY

1. If for an A.P. $a_1, a_2, a_3, ..., a_n, ...$

$a_1 + a_3 + a_5 = -12$ and $a_1 a_2 a_3 = 8$, then the value of $a_2 + a_4 + a_6$ equals

(A) – 12 (B) – 16 (C) – 18 (D) – 21

SS0021

2. If the sum of the first 11 terms of an arithmetical progression equals that of the first 19 terms, then the sum of its first 30 terms, is

(A) equal to 0 (B) equal to – 1 (C) equal to 1 (D) non unique

SS0022

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3. Let s_1, s_2, s_3 ..... and t_1, t_2, t_3 ..... are two arithmetic sequences such that $s_1 = t_1 \neq 0$; $s_2 = 2t_2$ and $\sum_{i=1}^{10} s_i = \sum_{i=1}^{15} t_i$. Then the value of $\frac{s_2 - s_1}{t_2 - t_1}$ is

(A) 8/3 (B) 3/2 (C) 19/8 (D) 2

SS0023

4. If $x \in R$, the numbers $(5^{1+x} + 5^{1-x})$, a/2, $(25^x + 25^{-x})$ form an A.P. then 'a' must lie in the interval

(A) [1, 5] (B) [2, 5] (C) [5, 12] (D) $[12, \infty)$

SS0024

5. Along a road lies an odd number of stones placed at intervals of 10 m. These stones have to be assembled around the middle stone. A person can carry only one stone at a time. A man carried out the job starting with the stone in the middle, carrying stones in succession, thereby covering a distance of 4.8 km. Then the number of stones is

(A) 15 (B) 29 (C) 31 (D) 35

SS0025

6. In an A.P. with first term 'a' and the common difference d $(a, d \neq 0)$, the ratio ' ρ ' of the sum of the first n terms to sum of n terms succeeding them does not depend on n. Then the ratio $\frac{a}{d}$ and the ratio 'ρ ', respectively are

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(A) $\frac{1}{2}, \frac{1}{4}$ (B) $2, \frac{1}{3}$ (C) $\frac{1}{2}, \frac{1}{3}$ (D) $\frac{1}{2}, 2$

SS0026

7. Let a_n, $n \in N$ is an A.P. with common difference 'd' and all whose terms are non-zero. If n approaches infinity, then the sum $\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + ... + \frac{1}{a_na_{n+1}}$ will approach

(A) $\frac{1}{a_1d}$ (B) $\frac{2}{a_1d}$ (C) $\frac{1}{2a_1d}$ (D) a_1d

SS0027

8. The arithmetic mean of the nine numbers in the given set {9, 99, 999, ... 999999999} is a 9 digit number N, all whose digits are distinct. The number N does not contain the digit

(A) 0 (B) 2 (C) 5 (D) 9

SS0028

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9. If $\dfrac{1+3+5+.....\text{upto n terms}}{4+7+10+....\text{upto n terms}} = \dfrac{20}{7\log_{10} x}$ and

$n = \log_{10} x + \log_{10} x^{\frac{1}{2}} + \log_{10} x^{\frac{1}{4}} + \log_{10} x^{\frac{1}{8}} + ... + \infty$, then x is equal to

(A) 10^3 (B) 10^5 (C) 10^6 (D) 10^7

SS0029

10. If $a \neq 1$ and $ln\, a^2 + (ln\, a^2)^2 + (ln\, a^2)^3 + ... = 3\left(ln\, a + (ln\, a)^2 + (ln\, a)^3 + (ln\, a)^4 + ...\right)$ then 'a' is equal to

(A) $e^{1/5}$ (B) $\sqrt{e}$ (C) $\sqrt[3]{e}$ (D) $\sqrt[4]{e}$

SS0030

11. If a, b, c are distinct positive real in H.P., then the value of the expression, $\dfrac{b+a}{b-a} + \dfrac{b+c}{b-c}$ is equal to

(A) 1 (B) 2 (C) 3 (D) 4

SS0031

12. An H.M. is inserted between the number 1/3 and an unknown number. If we diminish the reciprocal of the inserted number by 6, it is the G.M. of the reciprocal of 1/3 and that of the unknown number. If all the terms of the respective H.P. are distinct then

(A) the unknown number is 27 (B) the unknown number is 1/27

(C) the H.M. is 15 (D) the G.M. is 21

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SS0032

13. If abcd = 1 where a, b, c, d are positive reals then the minimum value of

$a^2 + b^2 + c^2 + d^2 + ab + ac + ad + bc + bd + cd$ is

(A) 6 (B) 10 (C) 12 (D) 20

SS0033

14. **Statement-1:** If $27\,abc \geq (a + b + c)^3$ and $3a + 4b + 5c = 12$ then $\dfrac{1}{a^2} + \dfrac{1}{b^3} + \dfrac{1}{c^5} = 10$; where a, b, c are positive real numbers.

Statement-2: For positive real numbers A.M. $\geq$ G.M.

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

SS0034

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15. If $x = \sum_{n=0}^{\infty} a^n$, $y = \sum_{n=0}^{\infty} b^n$, $z = \sum_{n=0}^{\infty} c^n$ where a, b, c are in A.P. and $|a| < 1$, $|b| < 1$, $|c| < 1$, then x, y, z are in-

(A) HP
(B) Arithmetic - Geometric Progression
(C) AP
(D) GP

SS0035

16. If $S = 1^2 + 3^2 + 5^2 + ... + (99)^2$ then the value of the sum $2^2 + 4^2 + 6^2 + ... + (100)^2$ is

(A) S + 2550
(B) 2S
(C) 4S
(D) S + 5050

SS0036

17. For which positive integers *n* is the ratio, $\dfrac{\sum_{k=1}^{n} k^2}{\sum_{k=1}^{n} k}$ an integer?

(A) odd *n* only
(B) even *n* only
(C) $n = 1 + 6k$ only, where $k \geq 0$ and $k \in I$
(D) $n = 1 + 3k$, integer $k \geq 0$

SS0037

MORE THAN ONE CORRECT :

18. Let a_1, a_2, a_3 ..... and b_1, b_2, b_3 ..... be arithmetic progressions such that $a_1 = 25$, $b_1 = 75$ and $a_{100} + b_{100} = 100$. Then

(A) the difference between successive terms in progression 'a' is opposite of the difference in progression 'b'.
(B) $a_n + b_n = 100$ for any *n*.
(C) $(a_1 + b_1), (a_2 + b_2), (a_3 + b_3)$, ... are in A.P.
(D) $\sum_{r=1}^{100} (a_r + b_r) = 10000$

SS0038

EXERCISE (S-1)

1. In an AP of which 'a' is the Ist term, if the sum of the Ist p terms is equal to zero, show that the sum of the next q terms is $-\left(\dfrac{aq(p+q)}{p-1}\right)$

SS0039

2. The interior angles of a convex polygon form an arithmetic progression with a common difference of 4°. Determine the number of sides of the polygon if its largest interior angle is 172°.

SS0040

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3. If $a > 0$, then minimum value of $a + 2a^2 + a^3 + 15 + a^{-1} + a^{-3} + a^{-4}$ is

SS0041

4. The sequence $a_1, a_2, a_3, \ldots\ldots a_{98}$ satisfies the relation $a_{n+1} = a_n + 1$ for $n = 1, 2, 3, \ldots\ldots 97$ and has the sum equal to 4949. Evaluate $\sum_{k=1}^{49} a_{2k}$.

SS0042

5. There are nAM's between 1 & 31 such that 7th mean : $(n-1)^{th}$ mean = 5 : 9, then find the value of n.

SS0043

6. The first term of an arithmetic progression is 1 and the sum of the first nine terms equal to 369. The first and the ninth term of a geometric progression with real common ratio coincide with the first and the ninth term of the arithmetic progression. Find the seventh term of the geometric progression.

SS0044

7. For an increasing G.P. $a_1, a_2, a_3 \ldots\ldots, a_n$, if $a_6 = 4a_4$, $a_9 - a_7 = 192$, then the value of $\sum_{i=1}^{\infty} \frac{1}{a_i}$ is

SS0045

8. Find three numbers a,b,c between 2 & 18 such that ;
(i) their sum is 25
(ii) the numbers 2,a,b are consecutive terms of an AP &
(iii) the numbers b,c,18 are consecutive terms of a G.P.

SS0046

9. If the 10th term of a HP is 21 and 21st term of the same HP is 10, then find the 210th term.

SS0047

10. The pth term T_p of H.P. is $q(p + q)$ and q^{th} term T_q is $p(p + q)$ when $p > 2$, $q > 2$ $(p \neq q)$. Prove that
(a) $T_{p+q} = pq$; (b) $T_{pq} = p + q$; (c) $T_{p+q} > T_{pq}$

SS0048

11. **(a)** The harmonic mean of two numbers is 4. The arithmetic mean A & the geometric mean G satisfy the relation $2A + G^2 = 27$. Find the two numbers.

SS0049

(b) The AM of two numbers exceeds their GM by 15 & HM by 27. Find the numbers.

SS0050

12. If a, b, c, d > 0 such that $a + 2b + 3c + 4d = 50$, then find the maximum value of $\left(\frac{a^2b^4c^3d}{16}\right)^{1/10}$

SS0051

13. If number of coins earned in n^{th} game is $n2^{n+2} - 2^n$ and total number of coins earned in first 10 games is $10(B.2^{10} + 1)$, where $B \in N$, then the value of B is

SS0052

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14. Find the nth term and the sum to n terms of the sequence :

(i) 1 + 5 + 13 + 29 + 61 +.....

SS0053

(ii) 6 + 13 + 22 + 33 +.....

SS0054

15. Sum the following series to n terms and to infinity (where it is finite and defined) :

(i) $\frac{1}{1.4.7}+\frac{1}{4.7.10}+\frac{1}{7.10.13}+.....$

SS0055

(ii) $\sum_{r=1}^{n} r(r+1)(r+2)(r+3)$

SS0056

(iii) $\sum_{r=1}^{n} \frac{1}{4r^2-1}$

SS0057

EXERCISE (S-2)

1. The sum of n terms of two arithmetic series are in the ratio of (7n + 1) : (4n + 27). Find the ratio of their n^{th} term.

SS0058

2. If the first 3 consecutive terms of a geometrical progression are the real roots of the equation $2x^3 - 19x^2 + 57x - 54 = 0$ find the sum to infinite number of terms of G.P.

SS0059

3. In a GP the ratio of the sum of the first eleven terms to the sum of the last eleven terms is 1/8 and the ratio of the sum of all the terms without the first nine to the sum of all the terms without the last nine is 2. Find the number of terms in the GP.

SS0060

4. If $A_1, A_2, A_3, ...A_{51}$ are arithmetic means inserted between the numbers a and b, then find the value of $\left(\frac{b+A_{51}}{b-A_{51}}\right)-\left(\frac{A_1+a}{A_1-a}\right)$.

SS0061

5. If one AM 'a' and two GM's p and q be inserted between any two given numbers then show that $p^3 + q^3 = 2apq$.

SS0062

6. If a,b,c,d,e be 5 numbers such that a,b,c are in AP ; b,c,d are in GP & c,d,e are in HP then :

(i) Prove that a,c,e are in GP.

(ii) Prove that $e = (2b - a)^2/a$

(iii) If a = 2 & e = 18, find all possible values of b,c,d.

SS0063

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7. In a set of four numbers, the first three are in GP & the last three are in A.P. with common difference 6. If the first number is the same as the fourth, find the four numbers.

SS0064

8. Given that the cubic $ax^3 - ax^2 + 9bx - b = 0$ $(a \neq 0)$ has all three positive roots. Find the harmonic mean of the roots independent of a and b, hence deduce that the root are all equal. Find also the minimum value of (a + b) if a and b $\in$ N.

SS0065

9. If the roots of $10x^3 - cx^2 - 54x - 27 = 0$ are in harmonic progression, then find c and all the roots.

SS0066

10. Find the sum of the n terms of the sequence $\frac{1}{1+1^2+1^4}+\frac{2}{1+2^2+2^4}+\frac{3}{1+3^2+3^4}+.....$

SS0067

11. Find the sum of the infinite series $\frac{1.3}{2}+\frac{3.5}{2^2}+\frac{5.7}{2^3}+\frac{7.9}{2^4}+.....\infty$.

SS0068

12. Let $S=\sum_{n=1}^{99}\frac{5^{100}}{(25)^n+5^{100}}$. Find [S].

Where [y] denotes largest integer less than or equal to y.

SS0069

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EXERCISE (JM)

1. If m is the A.M. of two distinct real numbers l and n(l, n > 1) and G_1, G_2 and G_3 are three geometric means between l and n, then $G_1^4+2G_2^4+G_3^4$ equals - **[JEE(Main)-2015]**

(1) 4 lmn^2 (2) 4 $l^2m^2n^2$ (3) 4 l^2mn (4) 4 lm^2n

SS0070

2. If the 2nd, 5th and 9th terms of a non-constant A.P. are in G.P., then the common ratio of this G.P. is:- **[JEE(Main)-2016]**

(1) $\frac{7}{4}$ (2) $\frac{8}{5}$ (3) $\frac{4}{3}$ (4) 1

SS0071

3. If the sum of the first ten terms of the series $\left(1\frac{3}{5}\right)^2+\left(2\frac{2}{5}\right)^2+\left(3\frac{1}{5}\right)^2+4^2+\left(4\frac{4}{5}\right)^2+...$, is $\frac{16}{5}m$, then m is equal to :- **[JEE(Main)-2016]**

(1) 99 (2) 102 (3) 101 (4) 100

SS0072

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4. If, for a positive integer n, the quadratic equation,

$x(x + 1) + (x + 1)(x + 2) + ... + (x + n - 1)(x + n) = 10n$

has two consecutive integral solutions, then n is equal to : **[JEE(Main)-2017]**

(1) 11 (2) 12 (3) 9 (4) 10

SS0073

5. For any three positive real numbers a, b and c, $9(25a^2 + b^2) + 25(c^2 - 3ac) = 15b(3a + c)$.

Then : **[JEE(Main)-2017]**

(1) a, b and c are in G.P. (2) b, c and a are in G.P.

(3) b, c and a are in A.P. (4) a, b and c are in A.P.

SS0074

6. Let $a_1, a_2, a_3, ... , a_{49}$ be in A.P. such that $\sum_{k=0}^{12} a_{4k+1} = 416$ and $a_9 + a_{43} = 66$.

If $a_1^2 + a_2^2 + ... + a_{17}^2 = 140m$, then m is equal to- **[JEE(Main)-2018]**

(1) 68 (2) 34 (3) 33 (4) 66

SS0075

7. Let A be the sum of the first 20 terms and B be the sum of the first 40 terms of the series

$1^2 + 2\cdot2^2 + 3^2 + 2\cdot4^2 + 5^2 + 2\cdot6^2 + ...$. If $B - 2A = 100\lambda$, then λ is equal to :

[JEE(Main)-2018]

(1) 248 (2) 464 (3) 496 (4) 232

SS0076

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8. If a, b and c be three distinct real numbers in G. P. and a + b + c = xb, then x cannot be :

[JEE(Main)-2019]

(1) 4 (2) –3 (3) –2 (4) 2

SS0077

9. Let $a_1, a_2, ..., a_{30}$ be an A. P., $S = \sum_{i=1}^{30} a_i$ and $T = \sum_{i=1}^{15} a_{(2i-1)}$. If $a_5 = 27$ and $S - 2T = 75$, then a_{10} is equal to : **[JEE(Main)-2019]**

(1) 57 (2) 47 (3) 42 (4) 52

SS0078

10. The sum of the follwing series

$$1 + 6 + \frac{9(1^2 + 2^2 + 3^2)}{7} + \frac{12(1^2 + 2^2 + 3^2 + 4^2)}{9} + \frac{15(1^2 + 2^2 + ... + 5^2)}{11} + ...$$ up to 15 terms, is :

[JEE(Main)-2019]

(1) 7820 (2) 7830 (3) 7520 (4) 7510

SS0079

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11. Let x, y be positive real numbers and m, n positive integers. The maximum value of the expression $\frac{x^m y^n}{(1+x^{2m})(1+y^{2n})}$ is :- **[JEE(Main)-2019]**

(1) $\frac{1}{2}$ (2) $\frac{1}{4}$ (3) $\frac{m+n}{6mn}$ (4) 1

SS0080

12. If α, β and γ are three consecutive terms of a non-constant G.P. such that the equations $\alpha x^2 + 2\beta x + \gamma = 0$ and $x^2 + x - 1 = 0$ have a common root, then $\alpha(\beta + \gamma)$ is equal to : **[JEE(Main)-2019]**

(1) $\beta\gamma$ (2) 0 (3) $\alpha\gamma$ (4) $\alpha\beta$

SS0081

13. The sum of all natural numbers 'n' such that $100 < n < 200$ and H.C.F. $(91, n) > 1$ is : **[JEE(Main)- 2019]**

(1) 3221 (2) 3121 (3) 3203 (4) 3303

SS0082

14. If three distinct numbers a,b,c are in G.P. and the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then which one of the following statements is correct? **[JEE(Main)-2019]**

(1) d,e,f are in A.P. (2) $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in G.P.

(3) $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P. (4) d,e,f are in G.P.

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SS0083

15. Let the sum of the first n terms of a non-constant A.P., $a_1, a_2, a_3, \ldots$ be $50n + \frac{n(n-7)}{2}A$, where A is a constant. If d is the common difference of this A.P., then the ordered pair (d, a_{50}) is equal to **[JEE(Main)- 2019]**

(1) (A, 50+46A) (2) (A, 50+45A) (3) (50, 50+46A) (4) (50, 50+45A)

SS0084

16. The sum $\frac{3\times1^3}{1^2} + \frac{5\times(1^3+2^3)}{1^2+2^2} + \frac{7\times(1^3+2^3+3^3)}{1^2+2^2+3^2} + \ldots\ldots$ upto 10th term, is : **[JEE(Main)- 2019]**

(1) 660 (2) 620 (3) 680 (4) 600

SS0085

17. If $a_1, a_2, a_3, \ldots\ldots, a_n$ are in A.P. and $a_1 + a_4 + a_7 + \ldots\ldots + a_{16} = 114$, then $a_1 + a_6 + a_{11} + a_{16}$ is equal to : **[JEE(Main)- 2019]**

(1) 38 (2) 98 (3) 76 (4) 64

SS0086

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18. Let a, b and c be in G. P. with common ratio r, where $a \neq 0$ and $0 < r \leq \frac{1}{2}$. If 3a, 7b and 15c are the first three terms of an A. P., then the 4th term of this A. P. is : **[JEE(Main)- 2019]**

(1) $\frac{7}{3}a$ (2) a (3) $\frac{2}{3}a$ (4) 5a

SS0087

19. If α and β are the roots of the equation $375x^2 - 25x - 2 = 0$, then $\lim_{n\to\infty}\sum_{r=1}^{n}\alpha^r + \lim_{n\to\infty}\sum_{r=1}^{n}\beta^r$ is equal to : **[JEE(Main)- 2019]**

(1) $\frac{21}{346}$ (2) $\frac{29}{358}$ (3) $\frac{1}{12}$ (4) $\frac{7}{116}$

SS0088

20. If the sum of the first 40 terms of the series, 3 + 4 + 8 + 9 + 13 + 14 + 18 +19 +… is (102)m, then m is equal to : **[JEE(Main)- 2020]**

(1) 20 (2) 5 (3) 10 (4) 25

SS0089

21. Let $a_1, a_2, a_3, \ldots$ be a G.P. such that $a_1 < 0$, $a_1 + a_2 = 4$ and $a_3 + a_4 = 16$. If $\sum_{i=1}^{9} a_i = 4\lambda$, then λ is equal to : **[JEE(Main)- 2020]**

(1) –171 (2) 171 (3) $\frac{511}{3}$ (4) –513

SS0090

22. Five numbers are in A.P., whose sum is 25 and product is 2520. If one of these five numbers is $-\frac{1}{2}$, then the greatest number amongst them is : **[JEE(Main)- 2020]**

(1) $\frac{21}{2}$ (2) 27 (3) 16 (4) 7

SS0091

23. The greatest positive integer k, fr which $49^k + 1$ is a factor of the sum $49^{125} + 49^{124} + \ldots\ 49^2 + 49 + 1$, is : **[JEE(Main)- 2020]**

(1) 32 (2) 60 (3) 63 (4) 65

SS0092

24. The sum, $\sum_{n=1}^{7}\frac{n(n+1)(2n+1)}{4}$ is equal to _____. **[JEE(Main)- 2020]**

SS0093

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25. The sum $\sum_{k=1}^{20}(1+2+3+...+k)$ is .... **[JEE(Main)- 2020]**

SS0094

26. The number of terms common to the two A.P.'s 3, 7, 11, ..., 407 and 2, 9, 16, ..., 709 is _____.
[JEE(Main)- 2020]
SS0095

27. The product $2^{\frac{1}{4}}\cdot 4^{\frac{1}{16}}\cdot 8^{\frac{1}{48}}\cdot 16^{\frac{1}{128}}\cdot$ .... to ∞ is equal to : **[JEE(Main)- 2020]**

(1) $2^{\frac{1}{2}}$ (2) $2^{\frac{1}{4}}$ (3) 2 (4) 1

SS0096

EXERCISE (JA)

1. If the sum of first n terms of an A.P. is cn^2, then the sum of squares of these n terms is

(A) $\frac{n(4n^2-1)c^2}{6}$ (B) $\frac{n(4n^2+1)c^2}{3}$ (C) $\frac{n(4n^2-1)c^2}{3}$ (D) $\frac{n(4n^2+1)c^2}{6}$

[JEE 2009, 3 (–1)]
SS0097

2. Let $a_1, a_2, a_3 ..... a_{11}$ be real numbers satisfying
$a_1 = 15$, $27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for $k = 3, 4 ..... 11$.
If $\frac{a_1^2 + a_2^2 + ... + a_{11}^2}{11} = 90$, then the value of $\frac{a_1 + a_2 + ... + a_{11}}{11}$ is equal to **[JEE 2010, 3+3]**

SS0098

3. The minimum value of the sum of real numbers a^{-5}, a^{-4}, $3a^{-3}$, 1, a^8 and a^{10} with $a > 0$ is
[JEE 2011, 4]
SS0099

4. Let $a_1, a_2, a_3, ..., a_{100}$ be an arithmetic progression with $a_1 = 3$ and $S_p = \sum_{i=1}^{p} a_i, 1 \le p \le 100$. For any integer n with $1 \le n \le 20$, let $m = 5n$. If $\frac{S_m}{S_n}$ does not depend on n, then a_2 is **[JEE 2011, 4]**

SS0100

5. Let $a_1, a_2, a_3, ...$ be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n < 0$ is **[JEE 2012, 3 (–1)]**

(A) 22 (B) 23 (C) 24 (D) 25

SS0101

6. Let $S_n = \sum_{k=1}^{4n}(-1)^{\frac{k(k+1)}{2}}k^2$. Then S_n can take value(s) **[JEE-Advanced 2013, 4, (–1)]**

(A) 1056 (B) 1088 (C) 1120 (D) 1332

SS0102

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7. A pack contains n cards numbered from 1 to n. Two consecutive numbered cards are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller to the numbers on the removed cards is k, then k – 20 = **[JEE-Advanced 2013, 4, (–1)]**

SS0103

8. Let a,b,c be positive integers such that $\frac{b}{a}$ is an integer. If a,b,c are in geometric progression and the arithmetic mean of a,b,c is b + 2, then the value of $\frac{a^2+a-14}{a+1}$ is **[JEE(Advanced)-2014, 3]**

SS0104

9. Suppose that all the terms of an arithmetic progression (A.P.) are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies in between 130 and 140, then the common difference of this A.P. is **[JEE 2015, 4M, –0M]**

SS0105

10. Let $b_i > 1$ for i = 1, 2, ..., 101. Suppose $\log_e b_1, \log_e b_2, \ldots, \log_e b_{101}$ are in Arithmetic Progression (A.P.) with the common difference $\log_e 2$. Suppose $a_1, a_2, \ldots, a_{101}$ are in A.P. such that $a_1 = b_1$ and $a_{51} = b_{51}$. If $t = b_1 + b_2 + \ldots + b_{51}$ and $s = a_1 + a_2 + \ldots + a_{51}$ then **[JEE(Advanced)-2016, 3(–1)]**

(A) $s > t$ and $a_{101} > b_{101}$
(B) $s > t$ and $a_{101} < b_{101}$
(C) $s < t$ and $a_{101} > b_{101}$
(D) $s < t$ and $a_{101} < b_{101}$

SS0106

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11. The sides of the right angled triangle are in arithmetic progression. If the triangle has area 24, then what is the length of its smallest side ? **[JEE(Advanced)-2017, 3]**

SS0107

12. Let X be the set consisting of the first 2018 terms of the arithmetic progression 1, 6, 11, ... , and Y be the set consisting of the first 2018 terms of the arithmetic progression 9, 16, 23, ... Then, the number of elements in the set $X \cup Y$ is _____ **[JEE(Advanced)-2018, 3]**

SS0108

13. Let m be the minimum possible value of $\log_3\left(3^{y_1}+3^{y_2}+3^{y_3}\right)$, where y_1, y_2, y_3 are real numbers for which $y_1 + y_2 + y_3 = 9$. Let M be the maximum possible value of $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$, where x_1, x_2, x_3 are positive real numbers for which $x_1 + x_2 + x_3 = 9$. Then the value of $\log_2(m^3) + \log_3(M^2)$ is _____. **[JEE(Advanced)-2020]**

SS0124

14. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers in arithmetic progression with common difference 2. Also, let $b_1, b_2, b_3, \ldots$ be a sequence of positive integers in geometric progression with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of c, for which the equality

$$2(a_1 + a_2 + \ldots + a_n) = b_1 + b_2 + \ldots + b_n$$

holds for some positive integer n, is _____ **[JEE(Advanced)-2020]**

SS0125

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ANSWERS

Do yourself-1

1. **(a)** $\frac{2}{1},\frac{4}{2},\frac{8}{3},\frac{16}{4},\ldots\ldots,$ **(b)** $\frac{2}{3},\frac{4}{9},\frac{2}{27},\frac{4}{81},\ldots\ldots;$ **3.** $-(p+q)$

4. 153 **5.** 0 **6.** –185 **7.** 820a – 1680b

Do yourself-2

1. 900 **2.** 14 **3.** 3 **4.** 5 **5.** 612 **6.** 4,9,14

7. 1,4,7 **8.** 495 **9.** n^2 **10.** 3,5,7,9 **11.** 6r – 1

Do yourself-3

1. 931 **2.** 4^5 **3.** 4, 12, 36 **4.** C **5.** 612 **6.** $191\frac{1}{4}$

7. $\frac{1}{4}\left(5^p-1\right)$ **8.** $364\left(\sqrt{3}+1\right)$ **9.** $\frac{64}{65}$ **10.** $\frac{27}{58}$ **11.** $\frac{3\left(3+\sqrt{3}\right)}{2}$ **12.** 2

13. 16,24,36,... **14.** 8,12,18 **15.** 2,6,18 **16.** $6,-3,1\frac{1}{2},\ldots$ **17.** $4,1,\frac{1}{4}\ldots$

18. $\frac{x\left(x^n-1\right)}{x-1}+\frac{\left(n\left(n+1\right)\right)a}{2}$

Do yourself-4

1. 2 **2.** 15 **4.** **(a)** 5 **(b)** 6 **(c)** –18

Do yourself-5

1. $\frac{1}{2}$ **3.** $-\frac{1}{4},-\frac{3}{4},\ldots,-9\frac{1}{4}$ **4.** $1,-1\frac{1}{2},\ldots,-39.$ **5.** $\frac{3}{2},1,\frac{2}{3}$

6. $\frac{16}{3},8,\ldots,27$ **7.** $6\frac{1}{9}.7\frac{6}{7}$ **8.** $\frac{2}{5},\frac{2}{7},\frac{2}{9},\frac{2}{11}$

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Do yourself-6

1. $4+\frac{8}{9}\left(1-\frac{1}{4^{n-1}}\right)-\left(\frac{2n+1}{3\times 4^{n-1}}\right)$

2. $\frac{1-a^n}{(1-a)^2}-\frac{na^n}{1-a}$

3. $\frac{8}{3}$

4. $\frac{9}{8}$ **5.** $n.2^{n+2}-2^{n+1}+2$

Do yourself-7

1. $\frac{n(n+3)}{4}$

2. **(a)** $\frac{n(n+1)(2n+3)}{2}$ **(b)** $n^2(n+1)$

(c) $\frac{1}{4}n(n+1)(n^2+n+3)$ **(d)** $\frac{1}{6}n(n+1)(2n+7)$

(e) $\frac{1}{2}n(n+1)(n^2+3n+1)$ **(f)** $\frac{1}{2}(3^{n+1}+1)-2^{n+1}$

(g) $4^{n+1}-4-n(n+1)(n^2-n-1)$

3. **(a)** $\frac{n}{4}(n+1)(n+6)(n+7)$ **(b)** $\frac{n}{4}(n+1)(n+8)(n+9)$

4. **(a)** $3n^2+n;\ n(n+1)^2$

(b) $3\,.\,2^n+n+2;\ 6(2^n-1)+\frac{n(n+5)}{2}$

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EXERCISE (O-1)

1. D	**2.** A	**3.** B	**4.** D	**5.** A	**6.** B	**7.** C
8. A	**9.** B	**10.** A	**11.** C	**12.** C	**13.** A	**14.** B
15. C	**16.** C	**17.** B	**18.** B	**19.** D	**20.** C	

EXERCISE (O-2)

1. D	**2.** A	**3.** C	**4.** D	**5.** C	**6.** C	**7.** A
8. A	**9.** B	**10.** D	**11.** B	**12.** B	**13.** B	**14.** D
15. A	**16.** D	**17.** D	**18.** A,B,C,D			

EXERCISE (S-1)

2. 12 **3.** 22 **4.** 2499 **5.** $n = 14$

6. 27 **7.** 2 **8.** $a = 5$, $b = 8$, $c = 12$ **9.** 1

11. (a) 6, 3 (b) 120, 30 **12.** 5 **13.** 7

14. (i) $2^{n+1} - 3$; $2^{n+2} - 4 - 3n$ (ii) $n^2 + 4n + 1$; $(1/6)\,n\,(n+1)\,(2n+13) + n$

15. **(i)** $s_n = (1/24) - [1/\{6(3n+1)(3n+4)\}]$; $s_\infty = 1/24$ **(ii)** $(1/5)\,n\,(n+1)\,(n+2)\,(n+3)\,(n+4)$
(iii) $n/(2n+1)$

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EXERCISE (S-2)

1. $(14n - 6)/(8n + 23)$ **2.** $\frac{27}{2}$ **3.** $n = 38$ **4.** 102

6. (iii) $b = 4, c = 6, d = 9$ **or** $b = -2, c = -6, d = -18$ **7.** $(8, -4, 2, 8)$ **8.** 28

9. $C = 9$; $(3, -3/2, -3/5)$ **10.** $\frac{n(n+1)}{2(n^2+n+1)}$ **11.** 23 **12.** 49

EXERCISE (JM)

1. 4	**2.** 3	**3.** 3	**4.** 1	**5.** 3	**6.** 2	**7.** 1
8. 4	**9.** 4	**10.** 1	**11.** 2	**12.** 1	**13.** 2	**14.** 3
15. 1	**16.** 1	**17.** 3	**18.** 2	**19.** 3	**20.** 1	**21.** 1
22. 3	**23.** 3	**24.** 504	**25.** 1540.00	**26.** 14	**27.** 1	

EXERCISE (JA)

1. C	**2.** 0	**3.** 8	**4.** 9 or 3	**5.** D	**6.** A,D	**7.** 5
8. 4	**9.** 9	**10.** B	**11.** 6	**12.** 3748	**13.** 8.00	**14.** 1.00

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Chapter 05 TRIGONOMETRIC EQUATION

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JEE (Main/Advanced) Syllabus

General solution of trigonometric equations.

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Important Notes

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TRIGONOMETRIC EQUATION

1. TRIGONOMETRIC EQUATION :

An equation involving one or more trigonometrical ratios of unknown angles is called a trigonometrical equation.

2. SOLUTION OF TRIGONOMETRIC EQUATION :

A value of the unknown angle which satisfies the given equation is called a solution of the trigonometric equation.

(a) **Principal solution :-** The solution of the trigonometric equation lying in the interval $[0, 2\pi)$.

(b) **General solution :-** Since all the trigonometric functions are many one & periodic, hence there are infinite values of θ for which trigonometric functions have the same value. All such possible values of θ for which the given trigonometric function is satisfied is given by a general formula. Such a general formula is called general solution of trigonometric equation.

(c) **Particular solution :-** The solution of the trigonometric equation lying in the given interval.

3. GENERAL SOLUTIONS OF SOME TRIGONOMETRIC EQUATIONS (TO BE REMEMBERED) :

(a) If $\sin\theta = 0$, then $\theta = n\pi$, $n \in I$ (set of integers)

(b) If $\cos\theta = 0$, then $\theta = (2n+1)\frac{\pi}{2}$, $n \in I$

(c) If $\tan\theta = 0$, then $\theta = n\pi$, $n \in I$

(d) If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha$ where $\alpha \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$

(e) If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$, $n \in I$, $\alpha \in [0,\pi]$

(f) If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$, $n \in I$, $\alpha \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

(g) If $\sin\theta = 1$, then $\theta = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}$, $n \in I$

(h) If $\cos\theta = 1$ then $\theta = 2n\pi$, $n \in I$

(i) If $\sin^2\theta = \sin^2\alpha$ or $\cos^2\theta = \cos^2\alpha$ or $\tan^2\theta = \tan^2\alpha$, then $\theta = n\pi \pm \alpha$, $n \in I$

(j) For $n \in I$, $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$, $n \in I$

$\sin(n\pi + \theta) = (-1)^n \sin\theta$ $\cos(n\pi + \theta) = (-1)^n \cos\theta$

(k) $\cos n\pi = (-1)^n$, $n \in I$

If n is an odd integer, then $\sin\frac{n\pi}{2} = (-1)^{\frac{n-1}{2}}, \cos\frac{n\pi}{2} = 0,$

$$\sin\left(\frac{n\pi}{2} + \theta\right) = (-1)^{\frac{n-1}{2}}\cos\theta$$

$$\cos\left(\frac{n\pi}{2} + \theta\right) = (-1)^{\frac{n+1}{2}}\sin\theta$$

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Illustration 1 : Find the set of values of x for which $\frac{\tan 3x - \tan 2x}{1+\tan 3x.\tan 2x} = 1$.

Solution : We have, $\frac{\tan 3x - \tan 2x}{1+\tan 3x.\tan 2x} = 1 \Rightarrow \tan(3x - 2x) = 1 \Rightarrow \tan x = 1$

$\Rightarrow \tan x = \tan\frac{\pi}{4} \Rightarrow x = n\pi + \frac{\pi}{4}, n \in I$ {using $\tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha$)

But for this value of x, tan 2x is not defined.

Hence the solution set for x is ϕ. **Ans.**

Do yourself-1 :

1. Find general solutions of the following equations :

(a) $\sin\theta = \frac{1}{2}$ (b) $\cos\left(\frac{3\theta}{2}\right) = 0$ (c) $\tan\left(\frac{3\theta}{4}\right) = 0$

(d) $\cos^2 2\theta = 1$ (e) $\sqrt{3}\sec 2\theta = 2$ (f) $\operatorname{cosec}\left(\frac{\theta}{2}\right) = -1$

2. cos 15 x = sin 5x if

(A) $x = -\frac{\pi}{20} + \frac{n\pi}{5}, n \in I$ (B) $x = \frac{\pi}{40} + \frac{n\pi}{10}, n \in I$

(C) $x = \frac{3\pi}{20} + \frac{n\pi}{5}, n \in I$ (D) $x = -\frac{3\pi}{40} + \frac{n\pi}{10}, n \in I$

3. tan (pπ/4) = cot (qπ/4) if :

(A) p + q = 0 (B) p + q = 2n + 1

(C) p + q = 2n (D) p + q = 2 (2n + 1)

4. Solve $\tan 2\theta = \tan\frac{2}{\theta}$.

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4. IMPORTANT POINTS TO BE REMEMBERED WHILE SOLVING TRIGONOMETRIC EQUATIONS :

(a) For equations of the type sin θ = k or cos θ = k, one must check that $|k| \le 1$.

(b) Avoid squaring the equations, if possible, because it may lead to extraneous solutions. Reject extra solutions if they do not satisfy the given equation.

(c) Do not cancel the common variable factor from the two sides of the equations which are in a product because we may loose some solutions.

(d) The answer should not contain such values of θ, which make any of the terms undefined or infinite.

(i) Check that denominator is not zero at any stage while solving equations.

(ii) If tan θ or sec θ is involved in the equations, θ should not be odd multiple of $\frac{\pi}{2}$.

(iii) If cot θ or cosec θ is involved in the equation, θ should not be multiple of π or 0.

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5. DIFFERENT STRATEGIES FOR SOLVING TRIGONOMETRIC EQUATIONS :

(a) Solving trigonometric equations by factorisation.

e.g. $(2 \sin x - \cos x)(1 + \cos x) = \sin^2 x$

$\therefore$ $(2 \sin x - \cos x)(1 + \cos x) - (1 - \cos^2 x) = 0$

$\therefore$ $(1 + \cos x)(2 \sin x - \cos x - 1 + \cos x) = 0$

$\therefore$ $(1 + \cos x)(2 \sin x - 1) = 0$

$\Rightarrow$ $\cos x = -1$ or $\sin x = \frac{1}{2}$

$\Rightarrow$ $\cos x = -1 = \cos\pi$ $\Rightarrow$ $x = 2n\pi + \pi = (2n+1)\pi,\ n \in I$

or $\sin x = \frac{1}{2} = \sin\frac{\pi}{6}$ $\Rightarrow$ $x = k\pi + (-1)^k \frac{\pi}{6},\ k \in I$

Illustration 2 : If $\frac{1}{6}\sin\theta$, $\cos\theta$ and $\tan\theta$ are in G.P. then the general solution for θ is -

(A) $2n\pi \pm \frac{\pi}{3}$ (B) $2n\pi \pm \frac{\pi}{6}$ (C) $n\pi \pm \frac{\pi}{3}$ (D) none of these

Solution : Since, $\frac{1}{6}\sin\theta$, $\cos\theta$, $\tan\theta$ are in G.P.

$\Rightarrow$ $\cos^2\theta = \frac{1}{6}\sin\theta \cdot \tan\theta$ $\Rightarrow$ $6\cos^3\theta + \cos^2\theta - 1 = 0$

$\therefore$ $(2\cos\theta - 1)(3\cos^2\theta + 2\cos\theta + 1) = 0$

$\Rightarrow$ $\cos\theta = \frac{1}{2}$ (other values of $\cos\theta$ are imaginary)

$\Rightarrow$ $\cos\theta = \cos\frac{\pi}{3}$ $\Rightarrow$ $\theta = 2n\pi \pm \frac{\pi}{3},\ n \in I.$ **Ans. (A)**

(b) Solving of trigonometric equation by reducing it to a quadratic equation.

e.g. $6 - 10\cos x = 3\sin^2 x$

$\therefore$ $6 - 10\cos x = 3 - 3\cos^2 x$ $\Rightarrow$ $3\cos^2 x - 10\cos x + 3 = 0$

$\Rightarrow$ $(3\cos x - 1)(\cos x - 3) = 0$ $\Rightarrow$ $\cos x = \frac{1}{3}$ or $\cos x = 3$

Since $\cos x = 3$ is not possible as $-1 \le \cos x \le 1$

$\therefore$ $\cos x = \frac{1}{3} = \cos\left(\cos^{-1}\frac{1}{3}\right)$ $\Rightarrow$ $x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right),\ n \in I$

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Illustration 3 : Solve $\sin^2\theta - \cos\theta = \frac{1}{4}$ for θ and write the values of θ in the interval $0 \le \theta \le 2\pi$.

Solution : The given equation can be written as

$1 - \cos^2\theta - \cos\theta = \frac{1}{4}$ $\Rightarrow$ $\cos^2\theta + \cos\theta - 3/4 = 0$

$\Rightarrow$ $4\cos^2\theta + 4\cos\theta - 3 = 0$ $\Rightarrow$ $(2\cos\theta - 1)(2\cos\theta + 3) = 0$

$\Rightarrow$ $\cos\theta = \frac{1}{2}, -\frac{3}{2}$

Since, $\cos\theta = -3/2$ is not possible as $-1 \le \cos\theta \le 1$

$\therefore$ $\cos\theta = \frac{1}{2}$ $\Rightarrow$ $\cos\theta = \cos\frac{\pi}{3}$ $\Rightarrow$ $\theta = 2n\pi \pm \frac{\pi}{3}, n \in I$

For the given interval, $n = 0$ and $n = 1$.

$\Rightarrow$ $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$ **Ans.**

Illustration 4 : Find the number of solutions of $\tan x + \sec x = 2\cos x$ in $[0, 2\pi]$.

Solution : Here, $\tan x + \sec x = 2\cos x$ $\Rightarrow$ $\sin x + 1 = 2\cos^2 x$

$\Rightarrow$ $2\sin^2 x + \sin x - 1 = 0$ $\Rightarrow$ $\sin x = \frac{1}{2}, -1$

But $\sin x = -1 \Rightarrow x = \frac{3\pi}{2}$ for which $\tan x + \sec x = 2\cos x$ is not defined.

Thus $\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\Rightarrow$ number of solutions of $\tan x + \sec x = 2\cos x$ is 2. **Ans.**

Illustration 5 : Solve the equation $5\sin^2 x - 7\sin x \cos x + 16\cos^2 x = 4$

Solution : To solve this equation we use the fundamental formula of trigonometric identities,

$\sin^2 x + \cos^2 x = 1$

writing the equation in the form,

$5\sin^2 x - 7\sin x \,.\, \cos x + 16\cos^2 x = 4(\sin^2 x + \cos^2 x)$

$\Rightarrow$ $\sin^2 x - 7\sin x \cos x + 12\cos^2 x = 0$

dividing by $\cos^2 x$ on both side we get,

$\tan^2 x - 7\tan x + 12 = 0$

Now it can be factorized as :

$(\tan x - 3)(\tan x - 4) = 0$

$\Rightarrow$ $\tan x = 3, 4$

i.e., $\tan x = \tan(\tan^{-1} 3)$ or $\tan x = \tan(\tan^{-1} 4)$

$\Rightarrow$ $x = n\pi + \tan^{-1} 3$ or $x = n\pi + \tan^{-1} 4$, $n \in I$. **Ans.**

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Illustration 6 : If $x \neq \frac{n\pi}{2}, n \in I$ and $(\cos x)^{\sin^2 x - 3\sin x + 2} = 1$, then find the general solutions of x.

Solution : As $x \neq \frac{n\pi}{2} \Rightarrow \cos x \neq 0, 1, -1$

So, $(\cos x)^{\sin^2 x - 3\sin x + 2} = 1 \quad \Rightarrow \sin^2 x - 3\sin x + 2 = 0$

$\therefore \quad (\sin x - 2)(\sin x - 1) = 0 \Rightarrow \sin x = 1, 2$

where sinx = 2 is not possible and sinx = 1 which is also not possible as $x \neq \frac{n\pi}{2}$

$\therefore$ no general solution is possible. **Ans.**

Illustration 7 : Solve the equation $\sin^4 x + \cos^4 x = \frac{7}{2}\sin x \,.\, \cos x$.

Solution : $\sin^4 x + \cos^4 x = \frac{7}{2}\sin x \,.\, \cos x \quad \Rightarrow \quad (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x = \frac{7}{2}\sin x \,.\, \cos x$

$\Rightarrow \quad 1 - \frac{1}{2}(\sin 2x)^2 = \frac{7}{4}(\sin 2x) \Rightarrow \quad 2\sin^2 2x + 7\sin 2x - 4 = 0$

$\Rightarrow \quad (2\sin 2x - 1)(\sin 2x + 4) = 0 \quad \Rightarrow \quad \sin 2x = \frac{1}{2}$ or $\sin 2x = -4$ (which is not possible)

$\Rightarrow \quad 2x = n\pi + (-1)^n \frac{\pi}{6}, n \in I$

i.e., $x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}, n \in I$ **Ans.**

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Do yourself-2 :

1. Solve the following equations :

(a) $3\sin x + 2\cos^2 x = 0$ (b) $\sec^2 2\alpha = 1 - \tan 2\alpha$

(c) $7\cos^2\theta + 3\sin^2\theta = 4$ (d) $4\cos\theta - 3\sec\theta = \tan\theta$

2. Solve the equation : $2\sin^2\theta + \sin^2 2\theta = 2$ for $\theta \in (-\pi, \pi)$.

3. Solve $\cos^3 x + \cos^2 x - 4\cos^2\frac{x}{2} = 0$

4. Solve $\cot^2\theta + 3\operatorname{cosec}\theta + 3 = 0$

5. The general solution of the equation, $\frac{1 - \sin x + \ldots + (-1)^n \sin^n x + \ldots\infty}{1 + \sin x + \ldots + \sin^n x + \ldots\infty} = \frac{1 - \cos 2x}{1 + \cos 2x}$ is

(A) $(-1)^n \pi/3 + n\pi$ (B) $(-1)^n \pi/6 + n\pi$ (C) $(-1)^{n+1} \pi/6 + n\pi$ (D) $(-1)^{n-1} \pi/3 + n\pi$

6. If sum of all solutions of the equation $6\sin\frac{x}{2} = \sec\frac{x}{2}$

7. If sum of all possible values of $x \in (0, 2\pi)$ satisfying the equation

$2\cos x \cdot \operatorname{cosec} x - 4\cos x - \operatorname{cosec} x = -2$, is $\frac{k\pi}{4}$, then k is equal to

(A) 9 (B) 12 (C) 16 (D) 32

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(c) Solving trigonometric equations by introducing an auxilliary argument.

Consider, $a \sin\theta + b\cos\theta = c$ ..... (i)

$$\therefore \quad \frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta = \frac{c}{\sqrt{a^2+b^2}}$$

equation (i) has a solution only if $|c| \le \sqrt{a^2+b^2}$

let $\frac{a}{\sqrt{a^2+b^2}} = \cos\phi$, $\frac{b}{\sqrt{a^2+b^2}} = \sin\phi$ & $\phi = \tan^{-1}\frac{b}{a}$

by introducing this auxillary argument ϕ, equation (i) reduces to

$\sin(\theta+\phi) = \frac{c}{\sqrt{a^2+b^2}}$ Now this equation can be solved easily.

Illustration 8 : Find the number of distinct solutions of $\sec x + \tan x = \sqrt{3}$, where $0 \le x \le 3\pi$.

Solution : Here, $\sec x + \tan x = \sqrt{3} \Rightarrow 1 + \sin x = \sqrt{3}\cos x$

or $\sqrt{3}\cos x - \sin x = 1$

dividing both sides by $\sqrt{a^2+b^2}$ i.e. $\sqrt{4} = 2$, we get

$$\Rightarrow \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = \frac{1}{2}$$

$$\Rightarrow \cos\frac{\pi}{6}\cos x - \sin\frac{\pi}{6}\sin x = \frac{1}{2} \Rightarrow \cos\left(x+\frac{\pi}{6}\right) = \frac{1}{2}$$

As $0 \le x \le 3\pi$

$$\frac{\pi}{6} \le x + \frac{\pi}{6} \le 3\pi + \frac{\pi}{6}$$

$$\Rightarrow x + \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3} \quad \Rightarrow \quad x = \frac{\pi}{6}, \frac{3\pi}{2}, \frac{13\pi}{6}$$

But at $x = \frac{3\pi}{2}$, tanx and secx is not defined.

$\therefore$ Total number of solutions are 2. **Ans.**

Illustration 9 : Prove that the equation $k\cos x - 3\sin x = k + 1$ possess a solution iff $k \in (-\infty, 4]$.

Solution : Here, $k\cos x - 3\sin x = k + 1$, could be re-written as :

$$\frac{k}{\sqrt{k^2+9}}\cos x - \frac{3}{\sqrt{k^2+9}}\sin x = \frac{k+1}{\sqrt{k^2+9}}$$

or $\cos(x+\phi) = \frac{k+1}{\sqrt{k^2+9}}$, where $\tan\phi = \frac{3}{k}$

which possess a solution only if $-1 \le \frac{k+1}{\sqrt{k^2+9}} \le 1$

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i.e., $\left|\dfrac{k+1}{\sqrt{k^2+9}}\right| \le 1$

i.e., $(k+1)^2 \le k^2+9$

i.e., $k^2 + 2k + 1 \le k^2 + 9$

or $k \le 4$

$\Rightarrow$ The interval of k for which the equation ($k\cos x - 3\sin x = k + 1$) has a solution is $(-\infty, 4]$. **Ans.**

Do yourself-3 :

1 Solve the following equations :

(a) $\sin x + \sqrt{2} = \cos x$.

(b) $\operatorname{cosec}\theta = 1 + \cot\theta$.

2. Solve: $\cos\theta + \sin\theta = \cos 2\theta + \sin 2\theta$.

3. Solve $8 \sin x = \dfrac{\sqrt{3}}{\cos x} + \dfrac{1}{\sin x}$

(d) Solving trigonometric equations by transforming sum of trigonometric functions into product.

e.g. $\cos 3x + \sin 2x - \sin 4x = 0$

$\cos 3x - 2 \sin x \cos 3x = 0$

$\Rightarrow$ $(\cos 3x)(1 - 2\sin x) = 0$

$\Rightarrow$ $\cos 3x = 0$ or $\sin x = \dfrac{1}{2}$

$\Rightarrow$ $\cos 3x = 0 = \cos\dfrac{\pi}{2}$ or $\sin x = \dfrac{1}{2} = \sin\dfrac{\pi}{6}$

$\Rightarrow$ $3x = 2n\pi \pm \dfrac{\pi}{2}$ or $x = m\pi + (-1)^m \dfrac{\pi}{6}$

$\Rightarrow$ $x = \dfrac{2n\pi}{3} \pm \dfrac{\pi}{6}$ or $x = m\pi + (-1)^m \dfrac{\pi}{6}$; $(n, m \in I)$

Illustration 10 : Solve : $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0$

Solution : We have $\cos\theta + \cos 7\theta + \cos 3\theta + \cos 5\theta = 0$

$\Rightarrow$ $2\cos 4\theta\cos 3\theta + 2\cos 4\theta\cos\theta = 0$ $\Rightarrow$ $\cos 4\theta(\cos 3\theta + \cos\theta) = 0$

$\Rightarrow$ $\cos 4\theta(2\cos 2\theta\cos\theta) = 0$

$\Rightarrow$ Either $\cos\theta = 0 \Rightarrow \theta = (2n_1 + 1)\pi/2,\ n_1 \in I$

or $\cos 2\theta = 0 \Rightarrow \theta = (2n_2 + 1)\dfrac{\pi}{4}, n_2 \in I$

or $\cos 4\theta = 0 \Rightarrow \theta = (2n_3 + 1)\dfrac{\pi}{8},\ n_3 \in I$ **Ans.**

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(e) Solving trigonometric equations by transforming a product into sum.

e.g. $\sin 5x.\cos 3x = \sin 6x.\cos 2x$

$\sin 8x + \sin 2x = \sin 8x + \sin 4x$

$\therefore \quad 2\sin 2x\,.\cos 2x - \sin 2x = 0$

$\Rightarrow \quad \sin 2x(2\cos 2x - 1) = 0$

$\Rightarrow \quad \sin 2x = 0 \quad \text{or} \quad \cos 2x = \frac{1}{2}$

$\Rightarrow \quad \sin 2x = 0 = \sin 0 \quad \text{or} \quad \cos 2x = \frac{1}{2} = \cos\frac{\pi}{3}$

$\Rightarrow \quad 2x = n\pi + (-1)^n \times 0,\ n \in I \ \text{or} \quad 2x = 2m\pi \pm \frac{\pi}{3},\ m \in I$

$\Rightarrow \quad x = \frac{n\pi}{2},\ n \in I \quad \text{or} \quad x = m\pi \pm \frac{\pi}{6},\ m \in I$

Illustration 11 : Solve : $\cos\theta \cos 2\theta \cos 3\theta = \frac{1}{4}$; where $0 \le \theta \le \pi$.

Solution : $\frac{1}{2}(2\cos\theta \cos 3\theta)\cos 2\theta = \frac{1}{4} \Rightarrow (\cos 2\theta + \cos 4\theta)\cos 2\theta = \frac{1}{2}$

$\Rightarrow \quad \frac{1}{2}[2\cos^2 2\theta + 2\cos 4\theta \cos 2\theta] = \frac{1}{2} \Rightarrow 1 + \cos 4\theta + 2\cos 4\theta \cos 2\theta = 1$

$\therefore \quad \cos 4\theta\,(1 + 2\cos 2\theta) = 0$

$\cos 4\theta = 0 \quad \text{or} \quad (1 + 2\cos 2\theta) = 0$

Now from the first equation : $2\cos 4\theta = 0 = \cos(\pi/2)$

$\therefore \quad 4\theta = \left(n + \frac{1}{2}\right)\pi \quad \Rightarrow \quad \theta = (2n+1)\frac{\pi}{8},\ n \in I$

for $\quad n = 0,\ \theta = \frac{\pi}{8};\ n = 1,\ \theta = \frac{3\pi}{8};\ n = 2,\ \theta = \frac{5\pi}{8};\ n = 3,\ \theta = \frac{7\pi}{8} \qquad (\because\ 0 \le \theta \le \pi)$

and from the second equation :

$\cos 2\theta = -\frac{1}{2} = -\cos(\pi/3) = \cos(\pi - \pi/3) = \cos(2\pi/3)$

$\therefore \quad 2\theta = 2k\pi \pm 2\pi/3 \quad \therefore \quad \theta = k\pi \pm \pi/3,\ k \in I$

again for $\quad k = 0,\ \theta = \frac{\pi}{3};\ k = 1,\ \theta = \frac{2\pi}{3} \qquad (\because\ 0 \le \theta \le \pi)$

$\therefore \quad \theta = \frac{\pi}{8}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{7\pi}{8}$ **Ans.**

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Illustration 12 : The general solution of the trigonometric equation $\sin x \cos 2x + \sin 2x \cos 5x = \sin 3x \cos 5x$, is

(A) $\frac{n\pi}{3}$ (B) $\frac{2n\pi}{9}$ (C) $2n\pi$ (D) $\frac{n\pi}{3} \cup \frac{2n\pi}{9}$

(where $n \in I$)

Solution : $2 \sin x \cos 2x + 2 \sin 2x \cos 5x = 2 \sin 3x \cos 5x$

$\sin 3x - \sin x + \sin 7x - \sin 3x = \sin 8x - \sin 2x$

$\sin 7x - \sin x = \sin 8x - \sin 2x$

$2 \cos 4x \sin 3x = 2 \cos 5x \sin 3x$

$2 \sin 3x [\cos 5x - \cos 4x] = 0$

$\sin 3x = 0 \Rightarrow x = \frac{n\pi}{3}$

if $\cos 5x - \cos 4x = 0$

$2 \sin \frac{9x}{2} \sin \frac{x}{2} = 0$

$\therefore \quad x = \frac{2n\pi}{9}$ or $2n\pi$

Hence general solution is $\frac{n\pi}{3} \cup \frac{2n\pi}{9}, n \in I$]

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Do yourself-4 :

1. Solve $4\sin\theta \sin 2\theta \sin 4\theta = \sin 3\theta$.

2. Solve for x : $\sin x + \sin 3x + \sin 5x = 0$.

3. Solve $\sin 2x + 5\sin x + 1 + 5\cos x = 0$

4. Solve $3\cos x + 3\sin x + \sin 3x - \cos 3x = 0$

5. Solve $(1 - \sin 2x)(\cos x - \sin x) = 1 - 2\sin^2 x$.

6. The number of integral values of a for which the equation $\cos 2x + a \sin x = 2a - 7$ possesses a solution is

(A) 2 (B) 3 (C) 4 (D) 5

(f) Solving equations by a change of variable :

(i) Equations of the form $P(\sin x \pm \cos x, \sin x . \cos x) = 0$, where $P(y,z)$ is a polynomial, can be solved by the substitution :

$\cos x \pm \sin x = t \quad \Rightarrow \quad 1 \pm 2 \sin x . \cos x = t^2$.

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Illustration 13 : Solve : $\sin x + \cos x = 1 + \sin x . \cos x$.

Solution : put $\sin x + \cos x = t$

$\Rightarrow \sin^2 x + \cos^2 x + 2\sin x . \cos x = t^2$

$\Rightarrow 2\sin x \cos x = t^2 - 1 \quad (\because \sin^2 x + \cos^2 x = 1)$

$\Rightarrow \sin x.\cos x = \left(\frac{t^2-1}{2}\right)$

Substituting above result in given equation, we get :

$t = 1 + \frac{t^2-1}{2}$

$\Rightarrow 2t = t^2 + 1 \Rightarrow t^2 - 2t + 1 = 0$

$\Rightarrow (t-1)^2 = 0 \quad \Rightarrow \quad t = 1$

$\Rightarrow \sin x + \cos x = 1$

Dividing both sides by $\sqrt{1^2+1^2}$ i.e. $\sqrt{2}$, we get

$\Rightarrow \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{1}{\sqrt{2}} \Rightarrow \cos x \cos\frac{\pi}{4} + \sin x.\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\Rightarrow \cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{4} \quad \Rightarrow \quad x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

$\Rightarrow x = 2n\pi$ or $x = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}, n \in I$

(ii) Equations of the form of $a\sin x + b\cos x + d = 0$, where a, b & d are real numbers can be solved by changing sin x & cos x into their corresponding tangent of half the angle.

Illustration 14 : Solve : $3\cos x + 4\sin x = 5$

Solution : $\Rightarrow 3\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right) + 4\left(\frac{2\tan x/2}{1+\tan^2 x/2}\right) = 5$

$\Rightarrow \frac{3 - 3\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} + \frac{8\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = 5$

$\Rightarrow 3 - 3\tan^2\frac{x}{2} + 8\tan\frac{x}{2} = 5 + 5\tan^2\frac{x}{2} \Rightarrow 8\tan^2\frac{x}{2} - 8\tan\frac{x}{2} + 2 = 0$

$\Rightarrow 4\tan^2\frac{x}{2} - 4\tan\frac{x}{2} + 1 = 0 \quad \Rightarrow \left(2\tan\frac{x}{2} - 1\right)^2 = 0$

$\Rightarrow 2\tan\frac{x}{2} - 1 = 0 \Rightarrow \tan\frac{x}{2} = \frac{1}{2} = \tan\left(\tan^{-1}\frac{1}{2}\right)$

$\Rightarrow \frac{x}{2} = n\pi + \tan^{-1}\left(\frac{1}{2}\right), n \in I \quad \Rightarrow \quad x = 2n\pi + 2\tan^{-1}\frac{1}{2}, n \in I$

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(g) Solving trigonometric equations with the use of the boundness of the functions involved.

Illustration 15 : Solve the equation $(\sin x + \cos x)^{1+\sin 2x} = 2$, when $0 \le x \le \pi$.

Solution : We know, $-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$ and $-1 \le \sin\theta \le 1$.

$\therefore$ $(\sin x + \cos x)$ admits the maximum value as $\sqrt{2}$

and $(1 + \sin 2x)$ admits the maximum value as 2.

Also $\left(\sqrt{2}\right)^2 = 2$.

$\therefore$ the equation could hold only when, $\sin x + \cos x = \sqrt{2}$ and $1 + \sin 2x = 2$

Now, $\sin x + \cos x = \sqrt{2}$ $\Rightarrow$ $\cos\left(x - \frac{\pi}{4}\right) = 1$

$\Rightarrow$ $x = 2n\pi + \pi/4,\ n \in I$ ..... (i)

and $1 + \sin 2x = 2$ $\Rightarrow$ $\sin 2x = 1 = \sin\frac{\pi}{2}$

$\Rightarrow$ $2x = m\pi + (-1)^m\frac{\pi}{2},\ m \in I$ $\Rightarrow$ $x = \frac{m\pi}{2} + (-1)^m\frac{\pi}{4}$ ..... (ii)

The value of x in $[0, \pi]$ satisfying equations (i) and (ii) is $x = \frac{\pi}{4}$ (when $n = 0$ & $m = 0$)

Ans.

Note : $\sin x + \cos x = -\sqrt{2}$ and $1 + \sin 2x = 2$ also satisfies but as $x \ge 0$, this solution is not in domain.

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Illustration 16 : Solve for x and y : $2^{\frac{1}{\cos^2 x}}\sqrt{y^2 - y + 1/2} \le 1$

Solution : $2^{\frac{1}{\cos^2 x}}\sqrt{y^2 - y + 1/2} \le 1$ ..... (i)

$$2^{\frac{1}{\cos^2 x}}\sqrt{\left(y - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \le 1$$

Minimum value of $2^{\frac{1}{\cos^2 x}} = 2$

Minimum value of $\sqrt{\left(y - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}$

$\Rightarrow$ Minimum value of $2^{\frac{1}{\cos^2 x}}\sqrt{y^2 - y + \frac{1}{2}}$ is 1

$\Rightarrow$ (i) is possible when $2^{\frac{1}{\cos^2 x}}\sqrt{\left(y - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1$

$\Rightarrow$ $\cos^2 x = 1$ and $y = 1/2$ $\Rightarrow \cos x = \pm 1 \Rightarrow x = n\pi$, where $n \in I$.

Hence $x = n\pi,\ n \in I$ and $y = 1/2$. **Ans.**

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Illustration 17 : The number of solution(s) of $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2+\frac{1}{x^2}$, $0 \le x \le \pi/2$, is/are -

(A) 0 (B) 1 (C) infinite (D) none of these

Solution : Let $y = 2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2+\frac{1}{x^2}$ $\Rightarrow$ $y = (1 + \cos x)\sin^2 x$ and $y = x^2 + \frac{1}{x^2}$

when $y = (1 + \cos x)\sin^2 x$ = (a number < 2)(a number ≤ 1) $\Rightarrow$ $y < 2$ ..... (i)

and when $y = x^2 + \frac{1}{x^2} = \left(x - \frac{1}{x}\right)^2 + 2 \ge 2$ $\Rightarrow$ $y \ge 2$ ..... (ii)

No value of y can be obtained satisfying (i) and (ii), simultaneously

$\Rightarrow$ No real solution of the equation exists. **Ans. (A)**

Note: If L.H.S. of the given trigonometric equation is always less than or equal to k and RHS is always greater than k, then no solution exists. If both the sides are equal to k for same value of θ, then solution exists and if they are equal for different values of θ, then solution does not exist.

Illustration 18 : The number of ordered pairs (x, y) satisfying $|x| + |y| = 3$ and $\sin\left(\frac{\pi x^2}{3}\right) = 1$ is less than or equal to

(A) 7 (B) 8 (C) 9 (D) 10

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Solution : $\sin\left(\frac{\pi}{3}x^2\right) = 1$

$\frac{\pi}{3}x^2 = 2n\pi + \frac{\pi}{2}$; $n \in I$

$\frac{\pi}{3}x^2 = (4n + 1)\frac{\pi}{2}$

$x^2 = \frac{3}{2}(4n + 1)$; $n \in I$

only n = 0 and n = 1 is possible.

$x^2 = \frac{3}{2}$ or $x^2 = \frac{15}{2}$

$\therefore x = \pm\sqrt{\frac{3}{2}}$ or $x = \pm\sqrt{\frac{15}{2}}$.

for each value of x will get 2 values of y, Hence 8 ordered pairs.]

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Do yourself-5 :

1. If $x^2 - 4x + 5 - \sin y = 0$, $y \in [0, 2\pi)$, then -
(A) $x = 1, y = 0$ (B) $x = 1, y = \pi/2$ (C) $x = 2, y = 0$ (D) $x = 2, y = \pi/2$

2. If $\sin x + \cos x = \sqrt{y + \frac{1}{y}}$, $y > 0$, $x \in [0, \pi]$, then find the least positive value of x satisfying the given condition.

3. Solve $\sin 3x + \cos 2x = -2$

4. Solve $\sqrt{3\sin 5x - \cos^2 x - 3} = 1 - \sin x$

5. The number of real solutions of the equation $\sin(e^x) = 5^x + 5^{-x}$ is-
(A) 0 (B) 1 (C) 2 (D) infinitely many

6. If $\sin^3\alpha + \cos^3\beta + 6\sin\alpha \cdot \cos\beta = 8$, where $\alpha, \beta \in [0, 2\pi]$ then number of ordered pairs (α, β) is equal to
(A) 0 (B) 1 (C) 2 (D) 3

6. TRIGONOMETRIC INEQUALITIES :

There is no general rule to solve trigonometric inequations and the same rules of algebra are valid provided the domain and range of trigonometric functions should be kept in mind.

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Illustration 19 : Find the solution set of inequality $\sin x > 1/2$.

Solution : When $\sin x = \frac{1}{2}$, the two values of x between 0 and 2π are $\pi/6$ and $5\pi/6$.

From the graph of $y = \sin x$, it is obvious that between 0 and 2π,

$\sin x > \frac{1}{2}$ for $\pi/6 < x < 5\pi/6$

Hence, $\sin x > 1/2$

$\Rightarrow \quad 2n\pi + \pi/6 < x < 2n\pi + 5\pi/6, n \in I$

Thus, the required solution set is $\bigcup_{n \in I}\left(2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right)$ **Ans.**

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Illustration 20 : Find the values of α lying between 0 and π for which the inequality : $\tan\alpha > \tan^3\alpha$ is valid.

Solution : We have : $\tan\alpha - \tan^3\alpha > 0 \Rightarrow \tan\alpha\,(1 - \tan^2\alpha) > 0$

$\Rightarrow$ $(\tan\alpha)(\tan\alpha + 1)(\tan\alpha - 1) < 0$

So $\tan\alpha < -1,\ 0 < \tan\alpha < 1$

$\therefore$ Given inequality holds for $\alpha \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$ **Ans.**

Do yourself - 6 :

(i) Find the solution set of the inequality : $\cos x \geq -1/2$.

(ii) Find the values of x in the interval $[0, 2\pi]$ for which $4\sin^2 x - 8\sin x + 3 \leq 0$.

Miscellaneous Illustration :

Illustration 21 : Solve the following equation : $\tan^2\theta + \sec^2\theta + 3 = 2(\sqrt{2}\sec\theta + \tan\theta)$

Solution : We have $\tan^2\theta + \sec^2\theta + 3 = 2\sqrt{2}\sec\theta + 2\tan\theta$

$\Rightarrow$ $\tan^2\theta - 2\tan\theta + \sec^2\theta - 2\sqrt{2}\sec\theta + 3 = 0$

$\Rightarrow$ $\tan^2\theta + 1 - 2\tan\theta + \sec^2\theta - 2\sqrt{2}\sec\theta + 2 = 0$

$\Rightarrow$ $(\tan\theta - 1)^2 + (\sec\theta - \sqrt{2})^2 = 0 \Rightarrow \tan\theta = 1$ and $\sec\theta = \sqrt{2}$

As the periodicity of $\tan\theta$ and $\sec\theta$ are not same, we get

$\theta = 2n\pi + \frac{\pi}{4},\ n \in I$

Ans.

Illustration 22 : Find the solution set of equation $5^{(1 + \log_5 \cos x)} = 5/2$.

Solution : Taking log to base 5 on both sides in given equation :

$(1 + \log_5 \cos x).\ \log_5 5 = \log_5(5/2) \Rightarrow \log_5 5 + \log_5 \cos x = \log_5 5 - \log_5 2$

$\Rightarrow \log_5 \cos x = -\log_5 2 \Rightarrow \cos x = 1/2 \quad \Rightarrow x = 2n\pi \pm \pi/3,\ n \in I$

Ans.

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Illustration 23 : If the set of all values of x in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ satisfying $|4\sin x+\sqrt{2}|<\sqrt{6}$ is $\left(\frac{a\pi}{24},\frac{b\pi}{24}\right)$ then find the value of $\left|\frac{a-b}{3}\right|$.

Solution : $|4\sin x+\sqrt{2}|<\sqrt{6}$

$\Rightarrow \quad -\sqrt{6}<4\sin x+\sqrt{2}<\sqrt{6} \qquad \Rightarrow \quad -\sqrt{6}-\sqrt{2}<4\sin x<\sqrt{6}-\sqrt{2}$

$\Rightarrow \quad \frac{-(\sqrt{6}+\sqrt{2})}{4}<\sin x<\frac{\sqrt{6}-\sqrt{2}}{4} \qquad \Rightarrow \quad -\frac{5\pi}{12}<x<\frac{\pi}{12}$ for $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Comparing with $\frac{a\pi}{24}<x<\frac{b\pi}{24}$, we get, $a=-10, b=2$

$\therefore \left|\frac{a-b}{3}\right|=\left|\frac{-10-2}{3}\right|=4$ **Ans.**

Illustration 24 : The number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3\sin^2x-7\sin x+2=0$ is - **[JEE 98]**

(A) 0 (B) 5 (C) 6 (D) 10

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Solution : $3\sin^2x-7\sin x+2=0$

$\Rightarrow \quad (3\sin x-1)(\sin x-2)=0$

$\because \quad \sin x\neq 2$

$\Rightarrow \quad \sin x=\frac{1}{3}=\sin\alpha$ (say)

where α is the least positive value of x such that $\sin\alpha=\frac{1}{3}$.

Clearly $0<\alpha<\frac{\pi}{2}$. We get the solution,

$x=\alpha, \pi-\alpha, 2\pi+\alpha, 3\pi-\alpha, 4\pi+\alpha$ and $5\pi-\alpha$.

Hence total six values in $[0, 5\pi]$ **Ans. (C)**

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EXERCISE (O-1)

1. If $2\tan^2\theta = \sec^2\theta$, then the general solution of θ -

(A) $n\pi + \frac{\pi}{4}(n\in I)$ (B) $n\pi - \frac{\pi}{4}(n\in I)$ (C) $n\pi \pm \frac{\pi}{4}(n\in I)$ (D) $2n\pi \pm \frac{\pi}{4}(n\in I)$

TE0001

2. If $\frac{1-\cos 2\theta}{1+\cos 2\theta} = 3$, then the general solution of θ is -

(A) $2n\pi \pm \pi/6$ (B) $n\pi \pm \pi/6$ (C) $2n\pi \pm \pi/3$ (D) $n\pi \pm \pi/3$

where $n \in I$

TE0002

3. The general value of θ satisfying $\sin^2\theta + \sin\theta = 2$ is-

(A) $n\pi\,(-1)^n\frac{\pi}{6}$ (B) $2n\pi + \frac{\pi}{4}$ (C) $n\pi + (-1)^n\frac{\pi}{2}$ (D) $n\pi + (-1)^n\frac{\pi}{3}$

TE0003

4. Let $A = \{\theta : \sin(\theta) = \tan(\theta)\}$ and $B = \{\theta : \cos(\theta) = 1\}$ be two sets. Then -

(A) $A = B$ (B) $A \subset B$ and $B - A \neq \phi$

(C) $A \not\subset B$ (D) $B \not\subset A$

TE0004

5. The solution set of $(5 + 4\cos\theta)(2\cos\theta + 1) = 0$ in the interval $[0, 2\pi]$ is :

(A) $\left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}$ (B) $\left\{\frac{\pi}{3}, \pi\right\}$ (C) $\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$ (D) $\left\{\frac{2\pi}{3}, \frac{5\pi}{3}\right\}$

TE0005

6. The general solution of equation $4\cos^2 x + 6\sin^2 x = 5$ is -

(A) $x = n\pi \pm \frac{\pi}{2}(n\in I)$ (B) $x = n\pi \pm \frac{\pi}{4}(n\in I)$

(C) $x = n\pi \pm \frac{3\pi}{2}(n\in I)$ (D) None of these

TE0006

7. If $\tan^2\theta - (1 + \sqrt{3})\tan\theta + \sqrt{3} = 0$, then the general value of θ is :

(A) $n\pi + \frac{\pi}{4}, n\pi + \frac{\pi}{3}$ (B) $n\pi - \frac{\pi}{4}, n\pi + \frac{\pi}{3}$ (C) $n\pi + \frac{\pi}{4}, n\pi - \frac{\pi}{3}$ (D) $n\pi - \frac{\pi}{4}, n\pi - \frac{\pi}{3}$

where $n \in I$

TE0007

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8. The general solution of the equation $\tan^2\alpha + 2\sqrt{3}\tan\alpha = 1$ is given by -

(A) $\alpha = \frac{n\pi}{2}(n\in I)$ (B) $\alpha = (2n+1)\frac{\pi}{2}(n\in I)$

(C) $\alpha = (6n+1)\frac{\pi}{12}(n\in I)$ (D) $\alpha = \frac{n\pi}{12}(n\in I)$

TE0008

9. The number of solutions of the equation $\sin 2x - 2\cos x + 4\sin x = 4$ in the interval $[0, 5\pi]$ is -

(A) 6 (B) 4 (C) 3 (D) 5

TE0009

10. The set of angles between 0 and 2π satisfying the equation $4\cos^2\theta - 2\sqrt{2}\cos\theta - 1 = 0$ is

(A) $\left\{\frac{\pi}{12}, \frac{5\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}\right\}$ (B) $\left\{\frac{\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}\right\}$

(C) $\left\{\frac{5\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}\right\}$ (D) $\left\{\frac{\pi}{12}, \frac{7\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}\right\}$

TE0010

11. If $\tan\theta + \tan4\theta + \tan7\theta = \tan\theta\tan4\theta\tan7\theta$, then $\theta =$

(A) $\frac{n\pi}{4}$ (B) $\frac{n\pi}{7}$ (C) $\frac{n\pi}{12}$ (D) $n\pi$

where $n \in I$

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TE0011

12. If $\frac{\tan 2\theta + \tan\theta}{1 - \tan\theta\tan 2\theta} = 0$, then the general value of θ is -

(A) $n\pi$; $n \in I$ (B) $\frac{n\pi}{3}$; $n \in I$ (C) $\frac{n\pi}{4}$ (D) $\frac{n\pi}{6}$; $n \in I$

where $n \in I$

TE0012

13. The smallest positive angle satisfying the equation $1 + \cos3x - 2\cos2x = 0$, is equal to

(A) 15° (B) 22.5° (C) 30° (D) 45°

TE0013

14. If $\tan\theta - \sqrt{2}\sec\theta = \sqrt{3}$, then the general solution of θ is -

(A) $n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{3}$ (B) $n\pi + (-1)^n\frac{\pi}{3} - \frac{\pi}{4}$

(C) $n\pi + (-1)^n\frac{\pi}{3} + \frac{\pi}{4}$ (D) $n\pi + (-1)^n\frac{\pi}{4} + \frac{\pi}{3}$

where $n \in I$

TE0014

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15. Number of principal solution(s) of the equation $4 \cdot 16^{\sin^2 x} = 2^{6 \sin x}$ is

(A) 1 (B) 2 (C) 3 (D) 4

TE0015

16. The most general values of x for which $\sin x + \cos x = \min_{a \in R} \{1, a^2 - 4a + 6\}$ is given by-

(A) $2n\pi$ (B) $2n\pi + \frac{\pi}{2}$ (C) $n\pi + (-1)^n . \frac{\pi}{4} - \frac{\pi}{4}$ (D) None of these

where $n \in I$

TE0016

17. If the equation $\sin^4 x - (k + 2) \sin^2 x - (k + 3) = 0$ has a solution then k must lie in the interval :

(A) $(-4, -2)$ (B) $[-3, 2)$ (C) $(-4, -3)$ (D) $[-3, -2]$

TE0017

18. Number of values of x satisfying the equation $\log_2(\sin x) + \log_{1/2}(-\cos x) = 0$ in the interval $(-\pi, \pi]$ is equal to-

(A) 0 (B) 1 (C) 2 (D) 3

TE0018

19. The equation $\sin x \cos x = 2$ has :

(A) one solution (B) two solutions (C) infinite solutions (D) no solution

TE0019

20. The number of solutions of the equation $\tan^2 x - \sec^{10} x + 1 = 0$ in $(0, 10)$ is -

(A) 3 (B) 6 (C) 10 (D) 11

TE0020

21. If $x \in \left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$, then the greatest positive solution of $1 + \sin^4 x = \cos^2 3x$ is -

(A) π (B) 2π (C) $\frac{5\pi}{2}$ (D) none of these

TE0021

22. The number of solutions of the equation $\sin x = x^2 + x + 1$ is-

(A) 0 (B) 1 (C) 2 (D) None

TE0022

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23. **Statement-1:** If $\sin\frac{3x}{2}\cos\frac{5y}{3} = k^8 - 4k^4 + 5$, where x, y $\in$ R then exactly four distinct real values of k are possible.

because

Statement-2: $\sin\frac{3x}{2}$ and $\cos\frac{5y}{3}$ both are less than or equal to one and greater than or equal to – 1.

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D) Statement-1 is false, statement-2 is true.

TE0023

24. The number of solutions of the equation $2\cos\left(\frac{x}{2}\right) = 3^x + 3^{-x}$ is-

(A) 1 (B) 2 (C) 3 (D) None

TE0024

25. The equation $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + x^{-2}$, $0 < x \le \frac{\pi}{2}$ has

(A) one real solutions (B) more than one real solutions
(C) no real solution (D) none of the above

TE0025

26. If $0 \le x \le 3\pi$, $0 \le y \le 3\pi$ and cos x. sin y =1, then the possible number of values of the ordered pair (x, y) is -

(A) 6 (B) 12 (C) 8 (D) 15

TE0026

27. Given $a^2 + 2a + \text{cosec}^2\left(\frac{\pi}{2}(a+x)\right) = 0$ then, which of the following holds good?

(A) $a = 1$; $\frac{x}{2} \in I$ (B) $a = -1$; $\frac{x}{2} \in I$
(C) $a \in R$; $x \in \phi$ (D) a , x are finite but not possible to find

TE0027

28. If the equation $\cot^4 x - 2\,\text{cosec}^2 x + a^2 = 0$ has atleast one solution then, sum of all possible integral values of 'a' is equal to

(A) 4 (B) 3 (C) 2 (D) 0

TE0028

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29. The complete solution set of the inequality $\tan^2 x - 2\sqrt{2}\tan x + 1 \le 0$ is-

(A) $n\pi + \frac{\pi}{8} \le x \le \frac{3\pi}{8} + n\pi,\ n \in I$

(B) $n\pi + \frac{\pi}{4} \le x \le \frac{3\pi}{4} + n\pi,\ n \in I$

(C) $n\pi + \frac{\pi}{16} \le x \le \frac{3\pi}{8} + n\pi,\ n \in I$

(D) $n\pi + \frac{\pi}{3} \le x \le \frac{2\pi}{3} + n\pi,\ n \in I$

TE0029

30. Number of integral solution(s) of the inequality $2\sin^2 x - 5\sin x + 2 > 0$ in $x \in [0, 2\pi]$, is-

(A) 3 (B) 4 (C) 5 (D) 6

TE0030

EXERCISE (O-2)

[MULTIPLE OBJECTIVE TYPE]

1. The equation $2\sin\frac{x}{2} . \cos^2 x + \sin^2 x = 2\sin\frac{x}{2} . \sin^2 x + \cos^2 x$ has a root for which

(A) $\sin 2x = 1$ (B) $\sin 2x = -1$ (C) $\cos x = \frac{1}{2}$ (D) $\cos 2x = -\frac{1}{2}$

TE0031

2. $\sin^2 x - \cos 2x = 2 - \sin 2x$ if

(A) $x = n\pi/2,\ n \in I$

(B) $\tan x = 3/2$

(C) $x = (2n + 1)\pi/2,\ n \in I$

(D) $x = n\pi + (-1)^n \sin^{-1}(2/3),\ n \in I$

TE0032

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3. $4\sin^4 x + \cos^4 x = 1$ if

(A) $x = n\pi$

(B) $x = n\pi \pm \frac{1}{2}\cos^{-1}\left(\frac{1}{5}\right)$

(C) $x = \frac{n\pi}{2}$

(D) none of these, $(n \in I)$

TE0033

4. Which of the following set of values of x satisfies the equation

$2^{(2\sin^2 x - 3\sin x + 1)} + 2^{(2 - 2\sin^2 x + 3\sin x)} = 9$ is

(A) $x = x\pi,\ n \in I$

(B) $x = 2n\pi + \frac{\pi}{2},\ n \in I$

(C) $x = n\pi \pm \frac{\pi}{6},\ n \in I$

(D) $x = n\pi \pm \frac{\pi}{3},\ n \in I$

TE0034

5. $5\sin^2 x + \sqrt{3}\sin x\cos x + 6\cos^2 x = 5$ if

(A) $\tan x = -1/\sqrt{3}$

(B) $\sin x = 0$

(C) $x = n\pi + \pi/2,\ n \in I$

(D) $x = n\pi + \pi/6,\ n \in I$

TE0035

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6. If sum of all the solution of the equation $\cot x + \csc x + \sec x = \tan x$ in $[0, 2\pi]$ is $\frac{k\pi}{2}$, then the value of k is greater than

(A) 1 (B) 2 (C) 3 (D) 4

TE0036

7. The equation $\sin x + \cos(k + x) + \cos(k - x) = 2$ has real solution(s), then $\sin k$ can be

(A) $\frac{-3}{4}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

TE0037

8. Which of the following equations have no solution?

(A) $2^{|x|} = \sin x^2$ (B) $3^{|\sin\sqrt{x}|} = |\cos x|$ (C) $x^2 = -\cos x$ (D) $3x^2 = 1 - 2\cos x$

TE0038

9. If $m > 0, n < 5, 0 < m + n < 10$ and $\frac{x^2+5}{2} = x - 2\cos(m + nx)$ has atleast one real root, then

(A) the greatest value of $(m + n)$ is 3π.
(B) the greatest value of $(m + n)$ is 2π.
(C) the least value of $(m + n)$ is π.
(D) the least value of $(m + n)$ is 2π.

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TE0039

10. If $x^4 + 3\cos(ax^2 + bx + c) = 2(x^2 - 2)$ has two solutions with $a, b, c \in (2, 5)$ then

(A) $a + b + c = \pi$ (B) $a - b + c = \pi$ (C) $a + b + c = 3\pi$ (D) $a - b + c = 3\pi$

TE0040

[COMPREHENSION TYPE]

Paragraph for question no. 11 & 12

Let $f(x) = \cos x + \sin x - 1$ and $g(x) = \sin 2x - 2$.

11. Number of solutions of the equation $f(x) = g(x)$ in $x \in \left[-\pi, \frac{7\pi}{2}\right]$ is equal to

(A) 4 (B) 5 (C) 6 (D) 8

TE0041

12. If the equation $f(x) = k$ has atleast one real solution then the number of possible integral values of k is equal to

(A) 1 (B) 2 (C) 3 (D) 4

TE0041

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Paragraph for Question 13 to 15

Let $f(x) = \sin^2 x - (a - 1)\sin x + 2(a - 3)$

On the basis of above information, answer the following questions :

13. If $x \in [0,\pi]$ and $f(x) = 0$ has exactly one real root, then 'a' lies in

(A) (3,5) (B) (2,4) (C) (4,5) (D) none of these

TE0042

14. If $f(x) = 0$ have two real roots in $(0,\pi)$, then $a \in$

(A) (1,2) (B) (3,4) (C) $(3,4) \cup \{5\}$ (D) (3,5)

TE0042

15. If $f(x) \geq 0\ \forall\ x \in R$ then range of 'a' is

(A) $[2,\infty)$ (B) $[4,\infty)$ (C) $(4,\infty)$ (D) none of these

TE0042

[MATRIX MATCH TYPE]

16.

	Column-I		Column-II
(A)	Number of common solutions of the equations $2\cos^2 x - 3\cos x + 1 = 0$ and $\tan\left(\frac{3x}{4}\right) + 1 = 0$, where $-\pi < x \leq 3\pi$, is equal to	(P)	2
(B)	Number of solutions of the equation $\sin^2 x + 2\sin x - 4\cos x - \sin 2x = 0$ in $(0, 2\pi)$ is equal to	(Q)	12
(C)	If the sum of all possible values of $x \in (0, 2\pi)$ satisfying the equation $2\cos x \operatorname{cosec} x - 4\cos x - \operatorname{cosec} x = -2$ is equal to $\frac{k\pi}{4}$ $(k \in N)$, then the value of k is	(R)	4
		(S)	3
(D)	Let $x - y = \frac{1}{4}$ and $\cos(\pi x) \cdot \cos(\pi y) = \frac{1}{2\sqrt{2}}$ where $x, y \in (0, 2)$ then the number of ordered pair(s) (x, y) is equal to	(T)	1

TE0043

TE0044

TE0045

TE0046

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17.

	Column I		Column II
(A)	The number of real roots of the equation $\cos^7 x + \sin^4 x = 1$ in $(-\pi, \pi)$ is	(P)	1
			TE0047
(B)	The number of solutions of the equation $\lvert\cot x\rvert = \cot x + \frac{1}{\sin x}\ (0 < x < \pi)$ is	(Q)	2
			TE0048
(C)	If $\sin\theta + \sin\phi = \frac{1}{2}$ and $\cos\theta + \cos\phi = 2$, then value of $\cot\left(\frac{\theta+\phi}{2}\right)$ is	(R)	0
		(S)	3
			TE0049
(D)	The number of values of $x \in [-2\pi, 2\pi]$, which satisfy $\operatorname{cosec} x = 1 + \cot x$	(T)	not exists
			TE0050

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EXERCISE (S-1)

1. Find all the values of θ satisfying the equation; $\sin\theta + \sin 5\theta = \sin 3\theta$ such that $0 \le \theta \le \pi$.

TE0051

2. Find the number of principal solution of the equation, $\sin x - \sin 3x + \sin 5x = \cos x - \cos 3x + \cos 5x$.

TE0052

3. Find all values of θ between $0°$ & $180°$ satisfying the equation ; $\cos 6\theta + \cos 4\theta + \cos 2\theta + 1 = 0$.

TE0053

4. Find all value of θ, between 0 & π, which satisfy the equation; $\cos\theta \,.\, \cos 2\theta \,.\, \cos 3\theta = 1/4$.

TE0054

5. Find the general solution of the equation, $\sin \pi x + \cos \pi x = 0$. Also find the sum of all solutions in $[0, 100]$.

TE0055

6. Find the range of y such that the equation, $y + \cos x = \sin x$ has a real solution. For $y = 1$, find x such that $0 < x < 2\pi$.

TE0056

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7. Find the general values of θ for which the quadratic function $(\sin\theta)\,x^2 + (2\cos\theta)x + \frac{\cos\theta+\sin\theta}{2}$ is the square of a linear function.

TE0057

8. Solve the equation for x, $5^{\frac{1}{2}} + 5^{\frac{1}{2}+\log_5(\sin x)} = 15^{\frac{1}{2}+\log_{15}\cos x}$

TE0058

9. Solve the equality: $2\sin 11x + \cos 3x + \sqrt{3}\sin 3x = 0$

TE0059

10. Solve for x, the equation $\sqrt{13 - 18\tan x} = 6\tan x - 3$, where $-2\pi < x < 2\pi$.

TE0060

EXERCISE (S-2)

1. Find all the solutions of $4\cos^2 x\ \sin x - 2\sin^2 x = 3\sin x$.

TE0061

2. Solve the equation: $1 + 2\,\text{cosec}\,x = -\frac{\sec^2\frac{x}{2}}{2}$.

TE0062

3. Solve: $\tan^2 x\,.\,\tan^2 3x\,.\,\tan 4x = \tan^2 x - \tan^2 3x + \tan 4x$.

TE0063

4. Find the values of x, between 0 & 2π, satisfying the equation $\cos 3x + \cos 2x = \sin\frac{3x}{2} + \sin\frac{x}{2}$.

TE0064

5. Find the general solution of the trigonometric equation $3^{\left(\frac{1}{2}+\log_3(\cos x+\sin x)\right)} - 2^{\log_2(\cos x-\sin x)} = \sqrt{2}$.

TE0065

6. Prove that the equations

(a) $\sin x \cdot \sin 2x \cdot \sin 3x = 1$ (b) $\sin x \cdot \cos 4x \cdot \sin 5x = -1/2$

have no solution.

TE0066

7. Let $f(x) = \sin^6 x + \cos^6 x + k(\sin^4 x + \cos^4 x)$ for some real number k. Determine

(a) all real numbers k for which $f(x)$ is constant for all values of x.

(b) all real numbers k for which there exists a real number 'c' such that $f(c) = 0$.

(c) If $k = -0.7$, determine all solutions to the equation $f(x) = 0$.

TE0067

8. Determine the smallest positive value of x which satisfy the equation, $\sqrt{1+\sin 2x} - \sqrt{2}\cos 3x = 0$.

TE0068

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9. Find the general solution of the following equation :
$2(\sin x - \cos 2x) - \sin 2x(1 + 2\sin x) + 2\cos x = 0.$

TE0069

10. Solve: $\tan^2 2x + \cot^2 2x + 2\tan 2x + 2\cot 2x = 6.$

TE0070

EXERCISE (JM)

1. Let A and B denote the statements

A : $\cos\alpha + \cos\beta + \cos\gamma = 0$

B : $\sin\alpha + \sin\beta + \sin\gamma = 0$

If $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\frac{3}{2}$, then - **[AIEEE 2009]**

(1) Both **A** and **B** are true
(2) Both **A** and **B** are false
(3) **A** is true and **B** is false
(4) **A** is false and **B** is true

TE0071

2. The possible values of $\theta \in (0, \pi)$ such that $\sin(\theta) + \sin(4\theta) + \sin(7\theta) = 0$ are: **[AIEEE 2011]**

(1) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{4\pi}{9}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{8\pi}{9}$
(2) $\frac{\pi}{4}, \frac{5\pi}{12}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{8\pi}{9}$
(3) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{35\pi}{36}$
(4) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{8\pi}{9}$

TE0072

3. In a ΔPQR, if $3\sin P + 4\cos Q = 6$ and $4\sin Q + 3\cos P = 1$, then the angle R is equal to :

[AIEEE-2012]

(1) $\frac{3\pi}{4}$
(2) $\frac{5\pi}{6}$
(3) $\frac{\pi}{6}$
(4) $\frac{\pi}{4}$

TE0073

4. If $0 \le x < 2\pi$, then the number of real values of x, which satisfy the equation
$\cos x + \cos 2x + \cos 3x + \cos 4x = 0$, is :- **[JEE(Main) 2016]**

(1) 9
(2) 3
(3) 5
(4) 7

TE0074

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5. If sum of all the solutions of the equation $8\cos x\cdot\left(\cos\left(\frac{\pi}{6}+x\right).\cos\left(\frac{\pi}{6}-x\right)-\frac{1}{2}\right)=1$ in $[0, \pi]$ is $k\pi$, then k is equal to : **[JEE(Main) 2018]**

(1) $\frac{13}{9}$ (2) $\frac{8}{9}$ (3) $\frac{20}{9}$ (4) $\frac{2}{3}$

TE0075

6. If $0 \le x < \frac{\pi}{2}$, then the number of values of x for which sin x-sin2x+sin3x = 0, is **[JEE(Main) 19]**

(1) 2 (2) 1 (3) 3 (4) 4

TE0076

7. The sum of all values of $\theta \in \left(0, \frac{\pi}{2}\right)$ satisfying $\sin^2 2\theta + \cos^4 2\theta = \frac{3}{4}$ is : **[JEE(Main) 19]**

(1) $\frac{\pi}{2}$ (2) π (3) $\frac{3\pi}{8}$ (4) $\frac{5\pi}{4}$

TE0077

8. Let S = $\{\theta \in [-2\pi, 2\pi] : 2\cos^2\theta + 3\sin\theta = 0\}$. Then the sum of the elements of S is **[JEE(Main) 19]**

(1) $\frac{13\pi}{6}$ (2) π (3) 2π (4) $\frac{5\pi}{3}$

TE0078

9. All the pairs (x, y) that satisfy the inequality $2\sqrt{\sin^2 x - 2\sin x + 5}\,.\,\frac{1}{4^{\sin^2 y}} \le 1$ also satisfy the eauation. **[JEE(Main) 19]**

(1) sin x = |siny| (2) sin x = 2 sin y (3) 2|sinx| = 3siny (4) 2sin x = siny

TE0079

10. The number of solutions of the equation $1+\sin^4 x = \cos^2 3x,\ x \in \left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$ is : **[JEE(Main) 19]**

(1) 5 (2) 4 (3) 7 (4) 3

TE0080

11. Let S be the set of all $\alpha \in R$ such that the equation, $\cos 2x + \alpha\sin x = 2\alpha - 7$ has a solution. Then S is equal to : **[JEE(Main) 19]**

(1) [2, 6] (2) [3,7] (3) R (4) [1,4]

TE0081

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12. If $\sin^4\alpha + 4\cos^4\beta + 2 = 4\sqrt{2}\sin\alpha\cos\beta$; $\alpha, \beta \in [0, \pi]$, then $\cos(\alpha + \beta) - \cos(\alpha - \beta)$ is equal to : **[JEE(Main) 2019]**

(1) 0 (2) $-\sqrt{2}$ (3) -1 (4) $\sqrt{2}$

TE0082

13. The number of distinct solutions of the equation $\log_{\frac{1}{2}}|\sin x| = 2 - \log_{\frac{1}{2}}|\cos x|$ in the interval $[0, 2\pi]$, is ———. **[JEE(Main) 20]**

TE0083

EXERCISE (JA)

1. The number of values of θ in the interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ such that $\theta \neq \frac{n\pi}{5}$ for n = 0, ±1,±2 and $\tan\theta = \cot 5\theta$ as well as $\sin 2\theta = \cos 4\theta$, is **[JEE 2010, 3]**

TE0084

2. The positive integer value of n > 3 satisfying the equation

$$\frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$$ is **[JEE 2011, 4]**

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TE0085

3. Let $\theta, \varphi \in [0, 2\pi]$ be such that

$$2\cos\theta(1 - \sin\varphi) = \sin^2\theta\left(\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\right)\cos\varphi - 1, \quad \tan(2\pi - \theta) > 0 \text{ and } -1 < \sin\theta < -\frac{\sqrt{3}}{2}.$$

Then φ **cannot** satisfy- **[JEE 2012, 4M]**

(A) $0 < \varphi < \frac{\pi}{2}$ (B) $\frac{\pi}{2} < \varphi < \frac{4\pi}{3}$

(C) $\frac{4\pi}{3} < \varphi < \frac{3\pi}{2}$ (D) $\frac{3\pi}{2} < \varphi < 2\pi$

TE0086

4. For $x \in (0, \pi)$, the equation $\sin x + 2\sin 2x - \sin 3x = 3$ has

(A) infinitely many solutions

(B) three solutions

(C) one solution

(D) no solution **[JEE(Advanced)-2014, 3(–1)]**

TE0087

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5. The number of distinct solutions of equation $\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$ in the interval $[0, 2\pi]$ is **[JEE 2015, 4M, –0M]**

TE0088

6. Let $S = \left\{x \in (-\pi, \pi) : x \neq 0, \pm\frac{\pi}{2}\right\}$. The sum of all distinct solution of the equation $\sqrt{3}\sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$ in the set S is equal to -

[JEE(Advanced)-2016, 3(–1)]

(A) $-\frac{7\pi}{9}$ (B) $-\frac{2\pi}{9}$ (C) 0 (D) $\frac{5\pi}{9}$

TE0089

7. Let a, b, c be three non-zero real numbers such that the equation

$$\sqrt{3}a\cos x + 2b\sin x = c, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

has two distinct real roots α and β with $\alpha + \beta = \frac{\pi}{3}$. Then the value of $\frac{b}{a}$ is _____

[JEE(Advanced)-2018, 3(0)]

TE0090

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Answer the following by appropriately matching the lists based on the information given in the paragraph

Let $f(x) = \sin(\pi\cos x)$ and $g(x) = \cos(2\pi\sin x)$ be two functions defined for $x > 0$. Define the following sets whose elements are written in the increasing order :

$$X = \{x : f(x) = 0\}, \quad Y = \{x : f'(x) = 0\}$$
$$Z = \{x : g(x) = 0\}, \quad W = \{x : g'(x) = 0\}.$$

List-I contains the sets X,Y,Z and W. List -II contains some information regarding these sets.

	List-I		List-II
(I)	X	(P)	$\supseteq \left\{\frac{\pi}{2}, \frac{3\pi}{2}, 4\pi, 7\pi\right\}$
(II)	Y	(Q)	an arithmetic progression
(III)	Z	(R)	NOT an arithmetic progression
(IV)	W	(S)	$\supseteq \left\{\frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}\right\}$
		(T)	$\supseteq \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \pi\right\}$
		(U)	$\supseteq \left\{\frac{\pi}{6}, \frac{3\pi}{4}\right\}$

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8. Which of the following is the only CORRECT combination ? **[JEE(Advanced)-2019, 3(–1)]**

(1) (II), (R), (S) (2) (I), (P), (R)

(3) (II), (Q), (T) (4) (I), (Q), (U)

TE0091

9. Which of the following is the only CORRECT combination ? **[JEE(Advanced)-2019, 3(–1)]**

(1) (IV), (Q), (T) (2) (IV), (P), (R), (S)

(3) (III), (R), (U) (4) (III), (P), (Q), (U)

TE0091

10. Let $f : [0, 2] \to \mathbb{R}$ be the function defined by

$$f(x) = (3 - \sin(2\pi x)) \sin\left(\pi x - \frac{\pi}{4}\right) - \sin\left(3\pi x + \frac{\pi}{4}\right)$$ **[JEE(Advanced)-2020]**

If $\alpha, \beta \in [0, 2]$ are such that $\{x \in [0, 2] : f(x) \geq 0\} = [\alpha, \beta]$, then the value of $\beta - \alpha$ is _____

TE0100

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ANSWERS

Do yourself-1

1. **(a)** $\theta = n\pi + (-1)^n \frac{\pi}{6}, n \in I$ **(b)** $\theta = (2n+1)\frac{\pi}{3}, n \in I$ **(c)** $\theta = \frac{4n\pi}{3}, n \in I$

(d) $\theta = \frac{n\pi}{2}, n \in I$ **(e)** $\theta = n\pi \pm \frac{\pi}{12}, n \in I$

(f) $\theta = 2n\pi + (-1)^{n+1}\pi, n \in I$

2. A,B,C,D **3.** D **4.** $\frac{n\pi}{4} \pm \sqrt{1+\frac{n^2\pi^2}{16}}, n \in I$

Do yourself-2

1. **(a)** $x = n\pi + (-1)^{n+1}\frac{\pi}{6}, n \in I$ **(b)** $\alpha = \frac{n\pi}{2}$ or $\alpha = \frac{k\pi}{2} + \frac{3\pi}{8}, n, k \in I$

(c) $\theta = n\pi \pm \frac{\pi}{3}, n \in I$

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(d) $\theta = n\pi + (-1)^n\alpha$, where $\alpha = \sin^{-1}\left(\frac{\sqrt{17}-1}{8}\right)$ or $\sin^{-1}\left(\frac{-1-\sqrt{17}}{8}\right), n \in I$

2. $\theta = \left\{-\frac{\pi}{4}, -\frac{3\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{2}\right\}$ **3.** $(2n+1)\pi, n \in I$

4. $2n\pi - \frac{\pi}{2}, n \in I$ or $n\pi + (-1)^{n+1}\frac{\pi}{6}, n \in I$ **5.** B **6.** 3

7. B

Do yourself-3

1. **(a)** $x = 2n\pi - \frac{\pi}{4}, n \in I$ **(b)** $2m\pi + \frac{\pi}{2}, m \in I$

2. $2n\pi, n \in I$ or $\frac{2n\pi}{3} + \frac{\pi}{6}, n \in I$ **3.** $x = n\pi + \frac{\pi}{6}, n \in I, x = \frac{n\pi}{2} - \frac{\pi}{12}, n \in I$

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Do yourself-4

1. $\theta = n\pi$ or $\theta = \frac{m\pi}{3} \pm \frac{\pi}{9}$; $n,m \in I$
2. $x = \frac{n\pi}{3}$, $n \in I$ and $k\pi \pm \frac{\pi}{3}$, $k \in I$
3. $n\pi - \frac{\pi}{4}$, $n \in I$
4. $n\pi - \frac{\pi}{4}$, $n \in I$
5. $2n\pi + \frac{\pi}{2}$, $n \in I$ or $2n\pi$, $n \in I$ or $n\pi + \frac{\pi}{4}$, $n \in I$
6. D

Do yourself-5

1. D
2. $x = \frac{\pi}{4}$
3. $(4p - 3)\frac{\pi}{2}$, $p \in I$
4. $2m\pi + \frac{\pi}{2}$, $m \in I$
5. A
6. C

Do yourself-6

1. $\bigcup_{n\in I}\left[2n\pi - \frac{2\pi}{3}, 2n\pi + \frac{2\pi}{3}\right]$
2. $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$

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EXERCISE (O-1)

1. C	**2.** D	**3.** C	**4.** C	**5.** C	**6.** B	**7.** A	**8.** C
9. C	**10.** B	**11.** C	**12.** B	**13.** C	**14.** D	**15.** C	**16.** C
17. D	**18.** B	**19.** D	**20.** A	**21.** B	**22.** A	**23.** D	**24.** A
25. C	**26.** A	**27.** B	**28.** D	**29.** A	**30.** C		

EXERCISE (O-2)

1. A,B,C,D	**2.** B,C	**3.** A,B	**4.** B,C	**5.** A,C
6. A,B,C,D	**7.** B,C	**8.** A,C,D	**9.** A,C	**10.** B,C
11. C	**12.** C	**13.** A	**14.** B	**15.** B

16. **(A)→(P); (B)→(P); (C)→(Q); (D)→(R)**

17. **(A) → (S), (B) → (P), (C) → (T), (D) → (Q)**

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EXERCISE (S-1)

1. $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}$ & π

2. 10 solutions

3. $30°, 45°, 90°, 135°, 150°$

4. $\frac{\pi}{8}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{7\pi}{8}$

5. $x = n - \frac{1}{4}$, $n \in I$; sum = 5025

6. $-\sqrt{2} \le y \le \sqrt{2}$; $\frac{\pi}{2}, \pi$

7. $2n\pi + \frac{\pi}{4}$ or $(2n+1)\pi - \tan^{-1}2$, $n \in I$

8. $x = 2n\pi + \frac{\pi}{6}$, $n \in I$

9. $x = \frac{n\pi}{7} - \frac{\pi}{84}$ or $x = \frac{n\pi}{4} + \frac{7\pi}{48}$, $n \in I$

10. $\alpha - 2\pi$; $\alpha - \pi$, α, $\alpha + \pi$, where $\tan\alpha = \frac{2}{3}$

EXERCISE (S-2)

1. $n\pi$; $n\pi + (-1)^n \frac{\pi}{10}$ **or** $n\pi + (-1)^n\left(-\frac{3\pi}{10}\right)$

2. $x = 2n\pi - \frac{\pi}{2}$

3. $\frac{(2n+1)\pi}{4}$, $k\pi$, where $n, k \in I$

4. $\frac{\pi}{7}, \frac{5\pi}{7}, \pi, \frac{9\pi}{7}, \frac{13\pi}{7}$

5. $x = 2n\pi + \frac{\pi}{12}$

7. (a) $-\frac{3}{2}$; (b) $k \in \left[-1, -\frac{1}{2}\right]$; (c) $x = \frac{n\pi}{2} \pm \frac{\pi}{6}$

8. $x = \pi/16$

9. $x = 2n\pi$ or $x = n\pi + (-1)^n\left(-\frac{\pi}{2}\right)$ **or** $x = n\pi + (-1)^n \frac{\pi}{6}$

10. $x = \frac{n\pi}{4} + (-1)^n \frac{\pi}{8}$ **or** $\frac{n\pi}{4} + (-1)^{n+1} \frac{\pi}{24}$

EXERCISE (JM)

1. 1 **2.** 1 **3.** 3 **4.** 4 **5.** 1 **6.** 1 **7.** 1 **8.** 3
9. 1 **10.** 1 **11.** 1 **12.** 2 **13.** 8.00

EXERCISE (JA)

1. 3 **2.** 7 **3.** A,C,D **4.** D **5.** 8 **6.** C **7.** 0.5
8. 3 **9.** 2 **10.** 1.00

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FUNDAMENTALS OF MATHEMATICS

Chapter Contents

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JEE (Main/Advanced) Syllabus

JEE (Main) Syllabus :
Sets and their representation ; Union, intersection and complement of sets and their algebraic properties; power sets ; polynomials.
JEE (Advanced) Syllabus :
Absolute value, polynomial.

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FUNDAMENTALS OF MATHEMATICS

1. SET THEORY

1.1 Sets : A set is a collection of well defined objects which are distinct from each other. Sets are generally denoted by capital letters A, B, C, ... etc. and the elements of the set by *a, b, c*,... etc.

If a is an element of a set A, then we write $a \in A$ (a belongs to A).

If a is **not** an element of a set A, then we write $a \notin A$ (a does not belong to A).

Ex. The collection of vowels in english alphabet is a set containing the elements *a, e, i, o, u*.

1.2 Methods to write a Set

(i) Roster Method : In this method a set is described by listing elements, separated by commas and enclosing them by curly brackets.

Ex. The set of vowels of English Alphabet may be described as $\{a, e, i, o, u\}$.

(ii) Set Builder Form : In this case we write down a property or rule or proposition, which gives us all the element of the set.

$A = \{x : P(x)\}$

Ex. $A = \{x : x \in \mathbb{N} \text{ and } x = 2n \text{ for } n \in \mathbb{N}\}$

i.e. $A = \{2, 4, 6, ...\}$

Ex. $B = \{x^2 : x \in \mathbb{N}\}$

i.e. $B = \{1, 4, 9, ...\}$

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Illustration-1 : The set $A = \{x : x \in \mathbb{R}, x^2 = 16 \text{ and } 2x = 6\}$ is equal to

(A) ϕ (B) $\{14, 3, 4\}$ (C) $\{3\}$ (D) $\{4\}$

Solution : $x^2 = 16 \Rightarrow x = \pm 4$

$2x = 6 \Rightarrow x = 3$

There is no value of x which satisfies both the above equations.

Thus, $A = \phi$

Hence, (A) is the correct answer.

1.3 Cardinality of a Finite Set :

The number of elements in a finite set is called the cardinality of the set A and is denoted |A| or n(A). It is also called cardinal number of the set.

Ex. $A = \{a, b, c, d\} \Rightarrow n(A) = 4$

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1.4 Types of Sets

(i) Null set or Empty set : A set having no element in it is called an Empty set or a Null set or Void set. It is denoted by ϕ or { }

Ex. $A = \{x \in \mathbb{N} : 5 < x < 6\} = \phi$

A set consisting of atleast one element is called a non-empty set or a non-void set.

(ii) Singleton set : A set consisting of a single element is called a singleton set.

Ex. set {0}, is a singleton set

(iii) Finite Set : A set which has only finite number of elements is called a finite set.

Ex. A = {a, b, c}

(iv) Infinite set : A set which has an infinite number of elements is called an infinite set.

Ex. A = {1, 2, 3, 4, ...} is an infinite set

(v) Subset : Let A and B be two sets, if every element of A is an element of B, then A is called a subset of B. If A is a subset of B, we write $A \subseteq B$

Ex : A = {1, 2, 3, 4} and B = {1, 2, 3, 4, 5, 6, 7} $\Rightarrow A \subseteq B$

The symbol "$\Rightarrow$" stands for "implies"

Note : $(x \in A \Rightarrow x \in B) \Leftrightarrow A \subseteq B$

(vi) Proper subset : If A is a subset of B and $A \neq B$ then A is a proper subset of B and we write $A \subset B$

Note :

- Every set is a subset of itself i.e. $A \subseteq A$ for all A
- Empty set ϕ is a subset of every set
- The total number of subsets of a finite set containing n elements is 2^n

(vii) Universal set : A set consisting of all possible elements which occur in the discussion is called a Universal set and is denoted by U

Note : All sets are contained in the universal set

Ex. If A = {1, 2, 3}, B = {2, 4, 5, 6}, C = {1, 3, 5, 7} then U = {1, 2, 3, 4, 5, 6, 7} can be taken as the Universal set.

(viii) Power set : Let A be any set. The set of all subsets of A is called power set of A and is denoted by P(A).

Note :

- If $A = \phi$ then P(A) has one element.
- Power set of a given set is always non empty

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Illustration-2 : If A = {1, 2} then find its power set.

Solution : A = {1, 2} then P(A) = {ϕ, {1}, {2}, {1, 2}}

Illustration-3 : If A = {x, y}, then the power set of A is-

(A) $\{x^y, y^x\}$ (B) {ϕ, x, y}

(C) {ϕ, {x} {2y}} (D) {ϕ, {x}, {y}, {x, y}}

Solution : Clearly P(A) = set of all subsets of A

= {ϕ, {x}, {y}, {x, y}}

∴ (D) holds.

1.5 Some Operation on Sets

(i) Union of two sets : $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Ex. A = {1, 2, 3}, B = {2, 3, 4} then $A \cup B$ = {1, 2, 3, 4}

Note : $x \in (A \cup B) \Leftrightarrow x \in A$ or $x \in B$

(ii) Intersection of two sets : $A \cap B = \{x : x \in A \text{ and } x \in B\}$

Ex. A = {1, 2, 3}, B = {2, 3, 4} then $A \cap B$ = {2, 3}

Note :

- $x \in (A \cap B) \Leftrightarrow x \in A$ and $x \in B$
- If $A \cap B = \phi$, then A, B are disjoint sets.

Ex. If A = {1, 2, 3}, B = {7, 8, 9} then $A \cap B = \phi$

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Illustration-4 : If aN = {ax : x ∈ N}, then the set $6N \cap 8N$ is equal to-

(A) 8N (B) 48N (C) 12N (D) 24N

Solution : 6N = {6, 12, 18, 24, 30, ...}

8N = {8, 16, 24, 32, ...}

∴ $6N \cap 8N$ = {24, 48, ...} = 24N

Short cut Method

$6N \cap 8N = 24N$ [24 is the L.C.M. of 6 and 8]

(iii) Difference of two sets : A – B or A\B = {x : x ∈ A and x ∉ B}

Ex. A = {1, 2, 3}, B = {2, 3, 4} ; A – B = {1}

Note : x ∈ (A – B) ⇔ x ∈ A and x ∉ B

(iv) Complement of a set : A' or A^c or $\overline{A}$ = {x : x ∉ A but x ∈ U} = U – A

Ex. U = {1, 2, ..., 10}, A = {1, 2, 3, 4, 5} then A' = {6, 7, 8, 9, 10}

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Note :

- $x \in A \Leftrightarrow x \notin A^c$
- $A \cap A' = \phi$ $\therefore$ A, A' are disjoint.
- $A \cup A' = U$
- $(A')' = A$.

(v) Cartesian Product of two sets

Cartersian product of two sets A and B, denoted as $A \times B$, is the set of all ordered pairs (a,b) where $a \in A$ and $b \in B$

$A \times B = \{(a,b) | a \in A \text{ and } b \in B\}$

Ex : Cartersian product $A \times B$ when $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$ is represented in the square grid

Note : $n(A \times B) = n(A) \times n(B)$

Illustration-5 : Find $A \times B$ when $A = \{x | x \text{ is prime number less than } 5\}$ and $B = \{1,2,3\}$

Solution : $A = \{2,3\}$

$B = \{1,2,3\}$

$A \times B = \{(2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

1.6 Equality of two Sets :

Two sets A and B are said to be equal if every element of A is an element of B, and every element of B is an element of A.

If sets A and B are equal, we write A = B and if A and B are not equal, then $A \neq B$

Ex. $A = \{1, 2, 6, 7\}$ and $B = \{6, 1, 2, 7\}$ $\Rightarrow$ $A = B$

Note : $A \subseteq B$ and $B \subseteq A \Leftrightarrow A = B$

Illustration-6 : Find the pairs of equal sets (if any), give reasons:

$A = \{0\}$, $B = \{x : x > 15 \text{ and } x < 5\}$,

$C = \{x : x - 5 = 0\}$, $D = \{x : x^2 = 25\}$,

$E = \{x : x \text{ is an integral positive root of the equation } x^2 - 2x - 15 = 0\}$.

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Solution : Since $0 \in A$ and 0 does not belong to any of the sets B, C, D and E, it follows that,

$A \neq B$, $A \neq C$, $A \neq D$, $A \neq E$.

Since $B = \phi$ but none of the other sets are empty. Therefore $B \neq C$, $B \neq D$ and $B \neq E$.

Also $C = \{5\}$ but $-5 \in D$, hence $C \neq D$.

Since $E = \{5\}$, $C = E$. Further, $D = \{-5, 5\}$ and $E = \{5\}$, we find that, $D \neq E$.

Thus, the only pair of equal sets is C and E.

Illustration-7 : Show that $A \cup B = A \cap B$ implies $A = B$

Solution : Let $a \in A$. Then $a \in A \cup B$. Since $A \cup B = A \cap B$, $a \in A \cap B$. So $a \in B$.

Therefore, $A \subseteq B$, Similarly, if $b \in B$, then $b \in A \cup B$. Since

$A \cup B = A \cap B$, $b \in A \cap B$. So, $b \in A$. Therefore, $B \subseteq A$. Thus, $A = B$

1.7 Venn Diagram

Venn diagram is a diagram representing sets pictorially as circles or closed curves within an enclosing rectangle/ a closed curve (the universal set), common elements of the sets being represented by intersections of the circles/closed curves.

See the following Examples :

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$A \cup B$

$A \cap B$

$A - B$

$B - A$

Clearly $(A - B) \cup (B - A) \cup (A \cap B) = A \cup B$

A'

Disjoint Sets

$(A \cap B \cap C)$

$(A \cap B) \cup (B \cap C) \cup (C \cap A)$

$(A \cup B \cup C)'$

$C - (A \cup B)$

$(A \cup B \cup C)$

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Do yourself -1 :

1. Check whether the following statements are true or false :

(a) $A \cap \phi = \phi$ **(b)** $A \cap U = A$ **(c)** $A \cup \phi = A$ **(d)** $A \cup U = U$

(e) $A \cap B \subseteq A$ **(f)** $A \cap B \subseteq B$ **(g)** $A \subseteq A \cup B$ **(h)** $B \subseteq A \cup B$

(i) $A \subseteq B \Rightarrow A \cap B = A$ **(j)** $A \subseteq B \Rightarrow A \cup B = B$

1.8 Some Important Laws

(i) **Commutative Law :** $A \cup B = B \cup A$; $A \cap B = B \cap A$

(ii) **Associative Law :** $(A \cup B) \cup C = A \cup (B \cup C)$; $(A \cap B) \cap C = A \cap (B \cap C)$

(iii) **Distributive Law :**

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

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(iv) **De-Morgan's Law :**

(a) $(A \cup B)' = A' \cap B'$; (b) $(A \cap B)' = A' \cup B'$

Do yourself-2 :

1. If A and B are any two sets, then verify the following using venn diagram or otherwise :

(i) $A - B = A \cap B'$ **(ii)** $B - A = B \cap A'$

(iii) $A - B = A \Leftrightarrow A \cap B = \phi$ **(iv)** $(A - B) \cup B = A \cup B$

(v) $(A - B) \cap B = \phi$

(vi) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

(vii) $A - (B \cup C) = (A - B) \cap (A - C)$

(viii) $A - (B \cap C) = (A - B) \cup (A - C)$

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1.9 Some Important Results on Cardinality of Sets

If A, B and C are finite sets, and U be the finite universal set, then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A \cup B) = n(A) + n(B) \Leftrightarrow$ A, B are disjoint sets.

(iii) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

Illustration-8 : In a group of 1000 people, there are 750 who can speak Hindi and 400 who can speak Bengali. How many can speak Hindi only ?How many can speak Bengali ? How many can speak both Hindi and Bengali?

Solution : Let A and B be the sets of persons who can speak Hindi and Bengali respectively.

then $n(A \cup B) = 1000$, $n(A) = 750$, $n(B) = 400$.

Number of persons whose can speak both Hindi and Bengali

$= n(A \cap B) = n(A) + n(B) - n(A \cup B)$

$= 750 + 400 - 1000 = 150$

Number of persons who can speak Hindi only

$= n(A - B) = n(A) - n(A \cap B) = 750 - 150 = 600$

Number of persons who can speak Bengali only

$= n(B - A) = n(B) - n(A \cap B) = 400 - 150 = 250$

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Illustration-9 : Each person in a group of 80 can speak either Hindi or English or both. If 55 persons can speak English and 40 can speak both, find the number of persons who can speak Hindi.

Solution : Let E = set of persons who can speak English.

H = set of persons who can speak Hindi.

$n(E) = 55$, $n(H) = x$, $n(E \cap H) = 40$, $n(H \cup E) = 80$

Using $n(H \cup E) = n(H) + n(E) - n(H \cap E)$

$\Rightarrow 80 = x + 55 - 40$

$\Rightarrow x = 80 - 55 + 40 = 65$

Alternate :

Using the Venn diagram

$n(U) = n(E - x) + n(x) + n(H - x)$

$80 = (55 - 40) + 40 + n(H) - 40$

$\Rightarrow n(H) = 80 - 15 = 65$

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Illustration-10 : A group of members know at least one of the two languages, Hindi or Kannada. In the group, 150 members know Hindi and 80 members know Kannada and 70 of them know both. How many members are there in the group ?

Solution : Let H = set of persons who know Hindi.

K = set of persons who know Kannada.

$n(H \cap K)$ = the number of persons who know both Hindi and Kannada is 70.

$n(H \cup K) = n(H) + n(K) - n(H \cap K)$

$= 150 + 80 - 70 = 160$

Illustration-11 : In a survey of 25 students, it was found that 15 had taken mathematics, 12 had taken physics, and 11 had taken chemistry, 5 had taken mathematics and chemistry, 9 had taken mathematics and physics, 4 had taken physics and chemistry, and 3 had taken all the three subjects. Find the number of students who had taken.

(A) Only mathematics

(B) Mathematics and physics, but not chemistry

(C) At least one of the three subjects

(D) Only one of the three subjects

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Solution : Let M denote the set of students who had taken mathematics. P the set of students who had taken physics and C the set of students who had taken chemistry. In the Venn diagram, let a, b, c, d, e, f, g denote the number of students in the respective regions.

Now, $n(M \cap P \cap C) = c = 3$

$n(M \cap C) = g + c = 5 \Rightarrow g = 2$

$n(M \cap P) = b + c = 9 \Rightarrow b = 6$

$n(P \cap C) = e + c = 4 \Rightarrow e = 1$

$n(M) = a + b + g + c = 15 \Rightarrow a = 4$

$n(P) = b + c + d + e = 12 \Rightarrow d = 2$

$n(C) = e + f + g + c = 11 \Rightarrow f = 5$

(a) The number of students who had taken only mathematics = a = 4.

(b) The number of students who had taken mathematics and physics, but not chemistry = b=6

(c) The number of students who has taken at least one of the three subjects

$= a + b + c + d + e + f + g = 23.$

(d) The number of students who had taken only one of the three subjects = a + d + f = 11.

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Do yourself-3 :

1. Verify the following using Venn diagram.

(i) $n(A - B) = n(A) - n(A \cap B)$ i.e. $n(A - B) + n(A \cap B) = n(A)$

(ii) $n(A' \cup B') = n((A \cap B)') = n(U) - n(A \cap B)$

(iii) $n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$

2. If A and B are two sets, then $A \cap (A \cup B)'$ is equal to-

(A) A (B) B (C) ϕ (D) none of these

3. If A is any set, then-

(A) $A \cup A' = \phi$ (B) $A \cup A' = U$ (C) $A \cap A' = U$ (D) none of these

4. If A, B be any two sets, then $(A \cup B)'$ is equal to-

(A) $A' \cup B'$ (B) $A' \cap B'$ (C) $A \cap B$ (D) $A \cup B$

5. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 5\}$, $B = \{6, 7\}$ then $A \cap B'$ is-

(A) B' (B) A (C) A' (D) B.

6. If A and B are two sets, then $A \cup B = A \cap B$ iff-

(A) $A \subseteq B$ (B) $B \subseteq A$ (C) $A = B$ (D) none of these

7. Two sets A, B are disjoint iff-

(A) $A \cup B = \phi$ (B) $A \cap B \neq \phi$ (C) $A \cap B = \phi$ (D) None of these

8. Which of the following is a null set ?

(A) $\{0\}$ (B) $\{x : x > 0 \text{ or } x < 0\}$

(C) $\{x : x^2 = 4 \text{ or } x = 3\}$ (D) $\{x : x^2 + 1 = 0, x \in \mathbb{R}\}$

9. If $A = \{2, 4, 5\}$, $B = \{7, 8, 9\}$ then $n(A \times B)$ is equal to-

(A) 6 (B) 9 (C) 3 (D) 0

10. Which set is the subset of all given sets ?

(A) $\{1, 2, 3, 4, ...\}$ (B) $\{1\}$ (C) $\{0\}$ (D) $\{\ \}$

11. If $Q = \left\{x : x = \frac{1}{y}, \text{ where } y \in \mathbb{N}\right\}$, then-

(A) $0 \in Q$ (B) $1 \in Q$ (C) $2 \in Q$ (D) $\frac{2}{3} \in Q$

12. $A = \{x : x \neq x\}$ represents-

(A) $\{0\}$ (B) $\{\ \}$ (C) $\{1\}$ (D) $\{x\}$

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2. THEORY OF NUMBERS

2.1 Types of Numbers :

(i) **Natural numbers :** The counting numbers 1, 2, 3, 4, ... are called natural numbers. The set of natural numbers is denoted by $\mathbb{N}$. Thus $\mathbb{N} = \{1, 2, 3, 4, ...\}$

(ii) **Whole numbers :** Natural numbers including zero are called whole numbers. The set of whole numbers is denoted by **W** or $\mathbb{N}_0$. Thus **W** = {0, 1, 2, 3, 4, ...}

(iii) **Integers :** The numbers ... –3, –2, –1, 0, 1, 2, 3, ... are called integers and the set is denoted by **I** or $\mathbb{Z}$. Thus (**I** or $\mathbb{Z}$) = {... –3, –2, –1, 0, 1, 2, 3, ...}

Note :

- Natural numbers are also called positive integers

  (some time denoted by $\mathbf{I}^+$ or $\mathbb{Z}^+$) = {1, 2, 3, ...}
- Whole numbers are also called non-negative integers

  (denoted by **W** or $\mathbf{I}_0^+$ or $\mathbb{Z}_0^+$) = {0, 1, 2, 3, ...}
- The set of negative integers, $\mathbf{I}^-$ or $\mathbb{Z}^-$ = {..., –3, –2, –1}
- The set of non positive integers in $\mathbf{I}_0^-$ or $\mathbb{Z}_0^-$ = {..., –3, –2, –1, 0}
- Zero is neither positive nor negative but 0 is a member of the set of non negative integers as well as of the set of non positive integers.

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(iv) **Even integers :** Integers which are divisible by 2 are called even integers.

e.g. 0, ±2, ±4, ...

(v) **Odd integers :** Integers which are not divisible by 2 are called as odd integers.

e.g. ±1, ±3, ±5, ±7, ...

(vi) **Prime number :** Natural number having exactly two positive divisors i.e. 1 and itself are called prime numbers.

e.g. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, ...

(vii) **Composite number :** Let 'a' be a natural number, 'a' is said to be composite if it has atleast three distinct positive divisors.

Note :

- 1 is the only natural number that is neither a prime number nor a composite number.
- 2 is the only prime number which is even.
- Numbers which are not prime are composite numbers (except 1).
- '4' is the smallest composite number.

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(viii) Co-prime number : Two natural numbers (not necessarily prime) are coprime, if their H.C.F (Highest common factor) is 1.

e.g. (1, 2), (1, 3), (3, 4), (3, 10), (3, 8), (5, 6), (7, 8) (15, 16) etc.

These numbers are also called relatively prime numbers.

Note :

- Two prime numbers are always co-prime but converse need not be true.
- Two consecutive natural numbers are always co-prime numbers.
- Two consecutive odd natural numbers are always co-prime numbers.

(ix) Rational numbers : The numbers, which can be reduced in the form p/q where p, q $\in \mathbb{Z}$ and $q \neq 0$ and H.C.F (p, q) = 1, are called rational numbers. '$\mathbb{Q}$' represents their set.

Note :

- All integers are rational numbers with q = 1
- When numbers are expressed in reduced form of $\frac{p}{q}$, $q \neq 1$, the rational numbers are called fractions.
- Rational numbers when represented in decimal form are either 'terminating' or 'non-terminating but repeating'.

  e.g., 5/4 = 1.25 (terminating)

  5/3 = 1.6666 ... or $1.\overline{6}$ or $1.\dot{6}$ (non terminating but repeating)
- $0.\overline{9} = 0.9999... = 1$

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Illustration-12 : Express the following rational numbers in the form of p/q, (where p, q $\in \mathbb{Z}$)

(i) $0.1\overline{2}$ (ii) $1.5\overline{23}$

Solution : **(i)** Let x = 0.1222...

$10x = 1.\overline{2}$ ... (i)

$100x = 12.\overline{2}$ ... (ii)

$\Rightarrow$ 90x = 11

$\Rightarrow$ $x = \frac{11}{90}$ (so x is a rational number)

(ii) Let $x = 1.5\overline{23}$

$10x = 15.\overline{23}$

$1000x = 1523.\overline{23}$

990x = 1508

$\Rightarrow x = \frac{1508}{990} = \frac{754}{495}$ (so x is a rational number)

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(x) **Irrational numbers :** Numbers, which cannot be represented in p/q form as above. In decimal representation, they are neither terminating nor repeating.

e.g., $\sqrt{2}, (15)^{1/3}, \pi$ etc.

Note :

- $\pi \neq \frac{22}{7}$. Infact $\pi < \frac{22}{7}$

  $\left(\frac{22}{7} = 3.142857...\right)$ is only an approximate value of π in terms of rational numbers, taken for the sake of convenience.

  Actually π = 3.14159265359...

(xi) **Real numbers :** All rational and irrational numbers taken together form the set of real numbers, represented by $\mathbb{R}$. This is the largest set in the real world of numbers.

Note :

- Division by 0 is not defined.
- Integers are rational number, but converse need not be true.
- Sum of a rational number and an irrational number is always an irrational number.

  e.g. $5 + \sqrt{6}$

- The product of a non zero rational number and an irrational number will always be an irrational number.
- If a $\in \mathbb{Q}$ and b $\notin \mathbb{Q}$ then ab = rational number, only if a = 0.
- Sum, difference, product and quotient of two irrational numbers need not be an irrational number or we can say, result may be a rational number also.
- Sum, difference, product and quotient of two rational numbers is always a rational number.

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(xii) **Complex number :** A number z of the form a + ib is called complex number, where a, b $\in \mathbb{R}$ and i stands for $\sqrt{-1}$. Here 'a' is called real part of z denoted by Re(z) and 'b' is called imaginary part of z denoted by Im(z).

The set of complex number is represented by $\mathbb{C}$.

It may be noted that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

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Complex numbers can be represented on the **complex plane,** in the same way as a point (x, y) is plotted on the Cartesian plane. We can plot the number x + iy by taking the real part 'x' as the horizontal coordinate and the imaginary part 'y' as the vertical coordinate. For example in the adjacent diagram, the point 3 – 4i is shown on the complex plane.

When graphing complex numbers, the horizontal axis is often referred as the **real axis** and the vertical axis as the **imaginary axis**.

The complex plane also gives us a way to visualize the magnitude of a complex number.

The magnitude of a complex number is its distance from the origin when plotted on the complex plane. We use |x + iy| to denote the distance between x + iy and the origin when plotted on the complex plane. So, for example, we have $|3-4i| = \sqrt{3^2+(-4)^2} = 5$. More generally, we have $|x+iy| = \sqrt{x^2+y^2}$

Conjugate of z :

If z = a + ib, a, b $\in \mathbb{R}$, then conjugate of z (denoted as $\bar{z}$), $\bar{z}$ = a – ib.

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Illustration-13 : Find all complex numbers z when

(i) $\bar{z} = z$ (ii) $\bar{z} = -z$

Solution : (i) Let z = a + i b, $\forall$ a, b $\in \mathbb{R}$

$z = \bar{z} \Rightarrow a + ib = a - ib \Rightarrow 2ib = 0$

$\Rightarrow b = 0$, so $z = a$ where $a \in \mathbb{R}$

(ii) for $\bar{z} = -z \Rightarrow a - ib = -a - ib \Rightarrow 2a = 0$

$\Rightarrow a = 0$, so z = ib, $\forall$ b $\in \mathbb{R}$

Numbers to Remember :

Number	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Square	4	9	16	25	36	49	64	81	100	121	144	169	196	225	256	289	324	361	400
Cube	8	27	64	125	216	343	512	729	1000	1331	1728	2197	2744	3375	4096	4913	5832	6859	8000
Sq. Root	1.41	1.73	2	2.24	2.45	2.65	2.83	3	3.16	← rounded upto two places of decimal									

Note :

- Square of a real number is always non negative. (i.e. $x^2 \geq 0$)
- Square root of a positive number is always positive. e.g. $\sqrt{4} = 2$

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Do yourself-4 :

1. If $a\sqrt{2}+b=3\sqrt{2}+4$, find the integral value of a, b and justify your answer.
2. Express the following in form of p/q, where $p, q \in \mathbb{Z}$ and $q \neq 0$

   (i) $0.\overline{18}$ (ii) $0.\overline{16}$ (iii) $0.4\overline{23}$
3. Prove that $\sqrt{2}$ is an irrational number.
4. Identify rational, irrational among x + y, x – y, xy and x/y. When x and y $(xy \neq 0)$
   (i) are both rational
   (ii) are both irrational
   (iii) one is rational other is irrational
5. $\sqrt{a}+\sqrt{b}$ and a – b are rational numbers. Prove that $\sqrt{a}, \sqrt{b}$, a and b all are rational.
6. Check x = 0.101001000100001... is rational or irrational, where the number of zeroes between units increased by 1.

2.2 Divisibility Test

Divisibility of	Test
2	The digit at the unit's place of the number is divisible by 2.
3	The sum of digits of the number is divisible by 3.
4	The last two digits of the number together are divisible by 4.
5	The digit at the unit's place is either 0 or 5.
6	The digit at the unit's place of the number is divisible by 2 & the sum of all digits of the number is divisible by 3.
8	The last 3 digits of the number all together are divisible by 8.
9	The sum of all digits is divisible by 9.
11	The difference between the sum of the digits at even places and the sum of digits at odd places is 0 or multiple of 11. e.g. 1298, 1221, 123321, 12344321, 1234554321, 123456654321

Illustration-14 : Consider a number N = 2 1 P 5 3 Q 4

(i) Number of ordered pairs (P, Q) so that the number 'N' is divisible by 9, is

(A) 11 (B) 12 (C) 10 (D) 8

(ii) Number of values of Q so that the number 'N' is divisible by 8, is

(A) 4 (B) 3 (C) 2 (D) 6

Solution : (i) Sum of digits = P + Q + 15

'N' is divisible by 9 if

P + Q + 15 = 18, 27

$\Rightarrow$ P + Q = 3 .... (i) or P + Q = 12 .... (ii)

From equation (i)

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$$\left.\begin{array}{ll} P=0, & Q=3 \\ P=1, & Q=2 \\ P=2, & Q=1 \\ P=3, & Q=0 \end{array}\right\} \text{Number of ordered pairs is 4}$$

From equation (ii)

$$\left.\begin{array}{ll} P=3, & Q=9 \\ P=4, & Q=8 \\ \cdots\cdots & \cdots\cdots \\ P=8, & Q=4 \\ P=9, & Q=3 \end{array}\right\} \text{Number of ordered pairs is 7}$$

Total number of ordered pairs is 11

(ii) N is divisible by 8 if

Q = 0, 4, 8

Number of values of Q is 3

Do yourself-5 :

1. If x, y $\in \mathbb{N}$, and $x \cdot y = 20$ then find all possible ordered pairs (x, y).

2. Find all (x, y) where x, y $\in \mathbb{N}$ such that

(i) $\frac{1}{x}+\frac{1}{y}=1$ **(ii)** $xy + 5x = 4y + 38$

3. Let x and y be positive integers such that $\frac{1}{x}+\frac{1}{y}=\frac{1}{7}$. Find all (x,y).

4. How many integers in between 500 to 2020 (both inclusive) are multiple of 3 or 7?

5. How many integers in between 1000 to 2020 (both inclusive) are divisible by 5 or 7 but not divisible by 35 ?

Paragraph for Q.6 to Q.8

Consider the number N = 7 7 4 9 5 8 P 9 6 Q

6. If P = 2 and the number N is divisible by 3, then find the number of possible values of Q.

7. If N is divisible by 4, then find the number of possible ordered pairs (P, Q).

8. If N is divisible by 8 and 9 both, then find the number of possible ordered pairs (P,Q).

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3. INTERVALS

Intervals are basically subsets of $\mathbb{R}$ and are very much important in mathematics as you will get to know shortly. If there are two numbers a, b $\in \mathbb{R}$ such that a < b, we can define four types of intervals as follows :

3.1 Closed Intervals

All numbers 'x' between a and b including both numbers is written in closed interval. It is denoted by [ ]. **i.e.** $a \le x \le b$ or $x \in [a,b]$ or $\{x : x \in \mathbb{R}$ and $a \le x \le b\}$

Graphical Representation :

3.2 Open Intervals

All numbers 'x' between a and b excluding both numbers is written in open interval. It is denoted by] [or ( ). **i.e.** $a < x < b$ or $x \in]a, b[$ or $x \in (a, b)$ or $\{x : x \in \mathbb{R}$ and $a < x < b\}$

Graphical Representation :

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3.3 Open-Closed Intervals

All numbers 'x' between a and b including b and excluding a is written in open - closed interval. It is denoted by]a, b] or (a, b] or $a < x \le b$ or $\{x : x \in \mathbb{R}$ and $a < x \le b\}$

Graphical Representation :

3.4 Closed-Open Intervals

All numbers 'x' between a and b including a but excluding b is written in closed-open interval. It is denoted by [a, b[or [a, b) or $a \le x < b$ or $\{x : x \in \mathbb{R}$ and $a \le x < b\}$

Graphical Representation :

3.5 The Infinite Intervals are Defined as follows :

- $(a, \infty) = \{x : x > a\}$
- $[a, \infty) = \{x : x \ge a\}$
- $(-\infty, b] = \{x : x \le b\}$
- $(-\infty, b) = \{x : x < b\}$

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Note : $x \in \{1, 2\}$ denotes some particular values of x, i.e. x = 1, 2

If there is no value of x, then we can say $x \in \phi$ (Null set)

Intervals are particularly important in solving inequalities.

Illustration-15 : Represent following sets on the number line

(i) $(-\infty, 3) \cup [5, \infty)$ (ii) $x \le 5$ or $x > 7$ (iii) $(-\infty, -2) \cup (-3, 5]$

(iv) $(-2, 5] \cap [-3, 2)$ (v) $-1 \le x \le 3$ and $-2 < x$

(vi) $[-2, 10) - (1, 5]$ (vii) $(-1, 2) - (0, 3]$

Solution :

(i)

(ii)

(iii)

Union of given two sets is $(-\infty, 5]$

(iv)

The intersection of given two sets is (–2, 2).

(v)

The intersection of given two sets is [–1, 3]

(vi)

Difference of given two sets is $[-2, 1] \cup (5, 10)$

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(vii)

Difference of given two sets is (–1, 0].

Do yourself-6 :

Represent following sets on the number line :

(i) $(-3, 2] \cup [5, 10) \cup (-1, 4)$

(ii) $(A \cup B) \cap C$ when

$A = (-\infty, -2)$, $B = [4, 10]$ & $C = [-10, 5)$

(iii) $[-3, 5] \cap (-2, 6) \cap [-4, 4)$

(iv) $(A \cup B) - C$ when

$A = (-\infty, -5)$, $B = [4, 10]$ & $C = (-10, 10)$

4. INDICES AND SURDS

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4.1 Indices

Definition of Indices :

If 'a' is any non zero real or imaginary number and 'm' is a positive integer, then a^m = a. a. a....a (m times). Here a is called the base and m is the index, power or exponent.

Law of indices :

(i) $a^0 = 1, (a \neq 0)$

(ii) $a^{-m} = \dfrac{1}{a^m}, (a \neq 0)$

(iii) $a^m.a^n = a^{m+n}$

(iv) $\dfrac{a^m}{a^n} = a^{m-n}, a \neq 0$

(v) $(a^m)^n = a^{mn} = (a^n)^m$

(vi) $\sqrt[q]{a^p} = a^{p/q}, \ q \in \mathbb{N}$ and $q \geq 2$

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(vii) If x = y, then $a^x = a^y$, but the converse may not be true. e.g. : $(1)^6 = (1)^8$, but $6 \neq 8$

For $a^x = a^y$ we have following possibilities

- If $a \neq \pm 1, 0$, then x = y
- If a = 1, then x, y may be any real number
- If a = –1, then x, y may be both even or both odd
- If a = 0, then x, y may be any positive real number

But if we have to solve the equations like $\left(f(x)\right)^{g(x)} = \left(f(x)\right)^{h(x)}$ (i.e same base, different indices) then we have to solve :

(a) f(x) = 1 (b) f(x) = –1 (c) f(x) = 0 (d) g (x) = h(x)

Verification should be done in (b) and (c) cases

Illustration-16 : Solve $(4-x)^{x^3-4x} = (4-x)^{(x^2-4)(x-1)}$

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Solution : **Case-1 :** 4 – x = 1 ⇒ x = 3

Case-2 : 4 – x = –1 ⇒ x = 5

for x = 5, $x^3 - 4x$ is odd

and $(x^2 - 4)(x - 1)$ is even

so x = 5 does not satisfy the given equation

Case-3 : 4 – x = 0 ⇒ x = 4

For x = 4, $x^3 - 4x > 0$ and $(x^2 - 4)(x - 1) > 0$

⇒ x = 4 satisfies the given equation

Case-4 : $x^3 - 4x = (x^2 - 4)(x - 1)$

$\Rightarrow x^3 - 4x = x^3 - x^2 - 4x + 4$

$\Rightarrow x^2 - 4 = 0$, ⇒ x = ±2 which satisfies the given equation.

From case-1, case-2, case-3 and case-4, solutions set of the given equation is {–2, 2, 3, 4}

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(viii) If $a^x = b^x$ then consider the following cases :

- If $a \neq \pm b$, then $x = 0$
- If $a = b \neq 0$, then x may have any real value for which a^x is well defined.
- If $a = -b \neq 0$, then x is even.
- If $a = b = 0$, then x can be any positive real.

If we have to solve the equation of the form $[f(x)]^{h(x)} = [g(x)]^{h(x)}$, i.e., same index, different bases, then we have to solve :

(a) $f(x) = g(x)$ (b) $f(x) = -g(x)$ (c) $h(x) = 0$

Verification should be done in (a), (b) and (c) cases.

Illustration-17 : Solve $(x^2 - 4)^{2x} = (x^2 + 2x)^{2x}$.

Solution : **Case-1 :** when $x = 0$

Base : $x^2 + 2x = 0$, so $x = 0$ does not satisfy given equation

Case-2 : $x^2 - 4 = x^2 + 2x \neq 0$

$\Rightarrow x \in \phi$

Case-3 : $x^2 - 4 = -x^2 - 2x \neq 0 \Rightarrow 2x^2 + 2x - 4 = 0$

$x^2 + x - 2 = 0 \Rightarrow x = 1$

satisfies the given equation

Case-4 : $x^2 - 4 = x^2 + 2x \Rightarrow x = -2$

$x = -2$ does not satisfy the given equation.

From all above cases, $x \in \{1\}$

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Illustration-18 : If $a^x = b$, $b^y = c$, $c^z = a$, prove that $xyz = 1$, where a,b,c are distinct numbers.

Solution : We have, $a^{xyz} = (a^x)^{yz}$

$\Rightarrow a^{xyz} = (b)^{yz}$ $[\because a^x = b]$

$\Rightarrow a^{xyz} = (b^y)^z$

$\Rightarrow a^{xyz} = c^z$ $[\because b^y = c]$

$\Rightarrow a^{xyz} = a$ $[\because c^z = a]$

$\therefore a^{xyz} = a^1$

$\Rightarrow xyz = 1$

4.2 Surds

If 'x' is a rational number, which is not the nth power ($n \in \mathbb{N}\backslash\{1\}$) of any rational number, then the number $x^{1/n}$ usually denoted by $\sqrt[n]{x}$ is called surd. The sign '$\sqrt[n]{\ }$' is called the radical sign. The number in the angular part of the sign, i.e., 'n' is called order of the surd. In case of $n = 2$ the expression $\sqrt[2]{x}$, simply written as $\sqrt{x}$.

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Note :

- If $\sqrt[n]{x}$ is a surd then $-\left(\sqrt[n]{x}\right)$ is also a surd.
- Every surd is an irrational number (but every irrational number is not a surd).
- To rationalize the denominator of a fraction of the form $\frac{a}{\sqrt{b}}$, multiply the numerator and denominator of the fraction by $\sqrt{b} \Rightarrow \frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \cdot \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{\sqrt{b^2}} = \frac{a\sqrt{b}}{b}$.

Eg.

(a) $\sqrt{3}$ is a surd and $\sqrt{3}$ is an irrational number.

(b) $\sqrt[3]{5}$ is surd and $\sqrt[3]{5}$ is an irrational number.

(c) π is an irrational number, but it is not a surd.

4.2.1 Conjugate of a Surd

If two binomial surds (surds containing two terms such as $2+\sqrt{3}$, $2\sqrt{5}-\sqrt{7}$ etc) are such that only the sign connecting the individual terms are different, then they are said to be conjugate of each other. If these surds are quadratic, then their product would always be rational. So in case of a binomial quadratic surd, we use its conjugate as its rationalizing factor.

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Ex. Conjugate of $3\sqrt{2}+\sqrt{5}$ is $3\sqrt{2}-\sqrt{5}$ or $-3\sqrt{2}+\sqrt{5}$

Illustration-19 : Rationalize the denominator of $\frac{1}{3\sqrt{2}+\sqrt{5}}$

Solution : A conjugate of $3\sqrt{2}+\sqrt{5}$ is $3\sqrt{2}-\sqrt{5}$

Therefore multiplying the conjugate in the numerator and denominator of the given fraction.

$$\frac{3\sqrt{2}-\sqrt{5}}{\left(3\sqrt{2}+\sqrt{5}\right)\left(3\sqrt{2}-\sqrt{5}\right)}$$

$$= \frac{3\sqrt{2}-\sqrt{5}}{\left(3\sqrt{2}\right)^2-\left(\sqrt{5}\right)^2}$$

$$= \frac{3\sqrt{2}-\sqrt{5}}{18-5}$$

$$= \frac{3\sqrt{2}-\sqrt{5}}{13}$$

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Illustration-20 : Using the fact that $\left(\sqrt{x}+\sqrt{y}\right)^2 = x+y+\sqrt{xy}$ to find the square root of $7+2\sqrt{10}$. If this number can be expressed in the form $\sqrt{a}+\sqrt{b}$, where a ≤ b, find the value of b – a.

(A) 3 (B) 4 (C) 0 (D) 2

Solution : Let $\sqrt{7+2\sqrt{10}} = \sqrt{a}+\sqrt{b}$, where a,b $\in \mathbb{Q}$

$\Rightarrow 7+2\sqrt{10} = a+b+2\sqrt{ab}$

$\Rightarrow$ a + b = 7 and $\sqrt{ab} = \sqrt{10}$

$(b-a)^2 = (a+b)^2 - 4ab$

$= 49 - 40 = 9$

$\Rightarrow \sqrt{(b-a)^2} = 3 \Rightarrow b-a = 3$ (since b > a)

OR

$7+2\sqrt{10} = 7+2\cdot\sqrt{2}\cdot\sqrt{5} = \left(\sqrt{2}\right)^2+\left(\sqrt{5}\right)^2+2\cdot\sqrt{2}\cdot\sqrt{5} = \left(\sqrt{2}+\sqrt{5}\right)^2$

So $\sqrt{7+2\sqrt{10}} = \sqrt{2}+\sqrt{5}$ $\therefore$ b – a = 3

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Illustration-21 : If $x = \dfrac{1}{2+\sqrt{3}}$, find the value of $x^3 - x^2 - 11x + 4$.

Solution : as $x = \dfrac{1}{2+\sqrt{3}} \times \dfrac{2-\sqrt{3}}{2-\sqrt{3}} = \dfrac{2-\sqrt{3}}{(2)^2-(\sqrt{3})^2}$

$x = \dfrac{2-\sqrt{3}}{4-3} = 2-\sqrt{3}$

$x - 2 = -\sqrt{3}$, squaring both sides, we get

$(x-2)^2 = \left(-\sqrt{3}\right)^2 \Rightarrow x^2+4-4x = 3$

$\Rightarrow x^2 - 4x + 1 = 0$

Now $x^3 - x^2 - 11x + 4$

$= x^3 - 4x^2 + x + 3x^2 - 12x + 4$

$= x(x^2-4x+1) + 3(x^2-4x+1) + 1 = x \times 0 + 3(0) + 1 = 0 + 0 = 0 = 1$

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Illustration-22 : If $x = 3 - 2\sqrt{2}$, find $x^2 + \frac{1}{x^2}$

Solution : We have, $x = 3 - 2\sqrt{2}$.

$$\therefore \frac{1}{x} = \frac{1}{3-2\sqrt{2}} = \frac{1}{3-2\sqrt{2}} \times \frac{3+2\sqrt{2}}{3+2\sqrt{2}}$$

$$= \frac{3+2\sqrt{2}}{(3)^2 - (2\sqrt{2})^2} = \frac{3+2\sqrt{2}}{9-8} = 3 + 2\sqrt{2}$$

Thus, $x^2 + \frac{1}{x^2} = (3 - 2\sqrt{2})^2 + (3 + 2\sqrt{2})^2$

$$= 2((3)^2 + (2\sqrt{2})^2) = 2(9 + 8) = 34$$

Illustration-23 : Rationalise the denominator of $\frac{1}{\sqrt{3}-\sqrt{2}-1}$

Solution : $\frac{1}{\sqrt{3}-\sqrt{2}-1} = \frac{1}{\sqrt{3}-\sqrt{2}-1} \times \frac{\sqrt{3}+\sqrt{2}+1}{\sqrt{3}+\sqrt{2}+1}$

$$= \frac{\sqrt{3}+\sqrt{2}+1}{(\sqrt{3}-\sqrt{2}-1)(\sqrt{3}+\sqrt{2}+1)} = \frac{\sqrt{3}+\sqrt{2}+1}{(\sqrt{3})^2-(\sqrt{2}+1)^2} = \frac{\sqrt{3}+\sqrt{2}+1}{-2\sqrt{2}} = -\left(\frac{\sqrt{6}+\sqrt{2}+2}{4}\right)$$

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Illustration-24 : Evaluate the following

(i) $(\sqrt[3]{64})^{\frac{-1}{2}}$ (ii) $\left(\frac{121}{169}\right)^{-3/2}$

Solution : (i) $(\sqrt[3]{64})^{\frac{-1}{2}} = \left[(64)^{\frac{1}{3}}\right]^{\frac{-1}{2}} = (64)^{\frac{1}{3}\times\frac{-1}{2}} = (64)^{\frac{-1}{6}}$

$$= (2^6)^{\frac{-1}{6}} = 2^{6\times\left(\frac{-1}{6}\right)} = 2^{-1} = \frac{1}{2}$$

(ii) $\left(\frac{11\times11}{13\times13}\right)^{-3/2} = \left(\frac{11^2}{13^2}\right)^{-3/2} = \left(\frac{11}{13}\right)^{2\times\frac{-3}{2}} = \left(\frac{11}{13}\right)^{-3} = \left(\frac{13}{11}\right)^{3} = \frac{2197}{1331}$

Illustration 25 : Simplify $\left[\sqrt[3]{\sqrt[6]{a^9}}\right]^4 \left[\sqrt[6]{\sqrt[3]{a^9}}\right]^4$

(A) a^{16} (B) a^{12} (C) a^8 (D) a^4

Solution. $a^{9(1/6)(1/3)4} \cdot a^{9(1/3)(1/6)4} = a^2 \cdot a^2 = a^4$.

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Illustration 26 : Simplify $a\left(\frac{\sqrt{a}+\sqrt{b}}{2b\sqrt{a}}\right)^{-1}+b\left(\frac{\sqrt{a}+\sqrt{b}}{2a\sqrt{b}}\right)^{-1}$

Solution. The given expression is equal to

$$a\left(\frac{2b\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right)+b\left(\frac{2a\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right)$$

$$=\frac{2ab}{\sqrt{a}+\sqrt{b}}\left(\sqrt{a}+\sqrt{b}\right)=2ab$$

Illustration 27 : Evaluate $\sqrt{3+\sqrt{3}+\sqrt{2+\sqrt{3}+\sqrt{7+\sqrt{48}}}}$

Solution. $\sqrt{3+\sqrt{3}+\sqrt{2+\sqrt{3}+\sqrt{7+\sqrt{48}}}}$

$$=\sqrt{3+\sqrt{3}+\sqrt{2+\sqrt{3}+\sqrt{4+3+2\sqrt{12}}}}=\sqrt{3+\sqrt{3}+\sqrt{2+\sqrt{3}+\sqrt{4}+\sqrt{3}}}$$

$$=\sqrt{3+\sqrt{3}+\sqrt{3}+1}=\sqrt{4+2\sqrt{3}}=\sqrt{3}+1=\sqrt{3}+1$$

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Illustration 28 : Find rational numbers a and b, such that $\frac{4+3\sqrt{5}}{4-3\sqrt{5}}=a+b\sqrt{5}$

Solution. $\frac{4+3\sqrt{5}}{4-3\sqrt{5}}\times\frac{4+3\sqrt{5}}{4+3\sqrt{5}}=a+b\sqrt{5}$

$$\frac{61+24\sqrt{5}}{-29}=a+b\sqrt{5}$$

$$a=-\frac{61}{29},\ b=-\frac{24}{29}$$

Do yourself-7 :
Only One Option is Correct for Q.1 to Q.5

1. $\left(\left((625)^{-1/2}\right)^{-1/4}\right)^{2}=$

(A) 4 (B) 5 (C) 2 (D) 3

2. $\left(5\left(8^{1/3}+27^{1/3}\right)^{3}\right)^{1/4}=$

(A) 3 (B) 6 (C) 5 (D) 4

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3. $\left\{\sqrt[4]{\left(\frac{1}{x}\right)^{-12}}\right\}^{-2/3} =$

(A) $\frac{1}{x^2}$ (B) $\frac{1}{x^4}$ (C) $\frac{1}{x^3}$ (D) $\frac{1}{x}$

4. $\frac{\sqrt{x^3}\times\sqrt[3]{x^5}}{\sqrt[5]{x^3}}\times\sqrt[30]{x^{77}} =$

(A) $x^{76/15}$ (B) $x^{78/15}$ (C) $x^{79/15}$ (D) $x^{77/15}$

5. If $\sqrt[4]{\sqrt[3]{x^2}} = x^k$, then $k =$

(A) $\frac{2}{6}$ (B) 6 (C) $\frac{1}{6}$ (D) 7

6. Simplify:

(i) $\frac{\left(5\sqrt{3}+\sqrt{50}\right)\left(5-\sqrt{24}\right)}{\left(\sqrt{75}-5\sqrt{2}\right)}$ (ii) $\frac{3\sqrt{2}}{\sqrt{6}+\sqrt{3}}-\frac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}$

(iii) $\frac{\sqrt{\left[6+2\sqrt{3}+2\sqrt{2}+2\sqrt{6}\right]}-1}{\sqrt{5+2\sqrt{6}}}$ (iv) $\frac{2.3^{n+1}-7.3^{n-1}}{3^{n+1}+2\left(\frac{1}{3}\right)^{1-n}}$

(v) $\left(\frac{1}{3}\right)^{-10}.27^{-3}+\left(\frac{1}{5}\right)^{-4}.(25)^{-2}+\left(64^{-\frac{1}{9}}\right)^{-3}$

7. If $\frac{\left(2^{n+1}\right)^m\left(2^{2n}\right)2^n}{\left(2^{m+1}\right)^n\left(2^{2m}\right)}=1$, then find the value of $\frac{m}{n}$

8. Find rational numbers a and b, such that $\frac{2+3\sqrt{5}}{1-3\sqrt{5}} = a+b\sqrt{5}$

9. The square root of $11+\sqrt{112}$ is $a+\sqrt{b}$, $a,b \in \mathbb{N}$ then $b-a$ is

10. $\sqrt{11+\sqrt{21}} = \sqrt{\frac{a}{b}}+\sqrt{\frac{c}{d}}$, where a, b, c and d are natural numbers with gcd(a, b) = gcd(c, d) = 1. Find a + b + c + d.

11. For $x \neq 0$ find the value of $\left(\frac{x^\ell}{x^m}\right)^{\ell^2+\ell m+m^2}.\left(\frac{x^m}{x^n}\right)^{m^2+mn+n^2}.\left(\frac{x^n}{x^\ell}\right)^{n^2+n\ell+\ell^2}$

12. For $a^x = (x + y + z)^y$, $a^y = (x + y + z)^z$, $a^z = (x + y + z)^x$, then find the values of x,y and z. Where $a > 0$ and $a \neq 1$.

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5. FACTORIZATIONS

5.1 $a^2 - b^2 = (a - b)(a + b)$

Illustration-29 : $(3x - y)^2 - (2x - 3y)^2$

Solution : Use $a^2 - b^2 = (a - b)(a + b)$

$(3x - y)^2 - (2x - 3y)^2 = (3x - y + 2x - 3y)(3x - y - 2x + 3y) = (5x - 4y)(x + 2y)$

5.2 Factorising the Quadratic expression

Illustration-30 : $x^2 + 6x - 187$

Solution : $x^2 + 6x - 187 = x^2 + 17x - 11x - 187$

$= x(x + 17) - 11(x + 17)$

$= (x + 17)(x - 11)$

5.3 Factorisation by converting the given expression into a perfect square.

Illustration-31 : $9x^4 - 10x^2 + 1$

Solution : $9x^4 - 10x^2 + 1 = (3x^2)^2 - 2\cdot 3x^2 + 1 - 4x^2$

$= (3x^2 - 1)^2 - (2x)^2$

$= (3x^2 - 1 - 2x)(3x^2 - 1 + 2x)$

$= (x - 1)(3x + 1)(x + 1)(3x - 1)$

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5.4 $a^3 \pm b^3 \equiv (a \pm b)(a^2 \mp ab + b^2)$

Illustration-32 : $a^6 - b^6$

Solution : $a^6 - b^6 = (a^2)^3 - (b^2)^3$

$= (a^2 - b^2)(a^4 + a^2b^2 + b^4)$

$= (a - b)(a + b)(a^2 - ab + b^2)(a^2 + ab + b^2)$

5.5 Using Factor Theorem :

Illustration-33 : $x^3 - 13x - 12$

Solution : As $x = -1$ makes given expression 0, $x + 1$ is a factor

$$
\begin{array}{r|l}
x+1 & x^3 - 13x - 12 \quad (x^2 - x - 12 \\
 & x^3 + x^2 \\
 & \overline{-x^2 - 13x - 12} \\
 & -x^2 - x \\
 & \overline{-12x - 12} \\
 & -12x - 12 \\
 & \overline{\quad 0}
\end{array}
$$

$\therefore\ x^3 - 13x - 12 = (x + 1)(x^2 - x - 12)$

$= (x + 1)(x - 4)(x + 3)$

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5.6 $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac)$

Illustration-34 : $8x^3 + y^3 + 27z^3 - 18xyz$

Solution : $8x^3 + y^3 + 27z^3 - 18xyz = (2x)^3 + (y)^3 + (3z)^3 - 3(2x)(y)(3z)$

$= (2x + y + 3z)(4x^2 + y^2 + 9z^2 - 2xy - 6xz - 3yz)$

5.7 Cyclic Expression and its Factorization :

An expression is said to be cyclic with regard to the variables $x_1, x_2, ..., x_n$ arranged in this order, when it is unchanged by changing x_1 into x_2, x_2 into x_3, x_3 into x_4, ..., x_n into x_1.

For three variables

E(x, y, z) is cyclic if E(x, y, z) = E(y, z, x) = E(z, x, y)

Ex. 1. $x + y + z$ is a cyclic expression

2. $x^2 + y^2 + z^2 + xy + yz + zx$ is a cyclic expression

3. $E(x, y, z) = x(y - z) + y(z - x) + z(x - y)$ is a cyclic expression because

$\Rightarrow E(y, z, x) = y(z - x) + z(x - y) + x(y - z)$

and $E(z, x, y) = z(x - y) + x(y - z) + y(z - x)$

$\Rightarrow E(x, y, z) = E(y, z, x) = E(z, x, y)$

Theorem :

If E(x, y, z) is a cyclic expression and $x - y$ is a factor of E(x, y, z) then $y - z$ and $z - x$ are also factors of E(x, y, z).

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Illustration-35 : Factorize $x^2(y - z) + y^2(z - x) + z^2(x - y)$

Solution : $x^2(y - z) + x(z^2 - y^2) + yz(y - z)$

$= (y - z)(x^2 - x(z + y) + yz)$

$= (y - z)(x^2 - xz - xy + yz)$

$= (y - z)(x(x - z) - y(x - y))$

$= (y - z)(x - z)(x - y)$

$= -(x - y)(y - z)(z - x)$

OR

Let $E(x, y, z) = x^2(y - z) + y^2(z - x) + z^2(x - y)$

when $y = x$, $E(x, y, z) = x^2(x - z) + x^2(z - x) + z^2(x - x) = x^3 - x^2z + x^2z - x^3 = 0$

$\Rightarrow x - y$ is factor of E(x, y, z)

So, $y - z$ and $z - x$ are also factors of E(x, y, z)

$E(x, y, z) = A(x - y)(y - z)(z - x)$ ...(i)

Since degree of E(x, y, z) is 3 so A is constant. We can find A by substituting values of x, y & z in(i)

Let $x = 0, y = 1, z = 2$

$E(0, 1, 2) = A(-1)(-1)(2)$

$\Rightarrow 1(2 - 0) + 2^2(0 - 1) = 2A \Rightarrow A = -1$

$E(x, y, z) = -(x - y)(y - z)(z - x)$

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5.8 Important Algebraic Identities

- $xy + ax + by + ab = (x + a)(y + b)$
- $x^2 + 2xy + y^2 = (x + y)^2$
- $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = (x + y + z)^2$
- $x^2 - y^2 = (x - y)(x + y)$
- $x^4 + x^2 + 1 = (x^2 + 1)^2 - x^2 = (x^2 + x + 1)(x^2 - x + 1)$
- $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
- $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$
- $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + ... + y^{n-1}), n \in \mathbb{N}$
- $x^{2n+1} + y^{2n+1} = (x + y)(x^{2n} - x^{2n-1} y + x^{2n-2} y^2 - ... + y^{2n}), n \in \mathbb{N}$
- $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

$$= \frac{1}{2}(x+y+z)\left\{(x-y)^2 + (y-z)^2 + (z-x)^2\right\}$$

- $x^3 + y^3 + z^3 = 3xyz$ if $x + y + z = 0$ or $x = y = z$
- $x^3 + y^3 + z^3 + 3(x + y)(y + z)(z + x) = (x + y + z)^3$

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Problem Solving Strategies :

- When facing a problem with gigantic numbers, try replacing them with smaller numbers and look for a pattern. You can often prove your pattern works and solve the problem by substituting variable expressions for the numbers.

Illustration-36 : Compute $\sqrt{2022 \times 2020 \times 2018 \times 2016 + 16}$ without using calculator.

Solution : Let x = 2019

$$\sqrt{2022 \times 2020 \times 2018 \times 2016 + 16}$$

$$= \sqrt{(x+3)(x+1)(x-1)(x-3)+16}$$

$$= \sqrt{(x^2-9)(x^2-1)+16}$$

$$= \sqrt{x^4 - 10x^2 + 25} = \sqrt{(x^2-5)^2} = x^2 - 5$$

$= 2019^2 - 5 = (2000)^2 + (19)^2 + 2 \times 2000 \times 19 - 5$

$= 4000000 + 361 + 76000 - 5$

$= 4{,}076{,}356$

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- Focusing on what makes a problem tricky helps identify what strategies might be best to solve the problem.
- If you have a problem that involves expressions of the form a + b and a – b, where a and/or b involve square roots, consider finding a way to multiply the expressions to get rid of the square roots.

Illustration-37 : Suppose $x = z - \sqrt{z^2 - 5}$ and $5y = z + \sqrt{z^2 - 5}$.

Find x when y = 2/3.

Solution : $x = z - \sqrt{z^2 - 5}$ and $5y = z + \sqrt{z^2 - 5}$, multiplying both equations

we get $5xy = z^2 - (z^2 - 5) = 5$

$\Rightarrow xy = 1 \Rightarrow x = \frac{3}{2}$.

- When making a substitution, take some time to look for the substitution that simplifies your work the most.
- If we have the product of two variables added to linear terms with both variables, such as mn + 3m + 5n, then there is a constant we can add that will allow us to factor. For example, adding 15 to mn + 3m + 5n gives us mn + 3m + 5n + 15 = (m + 5)(n + 3).

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Illustration-38 : Find all integral solutions of x and y when xy – y – 2x = 3.

Solution : $xy - y - 2x = 3$

$\Rightarrow xy - y - 2x + 2 = 5$

$\Rightarrow y(x - 1) - 2(x - 1) = 5$

$\Rightarrow (x - 1)(y - 2) = 5$

$x-1$	$y-2$	(x,y)
5	1	(6,3)
1	5	(2,7)
–5	–1	(–4,1)
–1	–5	(0,–3)

All possible (x, y) and (6, 3), (2, 7), (0, –3) and (–4, 1)

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- Guessing has a long and glorious history in mathematics and science. It is often a very important first step in many discoveries. Don't be afraid to guess! But make sure you test your guesses – a guess itself is only a first step.

Illustration-39 : Find all pairs of positive integers m and n such that m^2 is 105 greater than n^2.

Solution : Turning the words into math is easy :

$m^2 = n^2 + 105$.

$\Rightarrow (m - n)(m + n) = 105$.

$(m - n)(m + n) = 1.105 = 3.35 = 5.21 = 7.15$

Because m and n are positive, we know that m – n is smaller than m + n, so we only have these four cases to consider :

m – n = 1	m – n = 3	m – n = 5	m – n = 7
m + n = 105	m + n = 35	m + n = 21	m + n = 15

Each of these systems of equations gives us a solution (m, n). Adding the equations in the first case gives us 2m = 106, so m = 53. Substitution then gives n = 52. Similarly, we can work through each of the other three cases to find the four solutions (m, n) = (53, 52); (19, 16); (13, 8); (11, 4).

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Illustration-40 : The number 7,999,999,999 has two prime factors. Find them.

Solution : Let 7,999,999,999 = 8,000,000,000 – 1

$= (2000)^3 - 1^3 = (2000 - 1)(2000^2 + 2000 + 1)$

$= (1999)(4002001)$

According to question, 7,999,999,999 has two prime factors, they must be 1999 and 4002001.

Illustration-41 : Factor $x^4 + 4y^4$.

Solution : $x^4 + 4y^4 = (x^2)^2 + (2y^2)^2 - 2(x^2)(2y^2) + (2xy)^2$

$= (x^2 + 2y^2)^2 - (2xy)^2$

$= (x^2 + 2y^2 - 2xy)(x^2 + 2y^2 + 2xy)$

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Do yourself-8 :

1. Factorize following expressions

(i) $x^4 - y^4$ **(ii)** $9a^2 - (2x - y)^2$ **(iii)** $4x^2 - 9y^2 - 6x - 9y$

2. Factorize following expressions

(i) $8x^3 - 27y^3$ **(ii)** $8x^3 - 125y^3 + 2x - 5y$

3. Factorize following expressions

(i) $x^2 + 3x - 40$ **(ii)** $x^2 - 3x - 40$ **(iii)** $x^2 + 5x - 14$

(iv) $x^2 - 3x - 4$ **(v)** $x^2 - 2x - 3$ **(vi)** $3x^2 - 10x + 8$

(vii) $12x^2 + x - 35$ **(viii)** $3x^2 - 5x + 2$ **(ix)** $3x^2 - 7x + 4$

(x) $7x^2 - 8x + 1$ **(xi)** $2x^2 - 17x + 26$ **(xii)** $3a^2 - 7a - 6$

(xiii) $14a^2 + a - 3$

4. Factorize following expressions

(i) $a^2 - 4a + 3 + 2b - b^2$ **(ii)** $x^4 + 324$ **(iii)** $x^4 - y^2 + 2x^2 + 1$

(iv) $4a^4 - 5a^2 + 1$ **(v)** $4x^4 + 81$ **(vi)** $1 + x^4 + x^8$

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5. Factorize following expressions

(i) $x^3 - 6x^2 + 11x - 6$ **(ii)** $2x^3 + 9x^2 + 10x + 3$

(iii) $2x^3 - 9x^2 + 13x - 6$ **(iv)** $x^6 - 7x^2 - 6$

(v) $(x + y + z)^3 - x^3 - y^3 - z^3$

6. (i) Factorize the expressions $8a^6 + 5a^3 + 1$

(ii) Show that $(x - y)^3 + (y - z)^3 + (z - x)^3 = 3(x - y)(y - z)(z - x)$.

7. Factorize following expressions

(i) $(x + 1)(x + 2)(x + 3)(x + 4) - 15$

(ii) $4x(2x + 3)(2x - 1)(x + 1) - 54$

(iii) $(x - 3)(x + 2)(x + 3)(x + 8) + 56$

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5.9 Miscellaneous Algebraic Manipulations

Illustration-42 : Suppose $x+\frac{1}{x}=5$ find **(i)** $x^2+\frac{1}{x^2}$ **(ii)** $x^4-\frac{1}{x^4}$

Solution : (i) Given : $x+\frac{1}{x}=5$

By squaring both sides $\left(x+\frac{1}{x}\right)^2=(5)^2 \Rightarrow x^2+\frac{1}{x^2}+2=25 \Rightarrow x^2+\frac{1}{x^2}=23$

(ii) $\left(x-\frac{1}{x}\right)^2=\left(x+\frac{1}{x}\right)^2-4=21 \Rightarrow x-\frac{1}{x}=\pm\sqrt{21}$

$x^2-\frac{1}{x^2}=\pm 5\sqrt{21}$

from (i) we have $x^2+\frac{1}{x^2}=23$

So $x^4-\frac{1}{x^4}=\pm 23\times 5\sqrt{21}=\pm 115\sqrt{21}$

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Illustration-43 : Simplify $\sqrt{6+\sqrt{11}}+\sqrt{6-\sqrt{11}}$

Solution : Let $x=\sqrt{6+\sqrt{11}}+\sqrt{6-\sqrt{11}}$

By squaring both sides

$x^2=\left(\sqrt{6+\sqrt{11}}\right)^2+\left(\sqrt{6-\sqrt{11}}\right)^2+2\sqrt{36-11}$

$= 12 + 10 = 22$

Since x is positive, so $x=\sqrt{22}$.

OR

$x=\sqrt{6+\sqrt{11}}+\sqrt{6-\sqrt{11}}=\frac{1}{\sqrt{2}}\left(\sqrt{12+2\sqrt{11}}+\sqrt{12-2\sqrt{11}}\right)$

$=\frac{1}{\sqrt{2}}\left(\sqrt{\left(\sqrt{11}+1\right)^2}+\sqrt{\left(\sqrt{11}-1\right)^2}\right)=\frac{1}{\sqrt{2}}\left(\sqrt{11}+1+\sqrt{11}-1\right)$

$x=\sqrt{22}$

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Illustration-44 : If $\left(a+\frac{1}{a}\right)^2 = 3$, then $a^3 + \frac{1}{a^3}$ equals :

(A) $6\sqrt{3}$ (B) $3\sqrt{3}$ (C) 0 (D) $6\sqrt{3}$

Solution : $a+\frac{1}{a}=\pm\sqrt{3}$

$$a^3+\frac{1}{a^3}=\left(a+\frac{1}{a}\right)^3-3\left(a+\frac{1}{a}\right)=\pm 3\sqrt{3}\mp 3\sqrt{3}=0$$

Illustration-45 : If $x=\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ and $y=\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$, then find $x^3 + y^3$.

Solution : $x=\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}=\left(\sqrt{3}-\sqrt{2}\right)^2=5-2\sqrt{6}$

$y=5+2\sqrt{6}$

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

$= (x + y)[(x + y)^2 - 3xy] = 10 \times [100 - 3] = 970$

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Do yourself-9 :

1. Suppose $a+\frac{1}{a}=3$:

(a) Find $a^2 + \frac{1}{a^2}$ **(b)** Find $a^4 + \frac{1}{a^4}$ **(c)** Find $a^3 + \frac{1}{a^3}$

2. If $x+\frac{1}{x}=2$, then prove that : $x^2+\frac{1}{x^2}=x^4+\frac{1}{x^4}=x^8+\frac{1}{x^8}$.

3. If $2x + 3y + 4z = 0$, then prove that $8x^3 + 27y^3 + 64z^3 = 72xyz$.

4. I'm thinking of two numbers. The sum of my numbers is 14 and the product of my numbers is 46. What is the sum of the squares of my numbers ?

5. Simplify $\sqrt{7-\sqrt{13}}-\sqrt{7+\sqrt{13}}$.

6. Simplify the expression $\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}$.

7. If x, y, z are all different real numbers, then prove that

$$\frac{1}{(x-y)^2}+\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2}=\left(\frac{1}{x-y}+\frac{1}{y-z}+\frac{1}{z-x}\right)^2.$$

8. Solve for x :

(a) $\sqrt{2x+1}=\sqrt{x}+1$ **(b)** $\sqrt{x^2-1}=x-3$ **(c)** $\sqrt{x+1}+2\sqrt{2x-3}=-3$

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6. POLYNOMIAL IN ONE VARIABLE

An algebraic expression of the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ is called a polynomial function in 'x' where $a_i (i = 0, 1, 2, ..., n)$ is a constant which belongs to the set of real numbers and sometimes to the set of complex numbers and n is a whole number.

- a_i is the coefficient of x^i for i = 1, 2, 3, ..., n and a_0 is constant term of p(x).
- If $a_n \neq 0$, then $a_n x^n$ is called leading term and a_n is called **leading co-efficient.**
- If $a_n = 1$, then polynomial is called **monic polynomial.**
- If $a_n \neq 0$, then **degree of the polynomial** is n.
- $f(x) = a_0$ is called **constant polynomial.** Its degree is 0, if $a_0 \neq 0$. If $a_0 = 0$ the polynomial $f(x)$ is called **ZERO polynomial.** Its degree is defined as $-\infty$ to preserve following two properties listed below. Some people prefer not to define degree of zero polynomial.

If $f(x)$ is a polynomial of degree n and g(x) is a polynomial of degree m then

1. $f(x) \pm g(x)$ is a polynomial of degree $\leq \max\{n, m\}$
2. $f(x) \times g(x)$ is a polynomial of degree m + n.

6.1 Types of Polynomials (w.r.t. Degree)

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Degree of the polynomial in one variable is the largest exponent of the variable. For example, the degree of the polynomial $3x^7 - 4x^6 + x + 9$ is 7 and the degree of the polynomial $5x^6 - 4x^2 - 6$ is 6.

Polynomials classified by degree

Degree	Name	General form	Example
undefined or $-\infty$	Zero polynomial	0	0
0	(Non-zero) constant polynomial	a ; $(a \neq 0)$	4
1	Linear polynomial	$ax + b$; $(a \neq 0)$	$x + 1$
2	Quadratic polynomial	$ax^2 + bx + c$; $(a \neq 0)$	$x^2 + 1$
3	Cubic polynomial	$ax^3 + bx^2 + cx + d$; $(a \neq 0)$	$x^3 + 1$

Usually, a polynomial of degree n, for n greater than 3, is called a polynomial of degree n, although the phrases quartic polynomial for degree 4 and quintic polynomial for degree 5 are sometimes used.

Note :

Polynomials having only one term are called monomials. E.g. 2, 2x, $7y^5$, $12t^7$ etc. Polynomials having exactly two dissimilar terms are called binomials. E.g. $p(x) = 2x + 1$, $r(y) = 2y^7 + 5y^6$ etc. Polynomials having exactly three distinct terms are called trinomials. E.g. $p(x) = 2x^2 + x + 6$, $q(y) = 9y^6 + 4y^2 + 1$ etc.

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6.2 Division in Polynomials

Consider two polynomials P(x) & d(x) with d(x) being not identically zero and degree of d(x) ≤ degree of P(x) then there exists unique polynomials Q(x) and r(x) such that

$P(x) = Q(x) \,.\, d(x) + r(x)$

Here P(x) is called as dividend,

d(x) is called as divisor,

Q(x) is called as quotient,

& r(x) is called as remainder with degree of r(x) < degree of d(x)

Note : If d(x) is a divisor of P(x) then kd(x) will also be a divisor of P(x); $k \in \mathbb{R} - \{0\}$ and d(–x) will be a divisor of P(–x).

6.3 Remainder Theorem

Statement : Let p(x) be a polynomial of degree ≥ 1 and 'a' is any real number. If p(x) is divided by (x – a), then the remainder is p(a).

Illustration-46 : Let p(x) be $x^3 - 7x^2 + 6x + 4$. Divide p(x) with (x – 6) and find the remainder

Solution : Put x = 6 in p(x) i.e. p(6) will be the remainder.

∴ required remainder be

$p(6) = (6)^3 - 7\cdot 6^2 + 6\cdot 6 + 4 = 216 - 252 + 36 + 4 = 256 - 252 = 4$

$$
\begin{array}{l|l}
x - 6 & x^3 - 7x^2 + 6x + 4 \quad (x^2 - x \\
 & \underline{x^3 - 6x^2} \\
 & \quad\; - x^2 + 6x + 4 \\
 & \quad\; \underline{- x^2 + 6x} \\
 & \qquad\quad + \quad - \\
 & \text{Remainder} = 4
\end{array}
$$

Thus, p(a) is remainder on dividing p(x) by (x – a).

Remark : (i) p(–a) is remainder on dividing p(x) by (x + a)

[∵ x + a = 0 ⇒ x = – a]

(ii) $p\left(\frac{b}{a}\right)$ is remainder on dividing p(x) by (ax – b)

[∵ ax – b = 0 ⇒ x = b/a]

(iii) $p\left(\frac{-b}{a}\right)$ is remainder on dividing p(x) by (ax + b)

[∵ ax + b = 0 ⇒ x = –b/a]

(iv) $p\left(\frac{b}{a}\right)$ is remainder on dividing p(x) by (b – ax)

[∵ b – ax = 0 ⇒ x = b/a]

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Illustration-47 : Find the remainder when

$x^3 - ax^2 + 6x - a$ is divided by $x - a$

Solution : Let $p(x) = x^3 - ax^2 + 6x - a$

$p(a) = a^3 - a(a)^2 + 6(a) - a$

$= a^3 - a^3 + 6a - a = 5a$

So, by the Remainder theorem, remainder = 5a

6.4 Factor Theorem

Statement : Let f(x) be a polynomial of degree ≥ 1 and a be any real constant such that f(a) = 0, then (x – a) is a factor of f(x). Conversely, if (x – a) is a factor of f(x), then f(a) = 0.

Proof : By Remainder theorem, if f(x) is divided by (x – a), the remainder will be f(a). Let q(x) be the quotient. Then, we can write,

$f(x) = (x - a) \times q(x) + f(a)$ ($\because$ Dividend = Divisor × Quotient + Remainder)

If f(a) = 0, then $f(x) = (x - a) \times q(x)$

Thus, (x – a) is a factor of f(x).

Converse Let (x – a) is a factor of f(x).

Then we have a polynomial q(x) such that $f(x) = (x - a) \times q(x)$

Replacing x by a, we get f(a) = 0.

Hence, proved.

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Illustration-48 : Use the factor theorem to determine whether (x – 1) is a factor of $f(x) = 2\sqrt{2}x^3 + 5\sqrt{2}x^2 - 7\sqrt{2}$

Solution : By using factor theorem, (x – 1) is a factor of f(x), only when f(1) = 0

$f(1) = 2\sqrt{2}(1)^3 + 5\sqrt{2}(1)^2 - 7\sqrt{2} = 2\sqrt{2} + 5\sqrt{2} - 7\sqrt{2} = 0$

Hence, (x – 1) is a factor of f(x).

6.5 Fundamental Theorem of Algebra

Every polynomial function of degree ≥ 1 has atleast one zero in the complex numbers.

In other words, if we have

$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$, $a_i \in$ complex number $\forall$ i = 0, 1, 2, ..., n

with n ≥ 1, then there exists atleast one $h \in \mathbb{C}$, such that

$a_n h^n + a_{n-1}h^{n-1} + ... + a_1 h + a_0 = 0.$

From this, it is easy to deduce that a polynomial function of degree 'n' (≥ 1) has exactly n zeroes.

i.e., $f(x) = a(x - r_1)(x - r_2)...(x - r_n)$

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Illustration-49: Find the constants a, b, c such that $(2x^2+3x+7)(ax^2+bx+c)=2x^4+11x^3+9x^2+13x-35$

Solution : **Method : 1**

$(2x^2 + 3x + 7)(ax^2 + bx + c) = 2x^4 + 11x^3 + 9x^2 + 13x - 35$

By comparing coefficient of x^4 from both sides

$$2a = 2 \Rightarrow a = 1$$

By comparing coefficient of x^3 from both sides

$$2b + 3a = 11 \Rightarrow b = 4$$

By comparing coefficient of x^2 from both sides

$$2c + 3b + 7a = 9 \Rightarrow c = -5$$

By comparing coefficient of x from both sides

$$3c + 7b = 13 \qquad \text{(b \& c satisfy)}$$

By comparing constant

$$7c = -35 \Rightarrow c = -5$$

So $a = 1, b = 4, c = -5$

Method : 2

Given that $(2x^2 + 3x + 7)(ax^2 + bx + c) = 2x^4 + 11x^3 + 9x^2 + 13x - 35$

$$\Rightarrow \quad ax^2+bx+c=\frac{2x^4+11x^3+9x^2+13x-35}{2x^2+3x+7}$$

$$= x^2 + 4x - 5$$

By comparing

$$a = 1, b = 4 \text{ and } c = -5.$$

Illustration 50 : Show that $(x-3)$ is a factor of the polynomial $x^3-3x^2+4x-12$.

Solution. Let $p(x)=x^3-3x^2+4x-12$ be the given polynomial. By factor theorem, $(x-a)$ is a factor of a polynomial $p(x)$ if $p(a)=0$. Therefore, in order to prove that $x-3$ is a factor of $p(x)$, it is sufficient to show that $p(3)=0$.

Now, $p(x)=x^3-3x^2+4x-12$

$\Rightarrow p(3)=3^3-3\times3^2+4\times3-12=27-27+12-12=0$

Hence, $(x-3)$ is a factor of $p(x)=x^3-3x^2+4x-12$.

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Illustration 51 : Without actual division prove that $2x^4 - 6x^3 + 3x^2 + 3x - 2$ is exactly divisible by $x^2 - 3x + 2$.

Solution. Let $f(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$ and $g(x) = x^2 - 3x + 2$ be the given polynomials.

Then $g(x) = x^2 - 3x + 2 = x^2 - 2x - x + 2 = x(x-2) - 1(x-2) = (x-l)(x-2)$

In order to prove that f(x) is exactly divisible by g(x), it is sufficient to prove that $x - 1$ and $x - 2$ are factors of f(x). For this it is sufficient to prove that $f(l) = 0$ and $f(2) = 0$.

Now, $f(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$

$\Rightarrow \quad f(l) = 2 \times l^4 - 6 \times 1^3 + 3 \times l^2 + 3 \times 1 - 2 \Rightarrow f(l) = 0$

and, $f(2) = 2 \times 2^4 - 6 \times 2^3 \times 2^2 + 3 \times 2 - 2 \Rightarrow f(2) = 0$

Hence, $(x - 1)$ and $(x - 2)$ are factors of f(x).

$\Rightarrow$ $g(x) = (x-1)(x-2)$ is a factors of f(x).

Hence, f(x) is exactly divisible by g(x).

Illustration 52 : The polynomials $P(x) = kx^3 + 3x^2 - 3$ and $Q(x) = 2x^3 - 5x + k$, when divided by $(x - 4)$ leave the same remainder. The value of k is

(A) 2 (B) 1 (C) 0 (D) –1

Solution : $P(4) = 64k + 48 - 3 = 64k + 45$

$Q(4) = 128 - 20 + k = k + 108$

given $P(4) = Q(4)$

$64k + 45 = k + 108$

$\Rightarrow 63k = 63$ $\Rightarrow k = 1$ $\Rightarrow$ Option **(B)** is correct

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Illustration-53 : Let P(x) be a polynomial such that when P(x) is divided by $x - 19$, the remainder is 99, and when P(x) is divided by $x - 99$, the remainder is 19. What is the remainder when P(x) is divided by $(x - 19)(x - 99)$?

Solution : Because we are dividing by a quadratic, the degree of the remainder is not greater than 1. So, the remainder is $ax + b$, for some constants a and b. Therefore, we have

$$P(x) = (x - 19)(x - 99)Q(x) + ax + b,$$

where Q(x) is the quotient when P(x) is divided by $(x - 19)(x - 99)$. We eliminate the Q(x) term by letting $x = 19$ or by letting $x = 99$. Doing each in turn gives us the system of equations

$$P(19) = 19a + b = 99,$$

$$P(99) = 99a + b = 19.$$

Solving this system of equations gives us $a = -1$ and $b = 118$.

So, the remainder is $-x + 118$.

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Illustration-54 : The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has roots 1, 3, 5 and 7. Determine all the coefficients of $f(x)$.

Solution : By applying factor theorem

$f(x) = (x - 1)(x - 3)(x - 5)(x - 7)$

$= (x^2 - 4x + 3)(x^2 - 12x + 35)$

$= x^2(x^2 - 12x + 35) - 4x(x^2 - 12x + 35) + 3(x^2 - 12x + 35)$

$= x^4 - 16x^3 + 86x^2 - 176x + 105$

So $a = -16$, $b = 86$, $c = -176$ and $c = 105$

Illustration-55 : If $f(x)$ is monic polynomial of degree 6 such that $f(0) = 0$, $f(1) = -1$, $f(2) = -2$, $f(3) = -3$, $f(4) = -4$ and $f(5) = -5$, then find $f(x)$.

Solution : According to question

$f(0) = 0$, $f(1) = -1$, $f(2) = -2$, ..., $f(5) = -5$

$\Rightarrow f(x) + x = 0$ has the roots $x = 0, 1, 2, ..., 5$

$\Rightarrow f(x) + x = x(x - 1)(x - 2)(x - 3)(x - 4)(x - 5)$ (By factor theorem)

$\Rightarrow f(x) = x(x - 1)(x - 2)(x - 3)(x - 4)(x - 5) - x.$

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Illustration-56 : If $P(x) = 2x^3 + ax^2 + bx + c$, where $a, b, c \in \mathbb{Z}$. If $P(\sqrt{3}) = 10 - 2\sqrt{3}$.

Find (i) $P(-\sqrt{3})$ (ii) $3a + c$

Solution : $P(x) = 2x^3 + ax^2 + bx + c$

$P(\sqrt{3}) = 10 - 2\sqrt{3}$

$\Rightarrow 6\sqrt{3} + 3a + b\sqrt{3} + c = 10 - 2\sqrt{3}$

$\Rightarrow (3a + c) + \sqrt{3}(6 + b) = 10 - 2\sqrt{3}$

$\Rightarrow 3a + c = 10$ and $6 + b = -2$ $(a, b, c \in I)$

$3a + c = 10$ and $b = -8$

$P(-\sqrt{3}) = -6\sqrt{3} + 3a - \sqrt{3}b + c$

$= -6\sqrt{3} + (3a + c) + 8\sqrt{3}$

$P(-\sqrt{3}) = 2\sqrt{3} + 10$

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Illustration-57 : Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$, where a, b, c, d are constants. If $P(1) = 10$, $P(2) = 20$, $P(3) = 30$, then compute $P(4) + P(0)$.

Solution : $P(1) = 10$, $P(2) = 20$ and $P(3) = 30$

we can write these information as

$P(x) = 10x$ for $x = 1, 2, 3$

$\Rightarrow P(x) - 10x = 0$ has roots $x = 1, 2$ and 3

By factor theorem

$P(x) - 10x = (x - \alpha)(x - 1)(x - 2)(x - 3)$

$\Rightarrow P(x) = (x - \alpha)(x - 1)(x - 2)(x - 3) + 10x$

$P(4) + P(0) = (4 - \alpha)(3)(2)(1) + 40 + (-\alpha)(-1)(-2)(-3) + 0$

$= 24 - 6\alpha + 40 + 6\alpha$

$= 64$

Illustration-58 : Let a, b and c are roots of $2x^3 + x^2 + x + 1 = 0$

find (i) $a + b + c$ (ii) abc

Solution : By factor theorem

$2x^3 + x^2 + x + 1 = 2(x - a)(x - b)(x - c)$

$\Rightarrow 2x^3 + x^2 + x + 1 = 2(x^3 - (a + b + c)x^2 + (ab + bc + ca)x - abc)$

$\Rightarrow 2x^3 + x^2 + x + 1 = 2x^3 - 2(a + b + c)x^2 + 2(ab + bc + ca)x - 2abc$

By comparing coefficient of x^2 and constant term, we have

$-2(a + b + c) = 1$ and $-2abc = 1$

$\Rightarrow a + b + c = \frac{-1}{2}$ and $abc = \frac{-1}{2}$.

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Do yourself-10 :

1. Determine the remainder when the polynomial $P(x) = x^4 - 3x^2 + 2x + 1$ is divided by $(x - 1)$
2. Find the remainder when $f(x) = 3x^3 + 6x^2 - 4x - 5$ is divided by $(x + 3)$.
3. Determine the value of k for which $x^3 - 6x + k$ may be divisible by $(x - 2)$.
4. Find the value of a, if $(x - a)$ is a factor of $x^3 - a^2x + x + 2$.
5. Find remainder when $f(x) = x^5 - x^3 + 3x^2 + 3x + 1$ is divided by $(x^2 - 1)$.
6. Find the value of ℓ and m if $8x^3 + \ell x^2 - 27x + m$ is divisible by $2x^2 - x - 6$.
7. Find ℓ and m if $2x^3 - (2\ell + 1)x^2 + (\ell + m)x + m$ may be exactly divisible by $2x^2 - x - 3$.
8. f(x) when divided by $x^2 - 3x + 2$ leaves the remainder $ax + b$. If $f(l) = 4$ and $f(2) = 7$, determine a and b.
9. A polynomial in x of the third degree which will vanish when $x = 1$ & $x = -2$ and will have the values 4 & 28 when $x = -1$ and $x = 2$ respectively. Find the polynomial
10. If $f(x)$ is polynomial of degree 4 such that $f(1) = 1$, $f(2) = 2$, $f(3) = 3$, $f(4) = 4$ & $f(0) = 1$ find $f(5)$.

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7. EQUATIONS REDUCIABLE TO QUADRATIC EQUATIONS

There are certain equations which can be reduced to $ax^2 + bx + c = 0$ by some proper substitution.

7.1 $a(f(x))^2 + b(f(x)) + c = 0$, where $f(x)$ is expression of x

Method of solving : Put $f(x) = y$

Illustration-59 : **(a)** Solve $2^{x+3} + 2^{-x} - 6 = 0$ **(b)** Solve $\left(\frac{x}{x+1}\right)^2 + 6 - 5\left(\frac{x}{x+1}\right) = 0$

Solution : **(a)** Put $2^x = y$

$$8y + \frac{1}{y} - 6 = 0$$

$$8y^2 - 6y + 1 = 0$$

$$(4y - 1)(2y - 1) = 0$$

$$y = \frac{1}{4}, \frac{1}{2}$$

$\therefore\ 2^x = 2^{-2}$ and $2^x = 2^{-1}$ $\therefore\ x = -2,\ x = -1$

(b) Put $\frac{x}{x+1} = y$

$$y^2 - 5y + 6 = 0$$

$$(y - 2)(y - 3) = 0 \Rightarrow \frac{x}{x+1} = 2 \text{ and } \frac{x}{x+1} = 3$$

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7.2 $(x - a)^4 + (x - b)^4 = c$,

Method of Solving : Put $\frac{x - a + x - b}{2} = t \Rightarrow x = \frac{(a+b)}{2} + t$

Illustration-60 : $(x - 1)^4 + (x - 7)^4 = 272$

Solution : Put $x = \frac{7+1}{2} + t$

$$\Rightarrow x = t + 4$$

$$\Rightarrow (t + 3)^4 + (t - 3)^4 = 272$$

$$\Rightarrow 2(t^4 + 6.9t^2 + 81) = 272$$

$$\Rightarrow t^4 + 54t^2 + 81 = 136$$

$$\Rightarrow t^4 + 54t^2 - 55 = 0$$

$$\Rightarrow (t^2 - 1)(t^2 + 55) = 0$$

$$\Rightarrow t^2 = 1$$

$$\Rightarrow t = \pm 1$$

$$\Rightarrow x = 5, 3$$

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7.3 $ma^{2x} + n(ab)^x + rb^{2x} = 0$ (a & b > 0)

Method of Solving : Divide the equation by b^{2x} and put $\left(\frac{a}{b}\right)^x = t$ for $t > 0$

Illustration-61 : $3^{2x+2} + 5.6^x - 4^{x+1} = 0$

Solution : $3^{2x+2} + 5.6^x - 4^{x+1} = 0$

$\Rightarrow 9 \times (3)^{2x} + 5 \times (2 \times 3)^x - 4(2)^{2x} = 0$

$$\Rightarrow 9\left(\frac{3}{2}\right)^{2x} + 5\left(\frac{3}{2}\right)^x - 4 = 0 \qquad ...(1)$$

Let $\left(\frac{3}{2}\right)^x = t$ for $t > 0$

Equation 1 becomes

$9t^2 + 5t - 4 = 0$

$$\Rightarrow t = \frac{-5 \pm \sqrt{25+144}}{18} = -1 \text{ or } \frac{4}{9}$$

$t = -1$ (rejected)

$$t = \frac{4}{9} \Rightarrow \left(\frac{3}{2}\right)^x = \left(\frac{2}{3}\right)^2 = \left(\frac{3}{2}\right)^{-2}$$

$\Rightarrow x = -2$

Solution of the given equation is $x = -2$

7.4 $m \cdot a^{f(x)} + n \cdot b^{f(x)} + r = 0$, where ab = 1, a & b > 0 and $f(x)$ is expression of x

Method of Solving : put $a^{f(x)} = t$, then $b^{f(x)} = \frac{1}{t}$

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Illustration-62 : Solve $\left(\sqrt{5+2\sqrt{6}}\right)^x + \left(\sqrt{5-2\sqrt{6}}\right)^x = 10$

Solution : Let $a = \sqrt{5+2\sqrt{6}}$ and $b = \sqrt{5-2\sqrt{6}}$

ab = 1

Let $a^x = t$

Given equation become

$$t + \frac{1}{t} = 10 \Rightarrow t^2 - 10t + 1 = 0$$

$$\Rightarrow t = \frac{10 \pm \sqrt{96}}{2} = 5 \pm 2\sqrt{6}$$

$$\Rightarrow \left(5+2\sqrt{6}\right)^{x/2} = \left(5+2\sqrt{6}\right) \Rightarrow \frac{x}{2} = 1 \Rightarrow x = 2$$

or $\left(5+2\sqrt{6}\right)^{x/2} = \left(5-2\sqrt{6}\right) = \left(5+2\sqrt{6}\right)^{-1}$

$$\Rightarrow \frac{x}{2} = -1 \Rightarrow x = -2$$

Solutions of the given equation is x = 2 or –2.

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7.5 (x + a)(x + b)(x + c)(x + d) + e = 0 when b + c = a + d

Illustration-63 : Solve x(x + 1) (x + 2) (x + 3) – 8 = 0

Solution : x(x + 1) (x + 2) (x + 3) – 8 = 0

$\Rightarrow x(x + 3)(x + 1)(x + 2) - 8 = 0$

$\Rightarrow (x^2 + 3x)(x^2 + 3x + 2) - 8 = 0$

$\Rightarrow (x^2 + 3x)^2 + 2(x^2 + 3x) - 8 = 0$

$\Rightarrow (x^2 + 3x)^2 + 4(x^2 + 3x) - 2(x^2 + 3x) - 8 = 0$

$\Rightarrow (x^2 + 3x)(x^2 + 3x + 4) - 2(x^2 + 3x + 4) = 0$

$\Rightarrow (x^2 + 3x - 2)(x^2 + 3x + 4) = 0$

$\Rightarrow (x^2 + 3x - 2) = 0$ or $(x^2 + 3x + 4) = 0$

$$\Rightarrow x = \frac{-3 \pm \sqrt{17}}{2} \text{ or } x = \frac{-3 \pm \sqrt{7}\,i}{2}$$

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7.6 $(x + a)(x + b)(x + c)(x + d) + ex^2 = 0$, where ad = bc.

Method of solving : Divide given equation by x^2 and put $x + \frac{ad}{x} = t$

Illustration-64 : $(x + 1)(x + 2)(x + 3)(x + 6) = 3x^2$

Solution : $(x + 1)(x + 6)\,(x + 2)(x + 3) - 3x^2 = 0$

$\Rightarrow (x^2 + 7x + 6)(x^2 + 5x + 6) - 3x^2 = 0$

$\Rightarrow \left(x + \frac{6}{x} + 7\right)\left(x + \frac{6}{x} + 5\right) - 3 = 0$

Let $x + \frac{6}{x} = t$

$(t + 7)(t + 5) - 3 = 0$

$\Rightarrow t^2 + 12t + 32 = 0$

$t = -8$ or -4

when $x + \frac{6}{x} = -8 \Rightarrow x^2 + 8x + 6 = 0 \Rightarrow x = -4 \pm \sqrt{10}$

$x + \frac{6}{x} = -4 \Rightarrow x^2 + 4x + 6 = 0 \Rightarrow x = -2 \pm \sqrt{2}\,i$

Illustration-65 : Solve $(x^2 - 3x)(x^2 - 3x + 2) + 1 = 0$

Solution : Let $x^2 - 3x = y$

$\Rightarrow y(y + 2) + 1 = 0$

$\Rightarrow y^2 + 2y + 1 = 0$

$\Rightarrow (y + 1)^2 = 0$

$\Rightarrow y = -1$

Putting $y = 1$ in $x^2 - 3x = y$

we have $x^2 - 3x = -1$

$\Rightarrow x^2 - 3x + 1 = 0$

$\Rightarrow x = \frac{3 \pm \sqrt{5}}{2}$

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7.7 $ax^4 + bx^3 + cx^2 + dx + e = 0$, where $a = e$ & $b = \pm d$

Method of solving : Divide given equation by x^2 and put $x + \frac{1}{x} = t$ or $x - \frac{1}{x} = t$ whichever is applicable.

Illustration-66 : Solve $x^4 - 2x^3 + 3x^2 - 2x + 1 = 0$

Solution : $x^4 - 2x^3 + 3x^2 - 2x + 1 = 0$

By dividing x^2 both sides we have

$$x^2 - 2x + 3 - \frac{2}{x} + \frac{1}{x^2} = 0$$

$$\Rightarrow x^2 + \frac{1}{x^2} - 2\left(x + \frac{1}{x}\right) + 3 = 0$$

Let $x + \frac{1}{x} = t$

Above equation become

$t^2 - 2 - 2t + 3 = 0$

$\Rightarrow t^2 - 2t + 1 = 0 \Rightarrow (t-1)^2 = 0$

$\Rightarrow t = 1 \quad \Rightarrow x + \frac{1}{x} = 1 \Rightarrow x^2 - x + 1 = 0 \Rightarrow x = \frac{1 \pm \sqrt{3}\, i}{2}$

Roots are $\frac{1 \pm \sqrt{3}\, i}{2}$

7.8 By Guessing Rational Roots of Polynomial.

Illustration-67 : Solve : $x^4 + x^3 - 2x^2 - x + 1 = 0$

Solution : Let $P(x) = x^4 + x^3 - 2x^2 - x + 1$

$P(1) = 0$ and $P(-1) = 0$

$\Rightarrow (x-1)(x+1)$ factor of $P(x)$

We can find other factor of $P(x)$ by dividing $x^2 - 1$ from $P(x)$.

$P(x) = (x^2 - 1)(x^2 + x - 1) = 0$

$\Rightarrow x = \pm 1$ or $x^2 + x - 1 = 0$

$\Rightarrow x = \pm 1$ or $x = \frac{-1 \pm \sqrt{5}}{2}$

solution of $P(x) = 0$ are $x = \pm 1$ or $\frac{-1 \pm \sqrt{5}}{2}$.

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Do yourself-11 :

Solve the following equations for x :

1. $x^{2/3} + x^{1/3} - 2 = 0$

2. $x^{\frac{2}{5}} - 3x^{\frac{1}{5}} + 2 = 0$

3. $3x^4 - 8x^2 + 4 = 0$

4. $4^x - 3.2^{x+3} + 128 = 0$

5. $x^2 + \frac{1}{x^2} - 5\left(x + \frac{1}{x}\right) + 8 = 0$

6. $2\left(x^2 + \frac{1}{x^2}\right) - 3\left(x - \frac{1}{x}\right) - 4 = 0$

7. $2^{2x+1} - 7 \times 10^x + 5^{2x+1} = 0$

8. $(x-1)^4 + (x-5)^4 = 82$

9. $\left(\sqrt{3}+\sqrt{2}\right)^x + \left(\sqrt{3}-\sqrt{2}\right)^x - 2\sqrt{3} = 0$

10. $\left(3+2\sqrt{2}\right)^{x/2} + \left(3-2\sqrt{2}\right)^{x/2} - 34 = 0$

11. $x(x+1)(x+2)(x+3) = 24$

12. $(x+1)(x+2)(x+3)(x+4) = 120$

13. $(x+1)(x+2)^2(x+4) - 2x^2 = 0$

14. $x^4 - 2x^3 - 2x^2 + 2x + 1 = 0$

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15. Match the values of x given in **Column-II** satisfying the exponential equation in Column-I (Do not verify). Remember that for $a > 0$, the term a^x is always greater than zero $\forall\, x \in R$.

Column-I	Column-II
(A) $5^x - 24 = \frac{25}{5^x}$	**(P)** –3
(B) $(2^{x+1})(5^x) = 200$	**(Q)** –2
(C) $4^{2/x} - 5(4^{1/x}) + 4 = 0$	**(R)** –1
(D) $\frac{2^{x-1}.4^{x+1}}{8^{x-1}} = 16$	**(S)** 0
(E) $4^{x^2+2} - 9\left(2^{x^2+2}\right) + 8 = 0$	**(T)** 1
(F) $5^{2x} - 7^x - 5^{2x}(35) + 7^x(35) = 0$	**(U)** 2
	(V) 3
	(X) None

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8. SYSTEM OF EQUATIONS

Observe the following Illustrations :

Illustrations-68 : If x – y = 2 and xy = 24, find the value of $\frac{1}{x}+\frac{1}{y}$.

Solution : $(x + y)^2 = (x - y)^2 + 4xy = 4 + 4(24)$

$\Rightarrow (x + y)^2 = 100$

$\Rightarrow x + y = 10, -10$

$$\therefore \frac{x+y}{xy}=\frac{10}{24}=\frac{5}{12};\frac{x+y}{xy}=-\frac{10}{24}=-\frac{5}{12}$$

Illustrations-69 : If 2x – 3y – z = 0 and x + 3y – 14z = 0, then find $\frac{x^2+3xy}{y^2+z^2}$.

Solution : $\frac{2x}{z}-\frac{3y}{z}=1$ & $\frac{x}{z}+\frac{3y}{z}=14$

Solving $\frac{x}{z}=5;\frac{y}{z}=3$

$$\Rightarrow \frac{\left(\frac{x}{z}\right)^2+\frac{3x}{z}.\frac{y}{z}}{\left(\frac{y}{z}\right)^2+1}=\frac{25+3(5)(3)}{(3)^2+1}=\frac{70}{10}=7$$

Illustration-70 : Given a + b = 20 and $a^3 + b^3 = 800$, find $a^2 + b^2$.

Solution : a + b = 20

$\Rightarrow a^2 + b^2 + 2ab = 400$ .... (1)

$a^3 + b^3 = 800$

$\Rightarrow (a + b)(a^2 - ab + b^2) = 800$

$\Rightarrow a^2 + b^2 - ab = 40$ .... (2)

By adding twice of second equation with first equation.

$3(a^2 + b^2) = 480$

$\Rightarrow a^2 + b^2 = 160.$

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Illustrations-71 : $x(y + z) = 29$, $y(z + x) = 26$; $z(x + y) = 51$ find x,y,z.

Solution : $xy + zx = 29$ ...(1)

$yz + xy = 26$ ...(2)

$xz + yz = 51$ ...(3)

$\Rightarrow 2[xy + yz + zx] = 106$

$\Rightarrow xy + yz + zx = 53$

Now : $xy = 2$, $zx = 27$; $yz = 24$

$\Rightarrow x^2y^2z^2 = 24 \times 2 \times 27 = (36)^2$

$\Rightarrow xyz = \pm 36 \therefore (x, y, z) \equiv \left(\frac{3}{2}, \frac{4}{3}, 18\right)$ or $\left(-\frac{3}{2}, -\frac{4}{3}, -18\right)$

Illustrations-72 : If $x^3 + y^3 = 35$; and $x^2y + xy^2 = 30$, then find (x,y).

Solution : $\frac{x^3 + y^3}{x^2y + xy^2} = \frac{(x+y)(x^2 - xy + y^2)}{xy(x+y)} = \frac{35}{30} = \frac{7}{6}$

$\Rightarrow 6x^2 - 13xy + 6y^2 = 0 \Rightarrow (3x - 2y)(2x - 3y) = 0 \Rightarrow 3x = 2y$ or $2x = 3y$

Case -I : $3x = 2y$, $x^3 + \left(\frac{3x}{2}\right)^3 = 35 \Rightarrow 35x^3 = 8 \times 35 \Rightarrow x = 2, y = 3$

Case -II : $2x = 3y$, $x^3 + \left(\frac{2x}{3}\right)^3 = 35 \Rightarrow 35x^3 = 27 \times 35 \Rightarrow x = 3, y = 2$

Illustrations-73 : If x,y and z are real numbers such that $x^2 + y^2 + 2z^2 - 4x + 2z - 2yz + 5 = 0$, find the value of $(x - y - z)$.

Solution : $x^2 + y^2 + 2z^2 - 4x + 2z - 2yz + 5 = 0$

$\Rightarrow x^2 - 4x + 4 + y^2 + z^2 - 2yz + z^2 + 2z + 1 = 0$

$\Rightarrow (x - 2)^2 + (y - z)^2 + (z + 1)^2 = 0$

$\Rightarrow x = 2, y = z, z = -1$

$\therefore x - y - z = 2 + 1 + 1 = 4$

Illustrations-74 : $xy = 12$, $yz = 15$, $zx = 20$ find $x + y + z$; $x,y,z \in \mathbb{R}^+$

Solution : $(xyz)^2 = (3 \times 4 \times 5)^2$

$\Rightarrow xy\,z = 3 \times 4 \times 5$

$\Rightarrow z = 5, y = 3, x = 4$

$\Rightarrow x + y + z = 5 + 3 + 4 = 12$

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Do yourself-12 :

Solve the following systems of equations :

1. $\begin{cases} x^2 - y^2 = 16, \\ x + y = 8 \end{cases}$

2. $\begin{cases} x - y = 1, \\ x^3 - y^3 = 7 \end{cases}$

3. $x + y = 2$ and $x^3 + y^3 = 56$; $x, y \in \mathbb{R}$, then find x and y.

4. $\begin{cases} x^2 + y^2 + 6x + 2y = 0, \\ x + y + 8 = 0 \end{cases}$

5. $\begin{cases} \frac{x}{y} - \frac{y}{x} = \frac{5}{6}, \\ x^2 - y^2 = 5 \end{cases}$

6. $\begin{cases} \frac{x+y}{x-y} + \frac{x-y}{x+y} = \frac{13}{6}, \\ xy = 5 \end{cases}$

7. $\begin{cases} \frac{1}{x+1} + \frac{1}{y} = \frac{1}{3}, \\ \frac{1}{(x+1)^2} - \frac{1}{y^2} = \frac{1}{4} \end{cases}$

8. $\begin{cases} x^2 + y^2 = 25 - 2xy \\ y(x+y) = 10 \end{cases}$

9. $\begin{cases} \frac{1}{y-1} - \frac{1}{y+1} = \frac{1}{x}, \\ y^2 - x - 5 = 0 \end{cases}$

10. $\begin{cases} 2xy + y^2 - 4x - 3y + 2 = 0, \\ xy + 3y^2 - 2x - 14y + 16 = 0 \end{cases}$

11. $\begin{cases} x^3 + y^3 = 7, \\ xy(x+y) = -2 \end{cases}$

12. $\begin{cases} x^4 + y^4 = 82, \\ xy = 3 \end{cases}$

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9. INEQUALITIES

9.1 Basic Rules :

- If $a > b$ and $b > c$, then $a > c$.
- If $x > y$, then $x + c > y + c$ for any real number c. Additionally, if $a > b$, then $x + a > y + b$.
- If $x > y$ and $a > 0$, then $xa > ya$.
- If we multiply or divide an inequality by a negative number, we must reverse the sign. For example, if $x > y$ and $a < 0$, then $xa < ya$.
- If $x > y > 0$ and $a > b > 0$, then $xa > yb$.

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- If $x > y$ and x and y have the same sign (positive or negative), then $\frac{1}{x} < \frac{1}{y}$.
- If $x > y \geq 0$, then for any positive real number a, we have $x^a > y^a$.

  In particular, if $0 < a < b$, then $\sqrt[n]{a} < \sqrt[n]{b}$ for all positive integral values of $n > 1$.

  E.g. $\sqrt[4]{2} < \sqrt[4]{7}$, $\sqrt[3]{3} < \sqrt[3]{5}$, $\sqrt[5]{10} < \sqrt[5]{13}$ etc.

  If two simple surds of different orders viz. $\sqrt[n]{a}$ and $\sqrt[m]{b}$ have to be compared, they have to be expressed as surds of the same order i.e. LCM of n and m .

  Ex. compare $\sqrt[4]{6}$ and $\sqrt[3]{5}$,

  we express both as the surds of 12th order.

  $\therefore \sqrt[4]{6} = \sqrt[12]{6^3}$ and $\sqrt[3]{5} = \sqrt[12]{5^4}$. As $6^3 < 5^4 \Rightarrow \sqrt[4]{6} < \sqrt[3]{5}$

Illustration-75 : Solve :

(i) $7 - 3x \geq 8 + 2x$ (ii) $\frac{8-x}{7} \geq 4$ (iii) $\frac{1}{x} \geq 5$ (iv) $\frac{4}{x+1} \leq 2$

Solution : (i) $7 - 3x \geq 8 + 2x \Rightarrow 7 - 8 \geq 2x + 3x$

$\Rightarrow \frac{-1}{5} \geq x \Rightarrow x \in \left(-\infty, -\frac{1}{5}\right]$

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(ii) $\frac{8-x}{7} \geq 4 \Rightarrow 8 - x \geq 28 \Rightarrow -x \geq 20$

$\Rightarrow x \leq -20 \Rightarrow x \in (-\infty, -20]$

(iii) $\frac{1}{x} \geq 5$

$\Rightarrow \infty > \frac{1}{x} \geq 5 \quad \Rightarrow 0 < x \leq \frac{1}{5} \quad \Rightarrow x \in \left(0, \frac{1}{5}\right]$

(iv) $\frac{4}{x+1} \leq 2$

$\Rightarrow -\infty < \frac{4}{x+1} < 0$ or $0 < \frac{4}{x+1} \leq 2$ $\left(\frac{4}{x+1} = 0 \text{ is not possible}\right)$

$\Rightarrow 0 > \frac{x+1}{4} > -\infty$ or $\infty > \frac{x+1}{4} \geq \frac{1}{2}$

$\Rightarrow x + 1 < 0$ or $2 \leq x + 1$

$\Rightarrow x < -1$ or $1 \leq x \quad \Rightarrow x \in (-\infty, -1) \cup [1, \infty)$

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Do yourself : 13

1. Determine which of the following statements is true or false. If it is false, provide an example that shows the statement is false. Assume *a, b, c, x* and *y* are real numbers.

(a) If $a \le b$ and $b \le c$, then $a < c$.
(b) If $a \ge b \ge a$, then $a = b$.
(c) If $a > b$, then $ac > bc$.
(d) If $a > b$ and $c \le 0$, then $ac \le bc$.
(e) If $x + a \ge y + a$, then $x \ge y$.
(f) If $x + a \ge y + b$, then $x \ge y$ and $a \ge b$.

2. Which fraction is larger ?

(a) $\frac{13}{17}$ or $\frac{17}{21}$ (b) $\frac{31}{35}$ or $\frac{37}{41}$

3. Which of the following numbers is largest : $2^{36}, 3^{30}, 5^{18}, 6^{12}, 7^{8}, 8^{4}$? (No calculators!)

4. What values of x satisfy the inequality, $7 - 3x < x - 1 \le 2x + 9$?

5. Which of the following is greater $5+\sqrt{3}, 3+\sqrt{14}$? (Without calculating the values of $\sqrt{3}, \sqrt{14}$)

9.2 Trivial and Sum of squares (SOS) Inequality

The square of any real number is non-negative. So if x is real, then $x^2 \ge 0$. This is known as Trivial inequality. Equality holds only if x = 0.

Sum of squares (SOS) of real numbers is non negative. That is $\Sigma x_i^2 \ge 0$. This is know as SOS inequality. Equality holds if $x_i = 0 \ \forall \ i$

Ex. $x, y, z \in \mathbb{R}$ and $x^2 + y^2 + z^2 = 0 \Rightarrow x = y = z = 0$.

Note :

- $f(x) = [g(x)]^{2n}$ where $n \in \mathbb{N} \Rightarrow f(x) \ge 0$
- $f(x) = [g(x)]^{1/2n}$, $n \in \mathbb{N}$, $g(x) \ge 0 \Rightarrow f(x) \ge 0$

Illustration-76 : If $a, b, c \in \mathbb{R}$ and $a^2 + b^2 + c^2 - ab - bc - ca = 0$, prove that $a = b = c$.

Solution :

$a^2 + b^2 + c^2 - ab - bc - ca = 0$

$\Rightarrow 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = 0$

$\Rightarrow (a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ac + a^2) = 0$

$\Rightarrow (a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

$\Rightarrow a - b = b - c = c - a = 0$

$\Rightarrow a = b = c$.

Illustration-77 : For $x,y \in \mathbb{R}$, find the all possible values (range) of expression $4x^2 + 9y^2 - 12x + 6y$.

Solution : If $E(x,y) = 4x^2 + 9y^2 - 12x + 6y$

$= (2x)^2 - 2(2x) \times 3 + (3)^2 + (3y)^2 + 2(3y) + (1)^2 - 10$

$= (2x - 3)^2 + (3y + 1)^2 - 10$

By sum of square (SOS), $(2x - 3)^2 + (3y + 1)^2 \ge 0$

$\Rightarrow E(x,y) = (2x - 3)^2 + (3y + 1)^2 - 10 \ge 0 - 10$

So Range of $E(x,y) = [-10, \infty)$

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Illustration-78 : Find all ordered pairs of real numbers (x,y) such that $(4x^2 + 4x + 3)(y^2 - 6y + 13) = 8$

Solution : Rewriting the equation

$((2x + 1)^2 + 2)\ ((y - 3)^2 + 4) = 8$

from the SOS, $(2x + 1)^2 \geq 0$ and $(y - 3)^2 \geq 0$,

so we have

$(2x + 1)^2 + 2 \geq 2$ and $(y - 3)^2 + 4 \geq 4$

$\Rightarrow ((2x + 1)^2 + 2)\ ((y - 3)^2 + 4) \geq 8$

Equality holds only if

$$(2x + 1)^2 = (y - 3)^2 = 0 \Rightarrow (x, y) = \left(-\frac{1}{2}, 3\right)$$

Illustration-79 : If $b^2 - 4ac < 0$ and $a > 0$, then show that $ax^2 + bx + c > 0\ \forall\ x \in \mathbb{R}$

Solution : $ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x\right] + c$

$$= a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right] + c$$

$$= a\left[x + \frac{b}{2a}\right]^2 - \left(\frac{b^2 - 4ac}{4a}\right) > 0\ \forall\ x \in \mathbb{R}.$$ Hence proved

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9.3 Mean :

In any collection of data a specific value between two extremes (minimum/maximum) is called a mean of the data.

For Two Variables :

Let x, y be two **positive** real numbers with $x \leq y$.

- The **Arithmetic Mean** (**A**) of x and y is $\frac{x + y}{2}$

  Observe $x \leq \frac{x + y}{2} \leq y$

- The **Geometric Mean** (**G**) of x and y is $\sqrt{xy}$

  Observe $x \leq \sqrt{xy} \leq y$

We have $x \leq G \leq A \leq y$

Equality holds only when $x = y$.

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Proof For two positive numbers x and y, the Trivial inequality gives us $\left(\sqrt{x}-\sqrt{y}\right)^2 \geq 0$

$$\Rightarrow x+y-2\sqrt{xy} \geq 0$$

$$\Rightarrow \frac{x+y}{2} \geq \sqrt{xy} \Rightarrow A \geq G$$

Note :

- $x+\frac{1}{x} \geq 2 \ \forall \ x>0$ and $x+\frac{1}{x} \leq -2 \ \forall \ x<0$

Illustration : 80 Find all possible values (range) of the expression $x+\frac{4}{x}$, when $x \in \mathbb{R}-\{0\}$.

Sol. When $x>0$, $\frac{x+\frac{4}{x}}{2} \geq \sqrt{x \times \frac{4}{x}} = 2$ (By $A \geq G$)

$$\Rightarrow x+\frac{4}{x} \geq 4,$$

So, when $x>0$, range $=[4,\infty)$ ...(1)

for $x<0$, $\frac{x+\frac{4}{x}}{2} \leq -\sqrt{x \times \frac{4}{x}}$

$$\Rightarrow x+\frac{4}{x} \leq -4 \qquad ...(2)$$

From (1) and (2), range $=\left(-\infty,-4\right] \cup \left[4,\infty\right)$

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Do yourself : 14

1. Find the minimum value of $\frac{x^4+8}{x^2}$.

2. For $x<0$, find the maximum value of $\frac{3x^2+12}{x}$.

3. If $a, b \in \mathbb{R}^+$, find minimum possible value of $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)$.

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10. RATIO AND PROPORTION

If a and b be two quantities of the same kind, then their ratio is a : b; which may be denoted by the fraction $\frac{a}{b}$ (This may be an integer or fraction)

In the ratio a : b, a is the first term (Antecedent) and b is the second term (Consequent)

A ratio may represented in a number of ways e.g. $\frac{a}{b}=\frac{ma}{mb}=\frac{na}{nb}=\cdots$ where m, n, ... are non-zero numbers.

Let a, b, c, d be positive integers now to compare two ratios a : b and c : d we use following :

- (a : b) > (c : d) if ad > bc
- (a : b) = (c : d) if ad = bc
- (a : b) < (c : d) if ad < bc

To compare two or more ratio, reduce them to common denominator.

Note :

- If a > b > 0 and x > 0, then $\frac{a}{b}>\frac{a+x}{b+x}$, e.g. $\frac{41}{40}>\frac{45}{44}$
- If 0 < a < b and x > 0, then $\frac{a}{b}<\frac{a+x}{b+x}$

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Illustration-81 : What term must be added to each term of the ratio 5 : 37 to make it equal to 1 : 3 ?

Solution : Let x be added to each term of the ratio 5 : 37.

Then $\frac{x+5}{x+37}=\frac{1}{3}$ $\Rightarrow$ 3x + 15 = x + 37 i.e. x = 11

Illustration-82 : If x : y = 3 : 4; find the ratio of 7x – 4y : 3x + y

Solution : $\frac{x}{y}=\frac{3}{4}\Rightarrow x=\frac{3}{4}y$

Now $\frac{7x-4y}{3x+y}=\frac{7.\frac{3}{4}y-4y}{3.\frac{3}{4}y+y}$ (putting the value of x)

$=\frac{5y}{13y}=\frac{5}{13}$ i.e. 5 : 13

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10.1 Proportion :

When two ratios are equal, then the four quantities compositing them are said to be proportional.

so, if $\frac{a}{b}=\frac{c}{d}$, then it is written as a : b = c : d or a : b :: c : d

Where 'a' and 'd' are known as extremes and 'b and c' are known as means.

(i) An important property of proportion : Product of extremes = product of means.

(ii) If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$ then each is equal to $\frac{a+c+e}{b+d+f}$

(iii) If a : b = c : d, then b : a = d : c (Invertando)

i.e. $\frac{a}{b}=\frac{c}{d} \quad \Rightarrow \quad \frac{b}{a}=\frac{d}{c}$

(iv) If a : b = c : d, then a : c = b : d (Alternando)

i.e. $\frac{a}{b}=\frac{c}{d} \quad \Rightarrow \quad \frac{a}{c}=\frac{b}{d}$

(v) If a : b = c : d, then $\frac{a+b}{b}=\frac{c+d}{d}$ (Componendo)

i.e. $\frac{a}{b}=\frac{c}{d} \quad \Rightarrow \quad \frac{a}{b}+1=\frac{c}{d}+1$

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(vi) If a : b = c : d, then $\frac{a-b}{b}=\frac{c-d}{d}$ (Dividendo)

i.e. $\frac{a}{b}=\frac{c}{d} \quad \Rightarrow \quad \frac{a}{b}-1=\frac{c}{d}-1$

(vii) If a : b = c : d, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$ (Componendo and dividendo)

i.e. $\frac{a}{b}=\frac{c}{d} \quad \Rightarrow \quad \frac{a}{b}+1=\frac{c}{d}+1 \quad \Rightarrow \quad \frac{a+b}{b}=\frac{c+d}{d}$ ... (1)

$\frac{a}{b}-1=\frac{c}{d}-1 \quad \Rightarrow \quad \frac{a-b}{b}=\frac{c-d}{d}$ ... (2)

Dividing equation (1) by (2) we obtain

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}$$

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Illustration-83 : If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$ show that $\frac{x^3+a^3}{x^2+a^2}+\frac{y^3+b^3}{y^2+b^2}+\frac{z^3+c^3}{z^2+c^2}=\frac{(x+y+z)^3+(a+b+c)^3}{(x+y+z)^2+(a+b+c)^2}$

Solution : $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=k$ (constant)

x = ak; y = bk; z = ck

substituting these values of x, y, z in the given expression, we obtain

$$\text{L.H.S.} = \frac{a^3k^3+a^3}{a^2k^2+a^2}+\frac{b^3k^3+b^3}{b^2k^2+b^2}+\frac{c^3k^3+c^3}{c^2k^2+c^2}$$

$$=\frac{a^3(k^3+1)}{a^2(k^2+1)}+\frac{b^3(k^3+1)}{b^2(k^2+1)}+\frac{c^3(k^3+1)}{c^2(k^2+1)}=\frac{(k^3+1)}{(k^2+1)}.(a+b+c)$$

$$\text{Now R.H.S} = \frac{(ak+bk+ck)^3+(a+b+c)^3}{(ak+bk+ck)^2+(a+b+c)^2}$$

$$=\frac{k^3(a+b+c)^3+(a+b+c)^3}{k^2(a+b+c)^2+(a+b+c)^2}$$

$$=\frac{(k^3+1)(a+b+c)^3}{(k^2+1)(a+b+c)^2}=\frac{(k^3+1)}{(k^2+1)}.(a+b+c) = \text{L.H.S}$$

Illustration-84 : If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}$, prove that

$(ab + bc + cd + de)^2 = (a^2 + b^2 + c^2 + d^2)(b^2 + c^2 + d^2 + e^2)$

Solution : $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}$, then we have

$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}=\frac{\sqrt{(a^2+b^2+c^2+d^2)}}{\sqrt{(b^2+c^2+d^2+e^2)}}=k \text{ (say)}$$

i.e. $a = bk$ $\therefore ab = b^2k$

$b = ck$ $\therefore bc = c^2k$

$c = dk$ $\therefore cd = d^2k$

$d = ek$ $\therefore de = e^2k$

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so, $(a^2 + b^2 + c^2 + d^2) = k^2 (b^2 + c^2 + d^2 + e^2)$ ... (i)

Now L.H.S. $= (ab + bc + cd + de)^2$

$= (kb^2 + kc^2 + kd^2 + ke^2)^2$

$= k^2(b^2 + c^2 + d^2 + e^2)^2$

$= k^2(b^2 + c^2 + d^2 + e^2)\ (b^2 + c^2 + d^2 + e^2)$

$= (a^2 + b^2 + c^2 + d^2)\ (b^2 + c^2 + d^2 + e^2) =$ R.H.S (by use of (i))

Illustration-85 : Solve the equation $\dfrac{3x^4 + x^2 - 2x - 3}{3x^4 - x^2 + 2x + 3} = \dfrac{5x^4 + 2x^2 - 7x + 3}{5x^4 - 2x^2 + 7x - 3}$

Solution : $\dfrac{3x^4 + x^2 - 2x - 3}{3x^4 - x^2 + 2x + 3} = \dfrac{5x^4 + 2x^2 - 7x + 3}{5x^4 - 2x^2 + 7x - 3}$

Applying componendo and dividendo, we have

$$\frac{3x^4}{x^2 - 2x - 3} = \frac{5x^4}{2x^2 - 7x + 3}$$

or $3x^4 (2x^2 - 7x + 3) - 5x^4(x^2 - 2x - 3) = 0$

or $x^4 [6x^2 - 21x + 9 - 5x^2 + 10x + 15] = 0$

or $x^4 (x^2 - 11x + 24) = 0$

$\therefore$ $x = 0$ or $x^2 - 11x + 24 = 0$

$\therefore$ $x = 0$ or $(x - 8)(x - 3) = 0$

$\therefore$ $x = 0, 8, 3$

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Do yourself : 15

1. If $\dfrac{a}{b} = \dfrac{2}{3}$ and $\dfrac{b}{c} = \dfrac{4}{5}$, then find value of $\dfrac{a+b}{b+c}$

2. If $\dfrac{a}{b} = \dfrac{3}{5}$ and $\dfrac{b}{c} = \dfrac{7}{13}$, then find the value of a : b : c

3. If sum of two numbers is s and their quotient is $\dfrac{p}{q}$. Find number.

4. If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$, then find the value of $\dfrac{2a^4b^2 + 3a^2c^2 - 5e^4f}{2b^6 + 3b^2d^2 - 5f^5}$ in terms of a and b.

5. If x : a = y : b = z : c, then show that $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (ax + by + cz)^2$.

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11. SIGN-SCHEME (WAVY CURVE) METHOD

Given $f(x)$ and $g(x)$ are polynomials.

To solve the inequalities of the type $\frac{f(x)}{g(x)} * 0$, where '*' can be >, ≥, < or ≤, we take the following steps :

(i) Find all the roots of $f(x) = 0$ and $g(x) = 0$

(ii) Write all these roots on the real line in increasing order of values.

(iii) Check the sign of the expression $\frac{f(x)}{g(x)}$ at some x greater than the largest root. If it is positive, put + sign in rightmost interval. In case of negative, put -ve sign in rightmost interval and while moving from right to left change sign in accordance with step (iv).

(iv) If a root occurs even number of times, then sign of expression will be same on both sides of the root and if a root occurs odd number of times, then sign of the expression will be different on both sides of the root.

(v) Write the answer according to need of the question.

Note :

- We don't give equality sign on '±∞' in the solution as they are two improper points of number line.
- We can't take zeroes of denominator in the final answer as at these points expression is not defined (because division by '0' is not defined).
- In case of ≥ 0 or ≤ 0, zeroes of numerator will be part of the answer provided they are not appearing in denominator also.
- Do not cross multiply the terms in the inequalities

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Illustration-86 : Find the solution of $-x^2 + 6x + 7 \geq 0$

Solution : $-x^2 + 6x + 7 \geq 0$

$\Rightarrow x^2 - 6x - 7 \leq 0$

$\Rightarrow (x + 1)(x - 7) \leq 0$

$\Rightarrow x \in [-1,7]$

Illustration-87 : Find the solution of $2x^2 - 7x + 3 \geq 0$

Solution : $2x^2 - 7x + 3 \geq 0$

$\Rightarrow (2x - 1)(x - 3) \geq 0$

$x \in \left(-\infty, \frac{1}{2}\right] \cup [3, \infty)$

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Illustration-88 : Solve $2x^4 > 3x^3 + 9x^2$

Solution : $2x^4 - 3x^3 - 9x^2 > 0$

$x^2(2x^2 - 3x - 9) > 0$

$x^2(2x + 3)(x - 3) > 0$

$$x \in \left(-\infty, -\frac{3}{2}\right) \cup (3, \infty)$$

Illustration-89 : Find the solution of the inequalities $\frac{(x-1)(x-2)}{(x-3)} \geq 0$

Solution : $x - 1 = 0,\ x - 2 = 0,\ x - 3 = 0$

$\Rightarrow x = 1, 2, 3$

Since $x - 3 \neq 0,\ x \neq 3$

so, $x \in [1, 2] \cup (3, \infty)$

Illustration-90 : If $f(x) = \frac{x^3 + 4x^2 - 11x - 30}{x^2 - 6x - 7}$ then find x such that

(i) $f(x) > 0$ **(ii)** $f(x) < 0$.

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Solution : Given $f(x) = \frac{(x-3)(x+2)(x+5)}{(x+1)(x-7)}$

(i) $f(x) > 0 \Rightarrow x \in (-5, -2) \cup (-1, 3) \cup (7, \infty)$

(ii) $f(x) < 0 \Rightarrow x \in (-\infty, -5) \cup (-2, -1) \cup (3, 7)$

Illustration 91: Solve for real x : $\frac{1}{x+1} + \frac{2}{x+2} > \frac{3}{x+3}$

Solution : $\frac{3x+4}{(x+1)(x+2)} > \frac{3}{x+3}$

$$\Rightarrow \frac{3x+4}{(x+1)(x+2)} - \frac{3}{x+3} > 0 \qquad \Rightarrow \frac{4x+6}{(x+1)(x+2)(x+3)} > 0$$

so, $x \in (-\infty, -3) \cup \left(-2, -\frac{3}{2}\right) \cup (-1, \infty)$

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Illustration 92: Let $f(x)=\dfrac{(x-1)^3(x+2)^4(x-3)^5(x+6)}{x^2(x-7)^3}$. Solve the following inequality

(i) $f(x)>0$ (ii) $f(x)\ge 0$ (iii) $f(x)<0$ (iv) $f(x)\le 0$

Solution : We mark on the number line zeroes of numerator of expression : 1, –2, 3 and –6 (with black circles) and the zeroes of denominator 0 and 7 (with white circles), isolate the double points : –2 and 0 and draw the wavy curve :

From graph, we get

(i) $x\in(-\infty,-6)\cup(1,3)\cup(7,\infty)$

(ii) $x\in(-\infty,-6]\cup\{-2\}\cup[1,3]\cup(7,\infty)$

(iii) $x\in(-6,-2)\cup(-2,0)\cup(0,1)\cup(3,7)$

(iv) $x\in[-6,0)\cup(0,1]\cup[3,7)$

Do yourself-16 :

Solve following inequalities over the set of real numbers :

1. $(x-1)^2(x+1)^3(x-4)<0$

2. $\dfrac{6x-5}{4x+1}<0$

3. $\dfrac{(x-1)(x+2)^2}{-1-x}<0$

4. $\dfrac{(2x-1)(x-1)^2(x-2)^3}{(x-4)^4}>0$

5. $\dfrac{(x-1)^2(x+1)^3}{x^4(x-2)}\le 0$

6. $\dfrac{(x-2)^2(1-x)(x-3)^3(x-4)^2}{(x+1)}\le 0$

7. $\dfrac{x^2+4x+4}{2x^2-x-1}<0$

8. $\dfrac{x^3(x-2)(5-x)}{(x^2-4)(x+1)}>0$

9. $\dfrac{(2-x^2)(x-3)^3}{(x+1)(x^2-3x-4)}\ge 0$

10. $x^4-5x^2+4<0$

11. $\dfrac{15-4x}{x^2-x-12}<4$

12. $\dfrac{x^2+1}{4x-3}>2$

13. $\dfrac{1}{x-2}+\dfrac{1}{x-1}>\dfrac{1}{x}$

14. $(x-2)(x+3)\ge 0$

15 $\dfrac{x}{x+1}>2$

16. $\dfrac{3x-1}{4x+1}\le 0$

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12. MODULUS AND MODULUS EQUATIONS

For any real number x, modulus or absolute value of x is denoted by |x| and is defined as

$$|x| = \begin{cases} x, & \text{if} \quad x \geq 0 \\ -x, & \text{if} \quad x < 0 \end{cases}$$

- Graph of y = |x|

Note :

- $|x| = |-x| \geq 0$
- Geometrically |x| is distance of real number x from zero along the real number line
- More generally |x – a| is distance between 'x' and 'a' on the number line.

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- $|x| = \sqrt{x^2}$
- |xy| = |x| |y|

Illustration-93 : Sketch the graph of following equation and also find all possible values (Range) of y

(i) y = |x| + x (ii) y = |x – 2| + x + 1

(iii) y = |x – 3| – |x| (iv) y =2|x – 2| – |x + 1|

Soluiton :

(i) $y = |x| + x = \begin{cases} x + x & , \quad x \geq 0 \\ 0 & , \quad x < 0 \end{cases}$

$= \begin{cases} 2x & , \quad x \geq 0 \\ 0 & , \quad x < 0 \end{cases}$

From graph we can find all possible values (range) of y which is $[0,\infty)$

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(ii) $y = |x + 2| + x + 1 = \begin{cases} x+2+x+1 & , \quad x \ge -2 \\ -x-2+x+1 & , \quad x < -2 \end{cases}$

$$= \begin{cases} 2x+3 & , \quad x \ge -2 \\ -1 & , \quad x < -2 \end{cases}$$

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(iii) $y = |x + 3| - |x| = \begin{cases} x+3-x & , \quad x \ge 0 \\ x+3-(-x) & , \quad x \in [-3,0) \\ -x-3-(-x) & , \quad x < -3 \end{cases}$

$$= \begin{cases} 3 & , \quad x \ge 0 \\ 2x+3 & , \quad x \in [-3,0) \\ -3 & , \quad x < -3 \end{cases}$$

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(iv) $y = 2|x-2| - |x+1| = \begin{cases} 2(x-2)-(x+1) & , \quad x \ge 2 \\ 2(-x+2)-(x+1) & , \quad x \in [-1,2] \\ 2(-x+2)+(x+1) & , \quad x < -1 \end{cases}$

Illustration-94 : If $|x-1||x-2| = -(x^2-3x+2)$, then find the interval in which x lies ?

Solution : $|(x-1)(x-2)| = -(x-2)(x-1)$

$\Rightarrow (x-1)(x-2) \le 0$

$\Rightarrow 1 \le x \le 2$

Do yourself-17

(1) Sketch the graph of following

(i) $y = |x-2|$ (ii) $y = |x| - 2$ (iii) $y = 5 - |x|$

Solve for x

(2) $|2x+5| = 2$ **(3)** $|2x-5| = 7$ **(4)** $|x-3| = -1$

(5) $|2x-3| + 4 = 2$ **(6)** $\left|\frac{3x+4}{3}\right| = 7$ **(7)** $|x^2-3x+2| = 2$

(8) $|2x-3| = |3x+5|$ **(9)** $2|x+3| = 3|x-4|$ **(10)** $|x^2+x+1| = |x^2+x+2|$

(11) $|x^2-4x+3| = |x^2-5x+4|$ **(12)** $|x-6| + |x-3| = 1$ **(13)** $|2x-1| + |2x+3| = 6$

(14) $|3x+5| + |4x+7| = 12$ **(15)** $|x| + |x+1| + |x+2| = 3$

(16) $|2x-3| + |2x+1| + |2x+5| = 12$ **(17)** $|x| - 2|x+1| + 3|x+2| = 0$

(18) $|x| - x = 0$ **(19)** $|x^2+3x+2| + x + 1 = 0$

(20) $x^2 + 3|x| + 2 = 0$ **(21)** $|x^2+1| - x^2 - 1 = 0$

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Useful Mathematical Symbols for Reference

Symbol	How it is read
$\forall$	For all...
$\exists$	There exists...
$\wedge$	and
$\vee$	or
$<$	is less than
$>$	is greater than
$\leq$	is less than or equal to
$\geq$	is greater than or equal to
$\not<$	is not less than
$\not>$	is not greater than
$\in$	belongs to
$\notin$	does not belong to
$\mid$, : , s.t.	such that
$\Rightarrow$	implies (If... then...)
$\Leftrightarrow$	implies and implied by (if and only if / iff)
!	factorial
$\sqrt{\ }$	The square root of
$\sqrt[n]{\ }$	n^{th} root, $n \in N$, $n \geq 2$
Σ	The summation of
Π	The product of

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Greek Letters (Capital, Small) and there pronounciations

Symbol (Capital, Small)	How it is read	Symbol (Capital, Small)	How it is read
A,α	alpha	N,ν	nu
B,β	beta	Ξ,ξ	xi
Γ,γ	gamma	O,o	omicron
Δ,δ	delta	Π,π	pi
E,ε	epsilon	P,ρ	rho
Z,ζ	zeta	Σ,σ	sigma
H,η	eta	T,τ	tau
Θ,θ	theta	Y,υ	upsilon
I,ι	iota	Φ,ϕ	phi
K,κ	kappa	X,χ	chi
Λ,λ	lambda	Ψ,ψ	psi
M,μ	mu	Ω,ω	omega

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EXERCISE (O-1)

A⊆B A∩B

Straight Objective Type

1. If A and B be any two sets, then $(A \cap B)'$ is equal to-
(A) $A' \cap B'$ (B) $A' \cup B'$ (C) $A \cap B$ (D) $A \cup B$
FM0001

2. Let A and B be two sets in the universal set. Then A – B equals-
(A) $A \cap B'$ (B) $A' \cap B$ (C) $A \cap B$ (D) none of these
FM0002

3. If $A \subseteq B$, then $A \cap B$ is equal to-
(A) A (B) B (C) A' (D) B'
FM0003

4. If A and B are any two sets, then $A \cup (A \cap B)$ is equal to-
(A) A (B) B (C) A' (D) B'
FM0004

5. Which of the following statements is true ?
(A) $3 \subseteq \{1, 3, 5\}$ (B) $3 \in \{1, 3, 5\}$ (C) $\{3\} \in \{1, 3, 5\}$ (D) $\{3, 5\} \in \{1, 3, 5\}$
FM0005

6. Which of the following is a null set ?
(A) $A = \{x : x > 1 \text{ and } x < 1]$ (B) $B = \{x : x + 3 = 3\}$
(C) $C = \{\phi\}$ (D) $D = \{x : x \geq 1 \text{ and } x \leq 1\}$
FM0006

7. If A and B are not disjoint, then $n(A \cup B)$ is equal to-
(A) $n(A) + n(B)$ (B) $n(A) + n(B) - n(A \cap B)$
(C) $n(A) + n(B) + n(A \cap B)$ (D) $n(A).n(B)$
FM0007

8. $\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\ldots}}}} =$
(A) 3 (B) 2 (C) 1 (D) ±3
FM0008

9. If $x = 8 - \sqrt{60}$, then $\frac{1}{2}\left[\sqrt{x} + \frac{2}{\sqrt{x}}\right] =$
(A) $\sqrt{5}$ (B) $\sqrt{3}$ (C) $2\sqrt{5}$ (D) $2\sqrt{3}$
FM0009

10. If $\frac{4+3\sqrt{5}}{4-3\sqrt{5}} = a + b\sqrt{5}$, a, b are rational numbers, then (a, b) =
(A) $\left(\frac{61}{29}, \frac{-24}{29}\right)$ (B) $\left(\frac{-61}{29}, \frac{24}{29}\right)$ (C) $\left(\frac{61}{29}, \frac{24}{29}\right)$ (D) $\left(\frac{-61}{29}, \frac{-24}{29}\right)$
FM0010

11. The square root $5 + 2\sqrt{6}$ is :
(A) $\sqrt{3} + 2$ (B) $\sqrt{3} - \sqrt{2}$ (C) $\sqrt{2} - \sqrt{3}$ (D) $\sqrt{3} + \sqrt{2}$
FM0011

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12. If $\frac{4}{2+\sqrt{3}+\sqrt{7}} = \sqrt{a}+\sqrt{b}-\sqrt{c}$, then which of the following can be true -

(A) a = 1, b = 4/3, c = 7/3 (B) a = 1, b = 2/3, c = 7/9
(C) a = 2/3, b = 1, c = 7/3 (D) a = 7/9, b = 4/3, c = 1

FM0012

13. The numerical value of $\left(x^{1/a-b}\right)^{1/a-c} \times \left(x^{1/b-c}\right)^{1/b-a} \times \left(x^{1/c-a}\right)^{1/c-b}$ is (a, b, c are distinct real numbers)

(A) 1 (B) 8 (C) 0 (D) None

FM0013

14. $(1^3 + 2^3 + 3^3 + 4^3)^{-3/2} =$

(A) 10^{-3} (B) 10^{-2} (C) 10^{-4} (D) 10^{-1}

FM0014

15. $\left(7^{\left(-\frac{1}{2}\right)} \times 5^2\right)^2 \div \sqrt{25^3} =$

(A) $\frac{5}{7}$ (B) $\frac{7}{5}$ (C) 35 (D) $-\frac{5}{7}$

FM0015

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16. $(2d^2e^{-1})^3 \times \left(\frac{d^3}{e}\right)^{-2} =$

(A) $8e^{-2}$ (B) $8e^{-3}$ (C) $8e^{-1}$ (D) $8e^{-4}$

FM0016

17. If $x^y = y^x$ and x = 2y, then the values of x and y are (x, y > 0)

(A) x = 4, y = 2 (B) x = 3, y = 2 (C) x = 1, y = 1 (D) None of these

FM0017

18. If $(a^m)^n = a^{m^n}$, then express 'm' in the terms of n is ($a > 0, a \neq 1, m > 1, n > 1$)

(A) $n^{\left(\frac{1}{n-1}\right)}$ (B) $n^{\left(\frac{1}{n+1}\right)}$ (C) $n^{\left(\frac{1}{n}\right)}$ (D) None

FM0018

19. If $a = x + \frac{1}{x}$, then $x^3 + x^{-3} =$

(A) $a^3 + 3a$ (B) $a^3 - 3a$ (C) $a^3 + 3$ (D) $a^3 - 3$

FM0019

20. If $\left(\sqrt[3]{4}\right)^{2x+\frac{1}{2}} = \frac{1}{32}$, then x =

(A) –2 (B) 4 (C) –6 (D) –4

FM0020

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21. If $(5+2\sqrt{6})^{x^2-3}+(5-2\sqrt{6})^{x^2-3}=10$, then all possible values of x are

(A) –2, 2 (B) $\sqrt{2}$, $-\sqrt{2}$ (C) 2, $+\sqrt{2}$ (D) 2, –2, $\sqrt{2}$, $-\sqrt{2}$

FM0021

22. If $3^{2x^2}-2.3^{x^2+x+6}+3^{2(x+6)}=0$ then the value of x is

(A) – 2 (B) 3 (C) Both (A) and (B) (D) None of these

FM0022

23. How many integers in between 100 to 1500 (both inclusive) are multiples of 5 or 11 ?

(A) 408 (B) 26 (C) 382 (D) 380

FM0023

24. Square root of $4+\sqrt{15}$ is equal to

(A) $\frac{\sqrt{3}+\sqrt{5}}{2}$ (B) $\sqrt{\frac{3}{2}}+\sqrt{\frac{5}{2}}$ (C) $\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}$ (D) None of these

FM0024

25. If A = {(x, y) | xy = 8 and x, y ∈ ℤ}, then n(A) =

(A) 4 (B) 8 (C) 12 (D) 16

FM0025

26. The set of all real numbers x that satisfy $\frac{x-2}{x+2}>\frac{2x-3}{4x-1}$

(A) $(-\infty,-2)\cup\left(\frac{1}{4},\frac{3}{2}\right)\cup(2,\infty)$ (B) $\left(-2,\frac{1}{4}\right)\cup\left(\frac{3}{2},2\right)$

(C) $(-\infty,-2)\cup\left(\frac{1}{4},1\right)\cup(4,\infty)$ (D) $\left(-2,\frac{1}{4}\right)\cup(1,4)$

FM0026

27. The set of all real numbers x that satisfy $\frac{x^2-4}{x^2-5x+6}\le 0$

(A) [–2, 3] (B) [–2, 3)

(C) (–∞, –2] ∪ (3, ∞) (D) None of these

FM0027

28. If $\frac{(x+3)^2(x-1)^9(x+1)^5}{(x-3)(x-5)^4(x-6)^5}\le 0$, then number of possible integral values of x is -

(A) 6 (B) 3 (C) 4 (D) 5

FM0028

29. If $\left(\sqrt{2}+1\right)^x+\left(\sqrt{2}-1\right)^x-2\sqrt{2}=0$, then sum of all possible values of x is

(A) 0 (B) 1 (C) 2 (D) 3

FM0029

30. If $x^2+y^2+4z^2-6x-2y-4z+11=0$, then xyz is equal to

(A) 3/2 (B) 4 (C) 6 (D) 3

FM0030

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EXERCISE (O-2)

Straight objective Type

1. In a college of 300 students every student reads 5 newspapers and every newspaper is read by 60 students. The number of newspapers is

(A) at least 30 (B) at most 20 (C) exactly 25 (D) none of these

FM0031

2. If $x + y = 1$ and $x^2 + y^2 = 2$ then the value of $x^4 + y^4$ equals

(A) 7 (B) 6 (C) $\frac{7}{2}$ (D) $\frac{19}{4}$

FM0032

3. If $x = 2^{1/3} - 2^{-1/3}$ then the value of $2x^3 + 6x$ is equal to

(A) 1 (B) 2 (C) 3 (D) 4

FM0033

4. If $\frac{\sqrt{3}+4\sqrt{2}}{4\sqrt{2}-\sqrt{3}} = \frac{a+b\sqrt{6}}{c}$, then the value of a + b + c (where a, b, c $\in$ N and are relatively prime)

(A) 70 (B) 72 (C) 50 (D) 40

FM0034

5. If $x^2 + 4y^2 + z^2 - 2xy - 2yz - zx = 0$ then x : y : z equals

(A) 1 : 2 : 1 (B) 2 : 1 : 2 (C) 1 : 2 : 3 (D) 1 : 1 : 2

FM0035

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More than one correct

6. If $A \subseteq B$ then which of the following option(s) is/are correct ?

(A) $A' \subseteq B'$ (B) $B' \subseteq A'$ (C) $A \cap B' = \phi$ (D) $A' \cap B = \phi$

FM0036

7. If $x = \sqrt{7\sqrt{7\sqrt{7\sqrt{7...}}}}$ where x, y > 0

$y = \sqrt{20+\sqrt{20+\sqrt{20+...}}}$ then which of the following is/are correct.

(A) $x + y = 12$ (B) $x - y = 3$ (C) $x^2 + y^2 = 74$ (D) $x^2 - y^2 = 24$

FM0037

8. If complex number $-3 + ix^2y$ and $x^2 + y + 4i$ are conjugate of each other. Then real value of x & y can be

(A) x = 1, y = –4 (B) x = –1, y = –4 (C) x = 1, y = 4 (D) x = –2, y = –3

FM0038

9. If $8x^3 + 27x^2 - 9x - 50$ is divided by (x + 2) then remainder is λ then

(A) $\frac{3\lambda+4}{10}$ is equal to 4

(B) If $\lambda = \frac{p}{q}$ then (p + q) is divisible by 13 (where p & q are coprime)

(C) λ is a natural number

(D) $(\lambda - 2)$ is divisible by 3

FM0039

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10. If $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, d $\in \mathbb{R}$, $a \neq 0$ then which of the following option(s) is(are) correct ?

(A) if $a + b + c + d = 0$ then $x - 1$ is factor of $f(x)$.

(B) if $a + b = c + d$ then $x + 1$ is factor of $f(x)$.

(C) if $a + c = b + d$ then $x + 1$ is factor of $f(x)$.

(D) none of these

FM0040

11. If x = a and x = b are the two roots of the equation $9^x - 4 \times 3^{x+1} + 27 = 0$ then

(A) $a + b = 3$ (B) $(a - b)^2 = 1$ (C) $\frac{a}{b} + \frac{b}{a} = \frac{5}{2}$ (D) $a + b = 4$

FM0041

12. The real values of 'x' satisfying the equation $\left(4+\sqrt{15}\right)^{x^2-x-1} + \left(4-\sqrt{15}\right)^{x^2-x-1} = 8$ is/are :

(A) 0 (B) 1 (C) –1 (D) –2

FM0042

Linked Comprehension Type

Paragraph for question no. 13 to 15

If $(5+2\sqrt{6})^{x^2-8} + (5-2\sqrt{6})^{x^2-8} = 10, x \in \mathbb{R}$

On the basis of above information, answer the following questions :

13. Number of solution(s) of the given equation is/are-

(A) 1 (B) 2 (C) 4 (D) infinite

FM0043

14. Sum of positive solutions is

(A) 3 (B) $3+\sqrt{7}$ (C) $2+\sqrt{5}$ (D) 2

FM0043

15. If $x \in (-3,5]$, then number of possible values of x, is-

(A) 1 (B) 2 (C) 3 (D) 4

FM0043

Paragraph for question no. 16 & 17

A polynomial p(x) when divided by (x – 1), (x + 1) and (x + 2) gives remainder 5, 7 and 2 respectively. If p(x) is divided by $(x^2 - 1)$ and (x – 1) (x + 2) gives remainder as another polynomial R(x) and r(x) respectively. Then

16. The value of R(50) is :

(A) 34 (B) –44 (C) 44 (D) 104

FM0044

17. The value of r(100) is :

(A) 34 (B) 44 (C) 54 (D) 104

FM0044

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Matching list type

18. Match the list

	List-I		List-II
(P)	The units digit of $2^7 \times 3^{10}$ is	(1)	1
			FM0045
(Q)	The number of prime factors of $6^{100} \times 15^{200}$ is	(2)	2
			FM0046
(R)	Number of solutions of $xy = 35$ in natural numbers with $x > y > 1$ is	(3)	3
			FM0047
(S)	Remainder when $x^{100} - 3x^{99} + 2x^{98} + 3x - 2$ is divided by $x - 2$ is equal to	(4)	4
			FM0048

Codes :

(A) P-1, Q-2, R-3, S-4 (B) P-2, Q-3, R-1, S-4

(C) P-2, Q-3, R-2, S-3 (D) P-1, Q-4, R-1, S-2

19. Match the list

	List-I		List-II
(P)	If $x = \sqrt{5} - 2$ then value of $2x^3 + 11x^2 + 10x + 4$ is equal to	(1)	8
			FM0049
(Q)	If $x = \sqrt{5} + 2$ then value of $2x^3 - 7x^2 - 6x + 7$	(2)	7
			FM0050
(R)	If $\left(\left(\sqrt{11+2\sqrt{30}}\right) - \left(\sqrt{10+4\sqrt{6}}\right)\right)^2 = p + \sqrt{q} + \sqrt{r} + \sqrt{s}$ where p, q, r, s are integers & q, r, s are not perfect squares the $p + q + r + s$ is equal to	(3)	161
			FM0051
(S)	If $\left(\left(\sqrt{11+2\sqrt{30}}\right) - \left(\sqrt{10+4\sqrt{6}}\right)\right)^2 = \alpha - \sqrt{\beta}$ where α, β are integers & β is not a perfect square then $\alpha^2 + \beta$ is equal to	(4)	977
			FM0052

Codes :

(A) P-1, Q-2, R-3, S-4 (B) P-2, Q-3, R-1, S-4

(C) P-2, Q-1, R-3, S-4 (D) P-2, Q-1, R-4, S-3

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Answer the question question 20 and 21 by appropriately matching the lists based on the information given

	List-I		List-II
(I)	If $2\sqrt{x+5} = x+2$, then integral value of x	(P)	3
(II)	The number of real solution of the equation $x^2 + 5\|x\| + 4 = 0$, is	(Q)	–4
(III)	The real solution of the equation $(x-2)^2 + \|x-2\| - 2 = 0$, is	(R)	–1
(IV)	The number of real solution(s) $\|x+2\| = 2(3-x)$, is	(S)	0
		(T)	4
		(U)	1

20. Which of the following is only **CORRECT** option ?
(A) I $\to$ R, U (B) II $\to$ S, T (C) I $\to$ Q, S (D) II $\to$ S

FM0053

21. Which of the following is only **INCORRECT** option ?
(A) III $\to$ P, U (B) IV $\to$ U (C) III $\to$ Q, T (D) III $\to$ P

FM0053

Numerical Grid Type

22. If $a - \frac{1}{a} = \frac{1}{2}$ then $\left(4a^2 + \frac{4}{a^2}\right)$ is equal to

FM0054

23. If polynomial $Ax^3 + 4x^2 + Bx + 5$ leaves same remainder, when divided by $x - 1$ and $x + 2$ respectively then value of $3A + B$ is equal to

FM0055

24. If $9^x + 6^x = 2.4^x$ then the value of $\frac{x^3+64}{5}$

FM0056

25. If $a + b + c = 6$ & $a^2 + b^2 + c^2 = 14$ and $a^3 + b^3 + c^3 = 36$ then the value of $3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

FM0057

EXERCISE (S-1)

1. An investigator interviewed 100 students to determine their preferences for the three drinks : milk (M), coffee (C) and tea (T). He reported the following : 10 students had all the three drinks M, C and T; 20 had M and C; 30 had C and T; 25 had M and T; 12 had M only; 5 had C only; and 8 had T only. Using a Venn diagram find how many did not take any of the three drinks.

FM0058

20

2. Suppose $a + \frac{1}{a} = 3$. Find the values of

(a) $a^2 + \frac{1}{a^2}$ (b) $a^4 + \frac{1}{a^4}$ (c) $a^3 + \frac{1}{a^3}$

FM0059

3. Which of the following equation(s) has (have) only unity as the solution.
(A) $2(3^{x+1}) - 6(3^{x-1}) - 3^x = 9$ (B) $7(3^{x+1}) - 5^{x+2} = 3^{x+4} - 5^{x+3}$

FM0060

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4. Which of the following equation (s) has (have) only natural solution(s)

(A) $6.9^{1/x} - 13.6^{1/x} + 6.\ 4^{1/x} = 0$ (B) $4^x.\sqrt{8^{x-1}} = 4$ **FM0061**

5. If $\frac{29}{12}$ can be expressed as $a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d}}}$, (a, b, c, d $\in \mathbb{N}$) then find the value of $a^3 + b^3 + c^3 + d^3$ is equal to

FM0062

6. **Factorize the following expressions :**

(i) $(x - y)(x - y - 1) - 20$ **FM0063**

(ii) $(x + 2y)(x + 2y + 2) - 8$ **FM0064**

(iii) $3(4x + 5)^2 - 2(4x + 5) - 1$ **FM0065**

(iv) $x^{\frac{2}{3}} + x^{\frac{1}{3}} - 2$ **FM0066**

(v) $x^6 - 7x^3 - 8$ **FM0067**

(vi) $x^4 + 15x^2 - 16$ **FM0068**

(vii) $(x^2 - x - 3)(x^2 - x - 5) - 3$ **FM0069**

(viii) $(x^2 + 5x + 6)(x^2 + 5x + 4) - 120$ **FM0070**

(ix) $(a^2 + 1)^2 + (a^2 + 5)^2 - 4(a^2 + 3)^2$ **FM0071**

(x) $(x - 1)^2 + (y - 1)^2 + (z - 1)^2 + 2(x - 1)(y - 1) + 2(y - 1)(z - 1) + 2(z - 1)(x - 1)$ **FM0072**

(xi) $2x^4 - x^3 - 6x^2 - x + 2$ **FM0073**

(xii) $(x^4 + x^2 - 4)(x^4 + x^2 + 3) + 10$ **FM0074**

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7. The value of $(4 + 1)(4^2 + 1)(4^4 + 1)(4^8 + 1) + \frac{1}{3} = \frac{2^\lambda}{3}$ where $\lambda \in \mathbb{N}$ then find the value of 'λ' is equal to :

FM0075

8. Simplify the following expressions :

(a) $\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}$.

FM0076

(b) $\sqrt[3]{18+5\sqrt{13}} + \sqrt[3]{18-5\sqrt{13}}$.

FM0077

(c) $\sqrt{\frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{4}+\sqrt{3}} + \ldots \text{ upto 99 terms}}$

FM0078

9. Suppose a and b are constants such that
$(x^3 + bx^2 - 7x + 9)(x^2 + ax + 5) = x^5 + 13x^4 + 38x^3 - 22x^2 + 37x + 45\ \forall\ x \in \mathbb{R}$. Find a and b.

FM0079

10. If P(x) is a cubic polynomial such that P(1) = 1; P(2) = 2; P(3) = 3 with leading coefficient 3 then find the value of P(4).

FM0080

11. Find all conditions on a and b (a, b $\in \mathbb{R}$) when ax + b = 4x + 10 has

(i) exactly one solution in 'x'. (ii) no solution in 'x'.

(iii) has exactly two solutions in 'x' (iv) has infinite solutions in x

FM0081

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12. Let x, y, z are real numbers satisfying following equations :

$3x - 2y + 4z = 3$

$x - y + 2z = 10$ and

$2x - 3y + 3z = 5$ then the value of $(2x - 2y + 3z)^2$ is equal to

FM0082

13. How may ordered pairs of natural numbers (a, x) satisfy $ax = a + 4x$? **FM0083**

14. Let $x = \sqrt{3-\sqrt{5}}$ & $y = \sqrt{3+\sqrt{5}}$. If the value of the expression $x - y + 2x^2y + 2xy^2 - x^4y + xy^4$ can be expressed is the form $\sqrt{p}+\sqrt{q}$ (where $p, q \in \mathbb{N}$), then find the value of $(p + q)$

FM0084

15. **Solve following Inequalities over the set of real numbers -**

(i) $x^2 + 2x - 3 \le 0$ **FM0085**

(ii) $x^2 + 6x - 7 \le 0$ **FM0086**

(iii) $x^4 - 2x^2 - 63 \le 0$ **FM0087**

(iv) $\dfrac{x+1}{(x-1)^2} < 1$ **FM0088**

(v) $\dfrac{x^2-7x+12}{2x^2+4x+5} > 0$ **FM0089**

(vi) $\dfrac{(x-1)(x+2)^2}{-1-x} < 0$ **FM0090**

(vii) $\dfrac{x^4+x^2+1}{x^2-4x-5} < 0$ **FM0091**

(viii) $\dfrac{x+7}{x-5} + \dfrac{3x+1}{2} \ge 0$ **FM0092**

(ix) $\dfrac{1}{x+2} < \dfrac{3}{x-3}$ **FM0093**

(x) $\dfrac{14x}{x+1} - \dfrac{9x-30}{x-4} < 0$ **FM0094**

(xi) $\dfrac{x^2+2}{x^2-1} < -2$ **FM0095**

(xii) $\dfrac{5-4x}{3x^2-x-4} < 4$ **FM0096**

(xiii) $\dfrac{(x+2)(x^2-2x+1)}{4+3x-x^2} \ge 0$ **FM0097**

(xiv) $\dfrac{x^4-3x^3+2x^2}{x^2-x-30} > 0$ **FM0098**

(xv) $\dfrac{2x}{x^2-9} \le \dfrac{1}{x+2}$ **FM0099**

(xvi) $\dfrac{20}{(x-3)(x-4)} + \dfrac{10}{x-4} + 1 > 0$ **FM0100**

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EXERCISE (S-2)

1. A survey shows that 63% of the Americans like cheese whereas 76% like apples. If x% of the Americans like both cheese and apples, find the value of x.

39

FM0101

2. How many ordered pairs of integers which satisfy the equation $\frac{1}{m}+\frac{2}{n}=\frac{1}{2}$?

FM0102

3. Find the polynomial of 4th degree with integer co-efficients such that $x=\sqrt{3}+\sqrt{2}$ is root of polynomial.

FM0103

4. If $x+y+z=12$ & $x^2+y^2+z^2=96$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=36$. Find the value of $\frac{x^3+y^3+z^3}{4}$

FM0104

5. If $x=\sqrt{4-2\sqrt{3}}$ and $y=\sqrt{9-4\sqrt{5}}$ then the value of $\left(\sqrt{5}x-\sqrt{3}y\right)^2$ is equal to $\left(a-b\sqrt{c}\right)$ where a, b, c are coprime numbers then a + b + c is equal to (where 'c' is an odd integer)

FM0105

6. If $x, y, z \in R$ and $121x^2+4y^2+9z^2-22x+4y+6z+3=0$ then value of $x^{-1}-y^{-1}-z^{-1}$ is equal to

FM0106

7. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$ prove that $\frac{x^2-yz}{a^2-bc}=\frac{y^2-zx}{b^2-ca}=\frac{z^2-xy}{c^2-ab}$; $\forall$ a,b,c $\in$ R* and $a^2 \neq bc$, $b^2 \neq ca$, $c^2 \neq ab$.

FM0107

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8. If $\frac{p}{q}=\frac{x+2}{x-2}$, then evaluate $\frac{p^2-q^2}{p^2+q^2}$

FM0108

9. If $(a^2+b^2)(x^2+y^2)=(ax+by)^2$ prove that $\frac{a}{x}=\frac{b}{y}$ (for $x,y \in R^*$)

FM0109

10. $\frac{3x+\sqrt{9x^2-5}}{3x-\sqrt{9x^2-5}}=5$ find 'x'

FM0110

11. Value of $\left(\frac{\sqrt{6}}{\sqrt{3}-\sqrt{2}+\frac{\sqrt{6}}{\sqrt{3}-\sqrt{2}+\frac{\sqrt{6}}{(\sqrt{3}-\sqrt{2})+\ddots}}}\right)^2$ is

FM0111

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JEE-MAINS/ JEE-ADVANCE

JEE-MAINS

1. If A, B and C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then :-

(1) $B = C$ (2) $A \cap B = \phi$ (3) $A = B$ (4) $A = C$

FM0112

2. In a class of 140 students numbered 1 to 140, all even numbered students opted mathematics course, those whose number is divisible by 3 opted Physics course and those whose number is divisible by 5 opted Chemistry course. Then the number of students who did not opt for any of the three courses is :

(1) 102 (2) 42 (3) 1 (4) 38

FM0113

3. Two newspapers A and B are published in a city. It is known that 25% of the city populations reads A and 20% reads B while 8% reads both A and B. Further, 30% of those who read A but not B look into advertisements and 40% of those who read B but not A also look into advertisements, while 50% of those who read both A and B look into advertisements. Then the percentage of the population who look into advertisement is :-

(1) 12.8 (2) 13.5 (3) 13.9 (4) 13

FM0114

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4. Let $\mathbb{Z}$ be the set of integers. If $A = \left\{x \in \mathbb{Z} : 2^{(x+2)(x^2-5x+6)} = 1\right\}$ and $B = \{x \in \mathbb{Z} : -3 < 2x - 1 < 9\}$, then the number of subsets of the set $A \times B$, is:

(1) 2^{18} (2) 2^{10} (3) 2^{15} (4) 2^{12}

FM0115

JEE-ADVANCED

1. If X and Y are two sets, then $X \cap (X \cup Y)^c$ equals

(a) X (b) Y (c) ϕ (d) none of these

FM0116

2. The expression $\dfrac{12}{3+\sqrt{5}+2\sqrt{2}}$ is equal to

(a) $1-\sqrt{5}+\sqrt{2}+\sqrt{10}$ (b) $1+\sqrt{5}+\sqrt{2}-\sqrt{10}$

(c) $1+\sqrt{5}-\sqrt{2}+\sqrt{10}$ (d) $1-\sqrt{5}-\sqrt{2}+\sqrt{10}$

FM0117

3. If $x < 0$, $y < 0$, $x + y + \dfrac{x}{y} = \dfrac{1}{2}$ and $(x + y)\dfrac{x}{y} = -\dfrac{1}{2}$, then x = ... and y = ...

FM0118

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4. The equation $x-\frac{2}{x-1}=1-\frac{2}{x-1}$ has

(A) no root (B) one root (C) two equal roots (D) infinitely many roots

FM0119

5. If A = {1, 2, 3, 4, 5}, then the number of proper subsets of A is-

(1) 120 (2) 30 (3) 31 (4) 32

FM0120

6. If A, B, C be three sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then -

(1) A = B (2) B = C (3) A = C (4) A = B = C

FM0121

7. Sets A and B have 3 and 6 elements respectively. What can be the minimum number of elements in $A \cup B$?

(1) 3 (2) 6 (3) 9 (4) 18

FM0122

8. Find all real values of x which satisfy $x^2 - 3x + 2 > 0$ and $x^2 - 2x - 4 \le 0$.

FM0123

9. Find the set of all x for which $\frac{2x}{(2x^2+5x+2)} > \frac{1}{(x+1)}$.

FM0124

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10. The sum of all real roots of the equation $|x-2|^2 + |x-2| - 2 = 0$ is ...

FM0125

11. The number of real solutions of the equation $|x|^2 - 3|x| + 2 = 0$ is

(A) 4 (B) 1 (C) 2 (D) 3

FM0126

12. If S is the set of all real x such that $\frac{2x-1}{2x^3+3x^2+x}$ is positive, then S contains

(A) $\left(-\infty,-\frac{3}{2}\right)$ (B) $\left(-\frac{3}{2},-\frac{1}{4}\right)$ (C) $\left(-\frac{1}{4},\frac{1}{2}\right)$ (D) $\left(\frac{1}{2},3\right)$

(E) None of these

FM0127

13. Let $y=\sqrt{\frac{(x+1)(x-3)}{(x-2)}}$

Find all the real values of x for which y takes real values.

FM0128

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ANSWER KEY

Do yourself-1

All are true

Do yourself-3

2. C **3.** B **4.** B **5.** B **6.** C **7.** C **8.** D **9.** B
10. D **11.** B **12.** B

Do yourself-4

1. a = 3, b = 4

2. **(i)** $\frac{2}{11}$ **(ii)** $\frac{16}{99}$ **(iii)** $\frac{419}{990}$

4. **(i)** x + y, x – y, xy, x/y are rational numbers.
(ii) x + y, x – y, xy, x/y, all are may or may not be rational.
(iii) x + y, x – y, xy, x/y are irrational numbers.

5. x is irrational number (Non terminating and non-recurring)

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Do yourself-5

1. (x,y) ≡ (1,20); (2,10); (4,5); (5,4); (10,2); (20,1)

2. **(i)** (x,y) ≡ (2,2) **(ii)** (x,y) ≡ (5,13); (6,4) ; (7,1)

3. (x,y) ≡ (8,56); (14,14); (56,8)

4. 651 **5.** 293 **6.** 4 **7.** 30 **8.** 3

Do yourself-6

(i)

(ii)

(iii)

(iv)

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Do yourself-7

1. B **2.** C **3.** A **4.** D **5.** C

6. **(i)** 1 **(ii)** 0 **(iii)** 1 **(iv)** 1 **(v)** 8

7. 2 **8.** (a) $-\frac{47}{44}$, (b) $-\frac{9}{44}$ **9.** 5 **10.** 26

11. 1 **12.** $x = y = z = \frac{a}{3}$

Do yourself-8

1. **(i)** $(x-y)(x+y)(x^2+y^2)$ **(ii)** $(3a-2x+y)(3a+2x-y)$

(iii) $(2x+3y)(2x-3y-3)$

2. **(i)** $(2x-3y)(4x^2+6xy+9y^2)$ **(ii)** $(2x-5y)[4x^2+10xy+25y^2+1]$

3. **(i)** $(x+8)(x-5)$ **(ii)** $(x-8)(x+5)$ **(iii)** $(x+7)(x-2)$

(iv) $(x-4)(x+1)$ **(v)** $(x-3)(x+1)$ **(vi)** $(3x-4)(x-2)$

(vii) $(4x+7)(3x-5)$ **(viii)** $(3x-2)(x-1)$ **(ix)** $(x-1)(3x-4)$

(x) $(7x-1)(x-1)$ **(xi)** $(x-2)(2x-13)$ **(xii)** $(a-3)(3a+2)$

(xiii) $(2a+1)(7a-3)$

4. **(i)** $(a-b-1)(a+b-3)$ **(ii)** $(x^2-6x+18)(x^2+6x+18)$

(iii) $(x^2-y+1)(x^2+y+1)$ **(iv)** $(2a^2-a-1)(2a^2+a-1)$

(v) $(2x^2-6x+9)(2x^2+6x+9)$ **(vi)** $(x^4-x^2+1)(x^4+x^2+1)$

5. **(i)** $(x-1)(x-2)(x-3)$ **(ii)** $(x+1)(x+3)(2x+1)$

(iii) $(x-1)(x-2)(2x-3)$ **(iv)** $(x^2+2)(x^2+1)\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)$

(v) $3(x+y)(y+z)(z+x)$

6. **(i)** $(2a^2-a+1)(4a^4+2a^3-a^2+a+1)$

7. **(i)** $(x^2+5x+1)(x^2+5x+9)$

(ii) $2(2x^2+2x+3)(4x^2+4x-9)$

(iii) $(x^2+5x-22)(x+1)(x+4)$

Do yourself-9

1. (a) 7 (b) 47 (c) 18 **4.** 104 **5.** $-\sqrt{2}$ **6.** 1

8. (a) 0, 4 (b) no solution (c) no solution

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Do yourself-10

1. 1 **2.** –20 **3.** 4 **4.** –2 **5.** 3x + 4

6. $\ell = 2, m = -18$ **7.** $\ell = -1,\ m = -3$ **8.** a = 3, b = 1

9. $3x^3 + 4x^2 - 5x - 2$ **10.** 6

Do yourself-11

1. 1 or –8 **2.** 1,32 **3.** $\pm\sqrt{2},\ \pm\sqrt{\frac{2}{3}}$

4. 3 or 4 **5.** $x = 1, x = \frac{3 \pm \sqrt{5}}{2}$ **6.** $x = \pm 1,\ 2,\ -\frac{1}{2}$

7. 0,–1 **8.** 2,4 **9.** ±1

10. –4,4 **11.** –4,1 **12.** –6,1

13. $x = -3 \pm \sqrt{5}$ **14.** $\pm 1,\ 1 \pm \sqrt{2}$

15. (A)→(U); (B)→(U); (C)→(T); (D)→(P,Q,R,S,T,U,V);(E)→(R,T); (F)→(S)

Do yourself-12

1. x = 5, y = 3 **2.** x = 2, y = 1 or x = –1, y = –2

3. x = 4, y = –2 or x = –2, y = 4 **4.** x = –6, y = –2 or x = –4, y = –4

5. x = 3, y = 2 or x = –3, y = –2 **6.** x = 5, y = 1 or x = –5, y = –1

7. $x = \frac{11}{13},\ y = -\frac{24}{5}$ **8.** x = 3, y = 2 or x = –3, y = –2

9. x = 4, y = ±3 **10.** x = –1, y = 3 or $x \in \mathbb{R}$, y = 2

11. x = 2, y = –1or x = –1, y = 2 **12.** $(x,y) \cong (3,1),(-3,-1),(1,3),(-1,-3)$

Do yourself-13

1. (a) F (b) T (c) F (d) T (e) T (f) F

2. (a) $\frac{17}{21}$ (b) $\frac{37}{41}$ **3.** 3^{30} **4.** $(2,\infty)$ **5.** $3+\sqrt{14}$

Do yourself-14

1. $4\sqrt{2}$ **2.** –12 **3.** 4

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Do yourself-15

1. $\frac{20}{27}$ **2.** 21 : 35 : 65 **3.** $\frac{sp}{p+q}, \frac{sq}{p+q}$ **4.** $\frac{a^4}{b^4}$

Do yourself-16

1. $(-1,4)-\{1\}$

2. $\left(-\frac{1}{4},\frac{5}{6}\right)$

3. $(-\infty,-2)(-2,-1)\cup(1,\infty)$

4. $\left(-\infty,\frac{1}{2}\right)\cup(2,\infty)-\{4\}$

5. $[-1,2)-\{0\}$

6. $(-1,1]\cup[3,\infty)\cup\{2\}$

7. $\left(-\frac{1}{2},1\right)$

8. $(-2,-1)\cup(0,5)-\{2\}$

9. $\left[-\sqrt{2},\sqrt{2}\right]\cup[3,4)-\{-1\}$

10. $(-2,-1)\cup(1,2)$

11. $\left(-\infty,\frac{-\sqrt{63}}{2}\right)\cup\left(-3,\frac{\sqrt{63}}{2}\right)\cup(4,\infty)$

12. $\left(\frac{3}{4},1\right)\cup(7,\infty)$

13. $\left(-\sqrt{2},0\right)\cup\left(1,\sqrt{2}\right)\cup(2,\infty)$

14. $(-\infty,-3]\cup[2,\infty)$

15 $(-2,-1)$

16. $\left(-\frac{1}{4},\frac{1}{3}\right]$

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Do yourself-17

1. **(i)**

(ii)

(iii)

2. $\left\{-\frac{3}{2},-\frac{7}{2}\right\}$ **3.** $\{-1,6\}$ **4.** ϕ **5.** ϕ **6.** $\left\{-\frac{25}{3},\frac{17}{3}\right\}$

7. $\{0,3\}$ **8.** $\left\{-8,-\frac{2}{5}\right\}$ **9.** $\left\{\frac{6}{5},18\right\}$ **10.** ϕ

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11. $\left\{1,\frac{7}{2}\right\}$ **12.** ϕ **13.** $\{1,-2\}$ **14.** $\left\{0,-\frac{24}{7}\right\}$

15. $\{-2,0\}$ **16.** $\left\{-\frac{5}{2},\frac{3}{2}\right\}$ **17.** $\{-2\}$ **18.** $[0,\infty)$

19. $\{-3,-1\}$ **20.** ϕ **21.** $\mathbb{R}$

EXERCISE (O-1)

1. B	**2.** A	**3.** A	**4.** A	**5.** B	**6.** A	**7.** B
8. A	**9.** A	**10.** D	**11.** D	**12.** A	**13.** A	**14.** A
15. A	**16.** C	**17.** A	**18.** A	**19.** B	**20.** D	**21.** D
22. C	**23.** C	**24.** B	**25.** B	**26.** C	**27.** D	**28.** D
29. A	**30.** A					

EXERCISE (O-2)

1. C	**2.** C	**3.** C	**4.** B	**5.** B	**6.** B,C	**7.** A,C,D
8. A,B	**9.** A,B,C	**10.** A,C	**11.** A,B,C	**12.** A,B,C	**13.** C	**14.** B
15. C	**16.** B	**17.** D	**18.** B	**19.** D	**20.** D	**21.** C
22. 9.00	**23.** 4	**24.** 12.80	**25.** 5.5			

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EXERCISE (S-1)

1. 20 **2.** **(a)** 7 **(b)** 47 **(c)** 18 **3.** A **4.** B **5.** 32

6. **(i)** $(x - y - 5)(x - y + 4)$. **(ii)** $(x + 2y - 2)(x + 2y + 4)$.

(iii) $16(3x + 4)(x + 1)$. **(iv)** $(x^{\frac{1}{3}} + 2)(x^{\frac{1}{3}} - 1)$.

(v) $(x - 2)(x^2 + 2x + 4)(x + 1)(x^2 - x + 1)$.

(vi) $(x^2 + 16)(x - 1)(x + 1)$.

(vii) $(x + 1)(x - 2)(x - 3)(x + 2)$. **(viii)** $(x^2 + 5x + 16)(x + 6)(x - 1)$.

(ix) $-2(a^2 + 1)(a^2 + 5)$. **(x)** $(x + y + z - 3)^2$.

(xi) $(2x - 1)(x - 2)(x + 1)^2$. **(xii)** $(x^2 + 2)(x + 1)(x - 1)(x^4 + x^2 + 1)$.

7. 32 **8.** **(a)** 1 **(b)** 3 **(c)** 3

9. $a = 8$ and $b = 5$ **10.** 22

11. (i) $a \neq 4, b \in \mathbb{R}$ (ii) $a = 4, b \neq 10$ (iii) $a, b \in \phi$ (iv) $a = 4, b = 10$

12. 36 **13.** 3 **14.** 610

15. **(i)** $[-3,1]$ **(ii)** $[-7,1]$

(iii) $[-3, 3]$ **(iv)** $(-\infty, 0) \cup (3, +\infty)$

(v) $(-\infty, 3) \cup (4, +\infty)$ **(vi)** $(-\infty, -2) \cup (-2, -1) \cup (1, +\infty)$

(vii) $(-1, 5)$ **(viii)** $[1, 3] \cup (5, +\infty)$

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(ix) $\left(-\frac{9}{2},-2\right)\cup(3,\infty)$ **(x)** $(-1,\ 1)\cup(4,\ 6)$

(xi) $(-1,1)-\{0\}$ **(xii)** $\left(-\infty,-\frac{\sqrt{7}}{2}\right)\cup\left(-1,\frac{\sqrt{7}}{2}\right)\cup\left(\frac{4}{3},\infty\right)$

(xiii) $(-\infty,\ -2]\cup(-1,4)$ **(xiv)** $(-\infty,\ -5)\cup(1,\ 2)\cup(6,\ +\infty)$

(xv) $(-\infty,-3)\cup(-2,3)$ **(xvi)** $(-\infty,-2)\cup(-1,3)\cup(4,\infty)$

EXERCISE # (S-2)

1. $39\le x\le 63$ **2.** 7 **3.** $x^4-10x^2+1=0$ **4.** 216.50 **5.** 36 **6.** 16 **8.** $\frac{4x}{x^2+4}$

10. **x =1** **11.** **2**

EXERCISE # JEE-MAINS / JEE-ADVANCE

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JEE-MAINS

1. 1 **2.** 4 **3.** 3 **4.** 3

JEE-ADVANCED

1. **C** **2.** B **3.** $x=-\frac{1}{4},y=-\frac{1}{4}$ **4.** A **5.** 3

6. 2 **7.** 2 **8.** $x\in\left[1-\sqrt{5},1\right)\cup\left(2,1+\sqrt{5}\right]$

9. $(-2,-1)\cup\left(-\frac{2}{3},-\frac{1}{2}\right)$ **10.** 4 **11.** A **12.** A,D **13.** $x\in[-1,\ 2)\cup[3,\ \infty)$

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Chapter 01

FUNDAMENTALS OF MATHEMATICS # 02

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JEE (Main/Advanced) Syllabus

JEE (Main) Syllabus :

Sets and their representation ; Union, intersection and complement of sets and their algebraic properties; power sets ; polynomials.

JEE (Advanced) Syllabus :

Absolute value, polynomial.

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Important Notes

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FUNDAMENTALS OF MATHEMATICS # 02

SIMILAR TRIANGLES

We call two figures similar if one is simply a blown-up or scaled-up, and possibly rotated and/or flipped, version of the other.

Sufficient Conditions For Similarity of Triangles.

Two triangles are said to be similar if any one of the following conditions is satisfied.

(i) their corresponding angles are equal.

(ii) their corresponding sides are proportional.

Based on the above, there are three axioms for similarity of two triangles.

1. A.A. (Angle-Angle) Axiom of Similarity.
2. S.A.S (Side-Angle-Side) Axiom of Similarity.
3. S.S.S. (Side-Side-Side) Axiom of Similarity.

Note :

(i) Congruent triangles are necessarily similar but the similar triangles may not be congruent.

(ii) If ABC is a triangle, right-angled at B and $BD \perp AC$, then

$\Delta ABC \sim \Delta ADB \sim \Delta BDC$

(iii) If two triangles are similar to a third triangle, then they are similar to each other.

Similar Polygons

Two polygons with the same number of sides are said to be similar if their corresponding angles are equal as well as their corresponding sides are proportional.

Theorem 1 :

Basic Proportionality Theorem (B.P.T.)

If a line is drawn parallel to a side of a triangle intersecting the other two sides, then it divides the two sides in the same ratio. i.e., if in a ΔABC, $DE \| BC$ then, $\frac{AD}{DB} = \frac{AE}{EC}$.

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Theorem 2 :

Converse of B.P.T.

If a line divides any two sides of a triangle proportionally, then it must be parallel to the third side.

i.e., If in a ΔABC, $\frac{AD}{DB} = \frac{AE}{EC}$ then DE||BC.

Theorem 3 :

Angle Bisector Theorems

(i) Angle Bisector Theorem (Internal)

The internal bisector of an angle of a triangle divides the opposite side, internally, in the ratio of the sides containing the angle. i.e., If in a ΔABC, AD bisects $\angle A$,

then $\frac{AB}{AC} = \frac{BD}{DC}$.

(ii) Angle Bisector Theorem (External)

The bisector of an exterior angle of a triangle divides the opposite side (provided bisector and opposite side are not parallel) externally, in the ratio of the sides containing the angle.

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i.e., If in a ΔABC, AD bisects the exterior angle A and intersects side BC produced in D, then $\frac{BD}{CD} = \frac{AB}{AC}$

Converse of both the angle bisector theorems are also true.

Theorem 4 :

Proportion Applied to Area

The areas of similar triangles are proportional to the squares of corresponding sides.

$$\frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta PQR)} = \frac{AB^2}{PQ^2} = \frac{BC^2}{QR^2} = \frac{CA^2}{RP^2} \qquad (\Delta ABC \sim \Delta PQR)$$

Theorem 5 :

The areas of similar polygons are proportional to the squares of corresponding sides.

$$\frac{\text{Area } 1^{st} \text{ poly}}{\text{Area } 2^{nd} \text{ poly}} = \frac{AB^2}{PQ^2} = \frac{BC^2}{QR^2} = \frac{CD^2}{RS^2} = \ldots$$

(i) (ii)

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Illustration-1 : In the given figure, DEFG is a square and $\angle BAC = 90°$. Prove that :

(i) $\Delta AGF \sim \Delta DBG$

(ii) $\Delta AGF \sim \Delta EFC$

(iii) $\Delta DBG \sim \Delta EFC$

(iv) $DE^2 = BD \times EC$

Solution : (i) In Δs AGF and DBG, we have :

$\angle GAF = \angle BDG = 90°$ and $\angle AGF = \angle DBG$ [corresponding $\angle$s]

$\therefore \Delta AGF \sim \Delta DBG$ [by A.A. similarity]

(ii) In Δs AGF and EFC, we have :

$\angle FAG = \angle CEF = 90°$ and $\angle AFG = \angle ECF$ [corresponding $\angle$s]

$\therefore \Delta AGF \sim \Delta EFC$ [by A.A. similarity]

(iii) Since, $\Delta AGF \sim \Delta DBG$ and $\Delta AGF \sim \Delta EFC$, it follows that $\Delta DBG \sim \Delta EFC$.

Note : If two triangles are similar to a third triangle, then they are similar to each other.

(iv) Since, $\Delta DBG \sim \Delta EFC$, we have :

$\frac{BD}{EF} = \frac{DG}{EC}$ or $\frac{BD}{DE} = \frac{DE}{EC}$ [$\because$ EF = DE, DG = DE (Sides of a sq.)]

Hence, $DE^2 = BD \times EC$.

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Illustration-2 : Given the $\overline{DE} \parallel \overline{BC}$ and $\overline{AY} \parallel \overline{XC}$, prove that $\frac{EY}{EX} = \frac{AD}{DB}$.

Solution : The ratios of side lengths in the problem suggest we look for similar triangles.

Since $\overline{AY} \parallel \overline{XC}$, we have $\Delta AYE \sim \Delta CXE$.

$\Rightarrow$ EY/EX = AE/EC. All we have left is to show that AE/EC = AD/DB.

Since $\overline{DE} \parallel \overline{BC}$, we have $\Delta ADE \sim \Delta ABC$. Therefore, AD/AB = AE/AC, which is almost what we want! We break AB and AC into AD + DB and AE + EC, hoping we can do a little algebra to finish :

$$\frac{AD}{AD + DB} = \frac{AE}{AE + EC}$$

If only we could get rid of the AD and AE in the denominators-then we would have AD/DB = AE/EC.

$$\frac{AD + DB}{AD} = \frac{AE + EC}{AE}$$

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Therefore, $\frac{AD}{AD}+\frac{DB}{AD}=\frac{AE}{AE}+\frac{EC}{AE}$, so $1+\frac{DB}{AD}=1+\frac{EC}{AE}$, which gives us $\frac{DB}{AD}=\frac{EC}{AE}$

Flipping these fractions back over gives us AD/DB = AE/EC. Therefore, we have EY/EX = AE/EC = AD/DB, as desired.

Illustration-3 : In ΔABC, $\angle B = 2\angle C$ and the bisector of $\angle B$ intersects AC at D. Prove that $\frac{BD}{DA}=\frac{BC}{BA}$.

Solution : Given : In ΔABC, $\angle B = 2\angle C$ and BD is bisector of $\angle B$.

To Prove : $\frac{BD}{DA}=\frac{BC}{BA}$

BD is bisector of $\angle CBA$.

$\therefore \frac{BC}{BA}=\frac{CD}{AD}$ ...(1)

$2\angle C = \angle 1 + \angle 2$ [given]

But $\angle 1 = \angle 2$

$\therefore$ $2\angle C = 2\angle 1$

$\Rightarrow$ $\angle C = \angle 1 \Rightarrow BD = CD$ ...(2)

from (1) & (2) $\frac{BC}{BA}=\frac{BD}{DA}$

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Illustration-4 : In a ΔABC, AD is the bisector of $\angle BAC$.
If AB = 3.5 cm, AC = 4.2 cm and DC = 2.4 cm
Find BD.

Solution : Since, AD is the bisector $\angle A$, we have $\frac{BD}{DC}=\frac{AB}{AC}$

$\therefore \frac{BD}{2.4}=\frac{3.5}{4.2}\Rightarrow BD=\frac{2.4\times 3.5}{4.2}=2\text{cm}.$

Illustration-5 : O is any point inside a triangle ABC. The bisector of $\angle AOB$, $\angle BOC$ and $\angle COA$ meet the sides AB, BC and CA in points D,E,F respectively. Prove that
AD ×BE × CF = DB × EC × FA.

Solution : Given : O is any point inside a ΔABC. The bisectors of $\angle AOB$, $\angle BOC$ and $\angle COA$ meet the sides AB, BC and CA in points D, E, F respectively.
To prove : AD.BE.CF= DB.EC.FA
Proof : In ΔAOB, OD is the bisector of $\angle AOB$.

$\therefore \frac{OA}{OB}=\frac{AD}{DB}$ ...(i)

In ΔBOC, OE is the bisector of $\angle BOC$

$\therefore \frac{OB}{OC}=\frac{BE}{EC}$ ...(ii)

In ΔCOA, OF is the bisector of $\angle COA$.

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$\therefore \dfrac{OC}{OA} = \dfrac{CF}{FA}$ ...(iii)

Multiplying the corresponding sides of (i), (ii) and (iii), we get :

$$\frac{OA}{OB} \times \frac{OB}{OC} \times \frac{OC}{OA} = \frac{AD}{DB} \times \frac{BE}{EC} \times \frac{CF}{FA}$$

$\Rightarrow \quad 1 = \dfrac{AD}{DB} \times \dfrac{BE}{EC} \times \dfrac{CF}{FA}$

$\Rightarrow \quad DB \times EC \times FA = AD \times BE \times CF$

or $\quad AD \times BE \times CF = DB \times EC \times FA.$

Illustration-6 : All the triangles in the diagram below are similar to isosceles triangle ABC in which AB = AC. Each of the smaller triangle has area 1 unit^2 and ΔABC has area 40 unit^2, then area of trapezium DBCE is

(A) 16 (B) 8 (C) 24 (D) 32

Solution : $\Delta PDQ \sim \Delta ABC$

$$\frac{ar.\Delta PDQ}{ar.\Delta ABC} = \frac{DQ^2}{BC^2} = \frac{1}{40}$$

$\therefore \; DQ = \dfrac{BC}{2\sqrt{10}}$

Also DE = 4.DQ

$DE = \dfrac{4.BC}{2\sqrt{10}}$ Hence $\dfrac{DE}{BC} = \dfrac{2}{\sqrt{10}}$

$$\frac{ar\Delta ADE}{ar\Delta ABC} = \frac{DE^2}{BC^2} = \frac{4}{10} = \frac{2}{5}$$

area $\Delta ADE = 16$

area trap DBCE = 40 – 16 = 24

Illustration-7 : ABCD is a square and E is mid point of AD. If area of square is given by A_1. Prove following :

(A) $\dfrac{\text{area of } \Delta EFD}{\text{area of } \Delta BFC} = \dfrac{1}{4}$

(B) $\dfrac{\text{area of } \Delta BEF}{\text{area of } \Delta DFC} = 1$

(C) area of $\Delta EFD = \dfrac{A_1}{12}$

(D) area of $\Delta BEF = \dfrac{A_1}{6}$

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Solution : Let side of square = a

$A_1 = a^2$

area of $\Delta BED = \frac{1}{2} \times a \times \frac{a}{2} = \frac{1}{4}a^2 = \frac{A_1}{4}$

Since ΔEFD is similar to ΔCFB

$\Rightarrow \frac{ED}{BC} = \frac{1}{2} = \frac{EF}{FC} = \frac{BF}{FD}$

$\Rightarrow \frac{\text{area of } \Delta EFD}{\text{area of } \Delta BFC} = \frac{1}{4}$

Let area of $\Delta EFD = \Delta$

$\Rightarrow$ area of $\Delta BFC = 4\Delta$

Also let area of $\Delta BEF = x$ and let area of $\Delta DFC = y$

$\Rightarrow x + \Delta = \frac{A_1}{4}$ and $y + 4\Delta = \frac{A_1}{2}$

Area of ΔBED = Area of ΔCED

($\because$ triangles on the same base ED and between the same parallel lines ED and BC)

$\Rightarrow$ Area of ΔBEF + Area of ΔEFD = Area of ΔCDF + Area of ΔEFD

$\Rightarrow x + \Delta = y + \Delta \Rightarrow x = y$

Now x = y therfore $x + \Delta = \frac{A_1}{4}$ and $x + 4\Delta = \frac{A_1}{2}$ on solving these equations

$\Rightarrow \Delta = \frac{A_1}{12} \Rightarrow \quad x = \frac{A_1}{6}$

Also $y + \Delta = \frac{A_1}{4} \Rightarrow y = \frac{A_1}{6}$

Illustration-8 : In a right angle triangle ABC. Let k is maximum possible area of a square that can be inscribed when one of its vertices coincide with the vertex of right angle of triangle then find the value of k

Solution : ΔAMN & ΔNLB

are similar.

$\frac{4-x}{x} = \frac{x}{3-x}$

$\Rightarrow 12 - 7x + x^2 = x^2$

$x = \frac{12}{7}$

$k = \frac{144}{49}$

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SOME IMPORTANT POINTS AND THEIR MEANING :

MEDIANS AND CENTROID :

A **median** of a triangle is a line segment from a vertex to the midpoint of the opposite side. In the figure below, $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ are all medians.

- The medians of a triangle are concurrent at a point called the **centroid** of the triangle. The centroid of the triangle is usually labelled by G.

- The medians of a triangle divide the triangle into six little triangles of equal area.
- The centroid of a triangle cuts its medians into 2 : 1 ratio. For example, for the triangle shown, we have $\frac{AG}{GD} = \frac{BG}{GE} = \frac{CG}{GF} = \frac{2}{1}$.
- The median to the hypotenuse of a right triangle is equal in length to half the hypotenuse.

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PERPENDICULAR BISECTORS AND CIRCUMCENTRE :

The perpendicular bisector of a segment is the straight line consisting of all the points that are equidistant from the end points of the segment.

Circumcentre :

The perpendicular bisectors of the sides of a triangle are concurrent at a point called the **circumcenter**. The circle centered at the circumcenter that passes through the vertices of the triangle is called the **circumcircle** of the triangle, because it is **circumscribed** about the triangle (meaning it passes through all the vertices of the triangle).

Finally, the radius of this circle is called the **circumradius**, the circumcenter is usually labelled with the letter O, and the circumradius is usually denoted by R.

$$OA = OB = OC = R$$

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- The circumcenter of a right triangle is the midpoint of the hypotenuse and the circumradius equals half the length of the hypotenuse.

- Just as two points determine a line, we have now shown that three non-collinear points determine a circle. This means that given any three non-collinear points, there is exactly one circle that passes through all three.

ANGLE BISECTORS, INCENTRE & EXCENTRE :

- The angle bisector of an angle consists of all points that are equidistant from the lines forming the angle.
- The angle bisectors of interior angles of a triangle are concurrent at a point called the **incentre**. This point is equidistant from the sides of the triangle. This common distance from the incentre to the sides of a triangle is called the **inradius**. Thus the circle with centre I and this radius is tangent to all three sides of the triangle. This circle is called the **incircle** because it is inscribed in the triangle (meaning it is tangent to all the sides of the triangle). The incentre is usually denoted I, and the inradius is usually written as r.

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Escribed Circle : It is a circle touching one side of a triangle internally and the other two externally. A triangle has three escribed circles. The centre of an escribed circle is the point of concurrence of the bisectors of the two exterior angles and the bisector of the third interior angle.

The centres of these escribed circles are called excentres and are labelled with I_1, I_2 and I_3. Also, their corresponding radii called exradii are denoted by r_1, r_2 and r_3.

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ALTITUDES & ORTHOCENTRE :

Altitude : It is the perpendicular dropped on any side of a triangle from the opposite vertex.

Orthocentre : The three altitudes of a triangle are concurrent and the point of concurrence is the orthocentre, which is denoted by H.

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EULER'S LINE :

In the figure

- AD is a median and G the centroid.
  AG : GD = 2 : 1.
- BM is the altitude to AC from B.
  Here H is the orthocentre.
- OD is the perpendicular bisector of side BC.
  O is the circumcentre.

Note : O, G and H are collinear. This line is known as Euler's line and HG : GO = 2 : 1.

- In an equilateral triangle these four centres (centroid, orthocentre, incentre, circumcentre) are coincident.
- In an isosceles triangle they are collinear.
- The incentre and centroid are always within the triangle.
- In an acute angled triangle circumcentre and orthocentre also lie inside the triangle
- In a right triangle circumcentre will be the midpoint of the hypotenuse and the orthocentre is the vertex containing the right angle.

- In obtuse angled triangle orthocentre and circumcentre lie outside the triangle.

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Illustration-9 : For any triangle, the area of triangle formed by medians of the triangle is $\left(\frac{3}{4}\right)^{th}$ the area of given triangle.

Solution : Consider ΔABC, AD, BE and CF are length of median say ℓ_1, ℓ_2 and ℓ_3 respectively

We have to find area of triangle, the length of whose sides are ℓ_1, ℓ_2 and ℓ_3.

Let G be the centroid of ΔABC

Let us produce AG to K

Such that GD = DK

and join CK and BK

We know that

BD = DC and GD = DK

$\therefore$ BKCG is a parallelogram and hence

BK = GC and CK = BG

$BK = \frac{2}{3}CF$ and $CK = \frac{2}{3}BE$

$BK = \frac{2}{3}\ell_3$ and $CK = \frac{2}{3}\ell_2$

and $GK = 2GD = 2.\left(\frac{1}{3}AD\right) = \frac{2}{3}\ell_1$

Now say PQR is triangle formed by length ℓ_1, ℓ_2 and ℓ_3 as sides. $\Delta PQR \sim \Delta GKC$

$$\frac{\text{area } \Delta PQR}{\text{area } \Delta GKC} = \frac{(\ell_1)^2}{\left(\frac{2\ell_1}{3}\right)^2} = \frac{9}{4}$$

$$\text{area } \Delta PQR = \frac{9}{4} \text{ area } \Delta GKC = \frac{9}{4}\times\frac{1}{3} \text{ area } \Delta ABC$$

$$\text{area } \Delta PQR = \frac{3}{4} \text{ area } \Delta ABC$$

Illustration-10 : If I is the incenter of ΔABC then $\angle BIC = 90° + \frac{1}{2}A$.

Solution : We have

$\angle BIC = 180° - (\angle IBC + \angle ICB)$

$$= 180° - \frac{1}{2}(B + C)$$

$$= 180° - \frac{1}{2}(180° - A)$$

$$= 90° + \frac{1}{2}A.$$

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SOME THEOREMS ON SIDE LENGTH OF A TRIANGLE

Baudhayana Theorem (Pythagorean/Pythagoras' Theorem) :

In a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the other two sides.

In Right Angle Triangle ABC, right angled at B,

$AC^2 = AB^2 + BC^2$

Illustration-11 : ABCD is a quadrilateral (shown in figure) AD = BC, AB = 40, CD = 20. $\angle DAB + \angle ABC = 90°$. If area of quadrilateral is N then $\frac{N}{100}$ is

Solution : Given : AD = BC, AB = 40

CD = 20, $\angle DAB + \angle ABC = 90°$

Construction :

Extend BC to intersect AD in X.

$\Rightarrow \angle AXB = 180° - (\angle DAB + \angle ABC)$

$\Rightarrow \angle AXB = 180° - 90° = 90°$

Let CX = c, AX = a, XD = b

CB = d

BC = AD

BC = AX + XD

$\boxed{BC = a + b}$

$\therefore \boxed{d = a + b}$

$$\text{area (quad)} = \frac{1}{2}a(c + d) + \frac{1}{2}bc$$

$$= \frac{1}{2}[ac + ad + bc]$$

$$= \frac{1}{2}[ad + c(a + b)]$$

$$= \frac{1}{2}[ad + cd]$$

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In ΔBXA

$a^2 + (c + d)^2 = 40^2$ ...(1)

In ΔCXD

$c^2 + b^2 = 20^2$ ...(2)

On substracting (2) from (1)

$a^2 + 2cd + d^2 - b^2 = 1200$

$a^2 + 2cd + (a + b)^2 - b^2 = 1200$ $(d = a + b)$

$2a^2 + 2cd + 2ab = 1200$

$2a(a + b) + 2cd = 1200$

$2ad + 2cd = 1200$

$\frac{1}{2}(ad + cd) = 300$ = Area of quad. ABCD

Illustration-12 : In an equilateral triangle ABC, the side BC is trisected at D. Prove that :

$9\ AD^2 = 7\ AB^2$

Solution : Given : An equilateral ΔABC, in which the side BC is trisected at D.

To Prove : $9\ AD^2 = 7AB^2$

Const. : Draw $AE \perp BC$

Proof : We know that in an equilateral triangle perpendicular from a vertex bisects the base.

$\therefore\ BE = EC = \frac{1}{2}BC$ ... (i)

Also, the side BC is trisected at D. [given]

$\Rightarrow BD = \frac{1}{3}BC$ [from figure] ... (ii)

Now, in a right-triangle AED, We have :

$AD^2 = AE^2 + DE^2$

$= AE^2 + (BE - BD)^2$ $[\because DE = BE - BD]$

$= AE^2 + BE^2 + BD^2 - 2\ BE.BD$

$= AB^2 + BD^2 - 2\ BE.BD$ $[\because AB^2 = AE^2 + BE^2]$

$$= BC^2 + \left(\frac{1}{3}BC\right)^2 - 2\left(\frac{1}{2}BC\right).\left(\frac{1}{3}BC\right)$$

$[\because AB = BC,\ BD = \frac{1}{3}BC$ and $BE = \frac{1}{2}BC$ see (i) and (ii)$]$

$$= BC^2 + \frac{BC^2}{9} - \frac{BC^2}{3} = \frac{9BC^2 + BC^2 - 3BC^2}{9}$$

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$\Rightarrow AD^2 = \frac{7BC^2}{9}$ or $9AD^2 = 7BC^2$

or $9AD^2 = 7AB^2$ $\quad$ (AB = BC = CA)

Illustration-13 : The perpendicular AD on the base BC of a ΔABC intersects BC at D so that DB = 3 CD.

Prove that : $2AB^2 = 2\,AC^2 + BC^2$.

Solution : Given : In figure, $AD \perp BC$ and DB = 3CD.

To Prove : $2AB^2 = 2AC^2 + BC^2$.

Proof : From right triangles ADB and ADC, we have :

$AB^2 = AD^2 + BD^2$

and $AC^2 = AD^2 + CD^2$

$\therefore \quad AB^2 - AC^2 = BD^2 - CD^2$

$= (3\,CD)^2 - CD^2 \qquad [\because BD = 3\,CD]$

$= 9CD^2 - CD^2 = 8CD^2 = 8 \times \left(\frac{1}{4}BC\right)^2 = 8 \times \frac{BC^2}{16} = \frac{BC^2}{2}$

$\left(\because \frac{BD}{CD} = \frac{3}{1} \Rightarrow \frac{BD+CD}{CD} = \frac{3+1}{1} \Rightarrow \frac{BC}{CD} = \frac{4}{1} \Rightarrow CD = \frac{1}{4}BC\right)$

$\therefore\ 2AB^2 - 2AC^2 = BC^2$

Hence, $2AB^2 = 2AC^2 + BC^2$.

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CIRCLE

Theorem 1 :

If a straight line drawn from the center of a circle bisects a chord, not passing through the centre, then it cuts the chord at right angles. Conversely, if it cuts the chord at right angles, then it bisects the chord.

Theorem 2 :

Equal chords of a circle are equidistant from the center.

AB = CD, $OP \perp AB$ and $OQ \perp CD \Rightarrow OP = OQ$

Conversely,

Chords which are equidistant from the centre are equal.

OP = OQ, $OP \perp AB$ and $OQ \perp CD \Rightarrow AB = CD$

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Theorem 3 :

Of any two chords of a circle, that which is nearer to the centre is greater than one lying remote. Conversely, the greater of two chords is nearer to the centre.

$OP > OQ \Rightarrow CD < AB$

Theorem 4 :

If P be any point (interior, exterior or on the circle) then greatest distance between point P and circumference of the circle is PO +r and the smallest distance between point P and circumference of the circle is |PO –r| (where r is radius and O is centre)

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ANGLES IN A CIRCLE

Theorem 1 :

The angle at the centre of a circle is double of an angle at the circumference subtended by the same arc.

$\angle AOB = 2\angle APB$

Theorem 2 :

Angles in the same segment of a circle are equal.

$\angle APB = \angle AQB$

Converse of Theorem 2 :

If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, then four points lie on a circle (i.e. they are concyclic). If $\angle APB = \angle AQB \Rightarrow AB$ is a chord of circle and A, P, Q, B are concyclic points

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Theorem 3 :

The opposite angles of any quadrilateral inscribed in a circle are together equal to two right angles.

$\angle P + \angle R = 180°$ and $\angle Q + \angle S = 180°$

Converse of Theorem 3 :

If a pair of opposite angles of a quadrilateral are supplementary, then its vertices are concyclic.

If $\angle P + \angle R = 180°$ or $\angle Q + \angle S = 180°$

$\Rightarrow$ Then points P, Q, R and S are concyclic.

Theorem 4 :

The angle in a semi-circle is a right angle.

$\angle APB = 90°$

Illustration-14 : In quadrilateral WXYZ with perpendicular diagonals, we are given $\angle WZX = 30°$, $\angle XWY = 40°$, and $\angle WYZ = 50°$.

(a) Compute $\angle Z$.

(b) Compute $\angle X$.

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Solution: Let P be the intersection of the diagonals.

In ΔPZY

$\angle PZY = 90° - 50° = 40°$.

Now consider angles.

$\angle XWY$ and $\angle XZY$

Since $\angle XWY = \angle XZY = 40°$

Using theorem : WXYZ is cyclic quadrilateral.

$\angle Z = \angle WZX + \angle PZY$

$= 30° + 40° = 70°$

Now $\angle Z + \angle X = 180°$ {opposite angle of cyclic quadrilateral}

$\angle X = 180° - 70° = 110°$

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Illustration-15 : In the given figure, the chord ED is parallel to the diameter AC and $\angle CBE = 50°$. Find $\angle CED$.

Solution:

$\angle CBE = \angle CAE$ ($\because$ angles in the same segment)

$\angle CAE = \angle 1 = 50°$ ...(1) ($\because \angle CBE = 50°$)

$\angle AEC = 90°$ ...(2) ($\because$ Angle in a semi circle is a right angle)

Now in ΔAEC

$\angle 1 + \angle AEC + \angle 2 = 180°$ [$\because$ sum of angles of a $\Delta = 180°$]

$\therefore 50° + 90° + \angle 2 = 180°$

$\Rightarrow \angle 2 = 40°$ ...(3)

Also, ED || AC (Given)

$\therefore \angle 2 = \angle 3$ (Alternate angles)

$\therefore 40° = \angle 3$ i.e., $\angle 3 = 40°$

Hence $\angle CED = 40°$

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Illustration-16 : In the figure below, BA and BC are tangents to the circle at points A and C, respectively. If $\angle AEC = 110°$. Find $\angle ABC$.

Solution :

Consider the diagram to the right.

- Opposite angles of a cyclic quadrilateral sum up to 180°.
  So $\angle ADC = 180° - 110° = 70°$.

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• From the inscribed angle theorem, we can see that

$\angle AOC = 2\angle ADC = 2(70°) = 140°$

Since the sum of interior angle of a quadrilateral is 360°, we have

$\angle ABC = 360° - \angle BCO - \angle BAO - \angle AOC$

$= 360° - 90° - 90° - 140°$

$= 40°$

Illustration-17 : An acute isosceles triangle, ABC is inscribed in a circle. Through B and C, tangents to the circle are drawn, meeting at point D. If $\angle ABC = \angle ACB = 2\angle D$, find the measure of $\angle A$.

Solution : Let $\angle D = \theta$

$\Rightarrow \angle ABC = \angle ACB = 2\theta$

In ΔBCD.

$180° - 4\theta + 180° - 4\theta + \theta = 180°$

$\Rightarrow 7\theta = 180°$

$\Rightarrow \theta = \dfrac{180°}{7}$

$\therefore \angle A = 180° - 4\theta$

$= 180° - \dfrac{4 \times 180°}{7}$

$\angle A = \dfrac{3 \times 180°}{7}$

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Illustration-18 : In the given figure if C_1, C_2, C_3 are three concentric circles such that radius of C_1 and C_2 is 1 and 3 unit respectively, then radius of C_3 is

Solution : Let $\angle BDC = \theta$, then

$\angle DBC = \dfrac{\pi}{2} - \theta$

$\therefore \angle CBA = \theta$, So

$\Delta ABD \sim \Delta BCD$, so

$\dfrac{AD}{BD} = \dfrac{BD}{CD}$

$\Rightarrow R_3 = \dfrac{(3)^2}{1} = 9$

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Illustration-19 : In the given fig., O is the centre of the circle. Prove that $\angle x + \angle y = \angle z$.

Solution : Given : In fig., O is the centre of the circle.

To Prove : $\angle x + \angle y = \angle z$

Proof : $\angle z = 2\ \angle 2$ [angle at the centre is double to the angle in the remaining part.]

or $\angle z = \angle 2 + \angle 2$

But, $\angle 2 = \angle 3$ [angles in the same segment]

$\Rightarrow$ $\angle z = \angle 2 + \angle 3$ ... (i)

Now, we will determine the values of $\angle 2$ and $\angle 3$ in terms of $\angle x$ and $\angle y$.

$\because$ $\angle 3$ is an exterior angle of ΔAEB,

$\therefore$ $\angle 3 = \angle 1 + \angle x$ ... (ii) [ext. $\angle$ = sum of two int. opp. $\angle$s]

Again, $\angle y$ is an exterior angle of ΔDEP

$\therefore$ $\angle y = \angle 1 + \angle 2$ $\Rightarrow \angle 2 = \angle y - \angle 1$ ... (iii)

(i), (ii) and (iii)

$\Rightarrow$ $\angle z = (\angle y - \angle 1) + (\angle 1 + \angle x) = \angle x + \angle y$

Hence, $\angle x + \angle y = \angle z$.

TANGENT

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Theorem 1 :

The tangent at any point of a circle is perpendicular to the radius drawn to the point of contact.

Theorem 2 :

Two tangents can be drawn to a circle from an external point and length of these tangents are equal.

So, PM = PN

Theorem 3 :

If two circles touch one another, the centres and the point of contact are collinear.

$O_1O_2 = r_1 + r_2$

$O_1O_2 = |r_1 - r_2|$

Theorem 4 (Alternate Segments Theorem) :

The angles made by a tangent to a circle with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle.

$\angle MTB = \angle TPM$, $\angle MTA = \angle MQT$

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LENGTH OF COMMON TANGENTS

Direct Common Tangent (External Common Tangent) :

Consider two circle with centre A and B and line 'ℓ' touching the two circles at P and Q. Let radius of the circles be r_1 and r_2.

Now draw perpendicular from 'B' to AP meeting at 'C'.

Thus, CPQB is a rectangle and hence PQ = CB; PC = BQ

Now, in ΔACB

$AB^2 = AC^2 + BC^2$

$\Rightarrow BC^2 = AB^2 - AC^2$

$\Rightarrow PQ^2 = AB^2 - AC^2$ [BC = PQ]

$\Rightarrow PQ^2 = AB^2 - (r_1 - r_2)^2$ [$AC = AP - PC = r_1 - r_2$]

Hence, length of direct common tangent

$PQ = \sqrt{d^2 - (r_1 - r_2)^2}$ [where AB = d = distance between the centres]

Transverse Common Tangent (Internal Common Tangent) :

Drop perpendicular from A to meet BQ produced in 'C'

Now $AP = r_1$; $BQ = r_2$; $AB = d$

AP = QC and PQ = AC

Also BC = BQ + QC

So $BC = r_2 + r_1$

Now in ΔACB

$AB^2 = AC^2 + BC^2$

$\Rightarrow d^2 = AC^2 + (r_1 + r_2)^2$

$\Rightarrow AC^2 = d^2 - (r_1 + r_2)^2$

$\Rightarrow PQ^2 = d^2 - (r_1 + r_2)^2$

$\Rightarrow PQ = \sqrt{d^2 - (r_1 + r_2)^2}$

Length of transverse common tangent is $\sqrt{d^2 - (r_1 + r_2)^2}$

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POWER OF A POINT

Suppose a line through a point P intersects a circle in two points, A and B. The **power of a point theorem** states that for all such lines, the product (PA)(PB) is constant. We call this product the power of point P.

P lies outside the circle

$PT^2 = (PA)(PB) = (PC)(PD)$

= The power of point P

P lies inside the circle

$PA \cdot PB = PC \cdot PD$

Proof : Given : A secant PAB to a circle C(O, r) intersecting it in points A and B and PT is a tangent segment of the circle.

To Prove : $PA \times PB = PT^2$.

Construction : Draw $OD \perp AB$. Join OP, OT and OA.

Proof : $OD \perp AB \Rightarrow AD = DB$

Now, $PA \times PB = (PD - AD)(PD + DB)$

$= (PD - AD)(PD + AD)$ [$\because DB = AD$]

$= PD^2 - AD^2$

$= (OP^2 - OD^2) - AD^2$ [$OP^2 = OD^2 + PD^2$] [Pythagoras Theorem]

$= OP^2 - (OD^2 + AD^2)$

$= OP^2 - OA^2$ [$\because OA^2 = OD^2 + AD^2$]

$= OP^2 - OT^2$ [$\because OA = OT$] [radii of same circle]

$= PT^2$ [$\because \angle OTP = 90°$, $OP^2 = OT^2 + PT^2$]

Illustration-20 : Chords $\overline{TY}$ and $\overline{OP}$ meet at point K such that TK = 2, KY = 16 and KP = 2(KO). Find OP.

Solution : A quick sketch suggests how to apply power of a point.

From the power of point K, we have

$$(KP)(KO) = (KT)(KY).$$

Substituting the given information in this equation yields

$$2(KO)(KO) = 2(16),$$

from which we find KO = 4. Therefore, KP = 2KO = 8, so OP = KO + KP = 12.

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Illustration-21 : In the diagram, CB = 9, BA = 11 and CE = 18. Find DE.

Solution : We have two intersecting secants, so we apply power of a point, which gives

$$(CD)(CE) = (CB)(CA).$$

Therefore, (CD)(18) = 9(20), so CD = 10 and DE = CE – CD = 8.

Illustration-22 : Points R and M trisect $\overline{PS}$, so PR = RM = MS. Point U is the midpoint of $\overline{PQ}$, TM = 2 and MQ = 8. Find PU.

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Solution : The power of point M gives us

$$(MR)(MS) = (MT)(MQ).$$

We know that RM = MS, so substitution gives $MR^2 = (2)(8)$, i.e., MR = 4. Therefore, PR = MR = 4 and PS = 3(MR) = 12. Since U is the midpoint of $\overline{PQ}$, we have PQ = 2PU.

Now we can apply the power of point P to find :

$$(PU)(PQ) = (PR)(PS).$$

Substitution gives (PU)(2PU) = 4(12), so PU = $2\sqrt{6}$.

Illustration-23 : In the diagram,

we have BP = 8, AB = 10,

CD = 7 and ∠APC = 60°.

Find the area of the circle.

Solution : It's not immediately obvious how we will find the radius, so we start by finding what we can. The power of a point P gives us

(PC)(PD) = (PB)(PA),

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so (PC)(PC + 7) = 144. Therefore, $PC^2 + 7PC - 144 = 0$, so (PC + 16)(PC – 9) = 0. PC must be positive, so PC = 9.

Seeing that $\angle APC = 60°$ makes us wonder if there are any equilateral or 30-60-90 triangle lurking about. Since CP = AP/2 and the angle between these sides is 60°, the sides adjacent to the 60° angle in ΔACP are in the same ratio as the sides adjascent to the 60° angle in a 30-60-90 triangle with right angle at $\angle ACP$! $\Rightarrow \angle ACP = 90° \Rightarrow \angle ACD = 90°$

Since $\angle ACD$ is right and inscribed in $\overset{\frown}{AD}$, we know $\overset{\frown}{AD}$ is a semicircle. Therefore, $\overline{AD}$ is a diameter of the circle. Since $AC = CP\sqrt{3} = 9\sqrt{3}$ from our 30-60-90 triangle, we have $AD = \sqrt{AC^2 + CD^2} = \sqrt{243 + 49} = 2\sqrt{73}$. Finally, the radius of the circle is $AD/2 = \sqrt{73}$, so the area is $\left(\sqrt{73}\right)^2 \pi = 73\pi$.

Illustration-24 : Chords AB and CD of a circle intersect each other at P as shown in fig., if AB = 6 cm, PB = 2 cm, PD = 2.5 cm, find CD.

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Solution : Since chords AB and CD of a circle meet in P (when produced),

$\therefore$ PA.PB = PC.PD

$\Rightarrow$ (6 + 2).2 = PC (2.5)

$\Rightarrow$ $PC = \frac{16}{2.5} = \frac{16 \times 2}{5} = \frac{32}{5} = 6.4$ cm

$\therefore$ CD = PC – PD = 6.4 – 2.5 = 3.9 cm.

Illustration-25 : If two circles of radius 1 cm and 4 cm touch each other, then find the length of common tangent.

Solution : Distance between centres AB = $r_1 + r_2 = 5$

Length of tangent

$PQ = \sqrt{d^2 - (r_1 - r_2)^2}$

$= \sqrt{5^2 - (4-1)^2}$

PQ = 4 cm

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INEQUALITIES IN TRIANGLE

In any triangle, the longest side is opposite the largest angle and the shortest side is opposite the smallest angle. The middle side, of course, is therefore opposite the middle angle.

In other words, in ΔABC, $AB \geq AC \geq BC$ if and only if $\angle C \geq \angle B \geq \angle A$.

Important :

- $\angle C$ of ΔABC is acute if and only if $AB^2 < AC^2 + BC^2$.

  $\angle C$ of ΔABC is right if and only if $AB^2 = AC^2 + BC^2$.

  $\angle C$ of ΔABC is obtuse if and only if $AB^2 > AC^2 + BC^2$.

- The Triangular Inequality states that for any three points, A, B and C, we have

  $$AB + BC \geq AC,$$

  where equality holds if and only if B lies on $\overline{AC}$. Therefore, for nondegenerate triangles (i.e., those in which the vertices are not collinear), $AB + BC > AC$.

Illustration-26 : Given the angles as shown, order the lengths AB, BD, CD, BC and AC from greatest to least. (Note : the diagram is not drawn to scale!)

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Solution : First we focus on ΔABC. Since $\angle A > \angle B > \angle C$, we have $BC > AC > AB$. Then, we turn to ΔBDC. Since $\angle B > \angle C > \angle D$, we have $CD > BD > BC$. BC is the smallest in one inequality string and the largest in the other, so we can put the inequalities together :

$$CD > BD > BC > AC > AB.$$

Illustration-27 : In ΔABC the median $\overline{AM}$ is longer than BC/2. Prove that $\angle BAC$ is acute.

Solution : We start with a diagram. We'd like to prove something about an angle, but all we are given is an inequality regarding lengths. So, we use the length inequality to get some angle inequalities to work with. Specifically, since $AM > BM$ in ΔABM and $AM > MC$ in ΔACM (because $AM > BC/2$ and $BM = MC = BC/2$), we have

$$\angle B > \angle BAM$$
$$\angle C > \angle CAM.$$

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We want to prove something about $\angle BAC$, which equals $\angle BAM + \angle CAM$, so we add these two inequalities to give $\angle B + \angle C > \angle BAC$. Since $\angle B + \angle C + \angle BAC = 180°$, we can write $\angle B + \angle C > \angle BAC$ as

$$180° - \angle BAC > \angle BAC.$$

This gives us $180° > 2\angle BAC$, so $90° > \angle BAC$. Therefore, $\angle BAC$ is acute.

Illustration-28 : In ΔXYZ, we have $XY = 11$ and $YZ = 14$. For how many integer values of XZ is ΔXYZ acute ?

Solution : In order for ΔXYZ to be acute, all three of its angles must be acute. Using the given side lengths and our inequalities above, we see that we must have

$$121 + 196 > XZ^2$$

$$196 + XZ^2 > 121$$

$$121 + XZ^2 > 196$$

Simplifying these three yields :

$$317 > XZ^2$$

$$XZ^2 > -75$$

$$XZ^2 > 75$$

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The middle inequality is clearly always true. (We really didn't even have to include it – clearly 11 couldn't ever be the largest side!) Combining the other two add noting that we seek integer values of XZ, we have $9 \le XZ \le 17$ (since $8^2 < 75 < 9^2$ and $18^2 > 317 > 17^2$). So, there are 9 integer values of XZ such that ΔXYZ is acute.

Illustration-29 : In how many ways can we choose three different numbers from the set $\{1, 2, 3, 4, 5, 6\}$ such that the three could be the sides of a nondegenerate triangle ?

(Note : The order of the chosen numbers doesn't matter; we consider $\{3, 4, 5\}$ to be the same as $\{4, 3, 5\}$.)

Solution : We first notice that if we have three numbers to consider as possible side lengths of a triangle, we only need to make sure that the sum of the smallest two is greater than the third. (Make sure you see why!) We could just start listing all the ones we see that work, but we should take an organized approach to make sure we don't miss any. We can do so by classifying sets of three numbers by the smallest number.

Case 1 : Smallest side has length 1. No triangles can be made with three different lengths from our set if we include one of length 1.

Case 2 : Smallest side has length 2. The other two sides must be 1 apart, giving the sets $\{2, 3, 4\}$, $\{2, 4, 5\}$, and $\{2, 5, 6\}$.

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Case 3 : Smallest side has length 3. There are only three possibilities and they all work : {3, 4, 5}, {3, 4, 6}, {3, 5, 6}.

Case 4 : Smallest side has length 4. The only possibility is {4, 5, 6}, which works.

Adding them all up, we have 7 possibilities.

We sometimes have to use some other tools in addition to triangle inequality.

Illustration-30 : Can the lengths of the altitudes of a triangle be in the ratio 2 : 5 : 6 ? Why or why not ?

Solution : We don't know anything about how the lengths of the altitudes of a triangle are related to each other. We do, however, know a whole lot about how the lengths of the sides of a triangle are related to each other. Therefore, we turn the problem from one involving altitudes into one involving side lengths. We let the area be K, let the side lengths be a, b, c and the lengths of the altitudes to these sides be h_a, h_b, h_c, respectively. Therefore, we have $K = ah_a/2 = bh_b/2 = ch_c/2$, so the sides of the triangle have lengths

$$\frac{2K}{h_a}, \frac{2K}{h_b}, \frac{2K}{h_c}.$$

If our heights are in the ratio 2 : 5 : 6, then for some x, our heights are 2x, 5x and 6x. Then, our sides are

$$\frac{2K}{2x}, \frac{2K}{5x}, \frac{2K}{6x}.$$

However, the sum of the smallest two sides is then

$$\frac{2K}{5x} + \frac{2K}{6x} = \frac{12K}{30x} + \frac{10K}{30x} = \frac{22K}{30x},$$

which is definitely less than the largest side, which is 2K/2x = K/x. Therefore, the sides don't satisfy the triangle inequality, which means it is impossible to have a triangle with heights in the ratio 2 : 5 : 6.

Concept : When facing problems involving lengths of altitudes of a triangle, consider using area as a tool.

Illustration-31 : If a circle is inscribed in regular hexagon of side 6 cm, then area of inscribed circle.

Solution : Since we know that sum of angles of polygon = (n – 2)180°

Each interior angle of polygon = $\frac{(n-2)180°}{n}$

For hexagon n = 6

$\therefore$ Each angle = $\frac{(6-2)180°}{6} = 120°$

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In ΔOPA

$\frac{r}{3} = \tan 60° \therefore r = 3\sqrt{3}$

Area of circle $= \pi r^2 = \pi\left(3\sqrt{3}\right)^2 = 27\pi$

Illustration-32 : In given circle with centre O a triangle ABC is inscribed, $\angle A = 60°$. OD is perpendicular to BC, then what is $\angle BOD$?

Solution : In ΔOBD and ΔODC

$OD \perp BC$

$\therefore \angle ODB = \angle ODC = 90°$

$OB = OC$ (radius of circle)

$BD = DC$ [$\perp$ from centre bisects chord]

$\therefore \Delta OBD \cong \Delta OCD$

Hence $\angle BOD = \angle COD = \frac{1}{2}\angle BOC$ .... (1)

Again $\angle BOC = 2\angle BAC$

$\angle BOC = 2[\angle A]$ .... (2)

From (1) and (2)

$\angle BOD = \frac{1}{2}(2\angle A) = 60°$

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Illustration-33 : In right ΔABC right angled at 'B' (As in figure).
Prove that distance between circumcentre and centroid

$GD = \frac{1}{3}(BD) = \frac{1}{3}\left(\frac{1}{2}AC\right)$. If AC = 10, then find GD

Solution. : **Property :** We know that mid-point of hypotenuse is equidistance from all the three vertices of triangle

$\therefore AD = CD = BD$

$\therefore BD = \frac{1}{2}AC$

$BD = \frac{1}{2}(10) = 5$

Again centroid divides median in ratio 2 : 1

$\therefore GD = \frac{1}{3}BD = \frac{5}{3}$

[D is circumcentre of ΔABC since mid-point of hypotenuse is circumcentre of triangle]

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Illustration-34 : In ΔABC, AD and BE are medians such that they intersect at right angles. If AD = 6 and BE = $\frac{9}{2}$ then area of ΔABC is

Solution. : Let G be the centroid of ΔABC i.e. point of intersection of medians AD and BE

Now $AG = \frac{2}{3}AD = \frac{2}{3}[6] = 4$

$BG = \frac{2}{3}BE = \frac{2}{3} \times \frac{9}{2} = 3$

Thus Area of ΔAGB is $\frac{1}{2} \times 3 \times 4 = 6$ unit^2

Now we know that area of ΔABC

= 3 × Area (ΔAGB) = 3(6) = 18 unit^2

Illustration-35 : In, ΔABC, AD, BE and CF are altitudes and H is orthocentre. If $\angle A = 40°$ then find $\angle FDE$.

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Solution. : It is very clear that $\angle HFB = 90°$ and $\angle HDB = 90°$

(Since CF and AD are altitudes)

Thus for quadrilateral HFBD

sum of opposite angle is 180°

Hence it is cyclic

In ΔAEB

$\angle ABE = 90° - A = \angle FBH$

and using the property of circle

$\angle FBH = \angle FDH$

$\therefore \angle FDH = 90° - A$

Similarly $\angle HDE = 90° - A$

$\therefore \angle FDE = 180° - 2A$

$\angle FDE = 180° - 2(40°) = 100°$

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EXERCISE

1. In the diagram 'O' is the centre of the circle. The point E lies on the circumference of the circle such that the area of the ΔECD is maximum. If $\angle AOB$ is a right angle, then the ratio of the area of ΔECD to the area of the ΔAOB is :

(A) 2 : 1 (B) 3 : 2 (C) 4 : 3 (D) 4 : 1

GM0001

2. Let ΔXOY be a right angled triangle with $\angle XOY = 90°$. Let M and N be the midpoints of legs OX and OY, respectively. Given that XN = 19 and YM = 22, the length XY is equal to

(A) 24 (B) 26 (C) 28 (D) 34

GM0002

3. In the diagram below, if ℓ and m are two tangents and AB is a chord making an angle of 60° with the tangent ℓ, then the angle between ℓ and m is

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(A) 45° (B) 30° (C) 60° (D) 90°

GM0003

4. In the diagram, O is the centre of the circle and D, E and F are mid points of AB, BO and OA respectively. If $\angle DEF = 30°$, then $\angle ACB$ is

(A) 30° (B) 60° (C) 90° (D) 120°

GM0004

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5. In the below diagram, O is the centre of the circle, AC is the diameter and if $\angle APB = 120°$, then $\angle BQC$ is

(A) 30° (B) 150° (C) 90° (D) 120°

GM0005

6. In the adjoining figure, PT is a tangent at point C of the circle. O is the circumcentre of ΔABC. If $\angle ACP = 118°$, then the measure of $\angle x$ is

(A) 28° (B) 32° (C) 42° (D) 38°

GM0006

7. In the cyclic quadrilateral WXYZ on the circle centered at O, $\angle ZYW = 10°$ and $\angle YOW = 100°$. What is the measure of $\angle YWZ$?

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(A) 30° (B) 40° (C) 50° (D) 60°

GM0007

8. In the adjacent figure, if $\angle y + \angle z = 100°$ then the measure of $\angle x$ is :

(A) 50° (B) 40° (C) 45° (D) Cannot be determined

GM0008

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9. In a ΔABC if P be the point of intersection of the interior angle bisectors and Q be the point of intersection of the exterior angle bisectors of angles B and C respectively, then the figure BPCQ is a :

(A) Parallelogram (B) Rhombus (C) Trapezium (D) Cyclic quadrilateral

GM0009

10. D, E and F are points on sides BC, CA and AB respectively of ΔABC such that AD bisects $\angle A$, BE bisects $\angle B$ and CF bisects $\angle C$. If AB = 5 cm, BC = 8 cm and CA = 4 cm, determine AF, CE and BD.

GM0010

11. In the adjoining figure, AE is the bisector of exterior $\angle CAD$ meeting BC produced in E. If AB = 10 cm, AC = 6 cm and BC = 12 cm, find CE.

GM0011

12. In fig. ΔABC and ΔDBC are two triangles on the same base BC. Prove that $\frac{ar(\Delta ABC)}{ar(\Delta DBC)} = \frac{AO}{DO}$.

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GM0012

13. ABC is a right-triangle with $\angle ABC = 90°$, $BD \perp AC$, $DM \perp BC$ and $DN \perp AB$. Prove that

(i) $DM^2 = DN \times MC$

(ii) $DN^2 = DM \times AN$

GM0013

14. Prove that sum of squares of diagonals of a parallelogram is equal to sum of squares of its sides.

GM0014

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15. In fig., ABC is a triangle in which AB = AC. A circle through B touches AC at D and intersects AB at P. If D is the mid-point of AC, show that 4 AP = AB.

GM0015

16. In fig., CP is a tangent to a circle. If ∠PCB = 60° and ∠BCA = 45°, find ∠ABC.

GM0016

17. The inscribed circle of ΔABC touches BC, CA and AB at X, Y and Z respectively. If ∠A = 64°, ∠C = 52°, find ∠XYZ and ∠XZY.

GM0017

18. Let $\overline{AM}$ be a median of ΔABC. Prove that AM > (AB + AC – BC)/2.

GM0018

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ANSWER KEY

1.	A	**2.**	B	**3.**	C	**4.**	B	**5.**	B	**6.**	A
7.	B	**8.**	B	**9.**	D	**10.**	AF = $\frac{5}{3}$, CE = $\frac{32}{13}$, BD = $\frac{40}{9}$				
11.	CE = 18 cm	**16.**	∠ABC = 75°			**17.**	∠XYZ = 58°, ∠XZY = 64°				

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Chapter 02

SOLUTIONS OF TRIANGLE

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JEE (Main/Advanced) Syllabus

JEE (Advanced) Syllabus :

Solutions of Triangle : *Relations between sides and angles of a triangle, sine rule, cosine rule, half-angle formula and the area of a triangle.*

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SOLUTIONS OF TRIANGLE

The process of calculating the sides and angles of triangle using given information is called solution of triangle.

In a ΔABC, the angles are denoted by capital letters A, B and C and the length of the sides opposite these angle are denoted by small letter a, b and c respectively.

1. SINE FORMULAE :

In any triangle ABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \lambda = \frac{abc}{2\Delta} = 2R$$

where R is circumradius and Δ is area of triangle.

Illustration 1 : Angles of a triangle are in 4 : 1 : 1 ratio. The ratio between its greatest side and perimeter is

(A) $\frac{3}{2+\sqrt{3}}$ (B) $\frac{\sqrt{3}}{2+\sqrt{3}}$ (C) $\frac{\sqrt{3}}{2-\sqrt{3}}$ (D) $\frac{1}{2+\sqrt{3}}$

Solution : Angles are in ratio 4 : 1 : 1.

$\Rightarrow$ angles are 120°, 30°, 30°.

If sides opposite to these angles are a, b, c respectively, then a will be the greatest side. Now from sine formula $\frac{a}{\sin 120°} = \frac{b}{\sin 30°} = \frac{c}{\sin 30°}$

$$\Rightarrow \frac{a}{\sqrt{3}/2} = \frac{b}{1/2} = \frac{c}{1/2}$$

$$\Rightarrow \frac{a}{\sqrt{3}} = \frac{b}{1} = \frac{c}{1} = k \text{ (say)}$$

then $a = \sqrt{3}k$, perimeter $= (2+\sqrt{3})k$

$\therefore$ required ratio $= \frac{\sqrt{3}k}{(2+\sqrt{3})k} = \frac{\sqrt{3}}{2+\sqrt{3}}$ **Ans. (B)**

Illustration 2 : In triangle ABC, if b = 3, c = 4 and $\angle B = \pi/3$, then number of such triangles is -

(A) 1 (B) 2 (C) 0 (D) infinite

Solution : Using sine formulae $\frac{\sin B}{b} = \frac{\sin C}{c}$

$$\Rightarrow \frac{\sin \pi/3}{3} = \frac{\sin C}{4} \Rightarrow \frac{\sqrt{3}}{6} = \frac{\sin C}{4} \Rightarrow \sin C = \frac{2}{\sqrt{3}} > 1 \text{ which is not possible.}$$

Hence there exist no triangle with given elements. **Ans. (C)**

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Illustration 3 : The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle.

Solution : Let the sides be n, n + 1, n + 2 cms.

i.e. AC = n, AB = n + 1, BC = n + 2

Smallest angle is B and largest one is A.

Here, $\angle A = 2\angle B$

Also, $\angle A + \angle B + \angle C = 180°$

$\Rightarrow \quad 3\angle B + \angle C = 180° \Rightarrow \angle C = 180° - 3\angle B$

We have, sine law as,

$$\frac{\sin A}{n+2} = \frac{\sin B}{n} = \frac{\sin C}{n+1} \Rightarrow \frac{\sin 2B}{n+2} = \frac{\sin B}{n} = \frac{\sin(180-3B)}{n+1}$$

$$\Rightarrow \underset{(i)}{\frac{\sin 2B}{n+2}} = \underset{(ii)}{\frac{\sin B}{n}} = \underset{(iii)}{\frac{\sin 3B}{n+1}}$$

from (i) and (ii);

$$\frac{2\sin B\cos B}{n+2} = \frac{\sin B}{n} \Rightarrow \cos B = \frac{n+2}{2n} \quad \text{..... (iv)}$$

and from (ii) and (iii);

$$\frac{\sin B}{n} = \frac{3\sin B - 4\sin^3 B}{n+1} \Rightarrow \frac{\sin B}{n} = \frac{\sin B(3-4\sin^2 B)}{n+1}$$

$$\Rightarrow \frac{n+1}{n} = 3 - 4(1-\cos^2 B) \quad \text{..... (v)}$$

from (iv) and (v), we get

$$\frac{n+1}{n} = -1 + 4\left(\frac{n+2}{2n}\right)^2 \Rightarrow \frac{n+1}{n} + 1 = \left(\frac{n^2+4n+4}{n^2}\right)$$

$$\Rightarrow \frac{2n+1}{n} = \frac{n^2+4n+4}{n^2} \Rightarrow 2n^2 + n = n^2 + 4n + 4$$

$\Rightarrow \quad n^2 - 3n - 4 = 0 \quad \Rightarrow \quad (n-4)(n+1) = 0$

$n = 4$ or -1

where $n \neq -1$

$\therefore \quad n = 4$. Hence the sides are 4, 5, 6 **Ans.**

Do yourself - 1 :

(i) If in a ΔABC, $\angle A = \frac{\pi}{6}$ and $b : c = 2 : \sqrt{3}$, find $\angle B$.

(ii) Show that, in any ΔABC : $a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$.

(iii) If in a ΔABC, $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, show that a^2, b^2, c^2 are in A.P.

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(iv) Prove that $(b-c)\cos\frac{A}{2} = a\sin\left(\frac{B-C}{2}\right)$.

(v) Prove that $(b^2 - c^2)\cot A + (c^2 - a^2)\cot B + (a^2 - b^2)\cot C = 0$.

(vi) Prove that $b^2\cos 2A - a^2\cos 2B = b^2 - a^2$.

(vii) The perimeter of a triangle ABC is six times the arithmetic mean of the sines of its angles. If the side b is equal to $\sqrt{3}$, then find angle B.

(viii) In any triangle ABC, if $\frac{a^2-b^2}{a^2+b^2} = \frac{\sin(A-B)}{\sin(A+B)}$, then prove that the triangle is either right angled or isosceles.

(ix) ABC is a triangle such that $\sin(2A+B) = \sin(C-A) = -\sin(B+2C) = \frac{1}{2}$. If A, B, C are in A.P. Find A, B, C.

TS0019

2. COSINE FORMULAE :

(a) $\cos A = \frac{b^2+c^2-a^2}{2bc}$ or $a^2 = b^2 + c^2 - 2bc\cos A$

(b) $\cos B = \frac{c^2+a^2-b^2}{2ca}$

(c) $\cos C = \frac{a^2+b^2-c^2}{2ab}$

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Illustration 4 : In a triangle ABC, if B = 30° and c = $\sqrt{3}$ b, then A can be equal to -

(A) 45° (B) 60° (C) 90° (D) 120°

Solution : We have $\cos B = \frac{c^2+a^2-b^2}{2ca} \Rightarrow \frac{\sqrt{3}}{2} = \frac{3b^2+a^2-b^2}{2\times\sqrt{3}b\times a}$

$\Rightarrow$ $a^2 - 3ab + 2b^2 = 0 \Rightarrow (a-2b)(a-b) = 0$

$\Rightarrow$ Either $a = b \Rightarrow A = 30°$

or $a = 2b \Rightarrow a^2 = 4b^2 = b^2 + c^2 \Rightarrow A = 90°$. **Ans. (C)**

Illustration 5 : In a triangle ABC, $(a^2 - b^2 - c^2)\tan A + (a^2 - b^2 + c^2)\tan B$ is equal to -

(A) $(a^2 + b^2 - c^2)\tan C$ (B) $(a^2 + b^2 + c^2)\tan C$

(C) $(b^2 + c^2 - a^2)\tan C$ (D) none of these

Solution : Using cosine law :

The given expression is equal to $-2bc\cos A\tan A + 2ac\cos B\tan B$

$= 2abc\left(-\frac{\sin A}{a} + \frac{\sin B}{b}\right) = 0$ **Ans. (D)**

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Do yourself - 2 :

(i) If a : b : c = 4 : 5 : 6, then show that $\angle C = 2\angle A$.

(ii) In any ΔABC, prove that :

(a) $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}=\frac{a^2+b^2+c^2}{2abc}$

(b) $\frac{b^2}{a}\cos A+\frac{c^2}{b}\cos B+\frac{a^2}{c}\cos C=\frac{a^4+b^4+c^4}{2abc}$

(iii) In ΔABC, prove that :

(a) $\frac{\cos B}{\cos C}=\frac{c-b\cos A}{b-c\cos A}$ (b) $\sum\frac{\cos A}{c\cos B+b\cos C}=\frac{a^2+b^2+c^2}{2abc}$

(iv) In ΔABC, if A,B,C are in A.P., then prove that $2\cos\frac{A-C}{2}=\frac{a+c}{\sqrt{a^2+c^2-ac}}$.

(v) If in a triangle ABC, $\angle C = 60^\circ$, then prove that $\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$.

(vi) Triangle ABC has BC = 1 and AC = 2. Find the maximum possible value of angle A.

(vii) In a ΔABC, prove that $b^2 \sin 2C + c^2 \sin 2B = 2bc \sin A$.

(viii) In a ΔABC, if $9a^2 + 9b^2 = 19c^2$, then find the value of $\frac{\cot C}{\cot A+\cot B}$.

(ix) In a triangle ABC, the sides a, b, c are roots of $x^3-11x^2+38x-40=0$. If $\sum\left(\frac{\cos A}{a}\right)=\frac{p}{q}$, then find the least value of $(p+q)$ where $p,q \in N$.

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3. PROJECTION FORMULAE :

(a) b cos C + c cos B = a **(b)** c cos A + a cos C = b **(c)** a cos B + b cos A = c

Illustration 6 : In a ΔABC, $c\cos^2\frac{A}{2}+a\cos^2\frac{C}{2}=\frac{3b}{2}$, then show a, b, c are in A.P.

Solution : Here, $\frac{c}{2}(1+\cos A)+\frac{a}{2}(1+\cos C)=\frac{3b}{2}$

$\Rightarrow$ a + c + (c cos A + a cos C) = 3b

$\Rightarrow$ a + c + b = 3b {using projection formula}

$\Rightarrow$ a + c = 2b

which shows a, b, c are in A.P.

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Do yourself - 3 :

(i) In a ΔABC, if $\angle A = \frac{\pi}{4}$, $\angle B = \frac{5\pi}{12}$, show that $a + c\sqrt{2} = 2b$.

(ii) In a ΔABC, prove that : (a) $b(a\cos C - c\cos A) = a^2 - c^2$ (b) $2\left(b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2}\right) = a + b + c$.

(iii) In ΔABC, prove that $\frac{\cos A + \cos C}{a + c} + \frac{\cos B}{b} = \frac{1}{b}$.

(iv) In a ΔABC, prove that :

(a) $a(\cos B + \cos C) = 2(b + c)\sin^2 A/2$

(b) $a(\cos C - \cos B) = 2(b - c)\cos^2 A/2$

(v) In ΔABC, prove that $a(b^2 + c^2)\cos A + b(c^2 + a^2)\cos B + c(a^2 + b^2)\cos C = 3abc$.

4. NAPIER'S ANALOGY (TANGENT RULE) :

(a) $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}$ **(b)** $\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\frac{B}{2}$ **(c)** $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$

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Illustration 7 : In a ΔABC, the tangent of half the difference of two angles is one-third the tangent of half the sum of the angles. Determine the ratio of the sides opposite to the angles.

Solution : Here, $\tan\left(\frac{A-B}{2}\right) = \frac{1}{3}\tan\left(\frac{A+B}{2}\right)$ ..... (i)

using Napier's analogy, $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right)$ ..... (ii)

from (i) & (ii) ;

$$\frac{1}{3}\tan\left(\frac{A+B}{2}\right) = \frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right) \Rightarrow \frac{1}{3}\cot\left(\frac{C}{2}\right) = \frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right)$$

$$\left\{\text{as } A + B + C = \pi \quad \therefore \tan\left(\frac{B+C}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cot\frac{C}{2}\right\}$$

$$\Rightarrow \quad \frac{a-b}{a+b} = \frac{1}{3} \quad \text{or} \quad 3a - 3b = a + b$$

$$2a = 4b \quad \text{or} \quad \frac{a}{b} = \frac{2}{1} \Rightarrow \frac{b}{a} = \frac{1}{2}$$

Thus the ratio of the sides opposite to the angles is b : a = 1 : 2. **Ans.**

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Do yourself - 4 :

(i) In any ΔABC, prove that $\frac{b-c}{b+c}=\frac{\tan\left(\frac{B-C}{2}\right)}{\tan\left(\frac{B+C}{2}\right)}$.

(ii) If ΔABC is right angled at C, prove that : (a) $\tan\frac{A}{2}=\sqrt{\frac{c-b}{c+b}}$ (b) $\sin(A-B)=\frac{a^2-b^2}{a^2+b^2}$.

(iii) If in a ΔABC, $b = 3c$, then find the value of $\cot\frac{A}{2}.\cot\frac{B-C}{2}$.

(iv) In a triangle ABC, $\angle A = 60^o$ and $b:c=\sqrt{3}+1:2$, then find the value of $(\angle B-\angle C)$.

(v) ΔABC with $\angle A$ acute, $b = 2$ and $c=\left(\sqrt{3}-1\right)$ has area $\left(\sqrt{3}-1\right)/2$, then find the measures of angles of triangle.

5. HALF ANGLE FORMULAE :

$s=\frac{a+b+c}{2}$ = semi-perimeter of triangle.

(a) (i) $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}}$ (iii) $\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$

(b) (i) $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$ (ii) $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}}$ (iii) $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$

(c) (i) $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ (ii) $\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$ (iii) $\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

$=\frac{\Delta}{s(s-a)}$ $=\frac{\Delta}{s(s-b)}$ $=\frac{\Delta}{s(s-c)}$

(d) Area of Triangle

$$\Delta=\sqrt{s(s-a)(s-b)(s-c)}=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B=\frac{1}{2}ab\sin C=\frac{1}{2}ap_1=\frac{1}{2}bp_2=\frac{1}{2}cp_3,$$

where p_1, p_2, p_3 are altitudes from vertices A,B,C respectively.

Illustration 8 : If in a triangle ABC, CD is the angle bisector of the angle ACB, then CD is equal to-

(A) $\frac{a+b}{2ab}\cos\frac{C}{2}$ (B) $\frac{2ab}{a+b}\sin\frac{C}{2}$ (C) $\frac{2ab}{a+b}\cos\frac{C}{2}$ (D) $\frac{b\sin\angle DAC}{\sin(B+C/2)}$

Solution : $\Delta CAB = \Delta CAD + \Delta CDB$

$$\Rightarrow \frac{1}{2}ab\sin C=\frac{1}{2}b.CD.\sin\left(\frac{C}{2}\right)+\frac{1}{2}a.CD\sin\left(\frac{C}{2}\right)$$

$$\Rightarrow CD(a+b)\sin\left(\frac{C}{2}\right)=ab\left(2\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)\right)$$

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So $\quad CD = \dfrac{2ab\cos(C/2)}{(a+b)}$

and in ΔCAD, $\dfrac{CD}{\sin\angle DAC} = \dfrac{b}{\sin\angle CDA}$ (by sine rule)

$\Rightarrow \quad CD = \dfrac{b\sin\angle DAC}{\sin(B+C/2)}$ **Ans. (C,D)**

Illustration 9 : If Δ is the area and 2s the sum of the sides of a triangle, then show $\Delta \le \dfrac{s^2}{3\sqrt{3}}$.

Solution : We have, $2s = a + b + c$, $\Delta^2 = s(s-a)(s-b)(s-c)$

Now, A.M. $\ge$ G.M.

$$\frac{(s-a)+(s-b)+(s-c)}{3} \ge \{(s-a)(s-b)(s-c)\}^{1/3}$$

or $\quad \dfrac{3s-2s}{3} \ge \left(\dfrac{\Delta^2}{s}\right)^{1/3}$

or $\quad \dfrac{s}{3} \ge \left(\dfrac{\Delta^2}{s}\right)^{1/3}$

or $\quad \dfrac{\Delta^2}{s} \le \dfrac{s^3}{27} \Rightarrow \Delta \le \dfrac{s^2}{3\sqrt{3}}$ **Ans.**

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Do yourself - 5 :

(i) Given a = 6, b = 8, c = 10. Find

(a) sinA (b) tanA (c) $\sin\dfrac{A}{2}$ (d) $\cos\dfrac{A}{2}$ (e) $\tan\dfrac{A}{2}$ (f) Δ

(ii) Prove that in any ΔABC, (abcs) $\sin\dfrac{A}{2}.\sin\dfrac{B}{2}.\sin\dfrac{C}{2} = \Delta^2$.

(iii) If the cotangents of half the angles of a triangle are in A.P., then prove that the sides are in A.P.

(iv) If in a triangle ABC, $\Delta = a^2 - (b-c)^2$, then find the value of tanA.

(v) In any triangle ABC, prove that $(a+b+c)\left(\tan\dfrac{A}{2} + \tan\dfrac{B}{2}\right) = 2c\cot\dfrac{C}{2}$.

(vi) In a triangle ABC, if $\tan\dfrac{A}{2}.\tan\dfrac{B}{2} = \dfrac{1}{3}$ and ab = 4, then find the least value of c.

(vii) In ΔABC, if a,b,c (taken in that order) are in A.P. then find the value of $\cot\dfrac{A}{2}\ \cot\dfrac{C}{2}$.

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6. m-n THEOREM :

$(m + n) \cot \theta = m \cot \alpha - n \cot \beta$

$(m + n) \cot \theta = n \cot B - m \cot C.$

7. RADIUS OF THE CIRCUMCIRCLE 'R' :

Circumcentre is the point of intersection of perpendicular bisectors of the sides and distance between circumcentre & vertex of triangle is called circumradius 'R'.

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}.$$

8. RADIUS OF THE INCIRCLE 'r' :

Point of intersection of internal angle bisectors is incentre and perpendicular distance of incentre from any side is called inradius 'r'.

$$r = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

$$= a\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = b\frac{\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}} = c\frac{\sin\frac{B}{2}\sin\frac{A}{2}}{\cos\frac{C}{2}}$$

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Illustration 10 : In a triangle ABC, if a : b : c = 4 : 5 : 6, then ratio between its circumradius and inradius is-

(A) $\frac{16}{7}$ (B) $\frac{16}{9}$ (C) $\frac{7}{16}$ (D) $\frac{11}{7}$

Solution : $\frac{R}{r} = \frac{abc}{4\Delta} \Big/ \frac{\Delta}{s} = \frac{(abc)s}{4\Delta^2} \Rightarrow \frac{R}{r} = \frac{abc}{4(s-a)(s-b)(s-c)}$ ....(i)

$\because$ a : b : c = 4 : 5 : 6 $\Rightarrow \frac{a}{4} = \frac{b}{5} = \frac{c}{6} = k$ (say)

$\Rightarrow$ a = 4k, b = 5k, c = 6k

$\therefore$ $s = \frac{a+b+c}{2} = \frac{15k}{2}$, $s - a = \frac{7k}{2}$, $s - b = \frac{5k}{2}$, $s - c = \frac{3k}{2}$

using (i) in these values $\frac{R}{r} = \frac{(4k)(5k)(6k)}{4\left(\frac{7k}{2}\right)\left(\frac{5k}{2}\right)\left(\frac{3k}{2}\right)} = \frac{16}{7}$ **Ans. (A)**

Illustration 11 : If A, B, C are the angles of a triangle, prove that : $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$.

Solution : $\cos A + \cos B + \cos C = 2\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right) + \cos C$

$$= 2\sin\frac{C}{2}.\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\frac{C}{2} = 1 + 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\left(\frac{C}{2}\right)\right]$$

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$$= 1+2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right] \qquad \left\{\because \frac{C}{2}=90°-\left(\frac{A+B}{2}\right)\right\}$$

$$= 1+2\sin\frac{C}{2}.2\sin\frac{A}{2}.\sin\frac{B}{2} = 1+4\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}$$

$$= 1+\frac{r}{R} \qquad \{\text{as, } r = 4R\sin A/2\ .\ \sin B/2\ .\ \sin C/2\}$$

$$\Rightarrow \quad \cos A + \cos B + \cos C = 1+\frac{r}{R}.\ \text{Hence proved.}$$

Do yourself - 6 :

(i) If in ΔABC, $a = 3$, $b = 4$ and $c = 5$, find

(a) Δ (b) R (c) r

(ii) In a ΔABC, show that :

(a) $\dfrac{a^2-b^2}{c} = 2R\sin(A-B)$ (b) $r\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2} = \dfrac{\Delta}{4R}$ (c) $a+b+c = \dfrac{abc}{2Rr}$

(iii) Let Δ & Δ' denote the areas of a Δ and that of its incircle. Prove that

$$\Delta : \Delta' = \left(\cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}\right) : \pi$$

(iv) If the median AD of a ΔABC makes an angle $3\pi/4$ with the side BC, then evaluate $|\cot B - \cot C|$.

(v) If in a ΔABC, AB = 4 cm, AC = 8cm and $\cos\left(\dfrac{B-C}{2}\right) = \dfrac{1}{\sqrt{2}}$; then evaluate $|2\cot B - \cot C|$.

(vi) In ΔABC, if circumradius 'R' and inradius 'r' are connected by relation $R^2 - 4Rr + 8r^2 - 12r + 9 = 0$, then find semiperimeter of the ΔABC.

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9. RADII OF THE EX-CIRCLES :

Point of intersection of two external angles and one internal angle bisectors is excentre and perpendicular distance of excentre from any side is called exradius. If r_1 is the radius of escribed circle opposite to $\angle A$ of ΔABC and so on, then -

(a) $$r_1 = \frac{\Delta}{s-a} = s\tan\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$$

(b) $$r_2 = \frac{\Delta}{s-b} = s\tan\frac{B}{2} = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} = \frac{b\cos\frac{A}{2}\cos\frac{C}{2}}{\cos\frac{B}{2}}$$

(c) $$r_3 = \frac{\Delta}{s-c} = s\tan\frac{C}{2} = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} = \frac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$$

I_1, I_2 and I_3 are taken as ex-centre opposite to vertex A, B, C repsectively.

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Illustration 12 : Value of the expression $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}$ is equal to -

(A) 1 (B) 2 (C) 3 (D) 0

Solution : $\frac{(b-c)}{r_1}+\frac{(c-a)}{r_2}+\frac{(a-b)}{r_3}$

$$\Rightarrow \quad (b-c)\left(\frac{s-a}{\Delta}\right)+(c-a)\left(\frac{s-b}{\Delta}\right)+(a-b).\left(\frac{s-c}{\Delta}\right)$$

$$\Rightarrow \quad \frac{(s-a)(b-c)+(s-b)(c-a)+(s-c)(a-b)}{\Delta}$$

$$=\frac{s(b-c+c-a+a-b)-[ab-ac+bc-ba+ac-bc]}{\Delta}=\frac{0}{\Delta}=0$$

Thus, $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}=0$ **Ans. (D)**

Illustration 13 : If $r_1 = r_2 + r_3 + r$, prove that the triangle is right angled.

Solution : We have, $r_1 - r = r_2 + r_3$

$$\Rightarrow \quad \frac{\Delta}{s-a}-\frac{\Delta}{s}=\frac{\Delta}{s-b}+\frac{\Delta}{s-c} \qquad \Rightarrow \quad \frac{s-s+a}{s(s-a)}=\frac{s-c+s-b}{(s-b)(s-c)}$$

$$\Rightarrow \quad \frac{a}{s(s-a)}=\frac{2s-(b+c)}{(s-b)(s-c)} \qquad \{as,\ 2s = a + b + c\}$$

$$\Rightarrow \quad \frac{a}{s(s-a)}=\frac{a}{(s-b)(s-c)} \qquad \Rightarrow \quad s^2 - (b + c)\, s + bc = s^2 - as$$

$$\Rightarrow \quad s(-a + b + c) = bc \qquad \Rightarrow \quad \frac{(b+c-a)(a+b+c)}{2}=bc$$

$$\Rightarrow \quad (b + c)^2 - (a)^2 = 2bc \qquad \Rightarrow \quad b^2 + c^2 + 2bc - a^2 = 2bc$$

$$\Rightarrow \quad b^2 + c^2 = a^2$$

$$\therefore \quad \angle A = 90°.$$ **Ans.**

Do yourself - 7 :

(i) In an equilateral ΔABC, $R = 2$, find

(a) r (b) r_1 (c) a

(ii) In a ΔABC, show that :

(a) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$ (b) $\frac{1}{4}r^2s^2\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r}-\frac{1}{r_3}\right)=R$

(c) $\sqrt{rr_1r_2r_3}=\Delta$ (d) $r_1 + r_2 + r_3 - r = 4R$

(e) $\frac{r_1+r_2}{1+\cos C}=2R$

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10. ANGLE BISECTORS & MEDIANS :

An angle bisector divides the base in the ratio of corresponding sides.

$$\frac{BD}{CD}=\frac{c}{b} \quad \Rightarrow \quad BD=\frac{ac}{b+c} \quad \& \quad CD=\frac{ab}{b+c}$$

If m_a and β_a are the lengths of a median and an angle bisector from the angle A then,

$$m_a=\frac{1}{2}\sqrt{2b^2+2c^2-a^2} \text{ and } \beta_a=\frac{2bc\cos\frac{A}{2}}{b+c}$$

Note that $m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2)$

11. ORTHOCENTRE :

(a) Point of intersection of altitudes is orthocentre & the triangle KLM which is formed by joining the feet of the altitudes is called the orthic triangle (special case of pedal triangle).

(b) The distances of the orthocentre from the angular points of the ΔABC are 2R cosA, 2R cosB, & 2R cosC.

(c) The distance of P from sides are 2R cosB cosC, 2R cosC cosA and 2R cosA cosB.

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Do yourself - 8 :

(i) If x, y, z are the distance of the vertices of ΔABC respectively from the orthocentre, then prove that $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{xyz}$.

(ii) If p_1, p_2, p_3 are respectively the perpendiculars from the vertices of a triangle to the opposite sides, prove that

(a) $p_1p_2p_3=\frac{a^2b^2c^2}{8R^3}$ (b) $\Delta=\sqrt{\frac{1}{2}Rp_1p_2p_3}$

(iii) In a ΔABC, AD is altitude and H is the orthocentre prove that AH : DH = (tanB + tanC) : tanA.

(iv) In a ΔABC, the lengths of the bisectors of the angle A, B and C are x, y, z respectively. Show that $\frac{1}{x}\cos\frac{A}{2}+\frac{1}{y}\cos\frac{B}{2}+\frac{1}{z}\cos\frac{C}{2}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

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12. THE DISTANCES BETWEEN THE SPECIAL POINTS :

(a) The distance between circumcentre and orthocentre is $= R\sqrt{1-8\cos A\cos B\cos C}$

(b) The distance between circumcentre and incentre is $= \sqrt{R^2-2Rr}$

(c) The distance between incentre and orthocentre is $=\sqrt{2r^2-4R^2\cos A\cos B\cos C}$

(d) The distances between circumcentre & excentres are

$$OI_1 = R\sqrt{1+8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \sqrt{R^2+2Rr_1} \text{ \& so on.}$$

Illustration 14 : Prove that the distance between the circumcentre and the orthocentre of a triangle ABC is $R\sqrt{1-8\cos A\cos B\cos C}$.

Solution : Let O and P be the circumcentre and the orthocentre respectively. If OF is the perpendicular to AB, we have $\angle OAF = 90° - \angle AOF = 90° - C$. Also $\angle PAL = 90° - C$.

Hence, $\angle OAP = A - \angle OAF - \angle PAL = A - 2(90° - C) = A + 2C - 180°$

$= A + 2C - (A + B + C) = C - B$.

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Also OA = R and PA = 2RcosA.

Now in ΔAOP,

$OP^2 = OA^2 + PA^2 - 2OA.\ PA\cos OAP$

$= R^2 + 4R^2\cos^2 A - 4R^2\cos A\cos(C - B)$

$= R^2 + 4R^2\cos A[\cos A - \cos(C - B)]$

$= R^2 - 4R^2\cos A[\cos(B + C) + \cos(C - B)] = R^2 - 8R^2\cos A\cos B\cos C$.

Hence $OP = R\sqrt{1-8\cos A\cos B\cos C}$. **Ans.**

13. SOLUTION OF TRIANGLES :

The three sides a,b,c and the three angles A,B,C are called the elements of the triangle ABC. When any three of these six elements (except all the three angles) of a triangle are given, the triangle is known completely; that is the other three elements can be expressed in terms of the given elements and can be evaluated. This process is called the solution of triangles.

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* If the three sides a,b,c are given, angle A is obtained from $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

or $\cos A=\frac{b^2+c^2-a^2}{2bc}$. B and C can be obtained in the similar way.

* If two sides b and c and the included angle A are given, then $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$ gives $\frac{B-C}{2}$.

Also $\frac{B+C}{2}=90°-\frac{A}{2}$, so that B and C can be evaluated. The third side is given by

$$a=b\,\frac{\sin A}{\sin B}$$

or $a^2=b^2+c^2-2bc\cos A$.

* If two sides b and c and an angle opposite the one of them (say B) are given then

$\sin C=\frac{c}{b}\sin B$, $A=180°-(B+C)$ and $a=\frac{b\sin A}{\sin B}$ given the remaining elements.

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Case I :

$b<c\sin B$.

We draw the side c and angle B. Now it is obvious from the figure that there is no triangle possible.

Case II :

$b=c\sin B$ and B is an acute angle, there is only one triangle possible.

and it is right-angled at C.

Case III :

$b>c\sin B$, $b<c$ and B is an acute angle, then there are two triangles possible for two values of angle C.

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Case IV :

b > c sin B, c < b and B is an acute angle, then there is only one triangle.

Case V :

b > c sin B, c > b and B is an obtuse angle.

For any choice of point C, b will be greater than c which is a contradication as c > b (given). So there is no triangle possible.

Case VI :

b > c sin B, c < b and B is an obtuse angle.

We can see that the circle with A as centre and b as radius will cut the line only in one point. So only one triangle is possible.

Case VII :

b > c and B = 90°.

Again the circle with A as centre and b as radius will cut the line only in one point. So only one triangle is possible.

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Case VIII :

b ≤ c and B = 90°.

The circle with A as centre and b as radius will not cut the line in any point. So no triangle is possible.

This is, sometimes, called an ambiguous case.

Alternative Method :

By applying cosine rule, we have $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\Rightarrow \quad a^2 - (2c \cos B)a + (c^2 - b^2) = 0 \Rightarrow a = c \cos B \pm \sqrt{(c\cos B)^2 - (c^2 - b^2)}$

$\Rightarrow \quad a = c \cos B \pm \sqrt{b^2 - (c \sin B)^2}$

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This equation leads to following cases :

Case-I : If $b < c\sin B$, no such triangle is possible.

Case-II: Let $b = c\sin B$. There are further following case :

(a) B is an obtuse angle $\Rightarrow$ cosB is negative. There exists no such triangle.

(b) B is an acute angle $\Rightarrow$ cosB is positive. There exists only one such triangle.

Case-III: Let $b > c\sin B$. There are further following cases :

(a) B is an acute angle $\Rightarrow$ cosB is positive. In this case triangle will exist if and only if

$c\cos B > \sqrt{b^2 - (c\sin B)^2}$ or $c > b \Rightarrow$ Two such triangle is possible. If $c < b$, only one such triangle is possible.

(b) B is an obtuse angle $\Rightarrow$ cosB is negative. In this case triangle will exist if and only if

$\sqrt{b^2 - (c\sin B)^2} > |c\cos B| \Rightarrow b > c$. So in this case only one such triangle is possible. If $b < c$ there exists no such triangle.

This is called an ambiguous case.

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* If one side a and angles B and C are given, then $A = 180° - (B + C)$, and $b = \frac{a\sin B}{\sin A}, c = \frac{a\sin C}{\sin A}$.

* If the three angles A,B,C are given, we can only find the ratios of the sides a,b,c by using sine rule (since there are infinite similar triangles possible).

Illustration 15 : In the ambiguous case of the solution of triangles, prove that the circumcircles of the two triangles are of same size.

Solution : Let us say b,c and angle B are given in the ambiguous case. Both the triangles will have b and its opposite angle as B. so $\frac{b}{\sin B} = 2R$ will be given for both the triangles. So their circumradii and therefore their sizes will be same.

Illustration 16 : If a,b and A are given in a triangle and c_1, c_2 are the possible values of the third side, prove that $c_1^2 + c_2^2 - 2c_1c_2\cos 2A = 4a^2\cos^2 A$.

Solution : $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow c^2 - 2bc\cos A + b^2 - a^2 = 0.$

$c_1 + c_2 = 2b\cos A$ and $c_1c_2 = b^2 - a^2$.

$\Rightarrow c_1^2 + c_2^2 - 2c_1c_2\cos 2A = (c_1 + c_2)^2 - 2c_1c_2(1 + \cos 2A)$

$= 4b^2\cos^2 A - 2(b^2 - a^2)2\cos^2 A = 4a^2\cos^2 A.$

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Illustration 17 : If b,c,B are given and b < c, prove that $\cos\left(\frac{A_1 - A_2}{2}\right) = \frac{c\sin B}{b}$.

Solution : $\angle C_2AC_1$ is bisected by AD.

$\Rightarrow$ In ΔAC_2D, $\cos\left(\frac{A_1 - A_2}{2}\right) = \frac{AD}{AC_2} = \frac{c\sin B}{b}$

Hence proved.

Do yourself - 9 :

(i) In triangle ABC b,c,B are given and b < c. If third side has two values a_1 and a_2 ($a_1 > a_2$). where A_1 is angle opposite to side a_1 and A_2 is angle opposite to side a_2 then prove that $\sin\left(\frac{A_1 - A_2}{2}\right) = \frac{a_1 - a_2}{2b}$.

(ii) In a ΔABC, b,c,B (c > b) are gives. If the third side has two values a_1 and a_2 such that $a_1 = 3a_2$, show that $\sin B = \sqrt{\frac{4b^2 - c^2}{3c^2}}$.

(iii) In ΔABC, sides b, c and the angle B are given such that a has two values a_1 and a_2. Then prove that $|a_1 - a_2| = 2\sqrt{b^2 - c^2\sin^2 B}$.

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14. REGULAR POLYGON :

A regular polygon has all its sides equal. It may be inscribed or circumscribed.

(a) Inscribed in circle of radius r :

(i) $a = 2h\tan\frac{\pi}{n} = 2r\sin\frac{\pi}{n}$

(ii) Perimeter (P) and area (A) of a regular polygon of n sides inscribed in a circle of radius r are given by $P = 2nr\sin\frac{\pi}{n}$ and $A = \frac{1}{2}nr^2\sin\frac{2\pi}{n}$

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(b) Circumscribed about a circle of radius r :

(i) $a = 2r \tan \frac{\pi}{n}$

(ii) Perimeter (P) and area (A) of a regular polygon of n sides circumscribed about a given circle of radius r is given by $P = 2nr \tan \frac{\pi}{n}$ and

$A = nr^2 \tan \frac{\pi}{n}$

Do yourself - 10 :

(i) If the perimeter of a circle and a regular polygon of n sides are equal, then prove that $\frac{\text{area of the circle}}{\text{area of polygon}} = \frac{\tan \frac{\pi}{n}}{\frac{\pi}{n}}$.

(ii) The ratio of the area of n-sided regular polygon, circumscribed about a circle, to the area of the regular polygon of equal number of sides inscribed in the circle is 4 : 3. Find the value of n.

(iii) A regular pentagon and a regular decagon have the same perimeter, prove that their areas are in the ratio $2 : \sqrt{5}$.

(iv) If the area of circle is A_1 and area of regular pentagon inscribed in the circle is A_2, then find the ratio of areas A_1/A_2.

(v) Consider a quadrilateral ABCD in which AB = a, BC = b, CD = c and DA = d. A circle is inscibed in it, and another circle is circumscribed about it, then show that $\cos A = \frac{ad - bc}{ad + bc}$.

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15. SOME NOTES :

(a) (i) If $a \cos B = b \cos A$, then the triangle is isosceles.

(ii) If $a \cos A = b \cos B$, then the triangle is isosceles or right angled.

(b) In right angle triangle

(i) $a^2 + b^2 + c^2 = 8R^2$ (ii) $\cos^2 A + \cos^2 B + \cos^2 C = 1$

(c) In equilateral triangle

(i) $R = 2r$ (ii) $r_1 = r_2 = r_3 = \frac{3R}{2}$

(iii) $r : R : r_1 = 1 : 2 : 3$ (iv) area $= \frac{\sqrt{3}a^2}{4}$ (v) $R = \frac{a}{\sqrt{3}}$

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(d) (i) The circumcentre lies (1) inside an acute angled triangle (2) outside an obtuse angled triangle & (3) mid point of the hypotenuse of right angled triangle.

(ii) The orthocentre of right angled triangle is the vertex at the right angle.

(iii) The orthocentre, centroid & circumcentre are collinear & centroid divides the line segment joining orthocentre & circumcentre internally in the ratio 2 : 1 except in case of equilateral triangle. In equilateral triangle, all these centres coincide

(e) Area of a cyclic quadrilateral $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$

where a, b, c, d are lengths of the sides of quadrilateral and $s = \frac{a+b+c+d}{2}$.

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ANSWERS FOR DO YOURSELF

1 : **(i)** 90° **(vii)** 60° **(ix)** 45°,60°,75°

2 : **(viii)** $\frac{5}{9}$ **(ix)** 25

4 : **(iii)** 2 **(iv)** 30° **(v)** $\angle A = 30^\circ$, $\angle B = 135^\circ$, $\angle C = 15^\circ$

5 : **(i)** **(a)** $\frac{3}{5}$ **(b)** $\frac{3}{4}$ **(c)** $\frac{1}{\sqrt{10}}$ **(d)** $\frac{3}{\sqrt{10}}$ **(e)** $\frac{1}{3}$ **(f)** 24

(iv) $\frac{8}{15}$ **(vi)** 2 **(vii)** 3

6 : **(i)** **(a)** 6 **(b)** $\frac{5}{2}$ **(c)** 1 **(iv)** 2 **(v)** 3 **(vi)** $\frac{9\sqrt{3}}{2}$

7 : **(i)** **(a)** 1 **(b)** 3 **(c)** $2\sqrt{3}$

10 : **(ii)** 6 **(iv)** $\frac{8\pi}{5\sqrt{10+2\sqrt{5}}}$

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EXERCISE (O-1)

1. In triangle ABC, if $\sin^3 A + \sin^3 B + \sin^3 C = 3\sin A.\sin B.\sin C$, then triangle is

(A) obtuse angled (B) right angled (C) obtuse right angled (D) equilateral

TS0021

2. A triangle has vertices A, B and C, and the respective opposite sides have lengths a, b and c. This triangle is inscribed in a circle of radius R. If $b = c = 1$ and the altitude from A to side BC has length $\sqrt{\frac{2}{3}}$, then R equals

(A) $\frac{1}{\sqrt{3}}$ (B) $\frac{2}{\sqrt{3}}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{\sqrt{3}}{2\sqrt{2}}$

TS0022

3. In a triangle ABC, if $\angle C = 105°$, $\angle B = 45°$ and length of side AC = 2 units, then the length of the side AB is equal to

(A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) $\sqrt{2}+1$ (D) $\sqrt{3}+1$

TS0023

4. In a triangle ABC, if a = 13, b = 14 and c = 15, then angle A is equal to
(All symbols used have their usual meaning in a triangle.)

(A) $\sin^{-1}\frac{4}{5}$ (B) $\sin^{-1}\frac{3}{5}$ (C) $\sin^{-1}\frac{3}{4}$ (D) $\sin^{-1}\frac{2}{3}$

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TS0024

5. In triangle ABC, if AC = 8, BC = 7 and D lies between A and B such that AD = 2, BD = 4, then the length CD equals

(A) $\sqrt{46}$ (B) $\sqrt{48}$ (C) $\sqrt{51}$ (D) $\sqrt{75}$

TS0025

6. In triangle ABC, if $2b = a + c$ and $A - C = 90°$, then sin B equals
[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) $\frac{\sqrt{7}}{5}$ (B) $\frac{\sqrt{5}}{8}$ (C) $\frac{\sqrt{7}}{4}$ (D) $\frac{\sqrt{5}}{3}$

TS0026

7. In a triangle ABC, $a^3 + b^3 + c^3 = c^2(a + b + c)$
(All symbol used have usual meaning in a triangle.)
Statement–1: The value of $\angle C = 60°$.
Statement –2: ΔABC must be equilateral.
(A) Statement–1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D) Statement-1 is false, statement-2 is true.

TS0027

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8. A circle is inscribed in a right triangle ABC, right angled at C. The circle is tangent to the segment AB at D and length of segments AD and DB are 7 and 13 respectively. Area of triangle ABC is equal to

(A) 91 (B) 96 (C) 100 (D) 104

TS0028

9. In triangle ABC, If $\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$ then angle C is equal to

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) 30° (B) 45° (C) 60° (D) 90°

TS0029

10. In a triangle ABC, if $b=(\sqrt{3}-1)a$ and $\angle C=30°$, then the value of $(A-B)$ is equal to

(All symbols used have usual meaning in a triangle.)

(A) 30° (B) 45° (C) 60° (D) 75°

TS0030

11. Angles A, B and C of a triangle ABC are in A.P. If $\frac{b}{c}=\sqrt{\frac{3}{2}}$, then $\angle A$ is equal to

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{5\pi}{12}$ (D) $\frac{\pi}{2}$

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TS0001

12. The sides a, b, c (taken in that order) of triangle ABC are in A.P.

If $\cos\alpha=\frac{a}{b+c}, \cos\beta=\frac{b}{c+a}, \cos\gamma=\frac{c}{a+b}$ then $\tan^2\left(\frac{\alpha}{2}\right)+\tan^2\left(\frac{\gamma}{2}\right)$ is equal to

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{2}{3}$

TS0032

13. AD and BE are the medians of a triangle ABC. If $AD=4$, $\angle DAB=\frac{\pi}{6}$, $\angle ABE=\frac{\pi}{3}$, then area of triangle ABC equals

(A) $\frac{8}{3}$ (B) $\frac{16}{3}$ (C) $\frac{32}{3}$ (D) $\frac{32}{9}\sqrt{3}$

TS0033

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14. In a triangle ABC, if $(a+b+c)(a+b-c)(b+c-a)(c+a-b) = \frac{8a^2b^2c^2}{a^2+b^2+c^2}$, then the triangle is

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) isosceles (B) right angled (C) equilateral (D) obtuse angled

TS0034

15. For right angled isosceles triangle, $\frac{r}{R} =$

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) $\tan\frac{\pi}{12}$ (B) $\cot\frac{\pi}{12}$ (C) $\tan\frac{\pi}{8}$ (D) $\cot\frac{\pi}{8}$

TS0035

16. If K is a point on the side BC of an equilateral triangle ABC and if $\angle BAK = 15°$, then the ratio of lengths $\frac{AK}{AB}$ is

(A) $\frac{3\sqrt{2}\left(3+\sqrt{3}\right)}{2}$ (B) $\frac{\sqrt{2}\left(3+\sqrt{3}\right)}{2}$ (C) $\frac{\sqrt{2}\left(3-\sqrt{3}\right)}{2}$ (D) $\frac{3\sqrt{2}\left(3-\sqrt{3}\right)}{2}$

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TS0002

17. Let ABC be a right triangle with length of side AB = 3 and hypotenuse AC = 5.

If D is a point on BC such that $\frac{BD}{DC} = \frac{AB}{AC}$, then AD is equal to

(A) $\frac{4\sqrt{3}}{3}$ (B) $\frac{3\sqrt{5}}{2}$ (C) $\frac{4\sqrt{5}}{3}$ (D) $\frac{5\sqrt{3}}{4}$

TS0003

18. In ΔABC, if a = 2b and A = 3B, then the value of $\frac{c}{b}$ is equal to

(A) 3 (B) $\sqrt{2}$ (C) 1 (D) $\sqrt{3}$

TS0004

19. If the angle A, B and C of a triangle are in an arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of expression $E = \left(\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A\right)$, is

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) 1 (D) $\sqrt{3}$

TS0005

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20. The ratio of the sides of a triangle ABC is $1 : \sqrt{3} : 2$. Then ratio of A : B : C is

(A) 3 : 5 : 2 (B) $1 : \sqrt{3} : 2$ (C) 3 : 2 : 1 (D) 1 : 2 : 3

TS0006

21. If in a triangle sin A : sin C = sin (A – B) : sin (B – C), then a^2, b^2, c^2

(A) are in A.P. (B) are in G.P. (C) are in H.P. (D) none of these

TS0007

22. In a triangle tan A : tan B : tan C = 1 : 2 : 3, then $a^2 : b^2 : c^2$ equals

(A) 5 : 8 : 9 (B) 5 : 8 : 12 (C) 3 : 5 : 8 (D) 5 : 8 : 10

TS0008

23. In an acute triangle ABC, $\angle ABC = 45°$, AB = 3 and AC = $\sqrt{6}$. The angle $\angle BAC$, is

(A) 60° (B) 65° (C) 75° (D) 15° or 75°

TS0009

24. Consider a triangle ABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C respectively. If a = 1, b = 3 and C = 60°, then $\sin^2 B$ is equal to

(A) $\frac{27}{28}$ (B) $\frac{3}{28}$ (C) $\frac{81}{28}$ (D) $\frac{1}{3}$

TS0010

25. In triangle ABC, If $s = 3 + \sqrt{3} + \sqrt{2}$, 3B – C = 30°, A + 2B = 120°, then the length of longest side of triangle is

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) 2 (B) $2\sqrt{2}$ (C) $2(\sqrt{3}+1)$ (D) $\sqrt{3}-1$

TS0011

26. In ΔABC if a = 8, b = 9, c = 10, then the value of $\frac{\tan C}{\sin B}$ is

(A) $\frac{32}{9}$ (B) $\frac{24}{7}$ (C) $\frac{21}{4}$ (D) $\frac{18}{5}$

TS0012

27. If the sides of a triangle are $\sin\alpha, \cos\alpha, \sqrt{1+\sin\alpha\cos\alpha}, 0 < \alpha < \frac{\pi}{2}$, the largest angle is

(A) 60° (B) 90° (C) 120° (D) 150°

TS0013

28. In a triangle ABC, $\angle A = 60°$ and $b : c = \left(\sqrt{3}+1\right) : 2$ then $(\angle B - \angle C)$ has the value equal to

(A) 15° (B) 30° (C) 22.5 ° (D) 45°

TS0014

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29. In triangle ABC, if $\Delta = a^2 - (b-c)^2$, then $\tan A =$

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) $\frac{15}{16}$ (B) $\frac{1}{2}$ (C) $\frac{8}{17}$ (D) $\frac{8}{15}$

TS0015

30. In triangle ABC, if $\cot\frac{A}{2} = \frac{b+c}{a}$, then triangle ABC must be

[Note: All symbols used have usual meaning in ΔABC.]

(A) isosceles (B) equilateral

(C) right angled (D) isoceles right angled

TS0016

EXERCISE (O-2)

Multiple Correct Answer Type :

1. Given an acute triangle ABC such that $\sin C = \frac{4}{5}$, $\tan A = \frac{24}{7}$ and AB = 50. Then-

(A) centroid, orthocentre and incentre of ΔABC are collinear

(B) $\sin B = \frac{4}{5}$

(C) $\sin B = \frac{4}{7}$

(D) area of $\Delta ABC = 1200$

TS0036

2. In ΔABC, angle A is 120°, BC + CA = 20 and AB + BC = 21, then

(A) AB > AC (B) AB < AC

(C) ΔABC is isosceles (D) area of $\Delta ABC = 14\sqrt{3}$

TS0037

3. In which of the following situations, it is possible to have a triangle ABC?

(All symbols used have usual meaning in a triangle.)

(A) $(a + c - b)(a - c + b) = 4bc$ (B) $b^2 \sin 2C + c^2 \sin 2B = ab$

(C) $a = 3, b = 5, c = 7$ and $C = \frac{2\pi}{3}$ (D) $\cos\left(\frac{A-C}{2}\right) = \cos\left(\frac{A+C}{2}\right)$

TS0038

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4. If the lengths of the medians AD,BE and CF of triangle ABC are 6, 8,10 respectively, then-

(A) AD & BE are perpendicular
(B) BE and CF are perpendicular
(C) area of $\Delta ABC = 32$
(D) area of $\Delta DEF = 8$

TS0039

5. In ΔABC, angle A, B and C are in the ratio 1 : 2 : 3, then which of the following is (are) correct? (All symbol used have usual meaning in a triangle.)

(A) Circumradius of $\Delta ABC = c$
(B) $a : b : c = 1 : \sqrt{3} : 2$
(C) Perimeter of $\Delta ABC = 3 + \sqrt{3}$
(D) Area of $\Delta ABC = \frac{\sqrt{3}}{8} c^2$

TS0040

6. In a triangle ABC, let BC = 1, AC = 2 and measure of angle C is 30°. Which of the following statement(s) is (are) correct?

(A) $2 \sin A = \sin B$
(B) Length of side AB equals $5 - 2\sqrt{3}$
(C) Measure of angle A is less than 30°
(D) Circumradius of triangle ABC is equal to length of side AB

TS0041

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7. In a triangle ABC, if $\cos A \cos 2B + \sin A \sin 2B \sin C = 1$, then

(A) A,B,C are in A.P.
(B) B,A,C are in A.P.
(C) $\frac{r}{R} = 2$
(D) $\frac{r}{R} = \sqrt{2} \sin \frac{\pi}{12}$

TS0042

8. In a triangle ABC, let $2a^2 + 4b^2 + c^2 = 2a(2b + c)$, then which of the following holds good?
[**Note:** All symbols used have usual meaning in a triangle.]

(A) $\cos B = \frac{-7}{8}$
(B) $\sin (A - C) = 0$
(C) $\frac{r}{r_1} = \frac{1}{5}$
(D) $\sin A : \sin B : \sin C = 1 : 2 : 1$

TS0043

9. In a triangle ABC, which of the following quantities denote the area of the triangle?

(A) $\frac{a^2 - b^2}{2}\left(\frac{\sin A \sin B}{\sin(A - B)}\right)$
(B) $\frac{r_1 r_2 r_3}{\sqrt{\sum r_1 r_2}}$
(C) $\frac{a^2 + b^2 + c^2}{\cot A + \cot B + \cot C}$
(D) $r^2 \cot \frac{A}{2} \cdot \cot \frac{B}{2} \cot \frac{C}{2}$

TS0044

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10. Let one angle of a triangle be 60°, the area of triangle is $10\sqrt{3}$ and perimeter is 20 cm. If a > b > c where a, b and c denote lengths of sides opposite to vertices A, B and C respectively, then which of the following is (are) correct?

(A) Inradius of triangle is $\sqrt{3}$

(B) Length of longest side of triangle is 7

(C) Circumradius of triangle is $\frac{7}{\sqrt{3}}$

(D) Radius of largest escribed circle is $\frac{1}{12}$

TS0045

11. In triangle ABC, let b = 10, c = $10\sqrt{2}$ and R = $5\sqrt{2}$ then which of the following statement(s) is (are) correct?

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) Area of triangle ABC is 50.

(B) Distance between orthocentre and circumcentre is $5\sqrt{2}$

(C) Sum of circumradius and inradius of triangle ABC is equal to 10

(D) Length of internal angle bisector of ∠ACB of triangle ABC is $\frac{5}{2\sqrt{2}}$

TS0046

12. In a triangle ABC, if a = 4, b = 8 and ∠C = 60°, then which of the following relations is (are) correct?

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) The area of triangle ABC is $8\sqrt{3}$

(B) The value of $\sum \sin^2 A = 2$

(C) Inradius of triangle ABC is $\frac{2\sqrt{3}}{3+\sqrt{3}}$

(D) The length of internal angle bisector of angle C is $\frac{4}{\sqrt{3}}$

TS0047

13. In a triangle ABC, ∠A = 30°, b = 6. Let CB_1 and CB_2 are least and greatest integral value of side a for each of which two triangles can be formed. If it is also given angle B_1 is obtuse and angle B_2 is acute angle, then (All symbols used have usual meaning in a triangle.)

(A) $|CB_1 - CB_2| = 1$

(B) $CB_1 + CB_2 = 9$

(C) area of $\Delta B_1CB_2 = 6 + \frac{3}{2}\sqrt{7}$

(D) area of $\Delta AB_2C = 6 + \frac{9}{2}\sqrt{3}$

TS0048

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14. Let P be an interior point of ΔABC.

Match the correct entries for the ratios of the Area of ΔPBC : Area of ΔPCA : Area of ΔPAB depending on the position of the point P w.r.t. Δ ABC.

Column-I	Column-II
(A) If P is centroid (G)	(P) tanA : tanB : tanC
(B) If P is incentre (I)	(Q) sin2A : sin2B : sin2C
(C) If P is orthocentre (H)	(R) sinA : sinB : sinC
(D) If P is circumcentre	(S) 1 : 1 : 1
	(T) cos A : cosB : cosC

TS0049

15. As shown in the figure AD is the altitude on BC and AD produced meets the circumcircle of ΔABC at P where DP = x. Similarly EQ = y and FR = z. If a, b, c respectively denotes the sides BC, CA and AB then $\frac{a}{2x}+\frac{b}{2y}+\frac{c}{2z}$ has the value equal to

(A) tanA + tanB + tanC (B) cotA + cotB + cotC

(C) cosA + cosB + cosC (D) cosecA + cosecB + cosecC

TS0050

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Linked Comprehension Type :

Paragraph for Question 16 to 18

If the circumradius of a triangle ABC is $\sqrt{3}$ and the triangle is such that it's perimeter is maximum. On the basis of above information, answer the following questions :

16. Perimeter of ΔABC is -

(A) $4\sqrt{3}$ (B) 18 (C) 9 (D) $6\sqrt{3}$

TS0100

17. A point P lie inside the triangle ABC such that it is equidistant from all the sides then the length of AP is -

(A) 1 (B) 2 (C) $\sqrt{3}$ (D) $2\sqrt{3}$

TS0101

18. If I_1, I_2 and I_3 are excentres of triangle ABC, then the area of $\Delta I_1I_2I_3$ is -

(A) $\frac{9}{4}\sqrt{3}$ (B) $12\sqrt{3}$ (C) $9\sqrt{3}$ (D) 18

TS0102

Paragraph for Question 19 to 21

Consider a ΔABC, where x, y, z are the length of perpendicular drawn from the vertices of the triangle to the opposite sides a, b, c respectively let the letters R, r, s, Δ denote the circumradius, inradius, semiperimeter and area of the triangle respectively.

On the basis of above information answer the following :

19. If $\frac{bx}{c}+\frac{cy}{a}+\frac{az}{b}=\frac{a^2+b^2+c^2}{k}$ then the value of k is -

(A) R (B) s (C) 2R (D) 3/2 R

TS0103

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20. If $\cot A + \cot B + \cot C = k\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)$ then the value of k is -

(A) R^2 (B) rR (C) Δ (D) $a^2 + b^2 + c^2$

TS0104

21. The value of $\frac{c\sin B + b\sin C}{x} + \frac{a\sin C + c\sin A}{y} + \frac{b\sin A + a\sin B}{z}$ is equal to -

(A) $\frac{R}{r}$ (B) $\frac{s}{R}$ (C) 2 (D) 6

TS0105

Numerical Grid Type :

22. In triangle ABC, a = 7, b = 6, c = 5. Incircle touches the sides BC, AC and AB at D, E and F respectively and K, L, M are the feet of perpendiculars from circumcentre to sides BC, AC and AB respectively, then the value of DK + EL + FM is **TS0106**

23. If sides of a triangle ABC are given by 4,7,7 then length of the tangent drawn from vertex A to the ex-circle opposite to vertex A is equal to **TS0107**

24. If f, g and h are the lengths of the perpendiculars from the circumcentre on the sides a, b and c of a triangle ABC respectively then $\frac{a}{f}+\frac{b}{g}+\frac{c}{h} = K\frac{abc}{fgh}$ where K has the value equal to _____.

TS0108

25. In ΔABC with usual notations, if a = 3, b = 2 and $\cos(A - B) = \frac{12}{13}$, then area of ΔABC is

TS0109

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EXERCISE (S-1)

1. If a,b,c are the sides of triangle ABC satisfying $\log\left(1+\frac{c}{a}\right)+\log a - \log b = \log 2$.
Also $a(1-x^2)+2bx+c(1+x^2)=0$ has two equal roots. Find the value of sinA + sinB + sinC.

TS0051

2. Given a triangle ABC with sides a = 7, b = 8 and c = 5. If the value of the expression $\left(\sum \sin A\right)\left(\sum \cot\frac{A}{2}\right)$ can be expressed in the form $\frac{p}{q}$ where p, q $\in$ N and $\frac{p}{q}$ is in its lowest form find the value of (p + q).

TS0052

3. If two times the square of the diameter of the circumcircle of a triangle is equal to the sum of the squares of its sides then prove that the triangle is right angled.

TS0053

4. With usual notations, prove that in a triangle ABC

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2} = \frac{s^2}{\Delta}$$

TS0054

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5. With usual notations, prove that in a triangle ABC

$a \cot A + b \cot B + c \cot C = 2(R + r)$

TS0055

6. With usual notations, prove that in a triangle ABC

$Rr(\sin A + \sin B + \sin C) = \Delta$

TS0056

7. With usual notations, prove that in a triangle ABC

$\cot A + \cot B + \cot C = \dfrac{a^2 + b^2 + c^2}{4\Delta}$

TS0057

8. In acute angled triangle ABC, a semicircle with radius r_a is constructed with its base on BC and tangent to the other two sides. r_b and r_c are defined similarly. If r is the radius of the incircle of triangle ABC then prove that, $\dfrac{2}{r} = \dfrac{1}{r_a} + \dfrac{1}{r_b} + \dfrac{1}{r_c}$.

TS0058

9. If the length of the perpendiculars from the vertices of a triangle A, B, C on the opposite sides are p_1, p_2, p_3 then prove that $\dfrac{1}{p_1} + \dfrac{1}{p_2} + \dfrac{1}{p_3} = \dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3}$.

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TS0059

10. With usual notations, prove that in a triangle ABC

$\dfrac{b-c}{r_1} + \dfrac{c-a}{r_2} + \dfrac{a-b}{r_3} = 0$

TS0060

EXERCISE (S-2)

1. With usual notations, prove that in a triangle ABC

$\dfrac{r_1}{(s-b)(s-c)} + \dfrac{r_2}{(s-c)(s-a)} + \dfrac{r_3}{(s-a)(s-b)} = \dfrac{3}{r}$.

TS0061

2. With usual notation, if in a ΔABC, $\dfrac{b+c}{11} = \dfrac{c+a}{12} = \dfrac{a+b}{13}$; then prove that, $\dfrac{\cos A}{7} = \dfrac{\cos B}{19} = \dfrac{\cos C}{25}$.

TS0067

3. In a triangle ABC if $a^2 + b^2 = 101c^2$ then find the value of $\dfrac{\cot C}{\cot A + \cot B}$.

TS0068

4. For any triangle ABC, if $B = 3C$, show that $\cos C = \sqrt{\dfrac{b+c}{4c}}$ & $\sin\dfrac{A}{2} = \dfrac{b-c}{2c}$.

TS0069

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5. If in a Δ ABC , a = 6, b = 3 and cos(A – B) = 4/5 then find its area.

TS0070

6. In a Δ ABC, (i) $\frac{a}{\cos A} = \frac{b}{\cos B}$ (ii) $2 \sin A \cos B = \sin C$

(iii) $\tan^2 \frac{A}{2} + 2 \tan \frac{A}{2} \tan \frac{C}{2} - 1 = 0$, prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

TS0071

7. The triangle ABC (with side lengths a, b, c as usual) satisfies $\log a^2 = \log b^2 + \log c^2 - \log (2bc \cos A)$. What can you say about this triangle?

TS0072

8. Given a triangle ABC with AB = 2 and AC = 1. Internal bisector of $\angle BAC$ intersects BC at D. If AD = BD and Δ is the area of triangle ABC, then find the value of $12\Delta^2$.

TS0073

9. Two sides of a triangle are of lengths $\sqrt{6}$ and 4 and the angle opposite to smaller side is 30^0. How many such triangles are possible ? Find the length of their third side and area.

TS0074

10. The two adjacent sides of a cyclic quadrilateral are 2 & 5 and the angle between them is 60°. If the area of the quadrilateral is $4\sqrt{3}$, find the remaining two sides.

TS0075

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EXERCISE (JM)

1. If 5, 5r, $5r^2$ are the lengths of the sides of a triangle, then r cannot be equal to : **[JEE(Main)-Jan 2019]**

(1) $\frac{3}{2}$ (2) $\frac{3}{4}$ (3) $\frac{5}{4}$ (4) $\frac{7}{4}$

TS0076

2. With the usual notation, in ΔABC, if $\angle A + \angle B = 120°$, $a = \sqrt{3} + 1$ and $b = \sqrt{3} - 1$, then the ratio $\angle A : \angle B$, is : **[JEE(Main)-Jan 2019]**

(1) 7 : 1 (2) 5 : 3 (3) 9 : 7 (4) 3 : 1

TS0077

3. In a triangle, the sum of lengths of two sides is x and the product of the lengths of the same two sides is y. If $x^2 - c^2 = y$, where c is the length of the third side of the triangle, then the circumradius of the triangle is : **[JEE(Main)-Jan 2019]**

(1) $\frac{y}{\sqrt{3}}$ (2) $\frac{c}{\sqrt{3}}$ (3) $\frac{c}{3}$ (4) $\frac{3}{2}y$

TS0078

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4. Given $\frac{b+c}{11}=\frac{c+a}{12}=\frac{a+b}{13}$ for a ΔABC with usual notation. If $\frac{\cos A}{\alpha}=\frac{\cos B}{\beta}=\frac{\cos C}{\gamma}$, then the ordered triad (α, β, γ) has a value :- **[JEE(Main)-Jan 2019]**

(1) (3, 4, 5) (2) (19, 7, 25) (3) (7, 19, 25) (4) (5, 12, 13)

TS0079

5. If the lengths of the sides of a triangle are in A.P. and the greatest angle is double the smallest, then a ratio of lengths of the sides of this triangle is : **[JEE(Main)-Apr 2019]**

(1) 5 : 9 : 13 (2) 5 : 6 : 7 (3) 4 : 5 : 6 (4) 3 : 4 : 5

TS0080

6. The angles A, B and C of a triangle ABC are in A.P. and a : b = 1 : $\sqrt{3}$. If c = 4 cm, then the area (in sq. cm) of this triangle is : **[JEE(Main)-Apr 2019]**

(1) $4\sqrt{3}$ (2) $\frac{2}{\sqrt{3}}$ (3) $2\sqrt{3}$ (4) $\frac{4}{\sqrt{3}}$

TS0081

EXERCISE (JA)

1. Let ABC and ABC′ be two non-congruent triangles with sides AB = 4, AC = AC′ = $2\sqrt{2}$ and angle B = 30°. The absolute value of the difference between the areas of these triangles is **[JEE 2009, 5]**

TS0082

2. (a) If the angle A,B and C of a triangle are in an arithmetic progression and if a,b and c denote the length of the sides opposite to A,B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C+\frac{c}{a}\sin 2A$, is -

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) 1 (D) $\sqrt{3}$

TS0083

(b) Consider a triangle ABC and let a,b and c denote the length of the sides opposite to vertices A,B and C respectively. Suppose a = 6, b = 10 and the area of the triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then r^2 is equal to

TS0084

(c) Let ABC be a triangle such that $\angle ACB=\frac{\pi}{6}$ and let a,b and c denote the lengths of the sides opposite to A,B and C respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is/are **[JEE 2010, 3+3+3]**

(A) $-\left(2+\sqrt{3}\right)$ (B) $1+\sqrt{3}$ (C) $2+\sqrt{3}$ (D) $4\sqrt{3}$

TS0085

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3. Let PQR be a triangle of area Δ with $a = 2$, $b = \frac{7}{2}$ and $c = \frac{5}{2}$, where a, b and c are the lengths of the sides of the triangle opposite to the angles at P, Q and R respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals **[JEE 2012, 3M, –1M]**

(A) $\frac{3}{4\Delta}$ (B) $\frac{45}{4\Delta}$ (C) $\left(\frac{3}{4\Delta}\right)^2$ (D) $\left(\frac{45}{4\Delta}\right)^2$

TS0086

4. In a triangle PQR, P is the largest angle and $\cos P = \frac{1}{3}$. Further the incircle of the triangle touches the sides PQ, QR and RP at N, L and M respectively, such that the lengths of PN, QL and RM are consecutive even integers. Then possible length(s) of the side(s) of the triangle is (are)

[JEE(Advanced) 2013, 3, (–1)]

(A) 16 (B) 18 (C) 24 (D) 22

TS0087

5. In a triangle the sum of two sides is x and the product of the same two sides is y. If $x^2 - c^2 = y$, where c is a third side of the triangle, then the ratio of the in-radius to the circum-radius of the triangle is - **[JEE(Advanced)-2014, 3(–1)]**

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(A) $\frac{3y}{2x(x+c)}$ (B) $\frac{3y}{2c(x+c)}$

(C) $\frac{3y}{4x(x+c)}$ (D) $\frac{3y}{4c(x+c)}$

TS0088

6. In a triangle XYZ, let x,y,z be the lengths of sides opposite to the angles X,Y,Z, respectively and 2s = x + y + z. If $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$ and area of incircle of the triangle XYZ is $\frac{8\pi}{3}$, then-

(A) area of the triangle XYZ is $6\sqrt{6}$

(B) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$

(C) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} = \frac{4}{35}$

(D) $\sin^2\left(\frac{X+Y}{2}\right) = \frac{3}{5}$ **[JEE(Advanced)-2016, 4(–2)]**

TS0089

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7. In a triangle PQR, let $\angle PQR = 30°$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018, 4(–2)]**

(A) $\angle QPR = 45°$

(B) The area of the triangle PQR is $25\sqrt{3}$ and $\angle QRP = 120°$

(C) The radius of the incircle of the triangle PQR is $10\sqrt{3} - 15$

(D) The area of the circumcircle of the triangle PQR is 100π.

TS0090

8. In a non-right-angled triangle ΔPQR, let p, q, r denote the lengths of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S, the perpendicular from P meets the side QR at E, and RS and PE intersect at O. If $p = \sqrt{3}$, $q = 1$, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct ?

[JEE(Advanced)-2019, 4(–1)]

(1) Area of $\Delta SOE = \frac{\sqrt{3}}{12}$

(2) Radius of incircle of $\Delta PQR = \frac{\sqrt{3}}{2}(2-\sqrt{3})$

(3) Length of $RS = \frac{\sqrt{7}}{2}$

(4) Length of $OE = \frac{1}{6}$

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TS0091

9. Let x, y and z be positive real numbers. Suppose x, y and z are lengths of the sides of a triangle opposite to its angles X, Y and Z, respectively. If $\tan\frac{X}{2} + \tan\frac{Z}{2} = \frac{2y}{x+y+z}$, then which of the following statements is/are TRUE? **[JEE(Advanced)-2020]**

(A) $2Y = X + Z$

(B) $Y = X + Z$

(C) $\tan\frac{X}{2} = \frac{x}{y+z}$

(D) $x^2 + z^2 - y^2 = xz$

TS0110

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ANSWERS

EXERCISE (O-1)

1. D	**2.** D	**3.** D	**4.** A	**5.** C	**6.** C	**7.** C	**8.** A
9. C	**10.** C	**11.** C	**12.** D	**13.** D	**14.** B	**15.** C	**16.** C
17. B	**18.** D	**19.** D	**20.** D	**21.** A	**22.** A	**23.** C	**24.** A
25. C	**26.** A	**27.** C	**28.** B	**29.** D	**30.** C		

EXERCISE (O-2)

1. A,B,D	**2.** A,D	**3.** B,C	**4.** A,C,D	**5.** B,D	**6.** A,C,D	**7.** B,D	**8.** B,C
9. A,B,D	**10.** A,C	**11.** A,B,C	**12.** A,B	**13.** A,B,C,D	**14.** (A) S; (B) R; (C) P; (D) Q		
15. A	**16.** C	**17.** C	**18.** C	**19.** C	**20.** C	**21.** D	**22.** 2
23. 9	**24.** $\frac{1}{4}$	**25.** 3					

EXERCISE (S-1)

1. $\frac{12}{5}$ **2.** 107

EXERCISE (S-2)

3. 50 **5.** 9 sq. unit **7.** triangle is isosceles **8.** 9

9. Two triangle $\left(2\sqrt{3}-\sqrt{2}\right)$, $\left(2\sqrt{3}+\sqrt{2}\right)$, $\left(2\sqrt{3}-\sqrt{2}\right)$ & $\left(2\sqrt{3}+\sqrt{2}\right)$ sq. units **10.** 3 cms & 2 cms

EXERCISE (JM)

1. 4 **2.** 1 **3.** 2 **4.** 3 **5.** 3 **6.** 3

EXERCISE (JA)

1. 4 **2.** (a) D, (b) 3, (c) B **3.** C **4.** B,D **5.** B **6.** A,C,D

7. B,C,D **8.** 2,3,4 **9.** **B,C**

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Chapter 03

DETERMINANT

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JEE (Main/Advanced) Syllabus

JEE (Main) Syllabus :

Determinants of order two and three, Properties of determinants, evaluation of determinants, area of triangles using determinants. Test of consistency and solution of simultaneous linear equations in two or three variables using determinants.

JEE (Advanced) Syllabus :

Determinant of a square matrix of order up to three, solutions of simultaneous linear equations in two or three variables.

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Important Notes

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DETERMINANT

1. INTRODUCTION :

If the equations $a_1x + b_1 = 0$, $a_2x + b_2 = 0$ are satisfied by the same value of x, then $a_1b_2 - a_2b_1 = 0$. The expression $a_1b_2 - a_2b_1$ is called a determinant of the second order, and is denoted by :

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

A determinant of second order consists of two rows and two columns.

Next consider the system of equations $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$

If these equations are satisfied by the same values of x and y, then on eliminating x and y we get.

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$$

The expression on the left is called a determinant of the third order, and is denoted by

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

A determinant of third order consists of three rows and three columns.

2. VALUE OF A DETERMINANT :

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

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Note : Sarrus diagram to get the value of determinant of order three :

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \left|\begin{array}{ccc|cc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array}\right. = (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_3b_2c_1 + a_2b_1c_3 + a_1b_3c_2)$$

–ve –ve –ve

+ve +ve +ve

Note that the product of the terms in first bracket (i.e. $a_1a_2a_3b_1b_2b_3c_1c_2c_3$) is same as the product of the terms in second bracket.

Illustration 1 : The value of $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$ is -

(A) 213 (B) – 231 (C) 231 (D) 39

Solution :

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix} = 1\begin{vmatrix} 3 & 6 \\ -7 & 9 \end{vmatrix} - 2\begin{vmatrix} -4 & 6 \\ 2 & 9 \end{vmatrix} + 3\begin{vmatrix} -4 & 3 \\ 2 & -7 \end{vmatrix}$$

$$= (27 + 42) - 2(-36 - 12) + 3(28 - 6) = 231$$

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Alternative : By sarrus diagram

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix} \begin{matrix} 1 & 2 \\ -4 & 3 \\ 2 & -7 \end{matrix}$$

$= (27 + 24 + 84) - (18 - 42 - 72) = 135 - (18 - 114) = 231$ **Ans. (C)**

3. MINORS & COFACTORS :

The minor of a given element of determinant is the determinant obtained by deleting the row & the column in which the given element stands.

For example, the minor of a_1 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ & the minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$.

Hence a determinant of order three will have "9 minors".

If M_{ij} represents the minor of the element belonging to i^{th} row and j^{th} column then the cofactor of that element is given by : $C_{ij} = (-1)^{i+j}.M_{ij}$

Illustration 2 : Find the minors and cofactors of elements '–3', '5', '–1' & '7' in the determinant $\begin{vmatrix} 2 & -3 & 1 \\ 4 & 0 & 5 \\ -1 & 6 & 7 \end{vmatrix}$

Solution : Minor of $-3 = \begin{vmatrix} 4 & 5 \\ -1 & 7 \end{vmatrix} = 33$; Cofactor of $-3 = -33$

Minor of $5 = \begin{vmatrix} 2 & -3 \\ -1 & 6 \end{vmatrix} = 9$; Cofactor of $5 = -9$

Minor of $-1 = \begin{vmatrix} -3 & 1 \\ 0 & 5 \end{vmatrix} = -15$; Cofactor of $-1 = -15$

Minor of $7 = \begin{vmatrix} 2 & -3 \\ 4 & 0 \end{vmatrix} = 12$; Cofactor of $7 = 12$

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4. EXPANSION OF A DETERMINANT IN TERMS OF THE ELEMENTS OF ANY ROW OR COLUMN:

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(i) The sum of the product of elements of any row (column) with their corresponding cofactors is always equal to the value of the determinant.

D can be expressed in any of the six forms :

$a_1A_1 + b_1B_1 + c_1C_1$, $a_1A_1 + a_2A_2 + a_3A_3$,

$a_2A_2 + b_2B_2 + c_2C_2$, $b_1B_1 + b_2B_2 + b_3B_3$,

$a_3A_3 + b_3B_3 + c_3C_3$, $c_1C_1 + c_2C_2 + c_3C_3$,

where A_i, B_i & C_i (i = 1,2,3) denote cofactors of a_i, b_i & c_i respectively.

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(ii) The sum of the product of elements of any row (column) with the cofactors of other row (column) is always equal to zero.

Hence,

$a_2A_1 + b_2B_1 + c_2C_1 = 0$,

$b_1A_1 + b_2A_2 + b_3A_3 = 0$ and so on.

where A_i, B_i & C_i (i = 1,2,3) denote cofactors of a_i, b_i & c_i respectively.

Do yourself -1 :

(i) Find minors & cofactors of elements '6', '5', '0' & '4' of the determinant $\begin{vmatrix} 2 & 1 & 3 \\ 6 & 5 & 7 \\ 3 & 0 & 4 \end{vmatrix}$.

(ii) Calculate the value of the determinant $\begin{vmatrix} 5 & -3 & 7 \\ -2 & 4 & -8 \\ 9 & 3 & -10 \end{vmatrix}$

(iii) The value of the determinant $\begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix}$ is equal to -

(A) $a^3 - b^3$ (B) $a^3 + b^3$ (C) 0 (D) none of these

(iv) Find the value of 'k', if $\begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 3 & k & 2 \end{vmatrix} = 4$

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5. PROPERTIES OF DETERMINANTS :

(a) The value of a determinant remains unaltered, if the rows & columns are inter-changed,

e.g. if $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

(b) If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only. e.g.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D_1 = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Then $D_1 = -D$.

(c) If all the elements of a row (or column) are zero, then the value of the determinant is zero.

(d) If all the elements of any row (or column) are multiplied by the same number, then the determinant is multiplied by that number.

e.g. If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D_1 = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Then $D_1 = KD$

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(e) If all the elements of a row (or column) are proportional (or identical) to the element of any other row, then the determinant vanishes, i.e. its value is zero.

e.g. If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D = 0$; If $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D_1 = 0$

Illustration 3 : Prove that $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$

Solution : $D = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix}$ (By interchanging rows & columns)

$= -\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix}$ $(C_1 \leftrightarrow C_2)$

$= \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$ $(R_1 \leftrightarrow R_2)$

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Illustration 4 : Find the value of the determinant $\begin{vmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix}$

Solution : $D = \begin{vmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix} = a\begin{vmatrix} a & b & c \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix} = abc\begin{vmatrix} a & b & c \\ a & b & c \\ a & b & c \end{vmatrix} = 0$

Since all rows are same, hence value of the determinant is zero.

Do yourself -2 :

(i) Without expanding the determinant prove that $\begin{vmatrix} a & p & \ell \\ b & q & m \\ c & r & n \end{vmatrix} + \begin{vmatrix} r & n & c \\ q & m & b \\ p & \ell & a \end{vmatrix} = 0$

(ii) If $D = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}$, then $\begin{vmatrix} 2\alpha & 2\beta \\ 2\gamma & 2\delta \end{vmatrix}$ is equal to -

(A) D (B) 2D (C) 4D (D) 16D

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(f) If each element of any row (or column) is expressed as a sum of two (or more) terms, then the determinant can be expressed as the sum of two (or more) determinants.

e.g. $\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Note that : If $D_r = \begin{vmatrix} f(r) & g(r) & h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$

where $r \in N$ and a,b,c, a_1, b_1,c_1 are constants, then

$$\sum_{r=1}^{n} D_r = \begin{vmatrix} \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

(g) **Row - column operation :** The value of a determinant remains unaltered under a column (C_i) operation of the form $C_i \to C_i + \alpha C_j + \beta C_k$ (j, k ≠ i) or row (R_i) operation of the form $R_i \to R_i + \alpha R_j + \beta R_k$ (j, k ≠ i). In other words, the value of a determinant is not altered by adding the elements of any row (or column) to the same multiples of the corresponding elements of any other row (or column)

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e.g. Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$D = \begin{vmatrix} a_1+\alpha a_2 & b_1+\alpha b_2 & c_1+\alpha c_2 \\ a_2 & b_2 & c_2 \\ a_3+\beta a_1 & b_3+\beta b_1 & c_3+\beta c_1 \end{vmatrix} \quad (R_1 \to R_1 + \alpha R_2; R_3 \to R_3 + \beta R_2)$$

Note :

(i) By using the operation $R_i \to xR_i + yR_j + zR_k$ (j, k ≠ i), the value of the determinant becomes x times the original one.

(ii) While applying this property **ATLEAST ONE ROW (OR COLUMN)** must remain unchanged.

Illustration 5 : If $D_r = \begin{vmatrix} r & r^3 & 2 \\ n & n^3 & 2n \\ \frac{n(n+1)}{2} & \left(\frac{n(n+1)}{2}\right)^2 & 2(n+1) \end{vmatrix}$, find $\sum_{r=0}^{n} D_r$.

Solution : $\sum_{r=0}^{n} D_r = \begin{vmatrix} \sum_{r=0}^{n} r & \sum_{r=0}^{n} r^3 & \sum_{r=0}^{n} 2 \\ n & n^3 & 2n \\ \frac{n(n+1)}{2} & \left(\frac{n(n+1)}{2}\right)^2 & 2(n+1) \end{vmatrix} = \begin{vmatrix} \frac{n(n+1)}{2} & \left(\frac{n(n+1)}{2}\right)^2 & 2(n+1) \\ n & n^3 & 2n \\ \frac{n(n+1)}{2} & \left(\frac{n(n+1)}{2}\right)^2 & 2(n+1) \end{vmatrix} = 0$ **Ans.**

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Illustration 6 : If $\begin{vmatrix} 3^2+k & 4^2 & 3^2+3+k \\ 4^2+k & 5^2 & 4^2+4+k \\ 5^2+k & 6^2 & 5^2+5+k \end{vmatrix} = 0$, then the value of k is-

(A) 2 (B) 1 (C) –1 (D) 0

Solution : Applying $(C_3 \to C_3 - C_1)$

$$D = \begin{vmatrix} 3^2+k & 4^2 & 3 \\ 4^2+k & 5^2 & 4 \\ 5^2+k & 6^2 & 5 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 9+k & 16 & 3 \\ 7 & 9 & 1 \\ 9 & 11 & 1 \end{vmatrix} = 0 \quad (R_3 \to R_3 - R_2;\ R_2 \to R_2 - R_1)$$

$\Rightarrow$ $k - 1 = 0 \Rightarrow k = 1$ **Ans. (B)**

Do yourself - 3 :

(i) Find the value of $\begin{vmatrix} 53 & 106 & 159 \\ 52 & 65 & 91 \\ 102 & 153 & 221 \end{vmatrix}$.

(ii) Solve for x : $\begin{vmatrix} x & 2 & 0 \\ 2+x & 5 & -1 \\ 5-x & 1 & 2 \end{vmatrix} = 0$

(iii) If $D_r = \begin{vmatrix} 2r & 1 & n \\ 1 & -2 & 3 \\ 3 & 2 & 1 \end{vmatrix}$, then find the value of $\sum_{r=1}^{n} D_r$.

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(h) **Factor theorem :** If the elements of a determinant D are rational integral functions of x and two rows (or columns) become identical when x = a then (x – a) is a factor of D.

Note that if r rows become identical when a is substituted for x, then $(x - a)^{r-1}$ is a factor of D.

Illustration 7 : Prove that $\begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = m(x-a)(x-b)$

Solution : Using factor theorem,

Put x = a

$$D = \begin{vmatrix} a & a & a \\ m & m & m \\ b & a & b \end{vmatrix} = 0$$

Since R_1 and R_2 are proportional which makes D = 0, therefore (x – a) is a factor of D.

Similarly, by putting x = b, D becomes zero, therefore (x – b) is a factor of D.

$$D = \begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = \lambda(x-a)(x-b) \qquad \text{.....(i)}$$

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To get the value of λ, put x = 0 in equation (i)

$$\begin{vmatrix} a & a & 0 \\ m & m & m \\ b & 0 & b \end{vmatrix} = \lambda ab$$

$amb = \lambda ab \quad \Rightarrow \quad \lambda = m$

$\therefore \quad D = m(x-a)(x-b)$

Do yourself - 4 :

(i) Without expanding the determinant prove that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

(ii) Using factor theorem, find the solution set of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$

6. MULTIPLICATION OF TWO DETERMINANTS :

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1 l_1 + b_1 l_2 & a_1 m_1 + b_1 m_2 \\ a_2 l_1 + b_2 l_2 & a_2 m_1 + b_2 m_2 \end{vmatrix}$$

Similarly two determinants of order three are multiplied.

(a) Here we have multiplied row by column. We can also multiply row by row, column by row and column by column.

(b) If D_1 is the determinant formed by replacing the elements of determinant D of order n by their corresponding cofactors then $D_1 = D^{n-1}$

Illustration 8 : Let α & β be the roots of equation $ax^2 + bx + c = 0$ and $S_n = \alpha^n + \beta^n$ for $n \geq 1$. Evaluate the value of the determinant $\begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}$.

Solution :

$$D = \begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix} = \begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix}^2 = [(1-\alpha)(1-\beta)(\alpha-\beta)]^2$$

$D = (\alpha - \beta)^2 (\alpha + \beta - \alpha\beta - 1)^2$

$\because \quad \alpha$ & β are roots of the equation $ax^2 + bx + c = 0$

$$\Rightarrow \quad \alpha + \beta = \frac{-b}{a} \quad \& \quad \alpha\beta = \frac{c}{a} \quad \Rightarrow \quad |\alpha - \beta| = \frac{\sqrt{b^2 - 4ac}}{|a|}$$

$$D = \frac{(b^2 - 4ac)}{a^2}\left(\frac{a+b+c}{a}\right)^2 = \frac{(b^2 - 4ac)(a+b+c)^2}{a^4}$$ **Ans.**

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Do yourself - 5 :

(i) If the determinant $D=\begin{vmatrix}1 & 1 & 1\\ \alpha+\beta & \alpha^2+\beta^2 & 2\alpha\beta\\ \alpha+\beta & 2\alpha\beta & \alpha^2+\beta^2\end{vmatrix}$ and $D_1=\begin{vmatrix}1 & 0 & 0\\ 0 & \alpha & \beta\\ 0 & \beta & \alpha\end{vmatrix}$, then find the determinant D_2 such that $D_2=\dfrac{D}{D_1}$.

(ii) If $D_1=\begin{vmatrix}ab^2-ac^2 & bc^2-a^2b & a^2c-b^2c\\ ac-ab & ab-bc & bc-ac\\ c-b & a-c & b-a\end{vmatrix}$ & $D_2=\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ bc & ac & ab\end{vmatrix}$, then D_1D_2 is equal to -

(A) 0 (B) D_1^2 (C) D_2^2 (D) D_2^3

7. SPECIAL DETERMINANTS :

(a) Cyclic Determinant :

The elements of the rows (or columns) are in cyclic arrangement.

$$\begin{vmatrix}a & b & c\\ b & c & a\\ c & a & b\end{vmatrix}=-(a^3+b^3+c^3-3abc)=-(a+b+c)\,(a^2+b^2+c^2-ab-bc-ac)$$

$$=-\frac{1}{2}(a+b+c)\times\{(a-b)^2+(b-c)^2+(c-a)^2\}$$

$=-(a+b+c)\,(a+b\omega+c\omega^2)\,(a+b\omega^2+c\omega)$, where ω, ω^2 are cube roots of unity

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(b) Other Important Determinants :

(i) $\begin{vmatrix}0 & b & -c\\ -b & 0 & a\\ c & -a & 0\end{vmatrix}=0$

(ii) $\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ bc & ac & ab\end{vmatrix}=\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}=(a-b)(b-c)(c-a)$

(iii) $\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^3 & b^3 & c^3\end{vmatrix}=(a-b)(b-c)(c-a)(a+b+c)$

(iv) $\begin{vmatrix}1 & 1 & 1\\ a^2 & b^2 & c^2\\ a^3 & b^3 & c^3\end{vmatrix}=(a-b)(b-c)(c-a)(ab+bc+ca)$

(v) $\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^4 & b^4 & c^4\end{vmatrix}=(a-b)(b-c)(c-a)(a^2+b^2+c^2-ab-bc-ca)$

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Illustration 9 : Prove that $\begin{vmatrix} 1 & \alpha & \alpha^2 \\ \alpha & \alpha^2 & 1 \\ \alpha^2 & 1 & \alpha \end{vmatrix} = -(1-\alpha^3)^2$.

Solution : This is a cyclic determinant.

$$\Rightarrow \begin{vmatrix} 1 & \alpha & \alpha^2 \\ \alpha & \alpha^2 & 1 \\ \alpha^2 & 1 & \alpha \end{vmatrix} = -(1+\alpha+\alpha^2)(1+\alpha^2+\alpha^4-\alpha-\alpha^2-\alpha^3)$$

$$= -(1+\alpha+\alpha^2)(-\alpha+1-\alpha^3+\alpha^4) = -(1+\alpha+\alpha^2)(1-\alpha)^2(1+\alpha+\alpha^2)$$

$$= -(1-\alpha)^2(1+\alpha+\alpha^2)^2 = -(1-\alpha^3)^2$$

Do yourself - 6 :

(i) The value of the determinant $\begin{vmatrix} ka & k^2+a^2 & 1 \\ kb & k^2+b^2 & 1 \\ kc & k^2+c^2 & 1 \end{vmatrix}$ is

(A) $k(a+b)(b+c)(c+a)$ (B) $kabc(a^2+b^2+c^2)$

(C) $k(a-b)(b-c)(c-a)$ (D) $k(a+b-c)(b+c-a)(c+a-b)$

(ii) Find the value of the determinant $\begin{vmatrix} a^2+b^2 & a^2-c^2 & a^2-c^2 \\ -a^2 & 0 & c^2-a^2 \\ b^2 & -c^2 & b^2 \end{vmatrix}$.

(iii) Prove that $\begin{vmatrix} a & b & c \\ bc & ca & ab \\ b+c & c+a & a+b \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$.

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8. CRAMER'S RULE (SYSTEM OF LINEAR EQUATIONS) :

Simultaneous linear equations

- Consistent (at least one solution)
  - Exactly one solution or Unique solution
    - Trivial solution i.e. All variable zero is the only solution
  - Infinite solutions
    - Non trivial solution
    - At least one non zero variable satisfies the system
- Inconsistent (no solution)

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(a) Equations involving two variables :

(i) Consistent Equations : Definite & unique solution (Intersecting lines)

(ii) Inconsistent Equations : No solution (Parallel lines)

(iii) Dependent Equations : Infinite solutions (Identical lines)

Let, $a_1x + b_1y + c_1 = 0$

$a_2x + b_2y + c_2 = 0$ then :

(1) $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \Rightarrow$ Given equations are consistent with unique solution

(2) $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \Rightarrow$ Given equations are inconsistent

(3) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow$ Given equations are consistent with infinite solutions

(b) Equations Involving Three variables :

Let $a_1x + b_1y + c_1z = d_1$ ..... (i)

$a_2x + b_2y + c_2z = d_2$ ..... (ii)

$a_3x + b_3y + c_3z = d_3$ ..... (iii)

Then, $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$, $z = \frac{D_3}{D}$.

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Where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$; $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$; $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ & $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Note :

(i) If $D \neq 0$ and atleast one of $D_1, D_2, D_3 \neq 0$, then the given system of equations is consistent and has unique non trivial solution.

(ii) If $D \neq 0$ & $D_1 = D_2 = D_3 = 0$, then the given system of equations is consistent and has trivial solution only.

(iii) If $D = 0$ but atleast one of D_1, D_2, D_3 is not zero then the equations are inconsistent and have no solution.

(iv) If $D = D_1 = D_2 = D_3 = 0$, then the given system of equations may have infinite or no solution.

Note that In case $\left.\begin{array}{l} a_1x + b_1y + c_1z = d_1 \\ a_1x + b_1y + c_1z = d_2 \\ a_1x + b_1y + c_1z = d_3 \end{array}\right\}$ (Atleast two of d_1, d_2 & d_3 are not equal)

$D = D_1 = D_2 = D_3 = 0$. But these three equations represent three parallel planes. Hence the system is inconsistent.

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(c) Homogeneous system of linear equations :

If x, y, z are not all zero, the condition for

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$

to be consistent in x, y, z is that $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

Remember that if a given system of linear equations have **Only Zero** Solution for all its variables then the given equations are said to have **TRIVIAL SOLUTION**.

9. APPLICATION OF DETERMINANTS IN GEOMETRY :

(a) The lines : $a_1x + b_1y + c_1 = 0$ ..... (i)

$a_2x + b_2y + c_2 = 0$ ..... (ii)

$a_3x + b_3y + c_3 = 0$ ..... (iii)

are concurrent or all three parallel if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

This is the necessary condition for consistency of three simultaneous linear equations in 2 variables but may not be sufficient.

(b) Equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if :

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

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(c) Area of a triangle whose vertices are (x_r, y_r); r = 1, 2, 3 is $D = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

If D = 0, then the three points are collinear.

(d) Equation of a straight line passing through points (x_1, y_1) & (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Illustration 10 : Find the nature of solution for the given system of equations :

$x + 2y + 3z = 1;\ 2x + 3y + 4z = 3;\ 3x + 4y + 5z = 0$

Solution : $D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$

Now, $D_1 = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 4 \\ 0 & 4 & 5 \end{vmatrix} = 5$

$\because$ D = 0 but $D_1 \neq 0$

Hence no solution. **Ans.**

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Illustration 11 : Find the value of λ, if the following equations are consistent :
$x + y - 3 = 0$; $(1 + \lambda)x + (2 + \lambda)y - 8 = 0$; $x - (1 + \lambda)y + (2 + \lambda) = 0$

Solution : The given equations in two unknowns are consistent, then $\Delta = 0$

i.e. $\begin{vmatrix} 1 & 1 & -3 \\ 1+\lambda & 2+\lambda & -8 \\ 1 & -(1+\lambda) & 2+\lambda \end{vmatrix} = 0$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 + 3C_1$

$\therefore \begin{vmatrix} 1 & 0 & 0 \\ 1+\lambda & 1 & 3\lambda-5 \\ 1 & -2-\lambda & 5+\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (5+\lambda) - (3\lambda - 5)(-2-\lambda) = 0 \quad \Rightarrow \quad 3\lambda^2 + 2\lambda - 5 = 0$

$\therefore \quad \lambda = 1, -5/3$

Illustration 12 : If the system of equations $x + \lambda y + 1 = 0$, $\lambda x + y + 1 = 0$ & $x + y + \lambda = 0$. is consistent, then find the value of λ.

Solution : For consistency of the given system of equations

$D = \begin{vmatrix} 1 & \lambda & 1 \\ \lambda & 1 & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$

$\Rightarrow \quad 3\lambda = 1 + 1 + \lambda^3$ or $\lambda^3 - 3\lambda + 2 = 0$

$\Rightarrow \quad (\lambda-1)^2 (\lambda + 2) = 0 \Rightarrow \lambda = 1$ or $\lambda = -2$ **Ans.**

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Do yourself -7 :

(i) Find nature of solution for given system of equations
$2x + y + z = 3$; $x + 2y + z = 4$; $3x + z = 2$

(ii) If the system of equations $x + y + z = 2$, $2x + y - z = 3$ & $3x + 2y + kz = 4$ has a unique solution, then
(A) $k \neq 0$ (B) $-1 < k < 1$ (C) $-2 < k < 1$ (D) $k = 0$

(iii) The system of equations $\lambda x + y + z = 0$, $-x + \lambda y + z = 0 = 0$ & $-x - y + \lambda z = 0$ has a non-trivial solution, then possible values of λ are -
(A) 0 (B) 1 (C) -3 (D) $\sqrt{3}$

ANSWERS FOR DO YOURSELF

1. **(i)** minors : 4, –1, –4, 4 ; cofactors : –4, –1, 4, 4 **(ii)** –98 **(iii)** B **(iv)** 0
2. **(ii)** C
3. **(i)** 0 **(ii)** 2 **(iii)** 0
4. **(ii)** $x = -1, 2$
5. **(i)** $\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \beta & \alpha \end{vmatrix}$ **(ii)** D
6. **(i)** C **(ii)** 0
7. **(i)** infinite solutions **(ii)** A **(iii)** A

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EXERCISE (O-1)

1. $\begin{vmatrix} y+z & x & x \\ y & z+x & y \\ z & z & x+y \end{vmatrix}$ equals -

(A) $x^2y^2z^2$ (B) $4x^2y^2z^2$ (C) xyz (D) 4xyz

DT0001

2. If $\begin{vmatrix} 1 & 3 & 4 \\ 1 & x-1 & 2x+2 \\ 2 & 5 & 9 \end{vmatrix} = 0$, then x is equal to-

(A) 2 (B) 1 (C) 4 (D) 0

DT0002

3. If $px^4 + qx^3 + rx^2 + sx + t = \begin{vmatrix} x^2+3x & x-1 & x+3 \\ x+1 & 2-x & x-3 \\ x-3 & x+4 & 3x \end{vmatrix}$ then t is equal to -

(A) 33 (B) 0 (C) 21 (D) none

DT0003

4. There are two numbers x making the value of the determinant $\begin{vmatrix} 1 & -2 & 5 \\ 2 & x & -1 \\ 0 & 4 & 2x \end{vmatrix}$ equal to 86. The sum of these two numbers, is-

(A) –4 (B) 5 (C) –3 (D) 9

DT0004

5. If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and A_2, B_2, C_2 are respectively cofactors of a_2, b_2, c_2 then $a_1A_2 + b_1B_2 + c_1C_2$ is equal to-

(A) $-\Delta$ (B) 0 (C) Δ (D) none of these

DT0005

6. If in the determinant $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, A_1, B_1, C_1 etc. be the co-factors of a_1, b_1, c_1 etc., then which of the following relations is incorrect-

(A) $a_1A_1 + b_1B_1 + c_1C_1 = \Delta$

(B) $a_2A_2 + b_2B_2 + c_2C_2 = \Delta$

(C) $a_3A_3 + b_3B_3 + c_3C_3 = \Delta$

(D) $a_1A_2 + b_1B_2 + c_1C_2 = \Delta$

DT0006

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7. If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and A_1, B_1, C_1 denote the co-factors of a_1, b_1, c_1 respectively, then the value of the determinant $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$ is -

(A) Δ (B) Δ^2 (C) Δ^3 (D) 0

DT0007

8. If a, b, c are in AP, then $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$ equals -

(A) a + b + c (B) x + a + b + c (C) 0 (D) none of these

DT0008

9. For positive numbers x, y and z, the numerical value of the determinant $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ is-

(A) 0 (B) log xyz (C) log(x + y + z) (D) logx logy logz

DT0009

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10. Let a determinant is given by $A = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$ and suppose det. A = 6. If $B = \begin{vmatrix} p+x & q+y & r+z \\ a+x & b+y & c+z \\ a+p & b+q & c+r \end{vmatrix}$ then

(A) det. B = 6 (B) det. B = – 6 (C) det. B = 12 (D) det. B = – 12

DT0010

11. The value of an odd order determinant in which $a_{ij} + a_{ji} = 0 \ \forall$ i, j is -

(A) perfect square (B) negative (C) ± 1 (D) 0

DT0011

12. If $S_r = \begin{vmatrix} 2r & x & n(n+1) \\ 6r^2-1 & y & n^2(2n+3) \\ 4r^3-2nr & z & n^3(n+1) \end{vmatrix}$, then $\sum_{r=1}^{n} S_r$ does not depend on -

(A) x (B) y (C) n (D) all of these

DT0012

13. If a, b, c > 0 and x, y, z $\in$ R, then the determinant $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^y+b^{-y})^2 & (b^y-b^{-y})^2 & 1 \\ (c^z+c^{-z})^2 & (c^z-c^{-z})^2 & 1 \end{vmatrix}$ is equal to -

(A) $a^x b^y c^x$ (B) $a^{-x} b^{-y} c^{-z}$ (C) $a^{2x} b^{2y} c^{2z}$ (D) zero

DT0013

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14. If a, b, c are sides of a scalene triangle, then the value of $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is : **[JEE-MAIN Online 2013]**

(A) non-negative (B) negative (C) positive (D) non-positive

DT0014

15. The value of k for which the set of equations 3x+ky– 2z=0, x + ky + 3z = 0 and 2x+3y – 4z=0 has a non-trivial solution is-

(A) 15 (B) 16 (C) 31/2 (D) 33/2

DT0015

16. If the system of linear equations **[JEE-MAIN Online 2013]**

$$x_1 + 2x_2 + 3x_3 = 6$$
$$x_1 + 3x_2 + 5x_3 = 9$$
$$2x_1 + 5x_2 + ax_3 = b$$

is consistent and has infinite number of solutions, then :-

(A) $a \in R - \{8\}$ and $b \in R - \{15\}$ (B) a = 8, b can be any real number

(C) a = 8, b = 15 (D) b = 15, a can be any real number

DT0016

17. Consider the system of equations : x + ay = 0, y + az = 0 and z + ax = 0. Then the set of all real values of 'a' for which the system has a unique solution is : **[JEE-MAIN Online 2013]**

(A) {1, –1} (B) R – {–1} (C) {1, 0, –1} (D) R – {1}

DT0017

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18. Let a, b, c be any real numbers. Suppose that there are real numbers x, y, z not all zero such that x = cy + bz, y = az + cx and z = bx + ay, then $a^2 + b^2 + c^2 + 2abc$ is equal to **[AIEEE - 2008]**

(A) 2 (B) –1 (C) 0 (D) 1

DT0018

EXERCISE (O-2)

1. Let $f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$, then the maximum value of f(x) , is-

(A) 2 (B) 4 (C) 6 (D) 8

DT0019

2. If the determinant $\begin{vmatrix} a+p & 1+x & u+f \\ b+q & m+y & v+g \\ c+r & n+z & w+h \end{vmatrix}$ splits into exactly K determinants of order 3, each element of which contains only one term, then the value of K, is-

(A) 6 (B) 8 (C) 9 (D) 12

DT0020

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3. Let $D_1 = \begin{vmatrix} a & b & a+b \\ c & d & c+d \\ a & b & a-b \end{vmatrix}$ and $D_2 = \begin{vmatrix} a & c & a+c \\ b & d & b+d \\ a & c & a+b+c \end{vmatrix}$ then the value of $\frac{D_1}{D_2}$ where $b \neq 0$ and $ad \neq bc$, is

(A) – 2 (B) 0 (C) – 2b (D) 2b

DT0021

4. If $a^2 + b^2 + c^2 = -2$ and $f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$ then $f(x)$ is a polynomial of degree-

(A) 0 (B) 1 (C) 2 (D) 3

DT0022

5. The number of real values of x satisfying $\begin{vmatrix} x & 3x+2 & 2x-1 \\ 2x-1 & 4x & 3x+1 \\ 7x-2 & 17x+6 & 12x-1 \end{vmatrix} = 0$ is -

(A) 3 (B) 0 (C) 1 (D) infinite

DT0023

6. The determinant $\begin{vmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & \cos 2\phi \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix}$ is -

(A) 0 (B) independent of θ

(C) independent of ϕ (D) independent of θ & ϕ both

DT0024

7. If the system of equation, $a^2x - ay = 1 - a$ & $bx + (3 - 2b)y = 3 + a$ possess a unique solution $x = 1, y = 1$ than :

(A) $a = 1; b = -1$ (B) $a = -1, b = 1$ (C) $a = 0, b = 0$ (D) none

DT0025

[ONE OR MORE THAN ONE ARE CORRECT]

8. The determinant $\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$ is divisible by -

(A) $a + b + c$ (B) $(a + b)(b + c)(c + a)$

(C) $a^2 + b^2 + c^2$ (D) $(a - b)(b - c)(c - a)$

DT0026

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9. The value of θ lying between $-\frac{\pi}{4}$ & $\frac{\pi}{2}$ and $0 \le A \le \frac{\pi}{2}$ and satisfying the equation

$$\begin{vmatrix} 1+\sin^2 A & \cos^2 A & 2\sin 4\theta \\ \sin^2 A & 1+\cos^2 A & 2\sin 4\theta \\ \sin^2 A & \cos^2 A & 1+2\sin 4\theta \end{vmatrix} = 0 \text{ are -}$$

(A) $A = \frac{\pi}{4}, \quad \theta = -\frac{\pi}{8}$

(B) $A = \frac{3\pi}{8} = \theta$

(C) $A = \frac{\pi}{5}, \quad \theta = -\frac{\pi}{8}$

(D) $A = \frac{\pi}{6}, \quad \theta = \frac{3\pi}{8}$

DT0027

10. Which of the following determinant(s) vanish(es) ?

(A) $\begin{vmatrix} 1 & bc & bc(b+c) \\ 1 & ca & ca(c+a) \\ 1 & ab & ab(a+b) \end{vmatrix}$

(B) $\begin{vmatrix} 1 & ab & \frac{1}{a}+\frac{1}{b} \\ 1 & bc & \frac{1}{b}+\frac{1}{c} \\ 1 & ca & \frac{1}{c}+\frac{1}{a} \end{vmatrix}$

(C) $\begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix}$

(D) $\begin{vmatrix} \log_x xyz & \log_x y & \log_x z \\ \log_y xyz & 1 & \log_y z \\ \log_z xyz & \log_z y & 1 \end{vmatrix}$

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DT0028

11. The determinant $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}$ is equal to zero, if -

(A) a, b, c are in AP

(B) a, b, c are in GP

(C) α is a root of the equation $ax^2+bx+c=0$

(D) $(x-\alpha)$ is a factor of $ax^2 + 2bx + c$

DT0029

12. System of linear equations in x,y,z

$2x + y + z = 1$

$x - 2y + z = 2$

$3x - y + 2z = 3$ have infinite solutions which

(A) can be written as $(-3\lambda - 1, \lambda, 5\lambda + 3)\ \forall\ \lambda \in R$

(B) can be written as $(3\lambda - 1, -\lambda, -5\lambda + 3)\ \forall\ \lambda \in R$

(C) are such that every solution satisfy $x - 3y + 1 = 0$

(D) are such that none of them satisfy $5x + 3z = 1$

DT0030

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13. System of equation $x + y + az = b$, $2x + 3y = 2a$ & $3x + 4y + a^2z = ab + 2$ has

(A) unique solution when $a \neq 0$, $b \in R$

(B) no solution when $a = 0$, $b = 1$

(C) infinite solution when $a = 0$, $b = 2$

(D) infinite solution when $a = 1$, $b \in R$

DT0031

[MATRIX MATCH TYPE]

14. Consider a system of linear equations $a_ix + b_iy + c_iz = d_i$(where $a_i, b_i, c_i \neq 0$ and $i = 1,2,3$) & (α,β,γ) is its unique solution, then match the following conditions.

	Column-I		Column-II
(A)	If $a_i = k$, $d_i = k^2$, $(k \neq 0)$ and $\alpha + \beta + \gamma = 2$, then k is	(P)	1
(B)	If $a_i = d_i = k \neq 0$, then $\alpha + \beta + \gamma$ is	(Q)	2
(C)	If $a_i = k > 0$, $d_i = k + 1$, then $\alpha + \beta + \gamma$ can be	(R)	0
(D)	If $a_i = k < 0$, $d_i = k + 1$, then $\alpha + \beta + \gamma$ can be	(S)	3
		(T)	–1

DT0032

EXERCISE (S-1)

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1. (a) Let $f(x) = \begin{vmatrix} x & 1 & \frac{-3}{2} \\ 2 & 2 & 1 \\ \frac{1}{x-1} & 0 & \frac{1}{2} \end{vmatrix}$. Find the minimum value of f(x) (given $x > 1$).

DT0033

(b) If $a^2 + b^2 + c^2 + ab + bc + ca \leq 0 \ \forall\ a, b, c \in R$, then find the value of the determinant

$$\begin{vmatrix} (a+b+2)^2 & a^2+b^2 & 1 \\ 1 & (b+c+2)^2 & b^2+c^2 \\ c^2+a^2 & 1 & (c+a+2)^2 \end{vmatrix}.$$

DT0034

2. (a) Solve for x, $\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0.$

DT0035

(b) $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$

DT0036

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3. If $a+b+c=0$, solve for x : $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0.$

DT0037

4. Show that, $\begin{vmatrix} a^2+\lambda & ab & ac \\ ab & b^2+\lambda & bc \\ ac & bc & c^2+\lambda \end{vmatrix}$ is divisible by λ^2 and find the other factor.

DT0038

5. Prove that : $\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}.$

DT0039

6. Let a, b, c are the solutions of the cubic $x^3 - 5x^2 + 3x - 1 = 0$, then find the value of the determinant

$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}.$

DT0040

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7. If $\Delta(x) = \begin{vmatrix} 0 & 2x-2 & 2x+8 \\ x-1 & 4 & x^2+7 \\ 0 & 0 & x+4 \end{vmatrix}$ and $f(x) = \sum_{j=1}^{3}\sum_{i=1}^{3} a_{ij}c_{ij}$, where a_{ij} is the element of ith and j^{th} column in $\Delta(x)$ and c_{ij} is the cofactor a_{ij} $\forall$ i and j, then find the greatest value of f(x), where $x \in [-3, 18]$

DT0041

8. **(a)** On which one of the parameter out of a, p, d or x the value of the determinant

$\begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$ does not depend.

DT0042

(b) If $\begin{vmatrix} x^3+1 & x^2 & x \\ y^3+1 & y^2 & y \\ z^3+1 & z^2 & z \end{vmatrix} = 0$ and x, y, z are all different then, prove that $xyz = -1$.

DT0043

9. Prove that :

(a) $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$

DT0044

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(b) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = [(x-y)(y-z)(z-x)(x+y+z)]$

DT0045

10. If $D = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$ and $D' = \begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ c+a & a+b & b+c \end{vmatrix}$, then prove that $D' = 2D$.

DT0046

11. If $S_r = \alpha^r + \beta^r + \gamma^r$ then show that $\begin{vmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{vmatrix} = (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2$.

DT0047

12. Prove that $\begin{vmatrix} (\beta+\gamma-\alpha-\delta)^4 & (\beta+\gamma-\alpha-\delta)^2 & 1 \\ (\gamma+\alpha-\beta-\delta)^4 & (\gamma+\alpha-\beta-\delta)^2 & 1 \\ (\alpha+\beta-\gamma-\delta)^4 & (\alpha+\beta-\gamma-\delta)^2 & 1 \end{vmatrix} = -64(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta)$

DT0048

13. Solve the following sets of equations using Cramer's rule and remark about their consistency.

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(a) $x + y + z - 6 = 0$
$2x + y - z - 1 = 0$
$x + y - 2z + 3 = 0$

DT0049

(b) $x + 2y + z = 1$
$3x + y + z = 6$
$x + 2y = 0$

DT0050

(c) $7x - 7y + 5z = 3$
$3x + y + 5z = 7$
$2x + 3y + 5z = 5$

DT0051

14. For what value of K do the following system of equations $x + Ky + 3z = 0$, $3x + Ky - 2z = 0$, $2x + 3y - 4z = 0$ possess a non trivial (i.e. not all zero) solution over the set of rationals Q.
For that value of K, find all the solutions of the system.

DT0052

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15. If the equations $a(y + z) = x$, $b(z + x) = y$, $c(x + y) = z$ (where $a,b,c \neq -1$)have nontrivial solutions, then find the value of $\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}$.

DT0053

16. Show that the system of equations $3x - y + 4z = 3$, $x + 2y - 3z = -2$ and $6x + 5y + \lambda z = -3$ has alteast one solution for any real number λ. Find the set of solutions of $\lambda = -5$.

DT0054

EXERCISE (S-2)

1. In a Δ ABC, determine condition under which $\begin{vmatrix} \cot\frac{A}{2} & \cot\frac{B}{2} & \cot\frac{C}{2} \\ \tan\frac{B}{2}+\tan\frac{C}{2} & \tan\frac{C}{2}+\tan\frac{A}{2} & \tan\frac{A}{2}+\tan\frac{B}{2} \\ 1 & 1 & 1 \end{vmatrix} = 0$

DT0055

2. Prove that $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1 + a^2 + b^2)^3$.

DT0056

3. Prove that : $\begin{vmatrix} (a-p)^2 & (a-q)^2 & (a-r)^2 \\ (b-p)^2 & (b-q)^2 & (b-r)^2 \\ (c-p)^2 & (c-q)^2 & (c-r)^2 \end{vmatrix} = \begin{vmatrix} (1+ap)^2 & (1+aq)^2 & (1+ar)^2 \\ (1+bp)^2 & (1+bq)^2 & (1+br)^2 \\ (1+cp)^2 & (1+cq)^2 & (1+cr)^2 \end{vmatrix}$

DT0057

4. Given $x = cy + bz$; $y = az + cx$; $z = bx + ay$, where x, y, z are not all zero, then prove that $a^2 + b^2 + c^2 + 2abc = 1$.

DT0058

5. Investigate for what values of λ, μ the simultaneous equations $x+y+z=6$; $x+2y+3z=10$ & $x+2y+\lambda z=\mu$ have :

(a) A unique solution.

(b) An infinite number of solutions.

(c) No solution.

DT0059

6. For what values of p, the equations : $x+y+z=1$; $x+2y+4z=p$ & $x+4y+10z=p^2$ have a solution ? Solve them completely in each case.

DT0060

7. Solve the equations : $Kx+2y-2z=1$, $4x+2Ky-z=2$, $6x+6y+Kz=3$ considering specially the case when $K = 2$.

DT0061

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8. Find the sum of all positive integral values of a for which every solution to the system of equation $x + ay = 3$ and $ax + 4y = 6$ satisfy the inequalities $x > 1$, $y > 0$.

DT0062

9. Given $a = \frac{x}{y-z}$; $b = \frac{y}{z-x}$; $c = \frac{z}{x-y}$, where x, y, z are not all zero, prove that : $1 + ab + bc + ca = 0$.

DT0063

10. Solve the system of equations : $\left.\begin{array}{l} z + ay + a^2x + a^3 = 0 \\ z + by + b^2x + b^3 = 0 \\ z + cy + c^2x + c^3 = 0 \end{array}\right]$ where $a \neq b \neq c$.

DT0064

EXERCISE (JM)

1. Let a, b, c be such that $b(a + c) \neq 0$. If $\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix} = 0$,

then the value of n is :- **[AIEEE - 2009]**

(1) Any odd integer (2) Any integer (3) Zero (4) Any even integer

DT0065

2. Consider the system of linear equations : $x_1 + 2x_2 + x_3 = 3$, $2x_1 + 3x_2 + x_3 = 3$, $3x_1 + 5x_2 + 2x_3 = 1$

The system has **[AIEEE - 2010]**

(1) Infinite number of solutions (2) Exactly 3 solutions

(3) A unique solution (4) No solution

DT0066

3. The number of values of k for which the linear equations

$4x + ky + 2z = 0$, $kx + 4y + z = 0$, $2x + 2y + z = 0$ possess a non-zero solution is : **[AIEEE - 2011]**

(1) 1 (2) zero (3) 3 (4) 2

DT0067

4. If the trivial solution is the only solution of the system of equations

$x - ky + z = 0$, $kx + 3y - kz = 0$, $3x + y - z = 0$ Then the set of all values of k is: **[AIEEE - 2011]**

(1) $\{2, -3\}$ (2) $R - \{2, -3\}$ (3) $R - \{2\}$ (4) $R - \{-3\}$

DT0068

5. The number of values of k, for which the system of equations : **[JEE(Main)-2013]**

$(k + 1)x + 8y = 4k$, $kx + (k + 3)y = 3k - 1$ has no solution, is -

(1) infinite (2) 1 (3) 2 (4) 3

DT0069

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6. If $\alpha, \beta \neq 0$, and $f(n) = \alpha^n + \beta^n$ and $\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix} = K(1-\alpha)^2 (1-\beta)^2 (\alpha-\beta)^2$, then K is equal to : **[JEE(Main)-2014]**

(1) $\alpha\beta$ (2) $\frac{1}{\alpha\beta}$ (3) 1 (4) –1

DT0070

7. The set of all values of λ for which the system of linear equations :
$2x_1 - 2x_2 + x_3 = \lambda x_1$, $2x_1 - 3x_2 + 2x_3 = \lambda x_2$, $-x_1 + 2x_2 = \lambda x_3$ has a non-trivial solution **[JEE(Main)-2015]**

(1) contains two elements (2) contains more than two elements
(3) is an empty set (4) is a singleton

DT0071

8. The system of linear equations $x + \lambda y - z = 0, \lambda x - y - z = 0, x + y - \lambda z = 0$ has a non-trivial solution for : **[JEE(Main)-2016]**

(1) exactly three values of λ. (2) infinitely many values of λ.
(3) exactly one value of λ. (4) exactly two values of λ.

DT0072

9. If S is the set of distinct values of 'b' for which the following system of linear equations

$$x + y + z = 1$$
$$x + ay + z = 1$$
$$ax + by + z = 0$$

has no solution, then S is : **[JEE(Main)-2017]**

(1) a singleton (2) an empty set
(3) an infinite set (4) a finite set containing two or more elements

DT0073

10. If $\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A + Bx)(x - A)^2$, then the ordered pair (A, B) is equal to : **[JEE(Main)-2018]**

(1) (–4, 3) (2) (–4, 5) (3) (4, 5) (4) (–4, –5)

DT0074

11. If the system of linear equations

$$x + ky + 3z = 0$$
$$3x + ky - 2z = 0$$
$$2x + 4y - 3z = 0$$

has a non-zero solution (x, y, z), then $\frac{xz}{y^2}$ is equal to : **[JEE(Main)-2018]**

(1) 10 (2) – 30 (3) 30 (4) –10

DT0075

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12. If the system of linear equations

$$x - 4y + 7z = g$$
$$3y - 5z = h$$
$$-2x + 5y - 9z = k$$

is consistent, then : **[JEE(Main) 2019]**

(1) $g + h + k = 0$ (2) $2g + h + k = 0$

(3) $g + h + 2k = 0$ (4) $g + 2h + k = 0$

DT0076

13. Let $d \in R$, and $A = \begin{bmatrix} -2 & 4+d & (\sin\theta)-2 \\ 1 & (\sin\theta)+2 & d \\ 5 & (2\sin\theta)-d & (-\sin\theta)+2+2d \end{bmatrix}$, $\theta \in [0, 2\pi]$. If the minimum value of det(A) is 8, then a value of d is : **[JEE(Main) 2019]**

(1) -7 (2) $2\left(\sqrt{2}+2\right)$ (3) -5 (4) $2\left(\sqrt{2}+1\right)$

DT0077

14. Let $a_1, a_2, a_3, \ldots, a_{10}$ be in G.P. with $a_i > 0$ for $i = 1, 2, \ldots, 10$ and S be the set of pairs (r,k), r, k $\in$ N (the set of natural numbers) for which $\begin{vmatrix} \log_e a_1^r a_2^k & \log_e a_2^r a_3^k & \log_e a_3^r a_4^k \\ \log_e a_4^r a_5^k & \log_e a_5^r a_6^k & \log_e a_6^r a_7^k \\ \log_e a_7^r a_8^k & \log_e a_8^r a_9^k & \log_e a_9^r a_{10}^k \end{vmatrix} = 0$. Then the number of elements in S, is : **[JEE(Main) 2019]**

(1) Infinitely many (2) 4 (3) 10 (4) 2

DT0078

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15. The set of all values of λ for which the system of linear equations.

$$x - 2y - 2z = \lambda x$$
$$x + 2y + z = \lambda y$$
$$-x - y = \lambda z$$

has a non-trivial solution. **[JEE(Main) 2019]**

(1) contains more than two elements (2) is a singleton

(3) is an empty set (4) contains exactly two elements

DT0079

16. If the system of linear equations

$2x + 2ay + az = 0$

$2x + 3by + bz = 0$

$2x + 4cy + cz = 0$, **[JEE(Main) 2020]**

where a, b, c $\in$ R are non-zero and distinct; has a non-zero solution, then :

(1) a, b, c are in A.P. (2) $a + b + c = 0$

(3) a, b, c are in G.P. (4) $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

DT0080

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17. The system of linear equations

$\lambda x + 2y + 2z = 5$

$2\lambda x + 3y + 5z = 8$

$4x + \lambda y + 6z = 10$ has **[JEE(Main) 2020]**

(1) infinitely many solutions when $\lambda = 2$ (2) a unique solution when $\lambda = -8$

(3) no solution when $\lambda = 8$ (4) no solution when $\lambda = 2$

DT0081

18. For which of the following ordered pairs (μ,δ), the system of linear equations

$x + 2y + 3z = 1$

$3x + 4y + 5z = \mu$

$4x + 4y + 4z = \delta$

is inconsistent ? **[JEE(Main) 2020]**

(1) (1,0) (2) (4,6) (3) (3,4) (4) (4,3)

DT0082

19. Let $a - 2b + c = 1$. If $f(x) = \begin{vmatrix} x+a & x+2 & x+1 \\ x+b & x+3 & x+2 \\ x+c & x+4 & x+3 \end{vmatrix}$, then : **[JEE(Main) 2020]**

(1) $f(-50) = 501$ (2) $f(-50) = -1$ (3) $f(50) = 1$ (4) $f(50) = -501$

DT0083

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EXERCISE (JA)

1. The number of all possible values of θ, where $0 < \theta < \pi$, for which the system of equations

$(y + z)\cos 3\theta = (xyz)\sin 3\theta$

$x\sin 3\theta = \dfrac{2\cos 3\theta}{y} + \dfrac{2\sin 3\theta}{z}$

$(xyz)\sin 3\theta = (y + 2z)\cos 3\theta + y\sin 3\theta$

have a solution (x_0, y_0, z_0) with $y_0 z_0 \neq 0$, is **[JEE 2010, 3]**

DT0084

2. Which of the following values of α satisfy the equation $\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha$?

[JEE(Advanced)-2015, 4M, –2M]

(A) –4 (B) 9 (C) –9 (D) 4

DT0085

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3. The total number of distinct $x \in R$ for which $\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$ is

[JEE(Advanced)-2016, 3(0)]

DT0086

4. Let $a, \lambda, m \in \mathbb{R}$. Consider the system of linear equations

$$ax + 2y = \lambda$$
$$3x - 2y = \mu$$

Which of the following statement(s) is(are) correct ?

(A) If $a = -3$, then the system has infinitely many solutions for all values of λ and μ

(B) If $a \neq -3$, then the system has a unique solution for all values of λ and μ

(C) If $\lambda + \mu = 0$, then the system has infinitely many solutions for $a = -3$

(D) If $\lambda + \mu \neq 0$, then the system has no solution for $a = -3$ **[JEE(Advanced)-2016, 4(–2)]**

DT0087

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ANSWER KEY

EXERCISE (O-1)

1. D	**2.** A	**3.** C	**4.** A	**5.** B	**6.** D	**7.** B
8. C	**9.** A	**10.** C	**11.** D	**12.** D	**13.** D	**14.** B
15. D	**16.** C	**17.** B	**18.** D			

EXERCISE (O-2)

1. C	**2.** B	**3.** A	**4.** C	**5.** D	**6.** B	**7.** A
8. A,C,D	**9.** A,B,C,D	**10.** A,B,C,D	**11.** B,D	**12.** A,B,D	**13.** B,C,D	

14. (A)→(Q); (B)→(P); (C)→(Q,S); (D)→(R,T)

EXERCISE (S-1)

1. **(a)** 4, **(b)** 65 **2.** **(a)** $x = -1$ or $x = -2$; **(b)** $x = 4$ **3.** $x = 0$ or $x = \pm\sqrt{\frac{3}{2}\left(a^2+b^2+c^2\right)}$

4. $\lambda^2(a^2+b^2+c^2+\lambda)$ **6.** 80 **7.** 0 **8.** **(a)** p

13. **(a)** $x = 1, y = 2, z = 3$; consistent **(b)** $x = 2, y = -1, z = 1$; consistent **(c)** inconsistent

14. $K = \frac{33}{2}, x : y : z = -\frac{15}{2} : 1 : -3$ **15.** 2

16. If $\lambda \neq -5$ then $x = \frac{4}{7}; y = -\frac{9}{7}$ & $z = 0$; If $\lambda = -5$ then $x = \frac{4-5K}{7}; y = \frac{13K-9}{7}$ and $z = K$, where $K \in R$

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EXERCISE (S-2)

1. Triangle ABC is isosceles **5.** (a) $\lambda \neq 3$; (b) $\lambda = 3, \mu = 10$; (c) $\lambda = 3, \mu \neq 10$

6. $x = 1 + 2k, y = -3K, z = K$, when $p = 1$; $x = 2K, y = 1 - 3K, z = K$ when $p = 2$; where $K \in R$

7. If $K \neq 2, \frac{x}{2(K+6)} = \frac{y}{2K+3} = \frac{z}{6(K-2)} = \frac{1}{2\left(K^2+2K+15\right)}$ If $K = 2$, then $x = \lambda$, $y = \frac{1-2\lambda}{2}$ and $z = 0$ where $\lambda \in R$

8. 4 **10.** $x = -(a+b+c), y = ab+bc+ca, z = -abc$

EXERCISE (JM)

1. 1	**2.** 4	**3.** 4	**4.** 2	**5.** 2	**6.** 3	**7.** 1
8. 1	**9.** 1	**10.** 2	**11.** 1	**12.** 2	**13.** 3	**14.** 1
15. 2	**16.** 4	**17.** 4	**18.** 4	**19.** 3		

EXERCISE (JA)

1. 3 **2.** B,C **3.** 2 **4.** B,C,D

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Chapter 04 POINT & STRAIGHT LINE

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JEE (Main/Advanced) Syllabus

JEE (Main) Syllabus :

Various forms of equations of a line, intersection of lines, angles between two lines, conditions for concurrence of three lines, distance of a point from a line, equations of internal and external bisectors of angles between two lines, coordinates of centroid, orthocentre and circumcentre of a triangle, equation of family of lines passing through the point of intersection of two lines.

JEE (Advanced) Syllabus :

Cartesian coordinates, distance between two points, section formulae, shift of origin. Equation of a straight line in various forms, angle between two lines, distance of a point from a line; Lines through the point of intersection of two given lines, equation of the bisector of the angle between two lines, concurrency of lines; Centroid, orthocentre, incentre and circumcentre of a triangle.

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Important Notes

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POINT & STRAIGHT LINE

1. INTRODUCTION OF COORDINATE GEOMETRY :

Coordinate geometry is the combination of algebra and geometry. A systematic study of geometry by the use of algebra was first carried out by celebrated French philosopher and mathematician René Descartes. The resulting combination of analysis and geometry is referred as ***analytical geometry***.

2. CARTESIAN CO-ORDINATES SYSTEM :

In two dimensional coordinate system, two lines are used; the lines are at right angles, forming a rectangular coordinate system. The horizontal axis is the x-axis and the vertical axis is y-axis. The point of intersection O is the origin of the coordinate system. Distances along the x-axis to the right of the origin are taken as positive, distances to the left as negative.

Distances along the y-axis above the origin are positive; distances below are negative. The position of a point anywhere in the plane can be specified by two numbers, the coordinates of the point, written as (x, y). The x-coordinate (or abscissa) is the distance of the point from the y-axis in a direction parallel to the x-axis (i.e. horizontally). The y-coordinate (or ordinate) is the distance from the x-axis in a direction parallel to the y-axis (vertically). The origin O is the point (0, 0).

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3. POLAR CO-ORDINATES SYSTEM :

A coordinate system in which the position of a point is determined by the length of a line segment from a fixed origin together with the angle that the line segment makes with a fixed line. The origin is called the pole and the line segment is the radius vector (r).

The angle θ between the polar axis and the radius vector is called the vectorial angle. By convention, positive values of θ are measured in an anticlockwise sense, negative values in clockwise sense. The coordinates of the point are then specified as (r, θ).

If (x,y) are cartesian co-ordinates of a point P, then : $x = r\cos\theta$, $y = r\sin\theta$

and $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

4. DISTANCE FORMULA AND ITS APPLICATIONS :

If $A(x_1,y_1)$ and $B(x_2,y_2)$ are two points, then $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Note :

(i) Three given points A,B and C are collinear, when sum of any two distances out of AB,BC, CA is equal to the remaining third otherwise the points will be the vertices of a triangle.

(ii) Let A,B,C & D be the four given points in a plane. Then the quadrilateral will be :

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(a) Square if AB = BC = CD = DA & AC = BD ; $AC \perp BD$

(b) Rhombus if AB = BC = CD = DA and $AC \neq BD$; $AC \perp BD$

(c) Parallelogram if AB = DC, BC = AD; $AC \neq BD$; $AC \not\perp BD$

(d) Rectangle if AB = CD, BC = DA, AC = BD ; $AC \not\perp BD$

Illustration 1 : The number of points on x-axis which are at a distance $c (c < 3)$ from the point (2, 3) is

(A) 2 (B) 1 (C) infinite (D) no point

Solution : Let a point on x-axis is $(x_1, 0)$ then its distance from the point (2, 3)

$$= \sqrt{(x_1-2)^2+9} = c \quad \text{or} \quad (x_1-2)^2 = c^2-9$$

$$\therefore\ x_1 - 2 = \pm\sqrt{c^2-9} \ \text{ since } c<3 \Rightarrow c^2-9<0$$

$\therefore$ x_1 will be imaginary. **Ans. (D)**

Illustration 2 : The distance between the point $P(a\cos\alpha,\ a\sin\alpha)$ and $Q(a\cos\beta,\ a\sin\beta)$, where $a > 0$ &

$\alpha > \beta$, is -

(A) $4a\sin\frac{\alpha-\beta}{2}$ (B) $2a\sin\frac{\alpha+\beta}{2}$ (C) $2a\sin\frac{\alpha-\beta}{2}$ (D) $2a\cos\frac{\alpha-\beta}{2}$

Solution :

$$d^2 = (a\cos\alpha - a\cos\beta)^2 + (a\sin\alpha - a\sin\beta)^2 = a^2(\cos\alpha-\cos\beta)^2 + a^2(\sin\alpha-\sin\beta)^2$$

$$= a^2\left\{2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}\right\}^2 + a^2\left\{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right\}^2$$

$$= 4a^2\sin^2\frac{\alpha-\beta}{2}\left\{\sin^2\frac{\alpha+\beta}{2}+\cos^2\frac{\alpha+\beta}{2}\right\} = 4a^2\sin^2\frac{\alpha-\beta}{2} \Rightarrow d = 2a\sin\frac{\alpha-\beta}{2}$$

Ans. (C)

Do yourself - 1 :

(i) Find the distance between the points P(–3, 2) and Q(2, –1).

(ii) If the distance between the points P(–3, 5) and Q(–x, – 2) is $\sqrt{58}$, then find the value(s) of x.

(iii) A line segment is of the length 15 units and one end is at the point (3, 2), if the abscissa of the other end is 15, then find possible ordinates.

5. SECTION FORMULA :

The co-ordinates of a point dividing a line joining the points $P(x_1,y_1)$ and $Q(x_2,y_2)$ in the ratio m:n is given by :

(a) For internal division : P - R - Q $\Rightarrow$ R divides line segment PQ, internally.

$$(x, y) \equiv \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$

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(b) **For external division :** R - P - Q or P - Q - R $\Rightarrow$ R divides line segment PQ, externally.

$$(x, y) \equiv \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$

$\frac{(PR)}{(QR)} < 1 \Rightarrow$ R lies on the left of P & $\frac{(PR)}{(QR)} > 1 \Rightarrow$ R lies on the right of Q

(c) **Harmonic conjugate :** If P divides AB internally in the ratio m : n & Q divides AB externally in the ratio m : n then P & Q are said to be harmonic conjugate of each other w.r.t. AB. Mathematically ; $\frac{2}{AB} = \frac{1}{AP} + \frac{1}{AQ}$ i.e. AP, AB & AQ are in **H.P.**

Illustration 3 : Determine the ratio in which $y - x + 2 = 0$ divides the line joining (3, –1) and (8, 9).

Solution : Suppose the line $y - x + 2 = 0$ divides the line segment joining A(3, –1) and B(8, 9) in the ratio $\lambda : 1$ at a point P, then the co-ordinates of the point P are $\left(\frac{8\lambda+3}{\lambda+1}, \frac{9\lambda-1}{\lambda+1}\right)$

But P lies on $y - x + 2 = 0$ therefore $\left(\frac{9\lambda-1}{\lambda+1}\right) - \left(\frac{8\lambda+3}{\lambda+1}\right) + 2 = 0$

$\Rightarrow$ $9\lambda - 1 - 8\lambda - 3 + 2\lambda + 2 = 0$

$\Rightarrow$ $3\lambda - 2 = 0$ or $\lambda = \frac{2}{3}$

So, the required ratio is $\frac{2}{3} : 1$, i.e., 2 : 3 (internally) since here λ is positive.

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Do yourself - 2 :

(i) Find the co-ordinates of the point dividing the join of A(1, – 2) and B(4, 7) :

(a) Internally in the ratio 1 : 2 (b) Externally in the ratio of 2 : 1

(ii) In what ratio is the line joining A(8, 9) and B(– 7, 4) divided by

(a) the point (2, 7) (b) the x-axis (c) the y-axis.

6. CO-ORDINATES OF SOME PARTICULAR POINTS :

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of any triangle ABC, then

(a) **Centroid :**

The centroid is the point of intersection of the medians (line joining the mid point of sides and opposite vertices). Centroid divides each median in the ratio of 2 : 1.

Co-ordinates of centroid $G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

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(b) Incenter :

The incenter is the point of intersection of internal bisectors of the angles of a triangle. Also it is a centre of the circle touching all the sides of a triangle.

Co-ordinates of incenter $I\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$

where a, b, c are the sides of triangle ABC.

Note :

(i) Angle bisector divides the opposite sides in the ratio of remaining sides. e.g.

$\frac{BD}{DC}=\frac{AB}{AC}=\frac{c}{b}$

(ii) Incenter divides the angle bisectors in the ratio $(b+c):a$, $(c+a):b$, $(a+b):c$.

(c) Circumcenter :

It is the point of intersection of perpendicular bisectors of the sides of a triangle. If O is the circumcenter of any triangle ABC, then $OA^2=OB^2=OC^2$. Also it is a centre of a circle touching all the vertices of a triangle.

Note :

(i) If the triangle is right angled, then its circumcenter is the mid point of hypotenuse.

(ii) Co-ordinates of circumcenter $\left(\frac{x_1\sin 2A+x_2\sin 2B+x_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}, \frac{y_1\sin 2A+y_2\sin 2B+y_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}\right)$

(d) Orthocenter :

It is the point of intersection of perpendiculars drawn from vertices on the opposite sides of a triangle and it can be obtained by solving the equation of any two altitudes.

Note :

(i) If a triangle is right angled, then orthocenter is the point where right angle is formed.

(ii) Co-ordinates of circumcenter $\left(\frac{x_1\tan A+x_2\tan B+x_3\tan C}{\tan A+\tan B+\tan C}, \frac{y_1\tan A+y_2\tan B+y_3\tan C}{\tan A+\tan B+\tan C}\right)$

Remarks :

(i) If the triangle is equilateral, then centroid, incentre, orthocenter, circumcenter, coincide.

(ii) Orthocentre, centroid and circumcentre are always collinear and centroid divides the line joining orthocentre and circumcentre in the ratio 2 : 1

(iii) In an isosceles triangle centroid, orthocentre, incentre & circumcentre lie on the same line.

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(e) Ex-centers :

The centre of a circle which touches side BC and the extended portions of sides AB and AC is called the ex-centre of ΔABC with respect to the vertex A. It is denoted by I_1 and its coordinates

are $I_1\left(\dfrac{-ax_1+bx_2+cx_3}{-a+b+c}, \dfrac{-ay_1+by_2+cy_3}{-a+b+c}\right)$

Similarly ex-centers of ΔABC with respect to vertices B and C are denoted by I_2 and I_3 respectively , and

$I_2\left(\dfrac{ax_1-bx_2+cx_3}{a-b+c}, \dfrac{ay_1-by_2+cy_3}{a-b+c}\right), I_3\left(\dfrac{ax_1+bx_2-cx_3}{a+b-c}, \dfrac{ay_1+by_2-cy_3}{a+b-c}\right)$

Illustration 4 : If $\left(\dfrac{5}{3}, 3\right)$ is the centroid of a triangle and its two vertices are (0, 1) and (2, 3), then find its third vertex, circumcentre, circumradius & orthocentre.

Solution : Let the third vertex of triangle be (x, y), then

$\dfrac{5}{3} = \dfrac{x+0+2}{3} \Rightarrow x = 3$ and $3 = \dfrac{y+1+3}{3} \Rightarrow y = 5$. So third vertex is (3, 5).

Now three vertices are A(0, 1), B(2, 3) and C(3, 5)

Let circumcentre be P(h, k),

then AP = BP = CP = R (circumradius) $\Rightarrow$ $AP^2 = BP^2 = CP^2 = R^2$

$h^2 + (k-1)^2 = (h-2)^2 + (k-3)^2 = (h-3)^2 + (k-5)^2 = R^2$ ..... (i)

from the first two equations, we have

$h + k = 3$ ..... (ii)

from the first and third equation, we obtain

$6h + 8k = 33$ ..... (iii)

On solving, (ii) & (iii), we get

$h = -\dfrac{9}{2}, \; k = \dfrac{15}{2}$

Substituting these values in (i), we have

$R = \dfrac{5}{2}\sqrt{10}$

Let $O(x_1, y_1)$ be the orthocentre, then $\dfrac{x_1 + 2\left(-\dfrac{9}{2}\right)}{3} = \dfrac{5}{3} \Rightarrow x_1 = 14, \dfrac{y_1 + 2\left(\dfrac{15}{2}\right)}{3} = 3$

$\Rightarrow$ $y_1 = -6$. Hence orthocentre of the triangle is (14, –6).

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Illustration 5 : The vertices of a triangle are A(0, –6), B(–6, 0) and C(1,1) respectively, then coordinates of the ex-centre opposite to vertex A is :

(A) $\left(\frac{-3}{2},\frac{-3}{2}\right)$ (B) $\left(-4,\frac{3}{2}\right)$ (C) $\left(\frac{-3}{2},\frac{3}{2}\right)$ (D) (–4, 6)

Solution :

$$a = BC = \sqrt{(-6-1)^2+(0-1)^2} = \sqrt{50} = 5\sqrt{2}$$

$$b = CA = \sqrt{(1-0)^2+(1+6)^2} = \sqrt{50} = 5\sqrt{2}$$

$$c = AB = \sqrt{(0+6)^2+(-6-0)^2} = \sqrt{72} = 6\sqrt{2}$$

coordinates of ex-centre opposite to vertex A will be :

$$x = \frac{-ax_1+bx_2+cx_3}{-a+b+c} = \frac{-5\sqrt{2}.0+5\sqrt{2}(-6)+6\sqrt{2}(1)}{-5\sqrt{2}+5\sqrt{2}+6\sqrt{2}} = \frac{-24\sqrt{2}}{6\sqrt{2}} = -4$$

$$y = \frac{-ay_1+by_2+cy_3}{-a+b+c} = \frac{-5\sqrt{2}(-6)+5\sqrt{2}.0+6\sqrt{2}(1)}{-5\sqrt{2}+5\sqrt{2}+6\sqrt{2}} = \frac{36\sqrt{2}}{6\sqrt{2}} = 6$$

Hence coordinates of ex-centre is (–4, 6) **Ans. (D)**

Do yourself - 3 :

(i) The coordinates of the vertices of a triangle are (0, 1), (2, 3) and (3, 5) :

(a) Find centroid of the triangle.

(b) Find circumcentre & the circumradius.

(c) Find orthocentre of the triangle.

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7. AREA OF TRIANGLE :

Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of a triangle, then

$$\text{Area of } \Delta ABC = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right| = \frac{1}{2}\left|[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]\right|$$

To remember the above formula, take the help of the following method :

$$= \frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{bmatrix} = \left|\frac{1}{2}[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_1-x_1y_3)]\right|$$

Remarks :

(i) If the area of triangle joining three points is zero, then the points are collinear.

(ii) **Area of Equilateral triangle :** If altitude of any equilateral triangle is P, then its area $= \frac{P^2}{\sqrt{3}}$. If 'a' be the side of equilateral triangle, then its area $= \left(\frac{a^2\sqrt{3}}{4}\right)$.

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(iii) Area of quadrilateral with given vertices $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3), D(x_4, y_4)$

Area of quad. ABCD $= \frac{1}{2}\left|\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{bmatrix}\right|$

Note : Area of a polygon can be obtained by dividing the polygon into disjoined triangles and then adding their areas.

Illustration 6 : If the vertices of a triangle are (1, 2), (4, –6) and (3, 5) then its area is

(A) $\frac{25}{2}$ sq. units (B) 12 sq. units (C) 5 sq. units (D) 25 sq. units

Solution : $\Delta = \frac{1}{2}\left[1(-6-5)+4(5-2)+3(2+6)\right] = \frac{1}{2}\left[-11+12+24\right] = \frac{25}{2}$ square units **Ans. (A)**

Illustration 7 : The point A divides the join of the points (–5, 1) and (3, 5) in the ratio k:1 and coordinates of points B and C are (1, 5) and (7, –2) respectively. If the area of ΔABC be 2 units, then k equals -

(A) 7, 9 (B) 6, 7 (C) 7, $\frac{31}{9}$ (D) 9, $\frac{31}{9}$

Solution : $A \equiv \left(\frac{3k-5}{k+1}, \frac{5k+1}{k+1}\right)$

Area of $\Delta ABC = 2$ units $\Rightarrow \frac{1}{2}\left[\frac{3k-5}{k+1}(5+2)+1\left(-2-\frac{5k+1}{k+1}\right)+7\left(\frac{5k+1}{k+1}-5\right)\right] = \pm 2$

$\Rightarrow 14k - 66 = \pm 4(k+1) \Rightarrow k = 7$ or $\frac{31}{9}$ **Ans. (C)**

Illustration 8 : Prove that the co-ordinates of the vertices of an equilateral triangle can not all be rational.

Solution : Let $A(x_1\ y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle ABC. If possible let x_1, y_1, x_2, y_2, x_3, y_3 be all rational.

Now area of $\Delta ABC = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)| =$ Rational ..... (i)

Since ΔABC is equilateral

$\therefore$ Area of $\Delta ABC = \frac{\sqrt{3}}{4}(\text{side})^2 = \frac{\sqrt{3}}{4}(AB)^2 = \frac{\sqrt{3}}{4}\{(x_1-x_2)^2+(y_1-y_2)^2\} =$ Irrational ..... (ii)

From (i) and (ii),

Rational = Irrational

which is contradiction.

Hence x_1, y_1, x_2, y_2, x_3, y_3 cannot all be rational.

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8. CONDITIONS FOR COLLINEARITY OF THREE GIVEN POINTS :

Three given points A (x_1, y_1), B (x_2, y_2), C (x_3, y_3) are collinear if any one of the following conditions are satisfied.

(a) Area of triangle ABC is zero i.e. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(b) Slope of AB = slope of BC = slope of AC. i.e. $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} = \frac{y_3 - y_1}{x_3 - x_1}$

(c) Find the equation of line passing through 2 given points, if the third point satisfies the given equation of the line, then three points are collinear.

Do yourself - 4 :

(i) Find the area of the triangle whose vertices are A(1,1), B(7, – 3) and C(12, 2)

(ii) Find the area of the quadrilateral whose vertices are A(1,1) B(7, – 3), C(12,2) and D(7, 21)

(iii) Prove that the points A(a, b + c), B(b, c + a) and C(c, a + b) are collinear (By determinant method)

9. LOCUS :

The locus of a moving point is the path traced out by that point under one or more geometrical conditions.

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(a) Equation of Locus :

The equation to a locus is the relation which exists between the coordinates of any point on the path, and which holds for no other point except those lying on the path.

(b) Procedure for finding the equation of the locus of a point :

(i) If we are finding the equation of the locus of a point P, assign coordinates (h, k) to P.

(ii) Express the given condition as equations in terms of the known quantities to facilitate calculations. We sometimes include some unknown quantities known as parameters.

(iii) Eliminate the parameters, so that the eliminate contains only h, k and known quantities.

(iv) Replace h by x, and k by y, in the eliminate. The resulting equation would be the equation of the locus of P.

Illustration 9 : The ends of the rod of length ℓ moves on two mutually perpendicular lines, find the locus of the point on the rod which divides it in the ratio $m_1 : m_2$

(A) $m_1^2x^2 + m_2^2y^2 = \frac{\ell^2}{(m_1 + m_2)^2}$

(B) $(m_2x)^2 + (m_1y)^2 = \left(\frac{m_1m_2\ell}{m_1 + m_2}\right)^2$

(C) $(m_1x)^2 + (m_2y)^2 = \left(\frac{m_1m_2\ell}{m_1 + m_2}\right)^2$

(D) none of these

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Solution : Let (h,k) be the point that divide the rod $AB = \ell$, in the ratio $m_1 : m_2$, and OA = a, OB = b say

$\therefore \quad a^2 + b^2 = \ell^2$ ..... (i)

Now $h = \left(\dfrac{m_2 a}{m_1 + m_2}\right) \Rightarrow a = \left(\dfrac{m_1 + m_2}{m_2}\right) h$

$k = \left(\dfrac{m_1 b}{m_1 + m_2}\right) \Rightarrow b = \left(\dfrac{m_1 + m_2}{m_1}\right) k$

putting these values in (i) $\dfrac{(m_1 + m_2)^2}{m_2^2} h^2 + \dfrac{(m_1 + m_2)^2}{m_1^2} k^2 = \ell^2$

$\therefore$ Locus of (h,k) is $m_1^2 x^2 + m_2^2 y^2 = \left(\dfrac{m_1 m_2 \ell}{m_1 + m_2}\right)^2$ **Ans. (C)**

Illustration 10 : A(a, 0) and B(–a, 0) are two fixed points of ΔABC. If its vertex C moves in such a way that cotA + cotB = λ, where λ is a constant, then the locus of the point C is -

(A) yλ = 2a (B) y = λa (C) ya =2λ (D) none of these

Solution : Given that coordinates of two fixed points A and B are (a, 0) and (–a, 0) respectively. Let variable point C is (h, k). From the adjoining figure

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$\cot A = \dfrac{DA}{CD} = \dfrac{a-h}{k}$

$\cot B = \dfrac{BD}{CD} = \dfrac{a+h}{k}$

But cotA + cotB = λ, so we have

$\dfrac{a-h}{k} + \dfrac{a+h}{k} = \lambda \Rightarrow \dfrac{2a}{k} = \lambda$

Hence locus of C is yλ = 2a **Ans. (A)**

Do yourself - 5 :

(i) Find the locus of a variable point which is at a distance of 2 units from the y-axis.

(ii) Find the locus of a point which is equidistant from both the axes.

10. STRAIGHT LINE :

Introduction : A relation between x and y which is satisfied by co-ordinates of every point lying on a line is called equation of the straight line. Here, remember that every one degree equation in variable x and y always represents a straight line i.e. **ax + by + c = 0 ; a & b ≠ 0** simultaneously.

(a) Equation of a line parallel to x-axis at a distance 'a' is **y = a** or **y = – a.**

(b) Equation of x-axis is **y = 0.**

(c) Equation of a line parallel to y-axis at a distance 'b' is **x = b** or **x = – b.**

(d) Equation of y-axis is **x = 0.**

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Illustration 11 : Prove that every first degree equation in x, y represents a straight line.

Solution : Let ax + by + c = 0 be a first degree equation in x, y where a, b, c are constants.

Let $P(x_1, y_1)$ & $Q(x_2, y_2)$ be any two points on the curve represented by ax + by + c = 0.

Then $ax_1 + by_1 + c = 0$ and $ax_2 + by_2 + c = 0$

Let R be any point on the line segment joining P & Q

Suppose R divides PQ in the ratio λ : 1. Then, the coordinates of R are $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$

We have $a\left(\frac{\lambda x_2 + x_1}{\lambda + 1}\right) + b\left(\frac{\lambda y_2 + y_1}{\lambda + 1}\right) + c = \lambda\, 0 + 0 = 0$

$\therefore$ $R\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$ lies on the curve represented by ax + by + c = 0. Thus every point on the line segment joining P & Q lies on ax + by + c = 0.

Hence ax + by + c = 0 represents a straight line.

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11. SLOPE OF LINE :

If a given line makes an angle **θ(0° ≤ θ < 180°, θ ≠ 90°)** with the positive direction of x-axis, then slope of this line will be tanθ and is usually denoted by the letter **m** i.e. **m = tanθ**. If $A(x_1, y_1)$ and $B(x_2, y_2)$ & $x_1 \neq x_2$ then slope of line AB = $\frac{y_2 - y_1}{x_2 - x_1}$

Remark :

(i) If θ = 90°, **m does not exist** and line is parallel to **y-axis.**

(ii) If θ = 0°, **m = 0** and the line is parallel to **x-axis.**

(iii) Let m_1 and m_2 be slopes of two given lines (none of them is parallel to y-axis)

(a) If lines are parallel, $\mathbf{m_1 = m_2}$ and vice-versa.

(b) If lines are perpendicular, $\mathbf{m_1 m_2 = -1}$ and vice-versa

12. STANDARD FORMS OF EQUATIONS OF A STRAIGHT LINE :

(a) Slope Intercept form : Let m be the slope of a line and c its intercept on y-axis. Then the equation of this straight line is written as : y = mx + c

If the line passes through origin, its equation is written as y = mx

(b) Point Slope form : If m be the slope of a line and it passes through a point (x_1, y_1), then its equation is written as : $y - y_1 = m(x - x_1)$

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(c) **Two point form :** Equation of a line passing through two points (x_1,y_1) and (x_2,y_2) is written as :

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \text{ or } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(d) **Intercept form :** If a and b are the intercepts made by a line on the axes of x and y, its equation is written as : $\frac{x}{a} + \frac{y}{b} = 1$

(i) Length of intercept of line between the coordinate axes $= \sqrt{a^2 + b^2}$

(ii) Area of triangle AOB $= \frac{1}{2}OA.OB = \left|\frac{1}{2}ab\right|$

Illustration 12 : The equation of the lines which passes through the point (3, 4) and the sum of its intercepts on the axes is 14 is -

(A) $4x - 3y = 24$, $x - y = 7$ (B) $4x + 3y = 24$, $x + y = 7$

(C) $4x + 3y + 24 = 0$, $x + y + 7=0$ (D) $4x - 3y + 24 = 0$, $x - y + 7 = 0$

Solution : Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$ .....(i)

This passes through (3 , 4) , therefore $\frac{3}{a} + \frac{4}{b} = 1$ .....(ii)

It is given that $a + b = 14 \Rightarrow b = 14 - a$. Putting $b = 14 - a$ in (ii) , we get

$\frac{3}{a} + \frac{4}{14-a} = 1 \Rightarrow a^2 - 13a + 42 = 0 \Rightarrow (a - 7)(a - 6) = 0 \Rightarrow a = 7 , 6$

For $a = 7$, $b = 14 - 7 = 7$ and for $a = 6$, $b = 14 - 6 = 8$

Putting the values of a and b in (i) , we get the equations of the lines

$\frac{x}{7} + \frac{y}{7} = 1$ and $\frac{x}{6} + \frac{y}{8} = 1$ or $x + y = 7$ and $4x + 3y = 24$ **Ans. (B)**

Illustration 13 : Two points A and B move on the positive direction of x-axis and y-axis respectively, such that OA + OB = K. Show that the locus of the foot of the perpendicular from the origin O on the line AB is $(x + y)(x^2 + y^2) = Kxy$.

Solution : Let the equation of AB be $\frac{x}{a} + \frac{y}{b} = 1$ ..... (i)

given, $a + b = K$ ..... (ii)

now, $m_{AB} \times m_{OM} = -1 \Rightarrow ah = bk$ ..... (iii)

from (ii) and (iii),

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$a = \dfrac{kK}{h+k}$ and $b = \dfrac{hK}{h+k}$

$\therefore$ from (i) $\dfrac{x(h+k)}{k.K} + \dfrac{y(h+k)}{h.K} = 1$

as it passes through (h, k)

$\dfrac{h(h+k)}{k.K} + \dfrac{k(h+k)}{h.K} = 1 \Rightarrow \quad (h+k)(h^2+k^2) = Khk$

$\therefore$ locus of (h, k) is $(x+y)(x^2+y^2) = Kxy$.

(e) **Normal form :** If p is the length of perpendicular on a line from the origin, and α the angle which this perpendicular makes with positive x-axis, then the equation of this line is written as : $\mathbf{x\cos\alpha + y\sin\alpha = p}$ (p is always positive) where $\mathbf{0 \le \alpha < 2\pi}$.

Illustration 14 : Find the equation of the straight line on which the perpendicular from origin makes an angle 30° with positive x-axis and which forms a triangle of area $\left(\dfrac{50}{\sqrt{3}}\right)$ sq. units with the co-ordinates axes.

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Solution : $\angle NOA = 30°$

Let ON = p > 0, OA = a, OB = b

In ΔONA, $\cos 30° = \dfrac{ON}{OA} = \dfrac{p}{a} \Rightarrow \dfrac{\sqrt{3}}{2} = \dfrac{p}{a}$

or $a = \dfrac{2p}{\sqrt{3}}$

and in ΔONB, $\cos 60° = \dfrac{ON}{OB} = \dfrac{p}{b} \Rightarrow \dfrac{1}{2} = \dfrac{p}{b}$

or $b = 2p$

$\because$ Area of $\Delta OAB = \dfrac{1}{2}ab = \dfrac{1}{2}\left(\dfrac{2p}{\sqrt{3}}\right)(2p) = \dfrac{2p^2}{\sqrt{3}}$

$\therefore \quad \dfrac{2p^2}{\sqrt{3}} = \dfrac{50}{\sqrt{3}} \Rightarrow \quad p^2 = 25$

or $p = 5$

$\therefore$ Using $x\cos\alpha + y\sin\alpha = p$, the equation of the line AB is $x\cos 30° + y\sin 30° = 5$

or $x\sqrt{3} + y = 10$

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(f) **Parametric form :** To find the equation of a straight line which passes through a given point A(h, k) and makes a given angle θ with the positive direction of the x-axis. P(x, y) is any point on the line LAL'.

Let AP = r, then $\mathbf{x - h = r\cos\theta,\ y - k = r\sin\theta}$ & $\dfrac{x-h}{\cos\theta} = \dfrac{y-k}{\sin\theta} = \mathbf{r}$ is the equation of the straight line LAL'.

Any point P on the line will be of the form (h + r cosθ, k + r sinθ), where |r| gives the distance of the point P from the fixed point (h, k).

Illustration 15 : Equation of a line which passes through point A(2, 3) and makes an angle of 45° with x axis. If this line meet the line x + y + 1 = 0 at point P then distance AP is -

(A) $2\sqrt{3}$ (B) $3\sqrt{2}$ (C) $5\sqrt{2}$ (D) $2\sqrt{5}$

Solution : Here $x_1 = 2$, $y_1 = 3$ and $\theta = 45°$ hence $\dfrac{x-2}{\cos 45°} = \dfrac{y-3}{\sin 45°} = r$

from first two parts $\Rightarrow$ $x - 2 = y - 3 \Rightarrow x - y + 1 = 0$

Co-ordinate of point P on this line is $\left(2 + \dfrac{r}{\sqrt{2}}, 3 + \dfrac{r}{\sqrt{2}}\right)$.

If this point is on line x + y + 1 = 0 then

$\left(2 + \dfrac{r}{\sqrt{2}}\right) + \left(3 + \dfrac{r}{\sqrt{2}}\right) + 1 = 0 \Rightarrow r = -3\sqrt{2}$; $|r| = 3\sqrt{2}$ **Ans. (B)**

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Illustration 16 : A variable line is drawn through O, to cut two fixed straight lines L_1 and L_2 in A_1 and A_2, respectively. A point A is taken on the variable line such that $\dfrac{m+n}{OA} = \dfrac{m}{OA_1} + \dfrac{n}{OA_2}$.

Show that the locus of A is a straight line passing through the point of intersection of L_1 and L_2 where O is being the origin.

Solution : Let the variable line passing through the origin is $\dfrac{x}{\cos\theta} = \dfrac{y}{\sin\theta} = r_i$ ..... (i)

Let the equation of the line L_1 is $p_1x + q_1y = 1$ ..... (ii)

Equation of the line L_2 is $p_2x + q_2y = 1$ ..... (iii)

the variable line intersects the line (ii) at A_1 and (iii) at A_2.

Let $OA_1 = r_1$.

Then $A_1 = (r_1\cos\theta, r_1\sin\theta) \Rightarrow A_1$ lies on L_1

$\Rightarrow \quad r_1 = OA_1 = \dfrac{1}{p_1\cos\theta + q_1\sin\theta}$

Similarly, $r_2 = OA_2 = \dfrac{1}{p_2\cos\theta + q_2\sin\theta}$

Let OA = r

Let co-ordinate of A are (h, k) $\Rightarrow$ (h, k) ≡ (rcosθ, rsinθ)

Now $\dfrac{m+n}{r} = \dfrac{m}{OA_1} + \dfrac{n}{OA_2} \Rightarrow \dfrac{m+n}{r} = \dfrac{m}{r_1} + \dfrac{n}{r_2}$

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$\Rightarrow \quad m + n = m(p_1 r\cos\theta + q_1 r\sin\theta) + n(p_2 r\cos\theta + q_2 r\sin\theta)$

$\Rightarrow \quad (p_1 h + q_1 k - 1) + \frac{n}{m}(p_2 h + q_2 k - 1) = 0$

Therefore, locus of A is $(p_1 x + q_1 y - 1) + \frac{n}{m}(p_2 x + q_2 y - 1) = 0$

$\Rightarrow \quad L_1 + \lambda L_2 = 0$ where $\lambda = \frac{n}{m}$.

This is the equation of the line passing through the intersection of L_1 and L_2.

Illustration 17 : A straight line through P(–2, –3) cuts the pair of straight lines $x^2 + 3y^2 + 4xy - 8x - 6y - 9 = 0$ in Q and R. Find the equation of the line if PQ. PR = 20.

Solution : Let line be $\frac{x+2}{\cos\theta} = \frac{y+3}{\sin\theta} = r$

$\Rightarrow \quad x = r\cos\theta - 2,\ y = r\sin\theta - 3$ ..... (i)

Now, $x^2 + 3y^2 + 4xy - 8x - 6y - 9 = 0$ ..... (ii)

Taking intersection of (i) with (ii) and considering terms of r^2 and constant (as we need PQ . PR = $r_1 . r_2$ = product of the roots)

$r^2(\cos^2\theta + 3\sin^2\theta + 4\sin\theta\cos\theta) + (\text{some terms})r + 80 = 0$

$\therefore \quad r_1.r_2 = PQ.\ PR = \frac{80}{\cos^2\theta + 4\sin\theta\cos\theta + 3\sin^2\theta}$

$\therefore \quad \cos^2\theta + 4\sin\theta\cos\theta + 3\sin^2\theta = 4$ ($\because$ PQ . PR = 20)

$\therefore \quad \sin^2\theta - 4\sin\theta\cos\theta + 3\cos^2\theta = 0$

$\Rightarrow \quad (\sin\theta - \cos\theta)(\sin\theta - 3\cos\theta) = 0$

$\therefore \quad \tan\theta = 1,\ \tan\theta = 3$

hence equation of the line is $y + 3 = 1(x + 2) \Rightarrow x - y = 1$

and $y + 3 = 3(x + 2) \Rightarrow 3x - y + 3 = 0$.

Illustration 18 : If the line $y - \sqrt{3}x + 3 = 0$ cuts the parabola $y^2 = x + 2$ at A and B, then find the value of PA.PB {where $P \equiv (\sqrt{3}, 0)$}

Solution : Slope of line $y - \sqrt{3}x + 3 = 0$ is $\sqrt{3}$.

If line makes an angle θ with x-axis, then $\tan\theta = \sqrt{3}$

$\therefore \quad \theta = 60°$

$\frac{x - \sqrt{3}}{\cos 60°} = \frac{y - 0}{\sin 60°} = r \Rightarrow \left(\sqrt{3} + \frac{r}{2}, \frac{r\sqrt{3}}{2}\right)$ be a point on the parabola $y^2 = x + 2$

then $\frac{3}{4}r^2 = \sqrt{3} + \frac{r}{2} + 2 \Rightarrow 3r^2 - 2r - 4(2 + \sqrt{3}) = 0$

$\therefore \quad PA.PB = r_1 r_2 = \left|\frac{-4(2+\sqrt{3})}{3}\right| = \frac{4(2+\sqrt{3})}{3}$

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Do yourself - 6 :

(i) Reduce the line $2x - 3y + 5 = 0$,

(a) In slope- intercept form and hence find slope & Y-intercept

(b) In intercept form and hence find intercepts on the axes.

(c) In normal form and hence find perpendicular distance from the origin and angle made by the perpendicular with the positive x-axis.

(ii) Find distance of point A (2, 3) measured parallel to the line $x - y = 5$ from the line $2x + y + 6 = 0$

(g) **General form :** We know that a first degree equation in x and y, $ax + by + c = 0$ always represents a straight line. This form is known as general form of straight line.

(i) Slope of this line $= \frac{-a}{b} = -\frac{\text{coeff. of } x}{\text{coeff. of } y}$

(ii) Intercept by this line on x-axis $= -\frac{c}{a}$ and intercept by this line on y-axis $= -\frac{c}{b}$

(iii) To change the general form of a line to normal form, first take c to right hand side and make it positive, then divide the whole equation by $\sqrt{a^2 + b^2}$.

13. EQUATION OF LINES PARALLEL AND PERPENDICULAR TO A GIVEN LINE :

(a) Equation of line parallel to line $ax + by + c = 0$

$\mathbf{ax + by + \lambda = 0}$

(b) Equation of line perpendicular to line $ax + by + c = 0$

$\mathbf{bx - ay + k = 0}$

Here λ, k, are parameters and their values are obtained with the help of additional information given in the problem.

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14. ANGLE BETWEEN TWO LINES :

(a) If θ be the angle between two lines : $y = m_1x + c_1$ and $y = m_2x + c_2$, then $\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1m_2}\right)$

Note :

(i) There are two angles formed between two lines but usually the acute angle is taken as the angle between the lines. So we shall find θ from the above formula only by taking positive value of $\tan\theta$.

(ii) Let m_1, m_2, m_3 are the slopes of three lines $L_1 = 0$; $L_2 = 0$; $L_3 = 0$ where $m_1 > m_2 > m_3$ then the interior angles of the ΔABC found by these formulas are given by,

$$\tan A = \frac{m_1 - m_2}{1 + m_1 m_2} ; \tan B = \frac{m_2 - m_3}{1 + m_2 m_3} \ \& \ \tan C = \frac{m_3 - m_1}{1 + m_3 m_1}$$

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(b) If equation of lines are $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, then these lines are -

(i) Parallel $\Leftrightarrow$ $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$

(ii) Perpendicular $\Leftrightarrow$ $a_1a_2+b_1b_2=0$

(iii) Coincident $\Leftrightarrow$ $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

Illustration 19 : If $x + 4y - 5 = 0$ and $4x + ky + 7 = 0$ are two perpendicular lines then k is -

(A) 3 (B) 4 (C) –1 (D) –4

Solution : $m_1 = -\frac{1}{4}$ $m_2 = -\frac{4}{k}$

Two lines are perpendicular if $m_1 m_2 = -1$

$\Rightarrow \left(-\frac{1}{4}\right)\times\left(-\frac{4}{k}\right)=-1 \Rightarrow k=-1$ **Ans. (C)**

Illustration 20 : A line L passes through the points (1, 1) and (0, 2) and another line M which is perpendicular to L passes through the point (0, –1/2).The area of the triangle formed by these lines with y-axis is -

(A) 25/8 (B) 25/16 (C) 25/4 (D) 25/32

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Solution : Equation of the line L is $y - 1 = \frac{-1}{1}(x-1) \Rightarrow y = -x + 2$

Equation of the line M is $y = x - 1/2$.

If these lines meet y-axis at P and Q, then PQ = 5/2.

Also x-coordinate of their point of intersection R = 5/4

$\therefore$ area of the $\Delta PQR = \frac{1}{2}\left(\frac{5}{2}\times\frac{5}{4}\right)=25/16.$ **Ans. (B)**

Illustration 21 : If the straight line $3x + 4y + 5 - k(x + y + 3) = 0$ is parallel to y-axis, then the value of k is -

(A) 1 (B) 2 (C) 3 (D) 4

Solution : A straight line is parallel to y-axis, if its y - coefficient is zero, i.e. $4 - k = 0$ i.e. $k = 4$ **Ans. (D)**

15. STRAIGHT LINE MAKING A GIVEN ANGLE WITH A LINE :

Equation of line passing through a point (x_1,y_1) and making an angle α, with the line y=mx+c is written as :

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

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Illustration 22 : Find the equation to the sides of an isosceles right-angled triangle, the equation of whose hypotenuse is 3x + 4y = 4 and the opposite vertex is the point (2, 2).

Solution : The problem can be restated as :

Find the equations of the straight lines passing through the given point (2, 2) and making equal angles of 45° with the given straight line 3x + 4y – 4 = 0. Slope of the line 3x + 4y – 4 = 0 is $m_1 = -3/4$.

$\Rightarrow \quad \tan 45° = \pm\dfrac{m - m_1}{1 + m_1 m}$, i.e., $1 = \pm\dfrac{m + 3/4}{1 - \frac{3}{4}m}$

$m_A = \dfrac{1}{7}$, and $m_B = -7$

Hence the required equations of the two lines are

$y - 2 = m_A(x - 2)$ and $y - 2 = m_B(x - 2)$

$\Rightarrow \quad 7y - x - 12 = 0$ and $7x + y = 16$ **Ans.**

Do yourself - 7 :

(i) Find the angle between the lines 3x + y – 7 = 0 and x + 2y – 9 = 0.

(ii) Find the line passing through the point (2, 3) and perpendicular to the straight line 4x – 3y = 10.

(iii) Find the equation of the line which has positive y-intercept 4 units and is parallel to the line 2x – 3y – 7 = 0. Also find the point where it cuts the x-axis.

(iv) Classify the following pairs of lines as coincident, parallel or intersecting :

(a) x + 2y – 3 = 0 & –3x – 6y + 9 = 0

(b) x + 2y + 1 = 0 & 2x + 4y + 3 = 0

(c) 3x – 2y + 5 = 0 & 2x + y – 5 = 0

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16. LENGTH OF PERPENDICULAR FROM A POINT ON A LINE :

Length of perpendicular from a point (x_1, y_1) on the line ax + by + c = 0 is $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

In particular, the length of the perpendicular from the origin on the line ax + by + c = 0 is $P = \dfrac{|c|}{\sqrt{a^2 + b^2}}$

Illustration 23 : If the algebraic sum of perpendiculars from n given points on a variable straight line is zero then prove that the variable straight line passes through a fixed point.

Solution : Let n given points be (x_i, y_i) where i = 1, 2.... n and the variable straight line is ax + by + c = 0.

Given that $\displaystyle\sum_{i=1}^{n}\left(\frac{ax_i + by_i + c}{\sqrt{a^2 + b^2}}\right) = 0 \Rightarrow a\Sigma x_i + b\Sigma y_i + cn = 0 \Rightarrow a\frac{\Sigma x_i}{n} + b\frac{\Sigma y_i}{n} + c = 0$.

Hence the variable straight line always passes through the fixed point $\left(\dfrac{\Sigma x_i}{n}, \dfrac{\Sigma y_i}{n}\right)$.

Ans.

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Illustration 24 : Prove that no line can be drawn through the point (4, –5) so that its distance from (–2, 3) will be equal to 12.

Solution : Suppose, if possible.

Equation of line through (4, –5) with slope m is $y + 5 = m(x - 4)$

$\Rightarrow \quad mx - y - 4m - 5 = 0$

Then $\dfrac{|m(-2) - 3 - 4m - 5|}{\sqrt{m^2 + 1}} = 12$

$\Rightarrow \quad |-6m - 8| = 12\sqrt{(m^2 + 1)}$

On squaring, $\quad (6m + 8)^2 = 144(m^2 + 1)$

$\Rightarrow \quad 4(3m + 4)^2 = 144(m^2 + 1) \Rightarrow (3m + 4)^2 = 36(m^2 + 1)$

$\Rightarrow \quad 27m^2 - 24m + 20 = 0 \qquad$ ..... (i)

Since the discriminant of (i) is $(-24)^2 - 4.27.20 = -1584$ which is negative, there is no real value of m. Hence no such line is possible.

17. DISTANCE BETWEEN TWO PARALLEL LINES :

(a) The distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $= \dfrac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$

(**Note :** The coefficients of x & y in both equations should be same)

(b) The area of the parallelogram $= \dfrac{p_1 p_2}{\sin\theta}$, where p_1 & p_2 are distances between two pairs of opposite sides & θ is the angle between any two adjacent sides . Note that area of the parallelogram bounded by the lines $y = m_1x + c_1$, $y = m_1x + c_2$ and $y = m_2x + d_1$, $y = m_2x + d_2$ is given by $\left|\dfrac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$.

Illustration 25 : Three lines $x + 2y + 3 = 0$, $x + 2y - 7 = 0$ and $2x - y - 4 = 0$ form 3 sides of two squares. Find the equation of remaining sides of these squares.

Solution : Distance between the two parallel lines is $\dfrac{|7 + 3|}{\sqrt{5}} = 2\sqrt{5}$.

The equations of sides A and C are of the form $2x - y + k = 0$.

Since distance between sides A and B = distance between sides B and C

$\Rightarrow \quad \dfrac{|k - (-4)|}{\sqrt{5}} = 2\sqrt{5} \Rightarrow \dfrac{k + 4}{\sqrt{5}} = \pm 2\sqrt{5} \Rightarrow k = 6, -14.$

Hence the fourth sides of the two squares are (i) $2x - y + 6 = 0$ (ii) $2x - y - 14 = 0$.

Ans.

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Do yourself - 8 :

(i) Find the distances between the following pair of parallel lines :

(a) $3x + 4y = 13,\ 3x + 4y = 3$

(b) $3x - 4y + 9 = 0,\ 6x - 8y - 15 = 0$

(ii) Find the points on the x-axis such that their perpendicular distance from the line $\frac{x}{a} + \frac{y}{b} = 1$ is 'a', a, b > 0.

(iii) Show that the area of the parallelogram formed by the lines

$2x - 3y + a = 0,\ 3x - 2y - a = 0,\ 2x - 3y + 3a = 0$ and $3x - 2y - 2a = 0$ is $\frac{2a^2}{5}$ square units.

18. POSITION OF TWO POINTS WITH RESPECT TO A GIVEN LINE :

Let the given line be $ax + by + c = 0$ and $P(x_1, y_1)$, $Q(x_2, y_2)$ be two points. If the expressions $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same signs, then both the points P and Q lie on the same side of the line $ax + by + c = 0$. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite signs, then they lie on the opposite sides of the line.

19. CONCURRENCY OF LINES :

(a) Three lines $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent, if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_2 & c_3 \end{vmatrix} = 0$

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(b) To test the concurrency of three lines, first find out the point of intersection of any two of the three lines. If this point lies on the remaining line (i.e. coordinates of the point satisfy the equation of the line) then the three lines are concurrent otherwise not concurrent.

Illustration 27 : If the lines $ax + by + p = 0$, $x\cos\alpha + y\sin\alpha - p = 0$ $(p \neq 0)$ and $x\sin\alpha - y\cos\alpha = 0$ are concurrent and the first two lines include an angle $\frac{\pi}{4}$, then $a^2 + b^2$ is equal to -

(A) 1 (B) 2 (C) 4 (D) p^2

Solution : Since the given lines are concurrent,

$$\begin{vmatrix} a & b & p \\ \cos\alpha & \sin\alpha & -p \\ \sin\alpha & -\cos\alpha & 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad a\cos\alpha + b\sin\alpha + 1 = 0 \qquad \text{..... (i)}$$

As $ax + by + p = 0$ and $x\cos\alpha + y\sin\alpha - p = 0$ include an angle $\frac{\pi}{4}$.

$$\pm\tan\frac{\pi}{4} = \frac{-\frac{a}{b} + \frac{\cos\alpha}{\sin\alpha}}{1 + \frac{a}{b}\frac{\cos\alpha}{\sin\alpha}}$$

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$\Rightarrow$ $-a\sin\alpha + b\cos\alpha = \pm(b\sin\alpha + a\cos\alpha)$

$\Rightarrow$ $-a\sin\alpha + b\cos\alpha = \pm 1$ [from (i)] ..... (ii)

Squaring and adding (i) & (ii), we get

$a^2 + b^2 = 2.$ **Ans. (B)**

Do yourself - 9 :

(i) Examine the positions of the points (3, 4) and (2, – 6) w.r.t. $3x - 4y = 8$

(ii) If (2, 9), (– 2, 1) and (1, –3) are the vertices of a triangle, then prove that the origin lies inside the triangle.

(iii) Find the equation of the line joining the point (2, – 9) and the point of intersection of lines $2x + 5y - 8 = 0$ and $3x - 4y - 35 = 0$.

(iv) Find the value of λ, if the lines $3x - 4y - 13 = 0$, $8x - 11y - 33 = 0$ and $2x - 3y + \lambda = 0$ are concurrent.

20. REFLECTION OF A POINT :

Let P(x, y) be any point, then its image with respect to

(a) x-axis is Q(x, –y)

(b) y-axis is R(–x, y)

(c) origin is S(–x,–y)

(d) line y = x is T(y, x)

(e) Reflection of a point about any arbitrary line : The image (h,k) of a point $P(x_1, y_1)$ about the line $ax + by + c = 0$ is given by following formula.

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = -2\frac{(ax_1 + by_1 + c)}{a^2 + b^2}$$

and the foot of perpendicular (α,β) from a point (x_1, y_1) on the line $ax + by + c = 0$ is given by following formula.

$$\frac{\alpha - x_1}{a} = \frac{\beta - y_1}{b} = -\frac{ax_1 + by_1 + c}{a^2 + b^2}$$

21. TRANSFORMATION OF AXES

(a) Shifting of origin without rotation of axes :

Let P (x, y) with respect to axes OX and OY.

Let O' (α, β) is new origin with respect to axes OX and OY and let P (x', y') with respect to axes O'X' and O'Y' , where OX and O'X' are parallel and OY and O'Y' are parallel.

Then $\mathbf{x = x' + \alpha, \quad y = y' + \beta}$

or $\mathbf{x' = x - \alpha, \quad y' = y - \beta}$

Thus if origin is shifted to point (α, β) without rotation of axes, then new equation of curve can be obtained by putting $x + \alpha$ in place of x and $y + \beta$ in place of y.

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(b) Rotation of axes without shifting the origin :

Let O be the origin. Let P (x, y) with respect to axes OX and OY and let P (x', y') with respect to axes OX' and OY' where $\angle X'OX = \angle YOY' = \theta$, where θ is measured in anticlockwise direction.

then $x = x' \cos\theta - y' \sin\theta$

$y = x' \sin\theta + y' \cos\theta$

and $x' = x \cos\theta + y \sin\theta$

$y' = -x \sin\theta + y \cos\theta$

The above relation between (x, y) and (x', y') can be easily obtained with the help of following table

New \ Old	x ↓	y ↓
x' →	cos θ	sin θ
y' →	–sin θ	cos θ

Illustration 28 : Through what angle should the axes be rotated so that the equation $9x^2 - 2\sqrt{3}\,xy + 7y^2 = 10$ may be changed to $3x^2 + 5y^2 = 5$?

Solution : Let angle be θ then replacing (x, y) by $(x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$

then $9x^2 - 2\sqrt{3}xy + 7y^2 = 10$ becomes

$$9(x\cos\theta - y\sin\theta)^2 - 2\sqrt{3}\,(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + 7(x\sin\theta + y\cos\theta)^2 = 10$$

$$\Rightarrow \quad x^2(9\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta + 7\sin^2\theta) + 2xy(-9\sin\theta\cos\theta - \sqrt{3}\cos2\theta + 7\sin\theta\cos\theta)$$

$$+ y^2(9\cos^2\theta + 2\sqrt{3}\sin\theta\cos\theta + 7\cos^2\theta) = 10$$

On comparing with $3x^2 + 5y^2 = 5$ (coefficient of xy = 0)

We get $-9\sin\theta\cos\theta - \sqrt{3}\cos2\theta + 7\sin\theta\cos\theta = 0$

or $\sin2\theta = -\sqrt{3}\cos2\theta$ or $\tan2\theta = -\sqrt{3} = \tan(180° - 60°)$

or $2\theta = 120°$ $\therefore$ $\theta = 60°$

Do yourself - 10 :

(i) The point (4, 1) undergoes the following transformations, then the match the correct alternatives:

	Column-I	Column-II
(A)	Reflection about x-axis is	(p) (4, –1)
(B)	Reflection about y-axis is	(q) (–4, –1)
(C)	Reflection about origin is	(r) $\left(-\frac{12}{25}, -\frac{59}{25}\right)$
(D)	Reflection about the line y = x is	(s) (–4, 1)
(E)	Reflection about the line 4x + 3y – 5 = 0 is	(t) (1, 4)

(ii) To what point must the origin be shifted, so that the coordinates of the point (4, 5) become (– 3, 9).

(iii) If the axes be turned through an angle $\tan^{-1}2$ (in anticlockwise direction),what does the equation $4xy - 3x^2 = a^2$ become ?

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22. EQUATION OF BISECTORS OF ANGLES BETWEEN TWO LINES :

If equation of two intersecting lines are $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, then equation of bisectors of the angles between these lines are written as :

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}} \qquad .....(i)$$

(a) Equation of bisector of angle containing origin :

If the equation of the lines are written with constant terms $\mathbf{c_1}$ and $\mathbf{c_2}$ positive, then the equation of the bisectors of the angle containing the origin is obtained by taking positive sign in (i)

(b) Equation of bisector of acute/obtuse angles :

To find the equation of the bisector of the acute or obtuse angle :

(i) let ϕ be the angle between one of the two bisectors and one of two given lines. Then if $\mathbf{\tan\phi < 1}$ i.e. $\phi < 45°$ i.e. $2\phi < 90°$, the angle bisector will be bisector of acute angle.

(ii) See whether the constant terms c_1 and c_2 in the two equation are +ve or not. If not then multiply both sides of given equation by –1 to make the constant terms positive.

Determine the sign of $\mathbf{a_1a_2 + b_1b_2}$

If sign of $a_1a_2 + b_1b_2$	For obtuse angle bisector	For acute angle bisector
+	use + sign in eq. (1)	use – sign in eq. (1)
–	use – sign in eq. (1)	use + sign in eq. (1)

i.e. if $a_1a_2 + b_1b_2 > 0$, then the bisector corresponding to + sign gives obtuse angle bisector

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

(iii) Another way of identifying an acute and obtuse angle bisector is as follows :

Let $L_1 = 0$ & $L_2 = 0$ are the given lines & $u_1 = 0$ and $u_2 = 0$ are the bisectors between $L_1 = 0$ & $L_2 = 0$. Take a point P on any one of the lines $L_1 = 0$ or $L_2 = 0$ and drop perpendicular on $u_1 = 0$ & $u_2 = 0$ as shown . If,

$|p| < |q| \Rightarrow u_1$ is the acute angle bisector .

$|p| > |q| \Rightarrow u_1$ is the obtuse angle bisector .

$|p| = |q| \Rightarrow$ the lines L_1 & L_2 are perpendicular .

Note : Equation of straight lines passing through $P(x_1, y_1)$ & equally inclined with the lines $a_1x+b_1y+c_1=0$ & $a_2x+b_2y+c_2=0$ are those which are parallel to the bisectors between these two lines & passing through the point P.

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Illustration 29 : For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$, find the equation of the

(i) bisector of the obtuse angle between them.

(ii) bisector of the acute angle between them.

(iii) bisector of the angle which contains origin.

(iv) bisector of the angle which contains (1, 2).

Solution : Equations of bisectors of the angles between the given lines are

$$\frac{4x+3y-6}{\sqrt{4^2+3^2}} = \pm\frac{5x+12y+9}{\sqrt{5^2+12^2}} \Rightarrow 9x - 7y - 41 = 0 \text{ and } 7x + 9y - 3 = 0$$

If θ is the acute angle between the line $4x + 3y - 6 = 0$ and the bisector

$$9x - 7y - 41 = 0, \text{ then } \tan\theta = \left|\frac{-\frac{4}{3}-\frac{9}{7}}{1+\left(\frac{-4}{3}\right)\frac{9}{7}}\right| = \frac{11}{3} > 1$$

Hence

(i) bisector of the obtuse angle is $9x - 7y - 41 = 0$

(ii) bisector of the acute angle is $7x + 9y - 3 = 0$

(iii) bisector of the angle which contains origin

$$\frac{-4x-3y+6}{\sqrt{(-4)^2+(-3)^2}} = \frac{5x+12y+9}{\sqrt{5^2+12^2}} \Rightarrow 7x+9y-3=0$$

(iv) $L_1(1, 2) = 4 \times 1 + 3\times 2 - 6 = 4 > 0$

$L_2(1, 2) = 5 \times 1 + 12 \times 2 + 9 = 38 > 0$

+ve sign will give the required bisector, $\frac{4x+3y-6}{5} = +\frac{5x+12y+9}{13}$

$\Rightarrow$ $9x - 7y - 41 = 0$.

Alternative :

Making c_1 and c_2 positive in the given equation, we get $-4x - 3y + 6 = 0$ and $5x + 12y + 9 = 0$

Since $a_1a_2 + b_1b_2 = -20 - 36 = -56 < 0$, so the origin will lie in the acute angle.

Hence bisector of the acute angle is given by

$$\frac{-4x-3y+6}{\sqrt{4^2+3^2}} = \frac{5x+12y+9}{\sqrt{5^2+12^2}} \Rightarrow 7x+9y-3=0$$

Similarly bisector of obtuse angle is $9x - 7y - 41 = 0$.

Illustration 30 : A ray of light is sent along the line $x - 2y - 3 = 0$. Upon reaching the line mirror $3x - 2y - 5 = 0$, the ray is reflected from it. Find the equation of the line containing the reflected ray.

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Solution : Let Q be the point of intersection of the incident ray and the line mirror, then

$$x_1 - 2y_1 - 3 = 0 \ \& \ 3x_1 - 2y_1 - 5 = 0$$

on solving these equations, we get

$$x_1 = 1 \ \& \ y_1 = -1$$

Since P(–1, –2) be a point lies on the incident ray, so we can find the image of the point P on the reflected ray about the line mirror (by property of reflection).

Let P'(h, k) be the image of point P about line mirror, then

$$\frac{h+1}{3} = \frac{k+2}{-2} = \frac{-2(-3+4-5)}{13} \Rightarrow h = \frac{11}{13} \text{ and } k = \frac{-42}{13}.$$

So $P'\left(\frac{11}{13}, \frac{-42}{13}\right)$

Then equation of reflected ray will be

$$(y+1) = \frac{\left(\frac{-42}{13}+1\right)(x-1)}{\left(\frac{11}{13}-1\right)}$$

$\Rightarrow$ $2y - 29x + 31 = 0.$

---

23. FAMILY OF LINES :

If equation of two lines be $P \equiv a_1x + b_1y + c_1 = 0$ and $Q \equiv a_2x + b_2y + c_2 = 0$, then the equation of the lines passing through the point of intersection of these lines is : $P + \lambda Q = 0$ or $a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$. The value of λ is obtained with the help of the additional informations given in the problem.

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Illustration 31 : Prove that each member of the family of straight lines $(3\sin\theta + 4\cos\theta)x + (2\sin\theta - 7\cos\theta)y + (\sin\theta + 2\cos\theta) = 0$ (θ is a parameter) passes through a fixed point.

Solution : The given family of straight lines can be rewritten as

$(3x + 2y + 1)\sin\theta + (4x - 7y + 2)\cos\theta = 0$

or, $(4x - 7y + 2) + \tan\theta(3x + 2y + 1) = 0$ which is of the form $L_1 + \lambda L_2 = 0$

Hence each member of it will pass through a fixed point which is the intersection of

$4x - 7y + 2 = 0$ and $3x + 2y + 1 = 0$ i.e. $\left(\frac{-11}{29}, \frac{2}{29}\right)$.

Do yourself - 11 :

(i) Find the equations of bisectors of the angle between the lines $4x + 3y = 7$ and $24x + 7y - 31 = 0$. Also find which of them is (a) the bisector of the angle containing origin (b) the bisector of the acute angle.

(ii) Find the equations of the line which pass through the point of intersection of the lines $4x - 3y = 1$ and $2x - 5y + 3 = 0$ and is equally inclined to the coordinate axes.

(iii) Find the equation of the line through the point of intersection of the lines $3x - 4y + 1 = 0$ & $5x + y - 1 = 0$ and perpendicular to the line $2x - 3y = 10$.

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24. PAIR OF STRAIGHT LINES :

(a) Homogeneous equation of second degree :

(i) Let us consider the homogeneous equation of 2nd degree as

$ax^2 + 2hxy + by^2 = 0$ .....(i)

which represents pair of straight lines passing through the origin.

Now, we divide by x^2, we get

$$a + 2h\left(\frac{y}{x}\right) + b\left(\frac{y}{x}\right)^2 = 0$$

$$\frac{y}{x} = m \quad \text{(say)}$$

then $a + 2hm + bm^2 = 0$ .....(ii)

if m_1 & m_2 are the roots of equation (ii), then $m_1 + m_2 = -\frac{2h}{b}$, $m_1m_2 = \frac{a}{b}$

and also, $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1 + m_1m_2}\right| = \left|\frac{\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{1 + \frac{a}{b}}\right| = \pm\frac{2\sqrt{h^2 - ab}}{a+b}$

These line will be :

(1) **Real and different**, if $h^2 - ab > 0$ (2) **Real and coincident**, if $h^2 - ab = 0$

(3) **Imaginary**, if $h^2 - ab < 0$

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(ii) The condition that these lines are :

(1) At **right angles** to each other is $\mathbf{a + b = 0}$. i.e. coefficient of $\mathbf{x^2}$ + coefficient of $\mathbf{y^2} = 0$.

(2) **Coincident** is $\mathbf{h^2 = ab}$.

(3) **Equally inclined** to the axes of x is $\mathbf{h = 0}$. i.e. coefficient of $\mathbf{xy = 0}$.

(iii) Homogeneous equation of 2nd degree $\mathbf{ax^2 + 2hxy + by^2 = 0}$ always represent a pair of straight lines whose equations are

$$y = \left(\frac{-h \pm \sqrt{h^2 - ab}}{b}\right)x \equiv y = m_1x \ \& \ y = m_2x \ \text{ and } \ m_1 + m_2 = -\frac{2h}{b} \ ; m_1m_2 = \frac{a}{b}$$

These straight lines passes through the origin.

(iv) Pair of straight lines perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$ and through origin are given by $bx^2 - 2hxy + ay^2 = 0$.

(v) The product of the perpendiculars drawn from the point (x_1, y_1) on the lines $ax^2 + 2hxy + by^2 = 0$ is $\left|\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}\right|$

Note : A homogeneous equation of degree n represents n straight lines passing through **origin**.

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(b) The combined equation of angle bisectors :

The combined equation of angle bisectors between the lines represented by homogeneous equation of 2nd degree is given by $\frac{x^2-y^2}{a-b}=\frac{xy}{h}$, $\mathbf{a \neq b, h \neq 0}$.

Note :

(i) If $\mathbf{a = b}$, the bisectors are $\mathbf{x^2 - y^2 = 0}$ i.e. $\mathbf{x - y = 0, x + y = 0}$

(ii) If $\mathbf{h = 0}$, the bisectors are $\mathbf{xy = 0}$ i.e. $\mathbf{x = 0, y = 0}$.

(iii) The two bisectors are always at **right angles**, since we have coefficient of $\mathbf{x^2}$ + coefficient of $\mathbf{y^2 = 0}$

(c) General Equation and Homogeneous Equation of Second Degree :

(i) The general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, if $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ i.e. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

(ii) If θ be the angle between the lines, then $\tan\theta = \pm\frac{2\sqrt{h^2-ab}}{a+b}$

Obviously these lines are

(1) Parallel, if $\Delta = 0$, $h^2 = ab$ or if $h^2 = ab$ and $bg^2 = af^2$

(2) Perpendicular, if $a + b = 0$ i.e. coeff. of x^2 + coeff. of $y^2 = 0$.

(iii) The product of the perpendiculars drawn from the origin to the lines

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $\left|\frac{c}{\sqrt{(a-b)^2+4h^2}}\right|$

Illustration 32 : If $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ represents a pair of straight lines, then λ is equal to -

(A) 4 (B) 3 (C) 2 (D) 1

Solution : Here $a = \lambda$, $b = 12$, $c = -3$, $f = -8$, $g = 5/2$, $h = -5$

Using condition $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, we have

$\lambda(12)(-3) + 2(-8)(5/2)(-5) - \lambda(64) - 12(25/4) + 3(25) = 0$

$\Rightarrow -36\lambda + 200 - 64\lambda - 75 + 75 = 0 \quad \Rightarrow 100\lambda = 200$

$\therefore \lambda = 2$ **Ans. (C)**

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Do yourself - 12 :

(i) Prove that the equation $x^2 - 5xy + 4y^2 = 0$ represents two lines passing through the origin. Also find their equations.

(ii) If the equation $3x^2 + kxy - 10y^2 + 7x + 19y = 6$ represents a pair of lines, find the value of k.

(iii) If the equation $6x^2 - 11xy - 10y^2 - 19y - 6 = 0$ represents a pair of lines, find their equations. Also find the angle between the two lines.

(iv) Find the point of intersection and the angle between the lines given by the equation : $2x^2 - 3xy - 2y^2 + 10x + 5y = 0$.

25. EQUATIONS OF LINES JOINING THE POINTS OF INTERSECTION OF A LINE AND A CURVE TO THE ORIGIN :

(a) Let the equation of curve be :

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ...(i)

and straight line be

$lx + my + n = 0$ ...(ii)

Now joint equation of line OP and OQ joining the origin and points of intersection P and Q can be obtained by making the equation (i) homogenous with the help of equation of the line.

Thus required equation is given by

$$ax^2+2hxy+by^2+2(gx+fy)\left(\frac{\ell x+my}{-n}\right)+c\left(\frac{\ell x+my}{-n}\right)^2=0$$

$\Rightarrow$ $(an^2 + 2gln + cl^2)x^2 + 2(hn^2 + gmn + fln + clm)xy + (bn^2 + 2fmn + cm^2)y^2 = 0$ .....(iii)

All points which satisfy (i) and (ii) simultaneously, will satisfy (iii)

(b) Any second degree curve through the four points of intersection of $f(x, y) = 0$ & $xy = 0$ is given by $f(x, y) + \lambda\, xy = 0$ where $f(x, y) = 0$ is also a second degree curve.

Illustration 35 : The chord $\sqrt{6}y = \sqrt{8}px + \sqrt{2}$ of the curve $py^2 + 1 = 4x$ subtends a right angle at origin then find the value of p.

Solution : $\sqrt{3}y - 2px = 1$ is the given chord. Homogenizing the equation of the curve, we get,

$py^2 - 4x(\sqrt{3}\,y - 2px) + (\sqrt{3}\,y - 2px)^2 = 0$

$\Rightarrow (4p^2 + 8p)x^2 + (p + 3)y^2 - 4\sqrt{3}\,xy - 4\sqrt{3}\,pxy = 0$

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Now, angle at origin is 90°

$\therefore$ coefficient of x^2 + coefficient of $y^2 = 0$

$\therefore$ $4p^2 + 8p + p + 3 = 0 \Rightarrow 4p^2 + 9p + 3 = 0$

$\therefore$ $p = \dfrac{-9 \pm \sqrt{81-48}}{8} = \dfrac{-9 \pm \sqrt{33}}{8}$.

Do yourself - 13 :

(i) Find the angle subtended at the origin by the intercept made on the curve $x^2 - y^2 - xy + 3x - 6y + 18 = 0$ by the line $2x - y = 3$.

(ii) Find the equation of the lines joining the origin to the points of intersection of the curve $2x^2 + 3xy - 4x + 1 = 0$ and the line $3x + y = 1$.

ANSWERS FOR DO YOURSELF

1 : **(i)** $PQ = \sqrt{34}$ units ; **(ii)** $x = 6$ or $x = 0$ **(iii)** 11, –7

2 : **(i)** **(a)** (2,1) ; **(b)** (7,16) ; **(ii)** **(a)** 2 : 3 (internally) ; **(b)** 9 : 4 (externally) ; **(c)** 8 : 7 (internally)

3 : **(i)** **(a)** $\left(\frac{5}{3}, 3\right)$; **(b)** $\left(-\frac{9}{2}, \frac{15}{2}\right), \frac{5\sqrt{10}}{2}$, **(c)** (14, – 6)

4 : **(i)** 25 square units ; **(ii)** 132 square units ;

5 : **(i)** $x = \pm 2$; **(ii)** $y = \pm x$;

6 : **(i)** **(a)** $y = \frac{2x}{3} + \frac{5}{3}, \frac{2}{3}, \frac{5}{3}$; **(b)** $\frac{x}{(-5/2)} + \frac{y}{(5/3)} = 1, -\frac{5}{2}, \frac{5}{3}$;

(c) $-\frac{2x}{\sqrt{13}} + \frac{3y}{\sqrt{13}} = \frac{5}{\sqrt{13}}, \frac{5}{\sqrt{13}}$, $\alpha = \pi - \tan^{-1}\left(\frac{3}{2}\right)$; **(ii)** $13\sqrt{2}/3$ units

7 : **(i)** $\theta = 135°$ or $45°$; **(ii)** $3x + 4y = 18$; **(iii)** $2x - 3y + 12 = 0$, (–6, 0)

(iv) **(a)** Coincident, **(b)** Parallel, **(c)** Intersecting

8 : **(i)** **(a)** 2 (b) 33/10 ; **(ii)** $\left(\frac{a}{b}\left(b \pm \sqrt{a^2+b^2}\right), 0\right)$

9 : **(i)** opposite sides of the line; **(iii)** $-y + x = 11$; **(iv)** $\lambda = -7$

10 : **(i)** (A) → (p), (B) → (s), (C)→ (q), (D) → (t), (E) → (r),; **(ii)** (7, – 4) ; **(iv)** $x^2 - 4y^2 = a^2$

11 : **(i)** $x - 2y + 1 = 0$ & $2x + y - 3 = 0$; **(a)** $x - 2y + 1 = 0$; **(b)** $2x + y - 3 = 0$

(ii) $x + y = 2$, $x = y$; **(iii)** $69x + 46y - 25 = 0$

12 : **(i)** $x - y = 0$ & $x - 4y = 0$; **(ii)** $k = -1$, or $-\frac{127}{6}$;

(iii) $2x - 5y - 2 = 0$ & $3x + 2y + 3 = 0$; $\pm\tan^{-1}\left(\frac{19}{4}\right)$ **(iv)** (–1, 2), 90°

13 : **(i)** $\theta = \pm\tan^{-1}\frac{4}{7}$; **(ii)** $x^2 - y^2 - 5xy = 0$

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EXERCISE (O-1)

1. A particle begins at the origin and moves successively in the following manner as shown,

1 unit to the right, 1/2 unit up, 1/4 unit to the right, 1/8 unit down, 1/16 unit to the right etc.

The length of each move is half the length of the previous move and movement continues in the 'zigzag' manner infinitely. The co-ordinates of the point to which the 'zigzag' converges is -

(A) (4/3, 2/3) (B) (4/3,2/5) (C) (3/2,2/3) (D) (2,2/5)

SL0001

2. Suppose ABC is a triangle with 3 acute angles A,B and C. The point whose coordinates are (cosB – sinA, sinB – cosA) can be in the -

(A) first and 2nd quadrant (B) second the 3rd quadrant
(C) third and 4th quadrant (D) second quadrant only

SL0002

3. Coordinates of the vertices of a triangle ABC are (12,8), (–2,6) and (6,0) then the **correct** statement is -

(A) triangle is right but not isosceles
(B) triangle is isosceles but not right
(C) triangle is obtuse
(D) the product of the abscissa of the centroid, orthocenter and circumcenter is 160.

SL0003

4. Find the value of x_1 if the distance between the points $(x_1, 2)$ and (3, 4) be 8.

(A) $3 \pm 2\sqrt{15}$ (B) $3 \pm \sqrt{15}$ (C) $2 \pm 3\sqrt{15}$ (D) $2 \pm \sqrt{15}$

SL0004

5. The four points whose co-ordinates are (2,1),(1,4),(4,5),(5,2) form :

(A) a rectangle which is not a square (B) a trapezium which is not a parallelogram
(C) a square (D) a rhombus which is not a square

SL0005

6. The length of a line segment AB is 10 units. If the coordinates of one extremity are (2,–3) and the abscissa of the other extremity is 10 then the sum of all possible values of the ordinate of the other extremity is -

(A) 3 (B) –4 (C) 12 (D) –6

SL0006

7. Consider the points P(2,–4); Q(4,–2) and R(7,1). The points P,Q,R -

(A) form an equilateral triangle
(B) form a right angled triangle
(C) form an isosceles triangle which is not equilateral
(D) are collinear.

SL0007

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8. Area of the quadrilateral formed by the lines $|x| + |y| = 2$ is :

(A) 8 (B) 6 (C) 4 (D) none

SL0008

9. If P(1,2), Q(4,6), R(5,7) & S(a,b) are the vertices of a parallelogram PQRS, then :

(A) a = 2, b = 4 (B) a = 3, b = 4 (C) a = 2, b = 3 (D) a = 3, b = 5

SL0009

10. If A and B are the points (–3,4) and (2,1), then the co-ordinates of the point C on AB produced such that AC = 2BC are :

(A) (2,4) (B) (3,7) (C) (7,–2) (D) $\left(-\frac{1}{2},\frac{5}{2}\right)$

SL0010

11. If the two vertices of a triangle are (7,2) and (1,6) and its centroid is (4,6) then the coordinate of the third vertex are (a,b). The value of (a + b), is-

(A) 13 (B) 14 (C) 15 (D) 16

SL0011

12. The orthocenter of the triangle ABC is 'B' and the circumcenter is 'S' (a,b). If A is the origin then the co-ordinates of C are :

(A) (2a,2b) (B) $\left(\frac{a}{2},\frac{b}{2}\right)$ (C) $\left(\sqrt{a^2+b^2},0\right)$ (D) none

SL0012

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13. The medians of a triangle meet at (0,–3) and its two vertices are at (–1,4) and (5,2). Then the third vertex is at -

(A) (4,15) (B) (–4,–15) (C) (–4,15) (D) (4,–15)

SL0013

14. If in triangle ABC, A ≡ (1,10), circumcenter ≡ $\left(-\frac{1}{3},\frac{2}{3}\right)$ and orthocenter ≡ $\left(\frac{11}{3},\frac{4}{3}\right)$ then the co-ordinates of mid-point of side opposite to A is-

(A) (1,–11/3) (B) (1,5) (C) (1,–3) (D) (1,6)

SL0014

15. The co-ordinates of the orthocentre of the triangle bounded by the lines, $4x - 7y + 10 = 0$; $x + y = 5$ and $7x + 4y = 15$ is-

(A) (2,1) (B) (–1,2) (C) (1,2) (D) (1,–2)

SL0015

16. A triangle has two of its vertices at (0,1) and (2,2) in the cartesian plane. Its third vertex lies on the x-axis. If the area of the triangle is 2 square units then the sum of the possible abscissae of the third vertex, is-

(A) –4 (B) 0 (C) 5 (D) 6

SL0016

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17. A line passes through (2,2) and cuts a triangle of area 9 square units from the first quadrant. The sum of all possible values for the slope of such a line, is-

(A) –2.5 (B) –2 (C) –1.5 (D) –1

SL0017

18. Let A(2,–3) and B(–2,1) be vertices of a ΔABC. If the centroid of ΔABC moves on the line 2x + 3y =1, then the locus of the vertex C is-

(A) 2x + 3y = 9 (B) 2x – 3y = 7 (C) 3x + 2y = 5 (D) 3x – 2y = 3

SL0018

19. A stick of length 10 units rests against the floor and a wall of a room. If the stick begins to slide on the floor then the locus of its middle point is :

(A) $x^2 + y^2 = 2.5$ (B) $x^2 + y^2 = 25$ (C) $x^2 + y^2 = 100$ (D) none

SL0019

20. A and B are any two points on the positive x and y axis respectively satisfying 2(OA) + 3(OB) = 10. If P is the middle point of AB then the locus of P is-

(A) 2x + 3y = 5 (B) 2x + 3y = 10 (C) 3x + 2y = 5 (D) 3x + 2y = 10

SL0020

21. A point P(x,y) moves so that the sum of the distance from P to the coordinate axes is equal to the distance from P to the point A(1,1). The equation of the locus of P in the first quadrant is -

(A) (x + 1) (y + 1) = 1 (B) (x + 1) (y + 1) = 2

(C) (x – 1)(y – 1) = 1 (D) (x – 1)(y – 1) = 2

SL0021

22. Given the points A(0,4) and B(0,–4), the equation of the locus of the point P such that |AP – BP| = 6 is -

(A) $9x^2 - 7y^2 + 63 = 0$ (B) $9x^2 - 7y^2 - 63 = 0$

(C) $7x^2 - 9y^2 + 63 = 0$ (D) $7x^2 - 9y^2 - 63 = 0$

SL0022

23. The diagonals of a parallelogram PQRS are along the lines x + 3y = 4 and 6x – 2y = 7. Then PQRS must be a :

(A) rectangle (B) square

(C) cyclic quadrilateral (D) rhombus

SL0023

24. If the x intercept of the line y = mx + 2 is greater than 1/2 then the gradient of the line lies in the interval-

(A) (–1,0) (B) (–1/4,0) (C) (–∞,–4) (D) (–4,0)

SL0024

25. The extremities of the base of an isosceles triangle ABC are the points A(2,0) and B(0,1). If the equation of the side AC is x = 2 then the slope of the side BC is -

(A) $\frac{3}{4}$ (B) $\frac{4}{3}$ (C) $\frac{3}{2}$ (D) $\sqrt{3}$

SL0025

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26. Point 'P' lies on the line l $\{(x,y)\,|3x + 5y = 15\}$. If 'P' is also equidistant from the coordinate axes, then P can be located in which of the four quadrants -

(A) I only (B) II only (C) I or II only (D) IV only

SL0026

27. If x_1,y_1 are the roots of $x^2 + 8x - 20 = 0$, x_2,y_2 are the roots of $4x^2 + 32x - 57 = 0$ and x_3,y_3 are the roots of $9x^2 + 72x - 112 = 0$, then the points, (x_1,y_1), (x_2,y_2) and (x_3,y_3) -

(A) are collinear

(B) form an equilateral triangle

(C) form a right angled isosceles triangle

(D) are concyclic

SL0027

28. Two mutually perpendicular straight lines through the origin from an isosceles triangle with the line $2x + y = 5$. Then the area of the triangle is :

(A) 5 (B) 3 (C) 5/2 (D) 1

SL0028

29. The area of the parallelogram formed by the lines $3x + 4y = 7a$; $3x + 4y = 7b$; $4x + 3y = 7c$ and $4x + 3y = 7d$ is-

(A) $\dfrac{|(a-b)(c-d)|}{7}$ (B) $|(a-b)(c-d)|$

(C) $\dfrac{|(a-b)(c-d)|}{49}$ (D) $7|(a-b)(c-d)|$

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SL0029

30. Number of lines that can be drawn through the point(4,–5) so that its distance from (–2,3) will be equal to 12 is equal to-

(A) 0 (B) 1 (C) 2 (D) 3

SL0030

31. A ray of light passing through the point A(1,2) is reflected at a point B on the x-axis and then passes through (5,3). Then the equation of AB is :

(A) $5x + 4y = 13$ (B) $5x - 4y = -3$

(C) $4x + 5y = 14$ (D) $4x - 5y = -6$

SL0031

32. Suppose that a ray of light leaves the point (3,4), reflects off the y-axis towards the x-axis, reflects off the x-axis, and finally arrives at the point (8,2). The value of x, is -

(A) $x = 4\frac{1}{2}$ (B) $x = 4\frac{1}{3}$

(C) $x = 4\frac{2}{3}$ (D) $x = 5\frac{1}{3}$

SL0032

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33. m,n are integer with $0 < n < m$. A is the point (m,n) on the cartesian plane. B is the reflection of A in the line $y = x$. C is the reflection of B in the y-axis, D is the reflection of C in the x-axis and E is the reflection of D in the y-axis. The area of the pentagon ABCDE is -

(A) $2m(m + n)$ (B) $m(m + 3n)$ (C) $m(2m + 3n)$ (D) $2m(m + 3n)$

SL0033

34. If the lines $\left.\begin{array}{l}\lambda x+(\sin\alpha)y+\cos\alpha=0\\ x+(\cos\alpha)y+\sin\alpha=0\\ x-(\sin\alpha)y+\cos\alpha=0\end{array}\right]$ pass through the same point where $\alpha \in R$ then λ lies in the interval -

(A) $[-1,1]$ (B) $\left[-\sqrt{2},\sqrt{2}\right]$ (C) $[-2,2]$ (D) $(-\infty,\infty)$

SL0034

35. If the straight lines, $ax + amy + 1 = 0$, $bx + (m + 1) by + 1 = 0$ and $cx + (m + 2)cy + 1 = 0$, $m \neq 0$ are concurrent then a,b,c are in :

(A) A.P. only for $m = 1$ (B) A.P. for all m

(C) G.P. for all m (D) H.P. for all m

SL0035

36. Family of lines represented by the equation $(\cos\theta + \sin\theta)x + (\cos\theta - \sin\theta)y - 3(3\cos\theta + \sin\theta) = 0$ passes through a fixed point M for all real values of θ. The reflection of M in the line $x - y = 0$, is-

(A) (6,3) (B) (3,6) (C) (–6,3) (D) (3,–6)

SL0036

37. Consider a parallelogram whose sides are represented by the lines $2x + 3y = 0$; $2x + 3y - 5 = 0$; $3x - 4y = 0$ and $3x - 4y = 3$. The equation of the diagonal not passing through the origin, is-

(A) $21x - 11y + 15 = 0$ (B) $9x - 11y + 15 = 0$

(C) $21x - 29y - 15 = 0$ (D) $21x - 11y - 15 = 0$

SL0037

38. If the line $y = mx$ bisects the angle between the lines $ax^2 + 2h\,xy + by^2 = 0$ then m is a root of the quadratic equation :

(A) $hx^2 + (a - b)x - h = 0$ (B) $x^2 + h(a - b)x - 1 = 0$

(C) $(a - b)x^2 + hx - (a - b) = 0$ (D) $(a - b)x^2 - hx - (a - b) = 0$

SL0038

39. The greatest slope along the graph represented by the equation $4x^2 - y^2 + 2y - 1 = 0$, is-

(A) –3 (B) –2 (C) 2 (D) 3

SL0039

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40. If the equation $ax^2 - 6xy + y^2 + 2gx + 2fy + c = 0$ represents a pair of lines whose slopes are m and m^2, then sum of all possible values of a is-

(A) 17 (B) –19 (C) 19 (D) –17

SL0040

41. Let $S = \{(x,y) | x^2 + 2xy + y^2 - 3x - 3y + 2 = 0\}$, then S -

(A) consists of two coincident lines.

(B) consists of two parallel lines which are not coincident.

(C) consists of two intersecting lines.

(D) is a parabola.

SL0041

[MATRIX LIST TYPE]

42. Find the equation to the straight line :

	Column-I		Column-II
(P)	passing through the point (2, 3) and perpendicular to the straight line $4x - 3y = 10$.	(1)	$4y + 11x = 10$
(Q)	passing through the point (–6, 10) and perpendicular to the straight line $7x + 8y = 5$.	(2)	$4y + 3x = 18$
(R)	passing through the point (2, –3) and perpendicular to the straight line joining the points (5, 7) and (–6, 3).	(3)	$x + 4y + 16 = 0$
(S)	passing through the point (–4, –3) and perpendicular to the straight line joining (1, 3) and (2, 7).	(4)	$7y - 8x = 118$

Codes :

	P	Q	R	S
(A)	1	2	3	4
(B)	2	4	1	3
(C)	4	3	2	1
(D)	1	3	4	2

SL0042

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[MATRIX MATCH]

43.

	Column-I		Column-II
(A)	The points (2,–2), (8,4), (5,7) and (–1,1) taken in order constitute the vertices of a	(P)	square
			SL0043
(B)	The points (0,–1),(2,1),(0,3) and (–2,1) taken in order are the vertices of a	(Q)	rectangle
			SL0044
(C)	The points (3,–5), (–5,–4),(7,10),(15,9) taken in order are the vertices of a	(R)	trapezium
			SL0045
(D)	The points (–3,4), (–1,0), (1,0) and (3,4) taken in order are the vertices of a	(S)	parallelogram
		(T)	cyclic quadrilateral
			SL0046

44.

	Column-I		Column-II
(A)	The points (2,–2), (–2,1) and (5,2)	(P)	are the vertices of a right angled triangle
			SL0047
(B)	The points (1,–2),(–3,0) and (5,6)	(Q)	are the vertices of a right angle isosceles triangle
			SL0048
(C)	The points (3,7),(6,5) and (15,–1)	(R)	are the vertices of an equilateral triangle
			SL0049
(D)	The points (2,2), (–2,–2) and $\left(-2\sqrt{3}, 2\sqrt{3}\right)$	(S)	do not form a triangle
			SL0050

45.

	Column-I		Column-II
(A)	The ratio in which the point $\left(\frac{10}{7}, \frac{33}{7}\right)$ divides the line segment joining the points (1, 3) and (2, 7) is p : q, then (p + q) is (p & q are coprime)	(P)	$\frac{1}{5}$
			SL0051
(B)	The ratio in which the point (–2, –9) divides The line segment joining the points (1, 3) and (2, 7) is p : q, then (p + q) is (p & q are coprime)	(Q)	–1
			SL0052
(C)	If (p, q) divides internally the line joining (–1, 2) and (4, –5) in ratio 2 : 3 then (p + q) is	(R)	5
			SL0053
(D)	If (p, q) divides externally the line joining (–1, 2) (4, –5) in ratio 2 : 3 then (p + q) is	(S)	7
			SL0054

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46. Equation of Straight Line

Column-I	Column-II
(A) Which cuts-off an intercept 4 on the x-axis and passes through the point (2,–3).	(P) $2x + y + 1 = 0$
	SL0055
(B) Which cuts-off equal intercepts on the co-ordinate axes and passes through (2,5)	(Q) $x + y = 7$
	SL0056
(C) Which makes an angle of 135° with the axis of x and which cuts the axis of y at a distance -8 from the origin and	(R) $3x - 2y = 12$
	SL0057
(D) Through the point (4,1) and making with the axes in the first quadrant a triangle whose area is 8.	(S) $x + 4y = 8$ (T) $x + y + 8 = 0$
	SL0058

47.

Column-I	Column-II
(A) The four lines $3x - 4y + 11 = 0$; $3x - 4y - 9 = 0$; $4x + 3y + 3 = 0$ and $4x + 3y - 17 = 0$ enclose a figure which is	(P) a quadrilateral which is neither a parallelogram nor a trapezium nor a kite.
	SL0059
(B) The lines $2x + y = 1$, $x + 2y = 1$, $2x + y = 3$ and $x + 2y = 3$ form a figure which is	(Q) a parallelogram which is neither a rectangle nor a rhombus
	SL0060
(C) If 'O' is the origin, P is the intersection of the lines $2x^2 - 7xy + 3y^2 + 5x + 10y - 25 = 0$, A and B are the points in which these lines are cut by the line $x + 2y - 5 = 0$, then the points O,A,P,B (in some order) are the vertices of	(R) a rhombus which is not a square (S) a square
	SL0061

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EXERCISE (O-2)

1. The line x= c cuts the triangle with corners (0,0); (1,1) and (9,1) into two region. For the area of the two regions to be the same c must be equal to-

(A) 5/2 (B) 3 (C) 7/2 (D) 3 or 15

SL0062

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2. If m and b are real numbers and mb > 0, then the line whose equation is y = mx + b cannot contain the point-

(A) (0,2009) (B) (2009,0) (C) (0,–2009) (D) (20,–100)

SL0063

3. In a triangle ABC, side AB has the equation 2x + 3y = 29 and the side AC has the equation, x + 2y = 16. If the mid- point of BC is (5,6) then the equation of BC is :

(A) x – y = –1 (B) 5x – 2y = 13 (C) x + y = 11 (D) 3x – 4y = –9

SL0064

4. The vertex of the right angle of a right angled triangle lies on the straight line 2x – y – 10 = 0 and the two other vertices, at points (2,–3) and (4,1) then the area of triangle in sq. units is-

(A) $\sqrt{10}$ (B) 3 (C) $\frac{33}{5}$ (D) 11

SL0065

5. A triangle ABC is formed by the lines 2x – 3y – 6 = 0; 3x – y + 3 = 0 and 3x + 4y – 12 = 0. If the points P(α,0) and Q(0,β) always lie on or inside the ΔABC, then :

(A) $\alpha \in [-1,2]$ and $\beta \in [-2,3]$ (B) $\alpha \in [-1,3]$ and $\beta \in [-2,4]$

(C) $\alpha \in [-2,4]$ and $\beta \in [-3,4]$ (D) $\alpha \in [-1,3]$ and $\beta \in [-2,3]$

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SL0066

6. The co-ordinates of a point P on the line 2x – y + 5 = 0 such that |PA – PB| is maximum where A is (4,–2) and B is (2,–4) will be :

(A) (11,27) (B) (–11,–17) (C) (–11,17) (D) (0,5)

SL0067

7. The line $(k + 1)^2x + ky - 2k^2 - 2 = 0$ passes through a point regardless of the value k. Which of the following is the line with slope 2 passing through the point ?

(A) y = 2x – 8 (B) y = 2x – 5 (C) y = 2x – 4 (D) y = 2x + 8

SL0068

8. If the straight lines joining the origin and the points of intersection of the curve

$5x^2 + 12xy - 6y^2 + 4x - 2y + 3 = 0$ and $x + ky - 1 = 0$

are equally inclined to the x- axis then the value of k :

(A) is equal to 1 (B) is equal to –1

(C) is equal to 2 (D) does not exist in the set of real numbers

SL0069

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9. Through a point A on the x-axis a straight line is drawn parallel to y-axis so as to meet the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ in B and C. If AB = BC then-

(A) $h^2 = 4ab$ (B) $8h^2 = 9ab$ (C) $9h^2 = 8ab$ (D) $4h^2 = ab$

SL0070

[MULTIPLE CHOICE]

10. The x-coordinates of the vertices of a square of unit area are the roots of the equation $x^2 - 3|x| + 2 = 0$ and the y-coordinates of the vertices are the roots of the equation $y^2 - 3y + 2 = 0$ then the possible vertices of the square is/are-

(A) (1,1), (2,1), (2,2), (1,2) (B) (–1,1), (–2,1), (–2,2), (–1,2)
(C) (2,1), (1,–1),(1,2),(2,2) (D) (–2,1), (–1,–1),(–1,2), (–2,2)

SL0071

11. Three vertices of a triangle are A(4,3); B(1,–1) and C(7,k). Value(s) of k for which centroid, orthocentre, incentre and circumcentre of the ΔABC lie on the same straight line is/are-

(A) 7 (B) –1 (C) –19/8 (D) none

SL0072

12. Line $\frac{x}{a} + \frac{y}{b} = 1$ cuts the co-ordinate axes at A(a,0) and B(0,b) and the line $\frac{x}{a'} + \frac{y}{b'} = -1$ at A'(–a',0) and B'(0,–b'). If the points A,B,A',B' are concyclic then the orthocentre of the triangle ABA' is -

(A) (0,0) (B) (0,b') (C) $\left(0, \frac{aa'}{b}\right)$ (D) $\left(0, \frac{bb'}{a}\right)$

SL0073

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13. If the vertices P,Q,R of a triangle PQR are rational points, which of the following points of the triangle PQR is/are always rational point(s) ?

(A) centroid (B) incentre (C) circumcentre (D) orthocentre

SL0074

14. The area of triangle ABC is 20 square units. The co-ordinates of vertex A are (–5,0) and B are (3,0). The vertex C lies on the line, x – y = 2. The co-ordinates of C are -

(A) (5,3) (B) (–3,–5) (C) (–5,–7) (D) (7,5)

SL0075

15. Consider the equation $y - y_1 = m(x - x_1)$. If m and x_1 are fixed and different lines are drawn for different values of y_1, then :

(A) the lines will pass through a fixed point (B) there will be a set of parallel lines
(C) all the lines intersect the line $x = x_1$ (D) all the lines will be parallel to the line $y = x_1$.

SL0076

16. If one vertex of an equilateral triangle of side 'a' lies at the origin and the other lies on the line $x - \sqrt{3}y = 0$ then the co-ordinates of the third vertex are :

(A) (0,a) (B) $\left(\frac{\sqrt{3}a}{2}, -\frac{a}{2}\right)$ (C) (0,–a) (D) $\left(-\frac{\sqrt{3}a}{2}, \frac{a}{2}\right)$

SL0077

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17. Let B(1,–3) and D(0,4) represent two vertices of rhombus ABCD in (x,y) plane, then coordinates of vertex A if ∠BAD = 60° can be equal to-

(A) $\left(\frac{1-7\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}\right)$ (B) $\left(\frac{1+7\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}\right)$

(C) $\left(\frac{1-14\sqrt{3}}{2}, \frac{1-2\sqrt{3}}{2}\right)$ (D) $\left(\frac{1+14\sqrt{3}}{2}, \frac{1+2\sqrt{3}}{2}\right)$

SL0078

18. The sides of a triangle are the straight lines x + y = 1; 7y = x and $\sqrt{3}y + x = 0$. Then which of the following is an interior point of the triangle ?

(A) circumcentre (B) centroid

(C) incentre (D) orthocentre

SL0079

19. A line passes through the origin and makes an angle of π/4 with the line x – y + 1 = 0. Then :

(A) equation of the line is x = 0

(B) the equation of the line is y = 0

(C) the point of intersection of the line with the given line is (–1,0)

(D) the point of intersection of the line with the given line is (0,1)

SL0080

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20. Equation of a straight line passing through the point (2,3) and inclined at an angle of arc tan $\frac{1}{2}$ with the line y + 2x = 5 is-

(A) y = 3 (B) x = 2

(C) 3x + 4y – 18 = 0 (D) 4x + 3y – 17 = 0

SL0081

21. If $a^2 + 9b^2 - 4c^2 = 6ab$ then the family of lines ax + by + c = 0 are concurrent at :

(A) (1/2, 3/2) (B) (–1/2, –3/2)

(C) (–1/2, 3/2) (D) (1/2, –3/2)

SL0082

22. The lines L_1 and L_2 denoted by $3x^2 + 10xy + 8y^2 + 14x + 22y + 15 = 0$ intersect at the point P and have gradients m_1 and m_2 respectively. The acute angles between them is θ. Which of the following relations hold good ?

(A) $m_1 + m_2 = 5/4$

(B) $m_1m_2 = 3/8$

(C) acute angle between L_1 and L_2 is $\sin^{-1}\left(\frac{2}{5\sqrt{5}}\right)$.

(D) sum of the abscissa and ordinate of the point P is –1.

SL0083

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Paragraph for Question Nos. 23 to 25

Let ABCD is a square with sides of unit length. Points E and F are taken on sides AB and AD respectively so that AE= AF. Let P be a point inside the square ABCD.

23. The maximum possible area of quadrilateral CDFE is -

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{5}{8}$ (D) $\frac{3}{8}$

SL0084

24. The value of $(PA)^2 - (PB)^2 + (PC)^2 - (PD)^2$ is equal to-

(A) 3 (B) 2 (C) 1 (D) 0

SL0084

25. Let a line passing through point A divides the square ABCD into two parts so that area of one portion is double the other, then the length of portion of line inside the square is -

(A) $\frac{\sqrt{10}}{3}$ (B) $\frac{\sqrt{13}}{3}$ (C) $\frac{\sqrt{11}}{3}$ (D) $\frac{2}{\sqrt{3}}$

SL0084

Paragraph for Question Nos. 26 to 28

In the diagram, a line is drawn through the points A(0,16) and B(8,0). Point P is chosen in the first quadrant on the line through A and B. Points C and D are chosen on the x and y axis respectively, so that PDOC is a rectangle.

26. Perpendicular distance of the line AB from the point (2, 2) is -

(A) $\sqrt{4}$ (B) $\sqrt{10}$

(C) $\sqrt{20}$ (D) $\sqrt{50}$

SL0085

27. Sum of the coordinates of the point P if PDOC is a square is -

(A) $\frac{32}{3}$ (B) $\frac{16}{3}$ (C) 16 (D) 11

SL0085

28. Number of possible ordered pair(s) of all positions of the point P on AB so that the area of the rectangle PDOC is 30 sq. units, is-

(A) three (B) two (C) one (D) zero

SL0085

Paragraph for Question Nos. 29 to 31

Consider two points $A \equiv (1,2)$ and $B \equiv (3,-1)$. Let M be a point on the straight line $L \equiv x + y = 0$.

29. If M be a point on the line L = 0 such that AM + BM is minimum, then the reflection of M in the line x = y is -

(A) (1,–1) (B) (–1,1) (C) (2,–2) (D) (–2,2)

SL0086

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30. If M be a point on the line L = 0 such that |AM – BM| is maximum, then the distance of M from N ≡ (1,1) is-

(A) $5\sqrt{2}$ (B) 7 (C) $3\sqrt{5}$ (D) 10

SL0086

31. If M be a point on the line L = 0 such that |AM – BM| is minimum, then the area of ΔAMB equals-

(A) $\frac{13}{4}$ (B) $\frac{13}{2}$ (C) $\frac{13}{6}$ (D) $\frac{13}{8}$

SL0086

Paragraph for question nos. 32 and 33

An equilateral triangle ABC has its centroid at the origin and the base BC lies along the line x + y = 1.

32. Area of the equilateral ΔABC is -

(A) $\frac{3\sqrt{3}}{2}$ (B) $\frac{3\sqrt{3}}{4}$ (C) $\frac{3\sqrt{2}}{2}$ (D) $\frac{2\sqrt{3}}{4}$

SL0087

33. Gradient of the other two lines are -

(A) $\sqrt{3}, \sqrt{2}$ (B) $\sqrt{3}, \frac{1}{\sqrt{3}}$ (C) $\sqrt{2}+1, \sqrt{2}-1$ (D) $2+\sqrt{3}, 2-\sqrt{3}$

SL0087

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EXERCISE (S-1)

1. Determine the ratio in which the point P(3 , 5) divides the join of A(1, 3) & B(7, 9). Find the harmonic conjugate of P w.r.t. A & B.

SL0088

2. Line $\frac{x}{6}+\frac{y}{8}=1$ intersects the x and y axes at M and N respectively. If the coordinates of the point P lying inside the triangle OMN (where 'O' is origin) are (a, b) such that the areas of the triangle POM, PON and PMN are equal. Find

(a) the coordinates of the point P and

(b) the radius of the circle escribed opposite to the angle N.

SL0089

3. Two vertices of a triangle are (4, –3) & (–2, 5). If the orthocentre of the triangle is at (1, 2), find the coordinates of the third vertex.

SL0090

4. The point A divides the join of P(–5 , 1) & Q(3, 5) in the ratio K : 1. Find the two values of K for which the area of triangle ABC, where B is (1, 5) & C is (7, –2), is equal to 2 units in magnitude.

SL0091

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5. The area of a triangle is 5. Two of its vertices are (2, 1) & (3, –2). The third vertex lies on $y = x + 3$. Find the third vertex.

SL0092

6. A variable line, drawn through the point of intersection of the straight lines $\frac{x}{a}+\frac{y}{b}=1$ & $\frac{x}{b}+\frac{y}{a}=1$, meets the coordinate axes in A & B. Find the locus of the mid point of AB.

SL0093

7. Let O(0, 0), A(6, 0) and B(3, $\sqrt{3}$) be the vertices of ΔOAB. Let R be the region consisting of all those points P inside ΔOAB which satisfy $d(P, OA) \le$ minimum $(d(P,OB), d(P,AB))$ where d(P,OA), d(P,OB) and d(P,AB) represent the distance of P from the sides OA,OB and AB respectively. If the area of region R is $9(a-\sqrt{b})$ where a and b are coprime, then find the value of (a + b)

SL0094

8. A triangle has side lengths 18, 24 and 30. Find the area of the triangle whose vertices are the incentre, circumcentre and centroid of the triangle.

SL0095

9. Consider the family of lines $(x - y - 6) + \lambda(2x + y + 3) = 0$ and $(x + 2y - 4) + \lambda(3x - 2y - 4) = 0$. If the lines of these 2 families are at right angle to each other then find the locus of their point of intersection.

SL0096

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10. A line is such that its segment between the straight lines $5x - y - 4 = 0$ and $3x + 4y - 4 = 0$ is bisected at the point (1, 5). Obtain the equation.

SL0097

11. The line $3x + 2y = 24$ meets the y-axis at A & the x-axis at B. The perpendicular bisector of AB meets the line through (0, –1) parallel to x-axis at C. Find the area of the triangle ABC.

SL0098

12. If the straight line drawn through the point P($\sqrt{3}$, 2) & inclined at an angle $\frac{\pi}{6}$ with the x-axis, meets the line $\sqrt{3}x - 4y + 8 = 0$ at Q. Find the length PQ.

SL0099

13. The points (1, 3) & (5, 1) are two opposite vertices of a rectangle. The other two vertices lie on the line $y = 2x + c$. Find c & the remaining vertices.

SL0100

14. A straight line L is perpendicular to the line $5x - y = 1$. The area of the triangle formed by the line L & the coordinate axes is 5. Find the equation of the line.

SL0101

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15. Given vertices A (1, 1), B (4, –2) & C (5, 5) of a triangle, find the equation of the perpendicular dropped from C to the interior bisector of the angle A.

SL0102

16. Consider a triangle ABC with sides AB and AC having the equations $L_1 = 0$ and $L_2 = 0$. Let the centroid, orthocentre and circumcentre of the ΔABC are G, H and S respectively. L = 0 denotes the equation of sides BC.

(a) If $L_1 : 2x - y = 0$ and $L_2 : x + y = 3$ and G(2, 3) then find the slope of the line L = 0.

SL0103

(b) If $L_1 : 2x + y = 0$ and $L_2 : x - y + 2 = 0$ and H(2, 3) then find the y-intercept of L = 0.

SL0104

(c) If $L_1 : x + y - 1 = 0$ and $L_2 : 2x - y + 4 = 0$ and S(2, 1) then find the x-intercept of the line L = 0.

SL0105

17. The equations of the perpendicular bisectors of the sides AB & AC of a triangle ABC are $x - y + 5 = 0$ & $x + 2y = 0$, respectively. If the point A is (1, –2) find the equation of the line BC.

SL0106

18. Let P be the point (3, 2). Let Q be the reflection of P about the x-axis. Let R be the reflection of Q about the line y = –x and let S be the reflection of R through the origin. PQRS is a convex quadrilateral. Find the area of PQRS.

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SL0107

19. Two equal sides of an isosceles triangle are given by the equations $7x - y + 3 = 0$ and $x + y - 3 = 0$ & its third side passes through the point (1, –10). Determine the equation of the third side.

SL0108

20. Two consecutive sides of a parallelogram are $4x + 5y = 0$ & $7x + 2y = 0$. If the equation to one diagonal is $11x + 7y = 9$, find the equation to the other diagonal.

SL0109

21. A straight line is drawn from the point (1, 0) to the curve $x^2 + y^2 + 6x - 10y + 1 = 0$, such that the intercept made on it by the curve subtends a right angle at the origin. Find the equations of the line.

SL0110

22. Consider a line pair $2x^2 + 3xy - 2y^2 - 10x + 15y - 28 = 0$ and another line L passing through origin with gradient 3. The line pair and line L form a triangle whose vertices are A, B and C.

(a) Find the sum of the cotangents of the interior angles of the triangle ABC.

(b) Find the area of triangle ABC

(c) Find the radius of the circle touching all the 3 sides of the triangle.

SL0111

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EXERCISE (S-2)

1. Point O, A, B, C ..... are shown in figure where OA = 2AB = 4BC = ... so on. Let A is the centroid of a triangle whose orthocentre and circumcenter are (2, 4) and $\left(\frac{7}{2}, \frac{5}{2}\right)$ respectively. If an insect starts moving from the point O(0,0) along the straight line in zig-zag fashion and terminates ultimately at point P(α, β) then find the value of (α + β)

SL0112

2. P is the point (–1, 2), a variable line through P cuts the x & y axes at A & B respectively Q is the point on AB such that PA, PQ, PB are H.P. Find the locus of Q.

SL0113

3. The interior angle bisector of angle A for the triangle ABC whose coordinates of the vertices are A(–8, 5); B(–15, –19) and C(1, – 7) has the equation ax + 2y + c = 0. Find 'a' and 'c'.

SL0114

4. Let ABC be a triangle such that the coordinates of A are (–3, 1). Equation of the median through B is 2x + y – 3 = 0 and equation of the angular bisector of C is 7x – 4y – 1= 0. Then match the entries of column-I with their corresponding correct entries of column-II.

	Column-I		Column-II
(A)	Equation of the line AB is	(P)	2x + y – 3 = 0
(B)	Equation of the line BC is	(Q)	2x – 3y + 9 = 0
(C)	Equation of CA is	(R)	4x + 7y + 5 = 0
		(S)	18x – y – 49 = 0

SL0115

5. The equations of perpendiculars of the sides AB & AC of triangle ABC are x – y – 4 = 0 and 2x – y – 5 = 0 respectively. If the vertex A is (– 2, 3) and point of intersection of perpendiculars bisectors is $\left(\frac{3}{2}, \frac{5}{2}\right)$, find the equation of medians to the sides AB & AC respectively.

SL0116

6. Consider a ΔABC whose sides AB, BC and CA are represented by the straight lines 2x + y = 0, x + py = q and x – y = 3 respectively. The point P is (2, 3).

(a) If P is the centroid, then find the value of (p + q).

(b) If P is the orthocentre, then find the value of (p + q).

(c) If P is the circumcentre, then find the value of (p + q).

SL0117

7. Find the equations of the sides of a triangle having (4, –1) as a vertex, if the lines x – 1 = 0 and x – y –1 = 0 are the equations of two internal bisectors of its angles.

SL0118

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8. Find the equation of the straight lines passing through (–2, –7) & having an intercept of length 3 between the straight lines $4x + 3y = 12$, $4x + 3y = 3$.

SL0119

9. Two sides of a rhombus ABCD are parallel to the lines $y = x + 2$ & $y = 7x + 3$. If the diagonals of the rhombus intersect at the point (1, 2) & the vertex A is on the y-axis, find the possible coordinates of A.

SL0120

10. Show that all the chords of the curve $3x^2 - y^2 - 2x + 4y = 0$ which subtend a right angle at the origin are concurrent. Does this result also hold for the curve, $3x^2 + 3y^2 - 2x + 4y = 0$? If yes, what is the point of concurrency & if not, give reasons.

SL0121

11. The sides of a triangle have the combined equation $x^2 - 3y^2 - 2xy + 8y - 4 = 0$. The third side, which is variable always passes through the point (–5, –1). If the range of values of the slope of the third line so that the origin is an interior point of the triangle, lies in the interval (a, b), then find $\left(a + \frac{1}{b^2}\right)$.

SL0122

12. The two line pairs $y^2 - 4y + 3 = 0$ and $x^2 + 4xy + 4y^2 - 5x - 10y + 4 = 0$ enclose a 4 sided convex polygon find **(i)** area of the polygon; **(ii)** length of its diagonals.

SL0123

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EXERCISE (JM)

1. A line is drawn through the point (1, 2) to meet the coordinate axes at P and Q such that it forms a triangle OPQ, where O is the origin. If the area of the triangle OPQ is least, then the slope of the line PQ is : **[AIEEE 2012]**

(1) $-\frac{1}{2}$ (2) $-\frac{1}{4}$ (3) –4 (4) –2

SL0124

2. If the line $2x + y = k$ passes through the point which divides the line segment joining the points (1, 1) and (2, 4) in the ratio 3 : 2, then k equals : **[AIEEE 2012]**

(1) $\frac{11}{5}$ (2) $\frac{29}{5}$ (3) 5 (4) 6

SL0125

3. A ray of light along $x + \sqrt{3}y = \sqrt{3}$ gets reflected upon reaching x-axis, the equation of the reflected ray is : **[JEE-MAIN 2013]**

(1) $y = x + \sqrt{3}$ (2) $\sqrt{3}y = x - \sqrt{3}$ (3) $y = \sqrt{3}x - \sqrt{3}$ (4) $\sqrt{3}y = x - 1$

SL0126

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4. The x-coordinate of the incentre of the triangle that has the coordinates of mid points of its sides as (0, 1)(1, 1) and (1, 0) is : **[JEE-MAIN 2013]**

(1) $2+\sqrt{2}$ (2) $2-\sqrt{2}$ (3) $1+\sqrt{2}$ (4) $1-\sqrt{2}$

SL0127

5. A light ray emerging from the point source placed at P(1, 3) is reflected at a point Q in the axis of *x*. If the reflected ray passes through the point R(6, 7), then the abscissa of Q is : **[JEE-MAIN Online 2013]**

(1) 3 (2) $\frac{7}{2}$ (3) 1 (4) $\frac{5}{2}$

SL0128

6. If the three lines $x-3y = p$, $ax + 2y = q$ and $ax + y = r$ from a right – angled triangle then:

[JEE-MAIN Online 2013]

(1) $a^2 - 6a - 12 = 0$ (2) $a^2 - 9a + 12 = 0$

(3) $a^2 - 9a + 18 = 0$ (4) $a^2 - 6a - 18 = 0$

SL0129

7. If the x-intercept of some line L is double as that of the line, $3x + 4y = 12$ and the y-intercept of L is half as that of the same line, then the slope of L is :- **[JEE-MAIN Online 2013]**

(1) –3 (2) –3/2 (3) –3/8 (4) –3/16

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SL0130

8. If the extremities of the base of an isosceles triangle are the points (2a, 0) and (0, a) and the equation of one of the sides is $x = 2a$, then the area of the triangle, in square units, is :

[JEE-MAIN Online 2013]

(1) $\frac{5}{2}a^2$ (2) $\frac{5}{4}a^2$ (3) $\frac{25a^2}{4}$ (4) $5a^2$

SL0131

9. Let θ_1 be the angle between two lines $2x + 3y + c_1 = 0$ and $-x + 5y + c_2 = 0$, and θ_2 be the angle between two lines $2x + 3y + c_1 = 0$ and $-x + 5y + c_3 = 0$, where c_1, c_2, c_3 are any real numbers :

[JEE-MAIN Online 2013]

Statement–1 : If c_2 and c_3 are proportional, then $\theta_1 = \theta_2$.

Statement–2 : $\theta_1 = \theta_2$ for all c_2 and c_3.

(1) Statement-1 is true and Statement - 2 is true, Statement-2 is not a correct explanation for Statement-1.

(2) Statement-1 is false and Statement-2 is true.

(3) Statement-1 is true and Statement-2 is false.

(4) Statement-1 is true and Statement - 2 is true, Statement-2 is a correct explanation for Statement-1.

SL0132

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10. Let A (-3, 2) and B (-2,1) be the vertices of a triangle ABC. If the centroid of this triangle lies on the line $3x + 4y + 2 = 0$, then the vertex C lies on the line : **[JEE-MAIN Online 2013]**

(1) $4x + 3y + 5 = 0$ (2) $3x + 4y + 5 = 0$

(3) $3x + 4y + 3 = 0$ (4) $4x + 3y + 3 = 0$

SL0133

11. If the image of point P(2, 3) in a line L is Q (4, 5) then, the image of point R (0, 0) in the same line is : **[JEE-MAIN Online 2013]**

(1) (4, 5) (2) (2, 2) (3) (3, 4) (4) (7, 7)

SL0134

12. Let a, b, c and d be non-zero numbers. If the point of intersection of the lines $4ax + 2ay + c = 0$ and $5bx + 2by + d = 0$ lies in the fourth quadrant and is equidistant from the two axes then : **[JEE(Main)-2014]**

(1) $2bc - 3ad = 0$ (2) $2bc + 3ad = 0$ (3) $3bc - 2ad = 0$ (4) $3bc + 2ad = 0$

SL0135

13. Let PS be the median of the triangle with vertices P (2, 2), Q (6, –1) and R (7, 3). The equation of the line passing through (1, –1) and parallel to PS is : **[JEE(Main)-2014]**

(1) $4x - 7y - 11 = 0$ (2) $2x + 9y + 7 = 0$ (3) $4x + 7y + 3 = 0$ (4) $2x - 9y - 11 = 0$

SL0136

14. Locus of the image of the point (2, 3) in the line $(2x - 3y + 4) + k(x - 2y + 3) = 0$, $k \in R$, is a

(1) circle of radius $\sqrt{2}$ (2) circle of radius $\sqrt{3}$

(3) straight line parallel to x-axis (4) straight line parallel to y-axis

[JEE(Main)-2015]

SL0137

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15. Two sides of a rhombus are along the lines, $x - y + 1 = 0$ and $7x - y - 5 = 0$. If its diagonals intersect at (–1, –2), then which one of the following is a vertex of this rhombus ? **[JEE(Main)-2016]**

(1) $\left(-\frac{10}{3}, -\frac{7}{3}\right)$ (2) (–3, –9) (3) (–3, –8) (4) $\left(\frac{1}{3}, -\frac{8}{3}\right)$

SL0138

16. Let k be an integer such that triangle with vertices (k, –3k), (5, k) and (–k, 2) has area 28 sq. units. Then the orthocentre of this triangle is at the point : **[JEE(Main)-2017]**

(1) $\left(2, \frac{1}{2}\right)$ (2) $\left(2, -\frac{1}{2}\right)$ (3) $\left(1, \frac{3}{4}\right)$ (4) $\left(1, -\frac{3}{4}\right)$

SL0139

17. Let the orthocentre and centroid of a triangle be A(–3, 5) and B(3, 3) respectively. If C is the circumcentre of this triangle, then the radius of the circle having line segment AC as diameter, is: **[JEE(Main)-2018]**

(1) $2\sqrt{10}$ (2) $3\sqrt{\frac{5}{2}}$ (3) $\frac{3\sqrt{5}}{2}$ (4) $\sqrt{10}$

SL0140

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18. A straight line through a fixed point (2, 3) intersects the coordinate axes at distinct points P and Q. If O is the origin and the rectangle OPRQ is completed, then the locus of R is : **[JEE(Main)-2018]**

(1) $2x + 3y = xy$ (2) $3x + 2y = xy$ (3) $3x + 2y = 6xy$ (4) $3x + 2y = 6$

SL0141

19. Let S be the set of all triangles in the xy-plane, each having one vertex at the origin and the other two vertices lie on coordinate axes with integral coordinates. If each triangle in S has area 50sq. units, then the number of elements in the set S is : **[JEE(Main)-2019]**

(1) 9 (2) 18 (3) 32 (4) 36

SL0142

20. If the line $3x + 4y - 24 = 0$ intersects the x-axis at the point A and the y-axis at the point B, then the incentre of the triangle OAB, where O is the origin, is **[JEE(Main)-2019]**

(1) (3, 4) (2) (2, 2) (3) (4, 4) (4) (4, 3)

SL0143

21. Two vertices of a triangle are (0,2) and (4,3). If its orthocentre is at the origin, then its third vertex lies in which quadrant ? **[JEE(Main)-2019]**

(1) Fourth (2) Second (3) Third (4) First

SL0144

22. Two sides of a parallelogram are along the lines, $x + y = 3$ and $x - y + 3 = 0$. If its diagonals intersect at (2,4), then one of its vertex is : **[JEE(Main)-2019]**

(1) (2,6) (2) (2,1) (3) (3,5) (4) (3,6)

SL0145

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23. If a circle of radius R passes through the origin O and intersects the coordinate axes at A and B, then the locus of the foot of perpendicular from O on AB is : **[JEE(Main)-2019]**

(1) $(x^2 + y^2)^2 = 4Rx^2y^2$ (2) $(x^2 + y^2)(x + y) = R^2xy$

(3) $(x^2 + y^2)^3 = 4R^2x^2y^2$ (4) $(x^2 + y^2)^2 = 4R^2x^2y^2$

SL0146

24. Slope of a line passing through P(2, 3) and intersecting the line, $x + y = 7$ at a distance of 4 units from P, is **[JEE(Main)-2019]**

(1) $\frac{\sqrt{5}-1}{\sqrt{5}+1}$ (2) $\frac{1-\sqrt{5}}{1+\sqrt{5}}$ (3) $\frac{1-\sqrt{7}}{1+\sqrt{7}}$ (4) $\frac{\sqrt{7}-1}{\sqrt{7}+1}$

SL0147

25. A rectangle is inscribed in a circle with a diameter lying along the line $3y = x + 7$. If the two adjacent vertices of the rectangle are (–8, 5) and (6, 5), then the area of the rectangle (in sq. units) is :-

[JEE(Main)-2019]

(1) 72 (2) 84 (3) 98 (4) 56

SL0148

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26. A straight line L at a distance of 4 units from the origin makes positive intercepts on the coordinate axes and the perpendicular from the origin to this line makes an angle of 60° with the line x + y = 0. Then an equation of the line L is : **[JEE(Main)-2019]**

(1) $\left(\sqrt{3}+1\right)x+\left(\sqrt{3}-1\right)y=8\sqrt{2}$

(2) $\left(\sqrt{3}-1\right)x+\left(\sqrt{3}+1\right)y=8\sqrt{2}$

(3) $\sqrt{3}x+y=8$

(4) $x+\sqrt{3}y=8$

SL0149

27. The locus of the mid-points of the perpendiculars drawn from points on the line, x = 2y to the line x = y is : **[JEE(Main)-2020]**

(1) 2x – 3y = 0 (2) 7x – 5y = 0 (3) 5x – 7y = 0 (4) 3x – 2y = 0

SL0150

28. Let A(1, 0), B(6, 2) and $C\left(\frac{3}{2},6\right)$ be the vertices of a triangle ABC. If P is a point inside the triangle ABC such that the triangles APC, APB and BPC have equal areas, then the length of the line segment PQ, where Q is the point $\left(-\frac{7}{6},-\frac{1}{3}\right)$, is _____. **[JEE(Main)-2020]**

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SL0151

29. Let two points be A(1,–1) and B(0,2). If a point P(x',y') be such that the area of ΔPAB = 5 sq. units and it lies on the line, 3x + y – 4λ = 0, then a value of λ is **[JEE(Main)-2020]**

(1) 1 (2) 4 (3) 3 (4) –3

SL0152

30. Let C be the centroid of the triangle with vertices (3, –1), (1, 3) and (2, 4). Let P be the point of intersection of the lines x + 3y – 1 = 0 and 3x – y + 1 = 0. Then the line passing through the points C and P also passes through the point : **[JEE(Main)-2020]**

(1) (7, 6) (2) (–9, –6) (3) (–9, –7) (4) (9, 7)

SL0153

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EXERCISE (JA)

1. Consider the lines given by

$L_1 = x + 3y - 5 = 0$

$L_2 = 3x - ky - 1 = 0$

$L_3 = 5x + 2y - 12 = 0$

Match the statements / Expression in **Column-I** with the statements / Expressions in **Column-II** and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrix given in OMR.

	Column-I		**Column-II**
(A)	L_1, L_2, L_3 are concurrent, if	(P)	$k = -9$
(B)	One of L_1, L_2, L_3 is parallel to at least one of the other two, if	(Q)	$k = -\frac{6}{5}$
(C)	L_1, L_2, L_3 form a triangle, if	(R)	$k = \frac{5}{6}$
(D)	L_1, L_2, L_3 do not form a triangle, if	(S)	$k = 5$

[JEE 2008, 6]

SL0154

2. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{i} - \hat{j}$, $4\hat{i}, 3\hat{i} + 3\hat{j}$ and $-3\hat{i} + 2\hat{j}$ respectively. The quadrilateral PQRS must be a

(A) parallelogram, which is neither a rhombus nor a rectangle

(B) square

(C) rectangle, but not a square

(D) rhombus, but not a square

[JEE 2010, 3]

SL0155

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3. A straight line L through the point (3, –2) is inclined at an angle 60° to the line $\sqrt{3}x + y = 1$. If L also intersect the x-axis, then the equation of L is **[JEE 2011, 3 (–1)]**

(A) $y + \sqrt{3}x + 2 - 3\sqrt{3} = 0$

(B) $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$

(C) $\sqrt{3}y - x + 3 + 2\sqrt{3} = 0$

(D) $\sqrt{3}y + x - 3 + 2\sqrt{3} = 0$

SL0156

4. For a > b > c > 0, the distance between (1, 1) and the point of intersection of the lines ax + by + c = 0 and bx + ay + c = 0 is less than $2\sqrt{2}$. Then **[JEE-Advanced 2013, 2]**

(A) a + b – c > 0

(B) a – b + c < 0

(C) a – b + c > 0

(D) a + b – c < 0

SL0157

5. For a point P in the plane, let $d_1(P)$ and $d_2(P)$ be the distances of the point P from the lines x – y = 0 and x + y = 0 respectively. The area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2 \le d_1(P) + d_2(P) \le 4$, is

[JEE(Advanced)-2014, 3]

SL0158

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ANSWER KEY

EXERCISE (O-1)

1. B	**2.** D	**3.** D	**4.** A	**5.** C	**6.** D	**7.** D	**8.** A
9. C	**10.** C	**11.** B	**12.** A	**13.** B	**14.** A	**15.** C	**16.** A
17. A	**18.** A	**19.** B	**20.** A	**21.** B	**22.** A	**23.** D	**24.** D
25. A	**26.** C	**27.** A	**28.** A	**29.** D	**30.** A	**31.** A	**32.** B
33. B	**34.** B	**35.** D	**36.** B	**37.** D	**38.** A	**39.** C	**40.** B

41. B **42.** B **43.** (A) Q,S,T; (B) P,Q,S,T; (C) S; (D) R,T

44. (A) P,Q; (B) P; (C) S; (D) R **45.** (A) S; (B) S; (C) P; (D) R

46. (A) R, (B) Q (C) T (D) S **47.** (A) S; (B) R ; (C) Q

EXERCISE (O-2)

1. B	**2.** B	**3.** C	**4.** B	**5.** D	**6.** B	**7.** A	**8.** B
9. B	**10.** A,B	**11.** B,C	**12.** B,C	**13.** A,C,D	**14.** B,D	**15.** B,C	
16. A,B,C,D	**17.** A,B	**18.** B,C	**19.** A,B,C,D	**20.** B,C	**21.** C,D		
22. B,C,D	**23.** C	**24.** D	**25.** B	**26.** C	**27.** A	**28.** B	**29.** B
30. D	**31.** A	**32.** A	**33.** D				

EXERCISE (S-1)

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1. $1:2$; $Q(-5,-3)$ **2.** (a) $\left(2,\frac{8}{3}\right)$; (b) 4 **3.** $(33, 26)$ **4.** $K = 7$ or $\frac{31}{9}$

5. $\left(\frac{7}{2},\frac{13}{2}\right)$ or $\left(-\frac{3}{2},\frac{3}{2}\right)$ **6.** $2xy(a+b) = ab(x+y)$ **7.** 5 **8.** 3 units

9. $x^2 + y^2 - 3x + 4y - 3 = 0$ **10.** $83x - 35y + 92 = 0$ **11.** 91 sq.units **12.** 6 units

13. $c = -4$; $B(2, 0)$; $D(4, 4)$ **14.** $x + 5y + 5\sqrt{2} = 0$ or $x + 5y - 5\sqrt{2} = 0$ **15.** $x - 5 = 0$

16. (a) 5; (b) 2; (c) 3/2 **17.** $14x + 23y = 40$ **18.** 15 **19.** $x - 3y - 31 = 0$ or $3x + y + 7 = 0$

20. $x - y = 0$ **21.** $x + y = 1$; $x + 9y = 1$ **22.** (a) $\frac{50}{7}$; (b) $\frac{63}{10}$; (c) $\frac{3}{10}\left(8\sqrt{5} - 5\sqrt{10}\right)$

EXERCISE (S-2)

1. 8 **2.** $y = 2x$ **3.** $a = 11$, $c = 78$ **4.** (A) R; (B) S; (C) Q

5. $x + 4y = 4$; $5x + 2y = 8$ **6.** (a) 74; (b) 50; (c) 47

7. $2x - y + 3 = 0$, $2x + y - 7 = 0$, $x - 2y - 6 = 0$ **8.** $7x + 24y + 182 = 0$ or $x = -2$

9. $(0, 0)$ or $\left(0,\frac{5}{2}\right)$ **10.** $(1,-2)$, yes $\left(\frac{1}{3},-\frac{2}{3}\right)$ **11.** 24

12. (i) area = 6 sq. units, (ii) diagonals are $\sqrt{5}$ & $\sqrt{53}$

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EXERCISE (JM)

1.	4	**2.**	4	**3.**	2	**4.**	2	**5.**	4	**6.**	3	**7.**	4	**8.**	1	**9.**	4	**10.**	3	**11.**	4	**12.**	3
13.	2	**14.**	1	**15.**	4	**16.**	1	**17.**	2	**18.**	2	**19.**	4	**20.**	2	**21.**	2	**22.**	4	**23.**	3	**24.**	3
25.	2	**26.**	1	**27.**	3	**28.**	5	**29.**	3	**30.**	2												

EXERCISE (JA)

1. (A) S; (B) P,Q; (C) R ; (D) P,Q,S **2.** A **3.** B **4.** A or C or A,C **5.** 6

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ALLEN CAREER INSTITUTE KOTA (RAJASTHAN)

Chapter 05

CIRCLE

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JEE (Main/Advanced) Syllabus

JEE (Main) Syllabus :

Standard form of equation of a circle, general form of the equation of a circle, its radius and centre, equation of a circle when the end points of a diameter are given, points of intersection of a line and a circle with the centre at the origin and condition for a line to be tangent to a circle, equation of the tangent.

JEE (Advanced) Syllabus :

Equation of a circle in various forms, equations of tangent, normal and chord.

Parametric equations of a circle, intersection of a circle with a straight line or a circle, equation of a circle through the points of intersection of two circles and those of a circle and a straight line.

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Important Notes

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CIRCLE

1. DEFINITION :

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point (in the same given plane) remains constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

Equation of a circle :

The curve traced by the moving point is called its circumference i.e. the equation of any circle is satisfied by co-ordinates of all points on its circumference.

or

The equation of the circle means the equation of its circumference.

or

It is the set of all points lying on the circumference of the circle.

Chord and diameter - the line joining any two points on the circumference is called a chord. If any chord passing through its centre is called its diameter.

AB = chord, PQ = diameter

C = centre

2. STANDARD EQUATIONS OF THE CIRCLE :

(a) Central Form :

If (h, k) is the centre and r is the radius of the circle then its equation is

$(x-h)^2 + (y-k)^2 = r^2$

Special Cases :

(i) If centre is origin (0,0) and radius is 'r' then equation of circle is $x^2 + y^2 = r^2$ and this is called the standard form.

(ii) If radius of circle is zero then equation of circle is $(x - h)^2 + (y - k)^2 = 0$. Such circle is called zero circle or **point circle.**

(iii) When circle touches x-axis then equation of the circle is

$(x-h)^2 + (y-k)^2 = k^2$.

(iv) When circle touches y-axis then equation of circle is

$(x-h)^2 + (y-k)^2 = h^2$

(v) When circle touches both the axes (x-axis and y-axis) then equation of circle $(x-h)^2 + (y-h)^2 = h^2$.

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(vi) When circle passes through the origin and centre of the circle is (h,k) then radius $\sqrt{h^2+k^2}=r$ and intercept cut on x-axis OP =2h, and intercept cut on y-axis is OQ = 2k and equation of circle is **$(x-h)^2+(y-k)^2=h^2+k^2$** or **$x^2+y^2-2hx-2ky=0$**

Note : Centre of the circle may exist in any quadrant hence for general cases use ± sign before h & k.

(b) General equation of circle

$x^2+y^2+2gx+2fy+c=0$. where g,f,c are constants and centre is (–g,–f)

i.e. $\left(-\frac{\text{coefficient of } x}{2},-\frac{\text{coefficient of } y}{2}\right)$ and radius $r=\sqrt{g^2+f^2-c}$

Note :

(i) If $(g^2+f^2-c)>0$, then r is real and positive and the circle is a real circle.

(ii) If $(g^2+f^2-c)=0$, then radius r = 0 and circle is a point circle.

(iii) If $(g^2+f^2-c)<0$, then r is imaginary then circle is also an imaginary circle with real centre.

(iv) $x^2+y^2+2gx+2fy+c=0$, has three constants and to get the equation of the circle at least three conditions should be known ⇒ A unique circle passes through three non collinear points.

(v) **The general second degree in x and y, $ax^2+by^2+2hxy+2gx+2fy+c=0$ represents a circle if :**

- coefficient of x^2 = coefficient of y^2 or $a=b\neq 0$
- coefficient of xy = 0 or h = 0
- $(g^2+f^2-c)\geq 0$ (for a real circle)

(c) Intercepts cut by the circle on axes :

The intercepts cut by the circle $x^2+y^2+2gx+2fy+c=0$ on :

(i) x-axis = $2\sqrt{g^2-c}$ (ii) y-axis $=2\sqrt{f^2-c}$

Note :

(i) If the circle cuts the x-axis at two distinct point, then $g^2-c>0$

(ii) If the circle cuts the y-axis at two distinct point, then $f^2-c>0$

(iii) If circle touches x-axis then $g^2=c$.

(iv) If circle touches y-axis then $f^2=c$.

(v) Circle lies completely above or below the x-axis then $g^2<c$.

(vi) Circle lies completely to the right or left to the y-axis, then $f^2<c$.

(vii) Intercept cut by a line on the circle $x^2+y^2+2gx+2fy+c=0$ or length of chord of the circle $=2\sqrt{a^2-P^2}$ where a is the radius and P is the length of perpendicular from the centre to the chord.

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(d) Equation of circle in diameter form :

If $A(x_1,y_1)$ and $B(x_2,y_2)$ are the end points of the diameter of the circle and $P(x,y)$ is the point other then A and B on the circle then from geometry we know that $\angle APB = 90°$.

$\Rightarrow$ (Slope of PA) × (Slope of PB) = –1

$\Rightarrow \quad \therefore \left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_2}\right) = -1$

$\Rightarrow \quad \mathbf{(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0}$

Note : This will be the circle of least radius passing through (x_1, y_1) and (x_2, y_2)

(e) Equation of circle in parametric forms :

(i) The parametric equation of the circle $x^2+y^2 = r^2$ are $\mathbf{x = r\cos\theta, y = r\sin\theta ; \theta \in [0, 2\pi)}$ and $(r\cos\theta, r\sin\theta)$ are called the parametric co-ordinates.

(ii) The parametric equation of the circle $(x-h)^2 + (y-k)^2 = r^2$ is $\mathbf{x = h + r\cos\theta, y = k + r\sin\theta}$ where θ is parameter.

(iii) The parametric equation of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are

$\mathbf{x = -g + \sqrt{g^2+f^2-c}\ \cos\theta,}$

$\mathbf{y = -f + \sqrt{g^2+f^2-c}\ \sin\theta}$ where θ is parameter.

Note : Equation of a straight line joining two point α & β on the circle $x^2 + y^2 = a^2$ is

$x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = a\cos\frac{\alpha-\beta}{2}$.

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Illustration 1 : Find the centre and the radius of the circles

(a) $3x^2 + 3y^2 - 8x - 10y + 3 = 0$

(b) $x^2 + y^2 + 2x\sin\theta + 2y\cos\theta - 8 = 0$

(c) $2x^2 + \lambda xy + 2y^2 + (\lambda - 4)x + 6y - 5 = 0$, for some λ.

Solution : (a) We rewrite the given equation as

$x^2 + y^2 - \frac{8}{3}x - \frac{10}{3}y + 1 = 0 \Rightarrow g = -\frac{4}{3}, f = -\frac{5}{3}, c = 1$

Hence the centre is $\left(\frac{4}{3}, \frac{5}{3}\right)$ and the radius is $\sqrt{\frac{16}{9}+\frac{25}{9}-1} = \sqrt{\frac{32}{9}} = \frac{4\sqrt{2}}{3}$ units

(b) $x^2 + y^2 + 2x\sin\theta + 2y\cos\theta - 8 = 0$. Centre of this circle is $(-\sin\theta, -\cos\theta)$

Radius = $\sqrt{\sin^2\theta + \cos^2\theta + 8} = \sqrt{1+8} = 3$ units

(c) $2x^2 + \lambda xy + 2y^2 + (\lambda - 4)x + 6y - 5 = 0$

We rewrite the equation as

$x^2 + \frac{\lambda}{2}xy + y^2 + \left(\frac{\lambda-4}{2}\right)x + 3y - \frac{5}{2} = 0$ ..... (i)

Since, there is no term of xy in the equation of circle $\Rightarrow \frac{\lambda}{2} = 0 \Rightarrow \lambda = 0$

So, equation (i) reduces to $x^2 + y^2 - 2x + 3y - \frac{5}{2} = 0$

$\therefore$ centre is $\left(1, -\frac{3}{2}\right)$ Radius = $\sqrt{1+\frac{9}{4}+\frac{5}{2}} = \frac{\sqrt{23}}{2}$ units.

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Illustration 2 : If the lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to a circle, then the radius of the circle is -

(A) 3/2 (B) 3/4 (C) 1/10 (D) 1/20

Solution : The diameter of the circle is perpendicular distance between the parallel lines (tangents) $3x - 4y + 4 = 0$ and $3x - 4y - \frac{7}{2} = 0$ and so it is equal to $\frac{4+7/2}{\sqrt{9+16}} = \frac{3}{2}$.

Hence radius is $\frac{3}{4}$. **Ans. (B)**

Illustration 3 : If $y = 2x + m$ is a diameter to the circle $x^2 + y^2 + 3x + 4y - 1 = 0$, then find m

Solution : Centre of circle = $(-3/2, -2)$. This lies on diameter $y = 2x + m$

$\Rightarrow -2 = (-3/2) \times 2 + m \Rightarrow m = 1$

Illustration 4 : The equation of a circle which passes through the point $(1, -2)$ and $(4, -3)$ and whose centre lies on the line $3x + 4y = 7$ is

(A) $15(x^2 + y^2) - 94x + 18y - 55 = 0$ (B) $15(x^2 + y^2) - 94x + 18y + 55 = 0$

(C) $15(x^2 + y^2) + 94x - 18y + 55 = 0$ (D) none of these

Solution : Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ..... (i)

Hence, substituting the points, $(1, -2)$ and $(4, -3)$ in equation (i)

$5 + 2g - 4f + c = 0$ ..... (ii)

$25 + 8g - 6f + c = 0$ ..... (iii)

centre $(-g, -f)$ lies on line $3x + 4y = 7$

Hence $-3g - 4f = 7$

solving for g, f,c, we get

Here $g = \frac{-47}{15}$, $f = \frac{9}{15}$, $c = \frac{55}{15}$

Hence the equation is $15(x^2 + y^2) - 94x + 18y + 55 = 0$ **Ans. (B)**

Illustration 5 : A circle has radius equal to 3 units and its centre lies on the line $y = x - 1$. Find the equation of the circle if it passes through (7, 3).

Solution : Let the centre of the circle be (α, β). It lies on the line $y = x - 1$

$\Rightarrow \beta = \alpha - 1$. Hence the centre is $(\alpha, \alpha - 1)$.

$\Rightarrow$ The equation of the circle is $(x - \alpha)^2 + (y - \alpha + 1)^2 = 9$

It passes through (7, 3) $\Rightarrow (7 - \alpha)^2 + (4 - \alpha)^2 = 9$

$\Rightarrow 2\alpha^2 - 22\alpha + 56 = 0 \Rightarrow \alpha^2 - 11\alpha + 28 = 0$

$\Rightarrow (\alpha - 4)(\alpha - 7) = 0 \Rightarrow \alpha = 4, 7$

Hence the required equations are

$x^2 + y^2 - 8x - 6y + 16 = 0$ and $x^2 + y^2 - 14x - 12y + 76 = 0$. **Ans.**

Illustration 6 : Let L_1 be a straight line through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 & L_2 are equal , then which of the following equations can represent L_1?

(A) $x + y = 0$ (B) $x - y = 0$ (C) $x + 7y = 0$ (D) $x - 7y = 0$

Solution : Let L_1 be $y = mx$

lines L_1 & L_2 will be at equal distances from centre of the circle centre of the circle is $\left(\frac{1}{2}, -\frac{3}{2}\right)$

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$$\Rightarrow \quad \frac{\left|\frac{1}{2}m+\frac{3}{2}\right|}{\sqrt{1+m^2}} = \frac{\left|\frac{1}{2}-\frac{3}{2}-1\right|}{\sqrt{2}} \quad \Rightarrow \quad \frac{(m+3)^2}{(1+m^2)} = 8$$

$$\Rightarrow \quad 7m^2 - 6m - 1 = 0 \quad \Rightarrow \quad (m-1)(7m+1) = 0$$

$$\Rightarrow \quad m = 1,\ m = -\frac{1}{7} \quad \Rightarrow \quad y = x,\ 7y + x = 0$$

Ans. (B, C)

Do yourself - 1 :

(i) Find the centre and radius of the circle $2x^2 + 2y^2 = 3x - 5y + 7$

(ii) Find the equation of the circle whose centre is the point of intersection of the lines $2x - 3y + 4 = 0$ & $3x + 4y - 5 = 0$ and passes through the origin.

(iii) Find the parametric form of the equation of the circle $x^2 + y^2 + px + py = 0$

(iv) Find the equation of the circle the end points of whose diameter are the centres of the circles $x^2 + y^2 + 16x - 14y = 1$ & $x^2 + y^2 - 4x + 10y = 2$

3. POSITION OF A POINT W.R.T CIRCLE :

(a) Let the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ and the point is (x_1, y_1) then -

Point (x_1, y_1) lies out side the circle or on the circle or inside the circle according as

$\Rightarrow x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c >, =, < 0$ or $S_1 >, =, < 0$

(b) The greatest & the least distance of a point A from a circle with centre C & radius r is $AC + r$ & $|AC - r|$ respectively.

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4. POWER OF A POINT W.R.T. CIRCLE :

Theorem : The power of point $P(x_1, y_1)$ w.r.t. the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is S_1

where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Note : If P outside, inside or on the circle then power of point is positive, negative or zero respectively.

If from a point $P(x_1, y_1)$, inside or outside the circle, a secant be drawn intersecting the circle in two points A & B, then PA . PB = constant. The product PA . PB is called power of point $P(x_1, y_1)$ w.r.t. the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, i.e. for number of secants PA.PB = $PA_1 . PB_1 = PA_2 . PB_2 = ... = PT^2 = S_1$

Illustration 7 : If P(2, 8) is an interior point of a circle $x^2 + y^2 - 2x + 4y - p = 0$ which neither touches nor intersects the axes, then set for p is -

(A) $p < -1$ (B) $p < -4$ (C) $p > 96$ (D) ϕ

Solution : For internal point p(2, 8), $4 + 64 - 4 + 32 - p < 0 \Rightarrow p > 96$

and x intercept = $2\sqrt{1+p}$ therefore $1 + p < 0$

$\Rightarrow \quad p < -1$ and y intercept = $2\sqrt{4+p} \quad \Rightarrow \quad p < -4$

Ans. (D)

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Do yourself - 2 :

(i) Find the position of the points (1, 2) & (6, 0) w.r.t. the circle $x^2 + y^2 - 4x + 2y - 11 = 0$

(ii) Find the greatest and least distance of a point P(7, 3) from circle $x^2 + y^2 - 8x - 6y + 16 = 0$. Also find the power of point P w.r.t. circle.

5. TANGENT LINE OF CIRCLE :

When a straight line meet a circle on two coincident points then it is called the tangent of the circle.

(a) Condition of Tangency :

The line L = 0 touches the circle S = 0 if P the length of the perpendicular from the centre to that line and radius of the circle r are equal i.e. P = r.

Illustration 8 : Find the range of parameter 'a' for which the variable line y = 2x + a lies between the circles $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 16x - 2y + 61 = 0$ without intersecting or touching either circle.

Solution : The given circles are $C_1 : (x-1)^2 + (y-1)^2 = 1$ and $C_2 : (x-8)^2 + (y-1)^2 = 4$

The line $y - 2x - a = 0$ will lie between these circle if centre of the circles lie on opposite sides of the line, i.e. $(1 - 2 - a)(1 - 16 - a) < 0 \Rightarrow a \in (-15, -1)$

Line wouldn't touch or intersect the circles if, $\frac{|1-2-a|}{\sqrt{5}} > 1$, $\frac{|1-16-a|}{\sqrt{5}} > 2$

$\Rightarrow \quad |1 + a| > \sqrt{5}, |15 + a| > 2\sqrt{5}$

$\Rightarrow \quad a > \sqrt{5} - 1$ or $a < -\sqrt{5} - 1, a > 2\sqrt{5} - 15$ or $a < -2\sqrt{5} - 15$

Hence common values of 'a' are $(2\sqrt{5} - 15, -\sqrt{5} - 1)$.

Illustration 9 : The equation of a circle whose centre is (3, –1) and which cuts off a chord of length 6 on the line 2x– 5y+ 18 = 0

(A) $(x-3)^2 + (y+1)^2 = 38$ (B) $(x+3)^2 + (y-1)^2 = 38$

(C) $(x-3)^2 + (y+1)^2 = \sqrt{38}$ (D) none of these

Solution : Let AB(= 6) be the chord intercepted by the line 2x – 5y + 18 = 0 from the circle and let CD be the perpendicular drawn from centre (3, –1) to the chord AB.

i.e., AD = 3, CD = $\frac{2.3 - 5(-1) + 18}{\sqrt{2^2 + 5^2}} = \sqrt{29}$

Therefore, $CA^2 = 3^2 + (\sqrt{29})^2 = 38$

Hence required equation is $(x-3)^2 + (y+1)^2 = 38$ **Ans. (A)**

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Illustration 10 : The area of the triangle formed by line joining the origin to the points of intersection(s) of the line $x\sqrt{5}+2y=3\sqrt{5}$ and circle $x^2+y^2=10$ is

(A) 3 (B) 4 (C) 5 (D) 6

Solution : Length of perpendicular from origin to the line $x\sqrt{5}+2y=3\sqrt{5}$ is

$$OL=\frac{3\sqrt{5}}{\sqrt{(\sqrt{5})^2+2^2}}=\frac{3\sqrt{5}}{\sqrt{9}}=\sqrt{5}$$

Radius of the given circle = $\sqrt{10}$ = OQ = OP

$$PQ = 2QL = 2\sqrt{OQ^2-OL^2}=2\sqrt{10-5}=2\sqrt{5}$$

Thus area of $\Delta OPQ=\frac{1}{2}\times PQ\times OL=\frac{1}{2}\times 2\sqrt{5}\times\sqrt{5}=5$ **Ans. (C)**

(b) Equation of the tangent (T = 0) :

(i) Tangent at the point (x_1,y_1) on the circle $x^2+y^2=a^2$ is $\mathbf{xx_1+yy_1=a^2}$.

(ii) (1) The tangent at the point (acos t, asin t) on the circle $x^2+y^2=a^2$ is **xcos t + ysin t = a**

(2) The point of intersection of the tangents at the points P(α) and Q(β)

is $\left(\dfrac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\dfrac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$.

(iii) The equation of tangent at the point (x_1,y_1) on the circle $x^2+y^2+2gx+2fy+c=0$ is $\mathbf{xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0}$

(iv) If line $y=mx+c$ is a straight line touching the circle $x^2+y^2=a^2$, then $c=\pm a\sqrt{1+m^2}$ and contact **points are** $\left(\mp\dfrac{am}{\sqrt{1+m^2}},\pm\dfrac{a}{\sqrt{1+m^2}}\right)$ **or** $\left(\mp\dfrac{a^2m}{c},\pm\dfrac{a^2}{c}\right)$ and equation of tangent is $\mathbf{y=mx\pm a\sqrt{1+m^2}}$

(v) The equation of tangent with slope m of the circle $(x-h)^2+(y-k)^2=a^2$ is

$\mathbf{(y-k)=m(x-h)\pm a\sqrt{1+m^2}}$

Note : To get the equation of tangent at the point $(x_1\ y_1)$ on any second degree curve we replace xx_1 in place of x^2, yy_1 in place of y^2, $\frac{x+x_1}{2}$ in place of x, $\frac{y+y_1}{2}$ in place of y, $\frac{xy_1+yx_1}{2}$ in place of xy and c in place of c.

(c) Length of tangent ($\sqrt{S_1}$) :

The length of tangent drawn from point (x_1,y_1) out side the circle $S\equiv x^2+y^2+2gx+2fy+c=0$ is,

$$PT=\sqrt{S_1}=\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$

Note : When we use this formula the coefficient of x^2 and y^2 must be 1.

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(d) Equation of Pair of tangents ($SS_1 = T^2$) :

Let the equation of circle $S \equiv x^2 + y^2 = a^2$ and $P(x_1,y_1)$ is any point outside the circle. From the point we can draw two real and distinct tangent PQ & PR and combine equation of pair of tangents is -

$$(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2 \quad \text{or}$$

$$SS_1 = T^2$$

Illustration 11 : Let A be the centre of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ and B(1, 7) and D(4, –2) are points on the circle then, if tangents be drawn at B and D, which meet at C, then area of quadrilateral ABCD is -

(A) 150 (B) 75 (C) 75/2 (D) none of these

Solution :

Here centre A(1, 2) and Tangent at (1, 7) is

$x.1 + y.7 - 1(x + 1) - 2(y + 7) - 20 = 0$ or $y = 7$ ..... (i)

Tangent at D(4, –2) is $3x - 4y - 20 = 0$ ..... (ii)

Solving (i) and (ii), C is (16, 7)

Area ABCD = AB × BC = 5 × 15 = 75 units. **Ans. (B)**

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Do yourself - 3 :

(i) Find the equation of tangent to the circle $x^2 + y^2 - 2ax = 0$ at the point $(a(1 + \cos\alpha), a\sin\alpha)$.

(ii) Find the equations of tangents to the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ which are parallel to the line $4x - 3y + 6 = 0$

(iii) Find the equation of the tangents to the circle $x^2 + y^2 = 4$ which are perpendicular to the line $12x - 5y + 9 = 0$. Also find the points of contact.

(iv) Find the value of 'c' if the line $y = c$ is a tangent to the circle $x^2 + y^2 - 2x + 2y - 2 = 0$ at the point (1, 1)

6. NORMAL OF CIRCLE :

Normal at a point is the straight line which is perpendicular to the tangent at the point of contact.

Note : Normal at point of the circle passes through the centre of the circle.

(a) Equation of normal at point (x_1,y_1) of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$y - y_1 = \left(\frac{y_1 + f}{x_1 + g}\right)(x - x_1)$$

(b) The equation of normal on any point (x_1,y_1) of circle $x^2 + y^2 = a^2$ is $\frac{y}{x} = \frac{y_1}{x_1}$.

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(c) If $x^2 + y^2 = a^2$ is the equation of the circle then at any point 't' of this circle (a cos t, a sint), the equation of normal is **xsint – ycost = 0.**

Illustration 12 : Find the equation of the normal to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at the point (5, 6).

Solution : Since normal to the circle always passes through the centre so equation of the normal will be the line passing through (5, 6) & $\left(\frac{5}{2}, -1\right)$

i.e. $y + 1 = \frac{7}{5/2}\left(x - \frac{5}{2}\right) \Rightarrow 5y + 5 = 14x - 35$

$\Rightarrow$ $14x - 5y - 40 = 0$ **Ans.**

Illustration 13 : If the straight line ax + by = 2; a, b $\neq$ 0 touches the circle $x^2 + y^2 - 2x = 3$ and is normal to the circle $x^2 + y^2 - 4y = 6$, then the values of a and b are respectively

(A) 1, –1 (B) 1, 2 (C) $-\frac{4}{3}$, 1 (D) 2, 1

Solution : Given $x^2 + y^2 - 2x = 3$

$\therefore$ centre is (1, 0) and radius is 2

Given $x^2 + y^2 - 4y = 6$

$\therefore$ centre is (0, 2) and radius is $\sqrt{10}$. Since line ax + by = 2 touches the first circle

$\therefore$ $\frac{|a(1) + b(0) - 2|}{\sqrt{a^2 + b^2}} = 2$ or $|(a - 2)| = [2\sqrt{a^2 + b^2}]$ ..... (i)

Also the given line is normal to the second circle. Hence it will pass through the centre of the second circle.

$\therefore$ $a(0) + b(2) = 2$ or $2b = 2$ or $b = 1$

Putting this value in equation (i) we get $|a - 2| = 2\sqrt{a^2 + 1^2}$ or $(a - 2)^2 = 4(a^2 + 1)$

or $a^2 + 4 - 4a = 4a^2 + 4$ or $3a^2 + 4a = 0$ or $a(3a + 4) = 0$ or $a = 0, -\frac{4}{3}$ $(a \neq 0)$

$\therefore$ values of a and b are $\left(-\frac{4}{3}, 1\right)$. **Ans. (C)**

Illustration 14 : Find the equation of a circle having the lines $x^2 + 2xy + 3x + 6y = 0$ as its normal and having size just sufficient to contain the circle $x(x - 4) + y(y - 3) = 0$.

Solution : Pair of normals are $(x + 2y)(x + 3) = 0$

$\therefore$ Normals are $x + 2y = 0$, $x + 3 = 0$.

Point of intersection of normals is the centre of required circle i.e. $C_1(-3, 3/2)$ and centre of given circle is $C_2(2, 3/2)$ and radius $r_2 = \sqrt{4 + \frac{9}{4}} = \frac{5}{2}$

Let r_1 be the radius of required circle

$\Rightarrow$ $r_1 = C_1C_2 + r_2 = \sqrt{(-3-2)^2 + \left(\frac{3}{2} - \frac{3}{2}\right)^2} + \frac{5}{2} = \frac{15}{2}$

Hence equation of required circle is $x^2 + y^2 + 6x - 3y - 45 = 0$

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Do yourself - 4 :

(i) Find the equation of the normal to the circle $x^2 + y^2 = 2x$, which is parallel to the line $x + 2y = 3$.

7. CHORD OF CONTACT (T = 0) :

A line joining the two points of contacts of two tangents drawn from a point out side the circle, is called chord of contact of that point.

If two tangents PT_1 & PT_2 are drawn from the point $P(x_1, y_1)$ to the circle

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact T_1T_2 is :

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ (i.e. T = 0 same as equation of tangent).

Remember :

(a) Length of chord of contact $T_1T_2 = \dfrac{2LR}{\sqrt{R^2 + L^2}}$.

(b) Area of the triangle formed by the pair of the tangents & its chord of contact $= \dfrac{RL^3}{R^2 + L^2}$,

where R is the radius of the circle & L is the length of the tangent from (x_1, y_1) on $S = 0$.

(c) Angle between the pair of tangents from $P(x_1, y_1) = \tan^{-1}\left(\dfrac{2RL}{L^2 - R^2}\right)$

(d) Equation of the circle circumscribing the triangle PT_1T_2 or quadrilateral CT_1PT_2 is :

$(x - x_1)(x + g) + (y - y_1)(y + f) = 0$.

(e) The joint equation of a pair of tangents drawn from the point $A(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is : $SS_1 = T^2$.

Where $S \equiv x^2 + y^2 + 2gx + 2fy + c$; $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$.

Illustration 15 : The chord of contact of tangents drawn from a point on the circle $x^2 + y^2 = a^2$ to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$. Show that a, b, c are in GP.

Solution : Let $P(a\cos\theta, a\sin\theta)$ be a point on the circle $x^2 + y^2 = a^2$.

Then equation of chord of contact of tangents drawn from $P(a\cos\theta, a\sin\theta)$ to the circle $x^2 + y^2 = b^2$ is $ax\cos\theta + ay\sin\theta = b^2$ ..... (i)

This touches the circle $x^2 + y^2 = c^2$ ..... (ii)

$\therefore$ Length of perpendicular from (0, 0) to (i) = radius of (ii)

$$\therefore \quad \frac{|0 + 0 - b^2|}{\sqrt{(a^2\cos^2\theta + a^2\sin^2\theta)}} = c$$

or $b^2 = ac \Rightarrow$ a, b, c are in GP.

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Do yourself - 5 :

(i) Find the equation of the chord of contact of the point (1, 2) with respect to the circle $x^2 + y^2 + 2x + 3y + 1 = 0$

(ii) Tangents are drawn from the point P(4, 6) to the circle $x^2 + y^2 = 25$. Find the area of the triangle formed by them and their chord of contact.

8. EQUATION OF THE CHORD WITH A GIVEN MIDDLE POINT ($T = S_1$) :

The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point M (x_1, y_1) is $\mathbf{y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1)}$. This on simplification can be put in the form

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is designated by $\mathbf{T = S_1}$.

Note that : The shortest chord of a circle passing through a point 'M' inside the circle, is one chord whose middle point is M.

Illustration 16 : Find the locus of middle points of chords of the circle $x^2 + y^2 = a^2$, which subtend right angle at the point (c, 0).

Solution : Let N(h, k) be the middle point of any chord AB, which subtend a right angle at P(c, 0).

Since $\angle APB = 90°$

$\therefore$ NA = NB = NP

(since distance of the vertices from middle point of the hypotenuse are equal)

or $(NA)^2 = (NB)^2 = (h - c)^2 + (k - 0)^2$ ..... (i)

But also $\angle BNO = 90°$

$\therefore$ $(OB)^2 = (ON)^2 + (NB)^2$

$\Rightarrow$ $-(NB)^2 = (ON)^2 - (OB)^2$ $\Rightarrow$ $-[(h - c)^2 + (k - 0)^2] = (h^2 + k^2) - a^2$

or $2(h^2 + k^2) - 2ch + c^2 - a^2 = 0$

$\therefore$ Locus of N(h, k) is $2(x^2 + y^2) - 2cx + c^2 - a^2 = 0$ **Ans.**

Illustration 17 : Let a circle be given by $2x(x - a) + y(2y - b) = 0$ $(a \neq 0, b \neq 0)$

Find the condition on a and b if two chords, each bisected by the x-axis, can be drawn to the circle from (a, b/2).

Solution : The given circle is $2x(x - a) + y(2y - b) = 0$

or $x^2 + y^2 - ax - by/2 = 0$

Let AB be the chord which is bisected by x-axis at a point M. Let its co-ordinates be M(h, 0).

and $S \equiv x^2 + y^2 - ax - by/2 = 0$

$\therefore$ Equation of chord AB is $T = S_1$

$$hx + 0 - \frac{a}{2}(x + h) - \frac{b}{4}(y + 0) = h^2 + 0 - ah - 0$$

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Since its passes through (a, b/2) we have $ah-\frac{a}{2}(a+h)-\frac{b^2}{8}=h^2-ah \Rightarrow h^2-\frac{3ah}{2}+\frac{a^2}{2}+\frac{b^2}{8}=0$

Now there are two chords bisected by the x-axis, so there must be two distinct real roots of h.

$\therefore \quad B^2-4AC>0$

$\Rightarrow \quad \left(\frac{-3a}{2}\right)^2-4.1.\left(\frac{a^2}{2}+\frac{b^2}{8}\right)>0 \quad \Rightarrow \quad a^2>2b^2.$ **Ans.**

Do yourself - 6 :

(i) Find the equation of the chord of $x^2+y^2-6x+10-a=0$ which is bisected at (–2, 4).

(ii) Find the locus of mid point of chord of $x^2+y^2+2gx+2fy+c=0$ that pass through the origin.

9. DIRECTOR CIRCLE :

The locus of point of intersection of two perpendicular tangents to a circle is called director circle. Let P(h,k) is the point of intersection of two tangents drawn on the circle $x^2+y^2=a^2$. Then the equation of the pair of tangents is $SS_1=T^2$

i.e. $(x^2+y^2-a^2)(h^2+k^2-a^2)=(hx+ky-a^2)^2$

As lines are perpendicular to each other then, coefficient of x^2 + coefficient of $y^2=0$

$\Rightarrow \quad [(h^2+k^2-a^2)-h^2]+[(h^2+k^2-a^2)-k^2]=0$

$\Rightarrow \quad \mathbf{h^2+k^2=2a^2}$

$\therefore$ locus of (h,k) is $x^2+y^2=2a^2$ which is the equation of the director circle.

$\therefore$ director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the circle.

Note : The director circle of $x^2+y^2+2gx+2fy+c=0$ is $x^2+y^2+2gx+2fy+2c-g^2-f^2=0$

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Illustration 18 : Let P be any moving point on the circle $x^2+y^2-2x=1$, from this point chord of contact is drawn w.r.t. the circle $x^2+y^2-2x=0$. Find the locus of the circumcentre of the triangle CAB, C being centre of the circle and A, B are the points of contact.

Solution : The two circles are

$(x-1)^2+y^2=1$ ..... (i)

$(x-1)^2+y^2=2$ ..... (ii)

So the second circle is the director circle of the first. So $\angle APB=\pi/2$

Also $\angle ACB=\pi/2$

Now circumcentre of the right angled triangle CAB would lie on the mid point of AB

So let the point be M ≡ (h, k)

Now, $CM = CB\sin 45° = \frac{1}{\sqrt{2}}$

So, $(h-1)^2+k^2=\left(\frac{1}{\sqrt{2}}\right)^2$

So, locus of M is $(x-1)^2+y^2=\frac{1}{2}$.

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Do yourself - 7 :

(i) Find the equation of the director circle of the circle $(x-h)^2+(y-k)^2=a^2$.

(ii) If the angle between the tangents drawn to $x^2+y^2+4x+8y+c=0$ from (0, 0) is $\frac{\pi}{2}$, then find value of 'c'

(iii) If two tangents are drawn from a point on the circle $x^2+y^2=50$ to the circle $x^2+y^2=25$, then find the angle between the tangents.

10. POLE AND POLAR :

Let any straight line through the given point $A(x_1,y_1)$ intersect the given circle S =0 in two points P and Q and if the tangent of the circle at P and Q meet at the point R then locus of point R is called polar of the point A and point A is called the pole, with respect to the given circle.

(a) The equation of the polar of point (x_1,y_1) w.r.t. circle $x^2+y^2=a^2$ (T = 0).

Let PQR is a chord which passes through the point $P(x_1,y_1)$ which intersects the circle at points Q and R and the tangents are drawn at points Q and R meet at point S(h,k) then equation of QR the chord of contact is $x_1h+y_1k=a^2$ $\therefore$ locus of point S(h,k) is $\mathbf{xx_1+yy_1=a^2}$ which is the equation of the polar.

Note :

(i) The equation of the polar is the T=0, so the polar of point (x_1,y_1) w.r.t circle $x^2+y^2+2gx+2fy+c=0$ is $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

(ii) If point is outside the circle then equation of polar and chord of contact is same. So the chord of contact is polar.

(iii) If point is inside the circle then chord of contact does not exist but polar exists.

(iv) If point lies on the circle then polar , chord of contact and tangent on that point are same.

(v) If the polar of P w.r.t. a circle passes through the point Q, then the polar of point Q will pass through P and hence P & Q are conjugate points of each other w.r.t. the given circle.

(vi) If pole of a line w.r.t. a circle lies on second line. Then pole of second line lies on first line and hence both lines are conjugate lines of each other w.r.t. the given circle.

(vii) If O be the centre of a circle and P be any point, then OP is perpendicular to the polar of P.

(viii) If O be the centre of a circle and P any point, then if OP (produce, if necessary) meet the polar of P in Q, then OP. OQ = $(\text{radius})^2$

(b) Pole of a given line with respect to a circle

To find the pole of a line we assume the coordinates of the pole then from these coordinates we find the polar. This polar and given line represent the same line. Then by comparing the coefficients of similar terms we can get the coordinates of the pole. The pole of $\ell x+my+n=0$

w.r.t. circle $x^2+y^2=a^2$ will be $\left(\frac{-\ell a^2}{n},\frac{-ma^2}{n}\right)$

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11. FAMILY OF CIRCLES :

(a) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ & $S_2 = 0$ is : $S_1 + K\,S_2 = 0$ $(K \neq -1)$.

(b) The equation of the family of circles passing through the point of intersection of a circle $S = 0$ & a line $L = 0$ is given by $S + KL = 0$.

(c) The equation of a family of circles passing through two given points (x_1, y_1) & (x_2, y_2) can be written in the form :

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+K\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$ where K is a parameter.

(d) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + K[y - y_1 - m(x - x_1)] = 0$, where K is a parameter.

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(e) Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ & $L_3 = 0$ is given by ; $L_1L_2 + \lambda\, L_2L_3 + \mu\, L_3L_1 = 0$ provided coefficient of $xy = 0$ & coefficient of x^2 = coefficient of y^2.

(f) Equation of circle circumscribing a quadrilateral whose sides in order are represented by the lines $L_1 = 0, L_2 = 0, L_3 = 0$ & $L_4 = 0$ is $L_1L_3 + \lambda L_2L_4 = 0$ provided coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.

Illustration 19 : The equation of the circle through the points of intersection of $x^2 + y^2 - 1 = 0$, $x^2 + y^2 - 2x - 4y + 1 = 0$ and touching the line $x + 2y = 0$, is -

(A) $x^2 + y^2 + x + 2y = 0$ (B) $x^2 + y^2 - x + 20 = 0$

(C) $x^2 + y^2 - x - 2y = 0$ (D) $2(x^2 + y^2) - x - 2y = 0$

Solution : Family of circles is $x^2 + y^2 - 2x - 4y + 1 + \lambda(x^2 + y^2 - 1) = 0$

$(1 + \lambda)\,x^2 + (1 + \lambda)\,y^2 - 2x - 4y + (1 - \lambda) = 0$

$$x^2 + y^2 - \frac{2}{1+\lambda}x - \frac{4}{1+\lambda}y + \frac{1-\lambda}{1+\lambda} = 0$$

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Centre is $\left(\frac{1}{1+\lambda}, \frac{2}{1+\lambda}\right)$ and radius $= \sqrt{\left(\frac{1}{1+\lambda}\right)^2 + \left(\frac{2}{1+\lambda}\right)^2 - \frac{1-\lambda}{1+\lambda}} = \frac{\sqrt{4+\lambda^2}}{|1+\lambda|}$.

Since it touches the line $x + 2y = 0$, hence

Radius = Perpendicular distance from centre to the line.

i.e., $\left|\frac{\frac{1}{1+\lambda} + 2\frac{2}{1+\lambda}}{\sqrt{1^2+2^2}}\right| = \frac{\sqrt{4+\lambda^2}}{|1+\lambda|} \Rightarrow \sqrt{5} = \sqrt{4+\lambda^2} \Rightarrow \lambda = \pm 1$

$\lambda = -1$ cannot be possible in case of circle. So $\lambda = 1$.

Thus, we get the equation of circle. **Ans. (C)**

Do yourself - 8 :

(i) Prove that the polar of a given point with respect to any one of circles $x^2 + y^2 - 2kx + c^2 = 0$, where k is a variable, always passes through a fixed point, whatever be the value of k.

(ii) Find the equation of the circle passing through the points of intersection of the circle $x^2 + y^2 - 6x + 2y + 4 = 0$ & $x^2 + y^2 + 2x - 4y - 6 = 0$ and with its centre on the line $y = x$.

(iii) Find the equation of the circle through the points of intersection of the circles $x^2 + y^2 + 2x + 3y - 7 = 0$ and $x^2 + y^2 + 3x - 2y - 1 = 0$ and passing through the point (1, 2).

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12. DIRECT AND TRANSVERSE COMMON TANGENTS :

Let two circles having centre C_1 and C_2 and radii, r_1 and r_2 and C_1C_2 is the distance between their centres then :

(a) Both circles will touch :

(i) **Externally** if $C_1C_2 = r_1 + r_2$ i.e. the distance between their centres is equal to sum of their radii and point P & T divides C_1C_2 in the ratio $r_1 : r_2$ (internally & externally respectively). In this case there are **three common tangents.**

(ii) **Internally** if $C_1C_2 = |r_1 - r_2|$ i.e. the distance between their centres is equal to difference between their radii and point P divides C_1C_2 in the ratio $r_1 : r_2$ **externally** and in this case there will be only **one common tangent.**

(b) The circles will intersect :

when $|r_1 - r_2| < C_1C_2 < r_1 + r_2$ in this case there are **two common tangents.**

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(c) The circles will not intersect :

(i) One circle will lie inside the other circle if $C_1C_2 < |r_1 - r_2|$ In this case there will be no common tangent.

(ii) When circle are apart from each other then $C_1C_2 > r_1 + r_2$ and in this case there

will be **four common tangents.** Lines PQ and RS are called **transverse** or **indirect** or **internal** common tangents and these lines meet line C_1C_2 on T_1 and T_1 divides the line C_1C_2 in the ratio $r_1 : r_2$ internally and lines AB & CD are called **direct** or **external** common tangents. These lines meet C_1C_2 produced on T_2. Thus T_2 divides C_1C_2 externally in the ratio $r_1 : r_2$.

Note : Length of direct common tangent $= \sqrt{(C_1C_2)^2 - (r_1 - r_2)^2}$

Length of transverse common tangent $= \sqrt{(C_1C_2)^2 - (r_1 + r_2)^2}$

Illustration 20 : Prove that the circles $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch each other, if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.

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Solution : Given circles are $x^2 + y^2 + 2ax + c^2 = 0$ ..... (i)

and $x^2 + y^2 + 2by + c^2 = 0$ ..... (ii)

Let C_1 and C_2 be the centres of circles (i) and (ii), respectively and r_1 and r_2 be their radii, then

$C_1 = (-a, 0)$, $C_2 = (0, -b)$, $r_1 = \sqrt{a^2 - c^2}$, $r_2 = \sqrt{b^2 - c^2}$

Here we find the two circles touch each other internally or externally.

For touch, $|C_1C_2| = |r_1 \pm r_2|$

or $\sqrt{(a^2 + b^2)} = \left|\sqrt{(a^2 - c^2)} \pm \sqrt{(b^2 - c^2)}\right|$

On squaring $a^2 + b^2 = a^2 - c^2 + b^2 - c^2 \pm 2\sqrt{(a^2 - c^2)}\sqrt{(b^2 - c^2)}$

or $c^2 = \pm\sqrt{a^2b^2 - c^2(a^2 + b^2) + c^4}$

Again squaring, $c^4 = a^2b^2 - c^2(a^2 + b^2) + c^4$

or $c^2(a^2 + b^2) = a^2b^2$

or $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$

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Do yourself - 9 :

(i) Two circles with radius 5 touches at the point (1, 2). If the equation of common tangent is $4x + 3y = 10$ and one of the circle is $x^2 + y^2 + 6x + 2y - 15 = 0$. Find the equation of other circle.

(ii) Find the number of common tangents to the circles $x^2 + y^2 = 1$ and $x^2 + y^2 - 2x - 6y + 6 = 0$.

13. THE ANGLE OF INTERSECTION OF TWO CIRCLES :

Definition : The angle between the tangents of two circles at the point of intersection of the two circles is called angle of intersection of two circles. If two circles are $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ and θ is the acute angle between them

$$\text{then } \cos\theta = \left|\left(\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)\right| = \left|\frac{2g_1g_2 + 2f_1f_2 - c_1 - c_2}{2\sqrt{g_1^2 + f_1^2 - c_1}\ \sqrt{g_2^2 + f_2^2 - c_2}}\right|$$

Here r_1 and r_2 are the radii of the circles and d is the distance between their centres.

If the angle of intersection of the two circles is a right angle then such circles are called **"Orthogonal circles"** and conditions for the circles to be orthogonal is -

$$\mathbf{2g_1g_2 + 2f_1f_2 = c_1 + c_2}$$

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14. RADICAL AXIS OF THE TWO CIRCLES ($S_1 - S_2 = 0$) :

(a) **Definition :** The locus of a point, which moves in such a way that the length of tangents drawn from it to the circles are equal and is called the radical axis. If two circles are -

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

$$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

Let P(h,k) is a point and PA,PB are length of two tangents on the circles from point P, Then from definition -

$$\sqrt{h^2 + k^2 + 2g_1h + 2f_1k + c_1} = \sqrt{h^2 + k^2 + 2g_2h + 2f_2k + c_2} \text{ or } 2(g_1 - g_2)h + 2(f_1 - f_2)k + c_1 - c_2 = 0$$

$\therefore$ locus of (h,k)

$$2x(g_1 - g_2) + 2y(f_1 - f_2)k + c_1 - c_2 = 0$$

$$\mathbf{S_1 - S_2 = 0}$$

which is the equation of radical axis.

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Note :

(i) To get the equation of the radical axis first of all make the coefficient of x^2 and $y^2=1$

(ii) If circles touch each other then radical axis is the common tangent to both the circles.

(iii) When the two circles intersect on real points then common chord is the radical axis of the two circles.

(iv) The radical axis of the two circles is perpendicular to the line joining the centre of two circles but not always pass through mid point of it.

(v) Radical axis (if exist) bisects common tangent to two circles.

(vi) The radical axes of three circles (taking two at a time) meet at a point.

(vii) If circles are concentric then the radical axis does not always exist but if circles are not concentric then radical axis always exists.

(viii) If two circles are orthogonal to the third circle then radical axis of both circle passes through the centre of the third circle.

(ix) A system of circle, every pair of which have the same radical axis, is called a **coaxial system of circles.**

(b) Radical centre :

The radical centre of three circles is the point from which length of tangents on three circles are equal i.e. the point of intersection of radical axis of the circles is the radical centre of the circles.

To get the radical axis of three circles $S_1=0$, $S_2=0$, $S_3=0$ we have to solve any two

$S_1-S_2=0$, $S_2-S_3=0$, $S_3-S_1=0$

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Note :

(i) The circle with centre as radical centre and radius equal to the length of tangent from radical centre to any of the circle, will cut the three circles orthogonally.

(ii) If three circles are drawn on three sides of a triangle taking them as diameter then its orthocenter will be its radical centre.

(iii) Locus of the centre of a variable circle orthogonal to two fixed circles is the radical axis between the two fixed circles.

(iv) If two circles are orthogonal, then the polar of a point 'P' on first circle w.r.t. the second circle passes through the point Q which is the other end of the diameter through P. Hence locus of a point which moves such that its polars w.r.t. the circles $S_1 = 0$, $S_2 = 0$ & $S_3 = 0$ are concurrent is a circle which is orthogonal to all the three circles.

Illustration 21 : A and B are two fixed points and P moves such that PA = nPB where $n \neq 1$. Show that locus of P is a circle and for different values of n all the circles have a common radical axis.

Solution : Let $A \equiv (a, 0)$, $B \equiv (-a, 0)$ and $P(h, k)$

so PA = nPB

$\Rightarrow \quad (h-a)^2 + k^2 = n^2[(h+a)^2 + k^2]$

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$\Rightarrow \quad (1 - n^2)h^2 + (1 - n^2)k^2 - 2ah(1 + n^2) + (1 - n^2)a^2 = 0$

$\Rightarrow \quad h^2 + k^2 - 2ah\left(\frac{1+n^2}{1-n^2}\right) + a^2 = 0$

Hence locus of P is

$x^2 + y^2 - 2ax\left(\frac{1+n^2}{1-n^2}\right) + a^2 = 0$, which is a circle of different values of n.

Let n_1 and n_2 are two different values of n so their radical axis is x = 0 i.e. y-axis. Hence for different values of n the circles have a common radical axis.

Illustration 22 : Find the equation of the circle through the points of intersection of the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 4y - 12 = 0$ and cutting the circle $x^2 + y^2 - 2x - 4 = 0$ orthogonally.

Solution : The equation of the circle through the intersection of the given circles is

$x^2 + y^2 - 4x - 6y - 12 + \lambda(-10x - 10y) = 0$ ..... (i)

where $(-10x - 10y = 0)$ is the equation of radical axis for the circle

$x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 4y - 12 = 0$.

Equation (i) can be re-arranged as

$x^2 + y^2 - x(10\lambda + 4) - y(10\lambda + 6) - 12 = 0$

It cuts the circle $x^2 + y^2 - 2x - 4 = 0$ orthogonally.

Hence $2gg_1 + 2ff_1 = c + c_1$

$\Rightarrow \quad 2(5\lambda + 2)(1) + 2(5\lambda + 3)(0) = -12 - 4 \Rightarrow \lambda = -2$

Hence the required circle is

$x^2 + y^2 - 4x - 6y - 12 - 2(-10x - 10y) = 0$

i.e., $x^2 + y^2 + 16x + 14y - 12 = 0$

Illustration 23 : Find the radical centre of circles $x^2 + y^2 + 3x + 2y + 1 = 0$, $x^2 + y^2 - x + 6y + 5 = 0$ and $x^2 + y^2 + 5x - 8y + 15 = 0$. Also find the equation of the circle cutting them orthogonally.

Solution : Given circles are $S_1 \equiv x^2 + y^2 + 3x + 2y + 1 = 0$

$S_2 \equiv x^2 + y^2 - x + 6y + 5 = 0$

$S_3 \equiv x^2 + y^2 + 5x - 8y + 15 = 0$

Equations of two radical axes are $S_1 - S_2 \equiv 4x - 4y - 4 = 0$ or $x - y - 1 = 0$

and $S_2 - S_3 \equiv -6x + 14y - 10 = 0$ or $3x - 7y + 5 = 0$

Solving them the radical centre is (3, 2). Also, if r is the length of the tangent drawn from the radical centre (3, 2) to any one of the given circles, say S_1, we have

$r = \sqrt{S_1} = \sqrt{3^2 + 2^2 + 3.3 + 2.2 + 1} = \sqrt{27}$

Hence (3, 2) is the centre and $\sqrt{27}$ is the radius of the circle intersecting them orthogonally.

$\therefore$ Its equation is $(x - 3)^2 + (y - 2)^2 = r^2 = 27 \Rightarrow x^2 + y^2 - 6x - 4y - 14 = 0$

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Alternative Method :

Let $x^2 + y^2 + 2gx + 2fy + c = 0$ be the equation of the circle cutting the given circles orthogonally.

$\therefore$ $2g\left(\frac{3}{2}\right)+2f(1) = c + 1$ or $3g + 2f = c + 1$ ..... (i)

$2g\left(-\frac{1}{2}\right)+2f(3) = c + 5$ or $-g + 6f = c + 5$ ..... (ii)

and $2g\left(\frac{5}{2}\right)+2f(-4) = c + 15$ or $5g - 8f = c + 15$ ..... (iii)

Solving (i), (ii) and (iii) we get $g = -3$, $f = -2$ and $c = -14$

$\therefore$ equation of required circle is $x^2 + y^2 - 6x - 4y - 14 = 0$ **Ans.**

Do yourself - 10 :

(i) Find the angle of intersection of two circles
$S : x^2 + y^2 - 4x + 6y + 11 = 0$ & $S' : x^2 + y^2 - 2x + 8y + 13 = 0$

(ii) Find the equation of the radical axis of the circle $x^2 + y^2 - 3x - 4y + 5 = 0$ and $3x^2 + 3y^2 - 7x - 8y + 11 = 0$

(iii) Find the radical centre of three circles described on the three sides $4x - 7y + 10 = 0$, $x + y - 5 = 0$ and $7x + 4y - 15 = 0$ of a triangle as diameters.

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15. SOME IMPORTANT RESULTS TO REMEMBER :

(a) If the circle $S_1 = 0$, bisects the circumference of the circle $S_2 = 0$, then their common chord will be the diameter of the circle $S_2 = 0$.

(b) The radius of the director circle of a given circle is $\sqrt{2}$ times the radius of the given circle.

(c) The locus of the middle point of a chord of a circle subtend a right angle at a given point will be a circle.

(d) The length of side of an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$ is $\sqrt{3}\,a$

(e) If the lengths of tangents from the points A and B to a circle are ℓ_1 and ℓ_2 respectively, then if the points A and B are conjugate to each other, then $(AB)^2 = \ell_1^2 + \ell_2^2$.

(f) Length of transverse common tangent is less than the length of direct common tangent.

Do yourself - 11 :

(i) When the circles $x^2 + y^2 + 4x + 6y + 3 = 0$ and $2(x^2 + y^2) + 6x + 4y + c = 0$ intersect orthogonally, then find the value of c is

(ii) Write the condition so that circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ touch externally.

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Miscellaneous Illustrations :

Illustration 24 : Find the equation of a circle which passes through the point (2, 0) and whose centre is the limit of the point of intersection of the lines $3x + 5y = 1$ and $(2 + c)x + 5c^2y = 1$ as $c \to 1$.

Solution : Solving the equations $(2 + c)x + 5c^2y = 1$ and $3x + 5y = 1$

then $(2 + c)x + 5c^2\left(\frac{1-3x}{5}\right) = 1$ or $(2 + c)x + c^2(1 - 3x) = 1$

$\therefore \quad x = \frac{1-c^2}{2+c-3c^2}$ or $x = \frac{(1+c)(1-c)}{(3c+2)(1-c)} = \frac{1+c}{3c+2}$

$\therefore \quad x = \lim_{c\to 1}\frac{1+c}{3c+2}$ or $x = \frac{2}{5}$

$\therefore \quad y = \frac{1-3x}{5} = \frac{1-\frac{6}{5}}{5} = -\frac{1}{25}$

Therefore the centre of the required circle is $\left(\frac{2}{5}, \frac{-1}{25}\right)$ but circle passes through (2, 0)

$\therefore$ Radius of the required circle $= \sqrt{\left(\frac{2}{5}-2\right)^2 + \left(-\frac{1}{25}-0\right)^2} = \sqrt{\frac{64}{25}+\frac{1}{625}} = \sqrt{\frac{1601}{625}}$

Hence the required equation of the circle is $\left(x-\frac{2}{5}\right)^2 + \left(y+\frac{1}{25}\right)^2 = \frac{1601}{625}$

or $25x^2 + 25y^2 - 20x + 2y - 60 = 0$ **Ans.**

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Illustration 25 : Two straight lines rotate about two fixed points. If they start from their position of coincidence such that one rotates at the rate double that of the other. Prove that the locus of their point of intersection is a circle.

Solution : Let $A \equiv (-a, 0)$ and $B \equiv (a, 0)$ be two fixed points.

Let one line which rotates about B an angle θ with the x-axis at any time t and at that time the second line which rotates about A make an angle 2θ with x-axis.

Now equation of line through B and A are respectively

$y - 0 = \tan\theta(x - a)$ ..... (i)

and $y - 0 = \tan 2\theta(x + a)$ ..... (ii)

From (ii), $y = \frac{2\tan\theta}{1-\tan^2\theta}(x+a)$

$$= \left\{\frac{\frac{2y}{(x-a)}}{1-\frac{y^2}{(x-a)^2}}\right\}(x+a) \quad \text{(from (i))}$$

$\Rightarrow \quad y = \frac{2y(x-a)(x+a)}{(x-a)^2 - y^2} \quad \Rightarrow \quad (x-a)^2 - y^2 = 2(x^2 - a^2)$

or $x^2 + y^2 + 2ax - 3a^2 = 0$ which is the required locus.

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Illustration 26 : If the circle $x^2 + y^2 + 6x - 2y + k = 0$ bisects the circumference of the circle $x^2 + y^2 + 2x - 6y - 15 = 0$, then k =

(A) 21 (B) –21 (C) 23 (D) –23

Solution : $2g_2(g_1 - g_2) + 2f_2(f_1 - f_2) = c_1 - c_2$

$2(1)(3 - 1) + 2(-3)(-1 + 3) = k + 15$

$4 - 12 = k + 15$ or $-8 = k + 15 \Rightarrow k = -23$ **Ans. (D)**

Illustration 27 : Find the equation of the circle of minimum radius which contains the three circles.

$S_1 \equiv x^2 + y^2 - 4y - 5 = 0$

$S_2 \equiv x^2 + y^2 + 12x + 4y + 31 = 0$

$S_3 \equiv x^2 + y^2 + 6x + 12y + 36 = 0$

Solution : For S_1, centre = (0, 2) and radius = 3

For S_2, centre = (–6, –2) and radius = 3

For S_3, centre = (–3, –6) and radius = 3

let P(a, b) be the centre of the circle passing through the centres of the three given circles, then

$(a - 0)^2 + (b - 2)^2 = (a + 6)^2 + (b + 2)^2$

$\Rightarrow (a + 6)^2 - a^2 = (b - 2)^2 - (b + 2)^2$

$(2a + 6)6 = 2b(-4)$

$$b = \frac{2 \times 6(a+3)}{-8} = -\frac{3}{2}(a+3)$$

again $(a - 0)^2 + (b - 2)^2 = (a + 3)^2 + (b + 6)^2$

$\Rightarrow (a + 3)^2 - a^2 = (b - 2)^2 - (b + 6)^2$

$(2a + 3)3 = (2b + 4)(-8)$

$$(2a + 3)3 = -16\left[-\frac{3}{2}(a+3) + 2\right]$$

$6a + 9 = -8(-3a - 5)$

$6a + 9 = 24a + 40$

$18a = -31$

$$a = -\frac{31}{18},\ b = -\frac{23}{12}$$

radius of the required circle $= 3 + \sqrt{\left(-\frac{31}{18}\right)^2 + \left(-\frac{23}{12} - 2\right)^2} = 3 + \frac{5}{36}\sqrt{949}$

$\therefore$ equation of the required circle is $\left(x + \frac{31}{18}\right)^2 + \left(y + \frac{23}{12}\right)^2 = \left(3 + \frac{5}{36}\sqrt{949}\right)^2$

Illustration 28 : Find the equation of the image of the circle $x^2 + y^2 + 16x - 24y + 183 = 0$ by the line mirror $4x + 7y + 13 = 0$.

Solution : Centre of given circle = (–8, 12), radius = 5

the given line is $4x + 7y + 13 = 0$

let the centre of required circle is (h, k)

since radius will not change. so radius of required circle is 5.

Now (h, k) is the reflection of centre (–8, 12) in the line $4x + 7y + 13 = 0$

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Co-ordinates of A = $\left(\frac{-8+h}{2}, \frac{12+k}{2}\right)$

$\Rightarrow \quad \frac{4(-8+h)}{2} + \frac{7(12+k)}{2} + 13 = 0$

$-32 + 4h + 84 + 7k + 26 = 0$

$4h + 7k + 78 = 0$ .....(i)

Also $\frac{k-12}{h+8} = \frac{7}{4}$

$4k - 48 = 7h + 56$

$4k = 7h + 104$ .....(ii)

solving (i) & (ii)

$h = -16, k = -2$

$\therefore$ required circle is $(x + 16)^2 + (y + 2)^2 = 5^2$

Illustration 29 : The circle $x^2 + y^2 - 6x - 10y + k = 0$ does not touch or intersect the coordinate axes and the point (1, 4) is inside the circle. Find the range of the value of k.

Solution : Since (1, 4) lies inside the circle

$\Rightarrow \quad S_1 < 0$

$\Rightarrow \quad (1)^2 + (4)^2 - 6(1) - 10(4) + k < 0$

$\Rightarrow \quad k < 29$

Also centre of given circle is (3, 5) and circle does not touch or intersect the coordinate axes

$\Rightarrow \quad r < CA \quad \& \; r < CB$

$CA = 5$

$CB = 3$

$\Rightarrow \quad r < 5 \quad \& \quad r < 3$

$\Rightarrow \quad r < 3 \quad \text{or} \quad r^2 < 9$

$r^2 = 9 + 25 - k$

$r^2 = 34 - k \quad \Rightarrow \quad 34 - k < 9$

$k > 25$

$\Rightarrow \quad k \in (25, 29)$

Illustration 30 : The circle $x^2 + y^2 - 4x - 8y + 16 = 0$ rolls up the tangent to it at $(2 + \sqrt{3}, 3)$ by 2 units, find the equation of the circle in the new position.

Solution : Given circle is $x^2 + y^2 - 4x - 8y + 16 = 0$

let $P \equiv (2 + \sqrt{3}, 3)$

Equation of tangent to the circle at $P(2 + \sqrt{3}, 3)$ will be

$(2 + \sqrt{3})x + 3y - 2(x + 2 + \sqrt{3}) - 4(y + 3) + 16 = 0$

or $\quad \sqrt{3}x - y - 2\sqrt{3} = 0$

slope = $\sqrt{3} \quad \Rightarrow \quad \tan\theta = \sqrt{3}$

$\theta = 60°$

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line AB is parallel to the tangent at P

$\Rightarrow$ coordinates of point B = $(2 + 2\cos60°, 4 + 2\sin60°)$

thus B = $(3, 4 + \sqrt{3})$

radius of circle = $\sqrt{2^2 + 4^2 - 16} = 2$

$\therefore$ equation of required circle is $(x - 3)^2 + (y - 4 - \sqrt{3})^2 = 2^2$

Illustration 31 : A fixed circle is cut by a family of circles all of which, pass through two given points $A(x_1, y_1)$ and $B(x_2, y_2)$. Prove that the chord of intersection of the fixed circle with any circle of the family passes through a fixed point.

Solution : Let S = 0 be the equation of fixed circle

let $S_1 = 0$ be the equation of any circle through A and B which intersect S = 0 in two points.

$L \equiv S - S_1 = 0$ is the equation of the chord of intersection of S = 0 and $S_1 = 0$

let $L_1 = 0$ be the equation of line AB

let S_2 be the equation of the circle whose diametrical ends are $A(x_1, y_1)$ & $B(x_2, y_2)$

then $S_1 \equiv S_2 - \lambda L_1 = 0$

$\Rightarrow$ $L \equiv S - (S_2 - \lambda L_1) = 0$ or $L \equiv (S - S_2) + \lambda L_1 = 0$

or $L \equiv L' + \lambda L_1 = 0$ .....(i)

(i) implies each chord of intersection passes through the fixed point, which is the point of intersection of lines $L' = 0$ & $L_1 = 0$. Hence proved.

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EXERCISE (O-1)

[SINGLE CORRECT]

1. Centres of the three circles $x^2 + y^2 - 4x - 6y - 14 = 0$, $x^2 + y^2 + 2x + 4y - 5 = 0$ and $x^2 + y^2 - 10x - 16y + 7 = 0$

(A) are the vertices of a right triangle

(B) the vertices of an isosceles triangle which is not regular

(C) vertices of a regular triangle

(D) are collinear

CR0001

2. $y - 1 = m_1(x - 3)$ and $y - 3 = m_2(x - 1)$ are two family of straight lines, at right angled to each other. The locus of their point of intersection is

(A) $x^2 + y^2 - 2x - 6y + 10 = 0$ (B) $x^2 + y^2 - 4x - 4y + 6 = 0$

(C) $x^2 + y^2 - 2x - 6y + 6 = 0$ (D) $x^2 + y^2 - 4x - 4y - 6 = 0$

CR0002

3. Suppose that the equation of the circle having (–3, 5) and (5, –1) as end points of a diameter is $(x - a)^2 + (y - b)^2 = r^2$. Then $a + b + r$, $(r > 0)$ is

(A) 8 (B) 9 (C) 10 (D) 11

CR0003

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4. The area of an equilateral triangle inscribed in the circle $x^2 + y^2 - 2x = 0$ is

(A) $\frac{3\sqrt{3}}{4}$ (B) $\frac{3\sqrt{3}}{2}$ (C) $\frac{3\sqrt{3}}{8}$ (D) none

CR0004

5. The smallest distance between the circle $(x - 5)^2 + (y + 3)^2 = 1$ and the line $5x + 12y - 4 = 0$, is

(A) 1/13 (B) 2/13 (C) 3/15 (D) 4/15

CR0005

6. The equation of the image of the circle $x^2 + y^2 + 16x - 24y + 183 = 0$ by the line mirror $4x + 7y + 13 = 0$ is

(A) $x^2 + y^2 + 32x - 4y + 235 = 0$ (B) $x^2 + y^2 + 32x + 4y - 235 = 0$

(C) $x^2 + y^2 + 32x - 4y - 235 = 0$ (D) $x^2 + y^2 + 32x + 4y + 235 = 0$

CR0006

7. The radius of the circle passing through the vertices of the triangle ABC, is

(A) $\frac{8\sqrt{15}}{5}$ (B) $\frac{3\sqrt{15}}{5}$

(C) $3\sqrt{5}$ (D) $3\sqrt{2}$

CR0007

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8. (6, 0), (0, 6) and (7, 7) are the vertices of a triangle. The circle inscribed in the triangle has the equation

(A) $x^2 + y^2 - 9x + 9y + 36 = 0$ (B) $x^2 + y^2 - 9x - 9y + 36 = 0$

(C) $x^2 + y^2 + 9x - 9y + 36 = 0$ (D) $x^2 + y^2 - 9x - 9y - 36 = 0$

CR0008

9. The line joining (5, 0) to $(10\cos\theta, 10\sin\theta)$ is divided internally in the ratio 2 : 3 at P. If θ varies then the locus of P is :

(A) a pair of straight lines (B) a circle

(C) a straight line (D) a second degree curve which is not a circle

CR0009

10. The locus of the center of the circles such that the point (2, 3) is the mid point of the chord $5x + 2y = 16$ is

(A) $2x - 5y + 11 = 0$ (B) $2x + 5y - 11 = 0$

(C) $2x + 5y + 11 = 0$ (D) none

CR0010

11. In the xy-plane, the length of the shortest path from (0, 0) to (12, 16) that does not go inside the circle $(x - 6)^2 + (y - 8)^2 = 25$ is

(A) $10\sqrt{3}$ (B) $10\sqrt{5}$ (C) $10\sqrt{3} + \frac{5\pi}{3}$ (D) $10 + 5\pi$

CR0011

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12. Tangents PA and PB are drawn to the circle $x^2 + y^2 = 4$, then the locus of the point P if the triangle PAB is equilateral, is equal to-

(A) $x^2 + y^2 = 16$ (B) $x^2 + y^2 = 8$ (C) $x^2 + y^2 = 64$ (D) $x^2 + y^2 = 32$

CR0012

13. The points (x_1, y_1), (x_2, y_2), (x_1, y_2) and (x_2, y_1) are always

(A) collinear (B) concyclic

(C) vertices of a square (D) vertices of a rhombus

CR0013

14. Locus of all point P(x, y) satisfying $x^3 + y^3 + 3xy = 1$ consists of union of

(A) a line and an isolated point (B) a line pair and an isolated point

(C) a line and a circle (D) a circle and a isolated point.

CR0014

15. In the xy plane, the segment with end points (3, 8) and (–5, 2) is the diameter of the circle. The point (k, 10) lies on the circle for

(A) no value of k (B) exactly one integral k

(C) exactly one non integral k (D) two real values of k

CR0015

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16. If a circle of constant radius 3k passes through the origin 'O' and meets co-ordinate axes at A and B then the locus of the centroid of the triangle OAB is -

(A) $x^2 + y^2 = (2k)^2$ (B) $x^2 + y^2 = (3k)^2$ (C) $x^2 + y^2 = (4k)^2$ (D) $x^2 + y^2 = (6k)^2$

CR0016

17. Consider the points P(2, 1); Q(0, 0); R(4, –3) and the circle S : $x^2 + y^2 - 5x + 2y - 5 = 0$

(A) exactly one point lies outside S (B) exactly two points lie outside S

(C) all the three points lie outside S (D) none of the point lies outside S

CR0017

18. B and C are fixed points having co-ordinates (3, 0) and (–3, 0) respectively. If the vertical angle BAC is 90°, then the locus of the centroid of the ΔABC has the equation :

(A) $x^2 + y^2 = 1$ (B) $x^2 + y^2 = 2$ (C) $9(x^2 + y^2) = 1$ (D) $9(x^2 + y^2) = 4$

CR0018

19. The angle between the two tangents from the origin to the circle $(x - 7)^2 + (y + 1)^2 = 25$ equals

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$

CR0019

20. Tangents are drawn from (4, 4) to the circle $x^2 + y^2 - 2x - 2y - 7 = 0$ to meet the circle at A and B. The length of the chord AB is

(A) $2\sqrt{3}$ (B) $3\sqrt{2}$ (C) $2\sqrt{6}$ (D) $6\sqrt{2}$

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CR0020

21. The area of the quadrilateral formed by the tangents from the point (4, 5) to the circle $x^2 + y^2 - 4x - 2y - 11 = 0$ with the pair of radii through the points of contact of the tangents is :

(A) 4 sq. units (B) 8 sq. units (C) 6 sq. units (D) none

CR0021

22. If L_1 and L_2 are the length of the tangent from (0, 5) to the circles $x^2 + y^2 + 2x - 4 = 0$ and $x^2 + y^2 - 2x - y + 1 = 0$ then

(A) $L_1 = 2L_2$ (B) $L_2 = 2L_1$ (C) $L_1 = L_2$ (D) $L_1^2 = L_2$

CR0022

23. From (3, 4) chords are drawn to the circle $x^2 + y^2 - 4x = 0$. The locus of the mid points of the chords is :

(A) $x^2 + y^2 - 5x - 4y + 6 = 0$ (B) $x^2 + y^2 + 5x - 4y + 6 = 0$

(C) $x^2 + y^2 - 5x + 4y + 6 = 0$ (D) $x^2 + y^2 - 5x - 4y - 6 = 0$

CR0023

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24. Tangents are drawn to a unit circle with centre at the origin from each point on the line $2x + y = 4$. Then the equation to the locus of the middle point of the chord of contact is -

(A) $2(x^2 + y^2) = x + y$ (B) $2(x^2 + y^2) = x + 2y$ (C) $4(x^2 + y^2) = 2x + y$ (D) none

CR0024

25. Chord AB of the circle $x^2 + y^2 = 100$ passes through the point (7, 1) and subtends an angle of 60° at the circumference of the circle. If m_1 and m_2 are the slopes of two such chords then the value of m_1m_2, is

(A) –1 (B) 1 (C) 7/12 (D) –3

CR0025

26. Combined equation to the pair of tangents drawn from the origin to the circle $x^2 + y^2 + 4x + 6y + 9 = 0$ is

(A) $3(x^2 + y^2) = (x + 2y)^2$ (B) $2(x^2 + y^2) = (3x + y)^2$

(C) $9(x^2 + y^2) = (2x + 3y)^2$ (D) $x^2 + y^2 = (2x + 3y)^2$

CR0026

27. Sum of the abscissa and ordinate of the centre of the circle touching the line $3x + y + 2 = 0$ at the point (–1,1) and passing through the point (3,5) is-

(A) 2 (B) 3 (C) 4 (D) 5

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CR0027

28. A circle of radius 5 is tangent to the line $4x - 3y = 18$ at M(3,–2) and lies above the line. The equation of the circle, is-

(A) $x^2 + y^2 - 6x + 4y - 12 = 0$ (B) $x^2 + y^2 + 2x - 2y - 3 = 0$

(C) $x^2 + y^2 + 2x - 2y - 23 = 0$ (D) $x^2 + y^2 + 6x + 4y - 12 = 0$

CR0028

29. In the figure given, two circles with centres C_1 and C_2 are 35 units apart, i.e. $C_1C_2 = 35$. The radii of the circles with centres C_1 and C_2 are 12 and 9 respectively. If P is the intersection of C_1C_2 and a common internal tangent to the circles, then $l(C_1P)$ equals-

(A) 18 (B) 20 (C) 12 (D) 15

CR0029

30. Let C_1 and C_2 are circles defined by $x^2 + y^2 - 20x + 64 = 0$ and $x^2 + y^2 + 30x + 144 = 0$. The length of the shortest line segment PQ that is tangent to C_1 at P and to C_2 at Q is -

(A) 15 (B) 18 (C) 20 (D) 24

CR0030

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EXERCISE (O-2)

[SINGLE CHOICE]

1. As shown in the figure, three circles which have the same radius r, have centres at (0,0) ; (1,1) and (2,1). If they have a common tangent line, as shown then, their radius 'r' is -

(A) $\frac{\sqrt{5}-1}{2}$ (B) $\frac{\sqrt{5}}{10}$

(C) $\frac{1}{2}$ (D) $\frac{\sqrt{3}-1}{2}$

CR0031

2. Circle K is circle of largest radius, inscribed in the first quadrant touching the circle $x^2 + y^2 = 36$ internally. The length of the radius of the circle K, is-

(A) $\frac{6-\sqrt{2}}{2}$ (B) $\frac{3\sqrt{2}}{2}$ (C) 3 (D) $6\left(\sqrt{2}-1\right)$

CR0032

3. The angle at which the circle $(x-1)^2 + y^2 = 10$ and $x^2 + (y-2)^2 = 5$ intersect is -

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

CR0033

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4. Two circles whose radii are equal to 4 and 8 intersect at right angles. The length of their common chord is-

(A) $\frac{16}{\sqrt{5}}$ (B) 8 (C) $4\sqrt{6}$ (D) $\frac{8\sqrt{5}}{5}$

CR0034

5. Two circles of radii r_1 and r_2 are both touching the coordinate axes and intersecting each other orthogonally. The value of r_1/r_2 (where $r_1 > r_2$) equals -

(A) $2+\sqrt{3}$ (B) $\sqrt{3}+1$ (C) $2-\sqrt{3}$ (D) $2+\sqrt{5}$

CR0035

6. Two congruent circles with centres at (2,3) and (5,6) which intersect at right angles has radius equal to-

(A) $2\sqrt{2}$ (B) 3 (C) 4 (D) none

CR0036

7. The equation of a circle which touches the line $x + y = 5$ at N(–2,7) and cuts the circle $x^2 + y^2 + 4x - 6y + 9 = 0$ orthogonally, is -

(A) $x^2 + y^2 + 7x - 11y + 38 = 0$ (B) $x^2 + y^2 = 53$

(C) $x^2 + y^2 + x - y - 44 = 0$ (D) $x^2 + y^2 - x + y - 62 = 0$

CR0037

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[MULTIPLE CHOICE]

8. Which of the following lines have the intercepts of equal lengths on the circle, $x^2 + y^2 - 2x + 4y = 0$?

(A) $3x - y = 0$ (B) $x + 3y = 0$ (C) $x + 3y + 10 = 0$ (D) $3x - y - 10 = 0$

CR0038

9. $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$, represents : (Where x_1, y_1 are constant)

(A) equation of a straight line, if θ is constant and r is variable

(B) equation of a circle, if r is constant and θ is a variable

(C) a straight line passing through a fixed point and having a known slope

(D) a circle with a known centre and a given radius.

CR0039

10. A family of linear functions is given by $f(x) = 1 + c(x + 3)$ where $c \in R$. If a member of this family meets a unit circle centred at origin in two coincident points then 'c' can be equal to

(A) –3/4 (B) 0 (C) 3/4 (D) 1

CR0040

11. The equations of the tangents drawn from the origin to the circle, $x^2 + y^2 - 2rx - 2hy + h^2 = 0$ are :

(A) $x = 0$ (B) $y = 0$

(C) $(h^2 - r^2)x - 2rhy = 0$ (D) $(h^2 - r^2)x + 2rhy = 0$

CR0041

12. Tangents PA and PB are drawn to the circle $S \equiv x^2 + y^2 - 2y - 3 = 0$ from the point P(3,4). Which of the following alternative(s) is/are correct ?

(A) The power of point P(3,4) with respect to circle S = 0 is 14.

(B) The angle between tangents from P(3,4) to the circle S = 0 is $\frac{\pi}{3}$

(C) The equation of circumcircle of ΔPAB is $x^2 + y^2 - 3x - 5y + 4 = 0$

(D) The area of quadrilateral PACB is $3\sqrt{7}$ square units where C is the centre of circle S = 0.

CR0042

13. Consider the circles $C_1 : x^2 + y^2 = 16$ and $C_2 : x^2 + y^2 - 12x + 32 = 0$. Which of the following statement is/are correct ?

(A) Number of common tangent to these circles is 3.

(B) The point P with coordinates (4,1) lies outside the circle C_1 and inside the circle C_2.

(C) Their direct common tangent intersect at (12,0).

(D) Slope of their radical axis is not defined.

CR0043

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14. Which of the following is/are True ?

The circles $x^2 + y^2 - 6x - 6y + 9 = 0$ and $x^2 + y^2 + 6x + 6y + 9 = 0$ are such that -

(A) they do not intersect

(B) they touch each other

(C) their exterior common tangents are parallel.

(D) their interior common tangents are perpendicular.

CR0044

15. Consider the circles $S_1 : x^2 + y^2 = 4$ and $S_2 : x^2 + y^2 - 2x - 4y + 4 = 0$ which of the following statements are correct ?

(A) Number of common tangents to these circles is 2.

(B) If the power of a variable point P w.r.t. these two circles is same then P moves on the line $x + 2y - 4 = 0$

(C) Sum of the y-intercepts of both the circles is 6.

(D) The circles S_1 and S_2 are orthogonal.

CR0045

16. Two circles $x^2 + y^2 + px + py - 7 = 0$ and $x^2 + y^2 - 10x + 2py + 1 = 0$ intersect each other orthogonally then the value of p is -

(A) 1 (B) 2 (C) 3 (D) 5

CR0046

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[COMPREHENSION]

Paragraph for Question nos. 17 to 19

In the diagram as shown, a circle is drawn with centre C(1, 1) and radius 1 and a line L. The line L is tangential to the circle at Q. Further L meet the y-axis at R and the x-axis at P is such a way that the angle OPQ equals θ where $0 < \theta < \frac{\pi}{2}$.

17. The coordinates of Q are

(A) $(1 + \cos\theta, 1 + \sin\theta)$ (B) $(\sin\theta, \cos\theta)$

(C) $(1 + \sin\theta, \cos\theta)$ (D) $(1 + \sin\theta, 1 + \cos\theta)$

CR0047

18. Equation of the line PR is

(A) $x\cos\theta + y\sin\theta = \sin\theta + \cos\theta + 1$ (B) $x\sin\theta + y\cos\theta = \cos\theta + \sin\theta - 1$

(C) $x\sin\theta + y\cos\theta = \cos\theta + \sin\theta + 1$ (D) $x\tan\theta + y = 1 + \cot\left(\frac{\theta}{2}\right)$

CR0047

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19. If the area bounded by the circle, the x-axis and PQ is A(θ), then $A\left(\frac{\pi}{4}\right)$ equals

(A) $\sqrt{2}+1-\frac{3\pi}{8}$ (B) $\sqrt{2}-1+\frac{3\pi}{8}$ (C) $\sqrt{2}+1+\frac{\pi}{8}$ (D) $\sqrt{2}-1+\frac{\pi}{8}$

CR0047

Paragraph for question Nos. 20 to 23

Consider the circle S : $x^2 + y^2 - 4x - 1 = 0$ and the line L : $y = 3x - 1$. If the line L cuts the circle at A & B.

20. Length of the chord AB equal -

(A) $2\sqrt{5}$ (B) $\sqrt{5}$ (C) $5\sqrt{2}$ (D) $\sqrt{10}$

CR0048

21. The angle subtended by the chord AB in the minor arc of S is-

(A) $\frac{3\pi}{4}$ (B) $\frac{5\pi}{6}$ (C) $\frac{2\pi}{3}$ (D) $\frac{\pi}{4}$

CR0048

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22. Acute angle between the line L and the circle S is -

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

CR0048

23. If the equation of the circle on AB as diameter is of the form $x^2 + y^2 + ax + by + c = 0$ then the magnitude of the vector $\vec{V} = a\hat{i} + b\hat{j} + c\hat{k}$ has the value equal to-

(A) $\sqrt{8}$ (B) $\sqrt{6}$ (C) $\sqrt{9}$ (D) $\sqrt{10}$

CR0048

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EXERCISE (S-1)

1. Find the equation to the circle

(i) Whose radius is 10 and whose centre is (–5, –6).

CR0049

(ii) Whose radius is a + b and whose centre is (a, –b).

CR0050

2. Find the coordinates of the centres and the radii of the circles whose equations are **:**

(i) $x^2 + y^2 - 4x - 8y = 41$

CR0051

(ii) $\sqrt{1+m^2}\,(x^2 + y^2) - 2cx - 2mcy = 0$

CR0052

3. Find the equation to the circles which pass through the points :

(i) (0, 0), (a, 0) and (0, b)

CR0053

(ii) (1, 2), (3, –4) and (5, –6)

CR0054

(iii) (1, 1), (2, –1) and (3, 2)

CR0055

4. The line $lx + my + n = 0$ intersects the curve $ax^2 + 2hxy + by^2 = 1$ at the point P and Q. The circle on PQ as diameter passes through the origin. Prove that $n^2(a + b) = l^2 + m^2$.

CR0056

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5. Determine the nature of the quadrilateral formed by four lines $3x + 4y - 5 = 0$; $4x - 3y - 5 = 0$; $3x + 4y + 5 = 0$ and $4x - 3y + 5 = 0$. Find the equation of the circle inscribed and circumscribing this quadrilateral.

CR0057

6. One of the diameters of the circle circumscribing the rectangle ABCD is $4y = x + 7$. If A & B are the points (–3, 4) & (5,4) respectively, then find the area of the rectangle.

CR0058

7. Tangents OP and OQ are drawn from the origin O to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Find the equation of the circumcircle of the triangle OPQ.

CR0059

8. Find the equation to the circle which goes through the origin and cuts off intercepts equal to h and k from the positive parts of the axes.

CR0060

9. Find the equation to the circle which touches the axis of :

(a) x at a distance +3 from the origin and intercepts a distance 6 on the axis of y.

CR0061

(b) x, pass through the point (1, 1) and have line $x + y = 3$ as diameter.

CR0062

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10. Let L_1 be a straight line through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 & L_2 are equal, then find the equation(s) which represent L_1.

CR0063

11. Find the equation of a line with gradient 1 such that the two circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 10x - 14y + 65 = 0$ intercept equal length on it.

CR0064

12. Find the equations of straight lines which pass through the intersection of the lines $x - 2y - 5 = 0$, $7x + y = 50$ & divide the circumference of the circle $x^2 + y^2 = 100$ into two arcs whose lengths are in the ratio 2 : 1.

CR0065

13. (a) Find the shortest distance from the point M(–7, 2) to the circle $x^2 + y^2 - 10x - 14y - 151 = 0$.

CR0066

(b) Find the co-ordinate of the point on the circle $x^2 + y^2 - 12x - 4y + 30 = 0$, which is farthest from the origin.

CR0067

14. If the points $(\lambda, -\lambda)$ lies inside the circle $x^2 + y^2 - 4x + 2y - 8 = 0$, then find the range of λ.

CR0068

15. Given that $x^2 + y^2 = 14x + 6y + 6$, find the largest possible value of the expression $E = 3x + 4y$.

CR0069

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16. In the given figure, the circle $x^2 + y^2 = 25$ intersects the x-axis at the point A and B. The line $x = 11$ intersects the x-axis at the point C. Point P moves along the line $x = 11$ above the x-axis and AP intersects the circle at Q. Find

(i) The coordinates of the point P if the triangle AQB has the maximum area.

(ii) The coordinates of the point P if Q is the middle point of AP.

(iii) The coordinates of P if the area of the triangle AQB is $(1/4)^{th}$ of the area of the triangle APC.

CR0070

17. Show that the line $3x - 4y - c = 0$ will meet the circle having centre at (2, 4) and the radius 5 in real and distinct points if $-35 < c < 15$.

CR0071

18. **(i)** Write down the equation of the tangent to the circle $x^2 + y^2 - 3x + 10y = 15$ at the point (4, –11)

CR0072

(ii) Find the condition that the straight line $3x + 4y = k$ may touch the circle $x^2 + y^2 = 10x$.

CR0073

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19. Find the locus of the middle points of portions of the tangents to the circle $x^2 + y^2 = a^2$ terminated by the coordinate axes.

CR0074

20. If M and m are the maximum and minimum values of $\frac{y}{x}$ for pair of real number (x,y) which satisfy the equation $(x - 3)^2 + (y - 3)^2 = 6$, then find the value of (M + m).

CR0075

21. Find the equation of the tangent to the circle

(a) $x^2 + y^2 - 6x + 4y = 12$, which are parallel to the straight line $4x + 3y + 5 = 0$.

CR0076

(b) $x^2 + y^2 - 22x - 4y + 25 = 0$, which are perpendicular to the straight line $5x + 12y + 9 = 0$

CR0077

(c) $x^2 + y^2 = 25$, which are inclined at 30° to the axis of x.

CR0078

22. A line with gradient 2 is passing through the point P(1, 7) and touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at the point Q. If (a, b) are the coordinates of the point Q, then find the value of (7a + 7b + c).

CR0079

23. A circle passes through the points (–1, 1), (0, 6) and (5, 5). Find the points on the circle the tangents at which are parallel to the straight line joining origin to the centre.

CR0080

24. Tangents are drawn to the concentric circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ at right angle to one another. Show that the locus of their point of intersection is a 3rd concentric circle. Find its radius.

CR0081

25. A variable circle passes through the point A (a, b) & touches the x-axis. Show that the locus of the other end of the diameter through A is $(x - a)^2 = 4by$.

CR0082

26. Circles C_1 and C_2 are externally tangent and they are both internally tangent to the circle C_3. The radii of C_1 and C_2 are 4 and 10, respectively and the centres of the three circles are collinear. A chord of C_3 is also a common internal tangent of C_1 and C_2. Given that the length of the chord is $\frac{m\sqrt{n}}{p}$ where *m*, *n* and *p* are positive integers, *m* and *p* are relatively prime and *n* is not divisible by the square of any prime, find the value of $(m + n + p)$.

CR0083

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27. Consider a circle S with centre at the origin and radius 4. Four circles A, B, C and D each with radius unity and centres (–3, 0), (–1, 0), (1, 0) and (3, 0) respectively are drawn. A chord PQ of the circle S touches the circle B and passes through the centre of the circle C. If the length of this chord can be expressed as $\sqrt{x}$, find x.

CR0084

28. A point moving around circle $(x + 4)^2 + (y + 2)^2 = 25$ with centre C broke away from it either at the point A or point B on the circle and moved along a tangent to the circle passing through the point D (3, –3). Find the following.

(i) Equation of the tangents at A and B.

(ii) Coordinates of the points A and B.

(iii) Angle ADB and the maximum and minimum distances of the point D from the circle.

(iv) Area of quadrilateral ADBC and the ΔDAB.

(v) Equation of the circle circumscribing the ΔDAB and also the length of the intercepts made by this circle on the coordinate axes.

CR0085

29. Find the co-ordinates of the middle point of the chord which the circle $x^2 + y^2 - 2x + 2y - 2 = 0$ cuts off on the line $y = x - 1$.

Find also the equation of the locus of the middle point of all chords of the circle which are parallel to the line $y = x - 1$.

CR0086

30. The straight line $x - 2y + 1 = 0$ intersects the circle $x^2 + y^2 = 25$ in points T and T', find the co-ordinates of a point of intersection of tangents drawn at T and T' to the circle.

CR0087

31. Find the equation of the circle passing through the point of intersection of the circles $x^2 + y^2 - 6x + 2y + 4 = 0$, $x^2 + y^2 + 2x - 4y - 6 = 0$ and with its centre on the line $y = x$.

CR0088

32. Find the equation of the circle passing through the points of intersection of the circles $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 - 10x - 12y + 40 = 0$ and whose radius is 4.

CR0089

33. Find the equation of the circle through points of intersection of the circle $x^2 + y^2 - 2x - 4y + 4 = 0$ and the line $x + 2y = 4$ which touches the line $x + 2y = 0$.

CR0090

34. Find the equations of the circles which pass through the common points of the following pair of circles.

(a) $x^2 + y^2 + 2x + 3y - 7 = 0$ and $x^2 + y^2 + 3x - 2y - 1 = 0$ through the point (1,2)

CR0091

(b) $x^2 + y^2 + 4x - 6y - 12 = 0$ and $x^2 + y^2 - 5x + 17y = 19$ and having its centre on $x + y = 0$.

CR0092

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35. The line $2x - 3y + 1 = 0$ is tangent to a circle $S = 0$ at $(1, 1)$. If the radius of the circle is $\sqrt{13}$. Find the equation of the circle S.

CR0093

36. Find the equation of the circle which passes through the point (1, 1) & which touches the circle $x^2 + y^2 + 4x - 6y - 3 = 0$ at the point (2, 3) on it.

CR0094

37. A circle $S = 0$ is drawn with its centre at $(-1, 1)$ so as to touch the circle $x^2 + y^2 - 4x + 6y - 3 = 0$ externally. Find the intercept made by the circle $S = 0$ on the coordinate axes.

CR0095

38. Find the equation of the circle whose radius is 3 and which touches the circle $x^2 + y^2 - 4x - 6y - 12 = 0$ internally at the point $(-1, -1)$.

CR0096

39. Find the radical centre of the following set of circles
$x^2 + y^2 - 3x - 6y + 14 = 0$; $x^2 + y^2 - x - 4y + 8 = 0$; $x^2 + y^2 + 2x - 6y + 9 = 0$

CR0097

40. Find the equation to the circle, cutting orthogonally each of the following circles :
$x^2 + y^2 - 2x + 3y - 7 = 0$; $x^2 + y^2 + 5x - 5y + 9 = 0$; $x^2 + y^2 + 7x - 9y + 29 = 0$.

CR0098

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41. Find the equation to the circle orthogonal to the two circles
$x^2 + y^2 - 4x - 6y + 11 = 0$; $x^2 + y^2 - 10x - 4y + 21 = 0$ and has $2x + 3y = 7$ as diameter.

CR0099

42. Find the equation of the circle through the points of intersection of circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 4y - 12 = 0$ & cutting the circle $x^2 + y^2 - 2x - 4 = 0$ orthogonally.

CR0100

43. The centre of the circle $S = 0$ lie on the line $2x - 2y + 9 = 0$ & $S = 0$ cuts orthogonally the circle $x^2 + y^2 = 4$. Show that circle $S = 0$ passes through two fixed points & find their coordinates.

CR0101

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EXERCISE (S-2)

1. If the circle $x^2 + y^2 + 4x + 22y + a = 0$ bisects the circumference of the circle $x^2 + y^2 - 2x + 8y - b = 0$ (where a, b > 0), then find the maximum value of (ab).

CR0102

2. Real number x, y satisfies $x^2 + y^2 = 1$. If the maximum and minimum value of the expression $z = \frac{4-y}{7-x}$ are M and m respectively, then find the value (2M + 6m).

CR0103

3. A circle is drawn with its centre on the line $x + y = 2$ to touch the line $4x - 3y + 4 = 0$ and pass through the point (0, 1). Find its equation.

CR0104

4. A circle with center in the first quadrant is tangent to $y = x + 10$, $y = x - 6$, and the y-axis. Let (h, k) be the center of the circle. If the value of $(h + k) = a + b\sqrt{a}$ where $\sqrt{a}$ is a surd, find the value of a + b.

CR0105

5. Let $S_1 = 0$ and $S_2 = 0$ be two circles intersecting at P (6, 4) and both are tangent to x-axis and line y = mx (where m > 0). If product of radii of the circles $S_1 = 0$ and $S_2 = 0$ is $\frac{52}{3}$, then find the value of m.

CR0106

6. Through a given point P(5, 2), secants are drawn to cut the circle $x^2 + y^2 = 25$ at points $A_1(B_1)$, $A_2(B_2)$, $A_3(B_3)$, $A_4(B_4)$ and $A_5(B_5)$ such that $PA_1 + PB_1 = 5$, $PA_2 + PB_2 = 6$, $PA_3 + PB_3 = 7$, $PA_4 + PB_4 = 8$ and $PA_5 + PB_5 = 9$. Find the value of $\sum_{i=1}^{5} PA_i^2 + \sum_{i=1}^{5} PB_i^2$.

[**Note :** $A_r(B_r)$ denotes that the line passing through P(5, 2) meets the circle $x^2 + y^2 = 25$ at two points A_r and B_r.]

CR0107

7. Find the locus of the mid point of all chords of the circle $x^2 + y^2 - 2x - 2y = 0$ such that the pair of lines joining (0, 0) & the point of intersection of the chords with the circles make equal angle with axis of x.

CR0108

8. Consider a family of circles passing through two fixed points A(3, 7) & B(6, 5). The chords in which the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ cuts the members of the family are concurrent at a point. Find the coordinates of this point.

CR0109

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9. Find the equation of a circle which is co-axial with circles $2x^2 + 2y^2 - 2x + 6y - 3 = 0$ & $x^2 + y^2 + 4x + 2y + 1 = 0$. It is given that the centre of the circle to be determined lies on the radical axis of these two circles.

CR0110

10. (a) Find the equation of a circle passing through the origin if the line pair, $xy - 3x + 2y - 6 = 0$ is orthogonal to it. If this circle is orthogonal to the circle $x^2 + y^2 - kx + 2ky - 8=0$ then find the value of k.

CR0111

(b) Find the equation of the circle which cuts the circle $x^2 + y^2 - 14x - 8y + 64 = 0$ and the coordinate axes orthogonally.

CR0112

11. Find the equation of a circle which touches the line $x + y = 5$ at the point $(-2, 7)$ and cuts the circle $x^2 + y^2 + 4x - 6y + 9 = 0$ orthogonally.

CR0113

12. Consider two circles C_1 of radius 'a' and C_2 of radius 'b' $(b > a)$ both lying in the first quadrant and touching the coordinate axes. In each of the conditions listed in **column-I**, the ratio of b/a is given in **column-II.**

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Column-I		**Column-II**	
(A)	C_1 and C_2 touch each other	(P)	$2+\sqrt{2}$
(B)	C_1 and C_2 are orthogonal	(Q)	3
(C)	C_1 and C_2 intersect so that the common chord is longest	(R)	$2+\sqrt{3}$
(D)	C_2 passes through the centre of C_1	(S)	$3+2\sqrt{2}$
		(T)	$3-2\sqrt{2}$

CR0114

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EXERCISE (JM)

1. For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A false statement among the following is :- **[AIEEE-2010]**

(1) There is a regular polygon with $\frac{r}{R}=\frac{1}{2}$ (2) There is a regular polygon with $\frac{r}{R}=\frac{1}{\sqrt{2}}$

(3) There is a regular polygon with $\frac{r}{R}=\frac{2}{3}$ (4) There is a regular polygon with $\frac{r}{R}=\frac{\sqrt{3}}{2}$

CR0115

2. The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points if :- **[AIEEE-2010]**

(1) $-85 < m < -35$ (2) $-35 < m < 15$ (3) $15 < m < 65$ (4) $35 < m < 85$

CR0116

3. The two circles $x^2 + y^2 = ax$ and $x^2 + y^2 = c^2$ $(c > 0)$ touch each other if :- **[AIEEE-2011]**

(1) $a = 2c$ (2) $|a| = 2c$ (3) $2|a| = c$ (4) $|a| = c$

CR0117

4. The equation of the circle passing through the points (1, 0) and (0, 1) and having the smallest radius is - **[AIEEE-2011]**

(1) $x^2 + y^2 + x + y - 2 = 0$ (2) $x^2 + y^2 - 2x - 2y + 1 = 0$

(3) $x^2 + y^2 - x - y = 0$ (4) $x^2 + y^2 + 2x + 2y - 7 = 0$

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CR0118

5. The length of the diameter of the circle which touches the x-axis at the point (1, 0) and passes through the point (2, 3) is : **[AIEEE-2012]**

(1) 5/3 (2) 10/3 (3) 3/5 (4) 6/5

CR0119

6. The circle passing through (1, – 2) and touching the axis of x at (3, 0) also passes through the point : **[JEE (Main)-2013]**

(1) (–5, 2) (2) (2, –5) (3) (5, –2) (4) (–2, 5)

CR0120

7. If a circle C passing through (4, 0) touches the circle $x^2 + y^2 + 4x - 6y - 12 = 0$ externally at a point (1, –1), then the radius of the circle C is :- **[JEE-Main (on line)-2013]**

(1) $\sqrt{57}$ (2) $2\sqrt{5}$ (3) 4 (4) 5

CR0121

8. If the circle $x^2 + y^2 - 6x - 8y + (25 - a^2) = 0$ touches the axis of x, then a equals :- **[JEE-Main (on line)-2013]**

(1) ±4 (2) ±3 (3) 0 (4) ±2

CR0122

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9. Statement I : The only circle having radius $\sqrt{10}$ and a diameter along line $2x + y = 5$ is $x^2 + y^2 - 6x + 2y = 0$.
Statement II : $2x + y = 5$ is a normal to the circle $x^2 + y^2 - 6x + 2y = 0$. **[JEE-Main (on line)-2013]**
(1) Statement I is false, Statement II is true
(2) Statement I is true ; Statement II is false.
(3) Statement I is true, Statement II is true, Statement II is not a correct explanation for Statement I.
(4) Statement I is true : Statement II is true ; Statement II is a correct explanation for Statement I.

CR0123

10. Let C be the circle with centre at (1, 1) and radius = 1. If T is the circle centred at (0, y), passing through origin and touching the circle C externally, then the radius of T is equal to : **[JEE(Main)-2014]**

(1) $\frac{\sqrt{3}}{\sqrt{2}}$ (2) $\frac{\sqrt{3}}{2}$ (3) $\frac{1}{2}$ (4) $\frac{1}{4}$

CR0124

11. The number of common tangents to the circle
$x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$, is : **[JEE(Main)-2015]**
(1) 3 (2) 4 (3) 1 (4) 2

CR0125

12. If one of the diameters of the circle, given by the equation, $x^2 + y^2 - 4x + 6y - 12 = 0$, is a chord of a circle S, whose centre is at (–3, 2), then the radius of S is :- **[JEE(Main)-2016]**
(1) 10 (2) $5\sqrt{2}$ (3) $5\sqrt{3}$ (4) 5

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CR0126

13. The centres of those circles which touch the circle, $x^2 + y^2 - 8x - 8y - 4 = 0$, externally and also touch the x-axis, lie on :- **[JEE(Main)-2016]**
(1) A parabola (2) A circle
(3) An ellipse which is not a circle (4) A hyperbola

CR0127

14. Three circles of radii a, b, c($a < b < c$) touch each other externally. If they have x-axis as a common tangent, then : **[JEE(Main)-2019]**

(1) $\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}$ (2) a, b, c are in A.P.

(3) $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are in A.P. (4) $\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}}$

CR0128

15. If a circle C passing through the point (4,0) touches the circle $x^2 + y^2 + 4x - 6y = 12$ externally at the point (1, –1), then the radius of C is : **[JEE(Main)-2019]**

(1) $\sqrt{57}$ (2) 4 (3) $2\sqrt{5}$ (4) 5

CR0129

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16. If the area of an equilateral triangle inscribed in the circle, $x^2 + y^2 + 10x + 12y + c = 0$ is $27\sqrt{3}$ sq. units then c is equal to : **[JEE(Main)-2019]**

(1) 20 (2) 25 (3) 13 (4) –25

CR0130

17. A square is inscribed in the circle $x^2 + y^2 - 6x + 8y - 103 = 0$ with its sides parallel to the corrdinate axes. Then the distance of the vertex of this square which is nearest to the origin is :- **[JEE(Main)-2019]**

(1) 13 (2) $\sqrt{137}$ (3) 6 (4) $\sqrt{41}$

CR0131

18. A circle cuts a chord of length 4a on the x-axis and passes through a point on the y-axis, distant 2b from the origin. Then the locus of the centre of this circle, is :- **[JEE(Main)-2019]**

(1) A hyperbola (2) A parabola (3) A straight line (4) An ellipse

CR0132

19. If a variable line, $3x + 4y - \lambda = 0$ is such that the two circles $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 18x - 2y + 78 = 0$ are on its opposite sides, then the set of all values of λ is the interval :- **[JEE(Main)-2019]**

(1) [12, 21] (2) (2, 17) (3) (23, 31) (4) [13, 23]

CR0133

20. The sum of the squares of the lengths of the chords intercepted on the circle, $x^2 + y^2 = 16$, by the lines, $x + y = n$, $n \in N$, where N is the set of all natural numbers, is : **[JEE(Main)-2019]**

(1) 320 (2) 160 (3) 105 (4) 210

CR0134

21. If a tangent to the circle $x^2 + y^2 = 1$ intersects the coordinate axes at distinct points P and Q, then the locus of the mid-point of PQ is **[JEE(Main)-2019]**

(1) $x^2 + y^2 - 2xy = 0$ (2) $x^2 + y^2 - 16x^2y^2 = 0$

(3) $x^2 + y^2 - 4x^2y^2 = 0$ (4) $x^2 + y^2 - 2x^2y^2 = 0$

CR0135

22. The common tangent to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 + 6x + 8y - 24 = 0$ also passes through the point :- **[JEE(Main)-2019]**

(1) (–4, 6) (2) (6, –2) (3) (–6, 4) (4) (4, –2)

CR0136

23. Let the tangents drawn from the origin to the circle, $x^2 + y^2 - 8x - 4y + 16 = 0$ touch it at the points A and B. The $(AB)^2$ is equal to : **[JEE(Main)-2020]**

(1) $\frac{52}{5}$ (2) $\frac{32}{5}$ (3) $\frac{56}{5}$ (4) $\frac{64}{5}$

CR0137

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24. If a line, $y = mx + c$ is a tangent to the circle, $(x-3)^2 + y^2 = 1$ and it is perpendicular to a line L_1, where L_1 is the tangent to the circle, $x^2 + y^2 = 1$ at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, then **[JEE(Main)-2020]**

(1) $c^2 - 6c + 7 = 0$ (2) $c^2 + 6c + 7 = 0$ (3) $c^2 + 7c + 6 = 0$ (4) $c^2 - 7c + 6 = 0$

CR0138

25. If the curves, $x^2 - 6x + y^2 + 8 = 0$ and $x^2 - 8y + y^2 + 16 - k = 0$, $(k > 0)$ touch each other at a point, then the largest value of k is _____. **[JEE(Main)-2020]**

CR0139

EXERCISE (JA)

1. Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3}+1$ apart. If the chords subtend at the center, angles of $\frac{\pi}{k}$ and $\frac{2\pi}{k}$, where $k > 0$, then the value of [k] is **[JEE 10, 3M]**

[**Note :** [k] denotes the largest integer less than or equal to k]

CR0140

2. The circle passing through the point (–1,0) and touching the y-axis at (0,2) also passes through the point - **[JEE 2011, 3M, –1M]**

(A) $\left(-\frac{3}{2}, 0\right)$ (B) $\left(-\frac{5}{2}, 2\right)$ (C) $\left(-\frac{3}{2}, \frac{5}{2}\right)$ (D) (–4,0)

CR0141

3. The straight line $2x - 3y = 1$ divides the circular region $x^2 + y^2 \le 6$ into two parts. If

$$S = \left\{\left(2, \frac{3}{4}\right), \left(\frac{5}{2}, \frac{3}{4}\right), \left(\frac{1}{4}, -\frac{1}{4}\right), \left(\frac{1}{8}, \frac{1}{4}\right)\right\},$$

then the number of point(s) in S lying inside the smaller part is **[JEE 2011, 4M]**

CR0142

4. The locus of the mid-point of the chord of contact of tangents drawn from points lying on the straight line $4x - 5y = 20$ to the circle $x^2 + y^2 = 9$ is- **[JEE 2012, 3M, –1M]**

(A) $20(x^2 + y^2) - 36x + 45y = 0$ (B) $20(x^2 + y^2) + 36x - 45y = 0$

(C) $36(x^2 + y^2) - 20x + 45y = 0$ (D) $36(x^2 + y^2) + 20x - 45y = 0$

CR0143

Paragraph for Question 5 and 6

A tangent PT is drawn to the circle $x^2 + y^2 = 4$ at the point $P\left(\sqrt{3}, 1\right)$. A straight line L, perpendicular to PT is a tangent to the circle $(x-3)^2 + y^2 = 1$.

5. A common tangent of the two circles is **[JEE 2012, 3M, –1M]**

(A) $x = 4$ (B) $y = 2$ (C) $x + \sqrt{3}y = 4$ (D) $x + 2\sqrt{2}y = 6$

CR0144

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6. A possible equation of L is **[JEE 2012, 3M, –1M]**

(A) $x-\sqrt{3}y=1$ (B) $x+\sqrt{3}y=1$

(C) $x-\sqrt{3}y=-1$ (D) $x+\sqrt{3}y=5$

CR0144

7. Circle(s) touching x-axis at a distance 3 from the origin and having an intercept of length $2\sqrt{7}$ or y-axis is (are) **[JEE(Advanced) 2013, 3, (–1)]**

(A) $x^2 + y^2 - 6x + 8y + 9 = 0$ (B) $x^2 + y^2 - 6x + 7y + 9 = 0$

(C) $x^2 + y^2 - 6x - 8y + 9 = 0$ (D) $x^2 + y^2 - 6x - 7y + 9 = 0$

CR0145

8. A circle S passes through the point (0, 1) and is orthogonal to the circles $(x-1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$. Then :- **[JEE(Advanced)-2014, 3]**

(1) radius of S is 8 (B) radius of S is 7

(3) centre of S is (–7, 1) (D) centre is S is (–8, 1)

CR0146

9. Let RS be the diameter of the circle $x^2 + y^2 = 1$, where S is the point (1,0). Let P be a variable point (other than R and S) on the circle and tangents to the circle at S and P meet at the point Q. The normal to the circle at P intersects a line drawn through Q parallel to RS at point E. then the locus of E passes through the point(s)- **[JEE(Advanced)-2016, 4(–2)]**

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(A) $\left(\frac{1}{3},\frac{1}{\sqrt{3}}\right)$ (B) $\left(\frac{1}{4},\frac{1}{2}\right)$ (C) $\left(\frac{1}{3},-\frac{1}{\sqrt{3}}\right)$ (D) $\left(\frac{1}{4},-\frac{1}{2}\right)$

CR0147

10. For how many values of p, the circle $x^2 + y^2 + 2x + 4y - p = 0$ and the coordinate axes have exactly three common points ? **[JEE(Advanced)-2017, 3]**

CR0148

Paragraph "X"

Let S be the circle in the xy-plane defined by the equation $x^2 + y^2 = 4$.

(*There are two question based on Paragraph "X", the question given below is one of them*)

11. Let E_1E_2 and F_1F_2 be the chord of S passing through the point $P_0(1, 1)$ and parallel to the x-axis and the y-axis, respectively. Let G_1G_2 be the chord of S passing through P_0 and having slop –1. Let the tangents to S at E_1 and E_2 meet at E_3, the tangents of S at F_1 and F_2 meet at F_3, and the tangents to S at G_1 and G_2 meet at G_3. Then, the points E_3, F_3 and G_3 lie on the curve **[JEE(Advanced)-2018, 3(–1)]**

(A) $x + y = 4$ (B) $(x-4)^2 + (y-4)^2 = 16$

(C) $(x-4)(y-4) = 4$ (D) $xy = 4$

CR0149

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Paragraph "X"

Let S be the circle in the xy-plane defined by the equation $x^2 + y^2 = 4$.

(*There are two question based on Paragraph "X", the question given below is one of them*)

12. Let P be a point on the circle S with both coordinates being positive. Let the tangent to S at P intersect the coordinate axes at the points M and N. Then, the mid-point of the line segment MN must lie on the curve - **[JEE(Advanced)-2018, 3(–1)]**

(A) $(x+y)^2 = 3xy$ (B) $x^{2/3} + y^{2/3} = 2^{4/3}$

(C) $x^2 + y^2 = 2xy$ (D) $x^2 + y^2 = x^2y^2$

CR0150

13. Let T be the line passing through the points P(–2, 7) and Q(2, –5). Let F_1 be the set of all pairs of circles (S_1, S_2) such that T is tangents to S_1 at P and tangent to S_2 at Q, and also such that S_1 and S_2 touch each other at a point, say, M. Let E_1 be the set representing the locus of M as the pair (S_1, S_2) varies in F_1. Let the set of all straight line segments joining a pair of distinct points of E_1 and passing through the point R(1, 1) be F_2. Let E_2 be the set of the mid-points of the line segments in the set F_2. Then, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018, 4(–2)]**

(A) The point (–2, 7) lies in E_1

(B) The point $\left(\frac{4}{5}, \frac{7}{5}\right)$ does **NOT** lie in E_2

(C) The point $\left(\frac{1}{2}, 1\right)$ lies in E_2

(D) The point $\left(0, \frac{3}{2}\right)$ does **NOT** lie in E_1

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CR0151

14. A line y = mx + 1 intersects the circle $(x-3)^2 + (y+2)^2 = 25$ at the points P and Q. If the midpoint of the line segment PQ has x-coordinate $-\frac{3}{5}$, then which one of the following options is correct ?

[JEE(Advanced)-2019, 3(–1)]

(1) $6 \le m < 8$ (2) $2 \le m < 4$

(3) $4 \le m < 6$ (4) $-3 \le m < -1$

CR0152

15. Let the point B be the reflection of the point A(2, 3) with respect to the line 8x – 6y – 23 = 0. Let Γ_A and Γ_B be circles of radii 2 and 1 with centres A and B respectively. Let T be a common tangent to the circles Γ_A and Γ_B such that both the circles are on the same side of T. If C is the point of intersection of T and the line passing through A and B, then the length of the line segment AC is _____ **[JEE(Advanced)-2019, 3(0)]**

CR0153

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Answer the following by appropriately matching the lists based on the information given in the paragraph

Let the circles $C_1 : x^2 + y^2 = 9$ and $C_2 : (x-3)^2 + (y-4)^2 = 16$, intersect at the points X and Y. Suppose that another circle $C_3 : (x-h)^2 + (y-k)^2 = r^2$ satisfies the following conditions :

(i) centre of C_3 is collinear with the centres of C_1 and C_2

(ii) C_1 and C_2 both lie inside C_3, and

(iii) C_3 touches C_1 at M and C_2 at N.

Let the line through X and Y intersect C_3 at Z and W, and let a common tangent of C_1 and C_3 be a tangent to the parabola $x^2 = 8\alpha y$.

There are some expression given in the List-I whose values are given in List-II below :

	List-I		List-II
(I)	$2h + k$	(P)	6
(II)	$\dfrac{\text{Length of ZW}}{\text{Length of XY}}$	(Q)	$\sqrt{6}$
(III)	$\dfrac{\text{Area of triangle MZN}}{\text{Area of triangle ZMW}}$	(R)	$\dfrac{5}{4}$
(IV)	α	(S)	$\dfrac{21}{5}$
		(T)	$2\sqrt{6}$
		(U)	$\dfrac{10}{3}$

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16. Which of the following is the only INCORRECT combination ? **[JEE(Advanced)-2019, 3(–1)]**

(1) (IV), (S) (2) (IV), (U) (3) (III), (R) (4) (I), (P)

CR0154

17. Which of the following is the only CORRECT combination ? **[JEE(Advanced)-2019, 3(–1)]**

(1) (II), (T) (2) (I), (S) (3) (I), (U) (4) (II), (Q)

CR0154

18. Let O be the centre of the circle $x^2 + y^2 = r^2$, where $r > \dfrac{\sqrt{5}}{2}$. Suppose PQ is a chord of this circle and the equation of the line passing through P and Q is $2x + 4y = 5$. If the centre of the circumcircle of the triangle OPQ lies on the line $x + 2y = 4$, then the value of r is _____ **[JEE(Advanced)-2020]**

CR0159

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ANSWER KEY

Do yourself-1

(i) Centre $\left(\frac{3}{4}, -\frac{5}{4}\right)$, Radius $\frac{3\sqrt{10}}{4}$ **(ii)** $17(x^2 + y^2) + 2x - 44y = 0$

(iii) $x = \frac{p}{2}(-1+\sqrt{2}\cos\theta)$; $y = \frac{p}{2}(-1+\sqrt{2}\sin\theta)$ (iv) $x^2 + y^2 + 6x - 2y - 51 = 0$

Do yourself-2

(i) (1, 2) lie inside the circle and the point (6, 0) lies outside the circle

(ii) min = 0, max = 6, power = 0

Do yourself-3

(i) $x\cos\alpha + y\sin\alpha = a(1 + \cos\alpha)$ **(ii)** $4x - 3y + 7 = 0$ & $4x - 3y - 43 = 0$

(iii) $5x + 12y = \pm 26$; $\left(\mp\frac{10}{13}, \mp\frac{24}{13}\right)$ **(iv)** 1

Do yourself-4

(i) $x + 2y = 1$

Do yourself-5

(i) $4x + 7y + 10 = 0$ **(ii)** $\frac{405\sqrt{3}}{52}$ sq. units

Do yourself-6

(i) $5x - 4y + 26 = 0$ **(ii)** $x^2 + y^2 + gx + fy = 0$

Do yourself-7

(i) $(x - h)^2 + (y - k)^2 = 2a^2$ **(ii)** 10 **(iii)** angle between the tangents = 90°

Do yourself-8

(ii) $x^2 + y^2 - \frac{10x}{7} - \frac{10y}{7} - \frac{12}{7} = 0$ **(iii)** $x^2 + y^2 + 4x - 7y + 5 = 0$

Do yourself-9

(i) $(x - 5)^2 + (y - 5)^2 = 25$ **(ii)** 4

Do yourself-10

(i) 135° **(ii)** $x + 2y = 2$ **(iii)** (1, 2)

Do yourself-11

(i) 18 **(ii)** $a^{-2} + b^{-2} = c^{-1}$

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EXERCISE (O-1)

1. D	**2.** B	**3.** A	**4.** A	**5.** B	**6.** D	**7.** A	**8.** B
9. B	**10.** A	**11.** C	**12.** A	**13.** B	**14.** A	**15.** B	**16.** A
17. D	**18.** A	**19.** C	**20.** B	**21.** B	**22.** C	**23.** A	**24.** C
25. A	**26.** C	**27.** C	**28.** C	**29.** B	**30.** C		

EXERCISE (O-2)

1. B	**2.** D	**3.** B	**4.** A	**5.** A	**6.** B	**7.** A	**8.** A,B,C,D
9. A,B	**10.** A,B	**11.** A,C	**12.** A,C	**13.** A,C,D	**14.** A,C,D	**15.** A,B,D	**16.** B,C
17. D	**18.** C	**19.** A	**20.** D	**21.** A	**22.** C	**23.** B	

EXERCISE (S-1)

1. **(i)** $x^2 + y^2 + 10x + 12y = 39$; **(ii)** $x^2 + y^2 - 2ax + 2by = 2ab$

2. **(i)** $(2, 4)$; $\sqrt{61}$; **(ii)** $\left(\frac{c}{\sqrt{1+m^2}}, \frac{mc}{\sqrt{1+m^2}}\right)$; c

3. **(i)** $x^2 + y^2 - ax - by = 0$; **(ii)** $x^2 + y^2 - 22x - 4y + 25 = 0$; **(iii)** $x^2 + y^2 - 5x - y + 4 = 0$

5. square of side 2; $x^2 + y^2 = 1$; $x^2 + y^2 = 2$ **6.** 32 sq. unit

7. $x^2 + y^2 + gx + fy = 0$ **8.** $x^2 + y^2 - hx - ky = 0$

9. (a) $x^2 + y^2 - 6x \pm 6\sqrt{2}y + 9 = 0$; (b) $x^2 + y^2 + 4x - 10y + 4 = 0$; $x^2 + y^2 - 4x - 2y + 4 = 0$

10. $x - y = 0$; $x + 7y = 0$ **11.** $2x - 2y - 3 = 0$ **12.** $4x - 3y - 25 = 0$ **OR** $3x + 4y - 25 = 0$

13. (a) 2; (b) (9, 3) **14.** $\lambda \in (-1, 4)$ **15.** 73 **16.** **(i)** (11, 16), **(ii)** (11, 8), **(iii)** (11, 12)

18. **(i)** $5x - 12y = 152$, **(ii)** $k = 40$ or -10 **19.** $a^2(x^2 + y^2) = 4x^2y^2$ **20.** 6

21. (a) $4x + 3y + 19 = 0$ and $4x + 3y - 31 = 0$; (b) $12x - 5y + 8 = 0$ and $12x - 5y - 252 = 0$

(c) $x - \sqrt{3}y \pm 10 = 0$

22. 4 **23.** (5, 1) & (–1, 5) **24.** $x^2 + y^2 = a^2 + b^2$; $r = \sqrt{a^2 + b^2}$ **26.** 19 **27.** 63

28. (i) $3x - 4y = 21$; $4x + 3y = 3$; (ii) A(0, 1) and B (–1, – 6); (iii) 90°, $5(\sqrt{2} \pm 1)$ units

(iv) 25 sq. units, 12.5 sq. units; (v) $x^2 + y^2 + x + 5y - 6$, x-intercept = 5; y-intercept = 7

29. $\left(\frac{1}{2}, -\frac{1}{2}\right)$, $x + y = 0$ **30.** (–25, 50) **31.** $7x^2 + 7y^2 - 10x - 10y - 12 = 0$

32. $2x^2 + 2y^2 - 18x - 22y + 69 = 0$ and $x^2 + y^2 - 2y - 15 = 0$ **33.** $x^2 + y^2 - x - 2y = 0$

34. (a) $x^2 + y^2 + 4x - 7y + 5 = 0$, (b) $7(x^2 + y^2) + 19x - 19y - 91 = 0$

35. $x^2 + y^2 - 6x + 4y = 0$ **OR** $x^2 + y^2 + 2x - 8y + 4 = 0$ **36.** $x^2 + y^2 + x - 6y + 3 = 0$

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37. zero, zero **38.** $5x^2 + 5y^2 - 8x - 14y - 32 = 0$ **39.** (1,2)

40. $x^2 + y^2 - 16x - 18y - 4 = 0$ **41.** $x^2 + y^2 - 4x - 2y + 3 = 0$

42. $x^2 + y^2 + 16x + 14y - 12 = 0$ **43.** $(-4, 4)$; $(-1/2, 1/2)$

EXERCISE (S-2)

1. 625 **2.** 4 **3.** $x^2+y^2-2x-2y+1=0$ **OR** $x^2+y^2-42x+38y-39=0$

4. 10 **5.** $\sqrt{3}$ **6.** 215 **7.** $x + y = 2$ **8.** $\left(2, \frac{23}{3}\right)$ **9.** $4x^2 + 4y^2 + 6x + 10y - 1 = 0$

10. (a) $x^2 + y^2 + 4x - 6y = 0$; $k = 1$; (b) $x^2 + y^2 = 64$ **11.** $x^2 + y^2 + 7x - 11y + 38 = 0$

12. (A) S; (B) R; (C) Q; (D) P

EXERCISE (JM)

1. 3	**2.** 2	**3.** 4	**4.** 3	**5.** 2	**6.** 3	**7.** 4	**8.** 1
9. 1	**10.** 4	**11.** 1	**12.** 3	**13.** 1	**14.** 1	**15.** 4	**16.** 2
17. 4	**18.** 2	**19.** 1	**20.** 4	**21.** 3	**22.** 2	**23.** 4	**24.** 2

25 36

EXERCISE (JA)

1. 3	**2.** D	**3.** 2	**4.** A	**5.** D	**6.** A	**7.** A,C	**8.** B,C
9. A,C	**10.** 2	**11.** A	**12.** D	**13.** B,D	**14.** 2		
15. 10.00	**16.** 1	**17.** 4	**18.** 2				

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Important Notes

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BINOMIAL THEOREM

1. BINOMIAL EXPRESSION :

Any algebraic expression which contains two dissimilar terms is called binomial expression.

For example : $x - y$, $xy + \frac{1}{x}$, $\frac{1}{z} - 1$, $\frac{1}{(x-y)^{1/3}} + 3$ etc.

2. BINOMIAL THEOREM :

The formula by which any positive integral power of a binomial expression can be expanded in the form of a series is known as **BINOMIAL THEOREM.**

If $x, y \in R$ and $n \in N$, then :

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \ldots + {}^nC_rx^{n-r}y^r + \ldots + {}^nC_ny^n = \sum_{r=0}^{n} {}^nC_rx^{n-r}y^r$$

This theorem can be proved by induction.

Observations :

(a) The number of terms in the expansion is (n+1) i.e. one more than the index.

(b) The sum of the indices of x & y in each term is n.

(c) The binomial coefficients of the terms (${}^nC_0$, ${}^nC_1$.....) equidistant from the beginning and the end are equal. i.e. ${}^nC_r = {}^nC_{r-1}$

(d) Symbol ${}^nC_r$ can also be denoted by $\binom{n}{r}$, $C(n, r)$ or A^n_r.

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Some important expansions :

(i) $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots + {}^nC_nx^n$.

(ii) $(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 + \ldots\ldots + (-1)^n \cdot {}^nC_nx^n$.

Note : The coefficient of x^r in $(1+x)^n = {}^nC_r$ & that in $(1-x)^n = (-1)^r \cdot {}^nC_r$

Illustration 1 : Expand : $(y+2)^6$.

Solution : ${}^6C_0y^6 + {}^6C_1y^5.2 + {}^6C_2y^4.2^2 + {}^6C_3y^3.2^3 + {}^6C_4y^2.2^4 + {}^6C_5y^1.2^5 + {}^6C_6.2^6$.

$= y^6 + 12y^5 + 60y^4 + 160y^3 + 240y^2 + 192y + 64$.

Illustration 2 : Write first 4 terms of $\left(1 - \frac{2y^2}{5}\right)^7$

Solution : ${}^7C_0, {}^7C_1\left(-\frac{2y^2}{5}\right), {}^7C_2\left(-\frac{2y^2}{5}\right)^2, {}^7C_3\left(-\frac{2y^2}{5}\right)^3$

Illustration 3 : If in the expansion of $(1+x)^m(1-x)^n$, the coefficients of x and x^2 are 3 and – 6 respectively then m is - **[JEE 99]**

(A) 6 (B) 9 (C) 12 (D) 24

Solution : $(1+x)^m(1-x)^n = \left[1 + mx + \frac{(m)(m-1).x^2}{2} + \ldots\ldots\right]\left[1 - nx + \frac{n(n-1)}{2}x^2 + \ldots\ldots\right]$

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Coefficient of $x = m - n = 3$ .....(i)

Coefficient of $x^2 = -mn + \frac{n(n+1)}{2} + \frac{m(m-1)}{2} = -6$ .....(ii)

Solving (i) and (ii), we get

$m = 12$ and $n = 9$.

Do yourself - 1 :

(i) Expand $\left(3x^2 - \frac{x}{2}\right)^5$ **(ii)** Expand $(y + x)^n$

Pascal's triangle :

$(x+y)^0$	1	1
$(x+y)^1$	$x + y$	1 1
$(x+y)^2$	$x^2 + 2xy + y^2$	1 2 1
$(x+y)^3$	$x^3 + 3x^2y + 3xy^2 + y^3$	1 3 3 1
$(x+y)^4$	$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^3$	1 4 6 4 1

Pascal's triangle

(i) **Pascal's triangle -** A triangular arrangement of numbers as shown. The numbers give the binomial coefficients for the expansion of $(x + y)^n$. The first row is for $n = 0$, the second for $n = 1$, etc. Each row has 1 as its first and last number. Other numbers are generated by adding the two numbers immediately to the left and right in the row above.

(ii) Pascal triangle is formed by binomial coefficient.

(iii) The number of terms in the expansion of $(x+y)^n$ is $(n + 1)$ i.e. one more than the index.

(iv) The sum of the indices of x & y in each term is n.

(v) Power of first variable (x) decreases while of second variable (y) increases.

(vi) Binomial coefficients are also called **combinatorial coefficients.**

(vii) Binomial coefficients of the terms equidistant from the begining and end are equal.

(viii) r^{th} term from the beginning in the expansion of $(x + y)^n$ is same as r^{th} term from end in the expansion of $(y + x)^n$.

(ix) r^{th} term from the end in $(x + y)^n$ is **$(n - r + 2)^{th}$ term from the beginning**.

3. IMPORTANT TERMS IN THE BINOMIAL EXPANSION :

(a) General term: The general term or the $(r + 1)^{th}$ term in the expansion of $(x + y)^n$ is given by

$T_{r+1} = {}^nC_r\, x^{n-r}\, y^r$

Illustration 4 : Find : (a) The coefficient of x^7 in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$

(b) The coefficient of x^{-7} in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$

Also, find the relation between a and b, so that these coefficients are equal.

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Solution : (a) In the expansion of $\left(ax^2+\frac{1}{bx}\right)^{11}$, the general term is :

$$T_{r+1} = {}^{11}C_r(ax^2)^{11-r}\left(\frac{1}{bx}\right)^r = {}^{11}C_r.\frac{a^{11-r}}{b^r}.x^{22-3r}$$

putting $22 - 3r = 7$

$\therefore \quad 3r = 15 \quad \Rightarrow \quad r = 5$

$\therefore \quad T_6 = {}^{11}C_5\frac{a^6}{b^5}.x^7$

Hence the coefficient of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ is ${}^{11}C_5a^6b^{-5}$. **Ans.**

Note that binomial coefficient of sixth term is ${}^{11}C_5$.

(b) In the expansion of $\left(ax-\frac{1}{bx^2}\right)^{11}$, general term is :

$$T_{r+1} = {}^{11}C_r(ax)^{11-r}\left(\frac{-1}{bx^2}\right)^r = (-1)^r\,{}^{11}C_r\frac{a^{11-r}}{b^r}.x^{11-3r}$$

putting $11 - 3r = -7$

$\therefore \quad 3r = 18 \quad \Rightarrow \quad r = 6$

$\therefore \quad T_7 = (-1)^6.\,{}^{11}C_6\frac{a^5}{b^6}.x^{-7}$

Hence the coefficient of x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$ is ${}^{11}C_6a^5b^{-6}$. **Ans.**

Also given :

Coefficient of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ = coefficient of x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$

$\Rightarrow \quad {}^{11}C_5a^6b^{-5} = {}^{11}C_6a^5b^{-6}$

$\Rightarrow \quad ab = 1 \qquad (\because \ {}^{11}C_5 = {}^{11}C_6)$

which is the required relation between a and b. **Ans.**

Illustration 5 : Find the number of rational terms in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$.

Solution : The general term in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$ is

$$T_{r+1} = {}^{1000}C_r\left(9^{\frac{1}{4}}\right)^{1000-r}\left(8^{\frac{1}{6}}\right)^r = {}^{1000}C_r\,3^{\frac{1000-r}{2}}\,2^{\frac{r}{2}}$$

The above term will be rational if exponents of 3 and 2 are integers

It means $\frac{1000-r}{2}$ and $\frac{r}{2}$ must be integers

The possible set of values of r is {0, 2, 4, ..., 1000}

Hence, number of rational terms is 501 **Ans.**

(b) Middle term :

The middle term(s) in the expansion of $(x + y)^n$ is (are) :

(i) If n is even, there is only one middle term which is given by $T_{(n+2)/2} = {}^nC_{n/2}.\,x^{n/2}.\,y^{n/2}$

(ii) If n is odd, there are two middle terms which are $T_{(n+1)/2}$ & $T_{[(n+1)/2]+1}$

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Important Note :

Middle term has greatest binomial coefficient and if there are 2 middle terms their coefficients will be equal.

$\Rightarrow$ ${}^nC_r$ will be maximum $\begin{cases} \text{When } r=\frac{n}{2} \text{ if n is even} \\ \text{When } r=\frac{n-1}{2} \text{ or } \frac{n+1}{2} \text{ if n is odd} \end{cases}$

$\Rightarrow$ The term containing greatest binomial coefficient will be middle term in the expansion of $(1 + x)^n$

Illustration 6 : Find the middle term in the expansion of $\left(3x-\frac{x^3}{6}\right)^9$

Solution : The number of terms in the expansion of $\left(3x-\frac{x^3}{6}\right)^9$ is 10 (even). So there are two middle terms.

i.e. $\left(\frac{9+1}{2}\right)^{th}$ and $\left(\frac{9+3}{2}\right)^{th}$ are two middle terms. They are given by T_5 and T_6

$\therefore \quad T_5 = T_{4+1} = {}^9C_4(3x)^5\left(-\frac{x^3}{6}\right)^4 = {}^9C_4 3^5x^5.\ \frac{x^{12}}{6^4} = \frac{9.8.7.6}{1.2.3.4}.\frac{3^5}{2^4.3^4}x^{17} = \frac{189}{8}x^{17}$

and $T_6=T_{5+1}={}^9C_5(3x)^4\left(-\frac{x^3}{6}\right)^5 = -{}^9C_4 3^4.x^4.\frac{x^{15}}{6^5} = \frac{-9.8.7.6}{1.2.3.4}.\frac{3^4}{2^5.3^5}x^{19} = -\frac{21}{16}x^{19}$ **Ans.**

(c) Term independent of x :

Term independent of x does not contain x ; Hence find the value of r for which the exponent of x is zero.

Illustration 7 : The term independent of x in $\left[\sqrt{\frac{x}{3}}+\sqrt{\left(\frac{3}{2x^2}\right)}\right]^{10}$ is -

(A) 1 (B) $\frac{5}{12}$ (C) ${}^{10}C_1$ (D) none of these

Solution : General term in the expansion is

${}^{10}C_r\left(\frac{x}{3}\right)^{\frac{r}{2}}\left(\frac{3}{2x^2}\right)^{\frac{10-r}{2}} = {}^{10}C_r x^{\frac{3r}{2}-10}.\frac{3^{5-r}}{2^{\frac{10-r}{2}}}$ For constant term, $\frac{3r}{2}=10 \Rightarrow r=\frac{20}{3}$

which is not an integer. Therefore, there will be no constant term. **Ans. (D)**

Do yourself - 2 :

(i) Find the 7th term of $\left(3x^2-\frac{1}{3}\right)^{10}$

(ii) Find the term independent of x in the expansion : $\left(2x^2-\frac{3}{x^3}\right)^{25}$

(iii) Find the middle term in the expansion of : (a) $\left(\frac{2x}{3}-\frac{3}{2x}\right)^6$ (b) $\left(2x^2-\frac{1}{x}\right)^7$

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(d) Numerically greatest term :

Let numerically greatest term in the expansion of $(a + b)^n$ be T_{r+1}.

$$\Rightarrow \quad \begin{cases} |T_{r+1}| \geq |T_r| \\ |T_{r+1}| \geq |T_{r+2}| \end{cases} \text{ where } T_{r+1} = {}^nC_r a^{n-r} b^r$$

Solving above inequalities we get $\dfrac{n+1}{1+\left|\dfrac{a}{b}\right|} - 1 \leq r \leq \dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$

Case I : When $\dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$ is an integer equal to m, then T_m and T_{m+1} will be numerically greatest term.

Case II : When $\dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$ is not an integer and its integral part is m, then T_{m+1} will be the numerically greatest term.

Illustration 8 : Find numerically greatest term in the expansion of $(3 - 5x)^{11}$ when $x = \dfrac{1}{5}$

Solution : Using $\dfrac{n+1}{1+\left|\dfrac{a}{b}\right|} - 1 \leq r \leq \dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$

$$\frac{11+1}{1+\left|\dfrac{3}{-5x}\right|} - 1 \leq r \leq \frac{11+1}{1+\left|\dfrac{3}{-5x}\right|}$$

solving we get $2 \leq r \leq 3$

$\therefore$ r = 2, 3

so, the greatest terms are T_{2+1} and T_{3+1}.

$\therefore$ Greatest term (when r = 2)

$T_3 = {}^{11}C_2 . 3^9 (-5x)^2 = 55.3^9 = T_4$

From above we say that the value of both greatest terms are equal. **Ans.**

Illustration 9 : Given T_3 in the expansion of $(1 - 3x)^6$ has maximum numerical value. Find the range of 'x'.

Solution : Using $\dfrac{n+1}{1+\left|\dfrac{a}{b}\right|} - 1 \leq r \leq \dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$

$$\frac{6+1}{1+\left|\dfrac{1}{-3x}\right|} - 1 \leq 2 \leq \frac{7}{1+\left|\dfrac{1}{-3x}\right|}$$

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Let $|x| = t$

$$\frac{21t}{3t+1} - 1 \le 2 \le \frac{21t}{3t+1}$$

$$\begin{cases} \dfrac{21t}{3t+1} \le 3 \\ \dfrac{21t}{3t+1} \ge 2 \end{cases} \Rightarrow \begin{cases} \dfrac{4t-1}{3t+1} \le 0 \Rightarrow t \in \left[-\dfrac{1}{3}, \dfrac{1}{4}\right] \\ \dfrac{15t-2}{3t+1} \ge 0 \Rightarrow t \in \left(-\infty, -\dfrac{1}{3}\right] \cup \left[\dfrac{2}{15}, \infty\right) \end{cases}$$

Common solution $t \in \left[\dfrac{2}{15}, \dfrac{1}{4}\right] \Rightarrow x \in \left[-\dfrac{1}{4}, -\dfrac{2}{15}\right] \cup \left[\dfrac{2}{15}, \dfrac{1}{4}\right]$

Do yourself -3 :

(i) Find the numerically greatest term in the expansion of $(3 - 2x)^9$, when $x = 1$.

(ii) In the expansion of $\left(\dfrac{1}{2} + \dfrac{2x}{3}\right)^n$ when $x = -\dfrac{1}{2}$, it is known that 3rd term is the greatest term. Find the possible integral values of n.

4. PROPERTIES OF BINOMIAL COEFFICIENTS :

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots\ldots + C_nx^n = \sum_{r=0}^{n} {}^nC_r r^r \; ; n \in N \qquad \ldots(i)$$

where $C_0, C_1, C_2, \ldots\ldots\ldots\ldots C_n$ are called combinatorial (binomial) coefficients.

(a) The sum of all the binomial coefficients is 2^n.

Put $x = 1$, in (i) we get

$$C_0 + C_1 + C_2 + \ldots\ldots\ldots\ldots + C_n = 2^n \Rightarrow \sum_{r=0}^{n} {}^nC_r = 0 \qquad \ldots(ii)$$

(b) Put $x = -1$ in (i) we get

$$C_0 - C_1 + C_2 - C_3 \ldots\ldots\ldots\ldots + C_n = 0 \Rightarrow \sum_{r=0}^{n} (-1)^r \, {}^nC_r = 0 \qquad \ldots(iii)$$

(c) The sum of the binomial coefficients at odd position is equal to the sum of the binomial coefficients at even position and each is equal to 2^{n-1}.

From (ii) & (iii), $C_0 + C_2 + C_4 + \ldots\ldots\ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots\ldots = 2^{n-1}$

(d) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(e) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$

(f) ${}^nC_r = \dfrac{n}{r} \, {}^{n-1}C_{r-1} = \dfrac{n}{r} \cdot \dfrac{n-1}{r-1} \, {}^{n-2}C_{r-2} = \ldots\ldots\ldots = \dfrac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r(r-1)(r-2)\ldots\ldots\ldots 1}$

(g) ${}^nC_r = \dfrac{r+1}{n+1} \cdot {}^{n+1}C_{r+1}$

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Illustration 10 : Prove that : ${}^{25}C_{10} + {}^{24}C_{10} + ... + {}^{10}C_{10} = {}^{26}C_{11}$

Solution : LHS $= {}^{10}C_{10} + {}^{11}C_{10} + {}^{12}C_{10} + ... + {}^{25}C_{10}$

$\Rightarrow \quad {}^{11}C_{11} + {}^{11}C_{10} + {}^{12}C_{10} + ... + {}^{25}C_{10}$

$\Rightarrow \quad {}^{12}C_{11} + {}^{12}C_{10} + ... + {}^{25}C_{10}$

$\Rightarrow \quad {}^{13}C_{11} + {}^{13}C_{10} + ... {}^{25}C_{10}$

and so on. $\therefore$ LHS $= {}^{26}C_{11}$

Aliter :

LHS = coefficient of x^{10} in $\{(1 + x)^{10} + (1 + x)^{11} + ... (1+ x)^{25}\}$

$\Rightarrow$ coefficient of x^{10} in $\left[(1+x)^{10}\dfrac{\{1+x\}^{16}-1}{1+x-1}\right]$

$\Rightarrow$ coefficient of x^{10} in $\dfrac{\left[(1+x)^{26}-(1+x)^{10}\right]}{x}$

$\Rightarrow$ coefficient of x^{11} in $\left[(1+x)^{26}-(1+x)^{10}\right] = {}^{26}C_{11} - 0 = {}^{26}C_{11}$

Illustration 11 : A student is allowed to select at most n books from a collection of (2n + 1) books. If the total number of ways in which he can select books is 63, find the value of n.

Solution : Given student selects at most n books from a collection of (2n + 1) books. It means that he selects one book or two books or three books or ..... or n books. Hence, by the given condition-

${}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_n = 63$ ...(i)

But we know that

${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_{2n+1} = 2^{2n+1}$ ...(ii)

Since ${}^{2n+1}C_0 = {}^{2n+1}C_{2n+1} = 1$, equation (ii) can also be written as

$2 + ({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_n) +$

$({}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + {}^{2n+1}C_{n+3} + ... + {}^{2n+1}C_{2n-1} + {}^{2n+1}C_{2n}) = 2^{2n+1}$

$\Rightarrow 2 + ({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_n)$

$+ ({}^{2n+1}C_n + {}^{2n+1}C_{n-1} + ... + {}^{2n+1}C_2 + {}^{2n+1}C_1) = 2^{2n+1}$

$(\because {}^{2n+1}C_r = {}^{2n+1}C_{2n+1-r})$

$\Rightarrow 2 + 2\,({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_n) = 2^{2n+1}$ [from (i)]

$\Rightarrow 2 + 2.63 = 2^{2n+1}$ $\qquad \Rightarrow 1 + 63 = 2^{2n}$

$\Rightarrow 64 = 2^{2n} \Rightarrow 2^6 = 2^{2n}$ $\qquad \therefore 2n = 6$

Hence, n = 3. **Ans.**

Illustration 12 : Prove that :

(i) $C_1 + 2C_2 + 3C_3 + ... + nC_n = n \,.\, 2^{n-1}$

(ii) $C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + ... + \dfrac{C_n}{n+1} = \dfrac{2^{n+1}-1}{n+1}$

Solution : (i) L.H.S. $= \sum_{r=1}^{n} r.\,{}^{n}C_r = \sum_{r=1}^{n} r.\dfrac{n}{r}.\,{}^{n-1}C_{r-1}$

$= n\sum_{r=1}^{n} {}^{n-1}C_{r-1} = n.\left[{}^{n-1}C_0 + {}^{n-1}C_1 + ... + {}^{n-1}C_{n-1}\right]$

$= n \,.\, 2^{n-1}$

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Aliter : (Using method of differentiation)

$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$ .....(A)

Differentiating (A), we get

$n(1 + x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + ... + n.C_nx^{n-1}$.

Put x = 1,

$C_1 + 2C_2 + 3C_3 + ... + n.C_n = n.2^{n-1}$

(ii) L.H.S. $= \sum_{r=0}^{n} \frac{C_r}{r+1} = \frac{1}{n+1}\sum_{r=0}^{n} \frac{n+1}{r+1}\,{}^nC_r$

$= \frac{1}{n+1}\sum_{r=0}^{n} {}^{n+1}C_{r+1} = \frac{1}{n+1}\left[{}^{n+1}C_1 + {}^{n+1}C_2 + ... + {}^{n+1}C_{n+1}\right] = \frac{1}{n+1}\left[2^{n+1} - 1\right]$

Aliter : (Using method of integration)

Integrating (A), we get

$\frac{(1+x)^{n+1}}{n+1} + C = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + ... + \frac{C_nx^{n+1}}{n+1}$ (where C is a constant)

Put x = 0, we get, $C = -\frac{1}{n+1}$

$\therefore \quad \frac{(1+x)^{n+1} - 1}{n+1} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + ... + \frac{C_nx^{n+1}}{n+1}$

Put x = 1, we get $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... \frac{C_n}{n+1} = \frac{2^{n+1} - 1}{n+1}$

Put x = –1, we get $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - ..... = \frac{1}{n+1}$

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Illustration 13 : If $(1 + x)^n = \sum_{r=0}^{n} {}^nC_rx^r$, then prove that $C_1^2 + 2.C_2^2 + 3.C_3^2 + ... + n.C_n^2 = \frac{(2n-1)!}{((n-1)!)^2}$

Solution : $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_2x^3 + ... + C_n\,x^n$ .....(i)

Differentiating both the sides, w.r.t. x, we get

$n(1 + x)^{n-1} = C_1 + 2C_2x + 3C_2x^2 + ... + n.C_nx^{n-1}$ .....(ii)

also, we have

$(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_n$ .....(iii)

Multiplying (ii) & (iii), we get

$(C_1 + 2C_2x + 3C_3x^2 + ... + C_nx^{n-1})(C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ...+ C_n) = n(1 + x)^{2n-1}$

Equating the coefficients of x^{n-1}, we get

$C_1^2 + 2C_2^2 + 3C_3^2 + ... + n.C_n^2 = n.\,{}^{2n-1}C_{n-1} = \frac{(2n-1)!}{((n-1)!)^2}$ **Ans.**

Illustration 14 : Prove that : $C_0 - 3C_1 + 5C_2 - .....(-1)^n(2n + 1)C_n = 0$

Solution : $T_r = (-1)^r(2r + 1)^nC_r = 2(-1)^r r\,.\,{}^nC_r + (-1)^r\,{}^nC_r$

$\Sigma T_r = 2\sum_{r=1}^{n}(-1)^r.r.\frac{n}{r}.{}^{n-1}C_{r-1} + \sum_{r=0}^{n}(-1)^r\,{}^nC_r = 2\sum_{r=1}^{n}(-1)^r.{}^{n-1}C_{r-1} + \sum_{r=0}^{n}(-1)^r\,.\,{}^nC_r$

$= 2\left[{}^{n-1}C_0 - {}^{n-1}C_1 + ...\right] + \left[{}^nC_0 - {}^nC_1 + ...\right] = 0$

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Illustration 15 : Prove that $({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - .... + (-1)^n ({}^{2n}C_{2n})^2 = (-1)^n.\ {}^{2n}C_n$

Solution : $(1 - x)^{2n} = {}^{2n}C_0 - {}^{2n}C_1x + {}^{2n}C_2x^2 - ....+(-1)^n\ {}^{2n}C_{2n}x^{2n}$ ....(i)

and $(x + 1)^{2n} = {}^{2n}C_0x^{2n} + {}^{2n}C_1x^{2n-1} + {}^{2n}C_2x^{2n-2} +...+{}^{2n}C_{2n}$ ....(ii)

Multiplying (i) and (ii), we get

$(x^2 - 1)^{2n} = ({}^{2n}C_0 - {}^{2n}C_1x +....+ (-1)^n\ {}^{2n}C_{2n}x^{2n})\times({}^{2n}C_0x^{2n} + {}^{2n}C_1x^{2n-1} +.... + {}^{2n}C_{2n})$ ....(iii)

Now, coefficient of x^{2n} in R.H.S.

$= ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - ..... + (-1)^n ({}^{2n}C_{2n})^2$

$\because$ General term in L.H.S., $T_{r+1} = {}^{2n}C_r(x^2)^{2n-r}(-1)^r$

Putting $2(2n - r) = 2n$

$\therefore$ $r = n$

$\therefore$ $T_{n+1} = {}^{2n}C_nx^{2n}(-1)^n$

Hence coeffiecient of x^{2n} in L.H.S. $= (-1)^n.{}^{2n}C_n$

But (iii) is an identity, therefore coefficient of x^{2n} in R.H.S. = coefficient of x^{2n} in L.H.S.

$\Rightarrow$ $({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - .... + (-1)^n ({}^{2n}C_{2n})^2 = (-1)^n.\ {}^{2n}C_n$

Illustration 16 : Prove that : ${}^nC_0.{}^{2n}C_n - {}^nC_1.\ {}^{2n-2}Cn_n + {}^nC_2.{}^{2n-4}Cn_n + ... = 2^n$

Solution : L.H.S. = Coefficient of x^n in $[{}^nC_0(1 + x)^{2n} - {}^nC_1(1 + x)^{2n-2}$ .....]

= Coefficient of x^n in $[(1 + x)^2 - 1]^n$

= Coefficient of x^n in $x^n(x + 2)^n = 2^n$

Illustration 17 : If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ then show that the sum of the products of the C_i's taken two at a time represented by : $\sum\sum_{0\le i<j\le n} C_iC_j$ is equal to $2^{2n-1} - \dfrac{2n!}{2.n!n!}$

Solution : Since $(C_0 + C_1 + C_2 +.....+ C_{n-1} + C_n)^2$

$= C_0^2 + C_1^2 + C_2^2 + ... + C_{n-1}^2 + C_n^2 + 2(C_0C_1 + C_0C_2 + C_0C_3 + ... + C_0C_n + C_1C_2 + C_1C_3 +.....$

$+ C_1C_n + C_2C_3 + C_2C_4 +...+C_2C_n +.....+C_{n-1}C_n)$

$(2^n)^2 = {}^{2n}C_n + 2\sum\sum_{0\le i<j\le n} C_iC_j$

Hence $\sum\sum_{0\le i<j\le n} C_iC_j = 2^{2n-1} - \dfrac{2n!}{2.n!n!}$ **Ans.**

Illustration 18 : If $(1+x)^n = C_0 + C_1x + C_2x^2 +....+ C_nx^n$ then prove that $\sum\sum_{0\le i<j\le n}\left(C_i + C_j\right)^2 = (n-1)^{2n}C_n + 2^{2n}$

Solution : L.H.S. $\sum\sum_{0\le i<j\le n}\left(C_i + C_j\right)^2$

$= (C_0 + C_1)^2 + (C_0 + C_2)^2 +....+ (C_0 + C_n)^2 + (C_1 + C_2)^2 + (C_1 + C_3)^2 +....$

$+ (C_1 + C_n)^2 + (C_2 + C_3)^2 + (C_2 + C_4)^2 +... + (C_2 + C_n)^2 +....+ (C_{n-1} + C_n)^2$

$= n(C_0^2 + C_1^2 + C_2^2 + ... + C_n^2) + 2\sum\sum_{0\le i<j\le n} C_iC_j$

$= n.{}^{2n}C_n + 2.\left\{2^{2n-1} - \dfrac{2n!}{2.n!n!}\right\}$ {from Illustration 17}

$= n.{}^{2n}C_n + 2^{2n} - {}^{2n}C_n = (n - 1)\ .\ {}^{2n}C_n + 2^{2n}$ = R.H.S.

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Do yourself - 4 :

(i) $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+.....+\binom{n}{n}=$

(A) 2^{n-1} (B) ${}^{2n}C_n$ (C) 2^n (D) 2^{n+1}

(ii) If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, $n \in N$. Prove that

(a) $3C_0 - 8C_1 + 13C_2 - 18C_3 + ...$ upto $(n+1)$ terms $= 0$, if $n \geq 2$.

(b) $2C_0 + 2^2\frac{C_1}{2} + 2^3\frac{C_2}{3} + 2^4\frac{C_3}{4} + ... + 2^{n+1}\frac{C_n}{n+1} = \frac{3^{n+1}-1}{n+1}$

(c) $C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{3} + ... + \frac{C_n^2}{n+1} = \frac{(2n+1)!}{((n+1)!)^2}$

5. MULTINOMIAL THEOREM :

Using binomial theorem, we have $(x+a)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r}a^r$, $n \in N$

$= \sum_{r=0}^{n} \frac{n!}{(n-r)!r!}x^{n-r}a^r = \sum_{r+s=n} \frac{n!}{r!s!}x^s a^r$, where $s + r = n$

This result can be generalized in the following form.

$(x_1 + x_2 + ... + x_k)^n = \sum_{r_1+r_2+....+r_k=n} \frac{n!}{r_1!r_2!....r_k!}x_1^{r_1}x_2^{r_2}....x_k^{r_k}$

The general term in the above expansion $\frac{n!}{r_1!r_2!r_3!.....r_k!}.x_1^{r_1}x_2^{r_2}x_3^{r_3}.....x_k^{r_k}$

The number of terms in the above expansion is equal to the number of non-negative integral solution of the equation $r_1 + r_2 + ... + r_k = n$ because each solution of this equation gives a term in the above expansion. The number of such solutions is ${}^{n+k-1}C_{k-1}$

Particular cases :

(i) $(x+y+z)^n = \sum_{r+s+t=n} \frac{n!}{r!s!t!}x^r y^s z^t$

The above expansion has ${}^{n+3-1}C_{3-1} = {}^{n+2}C_2$ terms

(ii) $(x+y+z+u)^n = \sum_{p+q+r+s=n} \frac{n!}{p!q!r!s!}x^p y^q z^r u^s$

There are ${}^{n+4-1}C_{4-1} = {}^{n+3}C_3$ terms in the above expansion.

Illustration 19 : Find the coefficient of $x^2y^3z^4w$ in the expansion of $(x-y-z+w)^{10}$

Solution : $(x-y-z+w)^{10} = \sum_{p+q+r+s=10} \frac{n!}{p!q!r!s!}(x)^p(-y)^q(-z)^r(w)^s$

We want to get $x^2y^3z^4w$ this implies that $p = 2, q = 3, r = 4, s = 1$

$\therefore$ Coefficient of $x^2y^3z^4w$ is $\frac{10!}{2!\,3!\,4!\,1!}(-1)^3(-1)^4 = -12600$ **Ans.**

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Illustration 20 : Find the total number of terms in the expansion of $(1 + x + y)^{10}$ and coefficient of x^2y^3.

Solution : Total number of terms $= {}^{10+3-1}C_{3-1} = {}^{12}C_2 = 66$

Coefficient of $x^2y^3 = \dfrac{10!}{2! \times 3! \times 5!} = 2520$ **Ans.**

Illustration 21 : Find the coefficient of x^5 in the expansion of $(2 - x + 3x^2)^6$.

Solution : The general term in the expansion of $(2 - x + 3x^2)^6 = \dfrac{6!}{r!s!t!} 2^r (-x)^s (3x^2)^t$,

where $r + s + t = 6$.

$= \dfrac{6!}{r!s!t!} 2^r \times (-1)^s \times (3)^t \times x^{s+2t}$

For the coefficient of x^5, we must have $s + 2t = 5$.

But, $r + s + t = 6$,

$\therefore$ $s = 5 - 2t$ and $r = 1 + t$, where $0 \le r, s, t \le 6$.

Now $t = 0 \Rightarrow r = 1, s = 5$.

$t = 1 \Rightarrow r = 2, s = 3$.

$t = 2 \Rightarrow r = 3, s = 1$.

Thus, there are three terms containing x^5 and coefficient of x^5

$= \dfrac{6!}{1!\,5!\,0!} \times 2^1 \times (-1)^5 \times 3^0 + \dfrac{6!}{2!\,3!\,1!} \times 2^2 \times (-1)^3 \times 3^1 + \dfrac{6!}{3!\,1!\,2!} \times 2^3 \times (-1)^1 \times 3^2$

$= -12 - 720 - 4320 = -5052.$ **Ans.**

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Illustration 22 : If $(1+x+x^2)^n = \sum_{r=0}^{2n} a_r x^r$, then prove that (a) $a_r = a_{2n-r}$ (b) $\sum_{r=0}^{n-1} a_r = \dfrac{1}{2}(3^n - a_n)$

Solution : (a) We have

$$\left(1 + x + x^2\right)^n = \sum_{r=0}^{2n} a_r x^r \qquad ....(A)$$

Replace x by $\dfrac{1}{x}$

$$\therefore \quad \left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^n = \sum_{r=0}^{2n} a_r \left(\frac{1}{x}\right)^r$$

$$\Rightarrow \quad \left(x^2 + x + 1\right)^n = \sum_{r=0}^{2n} a_r x^{2n-r}$$

$$\Rightarrow \quad \sum_{r=0}^{2n} a_r x^r = \sum_{r=0}^{2n} a_r x^{2n-r} \qquad \{\text{Using (A)}\}$$

Equating the coefficient of x^{2n-r} on both sides, we get

$a_{2n-r} = a_r$ for $0 \le r \le 2n$.

Hence $a_r = a_{2n-r}$.

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(b) Putting x=1 in given series, then

$a_0 + a_1 + a_2 + ...+ a_{2n} = (1+1+1)^n$

$a_0 + a_1 + a_2 + ...+ a_{2n} = 3^n$ ....(1)

But $a_r = a_{2n-r}$ for $0 \le r \le 2n$

$\therefore$ series (1) reduces to

$2(a_0 + a_1 + a_2 + ...+ a_{n-1}) + a_n = 3^n$.

$\therefore$ $a_0 + a_1 + a_2 + ...+ a_{n-1} = \frac{1}{2}(3^n - a_n)$

Do yourself - 5 :

(i) Find the coefficient of x^2y^5 in the expansion of $(3 + 2x - y)^{10}$.

6. APPLICATION OF BINOMIAL THEOREM :

Illustration 23 : If $\left(6\sqrt{6}+14\right)^{2n+1}$ = [N] + F and F = N – [N]; where [.] denotes greatest integer function, then NF is equal to

(A) 20^{2n+1} (B) an even integer (C) odd integer (D) 40^{2n+1}

Solution : Since $\left(6\sqrt{6}+14\right)^{2n+1}$ = [N] + F

Let us assume that f = $\left(6\sqrt{6}-14\right)^{2n+1}$; where $0 \le f < 1$.

Now, [N] + F – f = $\left(6\sqrt{6}+14\right)^{2n+1} - \left(6\sqrt{6}-14\right)^{2n+1}$

$= 2\left[{}^{2n+1}C_1\left(6\sqrt{6}\right)^{2n}(14) + {}^{2n+1}C_3\left(6\sqrt{6}\right)^{2n-2}(14)^3 + ...\right]$

$\Rightarrow$ [N] + F – f = even integer.

Now $0 < F < 1$ and $0 < f < 1$

so $-1 < F - f < 1$ and F – f is an integer so it can only be zero

Thus NF = $\left(6\sqrt{6}+14\right)^{2n+1}\left(6\sqrt{6}-14\right)^{2n+1} = 20^{2n+1}$. **Ans. (A,B)**

Illustration 24 : Find the last three digits in 11^{50}.

Solution : Expansion of $(10 + 1)^{50} = {}^{50}C_0 10^{50} + {}^{50}C_1 10^{49} + ... + {}^{50}C_{48}10^2 + {}^{50}C_{49}10 + {}^{50}C_{50}$

$= \underbrace{{}^{50}C_0 10^{50} + {}^{50}C_1 10^{49} + ... + {}^{50}C_{47}10^3}_{1000K} + 49 \times 25 \times 100 + 500 + 1$

$\Rightarrow$ 1000 K + 123001

$\Rightarrow$ Last 3 digits are 001.

Illustration 25 : Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

Solution : When 2222 is divided by 7 it leaves a remainder 3.

So adding & subtracting 3^{5555}, we get :

$E = \underbrace{2222^{5555} - 3^{5555}}_{E_1} + \underbrace{3^{5555} + 5555^{2222}}_{E_2}$

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For E_1 : Now since 2222–3 = 2219 is divisible by 7, therefore E_1 is divisible by 7
($\because$ $x^n - a^n$ is divisible by x –a)
For E_2 : 5555 when devided by 7 leaves remainder 4.
So adding and subtracting 4^{2222}, we get :

$$E_2 = 3^{5555} + 4^{2222} + 5555^{2222} - 4^{2222}$$
$$= (243)^{1111} + (16)^{1111} + (5555)^{2222} - 4^{2222}$$

Again $(243)^{1111} + 16^{1111}$ and $(5555)^{2222} - 4^{2222}$ are divisible by 7
($\because$ $x^n + a^n$ is divisible by x + a when n is odd)
Hence $2222^{5555} + 5555^{2222}$ is divisible by 7.

Do yourself - 6 :

(i) Prove that $5^{25} - 3^{25}$ is divisible by 2.

(ii) Find the remainder when the number 9^{100} is divided by 8.

(iii) Find last three digits in 19^{100}.

(iv) Let $R = (8+3\sqrt{7})^{20}$ and [.] denotes greatest integer function, then prove that :

(a) [R] is odd (b) $R - [R] = 1 - \dfrac{1}{(8+3\sqrt{7})^{20}}$

(v) Find the digit at unit's place in the number $17^{1995} + 11^{1995} - 7^{1995}$.

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7. BINOMIAL THEOREM FOR NEGATIVE OR FRACTIONAL INDICES :

If $n \in Q$, then $(1 + x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots\infty$ provided $|x| < 1$.

Note :

(i) When the index n is a positive integer the number of terms in the expansion of $(1+x)^n$ is finite i.e. (n+1) & the coefficient of successive terms are : ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots {}^nC_n$

(ii) When the index is other than a positive integer such as negative integer or fraction, the number of terms in the expansion of $(1+x)^n$ is infinite and the symbol ${}^nC_r$ cannot be used to denote the coefficient of the general term.

(iii) Following expansion should be remembered ($|x| < 1$).

(a) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots\ \infty$

(b) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots\ \infty$

(c) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ \infty$

(d) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ \infty$

(e) $(1 + x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots\ldots + \dfrac{(-1)^r(r+1)(r+2)}{2!}x^r + \ldots\ldots$

(f) $(1 - x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots\ldots + \dfrac{(r+1)(r+2)}{2!}x^r + \ldots\ldots$

(iv) The expansions in ascending powers of x are only valid if x is 'small'. If x is large i.e. $|x| > 1$ then we may find it convenient to expand in powers of 1/x, which then will be small.

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8. APPROXIMATIONS :

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 .....$$

If **x < 1**, the terms of the above expansion go on decreasing and if x be very small, a stage may be reached when we may neglect the terms containing higher powers of x in the expansion. Thus, if x be so small that its square and higher powers may be neglected then $(1 + x)^n = 1 + nx$, approximately. This is an approximate value of $(1 + x)^n$

Illustration 26 : If x is so small such that its square and higher powers may be neglected then find the approximate value of $\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}}$

Solution : $\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}} = \frac{1-\frac{3}{2}x+1-\frac{5x}{3}}{2\left(1+\frac{x}{4}\right)^{1/2}} = \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1+\frac{x}{4}\right)^{-1/2} = \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1-\frac{x}{8}\right)$

$= \frac{1}{2}\left(2-\frac{x}{4}-\frac{19}{6}x\right) = 1-\frac{x}{8}-\frac{19}{12}x = 1-\frac{41}{24}x$ **Ans.**

Illustration 27 : The value of cube root of 1001 upto five decimal places is –

(A) 10.03333 (B) 10.00333 (C) 10.00033 (D) none of these

Solution : $(1001)^{1/3} = (1000+1)^{1/3} = 10\left(1+\frac{1}{1000}\right)^{1/3} = 10\left\{1+\frac{1}{3}\cdot\frac{1}{1000}+\frac{1/3(1/3-1)}{2!}\frac{1}{1000^2}+.....\right\}$

$= 10\{1 + 0.0003333 - 0.00000011 + ...\} = 10.00333$ **Ans. (B)**

Illustration 28 : The sum of $1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+....\infty$ is -

(A) $\sqrt{2}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\sqrt{3}$ (D) $2^{3/2}$

Solution : Comparing with $1 + nx + \frac{n(n-1)}{2!}x^2 + ...$

$nx = 1/4$ .....(i)

and $\frac{n(n-1)x^2}{2!} = \frac{1.3}{4.8}$

or $\frac{nx(nx-x)}{2!} = \frac{3}{32} \Rightarrow \frac{1}{4}\left(\frac{1}{4}-x\right) = \frac{3}{16}$ (by (i))

$\Rightarrow \left(\frac{1}{4}-x\right) = \frac{3}{4} \Rightarrow x = \frac{1}{4}-\frac{3}{4} = -\frac{1}{2}$ .....(ii)

putting the value of x in (i)

$n(-1/2) = 1/4 \Rightarrow n = -1/2$

$\therefore$ sum of series $= (1 + x)^n = (1 - 1/2)^{-1/2} = (1/2)^{-1/2} = \sqrt{2}$ **Ans. (A)**

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9. EXPONENTIAL SERIES :

(a) e is an irrational number lying between 2.7 & 2.8. Its value correct upto 10 places of decimal is 2.7182818284.

(b) Logarithms to the base 'e' are known as the Napierian system, so named after Napier, their inventor. They are also called **Natural Logarithm.**

(c) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\infty$; where x may be any real or complex number & $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

(d) $a^x = 1 + \frac{x}{1!}\ell na + \frac{x^2}{2!}\ell n^2 a + \frac{x^3}{3!}\ell n^3 a + ...\infty$, where $a > 0$

(e) $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\infty$

10. LOGARITHMIC SERIES :

(a) $\ell n\,(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$, where $-1 < x \le 1$

(b) $\ell n\,(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$, where $-1 \le x < 1$

Remember : (i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...\infty = \ell n\,2$ (ii) $e^{\ln x} = x$; for all $x > 0$

(iii) $\ell n2 = 0.693$ (iv) $\ell n10 = 2.303$

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EXERCISE (O-1)

[SINGLE CORRECT CHOICE TYPE]

1. If the coefficients of x^7 & x^8 in the expansion of $\left[2+\frac{x}{3}\right]^n$ are equal, then the value of n is :

(A) 15 (B) 45 (C) 55 (D) 56

BT0001

2. Set of value of r for which, ${}^{18}C_{r-2} + 2.\ {}^{18}C_{r-1} + {}^{18}C_r \geq {}^{20}C_{13}$ contains :

(A) 4 element (B) 5 elements (C) 7 elements (D) 10 elements

BT0002

3. If the constant term of the binomial expansion $\left(2x-\frac{1}{x}\right)^n$ is – 160, then n is equal to -

(A) 4 (B) 6 (C) 8 (D) 10

BT0003

4. The coefficient of x^{49} in the expansion of $(x-1)\left(x-\frac{1}{2}\right)\left(x-\frac{1}{2^2}\right)....\left(x-\frac{1}{2^{49}}\right)$ is equal to -

(A) $-2\left(1-\frac{1}{2^{50}}\right)$ (B) +ve coefficient of x

(C) –ve coefficient of x (D) $-2\left(1-\frac{1}{2^{49}}\right)$

BT0004

5. The largest real value for x such that $\sum_{k=0}^{4}\left(\frac{5^{4-k}}{(4-k)!}\right)\left(\frac{x^k}{k!}\right)=\frac{8}{3}$ is -

(A) $2\sqrt{2}-5$ (B) $2\sqrt{2}+5$ (C) $-2\sqrt{2}-5$ (D) $-2\sqrt{2}+5$

BT0005

6. The expression $[x + (x^3 - 1)^{1/2}]^5 + [x - (x^3 - 1)^{1/2}]^5$ is a polynomial of degree

(A) 5 (B) 6 (C) 7 (D) 8

BT0006

7. Number of rational terms in the expansion of $\left(\sqrt{2}+\sqrt[4]{3}\right)^{100}$ is :

(A) 25 (B) 26 (C) 27 (D) 28

BT0007

8. Given $(1 - 2x + 5x^2 - 10x^3)(1 + x)^n = 1 + a_1x + a_2x^2 + ...$ and that $a_1^2 = 2a_2$ then the value of n is-

(A) 6 (B) 2 (C) 5 (D) 3

BT0008

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9. The sum of the co-efficients of all the even powers of x in the expansion of $(2x^2 - 3x + 1)^{11}$ is -

(A) 2.6^{10} (B) 3.6^{10} (C) 6^{11} (D) none

BT0009

10. Co-efficient of α^t in the expansion of ,

$(\alpha + p)^{m-1} + (\alpha + p)^{m-2}(\alpha + q) + (\alpha + p)^{m-3}(\alpha + q)^2 + ...(\alpha + q)^{m-1}$ where $\alpha \neq -q$ and $p \neq q$ is :

(A) $\frac{{}^mC_t\left(p^t - q^t\right)}{p-q}$ (B) $\frac{{}^mC_t\left(p^{m-t} - q^{m-t}\right)}{p-q}$ (C) $\frac{{}^mC_t\left(p^t + q^t\right)}{p-q}$ (D) $\frac{{}^mC_t\left(p^{m-t} + q^{m-t}\right)}{p-q}$

BT0010

11. Let $\binom{n}{k}$ represents the combination of 'n' things taken 'k' at a time, then the value of the sum

$\binom{99}{97} + \binom{98}{96} + \binom{97}{95} + ... + \binom{3}{1} + \binom{2}{0}$ equals -

(A) $\binom{99}{97}$ (B) $\binom{100}{98}$ (C) $\binom{99}{98}$ (D) $\binom{100}{97}$

BT0011

[COMPREHENSION TYPE]

Paragraph for question nos. 12 to 14

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If $n \in N$ and if $(1 + 4x + 4x^2)^n = \sum_{r=0}^{2n} a_r x^r$, where $a_0, a_1, a_2, ..., a_{2n}$ are real numbers.

12. The value of $2\sum_{r=0}^{n} a_{2r}$, is

(A) $9^n - 1$ (B) $9^n + 1$ (C) $9^n - 2$ (D) $9^n + 2$

BT0012

13. The value of $2\sum_{r=1}^{n} a_{2r-1}$, is-

(A) $9^n - 1$ (B) $9^n + 1$ (C) $9^n - 2$ (D) $9^n + 2$

BT0012

14. The value of a_{2n-1} is -

(A) 2^{2n} (B) $n.\ 2^{2n}$ (C) $(n - 1)2^{2n}$ (D) $(n + 1)2^{2n}$

BT0012

15. If $n \in N$ & n is even, then $\frac{1}{1.(n-1)!} + \frac{1}{3!.(n-3)!} + \frac{1}{5!.(n-5)!} + ... + \frac{1}{(n-1)!1!} =$

(A) 2^n (B) $\frac{2^{n-1}}{n!}$ (C) $2^n n!$ (D) none of these

BT0013

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EXERCISE (O-2)

[ONE OR MORE THAN ONE CORRECT CHOICE TYPE]

1. If it is known that the third term of the binomial expansion $\left(x + x^{\log_{10} x}\right)^5$ is 10^6 then x is equal to-

(A) 10 (B) $10^{-5/2}$ (C) 100 (D) 5

BT0014

2. In the expansion of $\left(x^3 + 3.2^{-\log_{\sqrt{2}} \sqrt{x^3}}\right)^{11}$

(A) there appears a term with the power x^2
(B) there does not appear a term with the power x^2
(C) there appears a term with the power x^{-3}
(D) the ratio of the co-efficient of x^3 to that of x^{-3} is 1/3

BT0015

3. In the expansion of $\left(\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}}\right)^{10}$, the term which does not contain x is-

(A) ${}^{11}C_4 - {}^{10}C_3$ (B) ${}^{10}C_7$ (C) ${}^{10}C_4$ (D) ${}^{11}C_5 - {}^{10}C_5$

BT0016

4. Let $(1 + x^2)^2 (1 + x)^n = A_0 + A_1x + A_2x^2 + ...$ If A_0, A_1, A_2 are in A.P. then the value of n is-

(A) 2 (B) 3 (C) 5 (D) 7

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BT0017

5. Consider $E = \left(\sqrt[8]{x} + \sqrt[5]{y}\right)^z = I + f, 0 \le f < 1$

(A) If x = 5, y = 2, z = 100, then number of irrational terms in expansion of E is 98
(B) If x = 5, y = 2, z = 100, then number of rational terms in expansion of E is 4
(C) If x = 16, y = 1 & z = 6, then I = 197
(D) If x = 16, y = 1 & z = 6, then $f = \left(\sqrt{2} - 1\right)^6$

BT0018

6. Greatest term in the binomial expansion of $(a + 2x)^9$ when a = 1 & $x = \frac{1}{3}$ is :

(A) 3rd & 4th (B) 4th & 5th (C) only 4th (D) only 5th

BT0019

7. Let $(5 + 2\sqrt{6})^n = p + f$ where $n, p \in N$ and $0 < f < 1$ then the value of $f^2 - f + pf - p$ is -

(A) a natural number (B) a negative integer (C) a prime number (D) are irrational number

BT0020

8. If $(9 + \sqrt{80})^n = I + f$ where I, n are integers and $0 < f < 1$, then -

(A) I is an odd integer
(B) I is an even integer
(C) (I + f) (1 – f) = 1
(D) $1 - f = (9 - \sqrt{80})^n$

BT0021

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9. If $\sum_{r=1}^{10} r(r-1)\ {}^{10}C_r = k.\ 2^9$, then k is equal to-

(A) 10 (B) 45 (C) 90 (D) 100

BT0022

10. The sum $\frac{\binom{11}{0}}{1}+\frac{\binom{11}{1}}{2}+\frac{\binom{11}{2}}{3}+.....+\frac{\binom{11}{11}}{12}$ equals $\left(\text{where } \binom{n}{r} \text{ denotes } {}^nC_r\right)$

(A) $\frac{2^{11}}{12}$ (B) $\frac{2^{12}}{12}$

(C) $\frac{2^{11}-1}{12}$ (D) $\frac{2^{12}-1}{12}$

BT0023

11. **Statement-1 :** The sum of the series ${}^nC_0.\ {}^mC_r + {}^nC_1.\ {}^mC_{r-1} + {}^nC_2.\ {}^mC_{r-2} +.....+ {}^nC_r.{}^mC_0$ is equal to ${}^{n+m}C_r$, where ${}^nC_r$'s and ${}^mC_r$'s denotes the combinatorial coefficients in the expansion of $(1 + x)^n$ and $(1 + x)^m$ respectively.

Statement-2 : Number of ways in which r children can be selected out of (n + m) children consisting of n boys and m girls if each selection may consist of any number of boys and girls is equal to ${}^{n+m}C_r$.

(A) Statement-1 is true, statement-2 is true ; statement-2 is a correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true ; statement-2 is NOT a correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

BT0024

12. Which of the following statement(s) is/are correct ?

(A) $1+\frac{2}{2}+\frac{3}{2^2}+\frac{4}{2^3}+.....+\infty = 4$

(B) Integral part of $\left(9+4\sqrt{5}\right)^n$, $n \in N$ is even.

(C) $({}^nC_0 + {}^nC_1 + {}^nC_2 + ...+ {}^nC_n)^2 = 1 + {}^{2n}C_1 + {}^{2n}C_2+.....+ {}^{2n}C_{2n}$

(D) $\frac{1}{(3+2x)^2}$ can be expanded as infinite series in ascending powers of x only if $|x| < \frac{2}{3}$.

BT0025

13. If for $n \in I$, $n > 10$; $1 + (1 + x) + (1 + x)^2 + ... + (1 + x)^n = \sum_{k=0}^{n} a_k.x^k$, $x \neq 0$ then

(A) $\sum_{k=0}^{n} a_k = 2^{n+1}$ (B) $a_{n-2} = \frac{n(n+1)}{2}$

(C) $a_p > a_{p-1}$ for $p < \frac{n}{2}$, $p \in N$ (D) $(a_9)^2 - (a_8)^2 = {}^{n+2}C_{10}\ ({}^{n+1}C_{10} - {}^{n+1}C_9)$

BT0026

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14. Let $P(n)=\sum_{r=0}^{n}\frac{(-1)^r r}{r+1}\,{}^nC_r$. Now which of the following holds good ?

(A) $|P_{10}|$ is harmonic mean of $|P_9|$ & $|P_{11}|$

(B) $\sum_{r=5}^{10}P(r)P(r-1)=-\frac{6}{55}$

(C) $|P_{10}|$ is arithmetic mean of $|P_9|$ & $|P_{11}|$

(D) $\sum_{r=5}^{10}P(r)P(r-1)=\frac{6}{55}$

BT0027

15. Let $(1+x)^m = C_0 + C_1x + C_2x^2 + C_3x^3 + ... + C_mx^m$, where $C_r = {}^mC_r$ and $A = C_1C_3 + C_2C_4 + C_3C_5 + C_4C_6 + ...+ C_{m-2}C_m$, then -

(A) $A \geq {}^{2m}C_{m-2}$

(B) $A < {}^{2m}C_{m-2}$

(C) $A > C_0^2 + C_1^2 + C_2^2 + ...C_m^2$

(D) $A < C_0^2 + C_1^2 + C_2^2 + ...C_m^2$

BT0028

EXERCISE (S-1)

1. (a) If the coefficients of $(2r+4)^{th}$, $(r-2)^{th}$ terms in the expansion of $(1+x)^{18}$ are equal, find r.

BT0029

(b) If the coefficients of the rth, $(r+1)^{th}$ & $(r+2)^{th}$ terms in the expansion of $(1+x)^{14}$ are in AP, find r.

BT0030

(c) If the coefficients of 2nd, 3rd & 4th terms in the expansion of $(1+x)^{2n}$ are in AP, show that $2n^2 - 9n + 7 = 0$.

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BT0031

2. Find the term independent of x in the expansion of (i) $\left[\sqrt{\frac{x}{3}}+\frac{\sqrt{3}}{2x^2}\right]^{10}$ (ii) $\left[\frac{1}{2}x^{1/3}+x^{-1/5}\right]^8$

BT0032

3. Prove that the ratio of the coefficient of x^{10} in $(1-x^2)^{10}$ & the term independent of x in $\left(x-\frac{2}{x}\right)^{10}$ is 1 : 32.

BT0033

4. Find the term independent of x in the expansion of $(1+x+2x^3)\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$.

BT0034

5. Let $(1+x^2)^2 . (1+x)^n = \sum_{K=0}^{n+4}a_K . x^K$. If a_1, a_2 & a_3 are in AP, find n.

BT0035

6. Let $f(x) = 1 - x + x^2 - x^3 + ... + x^{16} - x^{17} = a_0 + a_1(1+x) + a_2(1+x)^2 + ... + a_{17}(1+x)^{17}$, find the value of a_2.

BT0036

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7. Find the coefficient of x^r in the expression :

$$(x+3)^{n-1} + (x+3)^{n-2}(x+2) + (x+3)^{n-3}(x+2)^2 + \ldots\ldots + (x+2)^{n-1}$$

BT0037

8. Find numerically greatest term in the expansion of :

(i) $(2+3x)^9$ when $x = \frac{3}{2}$ **BT0038**

(ii) $(3-5x)^{15}$ when $x = \frac{1}{5}$ **BT0039**

9. (a) Show that the integral part in each of the following is odd. $n \in N$

(A) $\left(5+2\sqrt{6}\right)^n$ (B) $\left(8+3\sqrt{7}\right)^n$

BT0040

(b) Show that the integral part in each of the following is even. $n \in N$

(A) $\left(3\sqrt{3}+5\right)^{2n+1}$ (B) $\left(5\sqrt{5}+11\right)^{2n+1}$

BT0041

10. Let $N = {}^{2000}C_1 + 2 \cdot {}^{2000}C_2 + 3 \cdot {}^{2000}C_3 + \ldots\ldots + 2000 \cdot {}^{2000}C_{2000}$. Prove that N is divisible by 2^{2003}.

BT0042

11. Prove the following identities using the theory of permutation where $C_0, C_1, C_2, \ldots\ldots, C_n$ are the combinatorial coefficients in the expansion of $(1+x)^n$, $n \in N$:

(a) $C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2 = \frac{(2n)!}{n!\,n!}$ **BT0043**

(b) $C_0 C_1 + C_1 C_2 + C_2 C_3 + \ldots + C_{n-1} C_n = \frac{(2n)!}{(n+1)!\,(n-1)!}$ **BT0044**

(c) $C_0 C_r + C_1 C_{r+1} + C_2 C_{r+2} + \ldots + C_{n-r} C_n = \frac{2n!}{(n-r)!\,(n+r)!}$ **BT0045**

(d) $\sum_{r=0}^{n-2} ({}^nC_r \cdot {}^nC_{r+2}) = \frac{(2n)!}{(n-2)!(n+2)!}$ **BT0046**

(e) ${}^{100}C_{10} + 5 \cdot {}^{100}C_{11} + 10 \cdot {}^{100}C_{12} + 10 \cdot {}^{100}C_{13} + 5 \cdot {}^{100}C_{14} + {}^{100}C_{15} = {}^{105}C_{90}$ **BT0047**

12. If $C_0, C_1, C_2, \ldots\ldots, C_n$ are the combinatorial coefficients in the expansion of $(1+x)^n$, $n \in N$, then prove the following :

(a) $C_1 + 2C_2 + 3C_3 + \ldots\ldots + n \cdot C_n = n \cdot 2^{n-1}$ **BT0048**

(b) $C_0 + 2C_1 + 3C_2 + \ldots\ldots + (n+1)C_n = (n+2)2^{n-1}$ **BT0049**

(c) $C_0 + 3C_1 + 5C_2 + \ldots\ldots + (2n+1)C_n = (n+1)\,2^n$ **BT0050**

(d) $(C_0+C_1)(C_1+C_2)(C_2+C_3) \ldots\ldots (C_{n-1}+C_n) = \frac{C_0 . C_1 . C_2 \ldots . C_{n-1} (n+1)^n}{n!}$ **BT0051**

(e) $1 \cdot C_0^2 + 3 \cdot C_1^2 + 5 \cdot C_2^2 + \ldots\ldots + (2n+1)\,C_n^2 = \frac{(n+1)\,(2n)!}{n!\,n!}$ **BT0052**

13. Prove that

(a) $\frac{C_1}{C_0} + \frac{2C_2}{C_1} + \frac{3C_3}{C_2} + \ldots\ldots + \frac{n.C_n}{C_{n-1}} = \frac{n(n+1)}{2}$ **BT0053**

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(b) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$ **BT0054**

(c) $2 . C_o + \frac{2^2.C_1}{2} + \frac{2^3.C_2}{3} + \frac{2^4.C_3}{4} + ... \frac{2^{n+1}.C_n}{n+1} = \frac{3^{n+1}-1}{n+1}$ **BT0055**

(d) $C_o - \frac{C_1}{2} + \frac{C_2}{3} - ..... + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$ **BT0056**

14. Given that $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$, find the values of :

(i) $a_0 + a_1 + a_2 + ... + a_{2n}$;

(ii) $a_0 - a_1 + a_2 - a_3 ..... + a_{2n}$;

(iii) $a_0^2 - a_1^2 + a_2^2 - a_3^2 + ... + a_{2n}^2$

BT0057

15. Find the sum of the series $\sum_{r=0}^{n}(-1)^r . {}^nC_r \left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + ...\text{up to m terms}\right]$

BT0058

16. Find the coefficient of

(a) x^4 in the expansion of $(1 + x + x^2 + x^3)^{11}$ **BT0059**

(b) x^4 in the expansion of $(2 - x + 3x^2)^6$ **BT0060**

17. Find the coefficient of

(a) $x^2 y^3 z^4$ in the expansion of $(ax - by + cz)^9$. **BT0061**

(b) $a^2 b^3 c^4 d$ in the expansion of $(a - b - c + d)^{10}$. **BT0062**

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EXERCISE (S-2)

1. Let a and b be the coefficient of x^3 in $(1 + x + 2x^2 + 3x^3)^4$ and $(1 + x + 2x^2 + 3x^3 + 4x^4)^4$ respectively. Find the value of $(a - b)$.

BT0063

2. Find the index n of the binomial $\left(\frac{x}{5} + \frac{2}{5}\right)^n$ if the 9th term of the expansion has numerically the greatest coefficient $(n \in N)$.

BT0064

3. Find the sum of the roots (real or complex) of the equation $x^{2001} + \left(\frac{1}{2} - x\right)^{2001} = 0$.

BT0065

4. Let $a = \left(4^{1/401} - 1\right)$ and let $b_n = {}^nC_1 + {}^nC_2 . a + {}^nC_3 . a^2 + ... + {}^nC_n . a^{n-1}$.
Find the value of $(b_{2006} - b_{2005})$

BT0066

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5. For which positive values of x, fourth term in the expansion of $(5 + 3x)^{10}$, is greatest.

BT0067

6. Let $P = \left(2+\sqrt{3}\right)^5$ and f = P – [P], where [P] denotes the greatest integer function.

Find the value of $\left(\frac{f^2}{1-f}\right)$.

BT0068

7. If $\left(7+4\sqrt{3}\right)^n = p+\beta$ where n & p are positive integers and β is a proper fraction show that $(1 - \beta)(p + \beta) = 1$.

BT0069

8. Find the coefficient of x^{49} in the polynomial

$$\left(x-\frac{C_1}{C_0}\right)\left(x-2^2\cdot\frac{C_2}{C_1}\right)\left(x-3^2\cdot\frac{C_3}{C_2}\right)\ldots\ldots\ldots\ldots\left(x-50^2\cdot\frac{C_{50}}{C_{49}}\right), \text{ where } C_r = {}^{50}C_r.$$

BT0070

9. Prove that $\sum_{K=0}^{n} {}^nC_K \sin Kx.\cos(n-K)x = 2^{n-1}\sin nx$.

BT0071

10. If $\binom{n}{r}$ denotes ${}^nC_r$, then

(a) Evaluate : $2^{15}\binom{30}{0}\binom{30}{15} - 2^{14}\binom{30}{1}\binom{29}{14} + 2^{13}\binom{30}{2}\binom{28}{13}\ldots\ldots - \binom{30}{15}\binom{15}{0}$

BT0072

(b) Prove that : $\sum_{r=1}^{n}\binom{n-1}{n-r}\binom{n}{r} = \binom{2n-1}{n-1}$

BT0073

(c) Prove that : $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$

BT0074

EXERCISE (JM)

1. Let $S_1 = \sum_{j=1}^{10} j(j-1)^{10}C_j$, $S_2 = \sum_{j=1}^{10} j^{10}C_j$ and $S_3 = \sum_{j=1}^{10} j^{2\,10}C_j$. **[AIEEE-2010]**

Statement–1 : $S_3 = 55 \times 2^9$.
Statement–2 : $S_1 = 90 \times 2^8$ and $S_2 = 10 \times 2^8$.
(1) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.
(2) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for Statement–1.
(3) Statement–1 is true, Statement–2 is false.
(4) Statement–1 is false, Statement–2 is true.

BT0075

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2. The coefficient of x^7 in the expansion of $(1 - x - x^2 + x^3)^6$ is :- **[AIEEE 2011]**

(1) – 144 (2) 132 (3) 144 (4) – 132

BT0076

3. If n is a positive integer, then $\left(\sqrt{3}+1\right)^{2n} - \left(\sqrt{3}-1\right)^{2n}$ is : **[AIEEE 2012]**

(1) a rational number other than positive integers (2) an irrational number

(3) an odd positive integer (4) an even positive integer

BT0077

4. The term independent of x in expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1} - \frac{x-1}{x-x^{1/2}}\right)^{10}$ is : **[JEE-Main 2013]**

(1) 4 (2) 120 (3) 210 (4) 310

BT0078

5. If the coefficients of x^3 and x^4 in the expansion of $(1 + ax + bx^2)(1 - 2x)^{18}$ in powers of x are both zero, then (a, b) is equal to :- **[JEE(Main)-2014]**

(1) $\left(16, \frac{251}{3}\right)$ (2) $\left(14, \frac{251}{3}\right)$ (3) $\left(14, \frac{272}{3}\right)$ (4) $\left(16, \frac{272}{3}\right)$

BT0079

6. The sum of coefficients of integral powers of x in the binomial expansion of $\left(1 - 2\sqrt{x}\right)^{50}$ is : **[JEE(Main)-2015]**

(1) $\frac{1}{2}\left(3^{50}-1\right)$ (2) $\frac{1}{2}\left(2^{50}+1\right)$ (3) $\frac{1}{2}\left(3^{50}+1\right)$ (4) $\frac{1}{2}\left(3^{50}\right)$

BT0080

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7. If the number of terms in the expansion of $\left(1 - \frac{2}{x} + \frac{4}{x^2}\right)^n$, $x \neq 0$, is 28, then the sum of the coefficients of all the terms in this expansion, is :- **[JEE(Main)-2016]**

(1) 729 (2) 64 (3) 2187 (4) 243

BT0081

8. The value of $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + ... + ({}^{21}C_{10} - {}^{10}C_{10})$ is :- **[JEE(Main)-2017]**

(1) $2^{20} - 2^{10}$ (2) $2^{21} - 2^{11}$ (3) $2^{21} - 2^{10}$ (4) $2^{20} - 2^9$

BT0082

9. The sum of the co-efficients of all odd degree terms in the expansion of $\left(x + \sqrt{x^3-1}\right)^5 + \left(x - \sqrt{x^3-1}\right)^5$, $(x > 1)$ is - **[JEE(Main)-2018]**

(1) 0 (2) 1 (3) 2 (4) –1

BT0083

10. If the fractional part of the number $\frac{2^{403}}{15}$ is $\frac{k}{15}$, then k is equal to : **[JEE(Main)- 2019]**

(1) 14 (2) 6 (3) 4 (4) 8

BT0084

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11. The coefficient of t^4 in the expansion of $\left(\frac{1-t^6}{1-t}\right)^3$ is **[JEE(Main)- 2019]**

(1) 12 (2) 15 (3) 10 (4) 14

BT0085

12. If $\sum_{r=0}^{25}\left\{{}^{50}C_r \cdot {}^{50-r}C_{25-r}\right\} = K\left({}^{50}C_{25}\right)$, then K is equal to : **[JEE(Main)- 2019]**

(1) $2^{25} - 1$ (2) $(25)^2$ (3) 2^{25} (4) 2^{24}

BT0086

13. The sum of the real values of x for which the middle term in the binomial expansion of $\left(\frac{x^3}{3} + \frac{3}{x}\right)^8$ equals 5670 is : **[JEE(Main)- 2019]**

(1) 6 (2) 8 (3) 0 (4) 4

BT0087

14. The value of r for which ${}^{20}C_r\ {}^{20}C_0 + {}^{20}C_{r-1}\ {}^{20}C_1 + {}^{20}C_{r-2}\ {}^{20}C_2 + \ldots\ {}^{20}C_0\ {}^{20}C_r$ is maximum, is **[JEE(Main)- 2019]**

(1) 20 (2) 15 (3) 11 (4) 10

BT0088

15. Let $(x + 10)^{50} + (x - 10)^{50} = a_0 + a_1x + a_2x^2 + \ldots + a_{50}x^{50}$, for all $x \in R$, then $\frac{a_2}{a_0}$ is equal to:- **[JEE(Main)- 2019]**

(1) 12.50 (2) 12.00 (3) 12.75 (4) 12.25

BT0089

16. Let $S_n = 1 + q + q^2 + \ldots + q^n$ and $T_n = 1 + \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 + \ldots + \left(\frac{q+1}{2}\right)^n$, where q is a real number and $q \neq 1$. If ${}^{101}C_1 + {}^{101}C_2.S_1 + \ldots + {}^{101}C_{101}.S_{100} = \alpha T_{100}$, then α is equal to :- **[JEE(Main)- 2019]**

(1) 2^{100} (2) 200 (3) 2^{99} (4) 202

BT0090

17. The total number of irrational terms in the binomial expansion of $\left(7^{1/5} - 3^{1/10}\right)^{60}$ is : **[JEE(Main)- 2019]**

(1) 55 (2) 49 (3) 48 (4) 54

BT0091

18. If some three consecutive in the binomial expansion of $(x + 1)^n$ is powers of x are in the ratio 2 : 15 : 70, then the average of these three coefficient is :- **[JEE(Main)- 2019]**

(1) 964 (2) 625 (3) 227 (4) 232

BT0092

19. The coefficient of x^{18} in the product $(1+x)(1-x)^{10}(1 + x + x^2)^9$ is : **[JEE(Main)- 2019]**

(1) –84 (2) 84 (3) 126 (4) –126

BT0093

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20. If ${}^{20}C_1 + (2^2)\,{}^{20}C_2 + (3^2)\,{}^{20}C_3 + \ldots\ldots + (20^2){}^{20}C_{20} = A(2^\beta)$, then the ordered pair (A, β) is equal to: **[JEE(Main)- 2019]**

(1) (420, 18) (2) (380, 19) (3) (380, 18) (4) (420, 19)

BT0094

21. The term independent of x in the expansion of $\left(\frac{1}{60}-\frac{x^8}{81}\right)\cdot\left(2x^2-\frac{3}{x^2}\right)^6$ is equal to : **[JEE(Main)- 2019]**

(1) 36 (2) – 108 (3) – 72 (4) – 36

BT0095

22. The coefficient of x^7 in the expression $(1 + x)^{10} + x(1 + x)^9 + x^2(1 + x)^8 + \ldots + x^{10}$ is :

(1) 120 (2) 330 (3) 210 (4) 420

[JEE(Main)- 2020]

BT0096

23. If the sum of the coefficients of all even powers of x in the product $(1 + x + x^2 + \ldots + x^{2n})(1 - x + x^2 - x^3 + \ldots + x^{2n})$ is 61, then n is equal to _____. **[JEE(Main)- 2020]**

BT0097

24. If α and β be the coefficients of x^4 and x^2 respectively in the expansion of $\left(x+\sqrt{x^2-1}\right)^6+\left(x-\sqrt{x^2-1}\right)^6$, then **[JEE(Main)- 2020]**

(1) $\alpha + \beta = 60$ (2) $\alpha + \beta = -30$ (3) $\alpha - \beta = -132$ (4) $\alpha - \beta = 60$

BT0098

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25. In the expansion of $\left(\frac{x}{\cos\theta}+\frac{1}{x\sin\theta}\right)^{16}$, if ℓ_1 is the least value of the term independent of x when $\frac{\pi}{8}\le\theta\le\frac{\pi}{4}$ and ℓ_2 is the least value of the term independent of x when $\frac{\pi}{16}\le\theta\le\frac{\pi}{8}$, then the ratio $\ell_2 : \ell_1$ is equal to : **[JEE(Main)- 2020]**

(1) 1 : 8 (2) 1 : 16 (3) 8 : 1 (4) 16 : 1

BT0099

26. If $C_r \equiv {}^{25}C_r$ and $C_0 + 5.C_1 + 9.C_2 + \ldots + (101).C_{25} = 2^{25}.k$, then k is equal to _____. **[JEE(Main)- 2020]**

BT0100

EXERCISE (JA)

1. For r = 0, 1,....,10, let A_r, B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1 + x)^{10}$, $(1 + x)^{20}$ and $(1 + x)^{30}$. Then $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ is equal to -

(A) $B_{10} - C_{10}$ (B) $A_{10}\left(B_{10}^2 - C_{10}A_{10}\right)$ (C) 0 (D) $C_{10} - B_{10}$

[JEE 2010, 5]

BT0101

2. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n =

[JEE (Advanced) 2013, 4M, –1M]

BT0102

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3. Coefficient of x^{11} in the expansion of $(1 + x^2)^4(1 + x^3)^7(1 + x^4)^{12}$ is -

(A) 1051 (B) 1106 (C) 1113 (D) 1120

[JEE(Advanced)-2014, 3(–1)]

BT0103

4. The coefficient of x^9 in the expansion of $(1 + x)(1 + x^2)(1 + x^3)\ldots(1 + x^{100})$ is **[JEE 2015, 4M, –0M]**

BT0104

5. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + \ldots\ldots + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)\,{}^{51}C_3$ for some positive integer n. Then the value of n is **[JEE(Advanced)-2016, 3(0)]**

BT0105

6. Let $X = \left({}^{10}C_1\right)^2 + 2\left({}^{10}C_2\right)^2 + 3\left({}^{10}C_3\right)^2 + \ldots + 10\left({}^{10}C_{10}\right)^2$, where ${}^{10}C_r$, $r \in \{1, 2, \ldots, 10\}$ denote binomial coefficients. Then, the value of $\frac{1}{1430}X$ is _____ . **[JEE(Advanced)-2018, 3(0)]**

BT0106

7. Suppose $\det\begin{bmatrix} \sum_{k=0}^{n} k & \sum_{k=0}^{n} {}^nC_k\, k^2 \\ \sum_{k=0}^{n} {}^nC_k\, k & \sum_{k=0}^{n} {}^nC_k\, 3^k \end{bmatrix} = 0$, holds for some positive integer n. Then $\sum_{k=0}^{n} \frac{{}^nC_k}{k+1}$ equals

[JEE(Advanced)-2019, 3(0)]

BT0107

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ANSWER KEY

Do yourself-1

(i) $^5C_0x(3x^2)^5+^5C_1(3x^2)^4\left(-\frac{x}{2}\right)+^5C_2(3x^2)^3\left(-\frac{x}{2}\right)^2+^5C_3(3x^2)^2\left(-\frac{x}{2}\right)^3+^5C_4(3x^2)^1\left(-\frac{x}{2}\right)^4+^5C_5\left(-\frac{x}{2}\right)^5$

(ii) $^nC_0y^n + {}^nC_1y^{n-1}.x + {}^nC_2.y^{n-2}.x^2 + ... +{}^nC_n.x^n$

Do yourself-2

(i) $\frac{70}{3}x^8$; **(ii)** $\frac{25!}{10!\ 5!}2^{15}3^{10}$;**(iii)** (a) –20; (b) $-560x^5, 280x^2$

Do yourself-3

(i) 4th & 5th i.e. 489888 **(ii)** n = 4, 5, 6

Do yourself-4

(i) C

Do yourself-5

(i) –272160 or $-{}^{10}C_5 \times {}^5C_2 \times 108$

Do yourself-6

(ii) 1 **(iii)** 001 **(v)** 1

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EXERCISE (O-1)

1.	C	**2.**	C	**3.**	B	**4.**	A	**5.**	A	**6.**	C	**7.**	B	
8.	A	**9.**	B	**10.**	B	**11.**	D	**12.**	B	**13.**	A	**14.**	B	**15.** B

EXERCISE (O-2)

1.	A,B	**2.**	B,C,D	**3.**	A,C,D	**4.**	A,B	**5.**	A,C	**6.**	B	**7.**	B	
8.	A,C,D	**9.**	B	**10.**	D	**11.**	A	**12.**	A,C	**13.**	B,C,D	**14.**	A,D	
15.	B,D													

EXERCISE (S-1)

1. **(a)** r = 6 **(b)** r = 5 or 9 **2.** (i) $\frac{5}{12}$ (ii) $T_6 = 7$ **4.** $\frac{17}{54}$ **5.** n = 2 or 3 or 4

6. 816 **7.** $^nC_r(3^{n-r} - 2^{n-r})$ **8.** **(i)** $T_7 = \frac{7.3^{13}}{2}$ **(ii)** 455×3^{12}

14. **(i)** 3^n **(ii)** 1, **(iii)** a_n **15.** $\frac{(2^{mn}-1)}{(2^n-1)(2^{mn})}$ **16.** **(a)** 990 **(b)** 3660

17. **(a)** $-1260 . a^2b^3c^4$ **; (b)** –12600

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EXERCISE (S-2)

1. 0 **2.** n = 12 **3.** 500 **4.** 2^{10} **5.** $\frac{5}{8} \le x \le \frac{20}{21}$

6. 722 **8.** – 22100 **10.** (a) $\binom{30}{15}$

EXERCISE (JM)

1. 3 **2.** 1 **3.** 2 **4.** 3 **5.** 4 **6.** 3

7. **Bonus**

Note : In the problem 'number of terms should be 13 instead of 28', then (1) will be the answer

8. 1 **9.** 3 **10.** 4 **11.** 2 **12.** 3 **13.** 3 **14.** 1

15. 4 **16.** 1 **17.** 4 **18.** 4 **19.** 2 **20.** 1 **21.** 4

22. **2** **23.** **30** **24.** **3** **25.** **4** **26.** **51**

EXERCISE (JA)

1. D **2.** 6 **3.** C **4.** 8 **5.** 5 **6.** 646 **7.** 6.20

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PERMUTATION & COMBINATION

1. FUNDAMENTAL PRINCIPLE OF COUNTING (counting without actual counting):

If an event A can occur in 'm' different ways and another event B can occur in 'n' different ways, then the total number of different ways of -

(a) simultaneous occurrence of both events in a definite order is $m \times n$. This can be extended to any number of events (known as multiplication principle).

(b) happening exactly one of the events is $m + n$ (known as addition principle).

Example : There are 15 IITs in India and let each IIT has 10 branches, then the IITJEE topper can select the IIT and branch in $15 \times 10 = 150$ number of ways.

Example : There are 15 IITs & 20 NITs in India, then a student who cleared both IITJEE & AIEEE exams can select an institute in $(15 + 20) = 35$ number of ways.

Illustration 1 : A college offers 6 courses in the morning and 4 in the evening. The possible number of choices with the student if he wants to study one course in the morning and one in the evening is-

(A) 24 (B) 2 (C) 12 (D) 10

Solution : The student has 6 choices from the morning courses out of which he can select one course in 6 ways.

For the evening course, he has 4 choices out of which he can select one in 4 ways.

Hence the total number of ways $6 \times 4 = 24$. **Ans.(A)**

Illustration 2 : A college offers 6 courses in the morning and 4 in the evening. The number of ways a student can select exactly one course, either in the morning or in the evening-

(A) 6 (B) 4 (C) 10 (D) 24

Solution : The student has 6 choices from the morning courses out of which he can select one course in 6 ways.

For the evening course, he has 4 choices out of which he can select one in 4 ways.

Hence the total number of ways $6 + 4 = 10$. **Ans. (C)**

Do yourself - 1 :

(i) There are 3 ways to go from A to B, 2 ways to go from B to C and 1 way to go from A to C. In how many ways can a person travel from A to C ?

(ii) There are 2 red balls and 3 green balls. All balls are identical except colour. In how many ways can a person select two balls ?

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Greatest Integer

For every real number x, there exist an unique integer k such that $k \le x < k + 1$. Then k is called integral part or greatest integer or floor of x.

It is usually denoted by $\lfloor x \rfloor$ or [x]

Here are some example

x	-2.1	$-\sqrt{2}$	3	$\sqrt{5}$	π	2	100	$-\sqrt{70}$
$\lfloor x \rfloor$	-3	-2	3	2	3	2	100	-9

Note :

$x - 1 < \lfloor x \rfloor \le x$

$\lfloor x \rfloor = x \Leftrightarrow x$ is integer

$\lfloor x \rfloor = n \Leftrightarrow x \in [n, n+1),\ n \in I$

For $n \in I$, $\lfloor x + n \rfloor = n + \lfloor x \rfloor$

$$\lfloor x \rfloor + \lfloor -x \rfloor = \begin{cases} 0, & x \in I \\ -1, & x \notin I \end{cases}$$

Graph of y = [x]

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Illustration 3 : Solve following

(i) $4[x] - 8 = 0$ (ii) $3\left[-\frac{x}{3}\right] + 9 = 0$

(iii) $[|x|] = 2$ (iv) $|[x]| = 2$ (v) $\left[\frac{5+x}{2}\right] + \left[\frac{3+x}{2}\right] = -9$

Here [.] denotes greatest integer function.

Solution : (i) $4[x] = 8 \Rightarrow [x] = 2 \Rightarrow x \in [2,3)$

(ii) $3\left[-\frac{x}{3}\right] + 9 = 0 \Rightarrow \left[-\frac{x}{3}\right] = -3 \Rightarrow -3 \le -\frac{x}{3} < -2 \Rightarrow 9 \ge x > 6 \Rightarrow x \in (6,9]$

(iii) $[|x|] = 2 \Rightarrow 2 \le |x| < 3 \Rightarrow x \in (-3,-2] \cup [2,3]$

(iv) $|[x]| = 2 \Rightarrow [x] = -2$ or $2 \Rightarrow x \in [-2,-1) \cup [2,3)$

(v) $\left[\frac{5+x}{2}\right] + \left[\frac{3+x}{2}\right] = -9 \Rightarrow \left[2 + \frac{1+x}{2}\right] + \left[1 + \frac{1+x}{2}\right] = -9$

$\Rightarrow 2 + \left[\frac{1+x}{2}\right] + 1 + \left[\frac{1+x}{2}\right] = -9 \Rightarrow 2\left[\frac{1+x}{2}\right] = -12 \Rightarrow \left[\frac{1+x}{2}\right] = -6$

$\Rightarrow -6 \le \frac{x+1}{2} < -5 \Rightarrow -12 \le x + 1 < -10 \Rightarrow x \in [-13,-11)$

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2. FACTORIAL NOTATION :

(i) A Useful Notation : n! (factorial n) = n.(n – 1).(n – 2).....3. 2. 1; n! = n. (n – 1)! where $n \in N$

(ii) 0! = 1! = 1

(iii) Factorials of negative integers are not defined.

(iv) n! is also denoted by $\lfloor n$

(v) $(2n)! = 2^n.n!\,[1.\,3.\,5.\,7.....(2n-1)]$

(vi) Prime factorisation of n! : Let p be a prime number and n be a positive integer, then exponent of p in n! is denoted by $E_p(n!)$ and is given by

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + ... + \left[\frac{n}{p^k}\right]$$

where, $p^k \le n < p^{k+1}$ and [x] denotes the integral part of x.

If we isolate the power of each prime contained in any number n, then n can be written as $n = 2^{\alpha_1}.3^{\alpha_2}.5^{\alpha_3}.7^{\alpha_4}....$, where α_i are whole numbers.

Illustration 4 : Find the exponent of 6 in 50!

Solution: $E_2(50!) = \left[\frac{50}{2}\right] + \left[\frac{50}{4}\right] + \left[\frac{50}{8}\right] + \left[\frac{50}{16}\right] + \left[\frac{50}{32}\right] + \left[\frac{50}{64}\right]$ (where [ ] denotes integral part)

$E_2(50!) = 25 + 12 + 6 + 3 + 1 + 0 = 47$

$E_3(50!) = \left[\frac{50}{3}\right] + \left[\frac{50}{9}\right] + \left[\frac{50}{27}\right] + \left[\frac{50}{81}\right]$

$E_3(50!) = 16 + 5 + 1 + 0 = 22$

$\Rightarrow$ 50! can be written as $50! = 2^{47}.\,3^{22}.....$

Therefore exponent of 6 in 50! = 22 **Ans.**

3. PERMUTATION & COMBINATION :

(a) **Permutation :** Each of the arrangements in a definite order which can be made by taking some or all of the things at a time is called a PERMUTATION. In permutation, order of appearance of things is taken into account; when the order is changed, a different permutation is obtained.

Generally, it involves the problems of arrangements (standing in a line, seated in a row), problems on digit, problems on letters from a word etc.

${}^nP_r$ denotes the number of permutations of n **different** things, taken r at a time ($n \in N$, $r \in W$, $r \le n$)

$${}^nP_r = n\,(n-1)\,(n-2)\,.....\,(n-r+1) = \frac{n!}{(n-r)!}$$

Note :

(i) ${}^nP_n = n!$, ${}^nP_0 = 1$, ${}^nP_1 = n$

(ii) Number of arrangements of n **distinct** things taken all at a time = n!

(iii) ${}^nP_r$ is also denoted by A_r^n or P(n,r).

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(b) Combination :

Each of the groups or selections which can be made by taking some or all of the things without considering the order of the things in each group is called a COMBINATION.

Generally, involves the problem of selections, choosing, distributed groups formation, committee formation, geometrical problems etc.

${}^nC_r$ denotes the number of combinations of n different things taken r at a time ($n \in N$, $r \in W$, $r \le n$)

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

Note :

(i) ${}^nC_r$ is also denoted by $\binom{n}{r}$ or C (n, r).

(ii) ${}^nP_r = {}^nC_r \cdot r!$

Illustration 5 : If *a* denotes the number of permutations of (x + 2) things taken all at a time, *b* the number of permutations of x things taken 11 at a time and *c* the number of permutations of (x – 11) things taken all at a time such that *a = 182 bc,* then the value of x is

(A) 15 (B) 12 (C) 10 (D) 18

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Solution : ${}^{x+2}P_{x+2} = a \Rightarrow a = (x+2)!$

$${}^xP_{11} = b \Rightarrow b = \frac{x!}{(x-11)!}$$

and ${}^{x-11}P_{x-11} = c \Rightarrow c = (x-11)!$

$\because a = 182bc$

$$(x+2)! = 182\frac{x!}{(x-11)!}(x-11)! \Rightarrow (x+2)(x+1) = 182 = 14 \times 13$$

$\therefore x + 1 = 13 \quad \Rightarrow x = 12$ **Ans. (B)**

Illustration 6 : A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be drawn so that there are atleast two balls of each colour ?

Solution : The selections of 6 balls, consisting of atleast two balls of each colour from 5 red and 6 white balls, can be made in the following ways

Red balls (5)	White balls (6)	Number of ways
2	4	${}^5C_2 \times {}^6C_4 = 150$
3	3	${}^5C_3 \times {}^6C_3 = 200$
4	2	${}^5C_4 \times {}^6C_2 = 75$

Therefore total number of ways = 425 **Ans.**

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Illustration 7 : How many 4 letter words can be formed from the letters of the word 'ANSWER' ? How many of these words start with a vowel ?

Solution : Number of ways of arranging 4 different letters from 6 different letters are ${}^6C_4\,4! = \frac{6!}{2!} = 360$.

There are two vowels (A & E) in the word 'ANSWER'.

Total number of 4 letter words starting with A : A _ _ _ = ${}^5C_3\,3! = \frac{5!}{2!} = 60$

Total number of 4 letter words starting with E : E _ _ _ = ${}^5C_3\,3! = \frac{5!}{2!} = 60$

$\therefore$ Total number of 4 letter words starting with a vowel = 60 + 60 = 120. **Ans.**

Illustration 8 : If all the letters of the word 'RAPID' are arranged in all possible manner as they are in a dictionary, then find the rank of the word 'RAPID'.

Solution : First of all, arrange all letters of given word alphabetically : 'ADIPR'

Total number of words starting with A _ _ _ _ = 4! = 24
Total number of words starting with D _ _ _ _ = 4! = 24
Total number of words starting with I _ _ _ _ = 4! = 24
Total number of words starting with P _ _ _ _ = 4! = 24
Total number of words starting with RAD _ _ = 2! = 2
Total number of words starting with RAI _ _ = 2! = 2
Total number of words starting with RAPD _ = 1
Total number of words starting with RAPI _ = 1

$\therefore$ Rank of the word RAPID = 24 + 24 + 24 + 24 + 2 + 2 + 1 + 1 = 102 **Ans.**

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Do yourself -2 :

(i) Find the exponent of 10 in ${}^{75}C_{25}$.

(ii) If ${}^{10}P_r = 5040$, then find the value of r.

(iii) Find the number of ways of selecting 4 even numbers from the set of first 100 natural numbers.

(iv) If all letters of the word 'RANK' are arranged in all possible manner as they are in a dictionary, then find the rank of the word 'RANK'.

(v) How many words can be formed using all letters of the word 'LEARN' ? In how many of these words vowels are together ?

(vi) **Sketch the graph of**

(a) $y = [2x]$ (b) $y = \left[\frac{x}{3}\right]$ (c) $y = [-x]$

Here [.] denotes greatest integer function.

(vii) **Solve following**

(a) $\left[\frac{x}{3}\right] + 2 = 0$ (b) $3\left[\frac{x}{3}\right] - 2 = 0$ (c) $\left[\frac{|x|}{3}\right] = 10$

(d) $\left|\left[-\frac{x}{3}\right]\right| = 4$ (e) $\left[\frac{-x-1}{2}\right] + \left[\frac{5-x}{2}\right] = 3$

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4. PROPERTIES OF ^nP_r and ^nC_r :

(a) The number of permutation of n different objects taken r at a time, when p particular objects are always to be included is $r!.^{n-p}C_{r-p}$ $(p \le r \le n)$

(b) The number of permutations of n different objects taken r at a time, when repetition is allowed any number of times is n^r.

(c) Following properties of ^nC_r should be remembered :

(i) $^nC_r = {}^nC_{n-r}$; $^nC_0 = {}^nC_n = 1$

(ii) $^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(iii) $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iv) $^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots\ldots\ldots + {}^nC_n = 2^n$

(v) $^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}$

(vi) ^nC_r is maximum when $r = \frac{n}{2}$ if n is even & $r = \frac{n-1}{2}$ or $r = \frac{n+1}{2}$, if n is odd.

(d) The number of combinations of n different things taking r at a time,

(i) when p particular things are always to be included $= {}^{n-p}C_{r-p}$

(ii) when p particular things are always to be excluded $= {}^{n-p}C_r$

(iii) when p particular things are always to be included and q particular things are to be excluded $= {}^{n-p-q}C_{r-p}$

Illustration 9 : There are 6 pockets in the coat of a person. In how many ways can he put 4 pens in these pockets?

(A) 360 (B) 1296 (C) 4096 (D) none of these

Solution : First pen can be put in 6 ways.

Similarly each of second, third and fourth pen can be put in 6 ways.

Hence total number of ways $= 6 \times 6 \times 6 \times 6 = 1296$ **Ans.(B)**

Illustration 10 : A delegation of four students is to be selected from a total of 12 students. In how many ways can the delegation be selected, if-

(a) all the students are equally willing ?

(b) two particular students have to be included in the delegation ?

(c) two particular students do not wish to be together in the delegation ?

(d) two particular students wish to be included together only ?

(e) two particular students refuse to be together and two other particular students wish to be together only in the delegation ?

Solution : (a) Formation of delegation means selection of 4 out of 12.

Hence the number of ways $= {}^{12}C_4 = 495$.

(b) If two particular students are already selected. Here we need to select only 2 out of the remaining 10. Hence the number of ways $= {}^{10}C_2 = 45$.

(c) The number of ways in which both are selected = 45. Hence the number of ways in which the two are not included together = 495 – 45 = 450

(d) There are two possible cases

(i) Either both are selected. In this case, the number of ways in which the selection can be made = 45.

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(ii) Or both are not selected. In this case all the four students are selected from the remaining ten students. This can be done in ${}^{10}C_4 = 210$ ways.

Hence the total number of ways of selection = 45 + 210 = 255

(e) We assume that students A and B wish to be selected together and students C and D do not wish to be together. Now there are following 6 cases.

(i) (A, B, C) selected, (D) not selected

(ii) (A, B, D) selected, (C) not selected

(iii) (A, B) selected, (C, D) not selected

(iv) (C) selected, (A, B, D) not selected

(v) (D) selected, (A, B, C) not selected

(vi) A, B, C, D not selected

For (i) the number of ways of selection = ${}^8C_1 = 8$

For (ii) the number of ways of selection = ${}^8C_1 = 8$

For (iii) the number of ways of selection = ${}^8C_2 = 28$

For (iv) the number of ways of selection = ${}^8C_3 = 56$

For (v) the number of ways of selection = ${}^8C_3 = 56$

For (vi) the number of ways of selection = ${}^8C_4 = 70$

Hence total number of ways = 8 + 8 + 28 + 56 + 56 + 70 = 226. **Ans.**

Illustration 11: In the given figure of squares, 6 A's should be written in such a manner that every row contains at least one 'A'. In how many number of ways is it possible ?

(A) 24 (B) 25 (C) 26 (D) 27

Solution : There are 8 squares and 6 'A' in given figure. First we can put 6 'A' in these 8 squares by ${}^8C_6$ number of ways.

(I)

(II)

According to question, atleast one 'A' should be included in each row. So after subtracting these two cases, number of ways are = $({}^8C_6 - 2) = 28 - 2 = 26$. **Ans. (C)**

Illustration 12: There are three coplanar parallel lines. If any p points are taken on each of the lines, the maximum number of triangles with vertices at these points is :

(A) $3p^2(p-1)+1$ (B) $3p^2(p-1)$ (C) $p^2(4p-3)$ (D) none of these

Solution : The number of triangles with vertices on different lines = ${}^pC_1 \times {}^pC_1 \times {}^pC_1 = p^3$

The number of triangles with two vertices on one line and the third vertex on any one of the other two lines = ${}^3C_1\{{}^pC_2 \times {}^{2p}C_1\} = 6p.\dfrac{p(p-1)}{2}$

So, the required number of triangles = $p^3 + 3p^2(p-1) = p^2(4p-3)$ **Ans. (C)**

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Illustration 13: There are 10 points in a row. In how many ways can 4 points be selected such that no two of them are consecutive ?

Solution : Total number of remaining non-selected points = 6

. . . . . .

Total number of gaps made by these 6 points = 6 + 1 = 7

If we select 4 gaps out of these 7 gaps and put 4 points in selected gaps then the new points will represent 4 points such that no two of them are consecutive.

x . . x . x . . x .

Total number of ways of selecting 4 gaps out of 7 gaps = ${}^{7}C_{4}$ **Ans.**

In general, total number of ways of selection of r points out of n points in a row such that no two of them are consecutive : ${}^{n-r+1}C_{r}$

Do yourself-3 :

(i) Find the number of ways of selecting 5 members from a committee of 5 men & 2 women such that all women are always included.

(ii) Out of first 20 natural numbers, 3 numbers are selected such that there is exactly one even number. How many different selections can be made ?

(iii) How many four letter words can be made from the letters of the word 'PROBLEM'. How many of these start as well as end with a vowel ?

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5. FORMATION OF GROUPS :

(a) (i) The number of ways in which (m + n) different things can be divided into two groups such that one of them contains m things and other has n things, is $\frac{(m+n)!}{m!\ n!}$ $(m \neq n)$.

(ii) If m = n, it means the groups are equal & in this case the number of divisions is $\frac{(2n)!}{n!\ n!\ 2!}$.

As in any one way it is possible to interchange the two groups without obtaining a new distribution.

(iii) If 2n things are to be divided equally between two persons then the number of ways :

$$\frac{(2n)!}{n!\ n!\ (2!)} \times 2!.$$

(b) (i) Number of ways in which (m + n + p) different things can be divided into three groups containing m, n & p things respectively is : $\frac{(m+n+p)!}{m!\ n!\ p!}$, $m \neq n \neq p$.

(ii) If m = n = p then the number of groups = $\frac{(3n)!}{n!\ n!\ n!\ 3!}$.

(iii) If 3n things are to be divided equally among three people then the number of ways in which it can be done is $\frac{(3n)!}{(n!)^3}$.

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(c) In general, the number of ways of dividing n distinct objects into ℓ groups containing p objects each and m groups containing q objects each is equal to $\dfrac{n!}{(p!)^{\ell}(q!)^{m}\,\ell!m!}$

Here $\ell p + mq = n$

Illustration 14 : In how many ways can 15 students be divided into 3 groups of 5 students each such that 2 particular students are always together ? Also find the number of ways if these groups are to be sent to three different colleges.

Solution : Here first we seperate those two particular students and make 3 groups of 5,5 and 3 of the remaining 13 so that these two particular students always go with the group of 3 students.

$\therefore$ Number of ways $= \dfrac{13!}{5!5!3!}.\dfrac{1}{2!}$.

Now if these groups are to be sent to three different colleges, total number of ways $= \dfrac{13!}{5!5!3!}.\dfrac{1}{2!}.3!$ **Ans.**

Illustration 15 : Find the number of ways of dividing 52 cards among 4 players equally such that each gets exactly one Ace.

Solution : Total number of ways of dividing 48 cards (Excluding 4Aces) in 4 groups $= \dfrac{48!}{(12!)^4 4!}$

Now, distribute exactly one Ace to each group of 12 cards. Total number of ways $= \dfrac{48!}{(12!)^4 4!} \times 4!$

Now, distribute these groups of cards among four players

$= \dfrac{48!}{(12!)^4 4!} \times 4!4! = \dfrac{48!}{(12!)^4} \times 4!$ **Ans.**

Illustration 16 : In how many ways can 8 different books be distributed among 3 students if each receives at least 2 books ?

Solution : If each receives at least two books, then the division trees would be as shown below :

The number of ways of division for tree in figure (i) is $\left[\dfrac{8!}{(2!)^2 4!2!}\right]$.

The number of ways of division for tree in figure (ii) is $\left[\dfrac{8!}{(3!)^2 2!2!}\right]$.

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The total number of ways of distribution of these groups among 3 students

is $\left[\frac{8!}{(2!)^2 4!2!}+\frac{8!}{(3!)^2 2!2!}\right]\times 3!$. **Ans.**

Do yourself-4 :

(i) Find the number of ways in which 16 constables can be assigned to patrol 8 villages, 2 for each.

(ii) In how many ways can 6 different books be distributed among 3 students such that none gets equal number of books and each gets atleast one book ?

(iii) n different toys are to be distributed among n children. Find the number of ways in which these toys can be distributed so that exactly one child gets no toy.

6. PRINCIPLE OF INCLUSION AND EXCLUSION :

(i)

(ii)

In the Venn's diagram (i), we get

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A' \cap B') = n(U) - n(A \cup B)$

In the Venn's diagram (ii), we get

$n(A \cup B \cup C)$

$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

$n(A' \cap B' \cap C') = n(U) - n(A \cup B \cup C)$

In general, we have $n(A_1 \cup A_2 \cup ..... \cup A_n)$

$= \sum n(A_i) - \sum_{i \neq j} n(A_i \cap A_j) + \sum_{i \neq j \neq k} n(A_i \cap A_j \cap A_k) + ... + (-1)^n \sum n(A_1 \cap A_2 \cap ... \cap A_n)$

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Illustration 17 : Find the number of permutations of letters a,b,c,d,e,f,g taken all at a time if neither 'beg' nor 'cad' pattern appear.

Solution : The total number of permutations without any restrictions; n(U) = 7!

(b e g) a c d f

Let A be the set of all possible permutations in which 'beg' pattern always appears : n(A) = 5!

(c a d) b e f g

Let B be the set of all possible permutations in which 'cad' pattern always appears : n(B) = 5!

(c a d) (b e g) f

$n(A \cap B)$: Number of all possible permutations when both 'beg' and 'cad' patterns appear.

$n(A \cap B) = 3!$.

Therefore, the total number of permutations in which 'beg' and 'cad' patterns do not appear

$n(A' \cap B') = n(U) - n(A \cap B) = n(U) - n(A) - n(B) + n(A \cap B)$

$= 7! - 5! - 5! + 3!$. **Ans.**

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Do yourself-5 :

(i) Find the number of n digit numbers formed using first 5 natural numbers, which contain the digits 2 & 4 essentially.

7. PERMUTATIONS OF ALIKE OBJECTS :

Case-I : Taken all at a time -

The number of permutations of n things taken all at a time : when p of them are similar of one type, q of them are similar of second type, r of them are similar of third type and the remaining $n-(p+q+r)$ are all different is : $\frac{n!}{p!\ q!\ r!}$.

Illustration 18 : In how many ways the letters of the word "ARRANGE" can be arranged without altering the relative position of vowels & consonants.

Solution : The consonants in their positions can be arranged in $\frac{4!}{2!} = 12$ ways.

The vowels in their positions can be arranged in $\frac{3!}{2!} = 3$ ways

$\therefore$ Total number of arrangements $= 12 \times 3 = 36$ **Ans.**

Illustration 19 : How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places?

(A) 17 (B) 18 (C) 19 (D) 20

Solution : There are 4 odd digits (1, 1, 3, 3) and 4 odd places (first, third, fifth and seventh). At these places the odd digits can be arranged in $\frac{4!}{2!2!} = 6$ ways

Then at the remaining 3 places, the remaining three digits (2, 2, 4) can be arranged in $\frac{3!}{2!} = 3$ ways

$\therefore$ The required number of numbers $= 6 \times 3 = 18$. **Ans. (B)**

Illustration 20 : (a) How many permutations can be made by using all the letters of the word HINDUSTAN ?

(b) How many of these permutations begin and end with a vowel ?

(c) In how many of these permutations, all the vowels come together ?

(d) In how many of these permutations, none of the vowels come together ?

(e) In how many of these permutations, do the vowels and the consonants occupy the same relative positions as in HINDUSTAN ?

Solution : (a) The total number of permutations = Arrangements of nine letters taken all at a time

$= \frac{9!}{2!} = 181440.$

(b) We have 3 vowels and 6 consonants, in which 2 consonants are alike. The first place can be filled in 3 ways and the last in 2 ways. The rest of the places can be filled in $\frac{7!}{2!}$ ways.

Hence the total number of permutations $= 3 \times 2 \times \frac{7!}{2!} = 15120.$

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(c) Assume the vowels (I, U, A) as a single letter. The letters (IUA), H, D, S, T, N, N can be arranged in $\frac{7!}{2!}$ ways. Also IUA can be arranged among themselves in $3! = 6$ ways.

Hence the total number of permutations $= \frac{7!}{2!} \times 6 = 15120$.

(d) Let us divide the task into two parts. In the first, we arrange the 6 consonants as shown below in $\frac{6!}{2!}$ ways.

$\times C \times C \times C \times C \times C \times C \times$ (Here C stands for a consonant and $\times$ stands for a gap between two consonants)

Now 3 vowels can be placed in 7 places (gaps between the consonants) in ${}^7C_3.3! = 210$ ways.

Hence the total number of permutations $= \frac{6!}{2!} \times 210 = 75600$.

(e) In this case, the vowels can be arranged among themselves in $3! = 6$ ways.

Also, the consonants can be arranged among themselves in $\frac{6!}{2!}$ ways.

Hence the total number of permutations $= \frac{6!}{2!} \times 6 = 2160$. **Ans.**

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Illustration 21 : If all the letters of the word 'PROPER' are arranged in all possible manner as they are in a dictionary, then find the rank of the word 'PROPER' .

Solution : First of all, arrange all letters of given word alphabetically : EOPPRR

Total number of words starting with-

E _ _ _ _ _ $= \frac{5!}{2!2!} = 30$

O _ _ _ _ _ $= \frac{5!}{2!2!} = 30$

PE _ _ _ _ $= \frac{4!}{2!} = 12$

PO _ _ _ _ $= \frac{4!}{2!} = 12$

PP _ _ _ _ $= \frac{4!}{2!} = 12$

PRE _ _ _ $= 3! = 6$

PROE _ _ $= 2! = 2$

PROPER $= 1 = 1$

Rank of the word PROPER = 105 **Ans.**

Case-II : Taken some at a time

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Illustration 22 : Find the total number of 4 letter words formed using four letters from the word "PARALLELOPIPED'.

Solution : Given letters are PPP, LLL, AA, EE, R, O, I, D.

Cases	No.of ways of selection	No.of ways of arrangements	Total
All distinct	${}^8C_4$	${}^8C_4 \times 4!$	1680
2 alike, 2 distinct	${}^4C_1 \times {}^7C_2$	${}^4C_1 \times {}^7C_2 \times \frac{4!}{2!}$	1008
2 alike, 2 other alike	${}^4C_2$	${}^4C_2 \times \frac{4!}{2!2!}$	36
3 alike, 1 distinct	${}^2C_1 \times {}^7C_1$	${}^2C_1 \times {}^7C_1 \times \frac{4!}{3!}$	56
		Total	2780

Ans.

Illustration 23 : Find the number of all 6 digit numbers such that all the digits of each number are selected from the set $\{1,2,3,4,5\}$ and any digit that appears in the number appears at least twice.

Solution :

Cases	No.of ways of selection	No.of ways of arrangements	Total
All alike	${}^5C_1$	${}^5C_1 \times 1$	5
4 alike + 2 other alike	${}^5C_2 \times 2!$	${}^5C_2 \times 2 \times \frac{6!}{2!4!}$	300
3 alike + 3 other alike	${}^5C_2$	${}^5C_2 \times \frac{6!}{3!3!}$	200
2 alike + 2 other alike +2 other alike	${}^5C_3$	${}^5C_3 \times \frac{6!}{2!2!2!}$	900
		Total	1405

Ans.

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Do yourself-6 :

(i) In how many ways can the letters of the word 'ALLEN' be arranged ? Also find its rank if all these words are arranged as they are in dictionary.

(ii) How many numbers greater than 1000 can be formed from the digits 1, 1, 2, 2, 3 ?

8. CIRCULAR PERMUTATION :

Let us consider that persons A,B,C,D are sitting around a round table. If all of them (A,B,C,D) are shifted by one place in anticlockwise order, then we will get Fig.(b) from Fig.(a). Now, if we shift A,B,C,D in anticlockwise order, we will get Fig.(c). Again, if we shift them, we will get Fig.(d) and in the next time, Fig.(a).

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Thus, we see that if 4 persons are sitting at a round table, they can be shifted four times and the four different arrangements, thus obtained will be the same, because anticlockwise order of A,B,C,D does not change.

But if A,B,C,D are sitting in a row and they are shifted in such an order that the last occupies the place of first, then the four arrangements will be different.

Thus, if there are 4 things, then for each circular arrangement number of linear arrangements is 4.

Similarly, if n different things are arranged along a circle, for each circular arrangement number of linear arrangements is n.

Therefore, the number of linear arrangements of n different things is n × (number of circular arrangements of n different things). Hence, the number of circular arrangements of n different things is -

1/n × (number of linear arrangements of n different things) $= \frac{n!}{n} = (n-1)!$

Therefore note that :

(i) The number of circular permutations of n different things taken all at a time is : $(n-1)!$.

If clockwise & anti-clockwise circular permutations are considered to be same, then it is : $\frac{(n-1)!}{2}$.

(ii) The number of circular permutations of n different things taking r at a time distinguishing clockwise & anticlockwise arrangements is : $\frac{{}^nP_r}{r}$

Illustration 24 : In how many ways can 5 boys and 5 girls be seated at a round table so that no two girls are together?

(A) $5! \times 5!$ (B) $5! \times 4!$ (C) $\frac{1}{2}(5!)^2$ (D) $\frac{1}{2}(5! \times 4!)$

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Solution : Leaving one seat vacant between two boys, 5 boys may be seated in 4! ways. Then at remaining 5 seats, 5 girls sit in 5! ways. Hence the required number of ways $= 4! \times 5!$

Ans. (B)

Illustration 25 : The number of ways in which 7 girls can stand in a circle so that they do not have same neighbours in any two arrangements ?

(A) 720 (B) 380 (C) 360 (D) none of these

Solution : Seven girls can stand in a circle by $\frac{(7-1)!}{2!}$ number of ways, because there is no difference in anticlockwise and clockwise order of their standing in a circle.

$\therefore \quad \frac{(7-1)!}{2!} = 360$ **Ans. (C)**

Illustration 26 : The number of ways in which 20 different pearls of two colours can be set alternately on a necklace, there being 10 pearls of each colour, is

(A) $9! \times 10!$ (B) $5(9!)^2$ (C) $(9!)^2$ (D) none of these

Solution : Ten pearls of one colour can be arranged in $\frac{1}{2}.(10-1)!$ ways. The number of arrangements of 10 pearls of the other colour in 10 places between the pearls of the first colour = 10!

$\therefore$ The required number of ways $= \frac{1}{2} \times 9! \times 10! = 5\,(9!)^2$ **Ans. (B)**

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Illustration 27 : A person invites a group of 10 friends at dinner. They sit
(i) 5 on one round table and 5 on other round table,
(ii) 4 on one round table and 6 on other round table.
Find the number of ways in each case in which he can arrange the guests.

Solution : (i) The number of ways of selection of 5 friends for first table is ${}^{10}C_5$. Remaining 5 friends are left for second table.
The total number of permutations of 5 guests at a round table is 4!. Hence, the total number of arrangements is ${}^{10}C_5 \times 4! \times 4! = \frac{10!4!4!}{5!5!} = \frac{10!}{25}$
(ii) The number of ways of selection of 6 guests is ${}^{10}C_6$.
The number of ways of permutations of 6 guests on round table is 5!. The number of permutations of 4 guests on round table is 3!
Therefore, total number of arrangements is : ${}^{10}C_6 5! \times 3! = \frac{(10)!}{6!4!} 5!3! = \frac{(10)!}{24}$ **Ans. (B)**

Do yourself-7 :
(i) In how many ways can 3 men and 3 women be seated around a round table such that all men are always together ?
(ii) Find the number of ways in which 10 different diamonds can be arranged to make a necklace.
(iii) Find the number of ways in which 6 persons out of 5 men & 5 women can be seated at a round table such that 2 men are never together.
(iv) In how many ways can 8 persons be seated on two round tables of capacity 5 & 3.

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9. TOTAL NUMBER OF COMBINATIONS :

(a) Given n different objects , the number of ways of selecting atleast one of them is,
${}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_n = 2^n - 1$. This can also be stated as the total number of combinations of n distinct things.

(b) (i) Total number of ways in which it is possible to make a selection by taking some or all out of p + q + r +.....things, where p are alike of one kind, q alike of a second kind, r alike of third kind & so on is given by : (p + 1) (q + 1) (r + 1).....–1.

(ii) The total number of ways of selecting one or more things from p identical things of one kind, q identical things of second kind, r identical things of third kind and n different things is given by :
$(p + 1)(q + 1)(r + 1)\, 2^n - 1$.

Illustration 28 : A is a set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen. The number of ways of choosing P and Q so that $P \cap Q = \phi$ is :-
(A) $2^{2n} - {}^{2n}C_n$ (B) 2^n (C) $2^n - 1$ (D) 3^n

Solution : Let $A = \{a_1, a_2, a_3, ... a_n\}$. For $a_i \in A$, we have the following choices :
(i) $a_i \in P$ and $a_i \in Q$ (ii) $a_i \in P$ and $a_i \notin Q$
(iii) $a_i \notin P$ and $a_i \in Q$ (iv) $a_i \notin P$ and $a_i \notin Q$

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Out of these only (ii), (iii) and (iv) imply $a_i \notin P \cap Q$. Therefore, the number of ways in which none of $a_1, a_2, ...a_n$ belong to $P \cap Q$ is 3^n. **Ans. (D)**

Illustration 29 : There are 3 books of mathematics, 4 of science and 5 of english. How many different collections can be made such that each collection consists of-

(i) one book of each subject ?

(ii) at least one book of each subject ?

(iii) at least one book of english ?

Solution : (i) ${}^3C_1 \times {}^4C_1 \times {}^5C_1 = 60$

(ii) $(2^3-1)(2^4-1)(2^5-1) = 7 \times 15 \times 31 = 3255$

(iii) $(2^5-1)(2^3)(2^4) = 31 \times 128 = 3968$ **Ans.**

Illustration 30 : Find the number of groups that can be made from 5 red balls, 3 green balls and 4 black balls, if at least one ball of all colours is always to be included. Given that all balls are identical except colours.

Solution : After selecting one ball of each colour, we have to find total number of combinations that can be made from 4 red. 2 green and 3 black balls. These will be $(4 + 1)(2 + 1)(3 + 1)$ $= 60$ **Ans.**

Do yourself-8 :

(i) There are p copies each of n different books. Find the number of ways in which atleast one book can be selected ?

(ii) There are 10 questions in an examination. In how many ways can a candidate answer the questions, if he attempts atleast one question.

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10. DIVISORS :

Let $N = p^a . q^b . r^c$ ..... where p, q, r..... are distinct primes & a, b, c..... are natural numbers then :

(a) The total numbers of divisors of N including 1 & N is = $(a + 1)(b + 1)(c + 1)$.....

(b) The sum of these divisors is

$$= (p^0 + p^1 + p^2 + ...+ p^a)(q^0 + q^1 + q^2 + ...+ q^b)(r^0 + r^1 + r^2 + ...+ r^c)...$$

(c) Number of ways in which N can be resolved as a product of two factor is =

$$\frac{1}{2}(a+1)(b+1)(c+1)..... \text{ if N is not a perfect square}$$

$$\frac{1}{2}\left[(a+1)(b+1)(c+1).....+1\right] \text{ if N is a perfect square}$$

(d) Number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N.

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Note :

(i) Every natural number except 1 has atleast 2 divisors. If it has exactly two divisors then it is called a prime. System of prime numbers begin with 2. All primes except 2 are odd.

(ii) A number having more than 2 divisors is called composite. 2 is the only even number which is not composite.

(iii) Two natural numbers are said to be relatively prime or coprime if their HCF is one. For two natural numbers to be relatively prime, it is not necessary that one or both should be prime. It is possible that they both are composite but still coprime, eg. 4 and 25.

(iv) 1 is neither prime nor composite however it is co-prime with every other natural number.

(v) Two prime numbers are said to be twin prime numbers if their non-negative difference is 2 (e.g.5 & 7, 19 & 17 etc).

(vi) All divisors except 1 and the number itself are called proper divisors.

Illustration 31: Find the number of proper divisors of the number 38808. Also find the sum of these divisors.

Solution : **(i)** The number 38808 = $2^3 . 3^2 . 7^2 . 11$

Hence the total number of divisors (excluding 1 and itself i.e.38808)

$= (3 + 1) (2 + 1) (2 + 1) (1 + 1) - 2 = 70$

(ii) The sum of these divisors

$=(2^0 + 2^1 + 2^2 + 2^3) (3^0 + 3^1 + 3^2) (7^0 + 7^1 + 7^2) (11^0 + 11^1) - 1 - 38808$

$= (15) (13) (57) (12) - 1 - 38808 = 133380 - 1 - 38808 = 94571.$ **Ans.**

Illustration 32: In how many ways the number 18900 can be split in two factors which are relative prime (or coprime) ?

Solution: Here N = 18900 $= 2^2 . 3^3 . 5^2 . 7^1$

Number of different prime factors in 18900 = n = 4

Hence number of ways in which 18900 can be resolved into two factors which are relative prime (or coprime) = $2^{4-1} = 2^3 = 8$. **Ans.**

Illustration 33: Find the total number of proper factors of the number 35700. Also find

(i) sum of all these factors,

(ii) sum of the odd proper divisors,

(iii) the number of proper divisors divisible by 10 and the sum of these divisors.

Solution: $35700 = 5^2 \times 2^2 \times 3^1 \times 7^1 \times 17^1$

The total number of factors is equal to the total number of selections from (5,5), (2,2), (3), (7) and (17), which is given by $3 \times 3 \times 2 \times 2 \times 2 = 72$.

These include 1 and 35700. Therefore, the number of proper divisors (excluding 1 and 35700) is 72 – 2 = 70

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(i) Sum of all these factors (proper) is :

$(5^\circ + 5^1 + 5^2)\,(2^\circ + 2^1 + 2^2)\,(3^\circ + 3^1)\,(7^\circ + 7^1)\,(17^\circ + 17^1) - 1 - 35700$

$= 31 \times 7 \times 4 \times 8 \times 18 - 1 - 35700 = 89291$

(ii) The sum of odd proper divisors is :

$(5^\circ + 5^1 + 5^2)\,(3^\circ + 3^1)\,(7^\circ + 7^1)\,(17^\circ + 17^1) - 1$

$= 31 \times 4 \times 8 \times 18 - 1 = 17856 - 1 = 17855$

(iii) The number of proper divisors divisible by 10 is equal to number of selections from (5,5), (2,2), (3), (7), (17) consisting of at least one 5 and at least one 2 and 35700 is to be excluded and is given by $2 \times 2 \times 2 \times 2 \times 2 - 1 = 31$.

Sum of these divisors is :

$(5^1 + 5^2)\,(2^1 + 2^2)\,(3^\circ + 3^1)\,(7^\circ + 7^1)\,(17^\circ + 17^1) - 35700$

$= 30 \times 6 \times 4 \times 8 \times 18 - 35700 = 67980$ **Ans.**

Do yourself-9 :

(i) Find the number of ways in which the number 94864 can be resolved as a product of two factors.

(ii) Find the number of different sets of solution of xy = 1440.

11. TOTAL DISTRIBUTION :

(a) Distribution of distinct objects : Number of ways in which n distinct things can be distributed to p persons if there is no restriction to the number of things received by them is given by : p^n

(b) Distribution of alike objects : Number of ways to distribute n alike things among p persons so that each may get none, one or more thing(s) is given by ${}^{n+p-1}C_{p-1}$.

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Illustration 34: In how many ways can 5 different mangoes, 4 different oranges & 3 different apples be distributed among 3 children such that each gets alteast one mango ?

Solution : 5 different mangoes can be distributed by following ways among 3 children such that each gets atleast 1 :

3 1 1
2 2 1

Total number of ways : $\left(\frac{5!}{3!1!1!2!} + \frac{5!}{2!2!2!}\right) \times 3!$

Now, the number of ways of distributing remaining fruits (i.e. 4 oranges + 3 apples) among 3 children = 3^7 (as each fruit has 3 options).

$\therefore$ Total number of ways $= \left(\frac{5!}{3!2!} + \frac{5!}{(2!)^3}\right) \times 3! \times 3^7$ **Ans.**

Illustration 35: In how many ways can 12 identical apples be distributed among four children if each gets atleast 1 apple and not more than 4 apples.

Solution : Let x,y,z & w be the number of apples given to the children.

$\Rightarrow x + y + z + w = 12$

Giving one-one apple to each

Now, $x + y + z + w = 8$ .....(i)

Here, $0 \le x \le 3,\ 0 \le y \le 3,\ 0 \le z \le 3,\ 0 \le w \le 3$

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$x = 3 - t_1$, $y = 3 - t_2$, $z = 3 - t_3$, $w = 3 - t_4$.
Putting value of x, y, z, w in equation (i)
Put $12 - 8 = t_1 + t_2 + t_3 + t_4$
$\Rightarrow t_1 + t_2 + t_3 + t_4 = 4$
(Here max. value that t_1, t_2, t_3 & t_4 can attain is 3, so we have to remove those cases when any of t_i getting value 4)
$= {}^7C_3 -$ (all cases when atleast one is 4)
$= {}^7C_3 - 4 = 35 - 4 = 31$ **Ans.**

Illustration 36: Find the number of non negative integral solutions of the inequation $x + y + z \le 20$.

Solution : Let w be any number ($0 \le w \le 20$), then we can write the equation as :
$x + y + z + w = 20$ (here $x, y, z, w \ge 0$)
Total ways $= {}^{23}C_3$ **Ans.**

Illustration 37: Find the number of integral solutions of $x + y + z + w < 25$, where $x > -2$, $y > 1$, $z \ge 2$, $w \ge 0$.

Solution : Given $x + y + z + w < 25$
$x + y + z + w + v = 25$ .....(i)
Let $x = -1 + t_1$, $y = 2 + t_2$, $z = 2 + t_3$, $w = t_4$, $v = 1 + t_5$ where ($t_1, t_2, t_3, t_4 \ge 0$)
Putting value of x, y, z, w, v in equation (i)
$\Rightarrow t_1 + t_2 + t_3 + t_4 + t_5 = 21$.
Number of solutions $= {}^{25}C_4$ **Ans.**

Illustration 38: Find the number of positive integral solutions of the inequation $x + y + z \ge 150$, where $0 < x \le 60$, $0 < y \le 60$, $0 < z \le 60$.

Solution : Let $x = 60 - t_1$, $y = 60 - t_2$, $z = 60 - t_3$ (where $0 \le t_1 \le 59$, $0 \le t_2 \le 59$, $0 \le t_3 \le 59$)
Given $x + y + z \ge 150$
or $x + y + z - w = 150$ (where $0 \le w \le 147$) .....(i)
Putting values of x, y, z in equation (i)
$60 - t_1 + 60 - t_2 + 60 - t_3 - w = 150$
$30 = t_1 + t_2 + t_3 + w$
Total solutions $= {}^{33}C_3$ **Ans.**

Illustration 39: Find the number of positive integral solutions of $xy = 12$

Solution : $xy = 12$
$xy = 2^2 \times 3^1$
(i) 3 has 2 ways either 3 can go to x or y
(ii) 2^2 can be distributed between x & y as distributing 2 identical things between 2 persons
(where each person can get 0, 1 or 2 things). Let two person be ℓ_1 & ℓ_2
$\Rightarrow \ell_1 + \ell_2 = 2$
$\Rightarrow {}^{2+1}C_1 = {}^3C_1 = 3$
So total ways $= 2 \times 3 = 6$.

Alternatively :
$xy = 12 = 2^2 \times 3^1$
$x = 2^{a_1}3^{a_2}$ $0 \le a_1 \le 2$

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$0 \le a_2 \le 1$

$y = 2^{b_1}3^{b_2}$ $\quad 0 \le b_1 \le 2$

$0 \le b_2 \le 1$

$2^{a_1+b_1}3^{a_2+b_2} = 2^2 3^1$

$\Rightarrow a_1 + b_1 = 2 \quad \rightarrow {}^3C_1$ ways

$a_2 + b_2 = 1 \quad \rightarrow {}^2C_1$ ways

Number of solutions $= {}^3C_1 \times {}^2C_1 = 3 \times 2 = 6$ **Ans.**

Illustration 40 : Find the number of solutions of the equation xyz = 360 when **(i)** $x,y,z \in N$ **(ii)** $x,y,z \in I$

Solution : **(i)** $xyz = 360 = 2^3 \times 3^2 \times 5$ $(x,y,z \in N)$

$x = 2^{a_1}3^{a_2}5^{a_3}$ (where $0 \le a_1 \le 3, 0 \le a_2 \le 2, 0 \le a_3 \le 1$)

$y = 2^{b_1}3^{b_2}5^{b_3}$ (where $0 \le b_1 \le 3, 0 \le b_2 \le 2, 0 \le b_3 \le 1$)

$z = 2^{c_1}3^{c_2}5^{c_3}$ (where $0 \le c_1 \le 3, 0 \le c_2 \le 2, 0 \le c_3 \le 1$)

$\Rightarrow 2^{a_1}3^{a_2}5^{a_3}.2^{b_1}3^{b_2}5^{b_3}.2^{c_1}3^{c_2}5^{c_3} = 2^3 \times 3^2 \times 5^1$

$\Rightarrow 2^{a_1+b_1+c_1}.3^{a_2+b_2+c_2}.5^{a_3+b_3+c_3} = 2^3 \times 3^3 \times 5^1$

$\Rightarrow a_1 + b_1 + c_1 = 3 \rightarrow {}^5C_2 = 10$

$a_2 + b_2 + c_2 = 2 \rightarrow {}^4C_2 = 6$

$a_3 + b_3 + c_3 = 1 \rightarrow {}^3C_2 = 3$

Total solutions $= 10 \times 6 \times 3 = 180$.

(ii) If $x,y,z \in I$ then, (a) all positive (b) 1 positive and 2 negative.

Total number of ways $= 180 + {}^3C_2 \times 180 = 720$ **Ans.**

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Do yourself -10 :

(i) In how many ways can 12 identical apples be distributed among 4 boys. (a) If each boy receives any number of apples. (b) If each boy receives atleast 2 apples.

(ii) Find the number of non-negative integral solutions of the equation $x + y + z = 10$.

(iii) Find the number of integral solutions of $x + y + z = 20$, if $x \ge -4$, $y \ge 1$, $z \ge 2$

12. DEARRANGEMENT :

There are n letters and n corresponding envelopes. The number of ways in which letters can be placed in the envelopes (one letter in each envelope) so that no letter is placed in correct envelope is

$$n!\left[1 - \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{(-1)^n}{n!}\right]$$

Proof : n letters are denoted by 1,2,3,.....,n. Let A_i denote the set of distribution of letters in envelopes (one letter in each envelope) so that the ith letter is placed in the corresponding envelope. Then, $n(A_i) = 1 \times (n-1)!$ [since the remaining n–1 letters can be placed in n –1 envelops in (n–1)! ways]

Then, $n(A_i \cap A_j)$ represents the number of ways where letters i and j can be placed in their corresponding envelopes. Then,

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$n(A_i \cap A_j) = 1 \times 1 \times (n-2)!$

Also $n(A_i \cap A_j \cap A_k) = 1 \times 1 \times 1 \times (n-3)!$

Hence, the required number is

$n(A_1' \cup A_2' \cup ..... \cup A_n') = n! - n(A_1 \cup A_2 \cup ..... \cup A_n)$

$$= n! - \left[\sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) + ... + (-1)^n \sum n(A_i \cap A_2 ..... \cap A_n)\right]$$

$$= n! - [{}^nC_1(n-1)! - {}^nC_2(n-2)! + {}^nC_3(n-3)! + ... + (-1)^{n-1} \times {}^nC_n 1]$$

$$= n! - \left[\frac{n!}{1!(n-1)!}(n-1)! - \frac{n!}{2!(n-2)!}(n-2)! + ... + (-1)^{n-1}\right] = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} + ... + \frac{(-1)^n}{n!}\right]$$

Illustration 41: A person writes letters to six friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that

(i) all the letters are in the wrong envelopes.

(ii) at least two of them are in the wrong envelopes.

Solution : (i) The number of ways is which all letters be placed in wrong envelopes

$$= 6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right) = 720\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}\right)$$

$= 360 - 120 + 30 - 6 + 1 = 265.$

(i) The number of ways in which at least two of them in the wrong envelopes

$$= {}^6C_4 \,.\, 2!\left(1 - \frac{1}{1!} + \frac{1}{2!}\right) + {}^6C_3 \,.\, 3!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right) + {}^6C_2 \,.\, 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$

$$+ {}^6C_1 .\, 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) + {}^6C_0\, 6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$$

$= 15 + 40 + 135 + 264 + 265 = 719.$ **Ans.**

Do yourself - 11 :

(i) There are four balls of different colours and four boxes of colours same as those of the balls. Find the number of ways in which the balls, one in each box, could be placed in such a way that a ball does not go to box of its own colour.

Miscellaneous Illustrations :

Illustration 42: In how many ways can a person go from point A to point B if he can travel only to the right or upward along the lines (Grid Problem) ?

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Solution : To reach the point B from point A, a person has to travel along 3 horizontal and 3 vertical strips. Therefore, we have to arrange 3H and 3V in a row. Total number of ways = $\frac{6!}{3!3!} = 20$ ways **Ans.**

Illustration 43: Find sum of all numbers formed using the digits 2,4,6,8 taken all at a time and no digit being repeated.

Solution : All possible numbers = 4! = 24

If 2 occupies the unit's place then total numbers = 6

Hence, 2 comes at unit's place 6 times.

Sum of all the digits occuring at unit's place

$= 6 \times (2 + 4 + 6 + 8)$

Same summation will occur for ten's, hundred's & thousand's place. Hence required sum

$= 6 \times (2 + 4 + 6 + 8) \times (1 + 10 + 100 + 1000) = 133320$ **Ans.**

Illustration 44: Find the sum of all the numbers greater than 1000 using the digits 0,1,2,2.

Solution : (i) When 1 is at thousand's place, total numbers formed will be $= \frac{3!}{2!} = 3$

(ii) When 2 is at thousand's place, total numbers formed will be = 3! = 6

(iii) When 1 is at hundred's, ten's or unit's place then total numbers formed will be-

Thousand's place is fixed i.e. only the digit 2 will come here, remaining two places can be filled in 2! ways.

So total numbers = 2!

(iv) When 2 is at hundred's, ten's or unit's place then total numbers formed will be-

Thousand's place has 2 options and other two places can be filled in 2 ways.

So total numbers $= 2 \times 2 = 4$

$\text{Sum} = 10^3(1 \times 3 + 2 \times 6) + 10^2(1 \times 2 + 2 \times 4) + 10^1(1 \times 2 + 2 \times 4) + (1 \times 2 + 2 \times 4)$

$= 15 \times 10^3 + 10^3 + 10^2 + 10$

$= 16110$ **Ans.**

Illustration 45 : Find the number of positive integral solutions of $x + y + z = 20$, if $x \neq y \neq z$.

Solution : $x \geq 1$

$y = x + t_1 \qquad t_1 \geq 1$

$z = y + t_2 \qquad t_2 \geq 1$

$x + x + t_1 + x + t_1 + t_2 = 20$

$3x + 2t_1 + t_2 = 20$

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(i) $x = 1$ $\quad 2t_1 + t_2 = 17$

$t_1 = 1,2 \ldots\ldots\ldots 8 \Rightarrow 8$ ways

(ii) $x = 2$ $\quad 2t_1 + t_2 = 14$

$t_1 = 1,2 \ldots\ldots\ldots 6 \Rightarrow 6$ ways

(iii) $x = 3$ $\quad 2t_1 + t_2 = 11$

$t_1 = 1,2 \ldots\ldots\ldots 5 \Rightarrow 5$ ways

(vi) $x = 4$ $\quad 2t_1 + t_2 = 8$

$t_1 = 1,2,3 \Rightarrow 3$ ways

(v) $x = 5$ $\quad 2t_1 + t_2 = 5$

$t_1 = 1, 2 \Rightarrow 2$ ways

Total $= 8 + 6 + 5 + 3 + 2 = 24$

But each solution can be arranged by 3! ways.

So total solutions $= 24 \times 3! = 144$. **Ans.**

Illustration 46: A regular polygon of 15 sides is constructed. In how many ways can a triangle be formed using the vertices of the polygon such that no side of triangle is same as that of polygon ?

Solution : Select one point out of 15 point, therefore total number of ways $= {}^{15}C_1$

Suppose we select point P_1. Now we have to choose 2 more point which are not consecutive.

since we can not select P_2 & P_{15}.

Total points left are 12.

Now we have to select 2 points out of 12 points which are not consecutive

Total ways $= {}^{12-2+1}C_2 = {}^{11}C_2$

Every select triangle will be repeated 3 times.

So total number of ways $= \dfrac{{}^{15}C_1 \times {}^{11}C_2}{3} = 275$

Alternative :

First of all let us cut the polygon between points P_1 & P_{15}. Now there are 15 points on a straight line and we have to select 3 points out of these, such that the selected points are not consecutive.

x O y O z O w

Here bubbles represents the selected points,

x represents the number of points before first selected point,

y represents the number of points between Ist & IInd selected point,

z represents the number of points between IInd & IIIrd selected point

and w represents the number of points after IIIrd selected point.

$x + y + z + w = 15 - 3 = 12$

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here $x \geq 0$, $y \geq 1$, $z \geq 1$, $w \geq 0$

Put $y = 1 + y'$ & $z = 1 + z'$ $(y' \geq 0, z' \geq 0)$

$\Rightarrow x + y' + z' + w = 10$

Total number of ways = ${}^{13}C_3$

These selections include the cases when both the points P_1 & P_{15} are selected. We have to remove those cases. Here a represents number of points between P_1 & 3rd selected point & b represents number of points between 3rd selected point and P_{15}

$\Rightarrow a + b = 15 - 3 = 12 \quad (a \geq 1, b \geq 1)$

put $a = 1 + t_1$ & $b = 1 + t_2$

$t_1 + t_2 = 10$

Total number of ways = ${}^{11}C_1 = 11$

Therefore required number of ways = ${}^{13}C_3 - {}^{11}C_1 = 286 - 11 = 275$ **Ans.**

Illustration 47: Find the number of ways in which three numbers can be selected from the set $\{5^1, 5^2, 5^3, ...5^{11}\}$ so that they form a G.P.

Solution : Any three selected numbers which are in G.P. have their powers in A.P.

Set of powers is = $\{1,2,.....6,7,.....11\}$

By selecting any two numbers from $\{1,3,5,7,9,11\}$, the middle number is automatically fixed. Total number of ways = ${}^6C_2$

Now select any two numbers from $\{2,4,6,8,10\}$ and again middle number is automatically fixed. Total number of ways = ${}^5C_2$

$\therefore$ Total number of ways are = ${}^6C_2 + {}^5C_2 = 15 + 10 = 25$ **Ans.**

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EXERCISE (O-1)

ONLY ONE CORRECT :

1. The number of different seven digit numbers that can be written using only three digits 1, 2 & 3 under the condition that the digit 2 occurs exactly twice in each number is

(A) 672 (B) 640 (C) 512 (D) none

PC0001

2. How many of the 900 three digit numbers have at least one even digit?

(A) 775 (B) 875 (C) 450 (D) 750

PC0002

3. Number of natural numbers between 100 and 1000 such that at least one of their digits is 7, is

(A) 225 (B) 243 (C) 252 (D) none

PC0003

4. Number of 4 digit numbers of the form N = abcd which satisfy following three conditions :

(i) $4000 \le N < 6000$ (ii) N is multiple of 5 (iii) $3 \le b < c \le 6$

is equal to

(A) 12 (B) 18 (C) 24 (D) 48

PC0004

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5. Consider the five points comprising of the vertices of a square and the intersection point of its diagonals. How many triangles can be formed using these points?

(A) 4 (B) 6 (C) 8 (D) 10

PC0005

6. The number of ways in which 5 different books can be distributed among 10 people if each person can get at most one book is :

(A) 252 (B) 10^5 (C) 5^{10} (D) ${}^{10}C_5.5!$

PC0006

7. A student has to answer 10 out of 13 questions in an examination . The number of ways in which he can answer if he must answer atleast 3 of the first five questions is :

(A) 276 (B) 267 (C) 80 (D) 1200

PC0007

8. A question paper on mathematics consists of twelve questions divided into three parts A, B and C, each containing four questions. In how many ways can an examinee answer five questions, selecting atleast one from each part.

(A) 624 (B) 208 (C) 1248 (D) 2304

PC0008

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9. 5 Indian & 5 American couples meet at a party & shake hands . If no wife shakes hands with her own husband & no Indian wife shakes hands with a male, then the number of hand shakes that takes place in the party is :

(A) 95 (B) 110 (C) 135 (D) 150

PC0009

10. The kindergarten teacher has 25 kids in her class . She takes 5 of them at a time, to zoological garden as often as she can, without taking the same 5 kids more than once. Then the number of visits, the teacher makes to the garden exceeds that of a kid by :

(A) ${}^{25}C_5 - {}^{24}C_5$ (B) ${}^{24}C_5$ (C) ${}^{24}C_4$ (D) none

PC0010

11. Number of cyphers at the end of ${}^{2002}C_{1001}$ is

(A) 0 (B) 1 (C) 2 (D) 200

PC0011

12. Three vertices of a convex n sided polygon are selected. If the number of triangles that can be constructed such that none of the sides of the triangle is also the side of the polygon is 30, then the polygon is a

(A) Heptagon (B) Octagon (C) Nonagon (D) Decagon

PC0012

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13. There are 12 books on Algebra and Calculus in our library , the books of the same subject being different. If the number of selections each of which consists of 3 books on each topic is greatest then the number of books of Algebra and Calculus in the library are respectively:

(A) 3 and 9 (B) 4 and 8 (C) 5 and 7 (D) 6 and 6

PC0013

14. Out of seven consonants and four vowels, the number of words of six letters, formed by taking four consonants and two vowels is (Assume that each ordered group of letter is a word):

(A) 210 (B) 462 (C) 151200 (D) 332640

PC0014

15. The number of six digit numbers that can be formed from the digits 1, 2, 3, 4, 5, 6 & 7 so that digits do not repeat and the terminal digits are even is :

(A) 144 (B) 72 (C) 288 (D) 720

PC0015

16. If the letters of the word "VARUN" are written in all possible ways and then are arranged as in a dictionary, then the rank of the word VARUN is :

(A) 98 (B) 99 (C) 100 (D) 101

PC0016

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17. A new flag is to be designed with six vertical strips using some or all of the colours yellow, green, blue and red. Then, the number of ways this can be done such that no two adjacent strips have the same colour is -

(A) 12×81 (B) 16×192 (C) 20×125 (D) 24×216

PC0017

18. Number of 5 digit numbers which are divisible by 5 and each number containing the digit 5, digits being all different is equal to k(4!), the value of k is

(A) 84 (B) 168 (C) 188 (D) 208

PC0018

19. A 5 digit number divisible by 3 is to be formed using the numerals 0, 1, 2, 3, 4 & 5 without repetition. The total number of ways this can be done is :

(A) 3125 (B) 600 (C) 240 (D) 216

PC0019

20. The number of n digit numbers which consists of the digits 1 & 2 only if each digit is to be used atleast once, is equal to 510 then n is equal to

(A) 7 (B) 8 (C) 9 (D) 10

PC0020

21. If m denotes the number of 5 digit numbers if each successive digits are in their descending order of magnitude and n is the corresponding figure, when the digits are in their ascending order of magnitude then $(m - n)$ has the value

(A) ${}^{10}C_4$ (B) ${}^{9}C_5$ (C) ${}^{10}C_3$ (D) ${}^{9}C_3$

PC0021

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22. A rack has 5 different pairs of shoes. The number of ways in which 4 shoes can be chosen from it , so that there will be no complete pair is :

(A) 1920 (B) 200 (C) 110 (D) 80

PC0022

23. Number of ways in which 8 people can be arranged in a line if A and B must be next each other and C must be somewhere behind D, is equal to

(A) 10080 (B) 5040 (C) 5050 (D) 10100

PC0023

24. An old man while dialing a 7 digit telephone number remembers that the first four digits consists of one 1's, one 2's and two 3's. He also remembers that the fifth digit is either a 4 or 5 while has no memorising of the sixth digit, he remembers that the seventh digit is 9 minus the sixth digit. Maximum number of distinct trials he has to try to make sure that he dials the correct telephone number, is

(A) 360 (B) 240 (C) 216 (D) none

PC0024

25. Number of 5 digit numbers divisible by 25 that can be formed using only the digits 1, 2, 3, 4, 5 & 0 taken five at a time is

(A) 2 (B) 32 (C) 42 (D) 52

PC0025

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26. Let P_n denotes the number of ways of selecting 3 people out of 'n' sitting in a row, if no two of them are consecutive and Q_n is the corresponding figure when they are in a circle. If $P_n - Q_n = 6$, then 'n' is equal to :

(A) 8 (B) 9 (C) 10 (D) 12

PC0026

27. Number of 7 digit numbers the sum of whose digits is 61 is :

(A) 12 (B) 24 (C) 28 (D) none

PC0027

28. In a unique hockey series between India & Pakistan, they decide to play on till a team wins 5 matches. The number of ways in which the series can be won by India, if no match ends in a draw is :

(A) 126 (B) 252 (C) 225 (D) none

PC0028

29. There are 100 different books in a shelf. Number of ways in which 3 books can be selected so that no two of which are neighbours is

(A) ${}^{100}C_3 - 98$ (B) ${}^{97}C_3$ (C) ${}^{96}C_3$ (D) ${}^{98}C_3$

PC0029

30. Let P_n denotes the number of ways in which three people can be selected out of 'n' people sitting in a row, if no two of them are consecutive. If, $P_{n+1} - P_n = 15$ then the value of 'n' is :

(A) 7 (B) 8 (C) 9 (D) 10

PC0030

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MATCH THE COLUMN :

31.

	Column-I		Column-II
(A)	Number of increasing permutations of *m* symbols are there from the *n* set numbers $\{a_1, a_2, \ldots, a_n\}$ where the order among the numbers is given by $a_1 < a_2 < a_3 < \ldots a_{n-1} < a_n$ is	(P)	n^m
			PC0031
(B)	There are *m* men and *n* monkeys. Number of ways in which every monkey has a master, if a man can have any number of monkeys	(Q)	${}^mC_n$
			PC0032
(C)	Number of ways in which *n* red balls and $(m-1)$ green balls can be arranged in a line, so that no two red balls are together, is (balls of the same colour are alike)	(R)	${}^nC_m$
			PC0033
(D)	Number of ways in which '*m*' different toys can be distributed in '*n*' children if every child may receive any number of toys, is	(S)	m^n
			PC0034

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32. Number of permutations of 1, 2, 3, 4, 5, 6, 7, 8 and 9 taken all at a time, such that the digit

1 appearing somewhere to the left of 2

3 appearing to the left of 4 and

5 somewhere to the left of 6, is

(*e.g.* 815723946 would be one such permutation)

(A) $9 \cdot 7!$ (B) $8!$ (C) $5! \cdot 4!$ (D) $8! \cdot 4!$

PC0035

33. Seven different coins are to be divided amongst three persons . If no two of the persons receive the same number of coins but each receives atleast one coin & none is left over, then the number of ways in which the division may be made is

(A) 420 (B) 630 (C) 710 (D) none

PC0036

34. Number of ways in which 9 different toys be distributed among 4 children belonging to different age groups in such a way that distribution among the 3 elder children is even and the youngest one is to receive one toy more, is :

(A) $\frac{(5!)^2}{8}$ (B) $\frac{9!}{2}$ (C) $\frac{9!}{3!\,(2!)^3}$ (D) none

PC0037

35. A gentleman invites a party of $m + n$ $(m \neq n)$ friends to a dinner & places m at one table T_1 and n at another table T_2 , the table being round . If not all people shall have the same neighbour in any two arrangement, then the number of ways in which he can arrange the guests, is

(A) $\frac{(m+n)!}{4\ mn}$ (B) $\frac{1}{2}\frac{(m+n)!}{mn}$ (C) $2\ \frac{(m+n)!}{mn}$ (D) none

PC0038

36. A team of 8 students goes on an excursion, in two cars, of which one can seat 5 and the other only 4. If internal arrangement inside the car does not matter then the number of ways in which they can travel, is

(A) 91 (B) 182 (C) 126 (D) 3920

PC0039

37. Let m denote the number of ways in which 4 different books are distributed among 10 persons, each receiving none or one only and let n denote the number of ways of distribution if the books are all alike. Then :

(A) $m = 4n$ (B) $n = 4m$ (C) $m = 24n$ (D) none

PC0040

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38. There are (p + q) different books on different topics in Mathematics. ($p \neq q$)
If L = The number of ways in which these books are distributed between two students X and Y such that X get p books and Y gets q books.
M = The number of ways in which these books are distributed between two students X and Y such that one of them gets p books and another gets q books.
N = The number of ways in which these books are divided into two groups of p books and q books then,
(A) L = M = N (B) L = 2M = 2N (C) 2L = M = 2N (D) L = M = 2N

PC0041

39. Number of ways in which 7 green bottles and 8 blue bottles can be arranged in a row if exactly 1 pair of green bottles is side by side, is (Assume all bottles to be alike except for the colour).
(A) 84 (B) 360 (C) 504 (D) none

PC0042

40. There are 10 red balls of different shades & 9 green balls of identical shades. Then the number of arranging them in a row so that no two green balls are together is
(A) $(10\,!)\,.\,{}^{11}P_9$ (B) $(10\,!)\,.\,{}^{11}C_9$
(C) 10 ! (D) 10 ! 9 !

PC0043

41. There are 2 identical white balls, 3 identical red balls and 4 green balls of different shades. The number of ways in which they can be arranged in a row so that atleast one ball is separated from the balls of the same colour, is :
(A) 6 (7 ! – 4 !) (B) 7 (6 ! – 4 !) (C) 8 ! – 5 ! (D) none

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PC0044

42. Number of ways in which 5 A's and 6 B's can be arranged in a row which reads the same backwards and forwards, is
(A) 6 (B) 8 (C) 10 (D) 12

PC0045

43. Number of ways in which four different toys and five indistinguishable marbles can be distributed between Amar, Akbar and Anthony, if each child receives atleast one toy and one marble, is
(A) 42 (B) 100 (C) 150 (D) 216

PC0046

44. There are counters available in x different colours. The counters are all alike except for the colour. The total number of arrangements consisting of y counters, assuming sufficient number of counters of each colour, if no arrangement consists of all counters of the same colour is :
(A) $x^y - x$ (B) $x^y - y$ (C) $y^x - x$ (D) $y^x - y$

PC0047

45. The number of ways in which 8 distinguishable apples can be distributed among 3 boys such that every boy should get atleast 1 apple & atmost 4 apples is $K \cdot {}^7P_3$ where K has the value equal to
(A) 14 (B) 66 (C) 44 (D) 22

PC0048

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46. There are six periods in each working day of a school. Number of ways in which 5 subjects can be arranged if each subject is allotted at least one period and no period remains vacant is

(A) 210 (B) 1800 (C) 360 (D) 3600

PC0049

47. Number of positive integral solutions satisfying the equation $(x_1 + x_2 + x_3)(y_1 + y_2) = 77$, is

(A) 150 (B) 270 (C) 420 (D) 1024

PC0050

48. There are counters available in 3 different colours (atleast four of each colour). Counters are all alike except for the colour. If 'm' denotes the number of arrangements of four counters if no arrangement consists of counters of same colour and 'n' denotes the corresponding figure when every arrangement consists of counters of each colour, then :

(A) m = 2n (B) 6m = 13n (C) 3m = 5n (D) 5m = 3n

PC0051

49. One hundred management students who read at least one of the three business magazines are surveyed to study the readership pattern. It is found that 80 read Business India, 50 read Business world, and 30 read Business Today. Five students read all the three magazines. How many read exactly two magazines?

(A) 50 (B) 10 (C) 95 (D) 65

PC0052

50. A person writes letters to his 5 friends and addresses the corresponding envelopes. Number of ways in which the letters can be placed in the envelope, so that atleast two of them are in the wrong envelopes,is,

(A) 1 (B) 2 (C) 118 (D) 119

PC0053

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EXERCISE (O-2)

ONLY ONE CORRECT :

1. A committee of 5 is to be chosen from a group of 9 people. Number of ways in which it can be formed if two particular persons either serve together or not at all and two other particular persons refuse to serve with each other, is

(A) 41 (B) 36 (C) 47 (D) 76

PC0054

2. There are m points on a straight line AB & n points on the line AC none of them being the point A. Triangles are formed with these points as vertices, when

(i) A is excluded (ii) A is included. The ratio of number of triangles in the two cases is:

(A) $\frac{m+n-2}{m+n}$ (B) $\frac{m+n-2}{m+n-1}$ (C) $\frac{m+n-2}{m+n+2}$ (D) $\frac{m(n-1)}{(m+1)(n+1)}$

PC0055

3. There are 10 straight lines in a plane, such that no 3 are concurrent and no 2 are parallel to each other. If points of intersection of above lines are joined, then maximum number of lines thus formed are (including old lines) -

(A) 610 (B) 620 (C) 630 (D) 640

PC0056

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4. Number of rectangles in the grid shown which are not squares is

(A) 160 (B) 162 (C) 170 (D) 185

PC0057

5. Given 11 points, of which 5 lie on one circle, other than these 5, no 4 lie on one circle. Then the maximum number of circles that can be drawn so that each contains atleast three of the given points is :

(A) 216 (B) 156 (C) 172 (D) none

PC0058

6. The number of ways of choosing a committee of 2 women & 3 men from 5 women & 6 men, if Mr. A refuses to serve on the committee if Mr. B is a member & Mr. B can only serve, if Miss C is the member of the committee, is

(A) 60 (B) 84 (C) 124 (D) none

PC0059

7. Product of all the even divisors of N = 1000, is

(A) $32 \cdot 10^2$ (B) $64 \cdot 2^{14}$ (C) $64 \cdot 10^{18}$ (D) $128 \cdot 10^6$

PC0060

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8. Two classrooms A and B having capacity of 25 and (n–25) seats respectively. A_n denotes the number of possible seating arrangements of room 'A', when 'n' students are to be seated in these rooms, starting from room 'A' which is to be filled up full to its capacity. If $A_n - A_{n-1} = 25!\,({}^{49}C_{25})$ then 'n' equals -

(A) 50 (B) 48 (C) 49 (D) 51

PC0061

MORE THAN ONE ARE CORRECT :

9. Lines y = x + i & y = –x + j are drawn in x – y plane such that $i \in \{1,2,3,4\}$ & $j \in \{1,2,3,4,5,6\}$. If m represents the total number of squares formed by the lines and n represents the total number of triangles formed by the given lines & x-axis, then correct option/s is/are-

(A) m + n = 50 (B) m – n = 2 (C) m + n = 48 (D) m – n = 4

PC0062

10. The combinatorial coefficient C(n, r) is equal to

(A) number of possible subsets of r members from a set of n distinct members.

(B) number of possible binary messages of length n with exactly r 1's.

(C) number of non decreasing 2-D paths from the lattice point (0, 0) to (r, n).

(D) number of ways of selecting r things out of n different things when a particular thing is always included plus the number of ways of selecting 'r' things out of n, when a particular thing is always excluded.

PC0063

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11. There are 10 questions, each question is either True or False. Number of different sequences of incorrect answers is also equal to

(A) Number of ways in which a normal coin tossed 10 times would fall in a definite order if both Heads and Tails are present.

(B) Number of ways in which a multiple choice question containing 10 alternatives with one or more than one correct alternatives, can be answered.

(C) Number of ways in which it is possible to draw a sum of money with 10 coins of different denominations taken some or all at a time.

(D) Number of different selections of 10 indistinguishable things taken some or all at a time.

PC0064

12. Number of ways in which 3 numbers in A.P. can be selected from 1, 2, 3, ... n is :

(A) $\left(\frac{n-1}{2}\right)^2$ if n is even

(B) $\frac{n(n-2)}{4}$ if n is odd

(C) $\frac{(n-1)^2}{4}$ if n is odd

(D) $\frac{n(n-2)}{4}$ if n is even

PC0065

13. The combinatorial coefficient ${}^{n-1}C_p$ denotes

(A) the number of ways in which *n* things of which p are alike and rest different can be arranged in a circle.

(B) the number of ways in which *p* different things can be selected out of *n* different thing if a particular thing is always excluded.

(C) number of ways in which *n* alike balls can be distributed in *p* different boxes so that no box remains empty and each box can hold any number of balls.

(D) the number of ways in which (n – 2) white balls and p black balls can be arranged in a line if black balls are separated, balls are all alike except for the colour.

PC0066

14. In a certain strange language, words are written with letters from the following six-letter alphabet : A, G, K, N, R, U. Each word consists of six letters and none of the letters repeat. Each combination of these six letters is a word in this language. The word "KANGUR" remains in the dictionary at,

(A) 248th (B) 247th (C) 246th (D) 253rd

PC0067

15. Six people are going to sit in a row on a bench. A and B are adjacent, C does not want to sit adjacent to D. E and F can sit anywhere. Number of ways in which these six people can be seated, is

(A) 200 (B) 144 (C) 120 (D) 56

PC0068

16. Six married couple are sitting in a room. Find the number of ways in which 4 people can be selected so that :

(A) they do not form a couple

(B) they form exactly one couple

(C) they form at least one couple

(D) they form atmost one couple

PC0069

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17. The number of three digit numbers having only two consecutive digits identical is :
(A) 153 (B) 162 (C) 180 (D) 161

PC0070

18. Number of 3 digit numbers in which the digit at hundredth's place is greater than the other two digit is
(A) 285 (B) 281 (C) 240 (D) 204

PC0071

19. All possible three digits even numbers which can be formed with the condition that if 5 is one of the digit, then 7 is the next digit is :
(A) 5 (B) 325 (C) 345 (D) 365

PC0072

Paragraph for Question Nos. 20 to 22

16 players $P_1, P_2, P_3, \ldots\ldots P_{16}$ take part in a tennis tournament. Lower suffix player is better than any higher suffix player. These players are to be divided into 4 groups each comprising of 4 players and the best from each group is selected for semifinals.

20. Number of ways in which 16 players can be divided into four equal groups, is

(A) $\frac{35}{27}\prod_{r=1}^{8}(2r-1)$ (B) $\frac{35}{24}\prod_{r=1}^{8}(2r-1)$ (C) $\frac{35}{52}\prod_{r=1}^{8}(2r-1)$ (D) $\frac{35}{6}\prod_{r=1}^{8}(2r-1)$

PC0073

21. Number of ways in which they can be divided into 4 equal groups if the players P_1, P_2, P_3 and P_4 are in different groups, is :

(A) $\frac{(11)!}{36}$ (B) $\frac{(11)!}{72}$ (C) $\frac{(11)!}{108}$ (D) $\frac{(11)!}{216}$

PC0073

22. Number of ways in which these 16 players can be divided into four equal groups, such that when the best player is selected from each group, P_6 is one among them, is (k) $\frac{12!}{(4!)^3}$. The value of k is :

(A) 36 (B) 24 (C) 18 (D) 20

PC0073

23. Consider the word W = MISSISSIPPI

(a) If N denotes the number of different selections of 5 letters from the word W = MISSISSIPPI then N belongs to the set
(A) {15, 16, 17, 18, 19} (B) {20, 21, 22, 23, 24}
(C) {25, 26, 27, 28, 29} (D) {30, 31, 32, 33, 34}

(b) Number of ways in which the letters of the word W can be arranged if atleast one vowel is separated from rest of the vowels

(A) $\frac{8!\cdot 161}{4!\cdot 4!\cdot 2!}$ (B) $\frac{8!\cdot 161}{4\cdot 4!\cdot 2!}$ (C) $\frac{8!\cdot 161}{4!\cdot 2!}$ (D) $\frac{8!}{4!\cdot 2!}\cdot\frac{165}{4!}$

(c) If the number of arrangements of the letters of the word W if all the S's and P's are separated is (K)$\left(\frac{10!}{4!\cdot 4!}\right)$, then K equals -

(A) $\frac{6}{5}$ (B) 1 (C) $\frac{4}{3}$ (D) $\frac{3}{2}$

PC0074

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24. The maximum number of permutations of 2n letters in which there are only a's & b's, taken all at a time is given by :

(A) ${}^{2n}C_n$

(B) $\frac{2}{1} \cdot \frac{6}{2} \cdot \frac{10}{3} \ldots\ldots \frac{4n-6}{n-1} \cdot \frac{4n-2}{n}$

(C) $\frac{n+1}{1} \cdot \frac{n+2}{2} \cdot \frac{n+3}{3} \cdot \frac{n+4}{4} \ldots\ldots \frac{2n-1}{n-1} \cdot \frac{2n}{n}$

(D) $\frac{2^n \cdot [1 \cdot 3 \cdot 5 \ldots\ldots (2n-3)(2n-1)]}{n!}$

PC0075

25. Number of ways in which the letters of the word 'B U L B U L' can be arranged in a line in a definite order is also equal to the

(A) number of ways in which 2 alike Apples and 4 alike Mangoes can be distributed in 3 children so that each child receives any number of fruits.

(B) Number of ways in which 6 different books can be tied up into 3 bundles, if each bundle is to have equal number of books.

(C) coefficient of $x^2y^2z^2$ in the expansion of $(x + y + z)^6$.

(D) number of ways in which 6 different prizes can be distributed equally in three children.

PC0076

26. Which of the following statements are correct?

(A) Number of words that can be formed with 6 only of the letters of the word "CENTRIFUGAL" if each word must contain all the vowels is $3 \cdot 7!$

(B) There are 15 balls of which some are white and the rest black. If the number of ways in which the balls can be arranged in a row, is maximum then the number of white balls must be equal to 7 or 8. Assume balls of the same colour to be alike.

(C) There are 12 things, 4 alike of one kind, 5 alike and of another kind and the rest are all different. The total number of combinations is 240.

(D) Number of selections that can be made of 6 letters from the word "COMMITTEE" is 35.

PC0077

MATCH THE COLUMN:

27.

	Column-I	Column-II
(A)	Four different movies are running in a town. Ten students go to watch these four movies. The number of ways in which every movie is watched by atleast one student, is (Assume each way differs only by number of students watching a movie)	(P) 11 (Q) 36

PC0078

(B) Consider 8 vertices of a regular octagon and its centre. If T denotes the number of triangles and S denotes the number of straight lines that can be formed with these 9 points then the value of (T – S) equals

PC0079

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(C) In an examination, 5 children were found to have their mobiles in their pocket. The Invigilator fired them and took their mobiles in his possession. Towards the end of the test, Invigilator randomly returned their mobiles. The number of ways in which at most two children did not get their own mobiles is (R) 52 (S) 60

PC0080

(D) The product of the digits of 3214 is 24. The number of 4 digit natural numbers such that the product of their digits is 12, is

PC0081

(E) The number of ways in which a mixed double tennis game can be arranged from amongst 5 married couple if no husband & wife plays in the same game, is (T) 84

PC0082

28. A guardian with 6 wards wishes everyone of them to study either Law or Medicine or Engineering. Number of ways in which he can make up his mind with regard to the education of his wards if every one of them be fit for any of the branches to study, and atleast one child is to be sent in each discipline is:

(A) 120 (B) 216 (C) 729 (D) 540

PC0083

EXERCISE (S-1)

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1. Four visitors A, B, C & D arrive at a town which has 5 hotels. In how many ways can they disperse themselves among 5 hotels, if 4 hotels are used to accommodate them.

PC0084

2. There are 6 roads between A & B and 4 roads between B & C.

(i) In how many ways can one drive from A to C by way of B ?

(ii) In how many ways can one drive from A to C and back to A, passing through B on both trips ?

(iii) In how many ways can one drive the circular trip described in (ii) without using the same road more than once.

PC0085

3. (i) Find the number of four letter word that can be formed from the letters of the word HISTORY. (each letter to be used atmost once)

(ii) How many of them contain only consonants?

(iii) How many of them begin & end in a consonant?

(iv) How many of them begin with a vowel?

(v) How many contain the letters Y?

(vi) How many begin with T & end in a vowel?

(vii) How many begin with T & also contain S?

(viii) How many contain both vowels?

PC0086

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4. If repetitions are not permitted

(i) How many 3 digit numbers can be formed from the six digits 2, 3, 5, 6, 7 & 9 ?

(ii) How many of these are less than 400 ?

(iii) How many are even ?

(iv) How many are odd ?

(v) How many are multiples of 5 ?

PC0087

5. How many two digit numbers are there in which the tens digit and the units digit are different and odd ?

PC0088

6. Every telephone number consists of 7 digits. How many telephone numbers are there which do not include any other digits but 2, 3, 5 & 7 ?

PC0089

7. (a) In how many ways can four passengers be accommodated in three railway carriages, if each carriage can accommodate any number of passengers.

PC0090

(b) In how many ways four persons can be accommodated in 3 different chairs if each person can occupy only one chair.

PC0091

8. How many odd numbers of five distinct digits can be formed with the digits 0,1,2,3,4 ?

PC0092

9. Number of ways in which 7 different colours in a rainbow can be arranged if green is always in the middle.

PC0093

10. Find the number of ways in which the letters of the word "MIRACLE" can be arranged if vowels always occupy the odd places.

PC0094

11. A letter lock consists of three rings each marked with 10 different letters. Find the number of ways in which it is possible to make an unsuccessful attempts to open the lock.

PC0095

12. (i) Prove that : ${}^{n}P_r = {}^{n-1}P_r + r.\ {}^{n-1}P_{r-1}$

PC0096

(ii) If ${}^{20}C_{r+2} = {}^{20}C_{2r-3}$, find ${}^{12}C_r$.

PC0097

(iii) Prove that : ${}^{n-1}C_3 + {}^{n-1}C_4 > {}^{n}C_3$ if $n > 7$.

PC0098

(iv) Find r if ${}^{15}C_{3r} = {}^{15}C_{r+3}$.

PC0099

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13. Find the number of ways in which two squares can be selected from an 8 by 8 chess board of size 1×1 so that they are not in the same row and in the same column.

PC0100

14. There are 10 seats in a double decker bus, 6 in the lower deck and 4 on the upper deck. Ten passengers board the bus, of them 3 refuse to go to the upper deck and 2 insist on going up. The number of ways in which the passengers can be accommodated is _____. (Assume all seats to be duly numbered)

PC0101

15. In a certain algebraical exercise book there are 4 examples on arithmetical progressions, 5 examples on permutation-combination and 6 examples on binomial theorem . Number of ways a teacher can select for his pupils atleast one but not more than 2 examples from each of these sets, is _____.

PC0102

16. In how many ways can a team of 6 horses be selected out of a stud of 16 , so that there shall always be 3 out of $A\ B\ C\ A'\ B'\ C'$, but never $A\ A'$, $B\ B'$ or $C\ C'$ together.

PC0103

17. There are 2 women participating in a chess tournament. Every participant played 2 games with the other participants. The number of games that the men played between themselves exceeded by 66 as compared to the number of games that the men played with the women. Find the number of participants & the total numbers of games played in the tournament.

PC0104

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18. Each of 3 committees has 1 vacancy which is to be filled from a group of 6 people. Find the number of ways the 3 vacancies can be filled if ;

(i) Each person can serve on atmost 1 committee.

(ii) There is no restriction on the number of committees on which a person can serve.

(iii) Each person can serve on atmost 2 committees.

PC0105

19. Find the number of ways in which 3 distinct numbers can be selected from the set $\{3^1, 3^2, 3^3, ... 3^{100}, 3^{101}\}$ so that they form a G.P.

PC0106

20. Find the number of ways in which letters of the word VALEDICTORY be arranged so that the vowels may never be separated.

PC0107

21. An examination paper consists of 12 questions divided into parts A & B.
Part-A contains 7 questions & Part-B contains 5 questions. A candidate is required to attempt 8 questions selecting atleast 3 from each part. In how many maximum ways can the candidate select the questions ?

PC0108

22. 5 boys & 4 girls sit in a straight line. Find the number of ways in which they can be seated if 2 girls are together & the other 2 are also together but separate from the first 2.

PC0109

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23. During a draw of lottery, tickets bearing numbers 1, 2, 3,....., 40, 6 tickets are drawn out & then arranged in the descending order of their numbers. In how many ways, it is possible to have 4th ticket bearing number 25.

PC0110

24. Find the number of distinct natural numbers upto a maximum of 4 digits and divisible by 5, which can be formed with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 each digit not occuring more than once in each number.

PC0111

25. A firm of Chartered Accountants in Bombay has to send 10 clerks to 5 different companies, two clerks in each. Two of the companies are in Bombay and the others are outside. Two of the clerks prefer to work in Bombay while three others prefer to work outside. In how many ways can the assignment be made if the preferences are to be satisfied.

PC0112

26. Find the number of permutations of the word "AUROBIND" in which vowels appear in an alphabetical order.

PC0113

27. Define a 'good word' as a sequence of letters that consists only of the letters A, B and C and in which A never immediately followed by B, B is never immediately followed by C, and C is never immediately followed by A. If the number of n-letter good words are 384, find the value of n.

PC0114

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28. In how many different ways a grandfather along with two of his grandsons and four grand daughters can be seated in a line for a photograph so that he is always in the middle and the two grandsons are never adjacent to each other.

PC0115

29. If as many more words as possible be formed out of the letters of the word "DOGMATIC" then find the number of words in which the relative order of vowels and consonants remain unchanged.

PC0116

30. There are 10 different books in a shelf. Find the number of ways in which 3 books can be selected so that exactly two of them are consecutive.

PC0117

31. In how many ways can you divide a pack of 52 cards equally among 4 players. In how many ways the cards can be divided in 4 sets, 3 of them having 17 cards each & the 4th with 1 card.

PC0118

32. Find the number of ways in which the letters of the word 'KUTKUT' can be arranged so that no two alike letters are together.

PC0119

33. How many 6 digits odd numbers greater than 60,0000 can be formed from the digits 5, 6, 7, 8, 9, 0 if

(i) repetitions are not allowed

(ii) repetitions are allowed.

PC0120

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34. In how many other ways can the letters of the word **MULTIPLE** be arranged;

(i) without changing the order of the vowels

(ii) keeping the position of each vowel fixed &

(iii) without changing the relative order/position of vowels & consonants.

PC0121

Paragraph for Question 35 & 36

Consider the number N = 2910600.

On the basis of above information, answer the following questions :

35. Total number of divisors of N, which are divisible by 15 but not by 36 are-

(A) 92 (B) 94 (C) 96 (D) 98

PC0122

36. Total number of ways, in which the given number can be split into two factors such that their highest common factor is a prime number is equal to-

(A) 16 (B) 32 (C) 48 (D) 64

PC0122

37. (a) How many divisors are there of the number x = 21600. Find also the sum of these divisors.

PC0123

(b) In how many ways the number 7056 can be resolved as a product of 2 factors.

PC0124

(c) Find the number of ways in which the number 300300 can be split into 2 factors which are relatively prime.

PC0125

(d) Find the number of positive integers that are divisors of atleast one of the numbers 10^{10} ; 15^{7} ; 18^{11}.

PC0126

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Subjetive :

38. A committee of 10 members is to be formed with members chosen from the faculties of Arts, Economics, Education, Engineering, Medicine and Science. Number of possible ways in which the faculties representation be distributed on this committee, is _____.

(Assume every department contains more than 10 members).

PC0127

39. If x_1, x_2, x_3 are the whole numbers and gives remainders 0,1,2 respectively, when divided by 3 then total number of different solutions of the equation $x_1 + x_2 + x_3 = 33$ are k, then $\frac{k}{11}$ is equal to

PC0128

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40. On the normal chess board as shown, I_1 & I_2 are two insects which starts moving towards each other. Each insect moving with the same constant speed . Insect I_1 can move only to the right or upward along the lines while the insect I_2 can move only to the left or downward along the lines of the chess board. Find the total number of ways the two insects can meet at same point during their trip.

PC0129

41. Determine the number of paths from the origin to the point (9, 9) in the cartesian plane which never pass through (5, 5) in paths consisting only of steps going 1 unit North and 1 unit East.

PC0130

42. There are 20 books on Algebra & Calculus in our library. Prove that the greatest number of selections each of which consists of 5 books on each topic is possible only when there are 10 books on each topic in the library.

PC0131

EXERCISE (S-2)

1. The straight lines l_1 , l_2 & l_3 are parallel & lie in the same plane. A total of m points are taken on the line l_1, n points on l_2 & k points on l_3. How many maximum number of triangles are there whose vertices are at these points ?

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PC0132

2. (a) How many five digits numbers divisible by 3 can be formed using the digits 0, 1, 2, 3, 4, 7 and 8 if each digit is to be used atmost once.

PC0133

(b) Find the number of 4 digit positive integers if the product of their digits is divisible by 3.

PC0134

3. There are 3 cars of different make available to transport 3 girls and 5 boys on a field trip. Each car can hold up to 3 children. Find

(a) the number of ways in which they can be accomodated.

(b) the numbers of ways in which they can be accomodated if 2 or 3 girls are assigned to one of the cars.

In both the cases internal arrangement of children inside the car is considered to be immaterial.

PC0135

4. Find the sum of all numbers greater than 10000 formed by using the digits 0 , 1 , 2 , 4 , 5 no digit being repeated in any number.

PC0136

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Paragraph for Question 5 & 6

If 10 vertical equispaced (1 cm) lines and 9 horizontal equispaced lines (1 cm) are drawn in a plane as shown in the given figure.

On the basis of above information, answer the following questions :

5. Total number of rectangles with one side odd & one side even are given by-

(A) 600 (B) 700 (C) 800 (D) 900

PC0137

6. If squares of odd side length are selected from the above grid, then sum of their areas is equal to-

(A) $\sum_{r=1}^{4}(11-2r)(10-2r)(2r-1)^2\ cm^2$ (B) $\sum_{r=1}^{8}(9-2r)(7-2r)(2r+1)^2\ cm^2$

(C) $\sum_{r=1}^{8}(11-2r)(9-2r)(2r+1)^2\ cm^2$ (D) $\sum_{r=1}^{5}(11+2r)(9+2r)(2r-1)^2\ cm^2$

PC0137

7. How many 4 digit numbers are there which contains not more than 2 different digits?

PC0138

8. Find the number of words each consisting of 3 consonants & 3 vowels that can be formed from the letters of the word **"Circumference".** In how many of these c's will be together.

PC0139

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9. Find the number of three elements sets of positive integers {a, b, c} such that $a \times b \times c = 2310$.

PC0140

Instruction for question nos. 10 to 12 :

2 American men; 2 British men; 2 Chinese men and one each of Dutch, Egyptial, French and German persons are to be seated for a round table conference.

10. If the number of ways in which they can be seated if exactly two pairs of persons of same nationality are together is $p(6!)$, then find p.

PC0141

11. If the number of ways in which only American pair is adjacent is equal to $q(6!)$, then find q.

PC0141

12. If the number of ways in which no two people of the same nationality are together given by $r\,(6!)$, find r.

PC0141

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13. For each positive integer k, let S_k denote the increasing arithmetic sequence of integers whose first term is 1 and whose common difference is k. For example, S_3 is the sequence 1, 4, 7, 10..... Find the number of values of k for which S_k contain the term 361.

PC0142

14. A shop sells 6 different flavours of ice-cream. In how many ways can a customer choose 4 ice-cream cones if

(i) they are all of different flavours

(ii) they are non necessarily of different flavours

(iii) they contain only 3 different flavours

(iv) they contain only 2 or 3 different flavours?

PC0143

15. How many different ways can 15 Candy bars be distributed between Ram, Shyam, Ghanshyam and Balram, if Ram can not have more than 5 candy bars and Shyam must have at least two. Assume all Candy bars to be alike.

PC0144

EXERCISE (JM)

1. There are two urns. Urn A has 3 distinct red balls and urn B has 9 distinct blue balls. From each urn two balls are taken out at random and then transferred to the other. The number of ways in which this can be done is - **[AIEEE-2010]**

(1) 3 (2) 36 (3) 66 (4) 108

PC0145

2. **Statement - 1 :** The number of ways of distributing 10 identical balls in 4 distinct boxes such that no box is empty is ${}^9C_3$. **[AIEEE-2011]**

Statement - 2 : The number of ways of choosing any 3 places from 9 different places is ${}^9C_3$.

(1) Statement-1 is true, Statement-2 is false.

(2) Statement-1 is false, Statement-2 is true

(3) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(4) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1.

PC0146

3. There are 10 points in a plane, out of these 6 are collinear. If N is the number of triangles formed by joining these points, then : **[AIEEE-2011]**

(1) $N > 190$ (2) $N \leq 100$ (3) $100 < N \leq 140$ (4) $140 < N \leq 190$

PC0147

4. Assuming the balls to be identical except for difference in colours, the number of ways in which one or more balls can be selected from 10 white, 9 green and 7 black balls is - **[AIEEE-2012]**

(1) 879 (2) 880 (3) 629 (4) 630

PC0148

5. Let A and B be two sets containing 2 elements and 4 elements respectively. The number of subsets of $A \times B$ having 3 or more elements is **[JEE (Main)-2013]**

(1) 256 (2) 220 (3) 219 (4) 211

PC0149

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6. Let T_n be the number of all possible triangles formed by joining vertices of an n-sided regular polygon. If $T_{n+1} - T_n = 10$, then the value of n is : **[JEE (Main)-2013]**

(1) 7 (2) 5 (3) 10 (4) 8

PC0150

7. The number of points, having both co-ordinates as integers, that lie in the interior of the triangle with vertices (0, 0), (0, 41) and (41, 0) is : **[JEE (Main)-2015]**

(1) 820 (2) 780 (3) 901 (4) 861

PC0151

8. Let A and B be two sets containing four and two elements respectively. Then the number of subsets of the set $A \times B$, each having at least three elements is : **[JEE (Main)-2015]**

(1) 275 (2) 510 (3) 219 (4) 256

PC0152

9. The number of integers greater than 6000 that can be formed, using the digits 3,5,6,7 and 8 without repetition, is : **[JEE (Main)-2015]**

(1) 120 (2) 72 (3) 216 (4) 192

PC0153

10. If all the words (with or without meaning) having five letters, formed using the letters of the word SMALL and arranged as in a dictionary; then the position of the word SMALL is : **[JEE (Main)-2016]**

(1) 58th (2) 46th (3) 59th (4) 52nd

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PC0154

11. A man X has 7 friends, 4 of them are ladies and 3 are men. His wife Y also has 7 friends, 3 of them are ladies and 4 are men. Assume X and Y have no common friends. Then the total number of ways in which X and Y together can throw a party inviting 3 ladies and 3 men, so that 3 friends of each of X and Y are in this party, is : **[JEE (Main)-2017]**

(1) 484 (2) 485 (3) 468 (4) 469

PC0155

12. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf so that the dictionary is always in the middle. The number of such arrangements is- **[JEE(Main)-2018]**

(1) less than 500 (2) at least 500 but less than 750

(3) at least 750 but less than 1000 (4) at least 1000

PC0156

13. Let S = {1,2,3, ..., 100}. The number of non-empty subsets A of S such that the product of elements in A is even is :- **[JEE(Main)-2019]**

(1) $2^{50}(2^{50}-1)$ (2) $2^{100}-1$ (3) $2^{50}-1$ (4) $2^{50}+1$

PC0157

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14. All possible numbers are formed using the digits 1, 1, 2, 2, 2, 2, 3, 4, 4 taken all at a time. The number of such numbers in which the odd digits occupy even places is : **[JEE(Main)-2019]**

(1) 175 (2) 162 (3) 160 (4) 180

PC0158

15. The number of four-digit numbers strictly greater than 4321 that can be formed using the digits 0,1,2,3,4,5 (repetition of digits is allowed) is : **[JEE(Main)-2019]**

(1) 288 (2) 306 (3) 360 (4) 310

PC0159

16. The number of 6 digit numbers that can be formed using the digits 0, 1, 2, 5, 7 and 9 which are divisible by 11 and no digit is repeated, is : **[JEE(Main)-2019]**

(1) 36 (2) 60 (3) 48 (4) 72

PC0160

17. Suppose that 20 pillars of the same height have been erected along the boundary of a circular stadium. If the top of each pillar has been connected by beams with the top of all its non-adjacent pillars, then the total number beams is : **[JEE(Main)-2019]**

(1) 210 (2) 190 (3) 170 (4) 180

PC0161

18. The number of ways of choosing 10 objects out of 31 objects of which 10 are identical and the remaining 21 are distinct, is : **[JEE(Main)-2019]**

(1) 2^{20} (2) $2^{20} - 1$ (3) $2^{20} + 1$ (4) 2^{21}

PC0162

19. Total number of 6-digit numbers in which only and all the five digits 1, 3, 5, 7 and 9 appear, is: **[JEE(Main)-2020]**

(1) $\frac{5}{2}(6!)$ (2) 5^6 (3) $\frac{1}{2}(6!)$ (4) 6!

PC0163

20. The number of 4 letter words (with or without meaning) that can be formed from the eleven letters of the word 'EXAMINATION' is _____. **[JEE(Main)-2020]**

PC0164

21. If a,b and c are the greatest value of ${}^{19}C_p, {}^{20}C_q$ and ${}^{21}C_r$ respectively, then **[JEE(Main)-2020]**

(1) $\frac{a}{11} = \frac{b}{22} = \frac{c}{21}$ (2) $\frac{a}{10} = \frac{b}{11} = \frac{c}{21}$ (3) $\frac{a}{10} = \frac{b}{11} = \frac{c}{42}$ (4) $\frac{a}{11} = \frac{b}{22} = \frac{c}{42}$

PC0165

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22. An urn contains 5 red marbles, 4 black marbles and 3 white marbles. Then the number of ways in which 4 marbles can be drawn so that at the most three of them are red is .... **[JEE(Main)-2020]**

PC0166

EXERCISE (JA)

1. Let S = {1,2,3,4}. The total number of unordered pairs of disjoint subsets of S is equal to -

[JEE 10, 5M, –2M]

(A) 25 (B) 34 (C) 42 (D) 41

PC0167

2. The total number of ways in which 5 balls of different colours can be distributed among 3 persons so that each person gets at least one ball is - **[JEE 2012, 3M, –1M]**

(A) 75 (B) 150 (C) 210 (D) 243

PC0168

Paragraph for Question 3 and 4 :

Let a_n denotes the number of all n-digit positive integers formed by the digits 0, 1 or both such that no consecutive digits in them are 0. Let b_n = the number of such n-digit integers ending with digit 1 and c_n = the number of such n-digit integers ending with digit 0.

3. The value of b_6 is **[JEE 2012, 3M, –1M]**

(A) 7 (B) 8 (C) 9 (D) 11

PC0169

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4. Which of the following is correct ? **[JEE 2012, 3M, –1M]**

(A) $a_{17} = a_{16} + a_{15}$ (B) $c_{17} \neq c_{16} + c_{15}$ (C) $b_{17} \neq b_{16} + c_{16}$ (D) $a_{17} = c_{17} + b_{16}$

PC0169

5. Let $n_1 < n_2 < n_3 < n_4 < n_5$ be positive integers such that $n_1 + n_2 + n_3 + n_4 + n_5 = 20$. The number of such distinct arrangements (n_1,n_2,n_3,n_4,n_5) is **[JEE(Advanced)-2014, 3]**

PC0170

6. Let $n \geq 2$ b an integer. Take n distinct points on a circle and join each pair of points by a line segment. Colour the line segment joining every pair of adjacent points by blue and the rest by red. If the number of red and blue line segments are equal, then the value of n is

[JEE(Advanced)-2014, 3]

PC0171

7. Six cards and six envelopes are numbered 1, 2, 3, 4, 5, 6 and cards are to be placed in envelopes so that each envelope contains exactly one card and no card is placed in the envelope bearing the same number and moreover the card numbered 1 in always placed in envelope numbered 2. Then the number of ways it can be done is - **[JEE(Advanced)-2014, 3(–1)]**

(A) 264 (B) 265 (C) 53 (D) 67

PC0172

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8. Let n be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that all the girls stand consecutively in the queue. Let m be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that exactly four girls stand consecutively in the queue. Then the value of $\frac{m}{n}$ is **[JEE (Advanced) 2015, 4M, –0M]**

PC0173

9. A debate club consists of 6 girls and 4 body. A team of 4 members is to be selected from this club including the selection of a captain (from among these 4 member) for the team. If the team has to include at most one boy, then the number of ways of selecting the team is

[JEE(Advanced)-2016, 3(–1)]

(A) 380 (B) 320 (C) 260 (D) 95

PC0174

10. Words of length 10 are formed using the letters A, B, C, D, E, F, G, H, I, J. Let x be the number of such words where no letter is repeated; and let y be the number of such words where exactly one letter is repeated twice and no other letter is repeated. Then $\frac{y}{9x} =$

[JEE(Advanced)-2017, 3]

PC0175

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11. Let S = {1, 2, 3,.....,9}. For k = 1,2, ..., 5, let N_k be the number of subsets of S, each containing five elements out of which exactly k are odd. Then $N_1 + N_2 + N_3 + N_4 + N_5 =$

[JEE(Advanced)-2017, 3(–1)]

(A) 125 (B) 252 (C) 210 (D) 126

PC0176

12. The number of 5 digit numbers which are divisible by 4, with digits from the set {1, 2, 3, 4, 5} and the repetition of digits is allowed, is _____ **[JEE(Advanced)-2018, 3(0)]**

PC0177

13. In a high school, a committee has to be formed from a group of 6 boys $M_1, M_2, M_3, M_4, M_5, M_6$ and 5 girls G_1, G_2, G_3, G_4, G_5.

(i) Let α_1 be the total number of ways in which the committee can be formed such that the committee has 5 members, having exactly 3 boy and 2 girls.

(ii) Let α_2 be the total number of ways in which the committe can be formed such that the committee has at least 2 members, and having an equal number of boys and girls.

(iii) Let α_3 be the total number of ways in which the committe can be formed such that the committee has 5 members, at least 2 of them being girls.

(iv) Let α_4 be the total number of ways in which the committee can be formed such that the commitee has 4 members, having at least 2 girls and such that both M_1 and G_1 are **NOT** in the committee together.

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LIST-I	LIST-II
P. The value of α_1 is	**1.** 136
Q. The value of α_2 is	**2.** 189
R. The value of α_3 is	**3.** 192
S. The value of α_4 is	**4.** 200
	5. 381
	6. 461

The correct option is :-

(A) **P $\rightarrow$ 4; Q $\rightarrow$ 6, R $\rightarrow$ 2; S $\rightarrow$ 1**

(B) **P $\rightarrow$ 1; Q $\rightarrow$ 4; R $\rightarrow$ 2; S $\rightarrow$ 3**

(C) **P $\rightarrow$ 4; Q $\rightarrow$ 6, R $\rightarrow$ 5; S $\rightarrow$ 2**

(D) **P $\rightarrow$ 4; Q $\rightarrow$ 2; R $\rightarrow$ 3; S $\rightarrow$ 1**

[JEE(Advanced)-2018, 3(–1)]

PC0178

14. Five person A,B,C,D and E are seated in a ciruclar arrangement. If each of them is given a hat of one of the three colours red, blue and green ,then the number of ways of distributing the hats such that the persons seated in adjacent seats get different coloured hats is

[JEE(Advanced)-2019, 3(0)]

PC0179

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ANSWER KEY

Do yourself-1

(i) 7 **(ii)** 3

Do yourself-2

(i) 0 **(ii)** r = 4 **(iii)** ${}^{50}C_4$ **(iv)** 20 **(v)** 120, 48

(vi) **Sketch the graph of**

(a)

(b)

(c)

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(vii) (a) $x \in [-6,-3)$ (b) $x \in \phi$ (c) $x \in (-33,-30] \cup [30,33)$

(d) $x \in (-15,-12] \cup (9,12]$ (e) $x \in (-3,-1]$

Do yourself-3

(i) 10 **(ii)** 450 **(iii)** 840, 40

Do yourself-4

(i) $\dfrac{16!}{(2!)^8 8!} \times 8!$ **(ii)** 360 **(iii)** ${}^{n}C_2.n!$

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Do yourself-5

(i) $5^n - 4^n - 4^n + 3^n$

Do yourself-6

(i) 60, 6th **(ii)** 60

Do yourself-7

(i) 36 **(ii)** $\frac{9!}{2} = 181440$ **(iii)** 5400 **(iv)** 2688

Do yourself-8

(i) $(p + 1)^n - 1$ **(ii)** $2^{10} - 1$

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Do yourself-9

(i) 23 **(ii)** 36

Do yourself-10

(i) (a) ${}^{15}C_3$ (b) ${}^{7}C_3$ **(ii)** ${}^{12}C_2$ **(iii)** ${}^{23}C_2$

Do yourself-11

(i) 9

EXERCISE (O-1)

1. A	2. A	3. C	4. C	5. C	6. D	7. A	8. A
9. C	10. B	11. B	12. C	13. D	14. C	15. D	16. C
17. A	18. B	19. D	20. C	21. B	22. D	23. B	24. B
25. C	26. C	27. C	28. A	29. D	30. B		
31. (A) R; (B) S; (C) Q; (D) P			32. A	33. B	34. C	35. A	36. C
37. C	38. C	39. C	40. B	41. A	42. C	43. D	44. A
45. D	46. B	47. C	48. B	49. A	50. D		

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EXERCISE (O-2)

1. A 2. A 3. D 4. A 5. B 6. C 7. C 8. A
9. A,B 10. A,B,D 11. B,C 12. C,D 13. B,D 14. A 15. B
16. 240, 240, 255, 480 17. B 18. A 19. D 20. A 21. C 22. D
23. (a) C ; (b) B ; (c) B 24. A,B,C,D 25. A,C,D 26. A,B,D
27. (A) T; (B) R; (C) P; (D) Q; (E) S 28. D

EXERCISE (S-1)

1. 120 2. (i) 24 ; (ii) 576; (iii) 360
3. (i) 840; (ii) 120; (iii) 400; (iv) 240; (v) 480; (vi) 40; (vii) 60; (viii) 240
4. (i) 120; (ii) 40; (iii) 40; (iv) 80; (v) 20
5. 20 6. 4^7 7. (a) 3^4 ; (b) 24 8. 36 9. 720 10. 576
11. 999 12. (ii) 792; (iv) r = 3 13. 1568 14. 172800
15. 3150 16. 960 17. 13, 156 18. 120, 216, 210 19. 2500
20. 967680 21. 420 22. 43200 23. ${}^{24}C_2 \cdot {}^{15}C_3$ 24. 1106
25. 5400 26. ${}^{8}C_4 \cdot 4!$ 27. n = 8 28. 528 29. 719
30. 56 31. $\frac{52!}{(13!)^4}$; $\frac{52!}{3!(17!)^3}$ 32. 30 33. 240, 15552
34. (i) 3359; (ii) 59; (iii) 359 35. C 36. C
37. (a) 72; 78120; (b) 23; (c) 32 ; (d) 435 38. 3003 39. 6
40. 12870 41. 30980

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EXERCISE (S-2)

1. ${}^{m+n+k}C_3 - ({}^{m}C_3 + {}^{n}C_3 + {}^{k}C_3)$ 2. (a) 744; (b) 7704 3. (a) 1680; (b) 1140
4. 3119976 5. C 6. A 7. 576 8. 22100 , 52
9. 40 10. 60 11. 64 12. 244 13. 24
14. (i) 15, (ii) 126, (iii) 60, (iv) 105 15. 440

EXERCISE (JM)

1. 4 **2.** 3 **3.** 2 **4.** 1 **5.** 3 **6.** 2 **7.** 2
8. 3 **9.** 4 **10.** 1 **11.** 2 **12.** 4 **13.** 1 **14.** 4
15. 4 **16.** 2 **17.** 3 **18.** 1 **19.** 1 **20.** 2454 **21.** 4 **22.** 490.00

EXERCISE (JA)

1. D **2.** B **3.** B **4.** A **5.** 7 **6.** 5 **7.** C
8. 5 **9.** A **10.** 5 **11.** D **12.** 625 **13.** C **14.** 30.00

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Chapter Contents

PROBABILITY

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JEE (Main/Advanced) Syllabus

JEE (Main) Syllabus :

Addition and multiplication rules of probability, conditional probability, Bayes Theorem, independence of events, computation of probability of events using permutations and combinations.

JEE (Advanced) Syllabus :

Addition and multiplication rules of probability, conditional probability, Bayes Theorem, independence of events, computation of probability of events using permutations and combinations.

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PROBABILITY

1 INTRODUCTION :

The theory of probability has been originated from the game of chance and gambling. In old days, gamblers used to gamble in a gambling house with a die to win the amount fixed among themselves. They were always desirous to get the prescribed number on the upper face of a die when it was thrown on a board. Shakuni of Mahabharat was perhaps one of them. People started to study the subject of probability from the middle of seventeenth century. The mathematicians Huygens, Pascal Fermat and Bernoulli contributed a lot to this branch of Mathematics. A.N. Kolmogorow proposed the set theoretic model to the theory of probability.

Probability gives us a measure of likelihood that something will happen. However probability can never predict the number of times that an occurrence actually happens. But being able to quantify the likely occurrence of an event is important because most of the decisions that affect our daily lives are based on likelihoods and not on absolute certainties.

2. DEFINITIONS :

(a) Experiment : An action or operation resulting in two or more well defined outcomes. e.g. tossing a coin, throwing a die, drawing a card from a pack of well shuffled playing cards etc.

(b) Sample space : A set S that consists of all possible outcomes of a random experiment is called a sample space and each outcome is called a sample point. Often, there will be more than one sample space that can describe outcomes of an experiment, but there is usually only one that will provide the most information.

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e.g. in an experiment of "throwing a die", following sample spaces are possible :

(i) {even number, odd number}

(ii) {a number less than 3, a number equal to 3, a number greater than 3}

(iii) {1,2,3,4,5,6}

Here 3rd sample space is the one which provides most information.

If a sample space has a finite number of points it is called finite sample space and if it has an infinite number of points, it is called infinite sample space. e.g. (i) "in a toss of coin" either a head (H) or tail (T) comes up, therefore sample space of this experiment is S = {H,T} which is a finite sample space. (ii) "Selecting a number from the set of natural numbers", sample space of this experiment is S = {1,2,3,4,.....} which is an infinite sample space.

(c) Event : An event is defined as an occurrence or situation, for example

(i) in a toss of a coin, it shows head,

(ii) scoring a six on the throw of a die,

(iii) winning the first prize in a raffle,

(iv) being dealt a hand of four cards which are all clubs.

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In every case it is set of some or all possible outcomes of the experiment. Therefore event (A) is subset of sample space (S). If outcome of an experiment is an element of A we say that event A has occurred.

- An event consisting of a single point of S is called a simple or elementary event.
- ϕ is called impossible event and S (sample space) is called sure event.

Note : Probability of occurrence of an event A is denoted by P(A).

(d) **Compound Event :** If an event has more than one sample points it is called **Compound Event**. If A & B are two given events then $A \cap B$ is called compound event and is denoted by $A \cap B$ or AB or A & B.

(e) **Complement of an event :** The set of all outcomes which are in S but not in A is called the complement of the event A & denoted by $\bar{A}$, A^c, A' or 'not A'.

(f) **Mutually Exclusive Events :** Two events are said to be **Mutually Exclusive** (or disjoint or incompatible) if the occurrence of one precludes (rules out) the simultaneous occurrence of the other. If A & B are two mutually exclusive events then $P(A \cap B) = 0$.

Consider, for example, choosing numbers at random from the set {3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

If, Event A is the selection of a prime number,

Event B is the selection of an odd number,

Event C is the selection of an even number,

then A and C are mutually exclusive as none of the numbers in this set is both prime and even. But A and B are not mutually exclusive as some numbers are both prime and odd (viz. 3, 5, 7, 11).

(g) **Equally Likely Events :** Events are said to be **Equally Likely** when each event is as likely to occur as any other event. Note that the term 'at random' or 'randomly' means that all possibilities are equally likely.

(h) **Exhaustive Events :** Events A,B,C..... N are said to be **Exhaustive Events** if no event outside this set can result as an outcome of an experiment. For example, if A & B are two events defined on a sample space S and A & B are exhaustive $\Rightarrow A \cup B = S \Rightarrow P(A \cup B) = 1$.

Note : Playing cards : A pack of playing cards consists of 52 cards of 4 suits, 13 in each, as shown in figure.

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Comparative study of Equally likely, Mutually Exclusive and Exhaustive events :

	Experiment	Events	E/L	M/E	Exhaustive
1.	Throwing of a die	A: throwing an odd face {1, 3, 5} B : throwing a composite {4,6}	No	Yes	No
2.	A ball is drawn from an urn containing 2White, 3Red and 4Green balls	E_1 : getting a White ball E_2 : getting a Red ball E_3 : getting a Green ball	No	Yes	Yes
3.	Throwing a pair of dice	A : throwing a doublet {11, 22, 33, 44, 55, 66} B : throwing a total of 10 or more { 46, 64, 55, 56, 65, 66 }	Yes	No	No
4.	From a well shuffled pack of cards a card is drawn	E_1 : getting a heart E_2 : getting a spade E_3 : getting a diamond E_4 : getting a club	Yes	Yes	Yes
5.	From a well shuffled pack of cards a card is drawn	A = getting a heart B = getting a face card	No	No	No

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Illustration 1 : A coin is tossed. If it shows head, we draw a ball from a bag consisting of 3 blue and 4 white balls; if it shows tail we throw a die. Describe the sample space of this experiment.

Solution : Let us denote blue balls by B_1, B_2, B_3 and the white balls by W_1, W_2, W_3, W_4. Then a sample space of the experiment is

S = {HB_1, HB_2, HB_3, HW_1, HW_2, HW_3, HW_4, T1, T2, T3, T4, T5, T6}.

Here HB_i means head on the coin and ball B_i is drawn, HW_i means head on the coin and ball W_i is drawn. Similarly, Ti means tail on the coin and the number i on the die.

Illustration 2 : Consider the experiment in which a coin is tossed repeatedly until a head comes up. Describe the sample space.

Solution : In the experiment head may come up on the first toss, or the 2nd toss, or the 3rd toss and so on. Hence, the desired sample space is S = {H, TH, TTH, TTTH, TTTTH,...}

Illustration 3 : A coin is tossed three times, consider the following events.

A : 'no head appears'

B : 'exactly one head appears'

C : 'at least two heads appear'

Do they form a set of mutually exclusive and exhaustive events ?

Solution : The sample space of the experiment is

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Events A, B and C are given by

A = {TTT}

B = {HTT, THT, TTH}

C = {HHT, HTH, THH, HHH}

Now,

$A \cup B \cup C$ = {TTT, HTT,THT,TTH,HHT,HTH,THH,HHH} = S

Therefore A,B and C are exhaustive events. Also, $A \cap B = \phi$, $A \cap C = \phi$ and $B \cap C = \phi$. Therefore, the events are pair-wise disjoint, i.e., they are mutually exclusive. Hence, A,B and C form a set of mutually exclusive and exhaustive events.

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Do yourself - 1 :

(i) Two balls are drawn from a bag containing 2 Red and 3 Black balls, write sample space of this experiment.

(ii) Out of 2 men and 3 women a team of two persons is to be formed such that there is exactly one man and one woman. Write the sample space of this experiment.

(iii) A coin in tossed and if head comes up, a die is thrown. But if tail comes up, the coin is tossed again. Write the sample space of this experiment.

(iv) In a toss of a die, consider following events :

A : An even number turns up. B : A prime number turns up.

These events are -

(A) Equally likely events (B) Mutually exclusive events

(C) Exhaustive events (D) None of these

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3. CLASSICAL DEFINITION OF PROBABILITY :

If n represents the total number of equally likely, mutually exclusive and exhaustive outcomes of an experiment and m of them are favourable to the happening of the event A, then the probability of happening of the event A is given by P(A) = m/n. There are (n–m) outcomes which are favourable to the event that A does not happen. 'The event A does not happen' is denoted by $\bar{A}$ (and is read as 'not A')

Thus $P(\bar{A}) = \frac{n-m}{n} = 1 - \frac{m}{n}$

i.e. $P(\bar{A}) = 1 - P(A)$

Note :

(i) $0 \le P(A) \le 1$

(ii) $P(A) + P(\bar{A}) = 1$,

(iii) If x cases are favourable to A & y cases are favourable to $\bar{A}$ then $P(A) = \frac{x}{(x+y)}$ and $P(\bar{A}) = \frac{y}{(x+y)}$. We say that Odds In Favour Of A are x: y & Odds Against A are y : x

OTHER DEFINITIONS OF PROBABILITY :

(a) Axiomatic probability : Axiomatic approach is another way of describing probability of an event. In this approach some axioms or rules are depicted to assign probabilities.

Let S be the sample space of a random experiment. The probability P is a real valued function whose domain is the power set of S and range is the interval [0, 1] satisfying the following axioms :

(i) For any event E, $P(E) \ge 0$ (ii) $P(S) = 1$

(iii) If E and F are mutually exclusive events, then $P(E \cup F) = P(E) + P(F)$.

It follows from (iii) that $P(E \cap F) = P(\phi) = 0$.

Let S be a sample space containing outcomes $\omega_1, \omega_2, \ldots, \omega_n$, i.e., $S = \{\omega_1, \omega_2, \ldots \omega_n\}$

It follows from the axiomatic definition of probability that :

(i) $0 \le P(\omega_i) \le 1$ for each $\omega_i \in S$

(ii) $P(\omega_1) + P(\omega_2) + \ldots + P(\omega_n) = 1$

(iii) For any event A, $P(A) = \Sigma P(\omega_i)$, $\omega_i \in A$.

(b) Empirical probability : The probability that you would hit the bull's-eye on a dartboard with one throw of a dart would depend on how much you had practised, how much natural talent for playing darts you had, how tired you were, how good a dart you were using etc. all of which are impossible to quantify. A method which can be adopted in the example given above is to throw the dart several times (each throw is a trial) and count the number of times you hit the bull's-eye (a success) and the number of times you miss (a failure). Then an empirical value of the probability that you hit the bull's-eye with any one throw is

$$\frac{\text{number of successes}}{\text{number of successes + number of failures}}.$$

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If the number of throws is small, this does not give a particular good estimate but for a large number of throws the result is more reliable.

When the probability of the occurrence of an event A cannot be worked out exactly, an empirical value can be found by adopting the approach described above, that is :

(i) making a large number of trials (i.e. set up an experiment in which the event may, or may not, occur and note the outcome),

(ii) counting the number of times the event does occur, i.e. the number of successes,

(iii) calculating the value of $\dfrac{\text{number of successes}}{\text{number of trials (i.e. successes + failures)}} = \dfrac{r}{n}$

The probability of event A occurring is defined as $P(A) = \lim\limits_{n\to\infty}\left(\dfrac{r}{n}\right)$

$n \to \infty$ means that the number of trials is large (but what should be taken as 'large' depends on the problem).

Illustration 4 : If the letters of INTERMEDIATE are arranged, then the odds in favour of the event that no two 'E's occur together, are-

(A) $\dfrac{6}{5}$ (B) $\dfrac{5}{6}$ (C) $\dfrac{2}{9}$ (D) none of these

Solution : $I \to 2, N \to 1, T \to 2, E \to 3, R \to 1, M \to 1, D \to 1, A \to 1$ (3'E's, Rest 9 letters)

First arrange rest of the letters $= \dfrac{9!}{2!\ 2!}$,

Now 3'E's can be placed by ${}^{10}C_3$ ways, so favourable cases $= \dfrac{9!}{2!\ 2!} \times {}^{10}C_3 = 3 \times 10!$

Total cases $= \dfrac{12!}{2!\ 2!\ 3!} = \dfrac{11}{2} \times 10!$; Non-favourable cases $= \left(\dfrac{11}{2} - 3\right) \times 10! = \dfrac{5}{2} \times 10!$

Odds in favour of the event $= \dfrac{3}{5/2} = \dfrac{6}{5}$ **Ans. (A)**

Illustration 5 : From a group of 10 persons consisting of 5 lawyers, 3 doctors and 2 engineers, four persons are selected at random. The probability that the selection contains at least one of each category is-

(A) $\dfrac{1}{2}$ (B) $\dfrac{1}{3}$ (C) $\dfrac{2}{3}$ (D) none of these

Solution : $n(S) = {}^{10}C_4 = 210.$

$n(E) = {}^5C_2 \times {}^3C_1 \times {}^2C_1 + {}^5C_1 \times {}^3C_2 \times {}^2C_1 + {}^5C_1 \times {}^3C_1 \times {}^2C_2 = 105$

$\therefore \quad P(E) = \dfrac{105}{210} = \dfrac{1}{2}.$ **Ans. (A)**

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Illustration 6 : If four cards are drawn at random from a pack of fifty-two playing cards, find the probability that at least one of them is an ace.

Solution : If A is a combination of four cards containing at least one ace (i.e. either one ace, or two aces, or three aces or four aces) then $\bar{A}$ is a combination of four cards containing no aces.

$$\text{Now } P(\bar{A}) = \frac{\text{Number of combinations of four cards with no aces}}{\text{Total number of combinations of four cards}} = {}^{48}C_4 / {}^{52}C_4 = 0.72$$

Using $P(A) + P(\bar{A}) = 1$ we have $P(A) = 1 - P(\bar{A}) = 1 - 0.72 = 0.28$

Illustration 7 : A bag contains n white and n red balls. Pairs of balls are drawn without replacement until the bag is empty. Show that the probability that each pair consists of one white and one red ball is $2^n/({}^{2n}C_n)$.

Solution : Let S be the sample space & E be the event that each of the n pairs of balls drawn consists of one white and one red ball.

$\therefore \quad n(S) = ({}^{2n}C_2)\,({}^{2n-2}C_2)\,({}^{2n-4}C_2)\ldots({}^{4}C_2)({}^{2}C_2)$

$$= \left\{\frac{(2n)(2n-1)}{1.2}\right\}\left\{\frac{(2n-2)(2n-3)}{1.2}\right\}\left\{\frac{(2n-4)(2n-5)}{1.2}\right\}\ldots\ldots\left\{\frac{4.3}{1.2}\right\}.\left\{\frac{2.1}{1.2}\right\}$$

$$= \frac{1.2.3.4\ldots.(2n-1)2n}{2^n} = \frac{2n!}{2^n}$$

and $n(E) = ({}^{n}C_1.{}^{n}C_1)\,({}^{n-1}C_1.{}^{n-1}C_1)\,({}^{n-2}C_1.{}^{n-2}C_1)\ldots({}^{2}C_1.{}^{2}C_1)\,({}^{1}C_1.{}^{1}C_1)$

$= n^2.(n-1)^2.(n-2)^2\ldots.2^2.1^2 = [1.2.3\ldots.(n-1)n]^2 = (n!)^2$

$$\therefore \quad \text{Required Probability, } P(E) = \frac{n(E)}{n(S)} = \frac{(n!)^2}{(2n)!/2^n} = \frac{2^n}{\frac{2n!}{(n!)^2}} = \frac{2^n}{{}^{2n}C_n}$$ **Ans.**

Do yourself - 2 :

(i) A coin is tossed successively three times. Find the probability of getting exactly one head or two heads.

(ii) A bag contains 5 red and 4 green balls. Four balls are drawn at random then find the probability that two balls are of red and two balls are of green colour.

(iii) Two natural numbers are selected at random, find the probability that their sum is divisible by 10.

(iv) Five card are drawn successively from a pack of 52 cards with replacement. Find the probability that there is at least one Ace.

4. VENN DIAGRAMS :

A diagram used to illustrate relationships between sets. Commonly, a rectangle represents the universal set and a circle within it represents a given set (all members of the given set are represented by points within the circle). A subset is represented by a circle within a circle and intersection is indicated by overlapping circles.

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Let S is the sample space of an experiment and A, B are two events corresponding to it :

Example : Let us conduct an experiment of tossing a pair of dice.

Two events defined on the experiment are

A : getting a doublet {11, 22, 33, 44, 55, 66}

B : getting total score of 10 or more {64, 46, 55, 56, 65, 66}

---

5. ADDITION THEOREM :

$A \cup B = A + B$ = A or B denotes occurrence of at least A or B.

For 2 events A & B :

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(a) $P(A \cup B)$, $P(A + B)$, $P(A \text{ or } B)$, P(occurence of atleast A or B) $= P(A) + P(B) - P(A \cap B)$ (This is known as generallised addition theorem)

$= P(A) + P(B \cap \bar{A})$

$= P(B) + P(A \cap \bar{B})$

$= P(A \cap \bar{B}) + P(A \cap B) + P(B \cap \bar{A})$

$= 1 - P(\bar{A} \cap \bar{B})$

$= 1 - P(\overline{A \cup B})$

Note :

(i) If A & B are mutually exclusive then $P(A \cup B) = P(A) + P(B)$.

(ii) If A & B are mutually exclusive and exhaustive, then $P(A \cup B) = P(A) + P(B) = 1$

(b) P(only A occurs) $= P(A \backslash B) = P(A - B) = P(A \cap B^C) = P(A) - P(A \cap B)$

(c) P(either A or B) = 1 – P(neither A nor B)

i.e. $P(A \cup B) = 1 - P(\bar{A} \cap \bar{B})$

(d) For any two events A & B

P(exactly one of A, B occurs) $= P(A \cap \bar{B}) + P(B \cap \bar{A})$

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$\Rightarrow$ **P(exactly one of A, B occurs) = P (A) + P (B) – 2P (A $\cap$ B)**

$= P(A \cup B) - P(A \cap B) = P(A^c \cup B^c) - P(A^c \cap B^c)$

(e) $P(A \cap B) \le P(A), P(B) \le P(A \cup B) \le P(A) + P(B)$

6. DE MORGAN'S LAW :

If A & B are two subsets of a universal set U, then

(i) $(A \cup B)^c = A^c \cap B^c$ & (ii) $(A \cap B)^c = A^c \cup B^c$

Note :

(a) $(A \cup B \cup C)^C = A^C \cap B^C \cap C^C$ & $(A \cap B \cap C)^C = A^C \cup B^C \cup C^C$

(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ & $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Illustration 8 : Given two events A and B. If odds against A are as 2 : 1 and those in favour of $A \cup B$ are as 3 : 1, then find the range of P(B).

Solution : Clearly $P(A) = 1/3$, $P(A \cup B) = 3/4$.

Now, $P(B) \le P(A \cup B)$

$\Rightarrow$ $P(B) \le 3/4$

Also, $P(B) = P(A \cup B) - P(A) + P(A \cap B)$

$\Rightarrow$ $P(B) \ge P(A \cup B) - P(A)$ $(\because P(A \cap B) \ge 0)$

$\Rightarrow$ $P(B) \ge 3/4 - 1/3$ $\Rightarrow$ $P(B) \ge \frac{5}{12}$

$\Rightarrow$ $\frac{5}{12} \le P(B) \le \frac{3}{4}$ **Ans.**

Illustration 9 : If A and B are two events such that $P(A \cup B) = \frac{3}{4}$, $P(A \cap B) = \frac{1}{4}$ and $P(A^c) = \frac{2}{3}$. Then find -

(i) P(A) (ii) P(B) (iii) $P(A \cap B^c)$ (iv) $P(A^c \cap B)$

Solution : $P(A) = 1 - P(A^c) = 1 - \frac{2}{3} = \frac{1}{3}$

$P(B) = P(A \cup B) + P(A \cap B) - P(A) = \frac{3}{4} + \frac{1}{4} - \frac{1}{3} = \frac{2}{3}$

$P(A \cap B^c) = P(A) - P(A \cap B) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

$P(A^c \cap B) = P(B) - P(A \cap B) = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$ **Ans.**

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Do yourself - 3 :

(i) Draw Venn diagram of (a) $(A^C \cap B^C) \cup (A \cap B)$ (b) $B^C \cup (A^C \cap B)$

(ii) If A and B are two mutually exclusive events, then-

(A) $P(A) \le P(\bar{B})$ (B) $P(\bar{A} \cap \bar{B}) = P(\bar{A}) - P(B)$ (C) $P(\bar{A} \cup \bar{B}) = 0$ (D) $P(\bar{A} \cap B) = P(B)$

(iii) A bag contains 6 white, 5 black and 4 red balls. Find the probability of getting either a white or a black ball in a single draw.

(iv) In a class of 125 students, 70 passed in English, 55 in mathematics and 30 in both. Find the probability that a student selected at random from the class has passed in

(a) at least one subject (b) only one subject.

7. CONDITIONAL PROBABILITY AND MULTIPLICATION THEOREM :

(a) Conditional Probability : Let A and B be two events such that P (A) >0. Then P(B|A) denote the conditional probability of B given that A has occurred. Since A is known to have occurred, it becomes the new sample space replacing the original S. From this we led to the definition

$$\mathbf{P(B|A)} = \frac{\mathbf{P(A \cap B)}}{\mathbf{P(A)}} = \text{which is called conditional probability of B given A}$$

(b) Multiplication Theorem : $P(A \cap B) = P(A)\ P(B|A)$ which is called compound probability or multiplication theorem. It says the probability that both A and B occur is equal to the probability that A occur times the probability that B occurs given that A has occurred.

Note : For any three events A_1, A_2, A_3 we have

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)\ P(A_2|A_1)\ P(A_3 | (A_1 \cap A_2))$$

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Illustration 10 : Two dice are thrown. Find the probability that the numbers appeared have a sum of 8 if it is known that the second die always exhibits 4

Solution : Let A be the event of occurrence of 4 always on the second die

$= \{(1,4),(2,4),(3,4),(4,4),(5,4),(6,4)\}$; $\therefore\ n(A) = 6$

and B be the event of occurrence of such numbers on both dice whose sum is 8 = {(6,2), (5,3), (4,4), (3,5), (2,6)}.

Thus, $A \cap B = A \cap \{(4,4)\} = \{(4,4)\}$

$\therefore\ n(A \cap B) = 1$

$$\therefore\ P(B/A) = \frac{n(A \cap B)}{n(A)} = \frac{1}{6} \text{ or } \frac{P(A \cap B)}{P(A)} = \frac{1/36}{6/36} = \frac{1}{6}$$

Illustration 11 : A bag contains 3 red, 6 white and 7 blue balls. Two balls are drawn one by one. What is the probability that first ball is white and second ball is blue when first drawn ball is not replaced in the bag?

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Solution : Let A be the event of drawing first ball white and B be the event of drawing second ball blue. Here A and B are dependent events.

$$P(A)=\frac{6}{16},\ P(B|A)=\frac{7}{15}$$

$$P(AB)=P(A).P(B|A)=\frac{6}{16}\times\frac{7}{15}=\frac{7}{40}$$

Illustration 12 : A bag contains 4 red and 4 blue balls. Four balls are drawn one by one from the bag, then find the probability that the drawn balls are in alternate colour.

Solution : E_1 : Event that first drawn ball is red, second is blue and so on.

E_2 : Event that first drawn ball is blue, second is red and so on.

$$\therefore\quad P(E_1)=\frac{4}{8}\times\frac{4}{7}\times\frac{3}{6}\times\frac{3}{5} \text{ and } P(E_2)=\frac{4}{8}\times\frac{4}{7}\times\frac{3}{6}\times\frac{3}{5}$$

$$P(E)=P(E_1)+P(E_2)=2\times\frac{4}{8}.\frac{4}{7}.\frac{3}{6}.\frac{3}{5}=\frac{6}{35}$$ **Ans.**

Illustration 13 : If two events A and B are such that $P(\bar{A})=0.3$, P(B) = 0.4 and $P(A\bar{B})=0.5$ then $P(B|(A\cup\bar{B}))$ equals -

(A) 1/2 (B) 1/3 (C) 1/4 (D) 1/5

Solution : We have $P(B|(A\cup\bar{B}))=\dfrac{P[B\cap(A\cup\bar{B})]}{P(A\cup\bar{B})}=\dfrac{P[(B\cap A)\cup(B\cap\bar{B})]}{P(A)+P(\bar{B})-P(A\cap\bar{B})}$

$$=\frac{P(AB)}{P(A)+P(\bar{B})-P(A\bar{B})}=\frac{P(A)-P(A\bar{B})}{P(A)+P(\bar{B})-P(A\bar{B})}=\frac{0.7-0.5}{0.7+0.6-0.5}=\frac{0.2}{0.8}=\frac{1}{4}$$ **Ans.(C)**

Illustration 14 : Three coins are tossed. Two of them are fair and one is biased so that a head is three times as likely as a tail. Find the probability of getting two heads and a tail.

Solution : E_1 : Event that head occurs on fair coin

E_2 : Event that, head occurs on biased coin

$$\therefore\ P(E_1)=\frac{1}{2},\ P(E_2)=\frac{3}{4}$$

E : HHT or HTH or THH

$$\Rightarrow P(E)=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{4}+\frac{1}{2}\times\frac{1}{2}\times\frac{3}{4}+\frac{1}{2}\times\frac{1}{2}\times\frac{3}{4}$$

$$=\frac{7}{16}$$

Illustration 15 : In a multiple choice test of three questions there are five alternative answers given to the first two questions each and four alternative answers given to the last question. If a candidate guesses answers at random, what is the probability that he will get-

(a) Exactly one right answer ? (b) At least one right answer ?

Solution : E_1 : Event that, candidate guesses a correct answer for I question

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E_2 : Event that, candidate guesses a correct answer for II question
E_3 : Event that, candidate guesses a correct answer for III question

$$\therefore\ P(E_1)=\frac{1}{5},\ P(E_2)=\frac{1}{5},\ P(E_3)=\frac{1}{4}$$

(a) E : Event that candidate get exactly one correct answer.

$$\therefore\ P(E)=P(E_1)P(\bar{E}_2)P(\bar{E}_3)+P(\bar{E}_1).P(E_2)P(\bar{E}_3)+P(\bar{E}_1)P(\bar{E}_2)P(E_3)$$

$$=\frac{1}{5}.\frac{4}{5}.\frac{3}{4}+\frac{4}{5}.\frac{1}{5}.\frac{3}{4}+\frac{4}{5}.\frac{4}{5}.\frac{1}{4}=\frac{2}{5}$$

(b) E : Event that candidate gets atleast one correct answer

$$\therefore\ P(E)=1-P(\bar{E}_1)P(\bar{E}_2)P(\bar{E}_3)=1-\frac{4}{5}.\frac{4}{5}.\frac{3}{4}=\frac{13}{25}$$

Illustration 16 : A speaks truth in 75% cases and B in 80% cases. What is the probability that they contradict each other in stating the same fact?

(A) 7/20 (B) 13/20 (C) 3/20 (D) 1/5

Solution : There are two mutually exclusive cases in which they contradict each other i.e. $\bar{A}B$ and $A\bar{B}$. Hence required probability = $P(A\bar{B}+\bar{A}B)=P(A\bar{B})+P(\bar{A}B)$

$$=P(A)P(\bar{B})+P(\bar{A})P(B)=\frac{3}{4}.\frac{1}{5}+\frac{1}{4}.\frac{4}{5}=\frac{7}{20}$$

Ans. (A)

Do yourself - 4 :

(i) A bag contains 2 black, 4 white and 3 red balls. One ball is drawn at random from the bag and kept aside. From the remaining balls another ball is drawn and kept aside the first. This process is repeated till all the balls are drawn. Then probability that the balls drawn are in sequence of 2 black, 4 white and 3 red is-

(A) $\frac{1}{1260}$ (B) $\frac{1}{7560}$ (C) $\frac{1}{210}$ (D) None of these

(ii) Three cards are drawn successively, without replacement from a pack of 52 well shuffled cards. What is the probability that the drawn cards are face cards of same suit ?

3. INDEPENDENT EVENTS :

Two events A & B are said to be independent if occurrence or non occurrence of one does not affect the probability of the occurrence or non occurrence of other.

(a) If the occurrence of one event affects the probability of the occurrence of the other event then the events are said to be **Dependent** or **Contingent.** For two independent events A and B : $P(A\cap B) = P(A) . P(B)$. Often this is taken as the definition of independent events.

Note : If A and B are independent events, then

(i) $P(\bar{A} \cap \bar{B}) = P(\bar{A}).P(\bar{B})$ (ii) $P(A \cap \bar{B}) = P(A).P(\bar{B})$

(b) Three events A, B & C are independent if & only if all the following conditions hold ;

$P(A \cap B) = P(A) . P(B)$; $P(B \cap C) = P(B). P(C)$

$P(C \cap A) = P(C) . P(A)$ and $P(A \cap B \cap C) = P(A) . P(B) . P(C)$

i.e. they must be pairwise as well as mutually independent.

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(c) If three events A, B and C are pair wise mutually exclusive then they must be mutually exclusive.

i.e. $P(A \cap B) = P(B \cap C) = P(C \cap A) = 0 \Rightarrow P(A \cap B \cap C) = 0$. However the converse of this is not true.

Note :

Independent events are not in general mutually exclusive & vice versa. Mutually exclusiveness can be used when the events are taken from the same experiment & independence can be used when the events are taken from different experiments.

Illustration 17 : If A & B are independent events such that $P(A \cap \bar{B}) = \frac{1}{3}$ & $P(A \cup B) = \frac{11}{15}$, then $P(A \cap B)$ is equal to

(A) $\frac{1}{2}$ (B) $\frac{5}{11}$ (C) $\frac{2}{9}$ (D) $\frac{7}{9}$

Solution : $P(A) - P(A \cap B) = \frac{1}{3}$ & $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{11}{15}$

$\Rightarrow \quad P(B) = \frac{6}{15} = \frac{2}{5}$

$P(A) - P(A)\,P(B) = \frac{1}{3} \Rightarrow P(A) = \frac{5}{9}$

$\Rightarrow \quad P(A \cap B) = P(A)\,P(B) = \frac{2}{5} \times \frac{5}{9} = \frac{2}{9}$

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Illustration 18 : A lot contains 50 defective and 50 non-defective bulbs. Two bulbs are drawn at random, one at a time, with replacement. The events A, B, C are defined as : **[IIT 1992]**

A = {The first bulb is defective}

B = {The second bulb is non-defective}

C = {The two bulbs are both defective or both non-defective}

Determine whether

(i) A, B, C are pairwise independent, (ii) A, B, C are independent.

Solution : We have $P(A) = \frac{50}{100}.1 = \frac{1}{2}$; $P(B) = 1.\frac{50}{100} = \frac{1}{2}$; $P(C) = \frac{50}{100}.\frac{50}{100} + \frac{50}{100}.\frac{50}{100} = \frac{1}{2}$

$A \cap B$ is the event that first bulb is defective and second is non-defective.

$\therefore \quad P(A \cap B) = \frac{1}{2}.\frac{1}{2} = \frac{1}{4}$

$A \cap C$ is the event that both bulbs are defective.

$\therefore \quad P(A \cap C) = \frac{1}{2}.\frac{1}{2} = \frac{1}{4}$

Similarly $P(B \cap C) = \frac{1}{4}$

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Thus we have $P(A \cap B) = P(A) . P(B)$; $P(A \cap C) = P(A) . P(C)$; $P(B \cap C) = P(B).P(C)$

$\therefore$ A, B and C are pairwise independent.

There is no element in $A \cap B \cap C$

$\therefore$ $P(A \cap B \cap C) = 0$

$\therefore$ $P(A \cap B \cap C) \neq P(A) . P(B) . P(C)$

Hence A, B and C are not mutually independent.

Do yourself - 5 :

(i) For two independent events A and B, the probability that both A & B occur is 1/8 and the probability that neither of them occur is 3/8. The probability of occurrence of A may be -

(A) 1/2 (B) 1/4 (C) 1/8 (D) 3/4

(ii) A die marked with numbers 1,2,2,3,3,3 is rolled three times. Find the probability of occurrence of 1,2 and 3 respectively.

9. TOTAL PROBABILITY THEOREM :

Let an event A of an experiment occurs with its n mutually exclusive & exhaustive events $B_1, B_2, B_3, ...B_n$

then total probability of occurrence of even A is

$$P(A) = P(AB_1) + P(AB_2) + ... + P(AB_n) = \sum_{i=1}^{n} P(AB_i)$$

$$P(A) = P(B_1)\,P(A|B_1) + P(B_2)\,P(A|B_2) + ... + P(B_n)\,P(A|B_n)$$
$$= \Sigma P(B_i)\,P(A|B_i)$$

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Illustration 19 : A purse contains 4 copper and 3 silver coins and another purse contains 6 copper and 2 silver coins. One coin is drawn from any one of these two purses. The probability that it is a copper coin is -

(A) $\frac{4}{7}$ (B) $\frac{3}{4}$ (C) $\frac{2}{7}$ (D) $\frac{37}{56}$

Solution : Let A ≡ event of selecting first purse

B ≡ event of selecting second purse

C ≡ event of drawing a copper coin

Then given event has two disjoint cases: AC and BC

$$\therefore P(C) = P(AC + BC) = P(AC) + P(BC) = P(A)P(C|A) + P(B)P(C|B)$$

$$= \frac{1}{2}.\frac{4}{7} + \frac{1}{2}.\frac{6}{8} = \frac{37}{56}$$

Ans. (D)

Illustration 20 : Three groups A, B, C are contesting for positions on the Board of Directors of a Company. The probabilities of their winning are 0.5, 0.3, 0.2 respectively. If the group A wins, the probability of introducing a new product is 0.7 and the corresponding probabilities for group B and C are 0.6 and 0.5 respectively. Find the probability that the new product will be introduced.

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Solution : Given P(A) = 0.5, P(B) = 0.3 and P(C) = 0.2

$\therefore$ P(A) + P(B) + P(C) =1

then events A, B, C are exhaustive.

If P(E) = Probability of introducing a new product, then as given

P(E|A) = 0.7, P(E|B) =0.6 and P(E|C) = 0.5

$\therefore$ P(E) = P(A). P(E|A) + P(B). P(E|B) + P(C). P(E|C)

$= 0.5 \times 0.7 + 0.3 \times 0.6 + 0.2 \times 0.5 = 0.35 + 0.18 + 0.10 = 0.63$

Illustration21 : A pair of dice is rolled together till a sum of either 5 or 7 is obtained. Find the probability that 5 comes before 7.

Solution : Let E_1 = the event of getting 5 in a roll of two dice = {(1, 4), (2, 3), (3, 2), (4, 1)}

$$\therefore \quad P(E_1) = \frac{n(E_1)}{n(S)} = \frac{4}{6\times 6} = \frac{1}{9}$$

Let E_2 = the event of getting either 5 or 7

= {(1, 4), (2, 3), (3, 2), (4, 1), (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}

$$\therefore \quad P(E_2) = \frac{n(E_2)}{n(S)} = \frac{10}{6\times 6} = \frac{5}{18}$$

$\therefore$ the probability of getting neither 5 nor 7 $= P(\overline{E}_2) = 1 - P(E_2) = 1 - \frac{5}{18} = \frac{13}{18}$

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The event of getting 5 before 7 = $E_1 \cup (\overline{E}_2 E_1) \cup (\overline{E}_2 \overline{E}_2 E_1) \cup$..... to ∞

$\therefore$ the probability of getting 5 before 7

$= P(E_1) + P(\overline{E}_2 E_1) + P(\overline{E}_2 \overline{E}_2 E_1) + ...$ to $\infty = P(E_1) + P(\overline{E}_2)P(E_1) + P(\overline{E}_2)P(\overline{E}_2)P(E_1) + ...$ to ∞

$$= \frac{1}{9} + \frac{13}{18}.\frac{1}{9} + \frac{13}{18}.\frac{13}{18}.\frac{1}{9} + ... \text{ to } \infty$$

$$= \frac{1}{9}\left[1 + \frac{13}{18} + \left(\frac{13}{18}\right)^2 + ... \text{ to } \infty\right] = \frac{1}{9}.\frac{1}{1-\frac{13}{18}} = \frac{1}{9}.\frac{18}{5} = \frac{2}{5}$$

Do yourself - 6 :

(i) An urn contains 6 white & 4 black balls. A die is rolled and the number of balls equal to the number obtained on the die are drawn from the urn. Find the probability that the balls drawn are all black.

(ii) There are n bags such that i^{th} bag $(1 \le i \le n)$ contains i black and 2 white balls. Two balls are drawn from a randomly selected bag out of given n bags. Find the probability that the both drawn balls are white.

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10. PROBABILITY OF THREE EVENTS :

For any three events A,B and C we have

(a) P(atleast one of A, B and C occurs)

$= P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) - P(A \cap B)$

$- P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

Note : $P(E_1 \cup E_2 .... \cup E_n) = 1 - P(\bar{E}_1 \cap \bar{E}_2 .... \cap \bar{E}_n)$

(b) P(at least two of A,B,C occur)

$= P(B \cap C) + P(C \cap A) + P(A \cap B) - 2P(A \cap B \cap C)$

(c) P(exactly two of A,B,C occur)

$= P(B \cap C) + P(C \cap A) + P(A \cap B) - 3P(A \cap B \cap C)$

(d) P(exactly one of A,B,C occurs)

$= P(A) + P(B) + P(C) - 2P(B \cap C) - 2P(C \cap A) - 2P(A \cap B) + 3P(A \cap B \cap C)$

Illustration 22 : Let A,B,C be three events. If the probability of occurring exactly one event out of A and B is 1 – a, out of B and C is 1 – 2a, out of C and A is 1 – a and that of occurring three events simultaneously is a^2, then prove that the probability that at least one out of A,B,C will occur is greater than 1/2.

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Solution : $P(A) + P(B) - 2P(A \cap B) = 1 - a$ .....(1)

and $P(B) + P(C) - 2P(B \cap C) = 1 - 2a$ .....(2)

and $P(C) + P(A) - 2P(C \cap A) = 1 - a$ .....(3)

and $P(A \cap B \cap C) = a^2$ .....(4)

$\because P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$= \frac{1}{2}\{P(A) + P(B) - 2P(A \cap B) + P(B) + P(C) - 2P(B \cap C) + P(C) + P(A) - 2P(C \cap A)\} + P(A \cap B \cap C)$

$= \frac{1}{2}\{1 - a + 1 - 2a + 1 - a\} + a^2$ {from (1), (2), (3) & (4)}

$= \frac{3}{2} - 2a + a^2 = (a-1)^2 + \frac{1}{2} > \frac{1}{2}$ **Ans.**

Do yourself - 7 :

(i) In a class, there are 100 students out of which 45 study mathematics, 48 study physics, 40 study chemistry, 12 study both mathematics & physics, 11 study both physics & chemistry, 15 study both mathematics & chemistry and 5 study all three subjects. A student is selected at random, then find the probability that the selected student studies

(a) only one subject (b) neither physics nor chemistry

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11. BINOMIAL PROBABILITY DISTRIBUTION :

Suppose that we have an experiment such as tossing a coin or die repeatedly or choosing a marble from an urn repeatedly. Each toss or selection is called a trial. In any single trial there will be a probability associated with a particular event such as head on the coin, 4 on the die, or selection of a red marble. In some cases this probability will not change from one trial to the next (as in tossing a coin or die). Such trials are then said to be independent and are often called Bernoulli trials after James Bernoulli who investigated them at the end of the seventeenth century.

Let p be the probability that an event will happen in any single Bernoulli trial (called the probability of success). Then q = 1 – p is the probability that the event will fail to happen in any single trial (called the probability of failure). The probability that the event will happen exactly x times in n trials (i.e., x successes and n – x failures will occur) is given by the probability function.

$$f(x) = P(X = x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x} \qquad \text{..... (i)}$$

where the random variable X denotes the number of successes in n trials and x = 0, 1,....., n.

Example : The probability of getting exactly 2 heads in 6 tosses of a fair coin is

$$P(X = 2) = \binom{6}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{6-2} = \frac{6!}{2!4!}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{6-2} = \frac{15}{64}$$

The discrete probability function (i) is often called the binomial distribution since for x = 0, 1, 2,...,n, it corresponds to successive terms in the binomial expansion

$$(q+p)^n = q^n + \binom{n}{1} q^{n-1}p + \binom{n}{2} q^{n-2}p^2 + ... + p^n = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$$

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Illustration 23 : If a fair coin is tossed 10 times, find the probability of getting

(i) exactly six heads (ii) atleast six heads (iii) atmost six heads

Solution : The repeated tosses of a coin are Bernoulli trials. Let X denotes the number of heads in an experiment of 10 trials.

Clearly, X has the binomial distribution with n = 10 and $p = \frac{1}{2}$

Therefore $P(X = x) = {}^nC_x q^{n-x} p^x,\ x = 0, 1, 2, ..., n$

Here $n = 10,\ p = \frac{1}{2},\ q = 1 - p = \frac{1}{2}$

Therefore $P(X = x) = {}^{10}C_x \left(\frac{1}{2}\right)^{10-x}\left(\frac{1}{2}\right)^x = {}^{10}C_x \left(\frac{1}{2}\right)^{10}$

Now (i) $P(X = 6) = {}^{10}C_6 \left(\frac{1}{2}\right)^{10} = \frac{10!}{6! \times 4! 2^{10}} = \frac{105}{512}$

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(ii) P(atleast six heads)

$= P(X \geq 6) = P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$

$$= {}^{10}C_6\left(\frac{1}{2}\right)^{10} + {}^{10}C_7\left(\frac{1}{2}\right)^{10} + {}^{10}C_8\left(\frac{1}{2}\right)^{10} + {}^{10}C_9\left(\frac{1}{2}\right)^{10} + {}^{10}C_{10}\left(\frac{1}{2}\right)^{10}$$

$$= \left[\left(\frac{10!}{6!\times 4!}\right)+\left(\frac{10!}{7!\times 3!}\right)+\left(\frac{10!}{8!\times 2!}\right)+\left(\frac{10!}{9!\times 1!}\right)+\left(\frac{10!}{10!}\right)\right]\frac{1}{2^{10}} = \frac{193}{512}$$

(iii) P(at most six heads) $= P(X \leq 6)$

$= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6)$

$$= \left(\frac{1}{2}\right)^{10} + {}^{10}C_1\left(\frac{1}{2}\right)^{10} + {}^{10}C_2\left(\frac{1}{2}\right)^{10} + {}^{10}C_3\left(\frac{1}{2}\right)^{10} + {}^{10}C_4\left(\frac{1}{2}\right)^{10} + {}^{10}C_5\left(\frac{1}{2}\right)^{10} + {}^{10}C_6\left(\frac{1}{2}\right)^{10}$$

$$= \frac{848}{1024} = \frac{53}{64}$$

Illustration 24 : India and Pakistan play a 5 match test series of hockey, the probability that India wins at least three matches is -

(A) $\frac{1}{2}$ (B) $\frac{3}{5}$ (C) $\frac{4}{5}$ (D) $\frac{5}{16}$

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Solution : India win atleast three matches

$$= {}^5C_3\left(\frac{1}{2}\right)^5 + {}^5C_4\left(\frac{1}{2}\right)^5 + {}^5C_5\left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)^5(16) = \frac{1}{2}$$ **Ans. (A)**

Illustration 25 : A coin is tossed 7 times. Each time a man calls head. The probability that he wins the toss on more than three occasions is -

(A) $\frac{1}{4}$ (B) $\frac{5}{8}$ (C) $\frac{1}{2}$ (D) none of these

Solution : The man has to win at least 4 times.

$\therefore$ the required probability

$$= {}^7C_4\left(\frac{1}{2}\right)^4\cdot\left(\frac{1}{2}\right)^3 + {}^7C_5\cdot\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^2 + {}^7C_6\left(\frac{1}{2}\right)^6\cdot\frac{1}{2} + {}^7C_7\left(\frac{1}{2}\right)^7$$

$$= ({}^7C_4 + {}^7C_5 + {}^7C_6 + {}^7C_7)\cdot\frac{1}{2^7} = \frac{64}{2^7} = \frac{1}{2}.$$ **Ans. (C)**

Do yourself - 8 :

(i) An experiment succeeds twice as often as it fails. Find the probability that in next 6 trials, there will be more than 3 successes.

(ii) Find the probability of getting 4 exactly thrice in 7 throws of a die.

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12. BAYE'S THEOREM :

Let an event A of an experiment occurs with its n mutually exclusive & exhaustive events $B_1, B_2, B_3, \ldots\ldots B_n$ & the probabilities $P(A/B_1)$, $P(A/B_2)$..... $P(A/B_n)$ are known, then

$$P(B_i/A) = \frac{P(B_i).P(A/B_i)}{\sum_{i=1}^{n} P(B_i).P(A/B_i)}$$

Explanation :

$A \equiv$ event what we have ; $B_i \equiv$ event what we want & remaining are alternative events.

Now, $P(AB_i) = P(A).\ P(B_i/A) = P(B_i).\ P(A/B_i)$

$$P(B_i/A) = \frac{P(B_i)\ .\ P(A/B_i)}{P(A)} = \frac{P(B_i)\ .\ P(A/B_i)}{\sum_{i=1}^{n} P(AB_i)}$$

$$P(B_i/A) = \frac{P(B_i)\ .\ P(A/B_i)}{\sum_{i=1}^{n} P(B_i)\ .\ P(A/B_i)}$$

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Illustration 26 : Given three identical boxes I, II and III, each containing two coins. In box I, both coins are gold coins, in box II, both are silver coins and in the box III, there is one gold and one silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold ?

Solution : Let E_1, E_2 and E_3 be the events that boxes I, II and III are chosen, respectively.

Then $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

Also, let A be the event that 'the coin drawn is of gold'

Then $P(A|E_1)$ = P(a gold coin from box I) = $\frac{2}{2} = 1$

$P(A|E_2)$ = P(a gold coin from box II) = 0

$P(A|E_3)$ = P(a gold coin from box III) = $\frac{1}{2}$

Now, the probability that the other coin in the box is of gold
= the probability that gold coin is drawn from the box I.
$= P(E_1|A)$
By Baye's theorem, we know that

$$P(E_1|A) = \frac{P(E_1)P(A|E_1)}{P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + P(E_3)P(A|E_3)} = \frac{\frac{1}{3}\times 1}{\frac{1}{3}\times 1 + \frac{1}{3}\times 0 + \frac{1}{3}\times\frac{1}{2}} = \frac{2}{3}$$

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Illustration 27 : A bag A contains 2 white and 3 red balls and a bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from the bag B.

Solution : Let E_1 = The event of ball being drawn from bag A

E_2 = The event of ball being drawn from bag B.

E = The event of ball being red.

Since, both the bags are equally likely to be selected, therefore

$$P(E_1) = P(E_2) = \frac{1}{2} \text{ and } P(E \mid E_1) = \frac{3}{5} \text{ and } P(E \mid E_2) = \frac{5}{9}$$

$\therefore$ Required probability

$$P(E_2 \mid E) = \frac{P(E_2)P(E \mid E_2)}{P(E_1).P(E \mid E_1) + P(E_2)P(E \mid E_2)} = \frac{\frac{1}{2}\times\frac{5}{9}}{\frac{1}{2}\times\frac{3}{5}+\frac{1}{2}\times\frac{5}{9}} = \frac{25}{52}$$

Do yourself - 9 :

(i) A pack of cards was found to contain only 51 cards. If first 13 cards, which are examined, are all red, then find the probability that the missing card is black.

(ii) A man has 3 coins A, B & C. A is fair coin. B is biased such that the probability of occurring head on it is 2/3. C is also biased with the probability of occurring head as 1/3. If one coin is selected and tossed three times, giving two heads and one tail, find the probability that the chosen coin was A.

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13. PROBABILITY THROUGH STATISTICAL (STOCHATIC) TREE DIAGRAM :

These tree diagrams are generally drawn by economist and give a simple approach to solve a problem.

Illustration 28 : A bag initially contains 1 red ball and 2 blue balls. A trial consists of selecting a ball at random, noting its colour and replacing it together with an additional ball of the same colour. Given that three trials are made, draw a tree diagram illustrating the various probabilities. Hence, or otherwise, find the probability that

(a) atleast one blue ball is drawn

(b) exactly one blue ball is drawn

(c) Given that all three balls drawn are of the same colour find the probability that they are all red.

Solution :

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Calculations :

$$P(A) = 1 - P(RRR) = 1 - \frac{1}{3}.\frac{1}{2}.\frac{3}{5} = 1 - \frac{1}{10} = \frac{9}{10}$$

$$P(\text{exactly one Blue}) = \frac{1}{3}.\frac{1}{2}.\frac{2}{5} + \frac{1}{3}.\frac{1}{2}.\frac{2}{5} + \frac{2}{3}.\frac{1}{4}.\frac{2}{5} = \frac{1}{15} + \frac{1}{15} + \frac{1}{15} = \frac{3}{15} = \frac{1}{5}$$

$$P(C) = P\left(\frac{RRR}{(RRR \cup BBB)}\right) = \frac{P(RRR)}{P(RRR) + P(BBB)} = \frac{\frac{1}{3}.\frac{1}{2}.\frac{3}{5}}{\frac{1}{3}.\frac{1}{2}.\frac{3}{5} + \frac{2}{3}.\frac{3}{4}.\frac{4}{5}} = \frac{\frac{1}{10}}{\frac{1}{10} + \frac{4}{10}} = \frac{1}{5}$$

14. PROBABILITY DISTRIBUTION (Not in JEE) :

(a) A Probability Distribution spells out how a total probability of 1 is distributed over several values of a random variable.

(b) Mean of any probability distribution of a random variable is given by :

$$\mu = \frac{\sum p_i x_i}{\sum p_i} = \sum p_i x_i \quad (\text{ Since } \Sigma p_i = 1)$$

(c) Variance of a random variable is given by, $\sigma^2 = \sum (x_i - \mu)^2 . p_i$

$\sigma^2 = \sum p_i x^2_i - \mu^2$ (Note that Standard Deviation (SD) = $+\sqrt{\sigma^2}$)

(d) The probability distribution for a binomial variate 'X' is given by ; $P(X = r) = {}^nC_r\, p^r q^{n-r}$ where: p = probability of success in a single trial, q = probability of failure in a single trial and p + q = 1.

(e) Mean of Binomial Probability Distribution (BPD) = np ; variance of BPD = npq.

(f) If p represents a person's chance of success in any venture and 'M' the sum of money which he will receive in case of success, then his expectations or probable value = pM

Illustration 29 : A random variable X has the probability distribution:

X :	1	2	3	4	5	6	7	8
p(X) :	0.15	0.23	0.12	0.10	0.20	0.08	0.07	0.05

For the events E = {X is a prime number} and F = {X < 4}, the probability $P(E \cup F)$ is -

(1) 0.35 (2) 0.77 (3) 0.87 (4) 0.50

Solution : E = x is a prime number

P(E) = P(2) + P(3) + P(5) + P(7) = 0.62

F = (x < 4), P(F) =P(1) + P(2) + P(3) = 0.50

$\therefore$ $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

= 0.62 + 0.50 – 0.35 = 0.77

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Illustration 30 : The mean and the variance of a binomial distribution are 4 and 2 respectively. Then the probability of 2 successes is -

(1) $\frac{128}{256}$ (2) $\frac{219}{256}$ (3) $\frac{37}{256}$ (4) $\frac{28}{256}$

Solution : $\left.\begin{array}{l} np = 4 \\ npq = 2 \end{array}\right\} \Rightarrow q = \frac{1}{2}, p = \frac{1}{2}, n = 8$

$$P(X = 2) = {}^8C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^6 = 28.\frac{1}{2^8} = \frac{28}{256}$$

<u>Miscellaneous Illustrations</u> :

Illustration 31 : Three persons A, B, C in order cut a pack of cards, replacing them after each cut, on the condition that the first who cuts a spade shall win a prize; find their respective chances.

Solution : Let p be the chance of cutting a spade and q the chance of not cutting a spade from a pack of 52 cards.

Then $p = \frac{{}^{13}C_1}{{}^{52}C_1} = \frac{1}{4}$ and $q = 1 - \frac{1}{4} = \frac{3}{4}$

Now A will win a prize if he cuts spade at 1st, 4th, 7th, 10th turns, etc. Note that A will get a second chance if A, B, C all fail to cut a spade once and then A cuts a spade at the 4th turn. Similarly he will cut a spade at the 7th turn when A, B, C fail to cut spade twice, etc.

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Hence A's chance of winning the prize $= p + q^3p + q^6p + q^9p + ... = \frac{p}{1-q^3} = \frac{\frac{1}{4}}{1-\left(\frac{3}{4}\right)^3} = \frac{16}{37}$

Similarly B's chance $= (qp + q^4p + q^7p + ...) = q(p + q^3p + q^6p + ...) = \frac{3}{4}.\frac{16}{37} = \frac{12}{37}$

and C's chance $= \frac{3}{4}$ of B's chance $= \frac{3}{4}.\frac{12}{37} = \frac{9}{37}$

Illustration 32 : (a) If p and q are chosen randomly from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, with replacement, determine the probability that the roots of the equation $x^2 + px + q = 0$ are real. **[IIT 1997]**

(b) Each coefficient in the equation $ax^2 + bx + c = 0$ is determined by throwing an ordinary die. Find the probability that the equation will have equal roots.

Solution : (a) If roots of $x^2 + px + q = 0$ are real, then $p^2 - 4q \geq 0$ ..... (i)

Both p, q belongs to set S = {1, 2, 3, ... 10} when p = 1, no value of q from S will satisfy (i)

p = 2	q = 1 will satisfy	1 value
p = 3	q = 1, 2	2 value
p = 4	q = 1, 2, 3, 4	4 value
p = 5	q = 1, 2, 3, 4, 5, 6	6 value
p = 6	q = 1, 2, 3, 4, 5, 6, 7, 8, 9	9 value

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For p = 7, 8, 9, 10 all the ten values of q will satisfy.

Sum of these selections is 1 + 2 + 4 + 6 + 9 + 10 + 10 + 10 + 10 = 62

But the total number of selections of p and q without any order is 10 × 10 = 100

Hence the required probability is = $\frac{62}{100} = 0.62$

(b) Roots equal $\Rightarrow$ $b^2 - 4ac = 0$

$\therefore \quad \left(\frac{b}{2}\right)^2 = ac$ ..... (i)

Each coefficient is an integer, so we consider the following cases :

b = 1 $\quad \therefore \quad \frac{1}{4} = ac$

No integral values of a and c

b = 2 $\quad$ 1 = ac $\quad \therefore \quad$ (1, 1)

b = 3 $\quad$ 9/2 = ac

No integral values of a and c

b = 4 $\quad$ 4 = ac $\quad \therefore \quad$ (1, 4), (2, 2), (4, 1)

b = 5 $\quad$ 25/2 = ac

No integral values of a and c

b = 6 $\quad$ 9 = ac $\quad \therefore \quad$ (3, 3)

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Thus we have 5 favourable ways for b = 2, 4, 6

Total number of equations is 6.6.6 = 216

$\therefore$ Required probability is $\frac{5}{216}$

Illustration 33: A set A has n elements. A subset P of A is selected at random. Returning the element of P, the set Q is formed again and then a subset Q is selected from it. Find the probability that P and Q have no common elements. **[IIT 1990]**

Solution : Let P be the empty set, or one element set or two elements set ..... or n elements set. Then the set Q will be chosen from amongst the remaining n elements or n – 1 elements or n – 2 elements ..... or no elements. The probability of P being an empty set is ${}^nC_0/2^n$, the probability of P being one element set is ${}^nC_1/2^n$ and in general, the probability of P being an r element set is ${}^nC_r/2^n$.

When the set P consisting of r elements is chosen from A, then the probability of choosing the set Q from amongst the remaining n – r elements is $2^{n-r}/2^n$. Hence the probability that P and Q have no common elements is given by

$$\sum_{r=0}^{n} \frac{{}^nC_r}{2^n} \cdot \frac{2^{n-r}}{2^n} = \frac{1}{4^n} \sum_{r=0}^{n} {}^nC_r . 2^{n-r} = \left(\frac{1}{4}\right)^n (1+2)^n = \left(\frac{3}{4}\right)^n \quad \text{[By binomial theorem]}$$

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Illustration 34 : The probabilities of three events A, B and C are P(A) = 0.6, P(B) = 0.4 and P(C) = 0.5. If $P(A \cup B) = 0.8$, $P(A \cap C) = 0.3$, $P(A \cap B \cap C) = 0.2$ and $P(A \cup B \cup C) \geq 0.85$, find $P(B \cap C)$.

Solution : $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.8 = 0.6 + 0.4 - P(A \cap B)$

$\therefore \quad P(A \cap B) = 0.2$

Now $P(A \cup B \cup C) = S_1 - S_2 + S_3 = (0.6 + 0.4 + 0.5) - (0.2 + P(B \cap C) + 0.3) + 0.2$

$= 1.5 - 0.3 - P(B \cap C)$

We know $0.85 \leq P(A \cup B \cup C) \leq 1$

or $\quad 0.85 \leq 1.2 - P(B \cap C) \leq 1$

$\therefore \quad 0.2 \leq P(B \cap C) \leq 0.35$

ANSWERS FOR DO YOURSELF

1 : **(i)** $\{B_1R_1, B_2R_1, B_3R_1, B_1R_2, B_2R_2, B_3R_2, B_1B_2, B_2B_3, B_1B_3, R_1R_2\}$

(ii) $\{M_1W_1, M_2W_1, M_1W_2, M_2W_2, M_1W_3, M_2W_3\}$

(iii) {H1, H2, H3, H4, H5, H6, TH, TT}

(iv) A

2 : **(i)** $\frac{3}{4}$ **(ii)** $\frac{10}{21}$ **(iii)** 1/10 **(iv)** $\frac{(13)^5 - (12)^5}{(13)^5}$

3 : **(i)** **(a)**

(b)

(ii) A,B,D **(iii)** 11/15

(iv) (a) $\frac{19}{25}$ (b) $\frac{13}{25}$

4 : **(i)** A **(ii)** 1/5525

5 : **(i)** A, B **(ii)** 1/36

6 : **(i)** 2/21 **(ii)** $\frac{1}{n+2}$

7 : **(i)** **(a)** 0.72 **(b)** 0.23

8 : **(i)** $\frac{496}{729}$ **(ii)** ${}^7C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^4$

9 : **(i)** 2/3 **(ii)** 9/25

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EXERCISE (O-1)

PART # 1

1. 6 married couples are standing in a room. If 4 people are chosen at random, then the chance that exactly one married couple is among the 4 is-

(A) $\frac{16}{33}$ (B) $\frac{8}{33}$ (C) $\frac{17}{33}$ (D) $\frac{24}{33}$

PB0001

2. The probability that a positive two digit number selected at random has its tens digit at least three more than its unit digit is -

(A) 14/45 (B) 7/45 (C) 36/45 (D) 1/6

PB0002

3. A 5 digit number is formed by using the digits 0,1,2,3,4 & 5 without repetition. The probability that the number is divisible by 6 is :

(A) 8% (B) 17% (C) 18% (D) 36%

PB0003

4. A cube with all six faces coloured is cut into 64 cubical blocks of the same size which are thoroughly mixed. Find the probability that the 2 randomly chosen blocks have 2 coloured faces each -

PB0004

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5. A card is drawn at random from a well shuffled deck of cards. Find the probability that the card is a-

(i) king or a red card (ii) club or a diamond (iii) king or a queen (iv) king or an ace

(v) spade or a club (vi) neither a heart nor a king

PB0005

6. A bag contain 5 white, 7 black, and 4 red balls, find the chance that three balls drawn at random are all white.

PB0006

7. If four coins are tossed, Two events A and B are defined as

A : No two consecutive heads occur

B : At least two consecutive heads occur.

Find P(A) and P(B). State whether the events are equally likely, mutually exclusive and exhaustive.

PB0007

8. Thirteen persons take their places at a round table, Find the odds against two particular persons sitting together.

PB0008

9. A has 3 shares in a lottery containing 3 prizes and 9 blanks, B has 2 shares in a lottery containing 2 prizes and 6 blanks. Compare their chances of success.

PB0009

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10. Mr. A forgot to write down a very important phone number. All he remembers is that it started with 713 and that the next set of 4 digit involved are 1,7 and 9 with one of these numbers appearing twice. He guesses a phone number and dials randomly. The odds in favour of dialing the correct telephone number, is -

(A) 1 : 35 (B) 1 : 71 (C) 1 : 23 (D) 1 : 36

PB0010

11. Consider a function $f(x)$ that has zeroes 4 and 9. Given that Mr. A randomly selects a number from the set {–10, –9,–8,.....8,9,10}, what is the probability that Mr. A chooses a zero of $f(x^2)$?

PB0011

12. (a) A fair die is tossed. If the number is odd, find the probability that it is prime.

(b) Three fair coins are tossed. If both heads and tails appear, determine the probability that exactly one head appears.

PB0012

13. Mr. A lives at origin on the cartesian plane and has his office at (4,5). His friend lives at (2,3) on the same plane. Mr. A can go to his office travelling one block at a time either in the +y or +x direction. If all possible paths are equally likely then the probability that Mr. A passed his friends house is -

(A) 1/2 (B) 10/21 (C) 1/4 (D) 11/21

PB0013

14. I have 3 normal dice, one red, one blue and one green and I roll all three simultaneously. Let P be the probability that the sum of the numbers on the red and blue dice is equal to the number on the green die. If P is the written in lowest terms as a/b then the value of (a + b) equals -

(A) 79 (B) 77 (C) 61 (D) 57

PB0014

15. There are three passengers on an airport shuttle bus that makes stops at four different hotels. The probability that all three passengers will be staying at different hotels, is -

(A) $\frac{1}{16}$ (B) $\frac{1}{4}$ (C) $\frac{3}{8}$ (D) $\frac{3}{4}$

PB0015

PART # 2

1. In throwing 3 dice, the probability that atleast 2 of the three numbers obtained are same is -

(A) 1/2 (B) 1/3 (C) 4/9 (D) none

PB0016

2. There are 4 defective items in a lot consisting of 10 items. From this lot we select 5 items at random. The probability that there will be 2 defective items among them is -

(A) $\frac{1}{2}$ (B) $\frac{2}{5}$ (C) $\frac{5}{21}$ (D) $\frac{10}{21}$

PB0017

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3. From a pack of 52 playing cards, face cards and tens are removed and kept aside then a card is drawn at random from the remaining cards. If

A: The event that the card drawn is an ace

H: The event that the card drawn is a heart

S : The event that the card drawn is a spade

then which of the following holds ?

(A) $9P(A) = 4P(H)$ (B) $P(S) = 4P(A \cap H)$

(C) $3P(H) = 4P(A \cup S)$ (D) $P(H) = 12P(A \cap S)$

PB0018

4. If two of the 64 squares are chosen at random on a chess board, the probability that they have a side in common is -

(A) 1/9 (B) 1/18 (C) 2/7 (D) none]

PB0019

5. Two red counters, three green counters and 4 blue counters are placed in a row in random order. The probability that no two blue counters are adjacent is -

(A) $\frac{7}{99}$ (B) $\frac{7}{198}$ (C) $\frac{5}{42}$ (D) none

PB0020

6. South African cricket captain lost the toss of a coin 13 times out of 14. The chance of this happening was

(A) $\frac{7}{2^{13}}$ (B) $\frac{1}{2^{13}}$ (C) $\frac{13}{2^{14}}$ (D) $\frac{13}{2^{13}}$

PB0021

7. There are ten prizes, five A's, three B's and two C's, placed in identical sealed envelopes for the top ten contestants in a mathematics contest. The prizes are awarded by allowing winners to select an envelope at random from those remaining. When the 8th contestant goes to select the prize, the probability that the remaining three prizes are one A, one B and one C, is

(A) 1/4 (B) 1/3 (C) 1/12 (D) 1/10

PB0022

8. A coin is tossed and a die is thrown. Find the probability that the outcome will be a head or a number greater than 4.

PB0023

9. A coin is biased so that heads is three times as likely to appear as tails. Find P(H) and P(T). If such a coin is tossed twice find the probability that head occurs at least once.

PB0024

10. Given two independent events A, B such that $P(A) = 0.3$, $P(B) = 0.6$. Determine

(i) P(A and B) (ii) P(A and not B) (iii) P(not A and B) (iv) P(neither A nor B)

(v) P(A or B)

PB0025

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11. The probabilities that a student will receive A, B, C or D grade are 0.40, 0.35, 0.15 and 0.10 respectively. Find the probability that a student will receive
(i) not an A grade (ii) B or C grade (iii) at most C grade

PB0026

12. In a single throw of three dice, determine the probability of getting
(i) a total of 5 (ii) a total of atmost 5 (iii) a total of at least 5.

PB0027

13. A natural number x is randomly selected from the set of first 100 natural numbers. Find the probability that it satisfies the inequality. $x + \frac{100}{x} > 50$

PB0028

14. 3 students A, B and C are in a swimming race. A and B have the same probability of winning and each is twice as likely to win as C. Find the probability that B or C wins. Assume no two reach the winning point simultaneously.

PB0029

15. A box contains 7 tickets, numbered from 1 to 7 inclusive. If 3 tickets are drawn from the box without replacement, one at a time, determine the probability that they are alternatively either odd-even-odd or even-odd-even.

PB0030

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16. Let a red die, a blue die, a green die and a white die are rolled once, the dice being fair. The outcomes on the red, blue, green and white die denote the numbers, a, b,c and d respectively. Let E denotes the event that absolute value of (a – 1) (b – 2) (c – 3) (d – 6) = 1, then P(E) is -

(A) $\frac{1}{324}$ (B) $\frac{1}{648}$ (C) $\frac{2}{324}$ (D) $\frac{1}{162}$

PB0031

17. 5 different marbles are placed in 5 different boxes randomly. Find the probability that exactly two boxes remain empty. Given each box can hold any number of marbles.

PB0032

18. Let A and B be events such that $P(\overline{A}) = 4/5$, P(B) = 1/3, P(A/B) = 1/6, then :
(a) $P(A \cap B)$
(b) $P(A \cup B)$
(c) P(B/A)
(d) Are A and B independent?

PB0033

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PART # 3

1. Let A & B be two events. Suppose P(A) = 0.4 , P(B) = p & P(A ∪ B) = 0.7. The value of p for which A & B are independent is :

(A) 1/3 (B) 1/4 (C) 1/2 (D) 1/5

PB0034

2. A pair of numbers is picked up randomly (without replacement) from the set {1, 2, 3, 5, 7, 11, 12, 13, 17, 19}. The probability that the number 11 was picked given that the sum of the numbers was even, is nearly :

(A) 0.1 (B) 0.125 (C) 0.24 (D) 0.18

PB0035

3. For a biased die the probabilities for the different faces to turn up are given below :

Faces :	1	2	3	4	5	6
Probabilities :	0.10	0.32	0.21	0.15	0.05	0.17

The die is tossed & you are told that either face one or face two has turned up. Then the probability that it is face one is :

(A) 1/6 (B) 1/10 (C) 5/49 (D) 5/21

PB0036

4. A determinant is chosen at random from the set of all determinants of order 2 with elements 0 or 1 only. The probability that the determinant chosen has the value non negative is :

(A) 3/16 (B) 6/16 (C) 10/16 (D) 13/16

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PB0037

5. A card is drawn & replaced in an ordinary pack of 52 playing cards. Minimum number of times must a card be drawn so that there is atleast an even chance of drawing a heart, is

(A) 2 (B) 3 (C) 4 (D) more than four

PB0038

6. A license plate is 3 capital letters (of English alphabets) followed by 3 digits. If all possible license plates are equally likely, the probability that a plate has either a letter palindrome or a digit palindrome (or both), is

(A) $\frac{7}{52}$ (B) $\frac{9}{65}$ (C) $\frac{8}{65}$ (D) none

PB0039

7. Whenever horses a, b, c race together, their respective probabilities of winning the race are 0.3, 0.5 and 0.2 respectively. If they race three times the probability that "the same horse wins all the three races" and the probability that a, b, c each wins one race, are respectively (Assume no dead heat)

(A) $\frac{8}{50}$; $\frac{9}{50}$ (B) $\frac{16}{100}$, $\frac{3}{100}$ (C) $\frac{12}{50}$; $\frac{15}{50}$ (D) $\frac{10}{50}$; $\frac{8}{50}$

PB0040

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8. Two cubes have their faces painted either red or blue. The first cube has five red faces and one blue face. When the two cubes are rolled simultaneously, the probability that the two top faces show the same colour is 1/2. Number of red faces on the second cube, is

(A) 1 (B) 2 (C) 3 (D) 4

PB0041

9. A committee of three persons is to be randomly selected from a group of three men and two women and the chair person will be randomly selected from the committee. The probability that the committee will have exactly two women and one man, and that the chair person will be a woman, is/are

(A) 1/5 (B) 8/15 (C) 2/3 (D) 3/10

PB0042

10. An urn contains 3 red balls and n white balls.
Mr. A draws two balls together from the urn. The probability that they have the same colour is 1/2.
Mr. B draws one ball from the urn, notes its colour and replaces it. He then draws a second ball from the urn and finds that both balls have the same colour is, 5/8. The possible value of n is

(A) 9 (B) 6 (C) 5 (D) 1

PB0043

11. The probability that an automobile will be stolen and found within one week is 0.0006. The probability that an automobile will be stolen is 0.0015. The probability that a stolen automobile will be found in one week is

(A) 0.3 (B) 0.4 (C) 0.5 (D) 0.6

PB0044

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12. A box contains 100 tickets numbered 1, 2, 3,.... ,100. Two tickets are chosen at random. It is given that the maximum number on the two chosen tickets is not more than 10. The minimum number on them is 5, with probability

(A) $\frac{1}{9}$ (B) $\frac{2}{11}$ (C) $\frac{3}{19}$ (D) none

PB0045

13. Two boys A and B find the jumble of n ropes lying on the floor. Each takes hold of one loose end randomly. If the probability that they are both holding the same rope is $\frac{1}{101}$ then the number of ropes is equal to

(A) 101 (B) 100 (C) 51 (D) 50

PB0046

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[REASONING TYPE]

14. For children A,B,C and D have 1,3,5 and 7 identical unbiased dice respectively and roll them with the condition that one who obtains an even score, wins. They keep playing till some one or the other wins.

Statement-1: All the four children are equally likely to win provided they roll their dice simultaneously.

Statement-2: The child A is most probable to win the game if they roll their dice in order of A,B,C and D respectively.

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

PB0047

15. In one day test match between India and Australia the umpire continues tossing a fair coin until the two consecutive throws either H T or T T are obtained for the first time. If it is H T, India wins and if it is T T, Australia wins.

Statement-1: Both India and Australia have equal probability of winning the toss.

Statement-2: If a coin is tossed twice then the events HT or TT are equiprobable.

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

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PB0048

PART # 4

1. If E & F are events with $P(E) \le P(F)$ & $P(E \cap F) > 0$, then :

(A) occurrence of E $\Rightarrow$ occurrence of F

(B) occurrence of F $\Rightarrow$ occurrence of E

(C) non – occurrence of E $\Rightarrow$ non – occurrence of F

(D) none of the above implications holds.

PB0049

2. Events A and C are independent. If the probabilities relating A, B and C are P (A) = 1/5; P (B) = 1/6 ;

$P(A \cap C) = 1/20$; $P(B \cup C) = 3/8$ then

(A) events B and C are independent

(B) events B and C are mutually exclusive

(C) events B and C are neither independent nor mutually exclusive

(D) events A and C are equiprobable

PB0050

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3. An unbiased cubic die marked with 1, 2, 2, 3, 3, 3 is rolled 3 times. The probability of getting a total score of 4 or 6 is

(A) $\frac{16}{216}$ (B) $\frac{50}{216}$ (C) $\frac{60}{216}$ (D) none

PB0051

4. A bag contains 3 R & 3 G balls and a person draws out 3 at random. He then drops 3 blue balls into the bag & again draws out 3 at random. The chance that the 3 later balls being all of different colours is

(A) 15% (B) 20% (C) 27% (D) 40%

PB0052

5. A biased coin with probability P, $0 < P < 1$, of heads, is tossed until a head appears for the first time. If the probability that the number of tosses required is even is 2/5, then the value of P is

(A) 1/4 (B) 1/6 (C) 1/3 (D) 1/2

PB0053

6. Two numbers a and b are selected from the set of natural number then the probability that $a^2 + b^2$ is divisible by 5 is

(A) $\frac{9}{25}$ (B) $\frac{7}{18}$ (C) $\frac{11}{36}$ (D) $\frac{17}{81}$

PB0054

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7. When a missile is fired from a ship, the probability that it is intercepted is 1/3. The probability that the missile hits the target, given that it is not intercepted is 3/4. If three missiles are fired independently from the ship, the probability that all three hits the target, is

(A) 1/12 (B) 1/8 (C) 3/8 (D) 3/4

PB0055

8. An urn contains 10 balls coloured either black or red. When selecting two balls from the urn at random, the probability that a ball of each colour is selected is $8/15$. Assuming that the urn contains more black balls than red balls, the probability that at least one black ball is selected, when selecting two balls, is

(A) $\frac{18}{45}$ (B) $\frac{30}{45}$ (C) $\frac{39}{45}$ (D) $\frac{41}{45}$

PB0056

9. An unbiased die with numbers 1, 2, 3, 4, 6 and 8 on its six faces is rolled. After this roll if an odd number appears on the top face, all odd numbers on the die are doubled. If an even number appears on the top face, all the even numbers are halved. If the given die changes in this way then the probability that the face 2 will appear on the second roll is -

(A) 2/18 (B) 3/18 (C) 2/9 (D) 5/18

PB0057

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10. A butterfly randomly lands on one of the six squares of the T-shaped figure shown and then randomly moves to an adjacent square. The probability that the butterfly ends up on the R square is

(A) 1/4 (B) 1/3

(C) 2/3 (D) 1/6

PB0058

11. A fair coin is tossed a large number of times. Assuming the tosses are independent which one of the following statement, is True?

(A) Once the number of flips is large enough, the number of heads will always be exactly half of the total number of tosses. For example, after 10,000 tosses one should have exactly 5,000 heads.

(B) The proportion of heads will be about 1/2 and this proportion will tend to get closer to 1/2 as the number of tosses increases

(C) As the number of tosses increases, any long run of heads will be balanced by a corresponding run of tails so that the overall proportion of heads is exactly 1/2

(D) All of the above

PB0059

12. The number 'a' is randomly selected from the set {0, 1, 2, 3, ... 98, 99}. The number 'b' is selected from the same set. Probability that the number $3^a + 7^b$ has a digit equal to 8 at the units place, is

(A) $\frac{1}{16}$ (B) $\frac{2}{16}$ (C) $\frac{4}{16}$ (D) $\frac{3}{16}$

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PB0060

PART # 5

1. An examination consists of 8 questions in each of which one of the 5 alternatives is the correct one. On the assumption that a candidate who has done no preparatory work chooses for each question any one of the five alternatives with equal probability, the probability that he gets more than one correct answer is equal to

(A) $(0.8)^8$ (B) $3(0.8)^8$ (C) $1 - (0.8)^8$ (D) $1 - 3(0.8)^8$

PB0061

2. An ant is situated at the vertex A of the triangle ABC. Every movement of the ant consists of moving to one of other two adjacent vertices from the vertex where it is situated. The probability of going to any of the other two adjacent vertices of the triangle is equal. The probability that at the end of the fourth movement the ant will be back to the vertex A, is :

(A) 4/16 (B) 6/16 (C) 7/16 (D) 8/16

PB0062

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3. A key to room number C_3 is dropped into a jar with five other keys, and the jar is throughly mixed. If keys are randomly drawn from the jar without replacement until the key to room C_3 is chosen, then what are the odds in favour that the key to room C_3 will be obtained on the 2nd try?

(A) 1 : 4 (B) 1 : 5 (C) 1 : 6 (D) 5 : 6

PB0063

4. Lot A consists of 3G and 2D articles. Lot B consists of 4G and 1D article. A new lot C is formed by taking 3 articles from A and 2 from B. The probability that an article chosen at random from C is defective, is

(A) 1/3 (B) 2/5 (C) 8/25 (D) none

PB0064

5. Mr. A and Mr. B each have a bag that contains one ball of each of the colours blue, green, orange, red and violet. 'A' randomly selects one ball from his bag and puts it into B's bag. 'B' then randomly selects one ball from his bag and puts it into A's bag. The probability that after this process the contents of the two bags are the same, is

(A) 1/6 (B) 1/5 (C) 1/3 (D) 1/2

PB0065

6. An instrument consists of two units. Each unit must function for the instrument to operate. The reliability of the first unit is 0.9 & that of the second unit is 0.8. The instrument is tested & fails. The probability that "only the first unit failed & the second unit is sound" is :

(A) 1/7 (B) 2/7 (C) 3/7 (D) 4/7

PB0066

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7. A box contains 10 tickets numbered from 1 to 10. Two tickets are drawn one by one without replacement. The probability that the "absolute value of difference between the first drawn ticket number and the second is not less than 4" is

(A) $\frac{7}{30}$ (B) $\frac{7}{15}$ (C) $\frac{11}{30}$ (D) $\frac{10}{30}$

PB0067

8. A purse contains 2 six sided dice. One is a normal fair die, while the other has 2 ones, 2 threes, and 2 fives. A die is picked up and rolled. Because of some secret magnetic attraction of the unfair die, there is 75% chance of picking the unfair die and a 25% chance of picking a fair die. The die is rolled and shows up the face 3. The probability that a fair die was picked up, is

(A) 1/7 (B) 1/4 (C) 1/6 (D) 1/24

PB0068

9. On a Saturday night 20% of all drivers is U.S.A. are under the influence of alcohol. The probability that a driver under the influence of alcohol will have an accident is 0.001. The probability that a sober driver will have an accident is 0.0001. If a car on a Saturday night smashed into a tree, the probability that the driver was under the influence of alcohol, is -

(A) 3/7 (B) 4/7 (C) 5/7 (D) 6/7

PB0069

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10. A box has four dice in it. Three of them are fair dice but the fourth one has the number five on all of its faces. A die is chosen at random from the box and is rolled three times and shows up the face five on all the three occasions. The chance that the die chosen was a rigged die, is

(A) $\frac{216}{217}$ (B) $\frac{215}{219}$ (C) $\frac{216}{219}$ (D) none

PB0070

11. On a normal standard die one of the 21 dots from any one of the six faces is removed at random with each dot equally likely to be chosen. The die is then rolled. The probability that the top face has an odd number of dots is

(A) $\frac{5}{11}$ (B) $\frac{5}{12}$ (C) $\frac{11}{21}$ (D) $\frac{6}{11}$

PB0071

Paragraph for question nos. 12 to 14

A JEE aspirant estimates that she will be successful with an 80 percent chance if she studies 10 hours per day, with a 60 percent chance if she studies 7 hours per day and with a 40 percent chance if she studies 4 hours per day. She further believes that she will study 10 hours, 7 hours and 4 hours per day with probabilities 0.1, 0.2 and 0.7, respectively

12. The chance she will be successful, is

(A) 0.28 (B) 0.38 (C) 0.48 (D) 0.58

PB0072

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13. Given that she is successful, the chance she studied for 4 hours, is

(A) $\frac{6}{12}$ (B) $\frac{7}{12}$ (C) $\frac{8}{12}$ (D) $\frac{9}{12}$

PB0072

14. Given that she does not achieve success, the chance she studied for 4 hour, is

(A) $\frac{18}{26}$ (B) $\frac{19}{26}$ (C) $\frac{20}{26}$ (D) $\frac{21}{26}$

PB0072

[REASONING TYPE]

15. A fair coin is tossed 3 times consider the events

A : first toss is head

B : second toss is head

C : exactly two consecutive heads or exactly two consecutive tails.

Statement-1: A, B, C are independent events.

Statement-2: A, B, C are pairwise independent.

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

PB0073

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PART # 6

1. A bowl has 6 red marbles and 3 green marbles. The probability that a blind folded person will draw a red marble on the second draw from the bowl without replacing the marble from the first draw, is

(A) 2/3 (B) 1/4 (C) 5/12 (D) 5/8

PB0074

2. The probability that a radar will detect an object in one cycle is p. The probability that the object will be detected in n cycles is :

(A) $1 - p^n$ (B) $1 - (1 - p)^n$ (C) p^n (D) $p(1 - p)^{n-1}$

PB0075

3. In a certain factory, machines A, B and C produce bolts. Of their production, machines A, B, and C produce 2%, 1% and 3% defective bolts respectively. Machine A produces 35% of the total output of bolts, machine B produces 25% and machine C produces 40%. A bolts is chosen at random from the factory's production and is found to be defective. The probability it was produced on machine C, is

(A) $\frac{6}{11}$ (B) $\frac{23}{45}$ (C) $\frac{24}{43}$ (D) $\frac{3}{11}$

PB0076

4. Three numbers are chosen at random without replacement from {1, 2, 3,..... , 10}. The probability that the minimum of the chosen numbers is 3 or their maximum is 7 is

(A) 1/2 (B) 1/3 (C) 1/4 (D) 11/40

PB0077

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5. Two buses A and B are scheduled to arrive at a town central bus station at noon. The probability that bus A will be late is 1/5. The probability that bus B will be late is 7/25. The probability that the bus B is late given that bus A is late is 9/10. Then the probabilities

(i) neither bus will be late on a particular day and

(ii) bus A is late given that bus B is late, are respectively

(A) 2/25 and 12/28 (B) 18/25 and 22/28

(C) 7/10 and 18/28 (D) 12/25 and 2/28

PB0078

6. If at least one child in a family with 3 children is a boy then the probability that exactly 2 of the children are boys, is

(A) 3/7 (B) 4/7 (C) 1/3 (D) 3/8

PB0079

7. From an urn containing six balls, 3 white and 3 black ones, a person selects at random an even number of balls (all the different ways of drawing an even number of balls are considered equally probable, irrespective of their number). Then the probability that there will be the same number of black and white balls among them

(A) 4/5 (B) 11/15 (C) 11/30 (D) 2/5

PB0080

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8. There are three main political parties namely 1, 2, 3. If in the adjoining table p_{ij} , (i, j=1, 2, 3) denote the probability that party j wins the general elections contested when party i is in the power. What is the probability that the party 2 will be in power after the next two elections, given that the party 1 is in the power?

$P_{11}=0.7$	$P_{12}=0.2$	$P_{13}=0.1$
$P_{21}=0.5$	$P_{22}=0.3$	$P_{23}=0.2$
$P_{31}=0.3$	$P_{32}=0.4$	$P_{33}=0.3$

(A) 0.27 (B) 0.24 (C) 0.14 (D) 0.06

PB0081

9. Shalu bought two cages of birds : Cage-I contains 5 parrots and 1 owl, and Cage-II contains 6 parrots, as shown

Birds like to fly

One day Shalu forgot to lock both cages and two birds flew from Cage-I to Cage-II. Then two birds flew back from Cage-II to Cage-I. Assume that all birds have equal chance of flying, the probability that the Owl is still in Cage-I, is

(A) 1/6 (B) 1/3 (C) 2/3 (D) 3/4

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PB0082

10. Suppose families always have one, two or three children, with probabilities $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{1}{4}$ respectively. Assume everyone eventually gets married and has children, the probability of a couple having exactly four grandchildren is

(A) $\frac{27}{128}$ (B) $\frac{37}{128}$ (C) $\frac{25}{128}$ (D) $\frac{20}{128}$

PB0083

11. Miss C has either Tea or Coffee at morning break. If she has tea one morning, the probability she has tea the next morning is 0.4. If she has coffee one morning, the probability she has coffee next morning is 0.3. Suppose she has coffee on a Monday morning. The probability that she has tea on the following Wednesday morning is

(A) 0.46 (B) 0.49 (C) 0.51 (D) 0.61

PB0084

12. In a maths paper there are 3 sections A, B & C. Section A is compulsory. Out of sections B & C a student has to attempt any one. Passing in the paper means passing in A & passing in B or C. The probability of the student passing in A, B & C are p, q & 1/2 respectively. If the probability that the student is successful is 1/2 then :

(A) p = q = 1 (B) p = q = 1/2 (C) p = 1, q = 0 (D) p = 1, q = 1/2

PB0085

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[REASONING TYPE]

13. From a well shuffled pack of 52 playing cards a card is drawn at random. Two events A and B are defined as

A : Red card is drawn.

B : Card drawn is either a Diamond or Heart

Statement-1: P(A + B) = P(AB)

Statement-2: $A \subseteq B$ and $B \subseteq A$

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

PB0086

EXERCISE (O-2)

[STRAIGHT OBJECTIVE TYPE]

1. n different books ($n \geq 3$) are put at random in a shelf. Among these books there is a particular book 'A' and a particular book B. The probability that there are exactly 'r' books between A and B is -

(A) $\frac{2}{n(n-1)}$ (B) $\frac{2(n-r-1)}{n(n-1)}$ (C) $\frac{2(n-r-2)}{n(n-1)}$ (D) $\frac{(n-r)}{n(n-1)}$

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PB0087

2. Of all the mappings that can be defined from the set $A : \{1, 2, 3, 4\} \rightarrow B(5, 6, 7, 8, 9\}$, a mapping is randomly selected. The chance that the selected mapping is strictly monotonic, is

(A) $\frac{1}{125}$ (B) $\frac{2}{125}$ (C) $\frac{5}{4096}$ (D) $\frac{5}{2048}$

PB0088

3. A fair die is thrown 3 times. The chance that sum of three numbers appearing on the die is less than 11, is equal to -

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{1}{6}$ (D) $\frac{5}{8}$

PB0089

4. One bag contains 3 white & 2 black balls, and another contains 2 white & 3 black balls. A ball is drawn from the second bag & placed in the first, then a ball is drawn from the first bag & placed in the second. When the pair of the operations is repeated, the probability that the first bag will contain 5 white balls is:

(A) 1/25 (B) 1/125 (C) 1/225 (D) 2/15

PB0090

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5. If a, b and c are three numbers (not necessarily different) chosen randomly and with replacement from the set {1, 2, 3, 4, 5}, the probability that (ab + c) is even, is

(A) $\frac{35}{125}$ (B) $\frac{59}{125}$ (C) $\frac{64}{125}$ (D) $\frac{75}{125}$

PB0091

6. A purse contains 100 coins of unknown value, a coin drawn at random is found to be a rupee. The chance that it is the only rupee in the purse, is (Assume all numbers of rupee coins in the purse to be equally likely.)

(A) $\frac{1}{5050}$ (B) $\frac{2}{5151}$ (C) $\frac{1}{4950}$ (D) $\frac{2}{4950}$

PB0092

7. Mr. Dupont is a professional wine taster. When given a French wine, he will identify it with probability 0.9 correctly as French, and will mistake it for a Californian wine with probability 0.1. When given a Californian wine, he will identify it with probability 0.8 correctly as Californian, and will mistake it for a French wine with probability 0.2. Suppose that Mr. Dupont is given ten unlabelled glasses of wine, three with French and seven with Californian wines. He randomly picks a glass, tries the wine, and solemnly says : "French". The probability that the wine he tasted was Californian, is nearly equal to

(A) 0.14 (B) 0.24 (C) 0.34 (D) 0.44

PB0093

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8. Sixteen players $s_1, s_2, \ldots, s_{16}$ play in a tournament. They are divided into eight pairs at random. From each pair a winner is decided on the basis of a game played between the two players of the pair. Assume that all the players are of equal strength. The probability that "exactly one of the two players s_1 & s_2 is among the eight winners" is

(A) $\frac{4}{15}$ (B) $\frac{7}{15}$ (C) $\frac{8}{15}$ (D) $\frac{9}{15}$

PB0094

9. A multiple choice test question has five alternative answers, of which only one is correct. If a student has done his home work, then he is sure to identify the correct answer; otherwise, he chooses an answer at random.

Let E : denotes the event that a student does his home work with P(E) = p and

F : denotes the event that he answer the question correctly.

(a) If p = 0.75 the value of P(E/F) equals

(A) $\frac{8}{16}$ (B) $\frac{10}{16}$ (C) $\frac{12}{16}$ (D) $\frac{15}{16}$

(b) The relation $P(E/F) \geq P(E)$ holds good for

(A) all values of p in [0, 1] (B) all values of p in (0, 1) only

(C) all values of p in [0.5, 1] only (D) no value of p.

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(c) Suppose that each question has *n* alternative answers of which only one is correct, and p is fixed but not equal to 0 or 1 then P(E/F)

(A) decreases as *n* increases for all $p \in (0, 1)$

(B) increases as *n* increases for all $p \in (0, 1)$

(C) remains constant for all $p \in (0, 1)$

(D) decreases if $p \in (0, 0.5)$ and increases if $p \in (0.5, 1)$ as *n* increases

PB0095

[MULTIPLE OBJECTIVE TYPE]

10. Which of the following statement(s) is/are correct?

(A) 3 coins are tossed once. Two of them atleast must land the same way. No matter whether they land heads or tails, the third coin is equally likely to land either the same way or oppositely. So, the chance that all the three coins land the same way is 1/2.

(B) Let $0 < P(B) < 1$ and $P(A/B) = P(A/B^c)$ then A and B are independent.

(C) Suppose an urn contains 'w' white and 'b' black balls and a ball is drawn from it and is replaced along with 'd' additional balls of the same colour. Now a second ball is drawn from it. The probability that the second drawn ball is white is independent of the value of 'd'.

(D) A, B, C simultaneously satisfy $P(ABC) = P(A)\cdot P(B)\cdot P(C)$ and $P(AB\overline{C}) = P(A)\cdot P(B)\cdot P(\overline{C})$ and $P(A\overline{B}C) = P(A)\cdot P(\overline{B})\cdot P(C)$ and $P(\overline{A}BC) = P(\overline{A})\cdot P(B)\cdot P(C)$ then A, B, C are independent.

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PB0096

11. Identify the correct statement :

(A) If the probability that a computer will fail during the first hour of operation is 0.01, then if we turn on 100 computers, exactly one will fail in the first hour of operation.

(B) A man has ten keys only one of which fits the lock. He tries them in a door one by one discarding the one he has tried. The probability that fifth key fits the lock is 1/10.

(C) Given the events A and B in a sample space. If P(A) = 1, then A and B are independent.

(D) When a fair six sided die is tossed on a table top, the bottom face can not be seen. The probability that the product of the numbers on the five faces that can be seen is divisible by 6 is one.

PB0097

12. Two whole numbers are randomly selected and multiplied. Consider two events E_1 and E_2 defined as

E_1 : Their product is divisible by 5

E_2 : Unit's place in their product is 5.

Which of the following statement(s) is/are correct?

(A) E_1 is twice as likely to occur as E_2.

(B) E_1 and E_2 are disjoint

(C) $P(E_2/E_1) = 1/4$

(D) $P(E_1/E_2) = 1$

PB0098

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13. A boy has a collection of blue and green marbles. The number of blue marbles belong to the sets {2, 3, 4, ... 13}. If two marbles are chosen simultaneously and at random from his collection, then the probability that they have different colour is $1/2$. Possible number of blue marbles is :

(A) 2 (B) 3 (C) 6 (D) 10

PB0099

14. If A & B are two events such that $P(B) \neq 1$, B^C denotes the event complementary to B, then

(A) $P\left(A/B^C\right) = \dfrac{P(A) - P(A \cap B)}{1 - P(B)}$

(B) $P(A \cap B) \geq P(A) + P(B) - 1$

(C) $P(A) > < P\left(A/B\right)$ according as $P\left(A/B^C\right) > < P(A)$

(D) $P\left(A/B^C\right) + P\left(A^C/B^C\right) = 1$

PB0100

15. For $P(A) = \dfrac{3}{8}$; $P(B) = \dfrac{1}{2}$; $P(A \cup B) = \dfrac{5}{8}$ which of the following do/does hold good?

(A) $P\left(A^c/B\right) = 2P\left(A/B^c\right)$ (B) $P(B) = P\left(A/B\right)$

(C) $15P\left(A^c/B^c\right) = 8P\left(B/A^c\right)$ (D) $P\left(A/B^c\right) = P\left(A \cap B\right)$

PB0101

16. If E_1 and E_2 are two events such that $P(E_1) = 1/4$, $P(E_2/E_1) = 1/2$ and $P(E_1/E_2) = 1/4$

(A) then E_1 and E_2 are independent

(B) E_1 and E_2 are exhaustive

(C) E_2 is twice as likely to occur as E_1

(D) Probabilities of the events $E_1 \cap E_2$, E_1 and E_2 are in G.P.

PB0102

17. Two events A and B are such that the probability that at least one of them occurs is 5/6 and both of them occurring simultaneously is 1/3. If the probability of not occurrence of B is 1/2 then

(A) A and B are equally likely (B) A and B are independent

(C) P(A/B) = 2/3 (D) 3 P(A) = 4 P(B)

PB0103

18. The probabilities of events, $A \cap B$, A, B & $A \cup B$ are respectively in A.P. with probability of second term equal to the common difference. Therefore the events A and B are

(A) mutually exclusive (B) independent

(C) such that one of them must occur (D) such that one is twice as likely as the other

PB0104

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19. A box contains 11 tickets numbered from 1 to 11. Six tickets are drawn simultaneously at random. Let E_1 denotes the event that the sum of the numbers on the tickets drawn is even and E_2 denotes the event that the sum of the numbers on the tickets drawn is odd. Which of the following hold good?

(A) E_1 and E_2 are equally likely
(B) E_1 and E_2 are exhaustive
(C) $P(E_2) > P(E_1)$
(D) $P(E_1/E_2) = P(E_2 / E_1)$

PB0105

20. If $\overline{E}$ & $\overline{F}$ are the complementary events of events E & F respectively & if $0 < P(F) < 1$, then :

(A) $P(E|F) + P(\overline{E}|F) = 1$
(B) $P(E|F) + P(E|\overline{F}) = 1$
(C) $P(\overline{E}|F) + P(E|\overline{F}) = 1$
(D) $P(E|\overline{F}) + P(\overline{E}|\overline{F}) = 1$

PB0106

21. Probability of **n** heads in **2n** tosses of a fair coin can be given by

(A) $\prod_{r=1}^{n}\left(\frac{2r-1}{2r}\right)$
(B) $\prod_{r=1}^{n}\left(\frac{n+r}{2r}\right)$
(C) $\sum_{r=0}^{n}\left(\frac{{}^nC_r}{2^n}\right)^2$
(D) $\frac{\sum_{r=0}^{n}\left({}^nC_r\right)^2}{\left(\sum_{r=0}^{n}{}^nC_r\right)^2}$

PB0107

22. Which of the following statements is/are True?

(A) A fair coin is tossed *n* times where *n* is a positive integer. The probability that n^{th} toss results in head is 1/2.
(B) The conditional probability that the nth toss results in head given that first (n – 1) tosses results in head is $1/2^n$
(C) Let E and F be the events such that F is neither impossible nor sure. If $P(E/F) > P(E)$ then $P(E/F^c) > P(E)$
(D) If A, B and C are independent then the events $(A \cup B)$ and C are independent.

PB0108

[MATRIX MATCH TYPE]

23.

	Column-I		Column-II
(A)	Two different numbers are taken from the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. The probability that their sum and positive difference, are both multiple of 4, is $x/55$ then *x* equals	(P)	4
		(Q)	6
(B)	There are two red, two blue, two white and certain number (greater than 0) of green socks in a drawer. If two socks are taken at random from the drawer without replacement, the probability that they are of the same colour is 1/5 then the number of green socks are	(R)	8
(C)	A drawer contains a mixture of red socks and blue socks, atmost 17 in all. It so happens that when two socks are selected randomly without replacement, there is a probability of exactly $1/2$ that both are red or both are blue. The largest possible number of red socks in the drawer that is consistent with this data, is	(S)	10

PB0109

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EXERCISE (S-1)

1. In a box, there are 8 alphabets cards with the letters: S, S, A,A,A,H,H,H. Find the probability that the word 'ASH' will form if :

(i) the three cards are drawn one by one & placed on the table in the same other that they are drawn.

(ii) the three cards are drawn simultaneously.

PB0110

2. There are 2 groups of subjects one of which consists of 5 science subjects & 3 engg. subjects & other consists of 3 science & 5 engg. subjects. An unbiased die is cast . If the number 3 or 5 turns up a subject is selected at random from first group, otherwise the subject is selected from 2nd group. Find the probability that an engg. subject is selected.

PB0111

3. A pair of fair dice is tossed. Find the probability that the maximum of the two numbers is greater than 4.

PB0112

4. In a given race, the odds in favour of four horses A, B, C & D are 1 : 3, 1 : 4, 1 : 5 and 1 : 6 respectively. Assuming that a dead heat is impossible, find the chance that one of them wins the race.

PB0113

5. A covered basket of flowers has some lilies and roses. In search of rose, Sweety and Shweta alternately pick up a flower from the basket but puts it back if it is not a rose. Sweety is 3 times more likely to be the first one to pick a rose. If sweety begin this 'rose hunt' and if there are 60 lilies in the basket, find the number of roses in the basket.

PB0114

6. A certain drug, manufactured by a Company is tested chemically for its toxic nature. Let the event "THE DRUG IS TOXIC" be denoted by H and the event "THE CHEMICAL TEST REVEALS THAT THE DRUG IS TOXIC" be denoted by S. Let P(H) = a, P(S/H) = $P(\bar{S}/\bar{H})$ = 1 – a. Then show that the probability that the drug is not toxic given that the chemical test reveals that it is toxic is free from 'a'.

PB0115

7. Players A and B alternately toss a biased coin, with A going first. A wins if A tosses a Tail before B tosses a Head; otherwise B wins. If the probability of a head is p, find the value of p for which the game is fair to both players.

PB0116

8. The entries in a two-by-two determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ are integers that are chosen randomly and independently, and , for each entry, the probability that the entry is odd is p. If the probability that the value of the determinant is even is 1/2, then find the value of p.

PB0117

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9. There are 4 urns. The first urn contains 1 white & 1 black ball, the second urn contains 2 white & 3 black balls, the third urn contains 3 white & 5 black balls & the fourth urn contains 4 white & 7 black balls. The selection of each urn is not equally likely. The probability of selecting i^{th} urn is $\frac{i^2+1}{34}$ (i = 1,2,3,4). If we randomly select one of the urns & draw a ball, then the probability of ball being white is p/q where p and q $\in$ N are in their lowest form. Find (p+q).

PB0118

10. A room has three electric lamps. From a collection of 10 electric bulbs of which 6 are good 3 are selected at random & put in the lamps. Find the probability that the room is lighted.

PB0119

11. Find the minimum number of tosses of a pair of dice so that the probability of getting the sum of the digits on the dice equal to 7 on at least one toss is greater than 0.95.
($\log_{10}2 = 0.3010$; $\log_{10}3 = 0.4771$)

PB0120

12. The probability that a person will get an electric contract is $2/5$ and the probability that he will not get plumbing contract is $4/7$. If the probability of getting at least one contract is $2/3$, what is the probability that he will get both?

PB0121

13. Five horses compete in a race. John picks two horses at random and bets on them. Find the probability that John picked the winner. Assume no dead heat.

PB0122

14. There are 6 red balls and 6 green balls in a bag. Five balls are drawn out at random and placed in a red box. The remaining seven balls are put in a green box. If the probability that the number of red balls in the green box plus the number of green balls in the red box is not a prime number, is $\frac{p}{q}$ where p and q are relatively prime, then find the value of $(p + q)$

PB0123

15. A lot contains 50 defective & 50 non defective bulbs. Two bulbs are drawn at random, one at a time, with replacement. The events A, B, C are defined as :
A = { the first bulb is defective}; B = { the second bulb is non defective}
C = { the two bulbs are both defective or both non defective}
Determine whether (i) A,B,C are pair wise independent (ii) A,B,C are independent

PB0124

16. An unbiased normal coin is tossed 'n' times
Let E_1 : event that both Heads and Tails are present in 'n' tosses.
E_2 : event that the coin shows up Heads atmost once.
Find the value of 'n' for which E_1 and E_2 are independent.

PB0125

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17. A bomber wants to destroy a bridge . Two bombs are sufficient to destroy it . If four bombs are dropped, what is the probability that it is destroyed, if the chance of a bomb hitting the target is 0.4.

PB0126

18. The chance of one event happening is the square of the chance of a 2nd event, but odds against the first are the cubes of the odds against the 2nd . Find the chances of each (assume that both events are neither sure nor impossible).

PB0127

19. A bag contains N balls, some of which are white, the others are black, white being more in number than back. Two balls are drawn at random from the bag, without replacement. It is found that the probability that the two balls are of the same colour is the same as the probability that they are of different colour. It is given that $180 < N < 220$. If K denotes the number of white balls, find the exact value of (K + N).

PB0128

20. An aircraft gun can take a maximum of four shots at an enemy's plane moving away from it. The probability of hitting the plane at first, second, third & fourth shots are 0.4, 0.3, 0.2 & 0.1 respectively. What is the probability that the gun hits the plane.

PB0129

21. In a batch of 10 articles, 4 articles are defective. 6 articles are taken from the batch for inspection. If more than 2 articles in this batch are defective, the whole batch is rejected Find the probability that the batch will be rejected.

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PB0130

22. An author writes a good book with a probability of 1/2. If it is good it is published with a probability of 2/3. If it is not, it is published with a probability of 1/4. Find the probability that he will get atleast one book published if he writes two.

PB0131

23. A uniform unbiased die is constructed in the shape of a regular tetrahedron with faces numbered 2, 2, 3 and 4 and the score is taken from the face on which the die lands. If two such dice are thrown together, find the probability of scoring.

(i) exactly 6 on each of 3 successive throws.

(ii) more than 4 on at least one of the three successive throws.

PB0132

24. A biased coin which comes up heads three times as often as tails is tossed. If it shows heads, a chip is drawn from urn-I which contains 2 white chips and 5 red chips. If the coin comes up tail, a chip is drawn from urn-II which contains 7 white and 4 red chips. Given that a red chip was drawn, what is the probability that the coin came up heads ?

PB0133

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25. Each of the 'n' passengers sitting in a bus may get down from it at the next stop with probability p. Moreover, at the next stop either no passenger or exactly one passenger boards the bus. The probability of no passenger boarding the bus at the next stop being p_0. Find the probability that when the bus continuous on its way after the stop, there will again be 'n' passengers in the bus.

PB0134

26. A normal coin is continued tossing unless a head is obtained for the first time. Find the probability that

(a) number of tosses needed are atmost 3.

(b) number of tosses are even.

PB0135

27. Before a race the chance of three runners, A, B, C were estimated to be proportional to 5, 3, 2, but during the race A meets with an accident which reduces his chance to 1/3. What are the respective chance of B and C now?

PB0136

28. A is one of the 6 horses entered for a race, and is to be ridden by one of two jockeys B or C. It is 2 to 1 that B rides A, in which case all the horses are equally likely to win; if C rides A, his chance is trebled, what are the odds against his winning?

PB0137

29. A real estate man has eight master keys to open several new houses. Only one master key will open a given house. If 40% of these homes are usually left unlocked, find the probability that the real estate man can get into a specific home if he selects three master keys at random.

PB0138

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30. A, B are two inaccurate arithmeticians whose chance of solving a given question correctly are (1/8) and (1/12) respectively. They solve a problem and obtained the same result. If it is 1000 to 1 against their making the same mistake, find the chance that the result is correct.

PB0139

31. During a power blackout, 100 persons are arrested on suspect of looting. Each is given a polygraph test. From past experience it is known that the polygraph is 90% reliable when administered to a guilty person and 98% reliable when given to some one who is innocent. Suppose that of the 100 persons taken into custody, only 12 were actually involved in any wrong doing. If the probability that a given suspect is innocent given that the polygraph says he is guilty is a/b where a and b are relatively prime, find the value of (a + b).

PB0140

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EXERCISE (S-2)

1. N fair coins are flipped once. The probability that at most 2 of the coins show up as heads is $\frac{1}{2}$. Find the value of N.

PB0141

2. To pass a test a child has to perform successfully in two consecutive tasks, one easy and one hard task. The easy task he can perform successfully with probability 'e' and the hard task he can perform successfully with probability 'h', where h < e. He is allowed 3 attempts, either in the order (Easy, Hard, Easy) (option A) or in the order (Hard, Easy, Hard)(option B) whatever may be the order, he must be successful twice in a row. Assuming that his attempts are independent, in what order he choses to take the tasks, in order to maximise his probability of passing the test.

PB0142

3. A box contains three coins two of them are fair and one two – headed. A coin is selected at random and tossed. If the head appears the coin is tossed again, if a tail appears, then another coin is selected from the remaining coins and tossed.

(i) Find the probability that head appears twice.

(ii) If the same coin is tossed twice, find the probability that it is two headed coin.

(iii) Find the probability that tail appears twice.

PB0143

4. Eight players P_1, P_2, P_3,.....P_8 play a knock out tournament. It is known that whenever the players P_i and P_j play, the player P_i will win if $i < j$. Assuming that the players are paired at random in each round, what is the probability that the players P_4 reaches the final ?

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PB0144

5. A doctor is called to see a sick child. The doctor knows (prior to the visit) that 90% of the sick children in that neighbourhood are sick with the flu, denoted by F, while 10% are sick with the measles, denoted by M.

A well known symptom of measles is rash, denoted by R. The probability of having a rash for a child sick with the measles is 0.95. However, occasionally children with the flu also develop a rash, with conditional probability 0.08.

Upon examination the child, the doctor finds a rash. What is the probability that the child has the measles ?

If the probability can be expressed in the form of p/q where $p,q \in N$ and are in their lowest form, find (p + q).

PB0145

6. A permutation of 5 digits from the set {1,2,3,4,5} where each digit is used exactly once, is chosen randomly. Let $\frac{p}{q}$ expressed as rational in lowest form be the probability that the chosen permutation changes from increasing to decreasing, or decreasing to increasing at most once e.g. the strings like 1 2 3 4 5, 5 4 3 2 1, 1 2 5 4 3 and 5 3 2 1 4 are acceptable but strings like 1 3 2 4 5 or 5 3 2 4 1 are not, find (p + q).

PB0146

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7. (a) Two natural numbers x and y are chosen at random. Find the probability that $x^2 + y^2$ is divisible by 10.

(b) Two numbers x & y are chosen at random from the set {1,2,3,4,....3n}. Find the probability that $x^2 - y^2$ is divisible by 3.

PB0147

8. A hotel packed breakfast for each of the three guests. Each breakfast should have consisted of three types of rolls, one each of nut, cheese and fruit rolls. The preparer wrapped each of the nine rolls and once wrapped, the rolls were indistinguishable from one another. She then randomly put three rolls in a bag for each of the guests. If the probability that each guest got one roll of each type is m/n where m and n are relatively prime integers, find the value of (m + n).

PB0148

9. A coin has probability 'p' of showing head when tossed. It is tossed 'n' times. Let p_n denote the probability that no two (or more) consecutive heads occur. Prove that,

$$p_1 = 1,\ p_2 = 1 - p^2\ \&\ p_n = (1-p)p_{n-1} + p(1-p)p_{n-2},\ \text{for all } n \geq 3.$$

PB0149

10. In a tournament, team X, plays with each of the 6 other teams once. For each match the probabilities of a win, drawn and loss are equal. Find the probability that the team X, finishes with more wins than losses.

PB0150

11. A pair of students is selected at random from a probability class. The probability that the pair selected will consist of one male and one female student is $\frac{10}{19}$. Find the maximum number of students the class can contain.

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PB0151

12. 3 students {A, B, C} tackle a puzzle together and offers a solution upon which majority of the 3 agrees. Probability of A solving the puzzle correctly is p. Probability of B solving the puzzle correctly is also p. C is a dumb student who randomly supports the solution of either A or B. There is one more student D, whose probability of solving the puzzle correctly is once again, p. Out of the 3 member team {A, B, C} and one member team {D}, which one is more likely to solve the puzzle correctly.

PB0152

13. In a knockout tournament 2^n equally skilled players; $S_1, S_2, \ldots\ldots S_{2^n}$ are participating. In each round players are divided in pair at random and winner from each pair moves in the next round. If S_2 reaches the semifinal then the probability that S_1 wins the tournament is 1/20. Find the value of 'n'.

PB0153

14. All the face cards from a pack of 52 plying cards are removed. From the remaining pack half of the cards are randomly removed without looking at them and then randomly drawn two cards simultaneously from the remaining. If the probability that, two cards drawn are both aces, is $\frac{p({}^{38}C_{20})}{{}^{40}C_{20}\cdot{}^{20}C_2}$, then find the value of p.

PB0154

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EXERCISE (JM)

1. If C and D are two events such that $C \subset D$ and $P(D) \neq 0$, then the correct statement among the following is :- **[AIEEE-2011]**

(1) $P(C|D) < P(C)$ (2) $P(C|D) = \frac{P(D)}{P(C)}$ (3) $P(C|D) = P(C)$ (4) $P(C|D) \geq P(C)$

PB0155

2. Consider 5 independent Bernoulli's trials each with probability of success p. If the probability of at least one failure is greater than or equal to $\frac{31}{32}$, then p lies in the interval :- **[AIEEE-2011]**

(1) $\left[0, \frac{1}{2}\right]$ (2) $\left(\frac{11}{12}, 1\right]$ (3) $\left(\frac{1}{2}, \frac{3}{4}\right]$ (4) $\left(\frac{3}{4}, \frac{11}{12}\right]$

PB0156

3. Let A, B, C be pairwise independent events with P(C)> 0 and $P(A \cap B \cap C) = 0$. Then $P(A^c \cap B^c | C)$ is equal to: **[AIEEE-2011]**

(1) $P(A^c) - P(B)$ (2) $P(A) - P(B^c)$ (3) $P(A^c) + P(B^c)$ (4) $P(A^c) - P(B^c)$

PB0157

4. Three numbers are chosen at random without replacement from {1, 2, 3, ... , 8}. The probability that their minimum is 3, given that their maximum is 6, is : **[AIEEE-2012]**

(1) $\frac{2}{5}$ (2) $\frac{3}{8}$ (3) $\frac{1}{5}$ (4) $\frac{1}{4}$

PB0158

5. A multiple choice examination has 5 questions. Each question has three alternative answers of which exactly one is correct. The probability that a student will get 4 or more correct answers just by guessing is : **[JEE-MAIN 2013]**

(1) $\frac{17}{3^5}$ (2) $\frac{13}{3^5}$ (3) $\frac{11}{3^5}$ (4) $\frac{10}{3^5}$

PB0159

6. Let A and B be two events such that $P(\overline{A \cup B}) = \frac{1}{6}$, $P(A \cap B) = \frac{1}{4}$ and $P(\bar{A}) = \frac{1}{4}$, where $\bar{A}$ stands for the complement of the event A. Then the events A and B are : **[JEE(Main)-2014]**

(1) mutually exclusive and independent. (2) equally likely but not independent.
(3) independent but not equally likely. (4) independent and equally likely.

PB0160

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7. Let two fair six-faced dice A and B be thrown simultaneously. If E_1 is the event that die A shows up four, E_2 is the event that die B shows up two and E_3 is the event that the sum of numbers on both dice is odd, then which of the following statements is **NOT true** ? **[JEE(Main)-2016]**

(1) E_1, E_2 and E_3 are independent. (2) E_1 and E_2 are independent.
(3) E_2 and E_3 are independent. (4) E_1 and E_3 are independent.

PB0161

8. A box contains 15 green and 10 yellow balls. If 10 balls are randomly drawn, one–by–one, with replacement, then the variance of the number of green balls drawn is :- **[JEE(Main)-2017]**

(1) $\frac{6}{25}$ (2) $\frac{12}{5}$ (3) 6 (4) 4

PB0162

9. If two different numbers are taken from the set {0, 1, 2, 3, ..., 10), then the probability that their sum as well as absolute difference are both multiple of 4, is :- **[JEE(Main)-2017]**

(1) $\frac{7}{55}$ (2) $\frac{6}{55}$ (3) $\frac{12}{55}$ (4) $\frac{14}{45}$

PB0163

10. For three events A, B and C, P(Exactly one of A or B occurs) = P(Exactly one of B or C occurs) = P(Exactly one of C or A occurs) = $\frac{1}{4}$ and P(All the three events occur simultaneously) = $\frac{1}{16}$. Then the probability that at least one of the events occurs, is :- **[JEE(Main)-2017]**

(1) $\frac{3}{16}$ (2) $\frac{7}{32}$ (3) $\frac{7}{16}$ (4) $\frac{7}{64}$

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PB0164

11. A bag contains 4 red and 6 black balls. A ball is drawn at random from the bag, its colour is observed and this ball along with two additional balls of the same colour are returned to the bag. If now a ball is drawn at random from the bag, then the probability that this drawn ball is red, is: **[JEE(Main)-2018]**

(1) $\frac{2}{5}$ (2) $\frac{1}{5}$ (3) $\frac{3}{4}$ (4) $\frac{3}{10}$

PB0165

12. An urn contains 5 red and 2 green balls. A ball is drawn at random from the urn. If the drawn ball is green, then a red ball is added to the urn and if the drawn ball is red, then a green ball is added to the urn; the original ball is not returned to the urn. Now, a second ball is drawn at random from it. The probability that the second ball is red, is : **[JEE(Main)-2019]**

(1) $\frac{26}{49}$ (2) $\frac{32}{49}$ (3) $\frac{27}{49}$ (4) $\frac{21}{49}$

PB0166

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13. An unbiased coin is tossed. If the outcome is a head then a pair of unbiased dice is rolled and the sum of the numbers obtained on them is noted. If the toss of the coin results in tail then a card from a well-shuffled pack of nine cards numbered 1,2,3,...,9 is randomly picked and the number on the card is noted. The probability that the noted number is either 7 or 8 is : **[JEE(Main)-2019]**

(1) $\frac{13}{36}$ (2) $\frac{19}{36}$ (3) $\frac{19}{72}$ (4) $\frac{15}{72}$

PB0167

14. If the probability of hitting a target by a shooter, in any shot, is 1/3, then the minimum number of independent shots at the target required by him so that the probability of hitting the target at least once is greater than $\frac{5}{6}$, is : **[JEE(Main)-2019]**

(1) 6 (2) 5 (3) 4 (4) 3

PB0168

15. Let S = {1, 2, ... , 20}. A subset B of S is said to be "nice", if the sum of the elements of B is 203. Then the probability that a randomly chosen subset of S is "nice" is :- **[JEE(Main)-2019]**

(1) $\frac{6}{2^{20}}$ (2) $\frac{5}{2^{20}}$ (3) $\frac{4}{2^{20}}$ (4) $\frac{7}{2^{20}}$

PB0169

16. In a random experiment, a fair die is rolled until two fours are obtained in succession. The probability that the experiment will end in the fifth throw of the die is equal to : **[JEE(Main)-2019]**

(1) $\frac{150}{6^5}$ (2) $\frac{175}{6^5}$ (3) $\frac{200}{6^5}$ (4) $\frac{225}{6^5}$

PB0170

17. In a game, a man wins Rs. 100 if he gets 5 of 6 on a throw of a fair die and loses Rs. 50 for getting any other number on the die. If he decides to throw the die either till he gets a five or a six or to a maximum of three throws, then his expected gain/loss (in rupees) is : **[JEE(Main)-2019]**

(1) $\frac{400}{3}$ gain (2) $\frac{400}{3}$ loss (3) 0 (4) $\frac{400}{9}$ loss

PB0171

18. Assume that each born child is equally likely to be a boy or a girl. If two families have two children each, then the conditional probability that all children are girls given that at least two are girls is : **[JEE(Main)-2019]**

(1) $\frac{1}{11}$ (2) $\frac{1}{17}$ (3) $\frac{1}{10}$ (4) $\frac{1}{12}$

PB0172

19. If three of the six vertices of a regular hexagon are chosen at random, then the probability that the triangle formed with these chosen vertices is equilateral is : **[JEE(Main)-2019]**

(1) $\frac{3}{10}$ (2) $\frac{1}{10}$ (3) $\frac{3}{20}$ (4) $\frac{1}{5}$

PB0173

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20. In a workshop, there are five machines and the probability of any one of them to be out of service on a day is $\frac{1}{4}$. If the probability that at most two machines will be out of service on the same day is $\left(\frac{3}{4}\right)^3 k$, then k is equal to : **[JEE(Main)-2020]**

(1) $\frac{17}{2}$ (2) 4 (3) $\frac{17}{8}$ (4) $\frac{17}{4}$

PB0174

21. An unbiased coin is tossed 5 times. Suppose that a variable X is assigned the value k when k consecutive heads are obtained for k = 3, 4, 5 otherwise X takes the value –1. Then the expected value of X, is : **[JEE(Main)-2020]**

(1) $\frac{3}{16}$ (2) $-\frac{3}{16}$ (3) $\frac{1}{8}$ (4) $-\frac{1}{8}$

PB0175

22. A random variable X has the following probability distribution : **[JEE(Main)-2020]**

X :	1	2	3	4	5
P(X) :	K^2	2K	K	2K	$5K^2$

Then P(X > 2) is equal to : **[JEE(Main)-2020]**

(1) $\frac{7}{12}$ (2) $\frac{23}{36}$ (3) $\frac{1}{36}$ (4) $\frac{1}{6}$

PB0176

23. If 10 different balls are to be placed in 4 distinct boxes at random, then the probability that two of these boxes contain exactly 2 and 3 balls is : **[JEE(Main)-2020]**

(1) $\frac{945}{2^{11}}$ (2) $\frac{965}{2^{11}}$ (3) $\frac{945}{2^{10}}$ (4) $\frac{965}{2^{10}}$

PB0177

24. In a box, there are 20 cards, out of which 10 are lebelled as A and the remaining 10 are labelled as B. Cards are drawn at random, one after the other and with replacement, till a second A-card is obtained. The probability that the second A-card appears before the third B-card is : **[JEE(Main)-2020]**

(1) $\frac{11}{16}$ (2) $\frac{13}{16}$ (3) $\frac{9}{16}$ (4) $\frac{15}{16}$

PB0178

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EXERCISE (JA)

1. (a) Let ω be a complex cube root of unity with $\omega \neq 1$. A fair die is thrown three times. If r_1, r_2 and r_3 are the numbers obtained on the die, then the probability that $\omega^{r_1} + \omega^{r_2} + \omega^{r_3} = 0$ is -

(A) $\frac{1}{18}$ (B) $\frac{1}{9}$ (C) $\frac{2}{9}$ (D) $\frac{1}{36}$

PB0179

(b) A signal which can be green or red with probability $\frac{4}{5}$ and $\frac{1}{5}$ respectively, is received by station A and then transmitted to station B. The probability of each station receiving the signal correctly is $\frac{3}{4}$. If the signal received at station B is green, then the probability that the original signal was green is - **[JEE 2010, 3+5]**

(A) $\frac{3}{5}$ (B) $\frac{6}{7}$ (C) $\frac{20}{23}$ (D) $\frac{9}{20}$

PB0180

Paragraph for Question 2 and 3

Let U_1 and U_2 be two urns such that U_1 contains 3 white and 2 red balls, and U_2 contains only 1 white ball. A fair coin is tossed. If head appears then 1 ball is drawn at random from U_1 and put into U_2. However, if tail appears then 2 balls are drawn at random from U_1 and put into U_2. Now 1 ball is drawn at random from U_2.

2. The probability of the drawn ball from U_2 being white is -

(A) $\frac{13}{30}$ (B) $\frac{23}{30}$ (C) $\frac{19}{30}$ (D) $\frac{11}{30}$

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PB0181

3. Given that the drawn ball from U_2 is white, the probability that head appeared on the coin is -

(A) $\frac{17}{23}$ (B) $\frac{11}{23}$ (C) $\frac{15}{23}$ (D) $\frac{12}{23}$

[JEE 2011, 3+3]

PB0181

4. Let E and F be two independent events. The probability that exactly one of them occurs is $\frac{11}{25}$ and the probability of none of them occurring is $\frac{2}{25}$. If P(T) denotes the probability of occurrence of the event T, then -

(A) $P(E) = \frac{4}{5}$, $P(F) = \frac{3}{5}$ (B) $P(E) = \frac{1}{5}, P(F) = \frac{2}{5}$

(C) $P(E) = \frac{2}{5}, P(F) = \frac{1}{5}$ (D) $P(E) = \frac{3}{5}, P(F) = \frac{4}{5}$

PB0182

[JEE 2011, 4M]

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5. A ship is fitted with three engines E_1, E_2 and E_3. The engines function independently of each other with respective probabilities $\frac{1}{2}, \frac{1}{4}$ and $\frac{1}{4}$. For the ship to be operational at least two of its engines must function. Let X denote the event that the ship is operational and X_1, X_2, X_3 denotes respectively the events that the engines E_1, E_2 and E_3 are functioning. Which of the following is (are) true?

[JEE 2012, 4M]

(A) $P\left[X_1^c \mid X\right] = \frac{3}{16}$

(B) P[Exactly two engines of ship are functioning $\mid X] = \frac{7}{8}$

(C) $P[X \mid X_2] = \frac{5}{16}$

(D) $P[X \mid X_1] = \frac{7}{16}$

PB0183

6. Four fair dice D_1, D_2, D_3 and D_4, each having six faces numbered 1, 2, 3, 4, 5 and 6 are rolled simultaneously. The probability that D_4 shows a number appearing on one of D_1, D_2 and D_3 is -

[JEE 2012, 4M]

(A) $\frac{91}{216}$ (B) $\frac{108}{216}$ (C) $\frac{125}{216}$ (D) $\frac{127}{216}$

PB0184

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7. Let X and Y be two events such that $P(X \mid Y) = \frac{1}{2}, P(Y \mid X) = \frac{1}{3}$ and $P(X \cap Y) = \frac{1}{6}$. Which of the following is(are) correct ?

[JEE 2012, 4M]

(A) $P(X \cup Y) = \frac{2}{3}$

(B) X and Y are independent

(C) X and Y are not independent

(D) $P\left(X^C \cap Y\right) = \frac{1}{3}$

PB0185

8. Four persons independently solve a certain problem correctly with probabilities $\frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{1}{8}$. Then the probability that the problem is solved correctly by at least one of them is

[JEE(Advanced) 2013, 2M]

(A) $\frac{235}{256}$ (B) $\frac{21}{256}$ (C) $\frac{3}{256}$ (D) $\frac{253}{256}$

PB0186

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9. Of the three independent events E_1, E_2 and E_3, the probability that only E_1 occurs is α, only E_2 occurs is β and only E_3 occurs is γ. Let the probability p that none of events E_1, E_2 or E_3 occurs satisfy the equations $(\alpha - 2\beta)\,p = \alpha\beta$ and $(\beta - 3\gamma)\,p = 2\beta\gamma$. All the given probabilities are assumed of lie in the interval (0,1).

Then $\dfrac{\text{Probability of occurrence of } E_1}{\text{Probability of occurrence of } E_3} =$ **[JEE-Advanced 2013, 4, (–1)]**

PB0187

Paragraph for Question 10 and 11

A box B_1 contains 1 white ball, 3 red balls and 2 black balls. Another box B_2 contains 2 white balls, 3 red balls and 4 black balls. A third box B_3 contains 3 white balls, 4 red balls and 5 black balls.

10. If 2 balls are drawn (without replacement) from a randomly selected box and one of the balls is white and the other ball is red, the probability that these 2 balls are drawn from box B_2 is

(A) $\frac{116}{181}$ (B) $\frac{126}{181}$ (C) $\frac{65}{181}$ (D) $\frac{55}{181}$

[JEE(Advanced) 2013, 3, (–1)]

PB0188

11. If 1 ball is drawn from each of the boxes B_1, B_2 and B_3, the probability that all 3 drawn balls are of the same colour is

(A) $\frac{82}{648}$ (B) $\frac{90}{648}$ (C) $\frac{558}{648}$ (D) $\frac{566}{648}$

[JEE(Advanced) 2013, 3, (–1)]

PB0188

12. Three boys and two girls stand in a queue. The probability, that the number of boys ahead of every girl is at least one more than the number of girls ahead of her, is - **[JEE(Advanced)-2014, 3(–1)]**

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{3}{4}$

PB0189

Paragraph For Questions 13 and 14

Box 1 contains three cards bearing numbers, 1,2,3 ; box 2 contains five cards bearing numbers 1,2,3,4,5; and box 3 contains seven cards bearing numbers 1,2,3,4,5,6,7. A card is drawn from each of the boxes. Let x_i be the number on the card drawn from the ith box, i = 1,2,3.

13. The probability that $x_1 + x_2 + x_3$ is odd, is -

(A) $\frac{29}{105}$ (B) $\frac{53}{105}$ (C) $\frac{57}{105}$ (D) $\frac{1}{2}$

[JEE(Advanced)-2014, 3(–1)]

PB0190

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14. The probability that x_1, x_2, x_3 are in an arithmetic progression, is -

(A) $\frac{9}{105}$ (B) $\frac{10}{105}$ (C) $\frac{11}{105}$ (D) $\frac{7}{105}$

[JEE(Advanced)-2014, 3(–1)]

PB0190

15. The minimum number of times a fair coin needs to be tossed, so that the probability of getting at least two heads is at least 0.96, is **[JEE 2015, 4M, –0M]**

PB0191

Paragraph For Questions 16 and 17

Let n_1 and n_2 be the number of red and black balls respectively, in box I. Let n_3 and n_4 be the number of red and black balls, respectively, in box II.

16. One of the two boxes, box I and box II, was selected at random and a ball was drawn randomly out of this box. The ball was found to be red. If the probability that this red ball was drawn from box II is $\frac{1}{3}$, then the correct option(s) with the possible values of n_1, n_2, n_3 and n_4 is(are)

[JEE 2015, 4M, –0M]

(A) $n_1 = 3, n_2 = 3, n_3 = 5, n_4 = 15$ (B) $n_1 = 3, n_2 = 6, n_3 = 10, n_4 = 50$

(C) $n_1 = 8, n_2 = 6, n_3 = 5, n_4 = 20$ (D) $n_1 = 6, n_2 = 12, n_3 = 5, n_4 = 20$

PB0192

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17. A ball is drawn at random from box I and transferred to box II. If the probability of drawing a red ball from box I, after this transfer, is $\frac{1}{3}$, then the correct option(s) with the possible values of n_1 and n_2 is(are) **[JEE 2015, 4M, –0M]**

(A) $n_1 = 4$ and $n_2 = 6$ (B) $n_1 = 2$ and $n_2 = 3$

(C) $n_1 = 10$ and $n_2 = 20$ (D) $n_1 = 3$ and $n_2 = 6$

PB0192

18. A computer producing factory has only two plants T_1 and T_2. Plant T_1 produces 20% and plant T_2 produces 80% of the total computers produced. 7% of computers produced in the factory turn out to be defective. It is known that

P(computer turns out to be defective given that is produced in plant T_1)

= 10P(computer turns out to be defective given that it is produced in plant T_2)

where P(E) denotes the probability of an event E. A computer produces in the factory is randomly selected and it does not turn out to be defective. Then the probability that it is produced in plant T_2 is **[JEE(Advanced)-2016, 3(–1)]**

(A) $\frac{36}{73}$ (B) $\frac{47}{79}$ (C) $\frac{78}{93}$ (D) $\frac{75}{83}$

PB0193

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Paragraph For Questions 19 and 20

Football teams T_1 and T_2 have to play two games against each other. It is assumed that the outcomes of the two games are independent. The probabilities of T_1 winning, drawing and losing a game against T_2 are $\frac{1}{2}$, $\frac{1}{6}$ and $\frac{1}{3}$, respectively. Each team gets 3 points for a win, 1 point for a draw and 0 point for a loss in a game. Let X and Y denote the total points scored by teams T_1 and T_2, respectively, after two games

19. P(X > Y) is- **[JEE(Advanced)-2016, 3(0)]**

(A) $\frac{1}{4}$ (B) $\frac{5}{12}$ (C) $\frac{1}{2}$ (D) $\frac{7}{12}$

PB0194

20. P(X = Y) is- **[JEE(Advanced)-2016, 3(0)]**

(A) $\frac{11}{36}$ (B) $\frac{1}{3}$ (C) $\frac{13}{36}$ (D) $\frac{1}{2}$

PB0194

21. Let X and Y be two events such that $P(X)=\frac{1}{3}$, $P(X|Y)=\frac{1}{2}$ and $P(Y|X)=\frac{2}{5}$. Then **[JEE(Advanced)-2017, 4(–2)]**

(A) $P(X'|Y)=\frac{1}{2}$ (B) $P(X\cap Y)=\frac{1}{5}$

(C) $P(X\cup Y)=\frac{2}{5}$ (D) $P(Y)=\frac{4}{15}$

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PB0195

22. Three randomly chosen nonnegative integers x, y and z are found to satisfy the equation x + y + z = 10. Then the probability that z is even, is **[JEE(Advanced)-2017, 3(–1)]**

(A) $\frac{36}{55}$ (B) $\frac{6}{11}$ (C) $\frac{5}{11}$ (D) $\frac{1}{2}$

PB0196

Paragraph For Questions 23 and 24

There are five students S_1, S_2, S_4 and S_5 in a music class and for them there are five sets R_1, R_2, R_3, R_4 and R_5 arranged in a row, where initially the seat R_i is allotted to the student S_i, i = 1, 2, 3, 4, 5. But, on the examination day, the five students are randomly allotted the five seats. (*There are two questions based on Paragraph "A". the question given below is one of them*)

23. The probability that, on the examination day, the student S_1 gets the previously allotted seat R_1 and **NONE** of the remaining students gets the seat previously allotted to him/her is - **[JEE(Advanced)-2018, 3(–1)]**

(A) $\frac{3}{40}$ (B) $\frac{1}{8}$ (C) $\frac{7}{40}$ (D) $\frac{1}{5}$

PB0197

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24. For i = 1, 2, 3, 4, let T_i denote the event that the students S_i and S_{i+1} do **NOT** sit adjacent to each other on the day of the examination. Then the probability of the event $T_1 \cap T_2 \cap T_3 \cap T_4$ is-

[JEE(Advanced)-2018, 3(–1)]

(A) $\frac{1}{15}$ (B) $\frac{1}{10}$ (C) $\frac{7}{60}$ (D) $\frac{1}{5}$

PB0197

25. There are three bags B_1, B_2 and B_3. The bag B_1 contains 5 red and 5 green balls, B_2 contains 3 red and 5 green balls, and B_3 contains 5 red and 3 green balls, Bags B_1, B_2 and B_3 have probabilities $\frac{3}{10}$, $\frac{3}{10}$ and $\frac{4}{10}$ respectively of being chosen. A bag is selected at random and a ball is chosen at random from the bag. Then which of the following options is/are correct ?

[JEE(Advanced)-2019, 4(–1)]

(1) Probability that the selected bag is B_3 and the chosen ball is green equals $\frac{3}{10}$

(2) Probability that the chosen ball is green equals $\frac{39}{80}$

(3) Probability that the chosen ball is green, given that the selected bag is B_3, equals $\frac{3}{8}$

(4) Probability that the selected bag is B_3, given that the chosen balls is green, equals $\frac{5}{13}$

PB0198

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26. Let S be the sample space of all 3×3 matrices with entries from the set {0, 1}. Let the events E_1 and E_2 be given by

$E_1 = \{A \in S : \det A = 0\}$ and

$E_2 = \{A \in S :$ sum of entries of A is 7$\}$.

If a matrix is chosen at random from S, then the conditional probability $P(E_1|E_2)$ equals ____

[JEE(Advanced)-2019, 3(0)]

PB0199

27. Let |X| denote the number of elements in set X. Let S = {1,2,3,4,5,6} be a sample space, where each element is equally likely to occur. If A and B are independent events associated with S, then the number of ordered pairs (A,B) such that $1 \le |B| < |A|$, equals **[JEE(Advanced)-2019, 3(0)]**

PB0200

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ANSWER KEY

EXERCISE (O-1)

PART # 1

1. A **2.** A **3.** C **4.** 23/168

5. (i) 7/13, (ii) 1/2, (iii) 2/13, (iv) 2/13, (v) 1/2, (vi) 9/13 **6.** 1/56 **7.** 1/2 ; 1/2

8. 5 : 1 **9.** 952 to 715 **10.** A **11.** 4/21 **12.** **(a)** 2/3, **(b)** 1/2

13. B **14.** B **15.** C

PART # 2

1. C **2.** D **3.** A **4.** B **5.** C **6.** A **7.** A

8. 2/3 **9.** 3/4, 1/4; 15/16 **10.** (i) 0.18, (ii) 0.12, (iii) 0.42, (iv) 0.28, (v) 0.72

11. (i) 0.6, (ii) 0.5, (iii) 0.25 **12.** (i) 1/36, (ii) 5/108, (iii) 53/54 **13.** 11/20

14. 3/5 **15.** 2/7 **16.** A **17.** 12/25 **18.** (a) 1/18, (b) 43/90, (c) 5/18, (d) NO

PART # 3

1. C **2.** C **3.** D **4.** D **5.** B **6.** A **7.** A

8. C **9.** A **10.** D **11.** B **12.** A **13.** C **14.** B

15. D

PART # 4

1. D **2.** A **3.** B **4.** C **5.** C **6.** A **7.** B

8. C **9.** C **10.** A **11.** B **12.** D

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PART # 5

1. D **2.** B **3.** B **4.** C **5.** C **6.** B **7.** B

8. A **9.** C **10.** C **11.** C **12.** C **13.** B **14.** D

15. B

PART # 6

1. A **2.** B **3.** C **4.** D **5.** C **6.** A **7.** B

8. B **9.** D **10.** A **11.** B **12.** D **13.** A

EXERCISE (O-2)

1. B **2.** B **3.** A **4.** C **5.** B **6.** A **7.** C

8. C **9.** **(a)** D, **(b)** A, **(c)** B **10.** B,C,D **11.** B,C,D **12.** C, D **13.** B,C,D

14. A,B,C,D **15.** A,B,D **16.** A,C,D **17.** B,C,D **18.** A,D **19.** B,C,D **20.** A,D

21. A,C,D **22.** A,D **23.** (A) Q; (B) P; (C) S

EXERCISE (S-1)

1. **(i)** 3/56; **(ii)** 9/28 **2.** 13/24 **3.** 5/9 **4.** 319/420 **5.** 120

6. $P(\bar{H}/S) = 1/2$ **7.** $\frac{\sqrt{5}-1}{2}$ **8.** $\frac{\sqrt{2}}{2}$ **9.** 2065 **10.** $\frac{29}{30}$

11. 17 **12.** 17/105 **13.** 2/5 **14.** 37

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15. **(i)** A,B,C are pairwise independent **(ii)** A,B,C are not independent **16.** 3

17. $\frac{328}{625}$ **18.** $\frac{1}{9}, \frac{1}{3}$ **19.** 301 **20.** 0.6976 **21.** 19/42 **22.** 407/576

23. (i) $\frac{125}{16^3}$; (ii) $\frac{63}{64}$ **24.** 165/193 **25.** $(1-p)^{n-1}.\,[p_0\,(1-p) + np(1-p_0)]$

26. (a) 7/8, (b) 1/3 **27.** B = 2/5 ; C = 4/15 **28.** 13 to 5 **29.** 5/8

30. 13/14 **31.** 179

EXERCISE (S-2)

1. 5 **2.** Option B **3.** 1/2, 1/2, 1/12 **4.** 4/35 **5.** 262

6. 5 **7.** **(a)** $\frac{9}{50}$ **(b)** $\frac{(5n-3)}{(9n-3)}$ **8.** 79 **9.** 79 **10.** $\frac{98}{243}$ **11.** 20

12. Both are equally likely **13.** 4 **14.** 6

EXERCISE (JM)

1. 4 **2.** 1 **3.** 1 **4.** 3 **5.** 3 **6.** 3 **7.** 1 **8.** 2

9. 2 **10.** 3 **11.** 1 **12.** 2 **13.** 3 **14.** 2 **15.** 2 **16.** 2

17. 3 **18.** 1 **19.** 2 **20.** 3 **21.** 3 **22.** 2

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23. NTA Ans. (3); Correct Ans. is $\frac{17\times 945}{2^{15}}$ **24.** 1

EXERCISE (JA)

1. **(a)** C; **(b)** C **2.** B **3.** D **4.** A,D **5.** B,D **6.** A **7.** A,B

8. A **9.** 6 **10.** D **11.** A **12.** A **13.** B **14.** C **15.** 8

16. A,B **17.** C,D **18.** **C** **19.** **B** **20.** **C** **21.** A,D **22.** B **23.** A

24. C **25.** 2,3 **26.** 0.50 **27.** 422.00

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MATRIX

1. INTRODUCTION :

A rectangular array of mn numbers (which may be **real or complex**) in the form of 'm' horizontal lines (called **rows**) and 'n' vertical lines (called **columns**), is called a matrix of order m by n, written as $m \times n$ matrix.

Such an array is enclosed by [ ] or ( ) or‖ An $m \times n$ matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

In compact form, the above matrix is represented by $A=[a_{ij}]_{m\times n}$. The number a_{11}, a_{12},.....etc are known as the elements of the matrix A, a_{ij} belongs to the ith row and j^{th} column and is called the **(i, j)th** element of the matrix $A = [a_{ij}]$.

e.g., $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 9 \end{bmatrix}$ is a matrix having 2 rows and 3 columns. Its order is 2×3 and it has 6 elements :

$a_{11} = 1$, $a_{12} = 2$, $a_{13} = 3$, $a_{21} = 0$, $a_{22} = -1$, $a_{23} = 9$.

2. SPECIAL TYPE OF MATRICES :

(a) Row Matrix (Row vector) : $A = [a_{11}, a_{12}, \dots\dots\dots a_{1n}]$ i.e. row matrix has exactly one row.

(b) Column Matrix (Column vector) : $A = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{m1} \end{bmatrix}$ i.e. column matrix has exactly one column.

(c) Zero or Null Matrix : ($A = \mathbf{O}_{m\times n}$) An $m \times n$ matrix whose all entries are zero.

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a 3×2 null matrix & $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is 3×3 null matrix

(d) Horizontal Matrix : A matrix of order $m \times n$ is a horizontal matrix if $\mathbf{n > m}$ e.g. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$

(e) Vertical Matrix : A matrix of order $m \times n$ is a vertical matrix if $\mathbf{m > n}$ e.g. $\begin{bmatrix} 2 & 5 \\ 1 & 1 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}$

(f) Square Matrix : If number of rows = number of columns $\Rightarrow$ matrix is a square matrix. If number of rows = number of columns = n then, matrix is of the **order 'n'**.

Note : The pair of elements a_{ij} & a_{ji} are called **Conjugate Elements.**

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3. TRACE OF MATRIX :

The sum of the elements of a **square matrix** A lying along the principal diagonal is called the trace of A i.e. (tr(A)). Thus, if $A = [a_{ij}]_{n \times n}$, then $tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + ...a_{nn}$

Properties of trace of a matrix :

Let $A = [a_{ij}]_{n \times n}$ and $B = [b_{ij}]_{n \times n}$ and λ be a scalar then

(i) $tr(\lambda A) = \lambda tr(A)$ (ii) $tr(A + B) = tr(A) + tr(B)$

4. SQUARE MATRICES

Triangular Matrix

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 4 & 3 & 3 \end{bmatrix}$$

Upper Triangular $a_{ij} = 0 \ \forall \ i > j$

Lower Triangular $a_{ij} = 0 \ \forall \ i < j$

Diagonal Matrix denoted as $A = diag\ (a_{11}, a_{22} ....., a_{nn})$ where $a_{ij} = 0$ for $i \neq j$

Scalar Matrix

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$

If $a_{11} = a_{22} = a_{33} = a$

Unit or Identity Matrix

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

If $a_{11} = a_{22} = a_{33} = 1$

Note :

(i) Minimum number of zeros in triangular matrix of order $n = n(n-1)/2$.

(ii) Minimum number of zero in a diagonal matrix of order $n = n(n-1)$.

5. EQUALITY OF MATRICES :

Let $A = [a_{ij}]$ & $B = [b_{ij}]$ are equal if

(a) both have the same order. (b) $a_{ij} = b_{ij}$ for each pair of i & j .

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Illustration 1 : Find the value of x, y, z and w which satisfy the matrix equation

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4w-8 \end{bmatrix} = \begin{bmatrix} -x-1 & 0 \\ 3 & 2w \end{bmatrix}$$

Solution : As the given matrices are equal so their corresponding elements are equal.

$x + 3 = -x - 1 \quad \Rightarrow \quad 2x = -4$

$\therefore \quad x = -2$ .....(i)

$2y + x = 0 \quad \Rightarrow \quad 2y - 2 = 0$ [from (i)]

$\Rightarrow \quad y = 1$ .....(ii)

$z - 1 = 3 \quad \Rightarrow \quad z = 4$ .....(iii)

$4w - 8 = 2w \quad \Rightarrow \quad 2w = 8$

$\therefore \quad w = 4$ .....(iv) **Ans.**

Do yourself -1 :

1 Find 2×3 matrix $[a_{ij}]_{2\times 3}$, where $a_{ij} = i + 2j$

2. Find the minimum number of zeroes in a triangular matrix of order 4.

3. Find minimum number of zeros in a diagonal matrix of order 6.

4. If $\begin{bmatrix} 2x+y & 2 & x-2y \\ a-b & 2a+b & -3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 4 \\ 4 & -1 & -3 \end{bmatrix}$, then find the values of x,y,a and b.

5. Let $A = \begin{bmatrix} \alpha^2(a-2\lambda) & 0 & 3 \\ -1 & \alpha(b-3\lambda) & 1 \\ 0 & 5 & (c-4\lambda) \end{bmatrix}$; $B = \begin{bmatrix} \beta^2(a-2\lambda) & -5 & -3 \\ 7 & \beta(b-3\lambda) & 7 \\ 8 & 9 & (c-4\lambda) \end{bmatrix}$ and

$C = \begin{bmatrix} \gamma^2(a-2\lambda) & -1 & 3 \\ 0 & \gamma(b-3\lambda) & 7 \\ 8 & -3 & (c-4\lambda) \end{bmatrix}$, where α,β,γ are real & distinct such that $tr(A) = tr(B)$

$= tr(C) = -1$, where $\lambda \in N$, then minimum possible value of $a + b + c$ is

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6. ALGEBRA OF MATRICES :

Addition : $A + B = [a_{ij} + b_{ij}]$ where A & B are of the same order.

(a) **Addition of matrices is commutative :** i.e. $A + B = B + A$

(b) **Matrix addition is associative :** $(A + B) + C = A + (B + C)$

(c) **Additive inverse :** If $A + B = \mathbf{O} = B + A$, then B is called **additive inverse** of A.

(d) **Existence of additive identity :** Let $A = [a_{ij}]$ be an $m \times n$ matrix and **O** be an $m \times n$ zero matrix, then $A + \mathbf{O} = \mathbf{O} + A = A$. In other words, **O** is the **additive identity** for matrix addition.

(e) **Cancellation laws** hold good in case of addition of matrices. If A,B,C are matrices of the same order, then $A + B = A + C \Rightarrow B = C$ (**left cancellation law**) and $B + A = C + A \Rightarrow B = C$ (**right-cancellation law**)

Note : The zero matrix plays the same role in matrix addition as the number zero does in addition of numbers.

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Illustration 2 : If $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 5 \end{bmatrix}$ & $B = \begin{bmatrix} -1 & -2 \\ 0 & 5 \\ 3 & 1 \end{bmatrix}$ and $A + B - D = \mathbf{O}$ (zero matrix), then D matrix will be-

(A) $\begin{bmatrix} 0 & 2 \\ 3 & 7 \\ 6 & 5 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & 2 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$ (C) $\begin{bmatrix} 0 & 1 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & -2 \\ -3 & -7 \\ -5 & -6 \end{bmatrix}$

Solution : Let $D = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$

$$\therefore \quad A + B - D = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ 0 & 5 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \Rightarrow \begin{bmatrix} 1-1-a & 3-2-b \\ 3+0-c & 2+5-d \\ 2+3-e & 5+1-f \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $-a = 0 \Rightarrow a = 0$, $1 - b = 0 \Rightarrow b = 1$,
$3 - c = 0 \Rightarrow c = 3$, $7 - d = 0 \Rightarrow d = 7$,
$5 - e = 0 \Rightarrow e = 5$, $6 - f = 0 \Rightarrow f = 6$

$$\therefore \quad D = \begin{bmatrix} 0 & 1 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$$

Ans. (C)

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Do yourself-2 :

1. If $A = \begin{bmatrix} 2 & 3 & 9 \\ 8 & -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -7 & 2 \\ 6 & 4 & 8 \end{bmatrix}$, then find a matrix C such that $A - B + C = \mathbf{O}$ and also find the order of the matrix C.

2. If $A = \begin{bmatrix} 8 & 9 \\ 7/2 & 8 \\ 1 & -1 \end{bmatrix}$, then find the additive inverse of A and show that additive inverse of additive inverse will be the matrix itself.

7. MULTIPLICATION OF A MATRIX BY A SCALAR :

If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$; $kA = \begin{bmatrix} ka & kb & kc \\ kb & kc & ka \\ kc & ka & kb \end{bmatrix}$

Properties of scalar multiplication :

(a) If A and B are two matrices of the same order and 'k' be a scalar then $k(A + B) = kA + kB$.

(b) If k_1 and k_2 are two scalars and 'A' is a matrix, then $(k_1 + k_2)A = k_1A + k_2A$.

(c) If k_1 and k_2 are two scalars and 'A' is a matrix, then $(k_1k_2)A = k_1(k_2A) = k_2(k_1A)$

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8. MULTIPLICATION OF MATRICES (Row by Column) :

Let A be a matrix of order $m \times n$ and B be a matrix of order $p \times q$, then the matrix multiplication AB is possible if and only if $n = p$ and matrices are said to be **conformable** for multiplication.

In the product AB, A is called pre-factor and B is called post factor.

$\Rightarrow$ AB is possible if and only if number of columns in pre-factor = number of rows in post-factor.

Let $A_{m\times n} = [a_{ij}]$ and $B_{n\times p} = [b_{ij}]$, then order of AB is $m \times p$ & $\boxed{(AB)_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj}}$

e.g. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}_{3\times 4}$

Then order of AB is 2×4.

$(AB)_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = \sum_{r=1}^{3} a_{1r}b_{r1}$

$(AB)_{23} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} = \sum_{r=1}^{3} a_{2r}b_{r3}$

In general $(ab)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} = \sum_{r=1}^{3} a_{ir}b_{rj}$

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Illustration 3 : If $[1 \ x \ 2]\begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & 2 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ 1 \\ -1 \end{bmatrix} = \mathbf{O}$, then the value of x is :-

(A) –1 (B) 0 (C) 1 (D) 2

Solution : The LHS of the equation

$= [2 \quad 4x + 9 \quad 2x + 5]\begin{bmatrix} x \\ 1 \\ -1 \end{bmatrix} = [2x + 4x + 9 - 2x - 5] = 4x + 4$

Thus $4x + 4 = 0 \Rightarrow x = -1$ **Ans. (A)**

Illustration 4 : If A, B are two matrices such that $A + B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $A - B = \begin{bmatrix} 3 & 2 \\ -2 & 0 \end{bmatrix}$, then find AB.

Solution : Given $A + B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ .....(i) & $A - B = \begin{bmatrix} 3 & 2 \\ -2 & 0 \end{bmatrix}$ .....(ii)

Adding (i) & (ii)

$2A = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$

Subtracting (ii) from (i)

$2B = \begin{bmatrix} -2 & 0 \\ 4 & 4 \end{bmatrix} \Rightarrow B = \begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix}$

Now $AB = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 4 \end{bmatrix}$ **Ans.**

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9. PROPERTIES OF MATRIX MULTIPLICATION :

(a) Matrix multiplication is not commutative : i.e. **AB ≠ BA**

Here both AB & BA exist and also they are of the same type but **AB ≠ BA.**

Example :

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; then $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $BA = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ AB ≠ BA (in general)

(b) AB = **O** $\not\Rightarrow$ A = **O** or B = **O** (in general)

Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, then $AB = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Note :

If A and B are two non - zero matrices such that AB = **O** then A and B are called the divisors of zero. If A and B are two matrices such that

(i) AB = BA then A and B are said to commute

(ii) AB = –BA then A and B are said to anticommute

(c) Matrix Multiplication Is Associative :

If A, B & C are conformable for the product AB & BC, then (AB) C = A(BC)

(d) Distributivity :

$\left.\begin{array}{l} A(B+C) = AB + AC \\ (A+B)C = AC + BC \end{array}\right]$ Provided A,B & C are conformable for respective products

(e) tr(AB) = tr(BA)

Illustration 5 : Let $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix} \& C = \begin{bmatrix} 1 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix}$ be the matrices then, prove that in matrix multiplication cancellation law does not hold.

Solution : We have to show that AB = AC; though B is not equal to C.

We have $AB = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix}_{3\times 4}$

Now, $AC = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix}_{3\times 4}$

Here, AB = AC though B is not equal to C. Thus cancellation law does not hold in general.

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Do yourself - 3 :

1. If $A = \begin{bmatrix} 2 & 9 \\ -4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 \\ 6 & 8 \end{bmatrix}$ and $C = \begin{bmatrix} 9 & -7 \\ -2 & 4 \end{bmatrix}$, then show that $A(B + C) = AB + AC$.

2. If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$, then prove that $(A - B)^2 \neq A^2 - 2AB + B^2$.

3. Find the value of x : $2\begin{bmatrix} 3 & 1 & -2 \\ -1 & -3 & 4 \end{bmatrix} + x\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 2 & -2 \\ -8 & -14 & -2 \end{bmatrix}$

4.. Let $A = \begin{bmatrix} -3 & 0 & 2 \\ 1 & x & 5 \\ -2 & 0 & x^2 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ b \\ -1 \end{bmatrix}$ and $C = [3\ \ 5\ \ 1]$, then the number of integral value(s) of 'b' for which $Tr(ABC) \leq -18\ \forall\ x \in R$ is (are)

5. The number of values of n such that at least one pair of two matrices (A, B) of order n exists such that $AB - BA = I$.

10. POSITIVE INTEGRAL POWERS OF A SQUARE MATRIX :

For a square matrix A, $\mathbf{A^n} = \underbrace{\mathbf{A.A.A.....A}}_{\text{upto n times}}$, where $n \in N$

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Note :

(i) $\mathbf{A^m.A^n = A^{m+n}}$

(ii) $\mathbf{(A^m)^n = A^{mn}}$, where $m,n \in N$

(iii) If A and B are square matrices of same order and AB = BA then

$(A + B)^n = {}^nC_0A^n + {}^nC_1A^{n-1}B + {}^nC_2A^{n-2}B^2 + ... + {}^nC_nB^n$

Note that for a unit matrix I of any order , $I^m = I$ for all $m \in N$.

Do yourself -4 :

1 If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$ where n is positive integer.

2. If $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, where $i = \sqrt{-1}$ and $x \in N$, then A^{4x} equals -

(A) $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

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3. For $\theta = \frac{3\pi}{5}$, let $B = [b_{ij}]$ be a square matrix of 2nd order such that

$$b_{ij} = \begin{cases} \cos\theta & , \ i = j \\ \cos\left(\frac{j\pi}{2} + \theta\right) & , \ i > j \\ \sin\left(\frac{j\pi}{2} - \theta\right) & , \ i < j \end{cases}$$, then trace of matrix B^5 is equal to-

(A) –2 (B) –1 (C) 1 (D) 2

4. If A and B are 3 × 3 matrices such that AB= A and BA = B, then-

(A) $A^2 = A, B^2 \neq B$ (B) $A^2 \neq A, B^2 = B$ (C) $A^2 = A, B^2 = B$ (D) $A^2 \neq A, B^2 \neq B$

5. If the matrix A is given by $A = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$ then determinant value of $(A^{27} - 2A^{26})$ is

(A) –5 (B) –7 (C) –10 (D) 0

6. If A, B are two square matrices such that AB = A, BA = B then $(A + B)^7$ equals

(A) A + B (B) $2^7(A + B)$ (C) $2^6(A + B)$ (D) $2^8(A + B)$

7. A square matrix P satisfies $P^2 = I - P$, where I is the identity matrix. If $P^n = 5I - 8P$ ($n \in N$), then minimum value of n is equal to-

(A) 4 (B) 5 (C) 6 (D) 7

8. Let A be a square matrix satisfying $A^2 = 2I$ and $\sum_{k=1}^{2n} A^k = 63A(A + I)$, then the value of n is (where I denotes identity matrix)

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11. SPECIAL SQUARE MATRICES :

(a) Idempotent Matrix : A square matrix is idempotent provided $A^2 = A$.

For idempotent matrix note the following :

(i) $A^n = A \ \forall \ n \geq 2, n \in N$.

(ii) determinant value of idempotent matrix is either 0 or 1

(b) Periodic Matrix : A square matrix which satisfies the relation $A^{k+1} = A$, for some positive integer K, is a periodic matrix. The period of the matrix is the least value of K for which this holds true.

Note that period of an idempotent matrix is 1.

(c) Nilpotent Matrix : A square matrix of the order 'n' is said to be nilpotent matrix of order m, $m \in N$, if $A^m = \mathbf{O}$ & $A^{m-1} \neq \mathbf{O}$.

(d) Involutary Matrix : If $A^2 = I$, the matrix is said to be an involutary matrix. i.e. square roots of identity matrix is involutary matrix.

Note : The determinant value of involutary matrix is 1 or –1.

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Illustration 6 : Show that the matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent.

Solution : $A^2 = A.A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \times \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 2.2+(-2).(-1)+(-4).1 & 2(-2)+(-2).3+(-4).(-2) & 2.(-4)+(-2).4+(-4).(-3) \\ (-1).2+3.(-1)+4.1 & (-1).(-2)+3.3+4.(-2) & (-1).(-4)+3.4+4.(-3) \\ 1.2+(-2).(-1)+(-3).1 & 1.(-2)+(-2).3+(-3).(-2) & 1.(-4)+(-2).4+(-3).(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Hence the matrix A is idempotent.

Illustration 7 : Show that $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent matrix of order 3.

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Solution : Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-37 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}$$

$\therefore$ $A^3 = \mathbf{O}$ i.e., $A^k = \mathbf{O}$

Here $k = 3$

Hence A is nilpotent of order 3.

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Illustration 8 : Show that the matrix $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is involutory.

Solution : $A^2 = A.A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 25-24+0 & 40-40+0 & 0+0+0 \\ -15+15+0 & -24+25+0 & 0+0+0 \\ -5+6-1 & -8+10-2 & 0+0+1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence the given matrix A is involutory.

Illustration 9: Show that a square matrix A is involutory, iff $(I - A)(I + A) = \mathbf{O}$

Solution : Let A be involutory

Then $A^2 = I$

$(I - A)(I + A) = I^2 + IA - AI - A^2 = I + A - A - A^2 = I - A^2 = \mathbf{O}$

Conversly, let $(I - A)(I + A) = \mathbf{O}$

$\Rightarrow$ $I^2 + IA - AI - A^2 = \mathbf{O}$ $\Rightarrow$ $I + A - A - A^2 = \mathbf{O}$

$\Rightarrow$ $I - A^2 = \mathbf{O}$ $\Rightarrow$ A is involutory

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Do yourself - 5 :

1 The matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$ is

(A) idempotent matrix (B) involutary matrix

(C) nilpotent matrix (D) periodic matrix

2. If $A = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ and A^2 is the identity matrix, then find the value of x

3. If A,B, A + I, A + B are idempotent matrices (where A and B are square matrices of same order & I is identity matrix of same order), then value of $|AB - BA|$ (where $|C|$ denotes determinant of matrix C) is-

(A) 0 (B) 1

(C) 2 (D) cannot be determined

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4. If A is nilpotent matrix of index 2, then $A(I_2+A)^{51}$ is equal to (where I_2 is the identity matrix of order 2)-

(A) A^{51} (B) I_2 (C) Null matrix (D) A

5. If A and B are two idempotent matrices of same order and A + B is idempotent, then -

(A) AB + BA = O (B) AB = I (C) A = B (D) none of these

6. If A and B are square matrices such that AB = B and BA = A, then which of the following is/are always true ?

(A) A is an idempotent matrix (B) B is an involutary matrix

(C) $A^6 + B^6 = A^8 + B^8$ (D) $A^2 + B^2 = A + I$

12. THE TRANSPOSE OF A MATRIX : (Changing rows & columns)

Let A be any matrix of order $m \times n$. Then A^T or $A' = [a_{ji}]$ for $1 \le i \le m$ & $1 \le j \le n$ of order $n \times m$

Properties of transpose :

If A^T & B^T denote the transpose of A and B ,

(a) $(A+B)^T = A^T+B^T$; note that A & B have the same order.

(b) $(A\,B)^T = B^T A^T$ (Reversal law) A & B are conformable for matrix product AB

Note : In general : $(A_1 . A_2, ... A_n)^T = A_n^T . ... A_2^T . A_1^T$ (reversal law for transpose)

(c) $(A^T)^T = A$

(d) $(kA)^T = kA^T$, k is a scalar.

Illustration 10 : If A and B are matrices of order $m \times n$ and $n \times m$ respectively, then order of matrix $B^T(A^T)^T$ is -

(A) $m \times n$ (B) $m \times m$ (C) $n \times n$ (D) Not defined

Solution : Order of B is $n \times m$ so order of B^T will be $m \times n$

Now $(A^T)^T = A$ & its order is $m \times n$. For the multiplication $B^T(A^T)^T$

Number of columns in prefactor $\neq$ Number of rows in post factor.

Hence this multiplication is not defined. **Ans. (D)**

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13. ORTHOGONAL MATRIX

A square matrix is said to be orthogonal matrix if $A\,A^T = I$

Note :

(i) The determinant value of orthogonal matrix is either 1 or –1.

(ii) Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$AA^T = \begin{bmatrix} a_1^2 + a_2^2 + a_3^2 & a_1b_1 + a_2b_2 + a_3b_3 & a_1c_1 + a_2c_2 + a_3c_3 \\ b_1a_1 + b_2a_2 + b_3a_3 & b_1^2 + b_2^2 + b_3^2 & b_1c_1 + b_2c_2 + b_3c_3 \\ c_1a_1 + c_2a_2 + c_3a_3 & c_1b_1 + c_2b_2 + c_3b_3 & c_1^2 + c_2^2 + c_3^2 \end{bmatrix}$$

If $AA^T = I$, then

$$\sum_{i=1}^{3} a_i^2 = \sum b_i^2 = \sum c_i^2 = 1 \quad \text{and} \quad \sum_{i=1}^{3} a_ib_i = \sum_{i=1}^{3} b_ic_i = \sum_{i=1}^{3} c_ia_i = 0$$

Illustration 11 : Determine the values of α, β, γ, when $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal.

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Solution : Let $A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$

$$\therefore \quad A' = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$$

But given A is orthogonal.

$\therefore \quad AA^T = I$

$$\Rightarrow \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}\begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4\beta^2 + \gamma^2 & 2\beta^2 - \gamma^2 & -2\beta^2 + \gamma^2 \\ 2\beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 \\ -2\beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we have

$4\beta^2 + \gamma^2 = 1$ .....(i)

$2\beta^2 - \gamma^2 = 0$ .....(ii)

$\alpha^2 + \beta^2 + \gamma^2 = 1$ .....(iii)

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From (i) and (ii), $6\beta^2 = 1 \quad \therefore \quad \beta^2 = \frac{1}{6}$ and $\gamma^2 = \frac{1}{3}$

From (iii) $\alpha^2 = 1 - \beta^2 - \gamma^2 = 1 - \frac{1}{6} - \frac{1}{3} = \frac{1}{2}$

Hence, $\alpha = \pm\frac{1}{\sqrt{2}}$, $\beta = \pm\frac{1}{\sqrt{6}}$ and $\gamma = \pm\frac{1}{\sqrt{3}}$ **Ans.**

Do yourself - 6 :

1. If $A = \begin{bmatrix} 4 & 2 & -5 \\ 1 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -7 & 0 \\ -1 & 2 & 5 \\ 1 & 0 & 3 \end{bmatrix}$, then show that $(AB)^T = B^T.A^T$.

2. If $A = \begin{bmatrix} 2 & 3 & -4 \\ -1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 3 & 4 \\ -5 & -6 \end{bmatrix}$, then find $A + B^T$.

3. If $A = \begin{bmatrix} 9 & -3 & 6 \\ 8 & ½ & 7 \\ -1 & 0 & 0 \end{bmatrix}$, then, show that $(A^T)^T = A$.

4. Show that the matrix $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ is an orthogonal matrix.

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5. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, then $(BB^TA)^5$ is equal to

(Where X^T denotes transpose of matrix X)

(A) $\begin{bmatrix} 2+\sqrt{3} & 1 \\ -1 & 2-\sqrt{3} \end{bmatrix}$ (B) $\frac{1}{2}\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 5 & 1 \\ 0 & 1 \end{bmatrix}$

6. Let B be a square matrix such that $|B| = 1$ and $2A + B = B^T - A^T$, then $|A + B|$ is -

(A) 0 (B) –1 (C) 1 (D) 2

7. Let A and B are non singular matrices of order 3 × 3, such that $A^TA = BB^T = |A|I$, where I is identity matrix of order 3. If b_{ij} is element of ith row and j^{th} column in matrix B, then $\sum_{r=1}^{3} b_{2r}^2$ is

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14. SYMMETRIC & SKEW SYMMETRIC MATRIX :

(a) Symmetric matrix :

A square matrix $A = [a_{ij}]$ is said to be, symmetric if, $a_{ij} = a_{ji}$ $\forall$ i & j (conjugate elements are equal). Hence for symmetric matrix $\mathbf{A = A^T}$.

Note : Max. number of distinct entries in any symmetric matrix of order n is $\frac{n(n+1)}{2}$.

(b) Skew symmetric matrix :

Square matrix $A = [a_{ij}]$ is said to be skew symmetric if $a_{ij} = -a_{ji}$ $\forall$ i & j (the pair of conjugate elements are additive inverse of each other). For a skew symmetric matrix $\mathbf{A = -A^T}$.

Note :

(i) If A is skew symmetric, then $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0 \ \forall \ i$. Thus the diagonal elements of a skew square matrix are all zero, but not the converse.

(ii) The determinant value of odd order skew symmetric matrix is zero.

(c) Properties of symmetric & skew symmetric matrix :

(i) A is symmetric if $A^T = A$ & A is skew symmetric if $A^T = -A$

(ii) Let A be any square matrix then, $A + A^T$ is a symmetric matrix & $A - A^T$ is a skew symmetric matrix.

(iii) The sum of two symmetric matrix is a symmetric matrix and the sum of two skew symmetric matrix is a skew symmetric matrix.

(iv) If A & B are symmetric matrices then,

(1) AB + BA is a symmetric matrix

(2) AB – BA is a skew symmetric matrix.

(v) Every square matrix can be uniquely expressed as a sum or difference of a symmetric and a skew symmetric matrix.

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{skew symmetric}} \quad \text{and} \quad A = \frac{1}{2}(A^T + A) - \frac{1}{2}(A^T - A)$$

Illustration 12 : If A is symmetric as well as skew symmetric matrix, then A is -

(A) diagonal matrix (B) null matrix (C) triangular matrix (D) none of these

Solution : Let $A = [a_{ij}]$ Since A is skew symmetric $a_{ij} = -a_{ji}$

for i = j, $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$

for $i \neq j$, $a_{ij} = -a_{ji}$ [$\because$ A is skew symmetric] & $a_{ij} = a_{ji}$ [$\because$ A is symmetric]

$\therefore$ $a_{ij} = 0$ for all $i \neq j$

so, $a_{ij} = 0$ for all 'i' and 'j' i.e. A is null matrix. **Ans. (B)**

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Do yourself - 7 :

1. If $A = \begin{bmatrix} -2 & -1 & 1 \\ -1 & 7 & 4 \\ 1 & -x & -3 \end{bmatrix}$ be symmetric matrix then find the value of x.

2. Express matrix $A = \begin{bmatrix} 2 & 5 & 7 \\ 9 & -7 & 2 \\ 1 & -1 & 0 \end{bmatrix}$ as a sum of a symmetric and a skew symmetric matrix.

3.. If A is a matrix of order 3×3 such that $A^2 + A = A^T$, then which of the following is wrong -

(A) $|A| = 0$ (B) $|A| \neq 0$

(C) A^2 is a skew symmetric matrix (D) A is singular matrix

4. Let $A = \begin{bmatrix} 0 & \tan\theta & 2 \\ -\tan\theta & 0 & \sec\theta \\ -2 & -\sec\theta & 0 \end{bmatrix}$, where $\theta \neq (2n+1)\frac{\pi}{2}$, $(n \in I)$ & $B = \sum_{r=1}^{5} A^{(2r-1)}$ then which of the following statement is correct

(A) A is skew symmetric matrix (B) B is skew symmetric matrix

(C) (A + B) is skew symmetric matrix (D) (A + B) is symmetric matrix

5. If A, B are symmetric matrices of the same order then AB – BA is -

(A) a symmetric matrix (B) a skew symmetric matrix

(C) unit matrix (D) a null matrix

6.. Let A and B be 3×3 matrices of real number where A is symmetric, B is skew symmetric and $(A + B)(A - B) = (A - B)(A + B)$ if $(AB)^T = (-1)^k AB$, then possible value of k is -

(A) 3 (B) 4 (C) 6 (D) 2

7. Which of the following matrices $A = [a_{ij}]_{n \times n}$ represents a symmetric matrix ?

(A) $a_{ij} = i - j + \sin(i + j).\sin(i - j)$

(B) $a_{ij} = |i - j| + \sin(|i + j|)\sin(i - j)$

(C) $a_{ij} = \tan^{-1}(\tan(i + j)) + i - j$

(D) $a_{ij} = \cos(i - j).\cos(i + j) - (i - j)^2$

8. Let A is symmetric and B is a skew symmetric matrix, such that $A - B = \begin{bmatrix} 2 & 5 \\ 7 & 6 \end{bmatrix}$, then $\frac{(|A|)^2}{96}$ is

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15. ADJOINT OF A SQUARE MATRIX :

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A then the adjoint of A, denoted by adjA, is defined as the transpose of the cofactor matrix.

Then, $\text{adj}A = [C_{ij}]^T \Rightarrow \text{adj}A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{23} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

Theorem : A (adj. A) = (adj. A) .A = $|A|\, I_n$.

Proof : $A.(\text{adj } A) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$

$$\begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix} = |A| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \quad A.(\text{Adj. } A) = |A|\, I$$

(whatever may be the value only |A| will come out as a common element)

If $|A| \neq 0$, then $\dfrac{\mathbf{A.(adj.A)}}{|\mathbf{A}|} = \mathbf{I}$ = unit matrix of the same order as that of A

Properties of adjoint matrix :

If **A** be a square matrix of order **n**, then

(i) $|\text{adj } A| = |A|^{n-1}$

(ii) $\text{adj}(\text{adj } A) = |A|^{n-2} A$, where $|A| \neq 0$

(iii) $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$, where $|A| \neq 0$

(iv) $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$

(v) $\text{adj}(KA) = K^{n-1}(\text{adj } A)$, K is a scalar

(vi) $\text{adj } A^T = (\text{adj } A)^T$

Method to find adjoint of a 2 × 2 square matrix, directly :

Let A be a 2 × 2 square matrix. In order to find the adjoint simply interchange the diagonal elements and reverse the sign of off diagonal elements (rest of the elements).

e.g. If $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \Rightarrow \text{adj}A = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$

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Illustration 13: If $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 1 \\ 5 & 1 & 3 \end{bmatrix}$, then adj A is equal to -

(A) $\begin{bmatrix} 14 & -4 & -22 \\ -4 & -22 & 14 \\ -22 & 14 & -4 \end{bmatrix}$ (B) $\begin{bmatrix} -14 & 4 & 22 \\ 4 & 22 & -14 \\ 22 & -14 & 4 \end{bmatrix}$

(C) $\begin{bmatrix} 14 & 4 & -22 \\ 4 & -22 & -14 \\ -22 & -14 & -4 \end{bmatrix}$ (D) none of these

Solution : adj. $A = \begin{bmatrix} 14 & -4 & -22 \\ -4 & -22 & 14 \\ -22 & 14 & -4 \end{bmatrix}^T = \begin{bmatrix} 14 & -4 & -22 \\ -4 & -22 & 14 \\ -22 & 14 & -4 \end{bmatrix}$ **Ans. (A)**

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Illustration 14: If $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$, then adj (adj A) is equal to -

(A) $8\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (B) $16\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (C) $64\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (D) none of these

Solution : $|A| = \begin{vmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{vmatrix} = 8$

Now adj (adj A) $= |A|^{3-2} A$

$= 8\begin{vmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{vmatrix} = 16\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ **Ans. (B)**

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Do yourself - 8 :

1. For any 2×2 matrix, if $A(AdjA) = \begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix}$, then $|A|$ is equal -

(A) 20 (B) 625 (C) 15 (D) 0

2. Which of the following is/are incorrect ?

(A) Adjoint of a symmetric matrix is skew symmetric matrix.

(B) Adjoint of a diagonal matrix is a diagonal matrix.

(C) $A(AdjA) = (AdjA)A = |A|I$

(D) Adjoint of a unit matrix is a diagonal matrix

3. If A be a square matrix of the order 5 and B = Adj(A) then find Adj (5A).

4. If A be a square matrix of order 4 and $|A| = 3$ then find adj(adjA).

5. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 3 \end{bmatrix}$, then $tr(A^3 adj(A^2))$, is

(where tr(A) and adj(A) denote trace and adjoint of matrix A)-

(A) 6 (B) 24 (C) 36 (D) 12

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6. Let A and B are two non-singular square matrices of order n with real entries such that adjA = adjB, then which of the following is necessarily true-

(A) A = B if n is even (B) A = –B if n is even

(C) A = – B if n is odd (D) A = B if n is odd

7. Let A be square matrix of order 3. If det A = 2, then value of det(adj A^3) is equal to (Here det A denotes determinant of matrix A)

(A) 8 (B) 28 (C) 32 (D) 64

8. If A is a square martix of order n with $|A| = 9$ and $|adj\ adj\ adjA| = 3^{16}$, then number of elements in A, is -

(A) 9 (B) 4 (C) 5 (D) 2

9. If A and B are non-singular matrices of order three such that $adj(AB) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & p & 1 \\ 1 & 1 & p \end{bmatrix}$ and

$|B^2\ adj(A)| = p^2 - 3$, then p is

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16. INVERSE OF A MATRIX (Reciprocal Matrix) :

A square matrix A said to be invertible if and only if it is non-singular (i.e. $|A| \neq 0$) and there exists a matrix B such that, AB = I = BA.

B is called the **inverse** (reciprocal) of A and is denoted by $\mathbf{A^{-1}}$. Thus

$A^{-1} = B \Leftrightarrow AB = I = BA$

We have, $A.(\text{adj } A) = |A| I_n$

$A^{-1}.\ A(\text{adj } A) = A^{-1}\ I_n |A|$

$I_n(\text{adj } A) = A^{-1} |A| I_n$

$$\therefore \quad \mathbf{A^{-1} = \frac{(adj\ A)}{|A|}}$$

Note : The necessary and sufficient condition for a square matrix A to be invertible is that $\mathbf{|A| \neq 0}$

Properties of inverse :

(i) If A & B are invertible matrices of the same order, then $\mathbf{(AB)^{-1} = B^{-1}A^{-1}}$.

Note: If $A_1, A_2, \ldots\ldots\ldots A_n$ are all invertible square matrices of order n then $(A_1A_2\ldots\ldots\ldots A_n)^{-1} = A_n^{-1} A_{n-1}^{-1} \ldots\ldots\ldots A_2^{-1} A_1^{-1}$

(ii) If A be an invertible matrix, then A^T is also invertible & $(A^T)^{-1} = (A^{-1})^T$.

(iii) If A is invertible, (a) $(A^{-1})^{-1} = A$ (b) $(A^k)^{-1} = (A^{-1})^k = A^{-k}$; $k \in N$

(iv) If A is non-singular matrix, then $|A^{-1}| = |A|^{-1}$

(v) If idempotent matrix is invertible then its inverse will be identity matrix.

(vi) A nilpotent matrix will not be invertible because its determinant value is zero.

(vii) Orthogonal matrix A is always invertible and $A^{-1} = A^T$.

(viii) $A = A^{-1}$ for an involutary matrix.

Cancellation law : Let A,B,C be square matrices of the same order 'n'.

If A is a non-singular matrix, then

(a) $AB = AC \Rightarrow B = C$ (Left cancellation law)

(b) $BA = CA \Rightarrow B = C$ (Right cancellation law)

Note that these cancellation laws hold only if the matrix 'A' is **non-singular** (i.e.$|A| \neq 0$).

Illustration 15 : Prove that if A is non-singular matrix such that A is symmetric then A^{-1} is also symmetric.

Solution : $A^T = A$ [$\because$ A is a symmetric matrix]

$(A^T)^{-1} = A^{-1}$ [since A is non-singular matrix]

$\Rightarrow (A^{-1})^T = A^{-1}$ Hence proved

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Illustration 16 : $\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}^{-1}$ is equal to -

(A) $\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$ (B) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ (C) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ (D) none of these

Solution : $\begin{bmatrix} 1 & \tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}^{-1} = \dfrac{1}{\sec^2\theta/2}\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}$

$\therefore$ Product $= \dfrac{1}{\sec^2\theta/2}\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}$

$= \dfrac{1}{\sec^2\theta/2}\begin{bmatrix} 1-\tan^2\theta/2 & -2\tan\theta/2 \\ 2\tan\theta/2 & 1-\tan^2\theta/2 \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta/2 & \sin^2\theta/2 & -2\sin\theta/2\cos\theta/2 \\ 2\sin\theta/2 & \cos\theta/2 & \cos^2\theta/2-\sin^2\theta/2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ **Ans. (C)**

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Illustration 17: If $A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and M = AB, then M^{-1} is equal to-

(A) $\begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 1/6 \end{bmatrix}$ (C) $\begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 1/6 \end{bmatrix}$ (D) $\begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 1/6 \end{bmatrix}$

Solution : $M = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix}$

$|M| = 6$, adj $M = \begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}$

$\therefore$ $M^{-1} = \dfrac{1}{6}\begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 1/6 \end{bmatrix}$ **Ans. (C)**

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Do yourself -9 :

1. If 'A' is a square matrix such that $A^2 = I$ then A^{-1} is equal to -
(A) A + I (B) A (C) 0 (D) 2A

2. If 'A' is an orthogonal matrix, then A^{-1} equals -
(A) A (B) A^T (C) A^2 (D) none of these

3. If $A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, then $(A^{-1})^3$ is equal to -

(A) $\frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & 27 \end{bmatrix}$ (B) $\frac{1}{27}\begin{bmatrix} 1 & 26 \\ 0 & 27 \end{bmatrix}$ (C) $\frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & -27 \end{bmatrix}$ (D) $\frac{1}{27}\begin{bmatrix} -1 & -26 \\ 0 & -27 \end{bmatrix}$

4. If A is a non-zero idempotent matrix satisfying $\left(I - \frac{2}{5}A\right)^{-1} = I - \alpha A$ where I is the unit matrix of the same order as that of A then the absolute value of 3α is

5. Let A, B and A + B are non-singular matrices of order 3×3 satisfying

$A^{-1} + B^{-1} = (A + B)^{-1}$ and $|AB^{-1}| \in R$ then value of $\frac{|A|}{|B|}$ is

6. Let P, Q, R be invertible matrices of second order such that $A = PQ^{-1}$, $B = QR^{-1}$, $C = RP^{-1}$, then det(ABC + BCA + CAB) is

7. An invertible matrix A of order 3 satisfies the relation $A = A^{-1} + 2I$, (where I denotes identity matrix). The value of $|A - I|.|A + I|.|A - 2I|$ is

8. If A is a square matrix such that $A^5 = O$ and $A^n \neq O$ for n = 1,2,3,4, then $(I - A)^{-1}$ is equal to (where O is a null matrix)-
(A) I + A (B) $I + A + A^2$
(C) $I + A^{-1}$ (D) $I + A + A^2 + A^3 + A^4$

9. If A and B are non-singular symmetric matrices of the same order such that AB = BA and $P = (AB^{-1})^T$ & $Q = (A^{-1}B)^T$ then $(PQ)^{-1}$ is equal to-
(A) $AB^{-1}AB$ (B) $A^{-1}B^{-1}AB$ (C) $A^{-1}BA^{-1}B$ (D) A^2B^2

17. MATRIX POLYNOMIAL :

If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots\dots + a_nx^0$, then we define a matrix polynomial

$f(A) = a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \dots\dots + a_nI^n$.

where A is the given square matrix. If $f(A)$ is the null matrix, then A is called the zero or root of the polynomial $f(x)$.

18. CHARACTERISTIC EQUATION :

Let A be a square matrix. Then the polynomial $|A - xI|$ is called as characteristic polynomial of A & the equation $|A - xI| = 0$ is called as characteristic equation of A. After solving the characterstic polynomial the values of 'x' are said to be characterstic roots of the polynomial.

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Note : (i) Sum of the roots of the characteristic equation is equal to trace of the matrix.

(ii) Product of the roots of the characteristic equation is equal to the determinant value.

(iii) The degree of characteristic equation is same as the order of the matrix.

Illustration 18 : If $f(x) = x^2 - 3x + 3$ and $A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ be a square matrix then prove that $f(A) = \mathbf{O}$.

Hence find A^4.

Solution : $A^2 = A.A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ -3 & 0 \end{bmatrix}$

Hence $A^2 - 3A + 3I = \begin{bmatrix} 3 & 3 \\ -3 & 0 \end{bmatrix} - 3\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O}$

Aliter : $\because |A - XI| = 0 \Rightarrow \begin{vmatrix} 2-x & 1 \\ -1 & 1-x \end{vmatrix} = 0$

$\Rightarrow$ $(2 - x)(1 - x) + 1 = 0 \Rightarrow x^2 - 3x + 3 = 0$ (characterstic polynomial)

by Cayley-Hamilton Theorem $A^2 - 3A + 3I = \mathbf{O}$. Hence proved.

Now $A^2 = 3A - 3I$

squaring on both the sides

$A^4 = 9(A^2 - 2A + I)$

$= 9\left(\begin{bmatrix} 3 & 3 \\ -3 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 9\begin{bmatrix} 3-4+1 & 3-2 \\ -3+2 & -2+1 \end{bmatrix}$

$= 9\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 9 \\ -9 & -9 \end{bmatrix}$

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19. CAYLEY - HAMILTON THEOREM :

Every square matrix A satisfy its characteristic equation

i.e. $a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0$ is the characteristic equation of A, then

$a_0A^n + a_1A^{n-1} + ... + a_{n-1}A + a_nI = \mathbf{O}$

Note : This theorem is helpful to find the inverse of any non-singular square matrix.

i.e. $a_0A^n + a_1A^{n-1} + ... + a_{n-1}A + a_nI = \mathbf{O}$

On multiplying by A^{-1} on both the sides of above equation, we get

$$A^{-1} = -\frac{1}{a_n}\left(a_0A^{n-1} + a_1A^{n-2} + ...a_{n-1}I\right)$$

Illustration 19 : If $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, show that $5A^{-1} = A^2 + A - 5I$

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Solution : We have the characteristic equation of A.

$|A - xI| = 0$

i.e. $\begin{vmatrix} 1-x & 2 & 0 \\ 2 & -1-x & 0 \\ 0 & 0 & -1-x \end{vmatrix} = 0$

i.e. $x^3 + x^2 - 5x - 5 = 0$

Using Cayley – Hamilton theorem

$A^3 + A^2 - 5A - 5I = \mathbf{O} \quad \Rightarrow \quad 5I = A^3 + A^2 - 5A$

Multiplying by A^{-1}, we get $5A^{-1} = A^2 + A - 5I$

Do yourself -10 :

1. Determine the characteristic roots of the matrix A. Hence find the trace and determinant value of A.

Where $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ and also prove that $A^3 - 18A^2 + 45A = \mathbf{O}$.

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2. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ is zero divisor of $f(x) = x^2 - 4x - 5$, then A^3 is equal to

(A) $\begin{bmatrix} 41 & 42 & 42 \\ 42 & 41 & 42 \\ 42 & 42 & 41 \end{bmatrix}$ (B) $\begin{bmatrix} 42 & 41 & 42 \\ 41 & 42 & 41 \\ 44 & 41 & 42 \end{bmatrix}$ (C) $\begin{bmatrix} 43 & 42 & 41 \\ 44 & 41 & 42 \\ 42 & 43 & 43 \end{bmatrix}$ (D) $\begin{bmatrix} 41 & 42 & 41 \\ 42 & 41 & 42 \\ 42 & 42 & 41 \end{bmatrix}$

3. Let A be a 3×3 matrix such that $a_{11} = a_{33} = 2$ & all the other $a_{ij} = 1$. Let $A^{-1} = xA^2 + yA + zI$, then the value of $(x + y + z)$ where I is a unit matrix of order 3, is-

(A) 0 (B) 1 (C) 2 (D) 3

20. SYSTEM OF EQUATION & CRITERIA FOR CONSISTENCY

Gauss - Jordan method :

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

$$\Rightarrow \begin{bmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

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$\Rightarrow$ A X = B ....(i)

Multiplying adjA on both the sides of (i)

$\Rightarrow$ (adjA) AX = (adjA)B $\Rightarrow$ |A|X = (adjA) B

$|A|X = (\text{adj}A)B$

- If $|A| \neq 0$: $\boxed{X = \frac{(\text{adj}A)B}{|A|}}$ or $\boxed{X = A^{-1}B}$ (unique solution)
- If $|A| = 0$:
  - $(\text{adj}A)B$ = null matrix → Infinite solutions or no solution
  - If $(\text{adj } A)B \neq O$ → Inconsistent (no solution)

Illustration 20: Solve the system $\begin{aligned} x+y+z&=16 \\ x-y+z&=2 \\ 2x+y-z&=1 \end{aligned}$ using matrix method.

Solution : Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$

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Then the system is AX = B.

$|A| = 6$, hence A is non singular,

$$\text{Cofactor } A = \begin{bmatrix} 0 & 3 & 3 \\ 2 & -3 & 1 \\ 2 & 0 & -2 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{6}\begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \quad \text{i.e.} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow$ x = 1, y = 2, z = 3 **Ans.**

node06\B0B0-BA\Kota\JEE(Advanced)\Enthuse\Maths\Sheet\Matrix\Eng\01_Matrix_(Theory).P65

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Do yourself -11 :

1. The system of equations $x + 2y - 3z = 1$, $x - y + 4z = 0$, $2x + y + z = 1$ has -

(A) only two solutions (B) only one solution

(C) no solution (D) infinitely many solutions

2. The system of equations $x + y + z = 8$, $x - y + 2z = 6$, $3x + 5y - 7z = 14$ has-

(A) Unique solution (B) infinite number of solutions

(C) no solution (D) none of these

3. If the system of linear equations $x + 2ay + az = 0$, $x + 3by + bz = 0$, $x + 4cy + cz = 0$, has a non-trival solution & $a, b, c > 0$ then the minimum value of $\left(\frac{a^3 + c^3}{b^3}\right)$ is

4. Square matrices B of second order are formed using some or all the possible values of k, when system of linear equations $kx + 4y + z = 0$, $4x + ky + 2z = 0$ & $2x + 2y + z = 0$, has infinite solution, then number of singular matrices B is

5. Consider the system of equations $x - 2y + 3z = -1$, $x - 3y + 4z = 1$ and $x + y + 2z = k$ has exactly three solutions, then -

(A) $k = 3$ (B) $k = 0$ (C) $k = -9$ (D) $k = \phi$

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EXERCISE (O-1)

1 Let $A + 2B = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{bmatrix}$ and $2A - B = \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$, then Tr (A) – Tr (B) has the value equal to

(A) 0 (B) 1 (C) 2 (D) none

MT0001

2. If $\begin{bmatrix} x & 3x-y \\ zx+z & 3y-w \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 7 \end{bmatrix}$, then

(A) $x = 3$, $y = 7$, $z = 1$, $w = 14$
(B) $x = 3$, $y = -5$, $x = -1$, $w = -4$
(C) $x = 3$, $y = 6$, $z = 2$, $w = 7$
(D) None of these

MT0002

3. The matrix $A^2 + 4A - 5I$, where I is identity matrix and $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$ equals :

(A) $32\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ (B) $4\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$ (C) $4\begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix}$ (D) $32\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$

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MT0003

4. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $A^2 - kA - I_2 = 0$, then value of k is-

(A) 4 (B) 2 (C) 1 (D) –4

MT0004

5. A and B are two given matrices such that the order of A is 3×4 , if A′ B and BA′ are both defined then

(A) order of B′ is 3 × 4
(B) order of B′A is 4 × 4
(C) order of B′A is 3 × 3
(D) B′A is undefined

MT0005

6. If p, q, r are 3 real number satisfying the matrix equation, $[p\ q\ r]\begin{bmatrix} 3 & 4 & 1 \\ 3 & 2 & 3 \\ 2 & 0 & 2 \end{bmatrix} = [3\ 0\ 1]$, then 2p + q – r equals :-

(A) –1 (B) 4 (C) –3 (D) 2

MT0006

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7. If A, B and C are $n \times n$ matrices and det(A) = 2, det(B) = 3 and det(C) = 5, then the value of the $\det(A^2BC^{-1})$ is equal to

(A) $\frac{6}{5}$ (B) $\frac{12}{5}$ (C) $\frac{18}{5}$ (D) $\frac{24}{5}$

MT0007

8. If $M = \begin{bmatrix} 0 & 2 \\ 5 & 0 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 5 \\ 2 & 0 \end{bmatrix}$, then M^{2011} is -

(A) $10^{1005}M$ (B) $10^{1005}N$ (C) $10^{2010}M$ (D) $10^{2011}M$

MT0008

9. Let three matrices are $A = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$, then

$t_r(A) + t_r\left(\frac{ABC}{2}\right) + t_r\left(\frac{A(BC)^2}{4}\right) + t_r\left(\frac{A(BC)^3}{8}\right) + ... + \infty$ is equal to-

(A) 6 (B) 9 (C) 12 (D) none

MT0009

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10. For a matrix $A = \begin{bmatrix} 1 & 2r-1 \\ 0 & 1 \end{bmatrix}$, the value of $\prod_{r=1}^{50} \begin{bmatrix} 1 & 2r-1 \\ 0 & 1 \end{bmatrix}$ is equal to -

(A) $\begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 4950 \\ 0 & 1 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 5050 \\ 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 2500 \\ 0 & 1 \end{bmatrix}$

MT0010

11. If the product of n matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.....\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ is equal to the matrix $\begin{bmatrix} 1 & 378 \\ 0 & 1 \end{bmatrix}$ then the value of n is equal to -

(A) 26 (B) 27 (C) 377 (D) 378

MT0011

12. Consider a matrix $A(\theta) = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$ then

(A) $A(\theta)$ is symmetric (B) $A(\theta)$ is skew symmetric

(C) $A^{-1}(\theta) = A(\pi - \theta)$ (D) $A^2(\theta) = A\left(\frac{\pi}{2} - 2\theta\right)$

MT0012

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13. Which of the following is an orthogonal matrix -

(A) $\begin{bmatrix} 6/7 & 2/7 & -3/7 \\ 2/7 & 3/7 & 6/7 \\ 3/7 & -6/7 & 2/7 \end{bmatrix}$ (B) $\begin{bmatrix} 6/7 & 2/7 & 3/7 \\ 2/7 & -3/7 & 6/7 \\ 3/7 & 6/7 & -2/7 \end{bmatrix}$

(C) $\begin{bmatrix} -6/7 & -2/7 & -3/7 \\ 2/7 & 3/7 & 6/7 \\ -3/7 & 6/7 & 2/7 \end{bmatrix}$ (D) $\begin{bmatrix} 6/7 & -2/7 & 3/7 \\ 2/7 & 2/7 & -3/7 \\ -6/7 & 2/7 & 3/7 \end{bmatrix}$

MT0013

14. The matrix $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is a

(A) non-singular (B) Idempotent (C) Nipotent (D) Orthogonal

MT0014

15. If $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$, then A is

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(A) Involutory matrix (B) Idempotent matrix (C) Nilpotent matrix (D) none of these

MT0015

16. If A and B are symmetric matrices, then ABA is -

(A) symmetric matrix (B) skew symmetric matrix

(C) diagonal matrix (D) scalar matrix

MT0016

17. Matrix $A = \begin{bmatrix} x & 3 & 2 \\ 1 & y & 4 \\ 2 & 2 & z \end{bmatrix}$, if xyz = 60 and 8x + 4y + 3z = 20, then A (adj A) is equal to -

(A) $\begin{bmatrix} 64 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 64 \end{bmatrix}$ (B) $\begin{bmatrix} 88 & 0 & 0 \\ 0 & 88 & 0 \\ 0 & 0 & 88 \end{bmatrix}$ (C) $\begin{bmatrix} 68 & 0 & 0 \\ 0 & 68 & 0 \\ 0 & 0 & 68 \end{bmatrix}$ (D) $\begin{bmatrix} 34 & 0 & 0 \\ 0 & 34 & 0 \\ 0 & 0 & 34 \end{bmatrix}$

MT0017

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18. Number of real values of λ for which the matrix $A = \begin{bmatrix} \lambda-1 & \lambda & \lambda+1 \\ 2 & -1 & 3 \\ \lambda+3 & \lambda-2 & \lambda+7 \end{bmatrix}$ has no inverse

(A) 0 (B) 1 (C) 2 (D) infinite

MT0018

19. Let $A = \begin{pmatrix} 0 & \sin\alpha & \sin\alpha\sin\beta \\ -\sin\alpha & 0 & \cos\alpha\cos\beta \\ -\sin\alpha\sin\beta & -\cos\alpha\cos\beta & 0 \end{pmatrix}$, then -

(A) $|A|$ is independent of α and β
(B) A^{-1} depends only on α
(C) A^{-1} depends only on β
(D) none of these

MT0019

20. If $A = [a_{ij}]_{2\times2}$ where $a_{ij} = \begin{cases} i+j & i \neq j \\ i^2 - 2j & i = j \end{cases}$, then A^{-1} is equal to -

(A) $\frac{1}{9}\begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix}$ (B) $\frac{1}{9}\begin{bmatrix} 0 & -3 \\ 3 & -1 \end{bmatrix}$ (C) $\frac{1}{9}\begin{bmatrix} 0 & -3 \\ -3 & -1 \end{bmatrix}$ (D) $\frac{1}{3}\begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix}$

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MT0020

EXERCISE (O-2)

[ONE OR MORE THAN ONE ARE CORRECT]

1. Let A, other than I or –I, be a 2×2 real matrix such that $A^2 = I$, I being the unit matrix. Let Tr(A) be the sum of diagonal elements of A.

Statement–1 : Tr (A) = 0

Statement–2 : det (A) = –1

(A) Statement-1 and Statement-2 are true and Statement-2 is not the correct explanation for Statement-1
(B) Statement-1 and Statement-2 are true and Statement-2 is a correct explanation for Statement-1.
(C) Statement-1 is true and Statement-2 is false.
(D) Statement-1 is false and Statement-2 is true.

MT0021

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2. If $A = \begin{bmatrix} \cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha \end{bmatrix}$; $B = \begin{bmatrix} \cos^2\beta & \sin\beta\cos\beta \\ \sin\beta\cos\beta & \sin^2\beta \end{bmatrix}$

are such that, AB is a null matrix, then which of the following should necessarily be an odd integral multiple of $\frac{\pi}{2}$.

(A) α (B) β (C) $\alpha - \beta$ (D) $\alpha + \beta$

MT0022

3. Identify the incorrect statement in respect of two square matrices A and B conformable for sum and product -

(A) $t_r(A + B) = t_r(A) + t_r(B)$ (B) $t_r(\alpha A) = \alpha\, t_r(A), \alpha \in R$

(C) $t_r(A^T) = t_r(A)$ (D) $t_r(AB) \neq t_r(BA)$

MT0023

4. If A and B are invertible matrices, which one of the following statement is/are incorrect -

(A) $\text{Adj}(A) = |A|A^{-1}$ (B) $\det(A^{-1}) = |\det(A)|^{-1}$

(C) $(A + B)^{-1} = B^{-1} + A^{-1}$ (D) $(AB)^{-1} = B^{-1}A^{-1}$

MT0024

5. Let $S = \left\{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : a_{ij} \in \{0, 1, 2\}, a_{11} = a_{22}\right\}$. Then the number of non-singular matrices in the set S is :

(A) 24 (B) 10 (C) 20 (D) 27

MT0025

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6. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ and $10B = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$. If B is the inverse of matrix A, then α is -

(A) –2 (B) –1 (C) 2 (D) 5

MT0026

7. Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -x & -y & z \\ 0 & y & 2z \\ x & -y & z \end{bmatrix}$ where $x, y, z \in R$. If $B^TAB = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 42 \end{bmatrix}$ then the number of ordered triplet (x,y,z) is-

(A) 2 (B) 6 (C) 8 (D) 9

MT0027

8. $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$ then let us define a function $f(x) = \det.(A^TA^{-1})$ then which of the following can not be the value of $\underbrace{f(f(f(f.....f(x))))}_{n \text{ times}}$ is $(n \geq 2)$

(A) $f^n(x)$ (B) 1 (C) $f^{n-1}(x)$ (D) $n f(x)$

MT0028

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[ONE OR MORE THAN ONE ARE CORRECT]

9. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & -1 \\ 3 & 0 & k \end{bmatrix}$ and $f(x) = x^3 - 2x^2 - \alpha x + \beta = 0$. If A satisfies $f(x) = 0$, then-

(A) $k = 1, \alpha = 14$ (B) $\alpha = 14, \beta = 22$ (C) $k = -1, \beta = 22$ (D) $\alpha = -14, \beta = -22$

MT0029

10. If A and B are 3×3 matrices and $|A| \neq 0$, then which of the following are true?

(A) $|AB| = 0 \Rightarrow |B| = 0$ (B) $|AB| = 0 \Rightarrow B = 0$

(C) $|A^{-1}| = |A|^{-1}$ (D) $|A + A| = 2|A|$

MT0030

11. If D_1 and D_2 are two 3×3 diagonal matrices where none of the diagonal element is zero, then -

(A) D_1D_2 is a diagonal matrix

(B) $D_1D_2 = D_2D_1$

(C) $D_1^2 + D_2^2$ is a diagonal matrix

(D) none of these

MT0031

12. If A and B are two 3×3 matrices such that their product AB is a null matrix then

(A) det. $A \neq 0 \Rightarrow$ B must be a null matrix.

(B) det. $B \neq 0 \Rightarrow$ A must be a null matrix.

(C) If none of A and B are null matrices then atleast one of the two matrices must be singular.

(D) If neither det. A nor det. B is zero then the given statement is not possible.

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MT0032

13. Let $A = a_{ij}$ be a matrix of order 3 where $a_{ij} = \begin{cases} x & \text{if } i = j, \ x \in R \\ 1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$, then which of the following hold(s) good ?

(A) for $x = 2$, A is a diagonal matrix. (B) A is a symmetric matrix

(C) for $x = 2$, det A has the value equal to 6

(D) Let $f(x) = \det A$, then the function $f(x)$ has both the maxima and minima.

MT0033

14. If A & B are square matrices of order 2 such that $A + \text{adj}(B^T) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ & $A^T - \text{adj}(B) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then-

(A) B is symmetric matrix (B) $A^n = A \ \forall \ n \in N$

(C) $|A + A^2 + A^3 + A^4 + A^5| = 0$ (D) $|B + B^2 + B^3 + B^4 + B^5| = 0$

MT0034

15. If A and B are two orthogonal matrices of order 3, then -

(A) A and B both will be invertible matrices (B) matrix ABA will also be orthogonal

(C) matrix A^2B^2 will also be orthogonal (D) maximum value of $\det\left(\frac{A}{2}\text{adj}(2B)\right)$ is 8.

MT0035

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16. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ & $A^n = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, (where $n \geq 2$ & $n \in N$), then -

(A) $a = d$ (B) $b = c$
(C) $b = a + 1$ if n is odd (D) $b = a - 1$ if n is even

MT0036

17. If A & B are two non singular matrices of order 3×3 such that $A^T + B = I$ & $BA^T = -B$, then which is/are always true (where X^T denotes transpose of X and I denotes unit matrix)-

(A) $|B| = 2$ (B) $|B| = 8$ (C) $|A| = -1$ (D) $|A| = 1$
(where $|X|$ denotes determinant value of X)

MT0037

18. Let $\det(\text{adj}(\text{adj}A)) = 14^4$ where $A = \begin{bmatrix} x & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$, $x \neq -\frac{11}{3}$, then

(A) $x = 1$ (B) $\det(2A) = 112$ (C) $x = 2$ (D) $\det(2A) = 256$

MT0038

19. Let $A^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, then -

(A) $7|A| = \frac{1}{2}$ (B) $|\text{adj } A| = \frac{1}{196}$

(C) trace $(\text{adj}A) = -\frac{1}{7}$ (D) Matrix A is a symmetric matrix

MT0039

20. If $A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 2 & 0 \\ 4 & 5 & 3 \end{bmatrix}$, then which of the following is(are) true ?

(trace of A denotes sum of principal diagonal elements of A)
(A) A is invertible (B) $\text{trace}(\text{adj}(\text{adj}(A))) = 144$
(C) $\text{trace}(\text{adj}(\text{adj}(A))) = 8$ (D) $|\text{adj } A|$ is less than 400

MT0040

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Paragraph for Question 21 to 22

Consider the system AX = B, where $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$.

21. Sum of elements of (adjA) B is-

(A) –1 (B) 2 (C) –2 (D) –4

MT0041

22. Value of $tr(XB^T)$ is (where tr(A) denotes trace of matrix A)-

(A) 0 (B) 1 (C) 2 (D) 3

MT0041

Paragraph for question nos. 23 to 25

If A is a symmetric and B skew symmetric matrix and A + B is non singular and $C = (A+B)^{-1}(A-B)$ then

23. $C^T(A+B)C =$

(A) A + B (B) A – B (C) A (D) B

MT0042

24. $C^T(A-B)C =$

(A) A + B (B) A – B (C) A (D) B

MT0042

25. C^TAC

(A) A + B (B) A – B (C) A (D) B

MT0042

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EXERCISE (S-1)

1. Find the number of 2×2 matrix satisfying following conditions :

(i) a_{ij} is 1 or –1 ; (ii) $a_{11}a_{21} + a_{12}a_{22} = 0$

MT0043

2. Find the value of x and y that satisfy the equations

$$\begin{bmatrix} 3 & -2 \\ 3 & 0 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} y & y \\ x & x \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3y & 3y \\ 10 & 10 \end{bmatrix}$$

MT0044

3. Let $M = \begin{bmatrix} a & -360 \\ b & c \end{bmatrix}$, where a, b and c are integers. Find the smallest positive value of b such that $M^2 = \mathbf{O}$, where **O** denotes 2×2 null matrix.

MT0045

4. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p \\ q \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Such that AB = B and a + d = 5050. Find the value of (ad – bc).

MT0046

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5. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, then show that the matrix A is a root of the polynomial $f(x) = x^3 - 6x^2 + 7x + 2$.

MT0047

6. Define $A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$. Find a vertical vector V such that $(A^8 + A^6 + A^4 + A^2 + I)V = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$ (where I is the 2×2 identity matrix).

MT0048

7. If the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(a, b, c, d not all simultaneously zero) commute, find the value of $\dfrac{d-b}{a+c-b}$. Also show that the matrix which commutes with A is of the form $\begin{bmatrix} \alpha-\beta & 2\beta/3 \\ \beta & \alpha \end{bmatrix}$

MT0049

8. $A = \begin{pmatrix} 3 & a & -1 \\ 2 & 5 & c \\ b & 8 & 2 \end{pmatrix}$ is Symmetric and $B = \begin{pmatrix} d & 3 & a \\ b-a & e & -2b-c \\ -2 & 6 & -f \end{pmatrix}$ is Skew Symmetric, then find AB.

Is AB a symmetric, Skew Symmetric or neither of them. Justify your answer.

MT0050

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9. Express the matrix $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & -6 \\ -1 & 0 & 4 \end{bmatrix}$ as a sum of a lower triangular matrix & an upper triangular matrix with zero in its leading diagonal. Also express the matrix as a sum of a symmetric and a skew symmetric matrix.

MT0051

10. (a) A is a square matrix of order n.

ℓ = maximum number of distinct entries if A is a triangular matrix

m = maximum number of distinct entries if A is a diagonal matrix

p = minimum number of zeroes if A is a triangular matrix.

If $\ell + 5 = p + 2m$, find the order of the matrix.

MT0052

(b) Let A be the set of all 3×3 skew symmetric matrices whose entries are either –1, 0 or 1. If there are exactly three 0's, three 1's and three (–1)'s, then find the number of such matrices.

MT0053

11. If A is an idempotent non-zero matrix and I is an identity matrix of the same order, find the value of n, $n \in N$, such that $(A + I)^n = I + 127\,A$.

MT0054

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12. Let $A = \begin{bmatrix} 3x^2 \\ 1 \\ 6x \end{bmatrix}$, $B = [a\ b\ c]$ and $C = \begin{bmatrix} (x+2)^2 & 5x^2 & 2x \\ 5x^2 & 2x & (x+2)^2 \\ 2x & (x+2)^2 & 5x^2 \end{bmatrix}$ be three given matrices, where a, b, c and $x \in R$. Given that tr(AB) = tr(C) $\forall$ $x \in R$, where tr(A) denotes trace of A. Find the value of (a + b + c)

MT0055

13. If $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ is an idempotent matrix. Find the value of f(a), where $f(x) = x - x^2$, when bc = 1/4. Hence otherwise evaluate a.

MT0056

14. If the matrix A is involutary, show that $\frac{1}{2}(I + A)$ and $\frac{1}{2}(I - A)$ are idempotent and $\frac{1}{2}(I + A).\ \frac{1}{2}(I - A) = \mathbf{O}$.

MT0057

15. Show that the matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ can be decomposed as a sum of a unit and a nilpotent matrix.

Hence evaluate the matrix $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{2007}$.

MT0058

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16. Let $A = \begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -x & 14x & 7x \\ 0 & 1 & 0 \\ x & -4x & -2x \end{bmatrix}$ are two matrices such that $AB = (AB)^{-1}$ and $AB \neq I$ (where I is an identity matrix of order 3×3).

Find the value of Tr. $\left(AB + (AB)^2 + (AB)^3 + ... + (AB)^{100}\right)$,

where Tr. (A) denotes the trace of matrix A.

MT0059

EXERCISE (S-2)

1. If A and B are square matrices of order 3, where $|A| = -2$ and $|B| = 1$, then find $\left|\left(A^{-1}\right)\text{adj}\left(B^{-1}\right)\text{adj}\left(2A^{-1}\right)\right|$.

MT0060

2. If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then show that $F(x).F(y) = F(x+y)$. Hence prove that $[F(x)]^{-1} = F(-x)$.

MT0061

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3. (a) Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$. Find P such that $BPA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

MT0062

(b) Find the matrix A satisfying the matrix equation, $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} . A . \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}$.

MT0063

4. Let A be a 3×3 matrix such that $a_{11} = a_{33} = 2$ and all the other $a_{ij} = 1$. Let $A^{-1} = xA^2 + yA + zI$, then find the value of $(x + y + z)$ where I is a unit matrix of order 3.

MT0064

5. Given that $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 1 & -1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 10 \\ 13 \\ 9 \end{bmatrix}$ and that $Cb = D$.

Solve the matrix equation $Ax = b$.

MT0065

6. Let X be the solution set of the equation $A^x = I$, where $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ and I is the corresponding unit matrix and $x \subseteq N$ then find the minimum value of $\sum (\cos^x \theta + \sin^x \theta), \theta \in R$.

MT0066

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7. Let $A = \begin{bmatrix} 1 & \frac{3}{2} \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -3 \\ -2 & 2 \end{bmatrix}$ and $C_r = \begin{bmatrix} r.3^r & 2^r \\ 0 & (r-1)3^r \end{bmatrix}$ be 3 given matrices.

Compute the value of $\sum_{r=1}^{50} tr.\left((AB)^r C_r\right)$. (where tr.(A) denotes trace of matrix A)

MT0067

8. For the matrix $A = \begin{bmatrix} 4 & -4 & 5 \\ -2 & 3 & -3 \\ 3 & -3 & 4 \end{bmatrix}$ find A^{-2}.

MT0068

9. Given $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$. I is a unit matrix of order 2. Find all possible matrix X in the following cases.

(a) $AX = A$ (b) $XA = I$ (c) $XB = \mathbf{O}$ but $BX \neq \mathbf{O}$.

MT0069

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10. Consider the two matrices A and B where $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$; $B = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$. Let n(A) denotes the number of elements in A and n(XY) = **O**, when the two matrices X and Y are not conformable for multiplication.

If C = (AB)(B'A); D = (B'A)(AB) then, find the value of $\left(\dfrac{n(C)\left((|D|^2 + n(D)\right)}{n(A) - n(B)}\right)$.

MT0070

11. $A_{3 \times 3}$ is a matrix such that |A|=a, B = (adj A) such that |B|= b. Find the value of $(ab^2 + a^2b + 1)S$ where $\frac{1}{2}S = \frac{a}{b} + \frac{a^2}{b^3} + \frac{a^3}{b^5} + \ldots\ldots$ up to ∞, and a = 3.

MT0071

12. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$, then solve the following matrix equation.

(a) AX = B – I (b) (B – I)X = IC (c) CX = A

MT0072

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13. Determine the values of a and b for which the system $\begin{bmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ 3 \\ -1 \end{bmatrix}$

(a) has a unique solution ; (b) has no solution and (c) has infinitely many solutions

MT0073

14. Find the product of two matrices A & B, where $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ and use it to solve the following system of linear equations,

x + y + 2z = 1; 3x + 2y + z = 7; 2x + y + 3z = 2.

MT0074

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15. Let A be the 2 × 2 matrices given by $A = \left[a_{ij}\right]$, where $a_{ij} \in \{0,1,2,3,4\}$ such that $a_{11} + a_{12} + a_{21} + a_{22} = 4$

(i) Find the number of matrices A such that the trace of A is equal to 4.

(ii) Find the number of matrices A such that A is invertible.

(iii) Find the absolute value of the difference between maximum value and minimum value of det (A).

(iv) Find the number of matrices A such that A is either symmetric or skew-symmetric or both and det(A) is divisible by 2.

MT0075

EXERCISE (JM)

1. Let A be a 2 × 2 matrix **[AIEEE- 2009]**

Statement–1 : adj (adj A) = A

Statement–2 : |adj A| = |A|

(1) Statement–1 is true, Statement–2 is false.

(2) Statement–1 is false, Statement–2 is true.

(3) Statement–1 is true, Statement–2 is true; Statement–2 is a correct explanation for Statement–1.

(4) Statement–1 is true, Statement–2 is true; Statement–2 is not a correct explanation for statement–1.

MT0076

2. The number of 3 × 3 non-singular matrices, with four entries as 1 and all other entries as 0, is :- **[AIEEE-2010]**

(1) Less than 4 (2) 5 (3) 6 (4) At least 7

MT0077

3. Let A be a 2 × 2 matrix with non-zero entries and let $A^2 = I$, where I is 2 × 2 identity matrix. Define Tr(A) = sum of diagonal elements of A and |A| = determinant of matrix A. **[AIEEE-2010]**

Statement–1 : Tr(A) = 0.

Statement–2 : |A| = 1.

(1) Statement–1 is true, Statement–2 is true; Statement–2 is a correct explanation for Statement–1.

(2) Statement–1 is true, Statement–2 is true; Statement–2 is not a correct explanation for statement–1.

(3) Statement–1 is true, Statement–2 is false.

(4) Statement–1 is false, Statement–2 is true.

MT0078

4. Let A and B be two symmetric matrices of order 3.

Statement-1 : A(BA) and (AB)A are symmetric matrices.

Statement-2 : AB is symmetric matrix if matrix multiplication of A with B is commutative. **[AIEEE-2011]**

(1) Statement-1 is true, Statement-2 is false.

(2) Statement-1 is false, Statement-2 is true

(3) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(4) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1.

MT0079

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5. **Statement-1 :** Determinant of a skew-symmetric matrix of order 3 is zero.
Statement-1 : For any matrix A, $\det(A^T) = \det(A)$ and $\det(-A) = -\det(A)$.
Where det(B) denotes the determinant of matrix B. Then : **[AIEEE-2011]**
(1) Statement-1 is true and statement-2 is false
(2) Both statements are true
(3) Both statements are false
(4) Statement-1 is false and statement-2 is true.

MT0080

6. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$. If u_1 and u_2 are column matrices such that $Au_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $Au_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then $u_1 + u_2$ is equal to : **[AIEEE-2012]**

(1) $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ (2) $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ (3) $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ (4) $\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$

MT0081

7. If $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is the adjoint of a 3×3 matrix A and $|A| = 4$, then α is equal to

[JEE(Main) - 2013]

(1) 4 (2) 11 (3) 5 (4) 0

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MT0082

8. If A is an 3×3 non-singular matrix such that $AA' = A'A$ and $B = A^{-1}A'$, the BB' equals :

[JEE(Main) - 2014]

(1) $I + B$ (2) I (3) B^{-1} (4) $(B^{-1})'$

MT0083

9. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where I is 3×3 identity matrix, then the ordered pair (a, b) is equal to : **[JEE(Main)-2015]**

(1) (2, 1) (2) (–2, –1) (3) (2, –1) (4) (–2, 1)

MT0084

10. If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and $A \text{ adj } A = A A^T$, then $5a + b$ is equal to : **[JEE(Main)-2016]**

(1) 13 (2) –1 (3) 5 (4) 4

MT0085

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11. If $A=\begin{bmatrix}2 & -3\\ -4 & 1\end{bmatrix}$, then adj $(3A^2 + 12A)$ is equal to :- **[JEE(Main)-2017]**

(1) $\begin{bmatrix}72 & -63\\ -84 & 51\end{bmatrix}$ (2) $\begin{bmatrix}72 & -84\\ -63 & 51\end{bmatrix}$ (3) $\begin{bmatrix}51 & 63\\ 84 & 72\end{bmatrix}$ (4) $\begin{bmatrix}51 & 84\\ 63 & 72\end{bmatrix}$

MT0086

12. Let A and B be two invertible matrices of order 3 × 3. If $\det(ABA^T) = 8$ and $\det(AB^{-1}) = 8$, then $\det(BA^{-1}B^T)$ is equal to :- **[JEE(Main) -2019]**

(1) 16 (2) $\frac{1}{16}$ (3) $\frac{1}{4}$ (4) 1

MT0087

13. Let $P=\begin{bmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ 9 & 3 & 1\end{bmatrix}$ and $Q = [q_{ij}]$ be two 3×3 matrices such that $Q-P^5 = I_3$. Then $\frac{q_{21}+q_{31}}{q_{32}}$ is equal to :

[JEE(Main) -2019]

(1) 15 (2) 9 (3) 135 (4) 10

MT0088

14. Let $A=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}, (\alpha \in R)$ such that $A^{32}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. Then a value of α is

[JEE(Main) -2019]

(1) $\frac{\pi}{16}$ (2) 0 (3) $\frac{\pi}{32}$ (4) $\frac{\pi}{64}$

MT0089

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15. If $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}\cdots\cdots\begin{bmatrix}1 & n-1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 78\\ 0 & 1\end{bmatrix}$, then the inverse of $\begin{bmatrix}1 & n\\ 0 & 1\end{bmatrix}$ is **[JEE(Main) -2019]**

(1) $\begin{bmatrix}1 & -13\\ 0 & 1\end{bmatrix}$ (2) $\begin{bmatrix}1 & 0\\ 12 & 1\end{bmatrix}$ (3) $\begin{bmatrix}1 & -12\\ 0 & 1\end{bmatrix}$ (4) $\begin{bmatrix}1 & 0\\ 13 & 1\end{bmatrix}$

MT0090

16. If a is A symmetric matrix and B is a skew-symmetric matrix such that $A+B=\begin{bmatrix}2 & 3\\ 5 & -1\end{bmatrix}$, then AB is equal to : **[JEE(Main) -2019]**

(1) $\begin{bmatrix}-4 & 2\\ 1 & 4\end{bmatrix}$ (2) $\begin{bmatrix}-4 & -2\\ -1 & 4\end{bmatrix}$ (3) $\begin{bmatrix}4 & -2\\ -1 & -4\end{bmatrix}$ (4) $\begin{bmatrix}4 & -2\\ 1 & -4\end{bmatrix}$

MT0091

17. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two 3 × 3 real matrices such that $b_{ij} = (3)^{(i+j-2)}a_{ji}$, where i, j = 1, 2, 3. If the determinant of B is 81, then the determinant of A is : **[JEE(Main) -2020]**

(1) 3 (2) 1/3 (3) 1/81 (4) 1/9

MT0092

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18. Let α be a root of the equation $x^2 + x + 1 = 0$ and the matrix $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha^4 \end{bmatrix}$, then the matrix A^{31} is equal to: **[JEE(Main) -2020]**

(1) A^3 (2) A (3) A^2 (4) I_3

MT0093

19. If $A = \begin{pmatrix} 2 & 2 \\ 9 & 4 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $10A^{-1}$ is equal to **[JEE(Main) -2020]**

(1) $4I - A$ (2) $A - 6I$ (3) $6I - A$ (4) $A - 4I$

MT0094

20. The number of all 3×3 matrices A, with enteries from the set $\{-1,0,1\}$ such that the sum of the diagonal elements of AA^T is 3, is **[JEE(Main) -2020]**

MT0095

21. If the matrices $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 1 & -1 & 3 \end{bmatrix}$, B = adjA and C = 3A, then $\frac{|\text{adj}B|}{|C|}$ is equal to : **[JEE(Main) -2020]**

(1) 72 (2) 2 (3) 8 (4) 16

MT0096

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EXERCISE (JA)

Comprehension (3 questions)

1. Let $\mathcal{A}$ be the set of all 3×3 symmetric matrices all of whose entries are either 0 or 1. Five of these entries are 1 and four of them are 0.

(a) The number of matrices in $\mathcal{A}$ is -

(A) 12 (B) 6 (C) 9 (D) 3

(b) The number of matrices A in $\mathcal{A}$ for which the system of linear equations

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

has a unique solution, is -

(A) less than 4 (B) at least 4 but less than 7

(C) at least 7 but less than 10 (D) at least 10

(c) The number of matrices A in $\mathcal{A}$ for which the system of linear equations

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is inconsistent, is -

(A) 0 (B) more than 2 (C) 2 (D) 1

[JEE 2009, 4+4+4]

MT0097

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2. (a) The number of 3×3 matrices A whose entries are either 0 or 1 and for which the system

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ has exactly two distinct solutions, is

(A) 0 (B) $2^9 - 1$ (C) 168 (D) 2

MT0098

(b) Let k be a positive real number and let

$$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}.$$

If det (adj A) + det(adj B) = 10^6, then [k] is equal to

[**Note :** adj M denotes the adjoint of a square matrix M and [k] denotes the largest integer less than or equal to k].

MT0099

(c) Let p be an odd prime number and T_p be the following set of 2×2 matrices :

$$T_p = \left\{ A = \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \{0, 1, 2, \ldots\ldots, p-1\} \right\}$$

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(i) The number of A in T_p such that A is either symmetric or skew-symmetric or both, and det(A) divisible by p is -

(A) $(p-1)^2$ (B) $2(p-1)$ (C) $(p-1)^2 + 1$ (D) $2p - 1$

(ii) The number of A in T_p such that the trace of A is not divisible by p but det (A) is divisible by p is -

[**Note :** The trace of a matrix is the sum of its diagonal entries.]

(A) $(p-1)(p^2 - p + 1)$ (B) $p^3 - (p-1)^2$

(C) $(p-1)^2$ (D) $(p-1)(p^2 - 2)$

(iii) The number of A in T_p such that det (A) is not divisible by p is -

(A) $2p^2$ (B) $p^3 - 5p$ (C) $p^3 - 3p$ (D) $p^3 - p^2$

[JEE 2010, 3+3+3+3+3]

MT0100

3. Let M and N be two 3×3 non-singular skew-symmetric matrices such that MN = NM. If P^T denotes the transpose of P, then $M^2N^2(M^TN)^{-1}(MN^{-1})^T$ is equal to - **[JEE 2011, 4]**

(A) M^2 (B) $-N^2$ (C) $-M^2$ (D) MN

MT0101

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4. Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$,

where each of a,b and c is either ω or ω^2. Then the number of distinct matrices in the set S is-

(A) 2 (B) 6 (C) 4 (D) 8

[JEE 2011, 3, (–1)]

MT0102

5. Let M be 3×3 matrix satisfying $M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$, $M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$

Then the sum of the diagonal entries of M is **[JEE 2011, 4]**

MT0103

6. Let $P = [a_{ij}]$ be a 3×3 matrix and let $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j}a_{ij}$ for $1 \le i, j \le 3$. If the determinant of P is 2, then the determinant of the matrix Q is - **[JEE 2012, 3M, –1M]**

(A) 2^{10} (B) 2^{11} (C) 2^{12} (D) 2^{13}

MT0104

7. If P is a 3×3 matrix such that $P^T = 2P + I$, where P^T is the transpose of P and I is the 3×3 identity matrix, then there exists a column matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that **[JEE 2012, 3M, –1M]**

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(A) $PX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (B) $PX = X$ (C) $PX = 2X$ (D) $PX = -X$

MT0105

8. If the adjoint of a 3×3 matrix P is $\begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$, then the possible value(s) of the determinant of P is (are) - **[JEE 2012, 4M]**

(A) –2 (B) –1 (C) 1 (D) 2

MT0106

9. For 3×3 matrices M and N, which of the following statement(s) is (are) **NOT** correct ?

(A) N^TMN is symmetric or skew symmetric, according as M is symmetric or skew symmetric

(B) MN – NM is skew symmetric for all symmetric matrices M and N

(C) MN is symmetric for all symmetric matrices M and N

(D) (adj M) (adj N) = adj (M N) for all invertible matrices M and N

[JEE-Advanced 2013, 4, (–1)]

MT0107

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10. Let M be a 2×2 symmetric matrix with integer entries. Then M is invertible if
(A) the first column of M is the transpose of the second row of M
(B) the second row of M is the transpose of the first column of M
(C) M is a diagonal matrix with nonzero entries in the main diagonal
(D) the product of entries in the main diagonal of M is not the square of an integer

[JEE(Advanced)-2014, 3]

MT0108

11. Let M and N be two 3×3 matrices such that MN = NM. Further, if $M \neq N^2$ and $M^2 = N^4$, then
(A) determinant of $(M^2 + MN^2)$ is 0
(B) there is a 3×3 non-zero matrix U such that $(M^2 + MN^2)U$ is zero matrix
(C) determinant of $(M^2 + MN^2) \geq 1$
(D) for a 3×3 matrix U, if $(M^2 + MN^2)U$ equals the zero matrix then U is the zero matrix

[JEE(Advanced)-2014, 3]

MT0109

12. Let X and Y be two arbitrary, 3×3, non-zero, skew-symmetric matrices and Z be an arbitrary 3×3, non-zero, symmetric matrix. Then which of the following matrices is (are) skew symmetric ? **[JEE(Advanced)-2015, 4M, –2M]**
(A) $Y^3Z^4 - Z^4Y^3$ (B) $X^{44} + Y^{44}$ (C) $X^4Z^3 - Z^3X^4$ (D) $X^{23} + Y^{23}$

MT0110

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13. Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$, where $\alpha \in \mathbb{R}$, Suppose $Q = [q_{ij}]$ is a matrix such that $PQ = kI$, where $k \in \mathbb{R}$, $k \neq 0$ and I is the identity matrix of order 3. If $q_{23} = -\frac{k}{8}$ and $\det(Q) = \frac{k^2}{2}$, then-
(A) $\alpha = 0, k = 8$
(B) $4\alpha - k + 8 = 0$
(C) $\det(P\,\text{adj}(Q)) = 2^9$
(D) $\det(Q\,\text{adj}(P)) = 2^{13}$

[JEE(Advanced)-2016, 4(–2)]

MT0111

14. Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3. If $Q = [q_{ij}]$ is a matrix such that $P^{50} - Q = I$, then $\frac{q_{31} + q_{32}}{q_{21}}$ equals **[JEE(Advanced)-2016, 3(–1)]**
(A) 52 (B) 103 (C) 201 (D) 205

MT0112

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15. Which of the following is(are) NOT the square of a 3×3 matrix with real entries ?

[JEE(Advanced)-2017, 4(–2)]

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

MT0113

16. How many 3×3 matrices M with entries from $\{0,1,2\}$ are there, for which the sum of the diagonal entries of M^TM is 5 ? **[JEE(Advanced)-2017, 3(–1)]**

(A) 198 (B) 126 (C) 135 (D) 162

MT0114

17. For a real number α, if the system

$$\begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

of linear equations, has infinitely many solutions, then $1 + \alpha + \alpha^2 =$ **[JEE(Advanced)-2017, 3]**

MT0115

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18. Let S be the set of all column matrices $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1, b_2, b_3 \in \mathbb{R}$ and the system of equations (in real variables)

$$-x + 2y + 5z = b_1$$
$$2x - 4y + 3z = b_2$$
$$x - 2y + 2z = b_3$$

has at least one solution. Then, which of the following system(s) (in real variables) has (have) at least one solution of each $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$? **[JEE(Advanced)-2018, 4(–2)]**

(A) $x + 2y + 3z = b_1$, $4y + 5z = b_2$ and $x + 2y + 6z = b_3$

(B) $x + y + 3z = b_1$, $5x + 2y + 6z = b_2$ and $-2x - y - 3z = b_3$

(C) $-x + 2y - 5z = b_1$, $2x - 4y + 10z = b_2$ and $x - 2y + 5z = b_3$

(D) $x + 2y + 5z = b_1$, $2x + 3z = b_2$ and $x + 4y - 5z = b_3$

MT0116

19. Let P be a matrix of order 3×3 such that all the entries in P are from the set $\{-1, 0, 1\}$. Then, the maximum possible value of the determinant of P is _____ **[JEE(Advanced)-2018, 3(0)]**

MT0117

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ANSWERS

Do yourself-1

1. $\begin{bmatrix} 3 & 5 & 7 \\ 4 & 6 & 8 \end{bmatrix}$ **2.** 6 **3.** 30 **4.** x = 2, y =–1, a = 1, b=–3

5. 8

Do yourself-2

1. $\begin{bmatrix} -7 & -10 & -7 \\ -2 & 6 & 3 \end{bmatrix}$ & 2×3 **2.** $\begin{bmatrix} -8 & -9 \\ -7/2 & -8 \\ -1 & 1 \end{bmatrix}$

Do yourself-3

3. x = –2 **4.** 5 **5.** 0

Do yourself-4

2. C **3.** A **4.** C **5.** B **6.** C **7.** C **8.** 6

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Do yourself-5

1. C **2.** x = 0 **3.** A **4.** D **5.** A **6.** A,C

Do yourself-6

2. $\begin{bmatrix} 1 & 6 & -9 \\ 1 & 6 & -3 \end{bmatrix}$ **5.** C **6.** C **7.** 1

Do yourself-7

1. –4 **2.** $\begin{bmatrix} 2 & 7 & 4 \\ 7 & -7 & \frac{1}{2} \\ 4 & \frac{1}{2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & \frac{3}{2} \\ -3 & -\frac{3}{2} & 0 \end{bmatrix}$

3. B **4.** A,B,C **5.** B **6.** A **7.** D **8.** 6

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Do yourself-8

1. C	**2.** A	**3.** 625 B	**4.** 9A	
5. B	**6.** A	**7.** D	**8.** A	**9.** 2

Do yourself-9

1. B	**2.** B	**3.** A	**4.** 2	
5. 1	**6.** 9	**7.** 8	**8.** D	**9.** B

Do yourself-10

1. $\lambda = 0,3$ and 15 tr(A) = 18, $|A| = 0$ **2.** A **3.** B

Do yourself-11

1. D	**2.** A	**3.** 2	**4.** 6
5. D			

EXERCISE (O-1)

1 C	**2.** A	**3.** B	**4.** A	**5.** B	**6.** C	**7.** B
8. A	**9.** A	**10.** D	**11.** B	**12.** C	**13.** A	**14.** B
15. C	**16.** A	**17.** C	**18.** D	**19.** A	**20.** A	

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EXERCISE (O-2)

1. A	**2.** C	**3.** D	**4.** C	**5.** C	**6.** D	**7.** C
8. D	**9.** B,C	**10.** A,C	**11.** A,B,C	**12.** A,B,C,D		**13.** B,D
14. A,C	**15.** **A,B,C,D**		**16.** **A,B,C,D**		**17.** **B,C**	**18.** A,B
19. B,C,D	**20.** A,B,D	**21.** C	**22.** A	**23.** A	**24.** B	**25.** C

EXERCISE (S-1)

1. 8 **2.** $x = \frac{3}{2}, y = 2$ **3.** 10 **4.** 5049

6. $V = \begin{bmatrix} 0 \\ 1 \\ \frac{1}{11} \end{bmatrix}$ **7.** 1 **8.** AB is neither symmetric nor skew symmetric

9. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 5 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & -3 \\ 2 & -3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \\ -3 & 3 & 0 \end{bmatrix}$

10. (a) 4, (b) 8 **11.** n = 7 **12.** 7

13. f(a) = 1/4, a = 1/2 **15.** $\begin{bmatrix} 1 & 0 \\ 4014 & 1 \end{bmatrix}$ **16.** 100

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EXERCISE (S-2)

1. – 8

3. (a) $\begin{bmatrix} -4 & 7 & -7 \\ 3 & -5 & 5 \end{bmatrix}$; (b) $\frac{1}{19}\begin{bmatrix} 48 & -25 \\ -70 & 42 \end{bmatrix}$

4. 1

5. $x_1 = 1,\ x_2 = -1,\ x_3 = 1$

6. 2

7. $3(49.3^{50} + 1)$

8. $\begin{bmatrix} 17 & 4 & -19 \\ -10 & 0 & 13 \\ -21 & -3 & 25 \end{bmatrix}$

9. (i) $X = \begin{bmatrix} a & b \\ 2-2a & 1-2b \end{bmatrix}$ for $a, b \in R$; (ii) X does not exist;

(iii) $X = \begin{bmatrix} a & -3a \\ c & -3c \end{bmatrix}$ $a, c \in R$ and $3a + c \neq 0$; $3b + d \neq 0$

10. 650

11. 225

12. (a) $X = \begin{bmatrix} -3 & -3 \\ \frac{5}{2} & 2 \end{bmatrix}$, (b) $X = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$, (c) no solution

13. (i) $a \neq -3, b \in R$; (ii) $a = -3$ and $b \neq 1/3$; (iii) $a = -3,\ b = 1/3$

14. $x = 2, y = 1, z = -1$

15. (i) 5, (ii) 18, (iii) 8, (iv) 5

EXERCISE (JM)

1. 4 **2.** 4 **3.** 3 **4.** 4 **5.** 1 **6.** 1 **7.** 2 **8.** 2 **9.** 2 **10.** 3
11. 3 **12.** 2 **13.** 4 **14.** 4 **15.** 1 **16.** 3 **17.** 4 **18.** 1 **19.** 2
20. 672. **21.** 3

EXERCISE (JA)

1. (a) A, (b) B,(c) B **2.** **(a)** A, **(b)** 4; **(c)** **(i)** D, **(ii)** C, **(iii)** D **3.** Bonus **4.** A
5. 9 **6.** D **7.** D **8.** A,D **9.** C,D **10.** C,D **11.** A,B
12. C,D **13.** B,C **14.** B **15.** A,B **16.** A **17.** 1 **18.** A,D **19.** 4

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FUNCTION

1. CARTESIAN PRODUCT OF TWO SETS :

Given two non-empty sets A and B. The cartesian product $A \times B$ is the set of all ordered pairs of the form (a, b) where the first entry comes from set A & second comes from set B.

$A \times B = \{(a, b) \mid a \in A, b \in B\}$

e.g. $A = \{1, 2, 3\}$ $B = \{p, q\}$

$A \times B = \{(1, p),(1, q), (2, p), (2, q), (3, p), (3, q)\}$

Note :

(i) If either A or B is the null set, then $A \times B$ will also be empty set, i.e. $A \times B = \phi$

(ii) If $n(A) = p, n(B) = q$, then $n(A \times B) = pq$, where n(X) denotes the number of elements in set X.

(iii) A **Relation** R from set A to B is any subset of $A \times B$. If $A R B$ & $(a, b) \in R$ then b is image of a under R and a is preimage of b under R.

Note : If $n(A) = m$, $n(B) = n$, then number of relations defined from set A to B are 2^{mn}.

2. FUNCTION :

A relation R from set A to set B is called a function if each element of A is uniquely associated with some element of B. It is denoted by the symbol :

$$f : A \to B \text{ or } A \xrightarrow{f} B$$

which reads 'f' is a function from A to B 'or' f maps A to B,

If an element $a \in A$ is associated with an element $b \in B$, then b is called 'the f image of a' or 'image of a under f' or 'the value of the function f at a'. Also a is called the 'pre-image of b' or 'argument of b under the function f'. We write it as

$b = f(a)$ or $f : a \to b$ or $f : (a, b)$

Thus a function 'f' from set A to set B is subset of $A \times B$ in which each a belonging to A appears in one and only one ordered pair belonging to f.

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Representation of Function :

(a) Ordered pair : Every function from $A \to B$ satisfies the following conditions :

(i) $f \subset A \times B$ (ii) $\forall\ a \in A$ there exist $b \in B$ and (iii) $(a, b) \in f$ & $(a, c) \in f \Rightarrow b = c$

(b) Formula based (uniformly/nonuniformly) :

(i) $f : \mathbb{R} \to \mathbb{R}$, $y = f(x) = 4x$, $f(x) = x^2$ (uniformly defined)

(ii) $f(x) = \begin{cases} x+1 & -1 \le x < 4 \\ -x & 4 \le x < 7 \end{cases}$ (non-uniformly defined)

(iii) $f(x) = \begin{cases} x^2 & x \ge 0 \\ -x-1 & x < 0 \end{cases}$ (non-uniformly defined)

(c) Graphical representation :

If a vertical line cuts a given graph at more than one point then it can not be the graph of a function.

Graph (1)

Graph (2)

Graph(1) represent a function but graph(2) does not represent a function.

Every function is a relation but every relation is not necessarily a function.

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3. DOMAIN, CO-DOMAIN & RANGE OF A FUNCTION :

Let $f : A \to B$, then the set A is known as the domain of f & the set B is known as co-domain of f. The set of f images of all the elements of A is known as the range of f.

Domain of $f = \{a \mid a \in A, (a, f(a)) \in f\}$

Range of $f = \{f(a) \mid a \in A, f(a) \in B\}$

(a) If only the rule of function is given then the domain of the function is the set of those real numbers, where function is defined.

(b) For a continuous function, the interval from minimum to maximum value of a function gives the range

(c) It should be noted that range is a subset of co-domain.

Note :

(i) The complete set of all positive real numbers is denoted by $\mathbb{R}^+$ or $\mathbb{R}_{>0}$

(ii) The complete set of all negative real numbers is denoted by $\mathbb{R}^-$ or $\mathbb{R}_{<0}$

(iii) The complete set of all real numbers other then zero is denoted by $\mathbb{R}^*$ or $\mathbb{R}_{\neq 0}$

(iv) The complete set of all integers is denoted by $\mathbb{Z}$.

Illustration 1 : Find the domain of following functions :

(i) $y = \sqrt{5-2x}$ (ii) $y = \dfrac{1}{\sqrt{x-|x|}}$

Solution : (i) $5 - 2x \geq 0 \Rightarrow x \leq \dfrac{5}{2}$ $\therefore$ Domain is $(-\infty, 5/2]$

(ii) $x - |x| > 0 \Rightarrow |x| < x \Rightarrow$ x cannot take any real values $\therefore$ Domain is ϕ

Illustration 2 : Find the range of following functions :

(i) $f(x) = \log_{\sqrt{2}}\left((x-1)^2+4\right)$ (ii) $f(x) = 3 - \cos x$

Solution : (i) $f(x) = \log_{\sqrt{2}}\left((x-1)^2+4\right)$

$4 \leq (x-1)^2 + 4 < \infty$

$\Rightarrow \log_{\sqrt{2}} 4 \leq \log_{\sqrt{2}}(x-1)^2 + 4 < \infty$

$\Rightarrow 4 \leq \log_{\sqrt{2}}(x-1)^2 + 4 < \infty$

$\therefore$ Range of $f(x) = [4, \infty)$

(ii) $f(x) = 3 - \cos x$

$-1 \leq \cos x \leq 1$

$2 \leq 3 - \cos x \leq 4$

$\therefore$ Range of $f(x) = [2, 4]$

Do yourself - 1 :

(i) Find the domain of following functions :

(a) $y = 1 - \log_{10} x$ (b) $y = \dfrac{1}{\sqrt{x^2 - 4x}}$

(ii) Find the range of the following function :

(a) $\log_4 \left|x + \dfrac{1}{x}\right|$ (b) $f(x) = \sin(3x^2 + 1)$

(c) $f(x) = 2\sin\left(2x + \dfrac{\pi}{4}\right)$ (d) $f(x) = \cos\left(2x + \dfrac{\pi}{4}\right)$

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4. IMPORTANT TYPES OF FUNCTIONS :

(a) Polynomial Function :

If a function f is defined by $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_{n-1} x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, ..., a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n. If n is odd, then polynomial is of odd degree, if n is even, then polynomial is of even degree.

Note :

(i) Range of odd degree polynomial is always $\mathbb{R}$.

(ii) Range of even degree polynomial is never $\mathbb{R}$.

(iii) A polynomial of degree one with no constant term is called an odd linear function. i.e. $f(x) = ax$, $a \neq 0$

(iv) $f(x) = ax + b$, $a \neq 0$ is a linear polynomial

(v) $f(x) = c$ is non linear polynomial (its degree is zero)

(vi) $f(x) = 0$ is a polynomial but its degree is **not** defined

(vii) There are four polynomial functions, satisfying the relation ; $f(x) . f(1/x) = f(x) + f(1/x)$. They are : (a) $f(x) = 0$ (b) $f(x) = 2$ (c) $f(x) = x^n + 1$ & (d) $f(x) = 1 - x^n$, where n is a positive integer.

(b) Algebraic Function :

y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form $P_0(x) y^n + P_1(x) y^{n-1} + ... + P_{n-1}(x) y + P_n(x) = 0$. Where n is a positive integer and $P_0(x)$, $P_1(x)$..... are Polynomials in x. e.g. $x^3 + y^3 - 3xy = 0$.

In other words a function 'f' is called an algebraic function if it can be constructed using finite number of algebraic operations (such as addition, subtraction, multiplication, division, and taking radicals) within polynomials.

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Example : $f(x) = \sqrt{x^2+1}$; $g(x) = \dfrac{x^4 - 16x^2}{x + \sqrt{x}} + (x-2)\sqrt[3]{x+1}$

Note :

(i) All polynomial functions are Algebraic but not the converse.

(ii) A function that is not algebraic is called **Transcendental Function.**

Basic algebraic function :

(i) $y = x^2$

Domain : $\mathbb{R}$

Range : $\mathbb{R}^+ \cup \{0\}$ or $[0, \infty)$

(ii) $y = \dfrac{1}{x}$

Domain : $\mathbb{R} - \{0\}$ or $\mathbb{R}_0$

Range : $\mathbb{R} - \{0\}$

(iii) $y = \dfrac{1}{x^2}$

Domain : $\mathbb{R}_0$

Range : $\mathbb{R}^+$ or $(0, \infty)$

(iv) $y = x^3$

Domain : $\mathbb{R}$

Range : $\mathbb{R}$

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(c) Rational function :

A rational function is a function of the form $y = f(x) = \dfrac{g(x)}{h(x)}$, where g(x) & h(x) are polynomials &
$h(x) \neq 0$, **Domain :** $\mathbb{R} - \{x \mid h(x)=0\}$

Any rational function is automatically an algebraic function.

(d) Trigonometric functions :

(i) Sine function

$f(x) = \sin x$

Domain : $\mathbb{R}$

Range : $[-1, 1]$, period 2π

(ii) Cosine function

$f(x) = \cos x$

Domain : $\mathbb{R}$

Range : $[-1, 1]$, period 2π

(iii) Tangent function

$f(x) = \tan x$

Domain : $\mathbb{R} - \left\{x \mid x = \dfrac{(2n+1)\pi}{2}, n \in I\right\}$

Range : $\mathbb{R}$, period π

(iv) Cosecant function

$f(x) = \operatorname{cosec} x$

Domain : $\mathbb{R} - \{x \mid x = n\pi, n \in I\}$

Range : $\mathbb{R} - (-1, 1)$, period 2π

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(v) Secant function

$f(x) = \sec x$

Domain : $\mathbb{R} - \{x | x = (2n + 1)\ \pi/2 : n \in I\}$

Range : $\mathbb{R} - (-1, 1)$, period 2π

(vi) Cotangent function

$f(x) = \cot x$

Domain : $\mathbb{R} - \{x | x = n\pi,\ n \in I\}$

Range : $\mathbb{R}$, period π

(e) Exponential and Logarithmic Function :

A function $f(x) = a^x (a > 0)$, $a \neq 1$, $x \in \mathbb{R}$ is called an exponential function. The inverse of the exponential function is called the logarithmic function, i.e. $g(x) = \log_a x$.

Note that f(x) & g(x) are inverse of each other & their graphs are as shown. (If functions are mirror image of each other about the line y = x)

Domain of a^x is $\mathbb{R}$ **Range** $\mathbb{R}^+$

Domain of $\log_a x$ is $\mathbb{R}^+$ **Range** $\mathbb{R}$

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Domain : $\mathbb{R}$

Range : $\mathbb{R}^+$

Note-1 : $f(x) = a^{1/x}$, $a > 0$ **Domain :** $\mathbb{R} - \{0\}$ **Range :** $\mathbb{R}^+ - \{1\}$

Note-2 : $f(x) = \log_x a = \dfrac{1}{\log_a x}$ **Domain :** $\mathbb{R}^+ - \{1\}$ **Range :** $\mathbb{R} - \{0\}$

$(a > 0)$ $(a \neq 1)$

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(f) Absolute Value Function :

A function $y = f(x) = |x|$ is called the absolute value function or Modulus function. It is defined as :

$$y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

For $f(x) = |x|$, domain is $\mathbb{R}$ and range is $[0,\infty)$

For $f(x) = \dfrac{1}{|x|}$, domain is $\mathbb{R} - \{0\}$ and range is $\mathbb{R}^+$.

(g) Signum Function :

A function $y = f(x) = \text{Sgn}(x)$ is defined as follows :

$$y = f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$

It is also written as $\text{Sgn } x = |x|/x$; $x \neq 0$;

$= 0$; $x = 0$

Note : $f(x) = (\text{sgn}(x))x \Rightarrow f(x) = |x|$

Domain : $\mathbb{R}$

Range : $\{-1, 0, 1\}$

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(h) Greatest integer or step up function :

The function y = f(x) = [x] is called the greatest integer function where [x] denotes the greatest integer less than or equal to x. Note that for :

x	[x]
[–2,–1)	–2
[–1,0)	–1
[0,1)	0
[1,2)	1

Domain : $\mathbb{R}$

Range : I

Properties of greatest integer function :

(i) $[x] \le x < [x] + 1$ and $x - 1 < [x] \le x$, $0 \le x - [x] < 1$

(ii) $[x + m] = [x] + m$, if m is an integer.

(iii) $[x] + [-x] = \begin{cases} 0, & x \in I \\ -1, & x \notin I \end{cases}$

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Illustration 3 : If y = 2[x] + 3 & y = 3[x – 2] + 5, then find [x + y] where [.] denotes greatest integer function.

Solution : y = 3[x – 2] + 5 = 3[x] – 1

so 3 [x] – 1 = 2 [x] + 3

$[x] = 4 \Rightarrow 4 \le x < 5$

then y = 11

so x + y will lie in the interval [15, 16)

so [x + y] = 15

Illustration 4 : Find the value of $\left[\frac{1}{2}\right]+\left[\frac{1}{2}+\frac{1}{1000}\right]+.....\left[\frac{1}{2}+\frac{2946}{1000}\right]$ where [.] denotes greatest integer function ?

Solution : $\left[\frac{1}{2}\right]+\left[\frac{1}{2}+\frac{1}{1000}\right]+.....\left[\frac{1}{2}+\frac{499}{1000}\right]+\left[\frac{1}{2}+\frac{500}{1000}\right]+.....\left[\frac{1}{2}+\frac{1499}{1000}\right]+\left[\frac{1}{2}+\frac{1500}{1000}\right]+.....$

$+\left[\frac{1}{2}+\frac{2499}{1000}\right]+\left[\frac{1}{2}+\frac{2500}{1000}\right]+.....\left[\frac{1}{2}+\frac{2946}{1000}\right]$

$= 0 + 1 \times 1000 + 2 \times 1000 + 3 \times 447 = 3000 + 1341 = 4341$ **Ans.**

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(i) Fractional part function :

It is defined as : $g(x) = \{x\} = x - [x]$

e.g. the fractional part of the number 2.1 is 2.1–2 = 0.1 and the fractional part of –3.7 is 0.3 The period of this function is 1 and graph of this function is as shown.

x	{x}
[–2,–1)	x+2
[–1,0)	x+1
[0,1)	x
[1,2)	x–1

Domain : $\mathbb{R}$

Range : $[0,1)$

Properties of fractional part function :

(i) $0 \le \{x\} < 1$ (ii) $\{[x]\} = [\{x\}] = 0$ (iii) $\{\{x\}\} = \{x\}$

(iv) $\{x+m\} = \{x\},\ m \in I$ (v) $\{x\} + \{-x\} = \begin{cases} 1, & x \notin I \\ 0, & x \in I \end{cases}$

Illustration 5 : Solve the equation $|2x - 1| = 3[x] + 2\{x\}$ where [.] denotes greatest integer and {.} denotes fractional part function.

Solution : We are given that, $|2x - 1| = 3[x] + 2\{x\}$

Let, $2x - 1 \le 0$ i.e. $x \le \frac{1}{2}$. The given equation yields.

$1 - 2x = 3[x] + 2\{x\}$

$\Rightarrow\ 1 - 2[x] - 2\{x\} = 3[x] + 2\{x\} \Rightarrow\ 1 - 5[x] = 4\{x\} \Rightarrow \{x\} = \frac{1-5[x]}{4}$

$\Rightarrow\ 0 \le \frac{1-5[x]}{4} < 1 \Rightarrow 0 \le 1 - 5[x] < 4 \Rightarrow -\frac{3}{5} < [x] \le \frac{1}{5}$

Now, $[x] = 0$ as zero is the only integer lying between $-\frac{3}{5}$ and $\frac{1}{5}$

$\Rightarrow\ \{x\} = \frac{1}{4} \Rightarrow x = \frac{1}{4}$ which is less than $\frac{1}{2}$, Hence $\frac{1}{4}$ is one solution.

Now, let $2x - 1 > 0$ i.e $x > \frac{1}{2}$

$\Rightarrow\ 2x - 1 = 3[x] + 2\{x\}\ \Rightarrow\ 2[x] + 2\{x\} - 1 = 3[x] + 2\{x\}$

$\Rightarrow\ [x] = -1 \Rightarrow -1 \le x < 0$ which is not a solution as $x > \frac{1}{2}$

$\Rightarrow\ x = \frac{1}{4}$ is the only solution.

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(j) Identity function :

The function $f: A \to A$ defined by $f(x) = x \ \forall \ x \in A$ is called the identity of A and is denoted by I_A. It is easy to observe that identity function defined on R is a bijection.

(k) Constant function :

A function $f: A \to B$ is said to be a constant function if every element of A has the same f image in B. Thus $f: A \to B$; $f(x) = c$, $\forall \ x \in A$, $c \in B$ is a constant function. Note that the range of a constant function is a singleton.

Do yourself - 2 :

(i) Let {x} & [x] denotes the fraction and integral part of a real number x respectively, then match the column.

	Column-I		Column-II
(A)	$[x^2] \geq 4$	(p)	$x \in [2, 4)$
(B)	$[x]^2 - 5[x] + 6 = 0$	(q)	$x \in (-\infty, -2] \cup [2, \infty)$
(C)	$x = \{x\}$	(r)	$x \in (-\infty, -5)$
(D)	$[x] < -5$	(s)	$x \in \{-2\}$
		(t)	$x \in [0, 1)$

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5. ALGEBRAIC OPERATIONS ON FUNCTIONS :

If f & g are real valued functions of x with domain set A, B respectively, f + g, f – g, (f. g) & (f/g) as follows :

(a) $(f \pm g)(x) = f(x) \pm g(x)$ domain in each case is $A \cap B$

(b) $(f.g)(x) = f(x).g(x)$ domain is $A \cap B$

(c) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ domain $A \cap B - \{x | g(x) = 0\}$

Illustration 6 : Find the domain of the following function :

(i) $y = \log_{(x-4)} (x^2 - 11x + 24)$ (ii) $f(x) = \log_2 \left(-\log_{1/2}\left(1 + \frac{1}{\sqrt[4]{x}}\right) - 1\right)$

Solution : (i) $y = \log_{(x-4)}(x^2 - 11x + 24)$

Here 'y' would assume real value if,

$x - 4 > 0$ and $\neq 1$, $x^2 - 11x + 24 > 0 \Rightarrow x > 4$ and $\neq 5$, $(x - 3)(x - 8) > 0$

$\Rightarrow x > 4$ and $\neq 5$, $x < 3$ or $x > 8 \Rightarrow x > 8 \Rightarrow$ Domain (y) = $(8, \infty)$

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(ii) We have $f(x) = \log_2\left(-\log_{1/2}\left(1+\frac{1}{\sqrt[4]{x}}\right)-1\right)$

f(x) is defined if $-\log_{1/2}\left(1+\frac{1}{\sqrt[4]{x}}\right)-1>0$

or if $\log_{1/2}\left(1+\frac{1}{\sqrt[4]{x}}\right)<-1$ or if $\left(1+\frac{1}{\sqrt[4]{x}}\right)>(1/2)^{-1}$

or if $1+\frac{1}{\sqrt[4]{x}}>2$ or if $\frac{1}{\sqrt[4]{x}}>1$ or if $x^{1/4}<1$ or if $0<x<1$

$\therefore$ D(f) = (0, 1)

Illustration 7 : Find the domain $f(x) = \frac{1}{\sqrt{|[|x|-5]|-11}}$ where [.] denotes greatest integer function.

Solution : $|[|x|-5]|>11$

so $[|x|-5]>11$ or $[|x|-5]<-11$

$[|x|]>16$ $[|x|]<-6$

$|x|\geq 17$ or $|x|<-6$ (Not Possible)

$\Rightarrow$ $x \leq -17$ or $x \geq 17$

so $x \in (-\infty, -17] \cup [17, \infty)$

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Illustration 8 : Find the range of following functions :

(i) $f(x) = \frac{1}{8-3\sin x}$ (ii) $f(x) = \log_{\sqrt{2}}\left(2-\log_2(16\sin^2 x+1)\right)$

Solution : (i) $f(x) = \frac{1}{8-3\sin x}$

$-1 \leq \sin x \leq 1$

$\therefore$ Range of f = $\left[\frac{1}{11}, \frac{1}{5}\right]$

(ii) $f(x) = \log_{\sqrt{2}}\left(2-\log_2(16\sin^2 x+1)\right)$

$1 \leq 16\sin^2 x + 1 \leq 17$

$\therefore$ $0 \leq \log_2(16\sin^2 x+1) \leq \log_2 17$

$\therefore$ $2-\log_2 17 \leq 2-\log_2(16\sin^2 x+1) \leq 2$

Now consider $0 < 2-\log_2(16\sin^2 x+1) \leq 2$

$\therefore$ $-\infty < \log_{\sqrt{2}}[2-\log_2(16\sin^2 x+1)] \leq \log_{\sqrt{2}} 2 = 2$

$\therefore$ the range is $(-\infty, 2]$

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Illustration 9 : Find the range of $f(x) = \dfrac{x-[x]}{1+x-[x]}$, where [.] denotes greatest integer function.

Solution : $y = \dfrac{x-[x]}{1+x-[x]} = \dfrac{\{x\}}{1+\{x\}}$

$\therefore \quad \dfrac{1}{y} = \dfrac{1}{\{x\}} + 1 \Rightarrow \dfrac{1}{\{x\}} = \dfrac{1-y}{y} \Rightarrow \{x\} = \dfrac{y}{1-y}$

$0 \le \{x\} < 1 \;\Rightarrow\; 0 \le \dfrac{y}{1-y} < 1$

Range = [0, 1/2)

Do yourself - 3 :

(i) Find domain of following functions :

(a) $f(x) = \sin\left(\sqrt{1-x^2}\right) + \sqrt{x+2} + \dfrac{1}{\log_{10}^{(x+1)}}$

(b) $f(x) = \sqrt{\dfrac{(2x+1)}{x^3-3x^2+2x}}$

(ii) Find range of following functions :

(a) $f(x) = \log_2\left(\log_{1/2}(x^2+4x+4)\right)$

(b) $f(x) = \dfrac{1}{2-\cos 3x}$

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6. EQUAL OR IDENTICAL FUNCTION :

Two function f & g are said to be equal if :

(a) The domain of f = the domain of g

(b) The co-domain of f = co-domain of g and

(c) f(x) = g (x), for every x belonging to their common domain (i.e. should have the same graph)

Illustration 10 : The functions $f(x) = \log(x-1) - \log(x-2)$ and $g(x) = \log\left(\dfrac{x-1}{x-2}\right)$ are identical when x lies in the interval

(A) [1, 2] (B) [2, ∞) (C) (2, ∞) (D) (–∞, ∞)

Solution : Since f(x) = log (x – 1) –log (x – 2).

Domain of f(x) is x > 2 or $x \in (2, \infty)$ .....(i)

$g(x) = \log\left(\dfrac{x-1}{x-2}\right)$ is defined if $\dfrac{x-1}{x-2} > 0 \;\Rightarrow\; x \in (-\infty, 1) \cup (2, \infty)$ .....(ii)

From (i) and (ii), $x \in (2, \infty)$. **Ans. (C)**

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Do yourself - 4 :

(i) Are the following functions identical ?

(a) $f(x) = \frac{x}{x^2}$ & $\phi(x) = \frac{x^2}{x}$ (b) $f(x) = x$ & $\phi(x) = \sqrt{x^2}$ (c) $f(x) = \log_{10}x^2$ & $\phi(x) = 2\log_{10}|x|$

7. HOMOGENEOUS FUNCTIONS :

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.

For examples $5x^2 + 3y^2 - xy$ is homogenous in x & y. Symbolically if, $f(tx, ty) = t^n f(x, y)$ then f(x, y) is homogeneous function of degree n.

Illustration 11 : Which of the following function is not homogeneous ?

(A) $x^3 + 8x^2y + 7y^3$ (B) $y^2 + x^2 + 5xy$ (C) $\frac{xy}{x^2+y^2}$ (D) $\frac{2x-y+1}{2y-x+1}$

Solution : It is clear that (D) does not have the same degree in each term. **Ans. (D)**

8. BOUNDED FUNCTION :

A function is said to be bounded if there exists a finite M such that $|f(x)| \le M, \forall x \in D_f$.

9. IMPLICIT & EXPLICIT FUNCTION :

A function defined by an equation not solved for the dependent variable is called an **implicit function**. e.g. the equations $x^3 + y^3 = 1$ & $x^y = y^x$, defines y as an implicit function. If y has been expressed in terms of x alone then it is called an **Explicit function.**

Illustration 12 : Which of the following function is implicit function ?

(A) $y = \frac{x^2+e^x+5}{\sqrt{1-\cos^{-1}x}}$ (B) $y = x^2$ (C) $xy - \sin(x+y) = 0$ (D) $y = \frac{x^2\log x}{\sin x}$

Solution : It is clear that in (C) y is not clearly expressed in x. **Ans. (C)**

Do yourself - 5 :

(i) Find the boundness of the function $f(x) = \frac{x^2}{x^4+1}$

(ii) Which of the following function is implicit function ?

(A) $xy - \cos(x+y) = 0$ (B) $y = x^3$

(C) $y = \log(x^2 + x + 1)$ (D) $y = |x|$

(iii) Convert the implicit form into the explicit function :

(a) $xy = 1$ (b) $x^2y = 1$.

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10. APPLICATIONS OF FUNCTIONAL RULE :

Illustration 13 : Determine all functions f satisfying the functional relation.

$$f(x)+f\left(\frac{1}{1-x}\right)=\frac{2(1-2x)}{x(1-x)} \text{ where } x\in\mathbb{R}-\{0,1\}$$

Solution : Given $f(x)+f\left(\frac{1}{1-x}\right)=\frac{2(1-2x)}{x(1-x)}=\frac{2}{x}-\frac{2}{1-x}$ ...(i)

Replacing x by $\frac{1}{1-x}$ we obtain

$$f\left(\frac{1}{1-x}\right)+f\left(\frac{1}{1-\frac{1}{1-x}}\right)=2(1-x)-\frac{2}{1-\frac{1}{1-x}}$$

$$\Rightarrow \quad f\left(\frac{1}{1-x}\right)+f\left(1-\frac{1}{x}\right)=-2x+\frac{2}{x} \qquad ...(ii)$$

Again replacing x by $\left(1-\frac{1}{x}\right)$ in (i) we obtain

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$$\Rightarrow \quad f\left(1-\frac{1}{x}\right)+f\left(\frac{1}{1-\left(1-\frac{1}{x}\right)}\right)=\frac{2}{1-\frac{1}{x}}-\frac{2}{1-\left(1-\frac{1}{x}\right)}$$

$$\Rightarrow \quad f\left(1-\frac{1}{x}\right)+f(x)=\frac{2x}{x-1}-2x \qquad ....(iii)$$

subtracting (ii) from (i) then

$$f(x)-f\left(1-\frac{1}{x}\right)=2x-\frac{2}{1-x}$$

Now adding (iii) and (iv) we get

$$2f(x)=\frac{2x}{x-1}-\frac{2}{1-x}$$

$$\Rightarrow \quad f(x)=\frac{x+1}{x-1}$$

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11. BASIC TRANSFORMATIONS ON GRAPHS :

(i) Drawing the graph of y = f(x) + b, b ∈ ℝ, from the known graph of y = f(x)

It is obvious that domain of f(x) and f(x) + b are the same. Let us take any point x_0 in the domain of f(x). $y\big|_{x=x_0} = f(x_0)$.

The corresponding point on f(x) + b would be $f(x_0) + b$.

For $b > 0 \Rightarrow f(x_0) + b > f(x_0)$ it means that the corresponding point on f(x) + b would be lying at a distance 'b' units above the point on f(x).

For $b < 0 \Rightarrow f(x_0) + b < f(x_0)$ it means that the corresponding point on f(x) + b would be lying at a distance 'b' units below the point on f(x).

Accordingly the graph of f(x) + b can be obtained by translating the graph of f(x) either in the positive y-axis direction (if b > 0) or in the negative y-axis direction (if b < 0), through a distance |b| units.

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(ii) Drawing the graph of y = –f(x) from the known graph of y = f(x)

To draw y = –f(x), take the image of the curve y = f(x) in the x-axis as plane mirror.

(iii) Drawing the graph of y = f(–x) from the known graph of y = f(x)

To draw y = f(–x), take the image of the curve y = f(x) in the y-axis as plane mirror.

(iv) Drawing the graph of y = |f(x)| from the known graph of y = f(x)

|f(x)| = f(x) if f(x) ≥ 0 and |f(x)| = –f(x) if f(x) < 0. It means that the graph of f(x) and |f(x)| would coincide if f(x) ≥ 0 and for the portions where f(x) < 0 graph of |f(x)| would be image of y = f(x) in x-axis.

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(v) Drawing the graph of y = f(|x|) from the known graph of y = f(x)

It is clear that, $f(|x|) = \begin{cases} f(x), & x \geq 0 \\ f(-x), & x < 0 \end{cases}$. Thus f(|x|) would be a even function, graph of f(|x|) and f(x) would be identical in the first and the fourth quadrants (as $x \geq 0$) and as such the graph of f(|x|) would be symmetric about the y-axis (as (|x|) is even).

(vi) Drawing the graph of |y| = f(x) from the known graph of y = f(x)

Clearly $|y| \geq 0$. If $f(x) < 0$, graph of $|y| = f(x)$ would not exist. And if $f(x) \geq 0$, $|y| = f(x)$ would give $y = \pm f(x)$. Hence graph of $|y| = f(x)$ would exist only in the regions where f(x) is non-negative and will be reflected about the x-axis only in those regions.

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(vii) Drawing the graph of y = f(x + a), a ∈ ℝ from the known graph of y = f(x)

(i) If $a > 0$, shift the graph of f(x) through 'a' units towards left of f(x).

(ii) If $a < 0$, shift the graph of f(x) through 'a' units towards right of f(x).

(viii) Drawing the graph of y = af(x) from the known graph of y = f(x)

It is clear that the corresponding points (points with same x co-ordinates) would have their ordinates in the ratio of 1 : a.

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(ix) Drawing the graph of y = f(ax) from the known graph of y = f(x).

Let us take any point $x_0 \in$ domain of f(x). Let $ax = x_0$ or $x = \frac{x_0}{a}$.

Clearly if $0 < a < 1$, then $x > x_0$ and $f(x)$ will stretch by $\frac{1}{a}$ units along the y-axis and if $a > 1$, $x < x_0$, then f(x) will compress by 'a' units along the y-axis.

Note :

(i) A function h(x) is defined as

$h(x) = \max.\{f(x), g(x)\}$ then

$$h(x) = \begin{cases} f(x) & f(x) \ge g(x) \\ g(x) & g(x) > f(x) \end{cases}$$

(ii) A function h(x) is defined as

$h(x) = \min.\{f(x), g(x)\}$ then

$$h(x) = \begin{cases} f(x) & f(x) \le g(x) \\ g(x) & g(x) < f(x) \end{cases}$$

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Illustration 14 : Find f(x) = max {1 + x, 1 – x, 2}.

Solution : From the graph it is clear that

$$f(x) = \begin{cases} 1-x\,; & x < -1 \\ 2\quad ; & -1 \le x \le 1 \\ 1+x\,; & x > 1 \end{cases}$$

Illustration 15 : Draw the graph of $y = |2 - |x - 1||$.

Solution :

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Illustration 16 : Draw the graph of $y = 2 - \dfrac{4}{|x-1|}$

Solution :

Illustration 17 : Draw the graph of $y = |e^{|x|} - 2|$

Solution :

Illustration 18 : Draw the graph of $f(x) = \cos x \cos(x + 2) - \cos^2(x + 1)$.

Solution : $f(x) = \cos x \cos(x + 2) - \cos^2(x + 1)$

$$= \frac{1}{2}[\cos(2x+2) + \cos 2] - \frac{1}{2}[\cos(2x+2) + 1]$$

$$= \frac{1}{2}\cos 2 - \frac{1}{2} < 0.$$

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Do yourself - 6 :

(i) Draw graph of following functions :

(a) $y = |\ell n|x||+1$ (b) $y = \min.\{x^2 + 1, 3-x\}$

12. CLASSIFICATION OF FUNCTIONS :

One-One Function (Injective mapping) :

A function $f : A \to B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B.

Thus there exist $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$, $\mathbf{f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2}$ **or** $\mathbf{x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)}$.

Diagramatically an injective mapping can be shown as

Many-one function (not injective) :

A function $f : A \to B$ is said to be a many one function if two or more elements of A have the same f image in B.

Thus $f : A \to B$ is many one there exist $x_1, x_2 \in A$, $\mathbf{f(x_1) = f(x_2)}$ **but** $\mathbf{x_1 \neq x_2}$.

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Diagramatically a many one mapping can be shown as

Note :

(i) If a line parallel to x-axis cuts the graph of the function atmost at one point, then the function is one-one.

(ii) If any line parallel to x-axis cuts the graph of the function atleast at two points, then f is many-one.

(iii) If continuous function $f(x)$ is always increasing or decreasing in whole domain, then $f(x)$ is one-one.

(iv) All linear functions are one-one.

(v) All trigonometric functions in their domain are many one

(vi) All even degree polynomials are many one

(vii) Linear by Linear is one-one

(viii) Quadratic by quadratic with no common factor is many one.

Onto function (Surjective mapping) :

If the function $f : A \to B$ is such that each element in B (co-domain) is the f image of atleast one element in A, then we say that f is a function from A 'onto' B. Thus $f : A \to B$ is surjective iff $\forall$ $b \in B$, $\exists$ some $a \in A$ such that $f(a) = b$.

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Diagramatically surjective mapping can be shown as

Note that : if range is same as co-domain, then f (x) is onto.

Into function :

If $f: A \to B$ is such that there exists atleast one element in co-domain which is not the image of any element in domain, then f (x) is into.

Diagramatically into function can be shown as

Note :

(i) A polynomial function of degree even define from $\mathbb{R} \to \mathbb{R}$ will always be into.

(ii) A polynomial function of degree odd defined from $\mathbb{R} \to \mathbb{R}$ will always be onto.

(iii) Quadratic by quadratic without any common factor define from $\mathbb{R} \to \mathbb{R}$ is always an into function.

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Thus a function can be one of these four types :

(i) one-one onto (injective & surjective)

(also known as **Bijective** mapping)

(ii) one-one into (injective but not surjective)

(iii) many-one onto (surjective but not injective)

(iv) many-one into (neither surjective nor injective)

Illustration19 : Let $A = \{x : -1 \le x \le 1\} = B$ be a mapping $f : A \to B$. For each of the following functions from A to B, find whether it is bijective or non-bijective.

(a) $f(x) = x|x|$ (b) $f(x) = x^3$ (c) $f(x) = \sin\frac{\pi x}{2}$

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Solution : (a) $f(x) = x|x| = \begin{cases} -x^2 & , -1 < x < 0 \\ x^2 & , 0 \le x < 1 \end{cases}$,

Graphically,
The graph shows f(x) is one-one, as the straight line parallel to x-axis cuts only at one point.
Here, range
$f(x) \in [-1, 1]$
Thus, range = co-domain
Hence, onto.
Therefore, f(x) is one-one onto or (Bijective).

(b) $f(x) = x^3$,
Graphically;
Graph shows f(x) is one-one onto
(i.e. Bijective)
[as explained in above example]

(c) $f(x) = \sin\frac{\pi x}{2}$
Graphically;
Which shows f(x) is one-one and onto as range = co-domain.
Therefore, f(x) is bijective.

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Illustration 20 : Let $f : \mathbb{N} \to \mathbb{Z}$ be a function defined as $f(x) = x–1000$. Show that $f(x)$ is an into function.

Solution : Let $f(x) = y = x - 1000 \quad \Rightarrow \quad x = y + 1000 = g(y)$ (say)
here g(y) is defined for each $y \in \mathbb{Z}$, but $g(y) \notin \mathbb{N}$ for $y \le -1000$.
Hence $f(x)$ is into.

Illustration 21 : Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x + \sqrt{x^2}$, then f is
(A) injective (B) surjective (C) bijective (D) None of these

Solution : We have, $f(x) = x + \sqrt{x^2} = x + |x|$
Clearly, f is not one-one as $f(-1) = f(-2) = 0$ and $-1 \ne -2$
Also, f is not onto as $f(x) \ge 0 \ \forall \ x \in \mathbb{R}$
$\therefore$ range of $f = (0, \infty) \subset \mathbb{R}$ **Ans.(D)**

Illustration 22 : Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as $f(x) = 2x^3 + 6x^2 + 12x + 3\cos x - 4\sin x$; then f is -
(A) Injective (B) Surjective (C) Bijective (D) Not Surjective

Solution : We have $f(x) = 2x^3 + 6x^2 + 12x + 3\cos x - 4\sin x$

$\Rightarrow \quad f'(x) = 6x^2 - 12x + 12 - 3\sin x - 4\cos x$

$$f'(x) = \underbrace{6(x-1)^2 + 6}_{g(x)} - \underbrace{(3\sin x + 4\cos x)}_{h(x)}$$

range of $g(x) = [6,\infty)$
range of $h(x) = [-5,5]$
hence $f'(x)$ always lies in the interval $[1,\infty)$
$\Rightarrow f'(x) > 0$

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Hence f (x) is increasing i.e. one-one function

Now $x \to \infty \Rightarrow f \to \infty$ & $x \to -\infty \Rightarrow f \to -\infty$ & $f(x)$ is continuous

hence its range is $\mathbb{R} \Rightarrow f$ is onto so f is bijective. **Ans. (C)**

Illustration 23 : Let $f(x) = \dfrac{x^2+3x+a}{x^2+x+1}$, where $f : \mathbb{R} \to \mathbb{R}$. Find the value of parameter 'a' so that the given function is one-one.

Solution : $f(x) = \dfrac{x^2+3x+a}{x^2+x+1}$

$$f'(x) = \frac{(x^2+x+1)(2x+3)-(x^2+3x+a)(2x+1)}{(x^2+x+1)^2} = \frac{-2x^2+2x(1-a)+(3-a)}{(x^2+x+1)^2}$$

Let, $g(x) = -2x^2 + 2x\,(1-a) + (3-a)$

$g(x)$ will be negative if $4\,(1-a)^2 + 8\,(3-a) < 0$

$\Rightarrow \quad 1 + a^2 - 2a + 6 - 2a < 0 \Rightarrow (a-2)^2 + 3 < 0$

which is not possible. Therefore function is not monotonic.

Hence, no value of a is possible.

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Do yourself - 7 :

(i) Is the function $f : \mathbb{N} \to \mathbb{N}$ (the set of natural numbers) defined by $f(x) = 2x + 3$ surjective ?

(ii) Let $A = R - \{3\}$, $\mathbb{R} = \mathbb{R} - \{1\}$ and let $f : A \to B$ defined by $f(x) = \dfrac{x-2}{x-3}$. Check whether the function $f(x)$ is bijective or not.

(iii) A mapping $f : A \to [-1,1]$ defined by $f(x) = \sin x$, $\forall\, x \in \mathbb{R}$, where A is a subset of $\mathbb{R}$ (the set of all real numbers) is one-one and onto if A is the interval, then A is belongs to

(A) $[0,2\pi]$ (B) $\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$ (C) $[-\pi,\pi]$ (D) $[0,\pi]$

13. COMPOSITE OF UNIFORMLY & NON-UNIFORMLY DEFINED FUNCTION:

Let $f : A \to B$ & $g : B \to C$ be two functions. Then the function $gof : A \to C$ defined by $(gof)(x) = g(f(x))$ $\forall\, x \in A$ is called the composite of the two functions f & g.

Diagramatically $\xrightarrow{x} \boxed{f} \xrightarrow{f(x)} \boxed{g} \longrightarrow g\,(f(x))$

Thus the image of every $x \in A$ under the function gof is the g-image of f-image of x.

Note that gof is defined only if $\forall\, x \in A$, f(x) is an element of the domain of 'g' so that we can take its g-image.

Properties of composite functions:

(a) In general composite of functions is not commutative i.e. $gof \neq fog$.

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(b) The composition of functions is associative i.e. if f, g, h are three functions such that fo(goh) & (fog)oh are defined, then fo(goh) = (fog)oh.

(c) The composition of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.

Illustration 24 : If $f(x) = x^2 + 1$, $g(x) = \dfrac{1}{x-1}$, then find (fog) (x) and (gof) (x).

Solution : Given, $f(x) = x^2 + 1$ .....(1) $g(x) = \dfrac{1}{x-1}$ ...(2)

Now (fog) (x) = f(g(x)) = $f\left(\dfrac{1}{x-1}\right) = f(z)$, where $z = \dfrac{1}{x-1}$

$= z^2 + 1$ $[\because f(x) = x^2 + 1]$

$= \left(\dfrac{1}{x-1}\right)^2 + 1 = \dfrac{1}{(x-1)^2} + 1$

Note : Domain of fog(x) is $x \in \mathbb{R} - \{1\}$

(gof) (x) = g(f(x)) = g(x^2 + 1) = g(u), where u = x^2+1

$= \dfrac{1}{u-1} = \dfrac{1}{x^2+1-1} = \dfrac{1}{x^2}$

Note : Domain of gof(x) is $x \in \mathbb{R} - \{0\}$

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Illustration 25 : If f be the greatest integer function and g be the modulus function, then

$(gof)\left(-\dfrac{5}{3}\right) - (fog)\left(-\dfrac{5}{3}\right) =$

(A) 1 (B) –1 (C) 2 (D) 4

Solution : Given $(gof)\left(\dfrac{-5}{3}\right) - (fog)\left(\dfrac{-5}{3}\right) = g\left\{f\left(\dfrac{-5}{3}\right)\right\} - f\left\{g\left(\dfrac{-5}{3}\right)\right\} = g(-2) - f\left(\dfrac{5}{3}\right) = 2 - 1 = 1$

Ans.(A)

Illustration 26 : Let $f(x) = \begin{cases} x+1, & x \le 1 \\ 2x+1, & 1 < x \le 2 \end{cases}$ and $g(x) = \begin{cases} x^2, & -1 \le x < 2 \\ x+2, & 2 \le x \le 3 \end{cases}$, find (fog)

Solution : $f(g(x)) = \begin{cases} g(x)+1, & g(x) \le 1 \\ 2g(x)+1, & 1 < g(x) \le 2 \end{cases}$

Here, g(x) becomes the variable that means we should draw the graph.

It is clear that $g(x) \le 1$; $\forall\ x \in [-1, 1]$

and $1 < g(x) \le 2$; $\forall\ x \in (1, \sqrt{2}]$

$\Rightarrow f(g(x)) = \begin{cases} x^2+1, & -1 \le x \le 1 \\ 2x^2+1, & 1 < x \le \sqrt{2} \end{cases}$

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Illustration 27: Find the domain and range of h(x) = g(f(x)), where

$f(x)=\begin{cases}[x], & -2\le x\le -1\\ |x|+1, & -1<x\le 2\end{cases}$ and $g(x)=\begin{cases}[x], & -\pi\le x\le 0\\ \sin x, & 0\le x\le \pi\end{cases}$, [.] denotes the greatest integer function.

Solution : $h(x) = g(f(x)) = \begin{cases}[f(x)], & -\pi\le f(x)<0\\ \sin(f(x)), & 0\le f(x)\le \pi\end{cases}$

From graph of f(x), we get

$h(x) = \begin{cases}[[x]], & -2\le x\le -1\\ \sin(|x|+1), & -1<x\le 2\end{cases}$

$\Rightarrow$ Domain of h(x) is [–2, 2]

and Range of h(x) is $\{-2, -1\} \cup [\sin 3, 1]$

Do yourself - 8 :

(i) $f(x) = x^3 - x$ & $g(x) = \sin 2x$, find

(a) f(f(1)) (b) f(f(–1)) (c) $f\left(g\left(\frac{\pi}{2}\right)\right)$

(d) $f\left(g\left(\frac{\pi}{4}\right)\right)$ (e) g(f(1)) (f) $g\left(g\left(\frac{\pi}{2}\right)\right)$

(ii) If $f(x)=\begin{cases}x+1; & 0\le x<2\\ |x|; & 2\le x<3\end{cases}$, then find fof(x).

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14. INVERSE OF A FUNCTION :

Let f : A → B be a one-one & onto function, then there exists a unique function g : B → A such that f(x) = y ⇔ g(y) = x, ∀ x ∈ A & y ∈ B. Then g is said to be inverse of f.

Thus $g = f^{-1} : B \to A = \{(f(x), x) | (x, f(x)) \in f\}$.

Properties of inverse function :

(a) The inverse of a bijection is unique.

(b) If f : A → B is a bijection & g : B → A is the inverse of f, then fog = I_B and gof = I_A, where I_A & I_B are identity functions on the sets A & B respectively. If fof = I, then f is inverse of itself.

(c) The inverse of a bijection is also a bijection.

(d) If f & g are two bijections f : A → B, g : B → C then the inverse of gof exists and $(gof)^{-1} = f^{-1} o g^{-1}$.

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(e) Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of 'f' if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from (a, b) by reflecting about the line $y = x$.

$\boxed{\text{The graph of } f^{-1} \text{ is obtained by reflecting the graph of f about the line } y = x.}$

Drawing the graph of $y = f^{-1}(x)$ from the known graph of $y = f(x)$

For drawing the graph of $y = f^{-1}(x)$ take the reflection of $y = f(x)$ about the line $y = x$. The reflected part would give us the graph of $y = f^{-1}(x)$.

e.g. let us draw the graph of $y = \sin^{-1}x$. We know that $y = f(x) = \sin x$ is invertible if $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$

$\Rightarrow$ the inverse mapping would be $f^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

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Illustration 28 : Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = (e^x - e^{-x})/2$. Is f(x) invertible ? If so, find its inverse.

Solution : Let us check for invertibility of f(x) :

(a) One-One :

$$f(x) = \frac{1}{2}\left(e^x - e^{-x}\right) \Rightarrow f'(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$$

$\Rightarrow f'(x) > 0$, $f(x)$ is increasing function

$\therefore f(x)$ is one-one function.

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(b) Onto :

As x tends to larger and larger values so does f(x) and

when $x \to \infty$, $f(x) \to \infty$.

Similarly as $x \to -\infty$, $f(x) \to -\infty$ i.e. $-\infty < f(x) < \infty$ so long as $x \in (-\infty, \infty)$

Hence the range of f is same as the set R. Therefore f(x) is onto.

Since f(x) is both one-one and onto, f(x) is invertible.

(c) To find $f^{-1}(x)$: Interchange x & y

$$\frac{e^y - e^y}{2} = x \Rightarrow e^{2y} - 2xe^y - 1 = 0$$

$$\Rightarrow \quad e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} \Rightarrow e^y = x \pm \sqrt{1+x^2}$$

Since $e^y > 0$, hence negative sign is ruled out and

Hence $e^y = x + \sqrt{1+x^2}$

Taking logarithm, we have $y = \ell n(x + \sqrt{1+x^2})$ or $f^{-1}(x) = \ell n(x + \sqrt{1+x^2})$

Illustration 29 : Find the inverse of the function $f(x) = \log_a\left(x + \sqrt{(x^2+1)}\right)$; $a > 1$ and assuming it to be an onto function.

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Solution : Given $f(x) = \log_a\left(x + \sqrt{x^2+1}\right)$

$$\therefore \quad f'(x) = \frac{\log_a e}{\sqrt{(1+x^2)}} > 0$$

which is a strictly increasing functions.

Thus, f(x) is injective, given that f(x) is onto. Hence the given function f(x) is invertible.

Interchanging x & y

$$\Rightarrow \log_a\left(y + \sqrt{(y)^2+1}\right) = x$$

$$\Rightarrow y + \sqrt{(y)^2+1} = a^x \qquad ...(i)$$

and $\sqrt{(y)^2+1} - y = a^{-x}$ ...(ii)

From (i) and (ii), we get $y = \frac{1}{2}(a^x - a^{-x})$ or $f^{-1}(x) = \frac{1}{2}(a^x - a^{-x})$

Illustration 30 : Find the inverse of the function $f(x) = \ell n(x^2+3x+1)$; $x \in [1,3]$ and assuming it to be an onto function.

Solution : Given $f(x) = \ell n(x^2 + 3x + 1)$

$$\therefore \quad f'(x) = \frac{2x+3}{(x^2+3x+1)} > 0 \forall x \in [1,3]$$

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Which is a strictly increasing function. Thus f(x) is injective, given that f(x) is onto. Hence the given function f(x) is invertible.

Interchanging x & y

$\Rightarrow (y)^2 + 3(y) + 1 - e^x = 0$

$$\therefore\ y = \frac{-3 \pm \sqrt{9 - 4.\left(1 - e^x\right)}}{2} = \frac{-3 \pm \sqrt{\left(5 + 4e^x\right)}}{2}$$

$$\Rightarrow\ y = \frac{-3 + \sqrt{\left(5 + 4e^x\right)}}{2} \qquad \text{(as } y \in [1,3]\text{)}$$

$$\text{Hence } f^{-1}(x) = \frac{-3 + \sqrt{\left(5 + 4e^x\right)}}{2}$$

Do yourself - 9 :

(i) Let $f : [-1, 1] \to [-1, 1]$ defined by $f(x) = x|x|$, find $f^{-1}(x)$.

(ii) $f(x) = 1 + \ell n(x + 2)$, find $f^{-1}(x)$.

15. ODD & EVEN FUNCTIONS :

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Consider a function $f(x)$ such that both x and –x are in its domain then

$$\text{If } \begin{cases} f(-x) = f(x) & \text{then } f \text{ is said to be an even function} \\ f(-x) = -f(x) & \text{then } f \text{ is said to be an odd function} \end{cases}$$

Note :

(i) $f(x) - f(-x) = 0 \Rightarrow f(x)$ is even & $f(x) + f(-x) = 0 \Rightarrow f(x)$ is odd.

(ii) A function may neither be odd nor even.

(iii) The only function which is defined on the entire number line & is even and odd at the same time is $f(x) = 0$.

(iv) Every constant function is even function.

(v) Inverse of an even function is not defined.

(vi) Every even function is symmetric about the y-axis & every odd function is symmetric about the origin.

Special Note : If a function $f(x)$ is defined as $f(a + x) = f(a - x)$ then this function is symmetric about line x = a

(vii) Every function which has '–x' in it's domain whenever 'x' is in it's domain, can be expressed as the sum of an even & an odd function .

$$\text{i.e. } f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{ODD}}$$

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(viii) If f (x) is odd and defined at x = 0, then f (0) = 0.

f(x)	g(x)	f (x) + g(x)	f(x) – g(x)	f (x) . g(x)	f (x)/g(x)	(g o f) (x)	(f o g)(x)
odd	odd	odd	odd	even	even	odd	odd
even	even	even	even	even	even	even	even
odd	even	neither odd nor even	neither odd nor even	odd	odd	even	even
even	odd	neither odd nor even	neither odd nor even	odd	odd	even	even

Illustration 31 : Which of the following functions is (are) even, odd or neither :

(i) $f(x) = x^2 \sin x$ (ii) $f(x) = \sin x - \cos x$ (iii) $f(x) = \dfrac{e^x + e^{-x}}{2}$

Solution : (i) $f(-x) = (-x)^2 \sin(-x) = -x^2 \sin x = -f(x)$. Hence f(x) is odd.

(ii) $f(-x) = \sin(-x) - \cos(-x) = -\sin x - \cos x$.

Hence f(x) is neither even nor odd.

(iii) $f(-x) = \dfrac{e^{-x} + e^{-(-x)}}{2} = \dfrac{e^{-x} + e^{x}}{2} = f(x)$. Hence f(x) is even

Illustration 32: Identify the given functions as odd, even or neither :

(i) $f(x) = \dfrac{x}{e^x - 1} + \dfrac{x}{2} + 1$ (ii) $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$

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Solution : (i) $f(x) = \dfrac{x}{e^x - 1} + \dfrac{x}{2} + 1$

Clearly domain of f(x) is $\mathbb{R} \sim \{0\}$. We have,

$$f(-x) = \frac{-x}{e^{-x} - 1} - \frac{x}{2} + 1 = \frac{-e^x . x}{1 - e^x} - \frac{x}{2} + 1 = \frac{(e^x - 1 + 1)x}{(e^x - 1)} - \frac{x}{2} + 1$$

$$= x + \frac{x}{e^x - 1} - \frac{x}{2} + 1 = \frac{x}{e^x - 1} + \frac{x}{2} + 1 = f(x)$$

Hence f(x) is an even function.

(ii) $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$

Replacing x, y by zero, we get $f(0) = 2f(0) \Rightarrow f(0) = 0$

Replacing y by –x, we get $f(x) + f(-x) = f(0) = 0 \Rightarrow f(x) = -f(-x)$

Hence f(x) is an odd function.

Do yourself - 10 :

(i) Which of the following functions is (are) even, odd or neither :

(a) $f(x) = x^3 \sin 3x$ (b) $f(x) = \dfrac{e^{x^2} + e^{-x^2}}{2x}$

(c) $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ (d) $f(x) = x^2 + 2^x$

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16. PERIODIC FUNCTION :

A function $f(x)$ is called periodic if there exists a positive number T such that $f(x+T) = f(x) = f(x-T)$, for all values of x within the domain of f. Smallest positive T (if exists) is called fundamental period of function $f(x)$.

Note :

(i) Odd powers of sinx, cosx, secx, cosecx are periodic with period 2π.

(ii) None zero integral powers of tanx, cotx are periodic with period π.

(iii) Non zero even powers or modulus of sinx, cosx, secx, cosecx are periodic with period π.

(iv) $f(T) = f(0) = f(-T)$, where 'T' is the period.

(v) if $f(x)$ has a period T then **$f(ax + b)$ has a period $T/|a|$** $(a \neq 0)$.

Proof : Let $f(x + T) = f(x)$ and $f[a(x + T') + b] = f(ax + b)$

$f(ax + b + aT') = f(ax + b)$

$$f(y + aT') = f(y) = f(y + T) \Rightarrow T = aT' \Rightarrow T' = \frac{T}{a}$$

(vi) If $f(x)$ & $g(x)$ are periodic with period T_1 & T_2 respectively, then a period (need not be fundamental) of $f(x) \pm g(x)$ is L.C.M. of (T_1, T_2).

(a) LCM of T_1 & T_2 is defined when T_1/T_2 is rational.

(b) LCM of $\left\{\frac{a}{b}, \frac{p}{q}\right\} = \frac{\text{LCM of } (a, p)}{\text{HCF of } (b, q)}$

In case if there exists a positive K such that K < LCM of T_1 and T_2 and overall function repeats itself after every K, then fundamental period of the function will be K.

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(vii) Every constant function is always periodic, whose fundamental period is undefined.

(viii) Periodic functions are non invertible.

Illustration 33 : Find the fundamental periods (if periodic) of the following functions, where [.] denotes the greatest integer function

(i) $f(x) = e^{\ell n(\sin x)} + \tan^3 x - \text{cosec}(3x - 5)$ (ii) $f(x) = x - [x - b],\ b \in \mathbb{R}$

(iii) $f(x) = \dfrac{|\sin x + \cos x|}{|\sin x| + |\cos x|}$ (iv) $f(x) = \tan\dfrac{\pi}{2}[x]$

(v) $f(x) = \cos(\sin x) + \cos(\cos x)$ (vi) $f(x) = \dfrac{(1+\sin x)(1+\sec x)}{(1+\cos x)(1+\text{cosec}\, x)}$

(vii) $f(x) = e^{x-[x]+|\cos \pi x|+|\cos 2\pi x|+.....+|\cos n\pi|}$

Solution : (i) $f(x) = e^{\ell n(\sin x)} + \tan^3 x - \text{cosec}(3x - 5)$

Period of $e^{\ell n \sin x} = 2\pi$, $\tan^3 x = \pi$

$\text{cosec}\,(3x - 5) = \dfrac{2\pi}{3}$

$\therefore$ Period $= 2\pi$

(ii) $f(x) = x - [x - b] = b + \{x - b\}$

$\therefore$ Period $= 1$

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(iii) $f(x) = \dfrac{|\sin x + \cos x|}{|\sin x| + |\cos x|}$

Since period of $|\sin x + \cos x| = \pi$ and period of $|\sin x| + |\cos x|$ is $\dfrac{\pi}{2}$. Hence f(x) is periodic with π as its period

(iv) $f(x) = \tan\dfrac{\pi}{2}[x]$

$\tan\dfrac{\pi}{2}[x + T] = \tan\dfrac{\pi}{2}[x] \Rightarrow \dfrac{\pi}{2}[x + T] = n\pi + \dfrac{\pi}{2}[x]$

$\therefore$ $T = 2$

$\therefore$ Period = 2

(v) Let f(x) is periodic then $f(x + T) = f(x)$

$\Rightarrow$ $\cos(\sin(x + T)) + \cos(\cos(x + T)) = \cos(\sin x) + \cos(\cos x)$

If x = 0 then $\cos(\sin T) + \cos(\cos T)$

$= \cos(0) + \cos(1) = \cos\left(\cos\dfrac{\pi}{2}\right) + \cos\left(\sin\dfrac{\pi}{2}\right)$

On comparing $T = \dfrac{\pi}{2}$

(vi) $f(x) = \dfrac{(1+\sin x)(1+\sec x)}{(1+\cos x)(1+\operatorname{cosec} x)} = \dfrac{(1+\sin x)(1+\cos x)\sin x}{\cos x(1+\sin x)(1+\cos x)}$

$\Rightarrow$ $f(x) = \tan x$

Hence f(x) has period π.

(vii) $f(x) = e^{x-[x]+|\cos\pi x|+|\cos 2\pi x|+.....+|\cos n\pi|}$

Period of $x - [x] = 1$

Period of $|\cos\pi x| = 1$

Period of $|\cos 2\pi x| = \dfrac{1}{2}$

.....

Period of $|\cos n\pi x| = \dfrac{1}{n}$

So period of f(x) will be L.C.M. of all period = 1

Illustration 34 : Find the fundamental periods (if periodic) of the following functions, where [.] denotes the greatest integer function

(i) $f(x) = e^{x-[x]} + \sin x$ (ii) $f(x) = \sin\dfrac{\pi x}{\sqrt{2}} + \cos\dfrac{\pi x}{\sqrt{3}}$ (iii) $f(x) = \sin\dfrac{\pi x}{\sqrt{3}} + \cos\dfrac{\pi x}{2\sqrt{3}}$

Solution : (i) Period of $e^{x-[x]} = 1$

period of $\sin x = 2\pi$

$\because$ L.C.M. of rational and an irrational number does not exist.

$\therefore$ not periodic.

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(ii) Period of $\sin\frac{\pi x}{\sqrt{2}} = \frac{2\pi}{\pi/\sqrt{2}} = 2\sqrt{2}$

Period of $\cos\frac{\pi x}{\sqrt{3}} = \frac{2\pi}{\pi/\sqrt{3}} = 2\sqrt{3}$

$\because$ L.C.M. of two different kinds of irrational number does not exist.

$\therefore$ not periodic.

(iii) Period of $\sin\frac{\pi x}{\sqrt{3}} = \frac{2\pi}{\pi/\sqrt{3}} = 2\sqrt{3}$

Period of $\cos\frac{\pi x}{2\sqrt{3}} = \frac{2\pi}{\pi/2\sqrt{3}} = 4\sqrt{3}$

$\because$ L.C.M. of two similar irrational number exist.

$\therefore$ Periodic with period = $4\sqrt{3}$ **Ans.**

Do yourself - 11 :

(i) Find the fundamental periods (if periodic) of the following functions.

(a) $f(x) = \ell n(\cos x) + \tan^3 x$.

(b) $f(x) = e^{x-[x]}$, [.] denotes greatest integer function

(c) $f(x) = \left|\sin\frac{x}{2}\right| + \left|\cos\frac{x}{2}\right|$

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17. GENERAL :

If x, y are independent variables and f is a continuous function, then :

(a) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k\ell n\, x$

(b) $f(xy) = f(x) \,.\, f(y) \Rightarrow f(x) = x^n,\ n \in \mathbb{R}$ or $f(x) = 0$

(c) $f(x + y) = f(x) \,.\, f(y) \Rightarrow f(x) = a^{kx}$ or $f(x) = 0$

(d) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

Miscellaneous Illustration :

Illustration 35 : ABCD is a square of side ℓ. A line parallel to the diagonal BD at a distance 'x' from the vertex A cuts two adjacent sides. Express the area of the segment of the square with A at a vertex, as a function of x. Find this area at $x = 1/\sqrt{2}$ and at x = 2, when $\ell = 2$.

Solution : There are two different situations

Case-I : when $x = AP \le OA$, i.e., $x \le \frac{\ell}{\sqrt{2}}$

$$ar(\Delta AEF) = \frac{1}{2}x.2x = x^2 \quad (\because PE = PF = AP = x)$$

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Case-II : when x = AP > OA, i.e., $x > \frac{\ell}{\sqrt{2}}$ but $x \le \sqrt{2}\ell$

ar(ABEFDA) = ar(ABCD) – ar(ΔCFE)

$$= \ell^2 - \frac{1}{2}\left(\sqrt{2}\ell - x\right).2\left(\sqrt{2}\ell - x\right) \quad [\because \ CP = \sqrt{2}\ell - x]$$

$$= \ell^2 - \left(2\ell^2 + x^2 - 2\sqrt{2}\ell x\right) = 2\sqrt{2}\ell x - x^2 - \ell^2$$

∴ the required function s(x) is as follows :

$$s(x) = \begin{cases} x^2, & 0 \le x \le \frac{\ell}{\sqrt{2}} \\ 2\sqrt{2}\,\ell x - x^2 - \ell^2, & \frac{\ell}{\sqrt{2}} < x \le \sqrt{2}\ell \end{cases} ; \text{ area of } s(x) = \begin{cases} \frac{1}{2} & \text{at } x = \frac{1}{\sqrt{2}} \\ 8(\sqrt{2} - 1) & \text{at } x = 2 \end{cases}$$

Ans.

Illustration 36 : If the function f(x) satisfies the functional rule, f(x + y) = f(x) + f(y) ∀ x,y ∈ ℝ & f(1) = 5, then find $\sum_{n=1}^{m} f(n)$.

Solution : Here, f(x + y) = f(x) + f(y); put x = t – 1, y = 1

f(t) = f(t – 1) + f(1) ....(1)

∴ f(t) = f(t – 1) + 5

⇒ f(t) = {f(t – 2) + 5} + 5

⇒ f(t) = f(t – 2) + 2(5)

⇒ f(t) = f(t – 3) + 3(5)

.....

.....

⇒ f(t) = f{t – (t –1)} + (t – 1)5

⇒ f(t) = f(1)+ (t – 1)5

⇒ f(t) = 5 + (t – 1)5

⇒ f(t) = 5t

$$\therefore \quad \sum_{n=1}^{m} f(n) = \sum_{n=1}^{m} (5n) = 5[1 + 2 + 3 + ... + m] = \frac{5m(m+1)}{2}$$

Hence, $\sum_{n=1}^{m} f(n) = \frac{5m(m+1)}{2}$.

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ANSWERS FOR DO YOURSELF

1 : **(i)** **(a)** $x \in (0, \infty)$ **(b)** $x \in (-\infty, 0) \cup (4, \infty)$

(ii) **(a)** $\left[\frac{1}{2}, \infty\right)$ **(b)** $[-1, 1]$ **(c)** $[-2, 2]$ **(d)** $[-1, 1]$

2 : **(i)** **(A)** $\rightarrow$ (p,q,r,s), **(B)** $\rightarrow$ (p), **(C)** $\rightarrow$ (t), **(D)** $\rightarrow$ (r)

3 : **(i)** **(a)** $(-1,0) \cup (0,1]$ **(b)** $\left(-\infty, -\frac{1}{2}\right] \cup (0,1) \cup (2, \infty)$

(ii) **(a)** $(-\infty, \infty)$ **(b)** $\left[\frac{1}{3}, 1\right]$

4 : **(i)** **(a)** no **(b)** no **(c)** yes

5 : **(i)** $\left[0, \frac{1}{2}\right]$ **(ii)** A **(iii)** **(a)** $y = \frac{1}{x}$ **(b)** $y = \frac{1}{x^2}$

6 : **(i)**

(ii)

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7 : **(i)** not onto **(ii)** yes **(iii)** B

8 : **(i)** **(a)** 0 **(b)** 0 **(c)** 0 **(d)** 0 **(e)** 0 **(f)** 0 **(ii)** $\begin{cases} x+2, & 0 \le x < 1 \\ x+1, & 1 \le x < 2 \\ x, & 2 \le x < 3 \end{cases}$

9 : **(i)** $f^{-1}(x) = \begin{cases} -\sqrt{-x}, & -1 \le x \le 0 \\ \sqrt{x}, & 0 \le x \le 1 \end{cases}$ **(ii)** $y = -2 + e^{x-1}]$

10: **(i)** **(a)** even **(b)** odd **(c)** odd **(d)** neither even nor odd

11: **(i)** **(a)** 2π **(b)** 1 **(c)** π

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EXERCISE (O-1)

Subjective Type Questions

1. Find the domain of definition of the given functions :

(i) $y=\sqrt{-px}\,(p>0)$ **FN0001**

(ii) $y=\dfrac{1}{x^2+1}$ **FN0002**

(iii) $y=\dfrac{1}{x^3-x}$ **FN0003**

(iv) $y=\dfrac{1}{\sqrt{x^2-4x}}$ **FN0004**

(v) $y=\sqrt{x^2-4x+3}$ **FN0005**

(vi) $y=\dfrac{x}{\sqrt{x^2-3x+2}}$ **FN0006**

(vii) $y=\sqrt{1-|x|}$ **FN0007**

(viii) $y=\log_x 2.$ **FN0008**

(ix) $y=\dfrac{1}{\log_{10}(1-x)}+\sqrt{x+2}$ **FN0009**

(x) $y=\sqrt{x}+\sqrt[3]{\dfrac{1}{x-2}}-\log_{10}(2x-3)$ **FN0010**

(xi) $y=\dfrac{3}{4-x^2}+\log_{10}(x^3-x)$ **FN0011**

(xii) $y=\dfrac{1}{\sqrt{\sin x}}+\sqrt[3]{\sin x}$ **FN0012**

(xiii) $y=\log_{10}\left(\sqrt{x-4}+\sqrt{6-x}\right)$ **FN0013**

(xiv) $y=\log_{10}[1-\log_{10}(x^2-5x+16)]$ **FN0014**

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2. **Find the range of the following functions :**

(i) $f(x)=\dfrac{x-1}{x+2}$ **FN0015**

(ii) $f(x)=\dfrac{2}{x}$ **FN0016**

(iii) $f(x)=\dfrac{1}{x^2-x+1}$ **FN0017**

(iv) $f(x)=\dfrac{x^2-x+1}{x^2+x+1}$ **FN0018**

(v) $f(x)=e^{(x-1)^2}$ **FN0019**

(vi) $f(x)=x^3-x^2+x+1$ **FN0020**

(vii) $f(x)=\log(x^8+x^4+x^2+1)$ **FN0021**

(viii) $f(x)=\sin^2x-2\sin x+4$ **FN0022**

(ix) $f(x)=\sin(\log_2 x)$ **FN0023**

(x) $f(x)=2^{x^2}+1$ **FN0024**

(xi) $f(x)=\dfrac{e^{2x}-e^x+1}{e^{2x}+e^x+1}$ **FN0025**

(xii) $f(x)=\dfrac{1}{8-3\sin x}$ **FN0026**

3. The number of integers lying in the domain of the function $f(x)=\sqrt{\log_{0.5}\left(\dfrac{5-2x}{x}\right)}$ is -

(A) 3 (B) 2 (C) 1 (D) 0

FN0027

4. The range of the function $f:\mathbb{N}\to\mathbb{Z};\ f(x)=(-1)^{x-1}$, is -

(A) $[-1, 1]$ (B) $\{-1, 1\}$ (C) $\{0, 1\}$ (D) $\{0, 1, -1\}$

FN0028

5. The range of the function $f(x)=e^{-x}+e^{x}$, is -

(A) $f(x)\ge 1$ (B) $f(x)\le 1$ (C) $f(x)\ge 2$ (D) $f(x)\le 2$

FN0029

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6. A function f has domain $[-1, 2]$ and range $[0, 1]$. The domain and range respectively of the function g defined by $g(x) = 1 - f(x+1)$ is

(A) $[-1, 1]$; $[-1, 0]$ (B) $[-2, 1]$; $[0, 1]$ (C) $[0, 2]$; $[-1, 0]$ (D) $[1, 3]$; $[-1, 0]$

FN0030

7. For the function $f(x) = \dfrac{e^x + 1}{e^x - 1}$, if n(d) denotes the number of integers which are not in its domain and n(r) denotes the number of integers which are not in its range, then n(d) + n(r) is equal to -

(A) 2 (B) 3 (C) 4 (D) Infinite

FN0031

8. Find a formula for each function graphed

(a)

(b)

FN0032

9. If [a] denotes the greatest integer less than or equal to a and $-1 \le x < 0$, $0 \le y < 1$, $1 \le z < 2$, then

$$\begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix}$$ is equal to -

(A) [x] (B) [y] (C) [z] (D) none of these

FN0033

10. The range of the function $f(x) = \text{sgn}\left(\dfrac{\sin^2 x + 2\sin x + 4}{\sin^2 x + 2\sin x + 3}\right)$ is (where sgn(.) denotes signum function)-

(A) $\{-1,0,1\}$ (B) $\{-1,0\}$ (C) $\{1\}$ (D) $\{0,1\}$

FN0034

11. The range of the function $f(x) = \sqrt{4 - x^2} + \sqrt{x^2 - 1}$ is

(A) $\left[\sqrt{3}, \sqrt{7}\right]$ (B) $\left[\sqrt{3}, \sqrt{5}\right]$ (C) $\left[\sqrt{2}, \sqrt{3}\right]$ (D) $\left[\sqrt{3}, \sqrt{6}\right]$

FN0035

12. If [x] and {x} denotes the greatest integer function less than or equal to x and fractional part function respectively, then the number of real x, satisfying the equation $(x-2)[x] = \{x\} - 1$, is-

(A) 0 (B) 1 (C) 2 (D) infinite

FN0036

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13. If $2f(x) - 3f\left(\frac{1}{x}\right) = x^2$, x is not equal to zero, then f(2) is equal to-

(A) $-\frac{7}{4}$ (B) $\frac{5}{2}$ (C) –1 (D) none of these

FN0037

14. If $f(x) = \frac{4^x}{4^x + 2}$, then f(x) + f(1 – x) is equal to-

(A) 0 (B) –1 (C) 1 (D) 4

FN0038

15. If $x^4 f(x) - \sqrt{1 - \sin 2\pi x} = |f(x)| - 2f(x)$, then f(–2) equals

(A) $\frac{1}{17}$ (B) $\frac{1}{11}$ (C) $\frac{1}{19}$ (D) 0

FN0039

16. The graph of a function f is given.

(a) State the value of $f(-1)$.

(b) For what values of x is $f(x) = 2$

(c) State the domain and range of f.

(d) On what interval is f increasing ?

(e) Estimated value of $f(2)$ is -

(A) 2.2 (B) 2.8 (C) 2.5 (D) 3

(f) Estimated value of x such that $f(x) = 0$, is -

(A) –2.5 (B) 0.8 (C) –2.9 (D) 0.3

FN0040

17. Graph the function $F(x) = \begin{cases} 3 - x, & x \le 1 \\ 2x, & x > 1 \end{cases}$

FN0041

18. The graphs of f and g are given.

(a) State the value of $f(-4)$ and g(3)

(b) For what value of x is $f(x) = g(x)$?

(c) Estimate the solution of the equation $f(x) = -1$.

(d) On what interval is f decreasing?

(e) State the domain and range of f.

(f) State the domain and range of g.

FN0042

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19. Solve the following inequalities using graph of f(x) :

(a) $0 \le f(x) \le 1$

(b) $-1 \le f(x) \le 2$

(c) $2 \le f(x) \le 3$

(d) $f(x) > -1$ & $f(x) < 0$

FN0043

20. Which of the following is the graph of $y = |x-1| + |x-3|$?

(A)

(B)

(C)

(D)

FN0044

21. Which of the following is the graph of $y = \dfrac{|\sin x|}{\sin x}$?

(A)

(B)

(C)

(D)

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FN0045

22. Let $f : \mathbb{R} - \left\{\dfrac{-15}{2}\right\} \to \mathbb{R} - \left\{\dfrac{1}{2}\right\}$ be defined by $f(x) = \dfrac{x+10}{2x+15}$ then $f(x)$ is-

(A) one-one but not onto
(B) many one but not-onto
(C) one-one and onto
(D) many one and onto

FN0046

23. $f : \mathbb{R} \to \mathbb{R}$ $f(x) = \dfrac{2x^2 - 5x + 3}{8x^2 + 9x + 11}$, then f is -

(A) one-one onto
(B) many-one onto
(C) one-one into
(D) many one into

FN0047

24. If $f : \mathbb{R} \to \mathbb{R}$ & $f(x) = \dfrac{\sin([x]\pi)}{x^2 + 2x + 3} + 2x - 1 + \sqrt{x(x-1) + \dfrac{1}{4}}$ (where [x] denotes integral part of x), then f(x) is -

(A) one-one but not onto
(B) one-one & onto
(C) onto but not one-one
(D) neither one-one nor onto

FN0048

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25. Which of the following function is surjective but not injective

(A) $f : \mathbb{R} \to \mathbb{R}$ $f(x) = x^4 + 2x^3 - x^2 + 1$

(B) $f : \mathbb{R} \to \mathbb{R}$ $f(x) = x^3 + x + 1$

(C) $f : \mathbb{R} \to \mathbb{R}^+$ $f(x) = \sqrt{1+x^2}$

(D) $f : \mathbb{R} \to \mathbb{R}$ $f(x) = x^3 + 2x^2 - x + 1$

FN0049

26. If $f : (-\infty,3] \to [7,\infty)$; $f(x) = x^2 - 6x + 16$, then which of the following is true -

(A) $f^{-1}(x) = 3+\sqrt{x-7}$

(B) $f^{-1}(x) = 3-\sqrt{x-7}$

(C) $f^{-1}(x) = \dfrac{1}{x^2-6x+16}$

(D) f is many-one

FN0050

27. $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \ell n\left(x+\sqrt{x^2+1}\right)$. Another function g(x) is defined such that $gof(x) = x \ \forall \ x \in R$. Then g(2) is -

(A) $\dfrac{e^2+e^{-2}}{2}$

(B) e^2

(C) $\dfrac{e^2-e^{-2}}{2}$

(D) e^{-2}

FN0051

28. If $f(x) = x|x|$ then $f^{-1}(x)$ equals-

(A) $\sqrt{|x|}$

(B) (sgn x). $\sqrt{|x|}$

(C) $-\sqrt{|x|}$

(D) Does not exist

(where sgn(x) denotes signum function of x)

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FN0052

29. Let $P(x) = kx^3 + 2k^2x^2 + k^3$. The sum of all real numbers k for which (x – 2) is a factor of P(x), is

(A) 4

(B) 8

(C) – 4

(D) –8

FN0053

30. Period of function $f(x) = \min\{\sin x, |x|\} + \dfrac{x}{\pi} - \left[\dfrac{x}{\pi}\right]$ (where [.] denotes greatest integer function) is -

(A) $\pi/2$

(B) π

(C) 2π

(D) 4π

FN0054

Multiple Correct Answer

31. Which of the following function(s) have the same domain and range ?

(A) $f(x) = \sqrt{1-x^2}$

(B) $g(x) = \dfrac{1}{x}$

(C) $h(x) = \sqrt{x}$

(D) $l(x) = \sqrt{4-x}$

FN0055

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32. Which of the following pair(s) of function have same graphs?

(A) $f(x) = \frac{\sec x}{\cos x} - \frac{\tan x}{\cot x}$, $g(x) = \frac{\cos x}{\sec x} + \frac{\sin x}{\operatorname{cosec} x}$

(B) $f(x) = \text{sgn}(x^2 - 4x + 5)$, $g(x) = \text{sgn}\left(\cos^2 x + \sin^2\left(x + \frac{\pi}{3}\right)\right)$ where sgn denotes signum function.

(C) $f(x) = e^{\ln(x^2+3x+3)}$, $g(x) = x^2 + 3x + 3$

(D) $f(x) = \frac{\sin x}{\sec x} + \frac{\cos x}{\operatorname{cosec} x}$, $g(x) = \frac{2\cos^2 x}{\cot x}$

FN0056

33. Let $f(x) = \sin^6 x + \cos^6 x$, then -

(A) $f(x) \in [0, 1]\ \forall\ x \in \mathbb{R}$

(B) $f(x) = 0$ has no solution

(C) $f(x) \in \left[\frac{1}{4}, 1\right] \forall x \in \mathbb{R}$

(D) $f(x)$ is an injective function

FN0057

34. Let $f(x) = \begin{cases} x^2 - 3x + 4 & ; \quad x < 3 \\ x + 7 & ; \quad x \geq 3 \end{cases}$ and $g(x) = \begin{cases} x + 6 & ; \quad x < 4 \\ x^2 + x + 2 & ; \quad x \geq 4 \end{cases}$, then which of the following is/are true -

(A) $(f + g)(1) = 9$

(B) $(f - g)(3.5) = 1$

(C) $(f g)(0) = 24$

(D) $\left(\frac{f}{g}\right)(5) = \frac{8}{3}$

FN0058

35. If a function is defined by an implicit equation $2^{|x|+|y|} + 2^{|x|-|y|} = 2$, then -

(A) Domain of function is singleton

(B) Range of function is singleton

(C) graph of the function intersects the line $y = x$

(D) maximum value of function is 2

FN0059

36. Let $f(x) = x^2 + 3x + 2$, then number of solutions to -

(A) $f(|x|) = 2$ is 1

(B) $f(|x|) = 2$ is 3

(C) $|f(x)| = 0.125$ is 4

(D) $|f(|x|)| = 0.125$ is 8

FN0060

37. For each real x, let $f(x) = \max\{x, x^2, x^3, x^4\}$, then $f(x)$ is -

(A) x^4 for $x \leq -1$

(B) x^2 for $-1 < x \leq 0$

(C) $f\left(\frac{1}{2}\right) = \frac{1}{2}$

(D) $f\left(\frac{1}{2}\right) = \frac{1}{4}$

FN0061

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Matrix Match Type

38. Match the functions given in column-I correctly with mappings given in column-II.

Column-I	Column-II
(A) $f:\left[-\frac{1}{2},\frac{1}{2}\right]\to\left[\frac{4}{7},\frac{4}{3}\right]$	(P) Injective mapping
$f(x)=\frac{1}{x^2+x+1}$	(Q) Non-injective mapping
(B) $f:[-2,2]\to[-1,1]$ $f(x)=\sin x$	(R) Surjective mapping (S) Non-surjective mapping
(C) $f:\mathbb{R}-I\to\mathbb{R}$ $f(x)=\ell n\{x\}$, (where $\{.\}$ represents fractional part function)	(T) Bijective mapping
(D) $f:(-\infty,0]\to[1,\infty)$, $f(x)=(1+\sqrt{-x})+(\sqrt{-x}-x)$	

FN0062

Linked Comprehension Type

Paragraph for Question 39 & 40

Let $f(x)=\begin{cases} x & ; \quad x<0 \\ 1-x & ; \quad x\ge 0\end{cases}$ & $g(x)=\begin{cases} x^2 & ; \quad x<-1 \\ 2x+3 & ; \quad -1\le x\le 1 \\ x & ; \quad x>1\end{cases}$

On the basis of above information, answer the following questions :

39. Range of $f(x)$ is -

(A) $(-\infty,1]$ (B) $(-\infty,\infty)$ (C) $(-\infty,0]$ (D) $(-\infty,2]$

FN0063

40. Range of $g(f(x))$ is -

(A) $(-\infty,\infty)$ (B) $[1,3)\cup(3,\infty)$ (C) $[1,\infty)$ (D) $[0,\infty)$

FN0063

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EXERCISE (O-2)

Straight Objective Type

1. Range of function $f(x)=\log_2\left(\frac{4}{\sqrt{x+2}+\sqrt{2-x}}\right)$ is given by

(A) $(0,\infty)$ (B) $\left[\frac{1}{2},1\right]$ (C) $[1,2]$ (D) $\left[\frac{1}{4},1\right]$

FN0064

2. Range of $f(x)=\frac{\sec x+\tan x-1}{\tan x-\sec x+1}$; $x\in\left(0,\frac{\pi}{2}\right)$ is-

(A) $(0,1)$ (B) $(1,\infty)$ (C) $(-1,0)$ (D) $(-\infty,-1)$

FN0065

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3. If $f(x,y) = \max(x,y) + \min(x,y)$ and $g(x,y) = \max(x,y) - \min(x,y)$, then the value of $f\left(g\left(-\frac{2}{3},-\frac{3}{2}\right), g(-3,-4)\right)$ is greater than -

(A) 1 (B) 2 (C) 3 (D) 4

FN0066

4. The number of integral values of x satisfying the inequality $[x-5][x-3]+2 < [x-5]+2[x-3]$ (where [.] represents greatest integer function) is -

(A) 0 (B) 1 (C) 2 (D) 3

FN0067

5. Suppose, $f(x, n) = \sum_{k=1}^{n} \log_x\left(\frac{k}{x}\right)$, then the value of x satisfying the equation $f(x, 10) = f(x, 11)$, is

(A) 9 (B) 10 (C) 11 (D) none

FN0068

6. A lion moves in the region given by the graph $y - |y| - x + |x| = 0$. Then on which of the following curve a person can move so that he does not encounter lion -

(A) $y = e^{-|x|}$ (B) $y = \frac{1}{x}$ (C) $y = \text{signum}(x)$ (D) $y = -|4 + |x||$

FN0069

7. Let $f(x) = \frac{x}{1-x}$ and let α be a real number. If $x_0 = \alpha$, $x_1 = f(x_0)$, $x_2 = f(x_1)$,..... & $x_{2011} = -\frac{1}{2012}$ then the value of α is -

(A) $\frac{2011}{2012}$ (B) 1 (C) 2011 (D) –1

FN0070

8. If $f_1(x) = 2^{f_2(x)}$, where $f_2(x) = 2012^{f_3(x)}$, where $f_3(x) = \left(\frac{1}{2013}\right)^{f_4(x)}$, where $f_4(x) = \log_{2013}\log_x 2012$, then the range of $f_1(x)$ is -

(A) $(2, \infty)$ (B) $(2012, \infty)$ (C) $(0, \infty)$ (D) $(-\infty, \infty)$

FN0071

9. If functions $f(x)$ and $g(x)$ are defined on $\mathbb{R} \to \mathbb{R}$ such that $f(x) = \begin{cases} x+3 & , \quad x \in \text{rational} \\ 4x & , \quad x \in \text{irrational} \end{cases}$,

$g(x) = \begin{cases} x+\sqrt{5} & , \quad x \in \text{irrational} \\ -x & , \quad x \in \text{rational} \end{cases}$ then $(f-g)(x)$ is-

(A) one-one & onto (B) neither one-one nor onto

(C) one-one but not onto (D) onto but not one-one

FN0072

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10. Let f: A $\to$ B be an onto function such that $f(x)=\sqrt{x-2-2\sqrt{x-3}}-\sqrt{x-2+2\sqrt{x-3}}$, then set 'B' is-

(A) [–2,0] (B) [0,2] (C) [–3,0] (D) [–1,0]

FN0073

11. Let $f:\mathbb{R}\to\mathbb{R}$ and $f(x) = x^3 + ax^2 + bx - 8$. If $f(x) = 0$ has three real roots & $f(x)$ is a bijective function, then (a + b) is equal to

(A) 0 (B) 6 (C) –6 (D) 12

FN0074

12. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x) = \ln(x+\sqrt{x^2+1})$, then number of solutions of $|f^{-1}(x)| = e^{-|x|}$ is :-

(A) 1 (B) 2 (C) 3 (D) Infinite

FN0075

13. $f(x) = [x-1] + \{x\}^{[x]}$, $x \in (1,3)$, then $f^{-1}(x)$ is -
(where [.] denotes greatest integer function and {.} denotes fractional part function)

(A) $\begin{cases} x+1 & x\in(1,2) \\ 2+\sqrt{x-1} & x\in[2,3) \end{cases}$ (B) $\begin{cases} x-1 & x\in(1,2) \\ 2-\sqrt{x-1} & x\in[2,3) \end{cases}$

(C) $\begin{cases} x-1 & x\in(0,1) \\ 2-\sqrt{x-1} & x\in[1,2) \end{cases}$ (D) $\begin{cases} x+1 & x\in(0,1) \\ 2+\sqrt{x-1} & x\in[1,2) \end{cases}$

FN0076

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14. Which of the following functions is an odd function ?

(A) $|x-2| + (x+2)\,\mathrm{sgn}(x+2)$ (B) $\dfrac{1}{x(e^x-1)}+\dfrac{1}{2x}$

(C) $\log\left(\sin x+\sqrt{1+\sin^2 x}\right)$ (D) $e^{-4x}\left(e^{2x}-1\right)^4$

(where sgn(x) denotes signum function of x)

FN0077

15. Period of $f(x) = \{x\} + \left\{x+\dfrac{1}{3}\right\} + \left\{x+\dfrac{2}{3}\right\}$ is equal to (where {.} denotes fractional part function)

(A) 1 (B) $\dfrac{2}{3}$ (C) $\dfrac{1}{2}$ (D) $\dfrac{1}{3}$

FN0078

16. Let $f(x) = 2x - \left\{\dfrac{x}{\pi}\right\}$ and $g(x) = \cos x$, where {.} denotes fractional part function, then period of $gof(x)$ is -

(A) $\dfrac{\pi}{2}$ (B) π (C) $\dfrac{3\pi}{2}$ (D) $\dfrac{\pi}{4}$

FN0079

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17. The period of the function $\frac{\sin x + \sin 5x}{\cos x + \cos 5x}$ is -

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C) π (D) 2π

FN0080

18. Let $f : \mathbb{R} \to \mathbb{R}$ be a real valued function such that $f(10 + x) = f(10 - x)\ \forall\ x \in \mathbb{R}$ and $f(20 + x) = -f(20 - x) \forall\ x \in \mathbb{R}$. Then which of the following statements is true -

(A) $f(x)$ is odd and periodic
(B) $f(x)$ is odd and aperiodic
(C) $f(x)$ is even and periodic
(D) $f(x)$ is even and aperiodic

FN0081

Multiple Correct Answer Type

19. The range of the function $f(\theta) = \sqrt{8\sin^2\theta + 4\cos^2\theta - 8\sin\theta\cos\theta}$ is -

(A) $\left[\sqrt{5}-1, \sqrt{5}+1\right]$
(B) $\left[0, \sqrt{5}+1\right]$
(C) $\left[\sqrt{6-\sqrt{20}}, \sqrt{6+\sqrt{20}}\right]$
(D) none of these

FN0082

20. For the function $f(x) = |x + 3| - |x + 1| - |x - 1| + |x - 3|$, identify correct option(s)

(A) Range of $f(x)$ is $(-\infty, 4]$
(B) maximum value of $f(x)$ is 4
(C) $f(x) = 4$ has infinite solutions
(D) $f(x) = 0$ has infinite solutions

FN0083

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21. The values of x in $[-2\pi, 2\pi]$, for which the graph of the function $y = \sqrt{\frac{1+\sin x}{1-\sin x}} - \sec x$ and $y = -\sqrt{\frac{1-\sin x}{1+\sin x}} + \sec x$, coincide are

(A) $\left[-2\pi, -\frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right]$
(B) $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$
(C) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(D) $[-2\pi, 2\pi] - \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}\right\}$

FN0084

22. Which of the following statement(s) is(are) correct ?

(A) If f is a one-one mapping from set A to A, then f is onto.
(B) If f is an onto mapping from set A to A, then f is one-one
(C) Let f and g be two functions defined from $\mathbb{R} \to \mathbb{R}$ such that gof is injective, then f must be injective.
(D) If set A contains 3 elements while set B contains 2 elements, then total number of functions from A to B is 8.

FN0085

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23. A parabola of the form $y = ax^2 + bx + c$ with $a > 0$ intersects the graph of $f(x) = \frac{1}{x^2 - 4}$. Number of possible distinct intersection(s) of these graph is

(A) 0 (B) 2 (C) 3 (D) 4

FN0086

24. Let $f(x) = \begin{cases} x^2 & ; \quad 0 < x < 2 \\ 2x - 3 & ; \quad 2 \le x < 3 \\ x + 2 & ; \quad x \ge 3 \end{cases}$ Then :-

(A) $f\left\{f\left(f\left(\frac{3}{2}\right)\right)\right\} = f\left(\frac{3}{2}\right)$

(B) $1 + f\left\{f\left(f\left(\frac{5}{2}\right)\right)\right\} = f\left(\frac{5}{2}\right)$

(C) $f\{f(1)\} = f(1) = 1$

(D) none of these

FN0087

25. If $\{x\} = \frac{2}{3}$ & $\left[x + \left\{x + \left[x + \left\{x + ...100 \text{ times}\right\}\right]\right\}\right] = 5$, then -

(A) $x = \frac{14}{3}$ (B) $[x] = 5$ (C) $x = \frac{17}{3}$ (D) $[x] = 4$

(where [.] & {.} denotes greatest integer function & fractional part function respectively)

FN0088

26. Given $f(1) = 2$ and $f(n+1) = \frac{f(n) - 1}{f(n) + 1} \forall n \in N$, then-

(A) $f(2015) = -\frac{1}{2}$

(B) $(f(2012))^{f(2013)} = 9$

(C) $f(1001) = 2$

(D) $f(2015) = -3$

FN0089

27. Which of the following is/are true?

(A) $f(x) = e^x$ and $g(x) = \ell nx$, then $f(g(x)) = x$ (wherever $f(g(x))$ is defined)

(B) $f(x) = \frac{2 + x}{3 - x}$ and $g(x) = \frac{(3x - 2)}{(x + 1)}$, then $f(g(x)) = x$ (wherever $f(g(x))$ is defined)

(C) $f(x) = 4x + 7$ and $g(x) = \frac{(x - 7)}{4}$, then $f(g(x)) = x$ (wherever $f(g(x))$ is defined)

(D) $f(x) = x^3 + 1$ and $g(x) = (x - 1)^{1/3}$, then $f(g(x)) = x$ (wherever $f(g(x))$ is defined)

FN0090

28. If $g(x) = x^2 - x + 1$ and $f(x) = \sqrt{\frac{1}{x} - x}$, then -

(A) Domain of $f(g(x))$ is $[0,1]$

(B) Range of $f(g(x))$ is $\left(0, \frac{7}{2\sqrt{3}}\right]$

(C) $f(g(x))$ is many-one function

(D) $f(g(x))$ is unbounded function

FN0091

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29. Let $f : \{1,2,3,4,5\} \to \{1,2,3,4,5\}$ is such that $f(x)$ is a one-one function satisfying following condition $f(x) = x + 1$ **if and only if** x is even (i.e. $f(3) \neq 4$, $f(4) = 5$ etc). Then $f^{-1}(2)$ can be-

(A) 1 (B) 3 (C) 5 (D) 2

FN0092

30. If $f(x) = ax + b$ and $f(f(f(x))) = 27x + 13$ where a and b are real numbers, then-

(A) $a + b = 3$ (B) $a + b = 4$ (C) $f'(x) = 3$ (D) $f'(x) = -3$

FN0093

Linked Comprehension Type

Paragraph for Question 31 to 33

$f(x) = \dfrac{x^2+1}{ax}$ $(a \neq 0)$; $g(x) = 3\sec x$; $h(x) = \dfrac{x+3}{x-4}$.

On the basis of above information, answer the following questions :

31. If range of $f(x)$ and $g(x)$ are equal sets then 'a' is equal to -

(A) 3 (B) –2/3 (C) 3/2 (D) –3/2

FN0094

32. $f(x)$ is one-one if-

(A) $x \in (0,\infty)$ (B) $x \in (-\infty,0)$ (C) $x \in (1,\infty)$ (D) $x \in (-\infty,1) - \{0\}$

FN0094

33. Which the following is always false ?

(A) h(x) is one-one

(B) $f(x)$ is one-one if $x > 10$

(C) g(x) is many-one if $x \in \left(0,\dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2},3\right)$

(D) The values of k for which $f(x) = k$ has exactly one solution is $k = 2$ or $k = -2$

FN0094

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Matrix Match Type

34.

	Column-I Number of integers in		**Column-II**
(A)	Domain of $f(x) = \ell n\{x\}$	(P)	0
(B)	Domain of $f(x) = \sqrt{\sec(\sin x)} + \sqrt{\left[x + \dfrac{1}{x}\right]} + \sqrt{10 - [x]^2}$	(Q)	2
(C)	Range of $f(x) = x^2 - 2x + 2$, $x \in [0,2]$	(R)	3
(D)	Range of $f(x) = \sqrt{25 - [x]^2}$	(S)	less than 3
		(T)	more than 3

(where [.] and {.} denote greatest integer function and fractional part function respectively)

FN0095

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35. Match the function mentioned in column-I with the respective classification given in column-II. (where [.] and {.} denote greatest integer function and fractional part function respectively)

Column-I	Column-II
(A) $f:\mathbb{R}\to\mathbb{R}^+$ $f(x)=\left(e^{[x]}\right)\left(e^{\{x\}}\right)$	(P) one-one
(B) $f:(-\infty,-2)\cup(0,\infty)\to\mathbb{R}$ $f(x)=\ell n(x^2+2x)$	(Q) many-one
(C) $f:[-2,2]\to[-1,1]$ $f(x)=\sin x$	(R) onto
(D) $f:\mathbb{R}\to\mathbb{R}$ $f(x)=x^3-3x^2+3x-7$	(S) periodic
	(T) aperiodic

FN0096

EXERCISE (S-1)

1. Find the domains of definitions of the following functions :
(Read the symbols [*] and {*} as greatest integers and fractional part functions respectively.)

(i) $f(x)=\sqrt{\cos 2x}+\sqrt{16-x^2}$ **FN0097**

(ii) $f(x)=\log_7\log_5\log_3\log_2(2x^3+5x^2-14x)$ **FN0098**

(iii) $f(x)=\ln\left(\sqrt{x^2-5x-24}-x-2\right)$ **FN0099**

(iv) $f(x)=\sqrt{\dfrac{1-5^x}{7^{-x}-7}}$ **FN0100**

(v) $y=\log_{10}\sin(x-3)+\sqrt{16-x^2}$ **FN0101**

(vi) $f(x)=\log_{100x}\left(\dfrac{2\log_{10}x+1}{-x}\right)$ **FN0102**

(vii) $f(x)=\sqrt{x^2-|x|}+\dfrac{1}{\sqrt{9-x^2}}$ **FN0103**

(viii) $f(x)=\sqrt{(x^2-3x-10)\ln^2(x-3)}$ **FN0104**

(ix) $f(x)=\sqrt{(5x-6-x^2)\left[\{\ell n\{x\}\}\right]}+\sqrt{(7x-5-2x^2)}+\left(\ln\left(\dfrac{7}{2}-x\right)\right)^{-1}$ **FN0105**

(x) $f(x)=\log_{\left[x+\frac{1}{x}\right]}\left|x^2-x-6\right|+{}^{16-x}C_{2x-1}+{}^{20-3x}P_{2x-5}$ **FN0106**

2. Find the domain & range of the following functions.

(i) $y=\log_{\sqrt{5}}\left(\sqrt{2}(\sin x-\cos x)+3\right)$ **FN0107**

(ii) $y=\dfrac{2x}{1+x^2}$ **FN0108**

(iii) $f(x)=\dfrac{x^2-3x+2}{x^2+x-6}$ **FN0109**

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(iv) $f(x) = \dfrac{x}{1+|x|}$ **FN0110**

(v) $y = \sqrt{2-x} + \sqrt{1+x}$ **FN0111**

(vi) $f(x) = \dfrac{\sqrt{x+4}-3}{x-5}$ **FN0112**

3. (i) Write explicitly, functions of y defined by the following equations and also find the domains of definition of the given implicit functions :

(a) $10^x + 10^y = 10$ (b) $x + |y| = 2y$

FN0113

(ii) The function f(x) is defined on the interval [0, 1]. Find the domain of definition of the functions.

(a) f(sinx) (b) f(2x + 3)

FN0114

(iii) Given that $y = f(x)$ is a function whose domain is [4,7] and range is [–1, 9]. Find the range and domain of

(a) $g(x) = \dfrac{1}{3}f(x)$ (b) $h(x) = f(x-7)$

FN0115

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4. The sum of integral values of the elements in the domain of $f(x) = \sqrt{\log_{\frac{1}{2}}|3-x|}$ is

FN0116

5. Number of integers in range of $f(x) = x(x+2)(x+4)(x+6)+7$, $x \in [-4,2]$ is

FN0117

6. The number of even integral value(s) in the range of the function $f(x) = \dfrac{\tan^2 x + 8\tan x + 15}{1+\tan^2 x}$ is

FN0118

7. (a) Draw graphs of the following function, where [ ] denotes the greatest integer function.

(i) $f(x) = x + [x]$ (ii) $y = (x)^{[x]}$ where $x = [x] + (x)$ & $x > 0$ & $x \le 3$

(iii) $y = \text{sgn}[x]$ (iv) $\text{sgn}(x - |x|)$

FN0119

(b) Identify the pair(s) of functions which are identical ?
(where [x] denotes greatest integer and {x} denotes fractional part function)

(i) $f(x) = \text{sgn}(x^2 - 3x + 4)$ and $g(x) = e^{[\{x\}]}$ (ii) $f(x) = \sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$ and $g(x) = \tan x$

(iii) $f(x) = \ln(1+x) + \ln(1-x)$ and $g(x) = \ln(1-x^2)$ (iv) $f(x) = \dfrac{\cos x}{1-\sin x}$ and $g(x) = \dfrac{1+\sin x}{\cos x}$

FN0120

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8. Solve the following problems from (a) to (d) on functional equation :

(a) The function f(x) defined on the real numbers has the property that f(f(x)).(1+f(x))=–f(x) for all x in the domain of f. If the number 3 is in the domain and range of f, compute the value of f(3).

FN0121

(b) Suppose f is a real function satisfying $f(x + f(x)) = 4f(x)$ and $f(1) = 4$. Find the value of f(21).

FN0122

(c) Let f be function defined from $\mathbb{R}^+ \to \mathbb{R}^+$. If $[f(xy)]^2 = x(f(y))^2$ for all positive numbers x and y and f(2) = 6, find the value of f(50).

FN0123

(d) Let f be a function such that f(3) = 1 and f(3x) = x + f(3x – 3) for all x. Then find the value of f(300).

FN0124

9. Classify the following functions f(x) defined in $\mathbb{R} \to \mathbb{R}$ as injective, surjective, both or none.

(a) $f(x) = \dfrac{x^2 + 4x + 30}{x^2 - 8x + 18}$ **FN0125**

(b) $f(x) = x^3 - 6x^2 + 11x - 6$ **FN0126**

(c) $f(x) = (x^2 + x + 5)(x^2 + x - 3)$ **FN0127**

10. Suppose f(x) = sinx and $g(x) = 1 - \sqrt{x}$. Then find the domain and range of the following functions.

(a) fog (b) gof (c) fof (d) gog

FN0128

11. A function $f : \mathbb{R} \to \mathbb{R}$ is such that $f\left(\dfrac{1-x}{1+x}\right) = x$ for all $x \neq -1$. Prove the following.

(a) $f(f(x)) = x$ (b) $f(1/x) = -f(x)$, $x \neq 0$ (c) $f(-x - 2) = -f(x) - 2$.

FN0129

12. (a) Find the formula for the function fogoh, given $f(x) = \dfrac{x}{x+1}$; $g(x) = x^{10}$ and h(x) = x + 3. Find also the domain of this function. Also compute (fogoh)(–1).

FN0130

(b) If f(x) = max(x, 1/x) for x > 0 where max (a, b) denotes the greater of the two real numbers a and b. Define the function g(x) = f(x) f(1/x) and plot its graph.

FN0131

13. $f(x) = \begin{cases} 1-x & \text{if } x \le 0 \\ x^2 & \text{if } x > 0 \end{cases}$ and $g(x) = \begin{cases} -x & \text{if } x < 1 \\ 1-x & \text{if } x \ge 1 \end{cases}$ find (fog)(x) and (gof)(x).

FN0132

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14. Compute the inverse of the functions :

(a) $f(x) = \ell n\left(x+\sqrt{x^2+1}\right)$ **FN0133**

(b) $f(x) = 2^{\frac{x}{x-1}}$ **FN0134**

(c) $y = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}}$ **FN0135**

15. Find the inverse of $f(x) = 2^{\log_{10} x} + 8$ and hence solve the equation $f(x) = f^{-1}(x)$.

FN0136

16. Let $f(x) = \dfrac{2x-1}{x+3}$. If $f^{-1} = \dfrac{ax+b}{c-x}$, then a + b + c is

FN0137

17. Let f be a one-one function with domain {x, y, z} and range {1, 2, 3}. It is given that exactly one of the following statements is true and the remaining two are false.

$f(x) = 1$; $f(y) \neq 1$; $f(z) \neq 2$. Determine $f^{-1}(1)$

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FN0138

18. Find whether the following functions are even or odd or none :

(a) $f(x) = \log\left(x+\sqrt{1+x^2}\right)$ **FN0139**

(b) $f(x) = \dfrac{x(a^x+1)}{a^x-1}$ **FN0140**

(c) $f(x) = \sin x + \cos x$ **FN0141**

(d) $f(x) = x\sin^2 x - x^3$ **FN0142**

(e) $f(x) = \sin x - \cos x$ **FN0143**

(f) $f(x) = \dfrac{(1+2^x)^2}{2^x}$ **FN0144**

(g) $f(x) = \dfrac{x}{e^x-1} + \dfrac{x}{2} + 1$ **FN0145**

(h) $f(x) = [(x+1)^2]^{1/3} + [(x-1)^2]^{1/3}$ **FN0146**

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19. (a) Function f & g are defined by f(x) = sinx, x $\in \mathbb{R}$; g(x) = tanx, x $\in \mathbb{R} - \left(K + \frac{1}{2}\right)\pi$ where K $\in \mathbb{Z}$. Find (i) periods of fog & gof (ii) range of the function fog & gof

FN0147

(b) Suppose that f is an even, periodic function with period 2, and that f(x) = x for all x in interval [0, 1]. Find the value of f(3.14).

FN0148

20. Let $f(x)$ be a periodic function with period 'p' satisfying $f(x) + f(x+3) + f(x+6) + ... + f(x+42)$ = constant $\forall$ x $\in \mathbb{R}$, then sum of digits of 'p' is

FN0149

21. Let 'f' be an even periodic function with period '4' such that $f(x) = 2^x - 1,\ 0 \le x \le 2$. The number of solutions of the equation $f(x) = 1$ in [–10, 20] are

FN0150

22. If $f(x) = \begin{cases} 0 & x < 1 \\ 2x - 2 & x \ge 1 \end{cases}$; then the number of solutions of the equation $f(f(f(x))) = x$ is

FN0151

23. Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$ be a polynomial such that $P(1) = 1, P(2) = 8, P(3) = 27, P(4) = 64$, then find P(10).

FN0152

24.

Column-I		**Column-II**	
	f(x)		**Range**
(A)	$\dfrac{\cos^2 x + \cos x + 2}{\cos^2 x + \cos x + 1}$	(P)	$\left(0, \dfrac{7}{3}\right]$
(B)	$\left\lvert \dfrac{(\sqrt{\cos x} - \sqrt{\sin x})(\sqrt{\cos x} + \sqrt{\sin x})}{3(\cos x + \sin x)} \right\rvert$	(Q)	$\left[\dfrac{4}{3}, \dfrac{7}{3}\right]$
(C)	$\dfrac{7}{3(x^6 + 2x^4 + 3x^2 + 1)}$	(R)	$\left[0, \dfrac{1}{3}\right]$
(D)	$\log_8(x^2 + 2x + 2)$	(S)	$[0, \infty)$

FN0153

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25. The graph of the function y = f(x) is as follows :

Match the function mentioned in **Column-I** with the respective graph given in **Column-II**.

Column-I	**Column-II**
(A) $y = \|f(x)\|$	(P)
(B) $y = f(\|x\|)$	(Q)
(C) $y = f(-\|x\|)$	(R)
(D) $y = \frac{1}{2}(\|f(x)\| - f(x))$	(S)

FN0154

26. If $f(x) = a\log\left(\frac{1+x}{1-x}\right) + bx^3 + c\sin x + 5$ and $f(\log_3 2) = 4$, then $f\left(\log_3\left(\frac{1}{2}\right)\right)$ is equal to

FN0155

EXERCISE (S-2)

1. Find the domains of definitions of the following functions :

(Read the symbols [*] as greatest integers function)

(i) $f(x) = \sqrt{\log_x(\cos 2\pi x)}$ **FN0156**

(ii) $f(x) = \frac{[x]}{2x - [x]}$ **FN0157**

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2. (a) Let $P(x) = x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ be a polynomial such that $P(1) = 1; P(2) = 2; P(3) = 3; P(4) = 4; P(5) = 5$ and $P(6) = 6$, then find the value of $P(7)$.

FN0158

(b) Let a and b be real numbers and let $f(x) = a\sin x + b\sqrt[3]{x} + 4, \forall x \in R$. If $f(\log_{10}(\log_3 10)) = 5$ then find the value of $f(\log_{10}(\log_{10} 3))$.

FN0159

3. Suppose p(x) is a polynomial with integer coefficients. The remainder when p(x) is divided by $x - 1$ is 1 and the remainder when p(x) is divided by $x - 4$ is 10. If r(x) is the remainder when p(x) is divided by $(x - 1)(x - 4)$, find the value of r(2006).

FN0160

4. A function f, defined for all $x, y \in \mathbb{R}$ is such that $f(1) = 2$ & $f(x + y) - kxy = f(x) + 2y^2$, where k is some constant. Find f(x) & show that :

$$f(x+y)f\left(\frac{1}{x+y}\right) = k \text{ for } x + y \neq 0.$$

FN0161

5. If $f(x) = -1 + |x - 2|, 0 \le x \le 4$

$g(x) = 2 - |x|, -1 \le x \le 3$

Then find fog(x) & gof(x). Draw rough sketch of the graphs of fog(x) & gof(x).

FN0162

6. Let $f(x) = x^{135} + x^{125} - x^{115} + x^5 + 1$. If $f(x)$ is divided by $x^3 - x$ then the remainder is some function of x say g(x). Find the value of g(10).

FN0163

7. Let {x} & [x] denote the fractional and integral part of a real number x respectively. Solve $4\{x\} = x + [x]$.

FN0164

8. Let $f(x) = \frac{9^x}{9^x + 3}$ then find the value of the sum $f\left(\frac{1}{2006}\right) + f\left(\frac{2}{2006}\right) + f\left(\frac{3}{2006}\right) + ...f\left(\frac{2005}{2006}\right)$

FN0165

9. Let $f(x) = (x + 1)(x + 2)(x + 3)(x + 4) + 5$ where $x \in [-6, 6]$. If the range of the function is [a, b] where $a, b \in N$ then find the value of $(a + b)$.

FN0166

10. The set of real values of 'x' satisfying the equality $\left[\frac{3}{x}\right] + \left[\frac{4}{x}\right] = 5$ (where [ ] denotes the greatest integer function) belongs to the interval $\left(a, \frac{b}{c}\right]$ where $a, b, c \in N$ and $\frac{b}{c}$ is in its lowest form. Find the value of $a + b + c + abc$.

FN0167

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11. $f(x)$ and g(x) are linear function such that for all x, $f(g(x))$ and $g(f(x))$ are Identity functions. If $f(0) = 4$ and $g(5) = 17$, compute f(2006).

FN0168

Comprehension

Let $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying $f(2 - x) = f(2 + x)$ and $f(20 - x) = f(x)$, $\forall\ x \in \mathbb{R}$.

On the basis of above information, answer the following questions :

12. If f(0) = 5, then minimum possible number of values of x satisfying f(x) = 5, for $x \in [0, 170]$ is-

(A) 21 (B) 12 (C) 11 (D) 22

FN0169

13. Graph of y = f(x) is -

(A) symmetrical about x = 18 (B) symmetrical about x = 5

(C) symmetrical about x = 8 (D) symmetrical about x = 20

FN0169

14. If $f(2) \neq f(6)$, then

(A) fundamental period of f(x) is 1 (B) fundamental period of f(x) may be 1

(C) period of f(x) can't be 1 (D) fundamental period of f(x) is 8

FN0169

15. The function $f(x)$ has the property that for each real number x in its domain, 1/x is also in its domain and $f(x) + f(1/x) = x$. Find the largest set of real numbers that can be in the domain of $f(x)$?

FN0170

EXERCISE (JM)

1. The domain of the function $f(x) = \dfrac{1}{\sqrt{|x| - x}}$ is :- **[AIEEE - 2011]**

(1) $(-\infty, 0)$ (2) $(-\infty, \infty) - \{0\}$ (3) $(-\infty, \infty)$ (4) $(0, \infty)$

FN0171

2. Let f be a function defined by $f(x) = (x - 1)^2 + 1$, $(x \geq 1)$ **[AIEEE - 2009, 2011]**

Statement - 1 : The set $\{x : f(x) = f^{-1}(x)\} = \{1, 2\}$

Statement - 2 : f is bijection and $f^{-1}(x) = 1 + \sqrt{x - 1}$, $x \geq 1$.

(1) Statement–1 is true, Statement–2 is false.

(2) Statement–1 is false, Statement–2 is true.

(3) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.

(4) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for statement–1.

FN0172

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3. If $a \in \mathbb{R}$ and the equation $-3(x-[x])^2 + 2(x-[x]) + a^2 = 0$ (where [x] deontes the greatest integer $\le x$) has no integral solution, then all possible values of a lie in the interval : **[JEE(Main)-2014]**

(1) $(-1, 0) \cup (0, 1)$ (2) $(1, 2)$ (3) $(-2, -1)$ (4) $(-\infty, -2) \cup (2, \infty)$

FN0173

4. If $f(x) + 2f\left(\frac{1}{x}\right) = 3x$, $x \neq 0$, and **[JEE(Main)-2016]**

$S = \{x \in \mathbb{R} : f(x) = f(-x)\}$; then S :

(1) contains more than two elements. (2) is an empty set.

(3) contains exactly one element (4) contains exactly two elements

FN0174

5. The function $f : \mathbb{R} \to \left[-\frac{1}{2}, \frac{1}{2}\right]$ defined as $f(x) = \frac{x}{1+x^2}$, is : **[JEE(Main)-2017]**

(1) neither injective nor surjective. (2) invertible.

(3) injective but not surjective. (4) surjective but not injective

FN0175

6. Let $S = \{x \in \mathbb{R} : x \ge 0 \text{ and } 2|\sqrt{x} - 3| + \sqrt{x}(\sqrt{x} - 6) + 6 = 0\}$. Then S : **[JEE(Main)-2018]**

(1) contains exactly one element. (2) contains exactly two elements.

(3) contains exactly four elements. (4) is an empty set.

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FN0176

7. For $x \in \mathbb{R} - \{0, 1\}$, let $f_1(x) = \frac{1}{x}$, $f_2(x) = 1 - x$ and $f_3(x) = \frac{1}{1-x}$ be three given functions. If a function, J(x) satisfies $(f_2 o J o f_1)(x) = f_3(x)$ then J(x) is equal to :- **[JEE(Main)-2019]**

(1) $f_3(x)$ (2) $f_1(x)$ (3) $f_2(x)$ (4) $\frac{1}{x} f_3(x)$

FN0177

8. Let $\mathbb{N}$ be the set of natural numbers and two functions f and g be defined as $f, g : \mathbb{N} \to \mathbb{N}$ such that : $f(n) = \begin{cases} \frac{n+1}{2} & \text{if n is odd} \\ \frac{n}{2} & \text{if n is even} \end{cases}$ and $g(n) = n - (-1)^n$. The fog is : **[JEE(Main)-2019]**

(1) Both one-one and onto (2) One-one but not onto

(3) Neither one-one nor onto (4) onto but not one-one

FN0178

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9. Let f : $\mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{x}{1+x^2}$, $x \in \mathbb{R}$. Then the range of f is : **[JEE(Main)-2019]**

(1) $(-1, 1) - \{0\}$ (2) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (3) $R - \left[-\frac{1}{2}, \frac{1}{2}\right]$ (4) $R - [-1, 1]$

FN0179

10. The number of functions f from {1, 2, 3, ..., 20} onto {1, 2, 3, ..., 20} such that f(k) is a multiple of 3, whenever k is a multiple of 4, is :- **[JEE(Main)-2019]**

(1) $(15)! \times 6!$ (2) $5^6 \times 15$ (3) $5! \times 6!$ (4) $6^5 \times (15)!$

FN0180

11. If $f(x) = \log_e\left(\frac{1-x}{1+x}\right), |x| < 1$, then $f\left(\frac{2x}{1+x^2}\right)$ is equal to : **[JEE(Main)-2019]**

(1) $2f(x)$ (2) $2f(x^2)$ (3) $(f(x))^2$ (4) $-2f(x)$

FN0181

12. Let $\sum_{k=1}^{10} f(a+k) = 16\left(2^{10}-1\right)$, where the function f satisfies $f(x+y) = f(x)f(y)$ for all natural numbers x, y and $f(1) = 2$. then the natural number 'a' is **[JEE(Main)-2019]**

(1) 4 (2) 3 (3) 16 (4) 2

FN0182

13. Le $f(x) = x^2$, $x \in \mathbb{R}$. For any $A \subseteq R$, define $g(A) = \{x \in \mathbb{R}, f(x) \in A\}$. If S = [0, 4], then which one of the following statements is not true ? **[JEE(Main)-2019]**

(1) $f(g(S)) \neq f(S)$ (2) $f(g(S)) = S$ (3) $g(f(S)) = g(S)$ (4) $g(f(S)) \neq S$

FN0183

14. The number of real roots of the equation $5 + |2^x - 1| = 2^x(2^x - 2)$ is : **[JEE(Main)-2019]**

(1) 2 (2) 3 (3) 4 (4) 1

FN0184

15. For $x \in \left(0, \frac{3}{2}\right)$, let $f(x) = \sqrt{x}$, $g(x) = \tan x$ and $h(x) = \frac{1-x^2}{1+x^2}$. If $\phi(x) = ((hof)og)(x)$, then $\phi = \left(\frac{\pi}{3}\right)$ is equal to : **[JEE(Main)-2019]**

(1) $\tan\frac{\pi}{12}$ (2) $\tan\frac{7\pi}{12}$ (3) $\tan\frac{11\pi}{12}$ (4) $\tan\frac{5\pi}{12}$

FN0185

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16. For $x \in \mathbb{R}$, let [x] denote the greatest integer $\le x$, then the sum of the series

$\left[-\frac{1}{3}\right]+\left[-\frac{1}{3}-\frac{1}{100}\right]+\left[-\frac{1}{3}-\frac{2}{100}\right]+.....+\left[-\frac{1}{3}-\frac{99}{100}\right]$ is **[JEE(Main)-2019]**

(1) –153 (2) –133 (3) –131 (4) –135

FN0186

17. If $g(x) = x^2 + x - 1$ and $(gof)(x) = 4x^2 - 10x + 5$, then $f\left(\frac{5}{4}\right)$ is equal to **[JEE(Main)-2020]**

(1) $\frac{3}{2}$ (2) $-\frac{1}{2}$ (3) $-\frac{3}{2}$ (4) $\frac{1}{2}$

FN0187

18. Let $f : (1,3) \to R$ be a function defined by $f(x)=\frac{x[x]}{1+x^2}$, where [x] denotes the greatest integer $\le$ x. Then the range of f is **[JEE(Main)-2020]**

(1) $\left(\frac{3}{5},\frac{4}{5}\right)$ (2) $\left(\frac{2}{5},\frac{3}{5}\right]\cup\left(\frac{3}{4},\frac{4}{5}\right)$ (3) $\left(\frac{2}{5},\frac{4}{5}\right]$ (4) $\left(\frac{2}{5},\frac{1}{2}\right)\cup\left(\frac{3}{5},\frac{4}{5}\right]$

FN0188

19. Let $f : R \to R$ be such that for all $x \in R$ $(2^{1+x} + 2^{1-x})$, $f(x)$ and $(3^x + 3^{-x})$ are in A.P., then the minimum value of $f(x)$ is **[JEE(Main)-2020]**

(1) 0 (2) 3 (3) 2 (4) 4

FN0189

20. The inverse function of $f(x)=\frac{8^{2x}-8^{-2x}}{8^{2x}+8^{-2x}}, x\in(-1,1)$, is **[JEE(Main)-2020]**

(1) $\frac{1}{4}(\log_8 e)\log_e\left(\frac{1-x}{1+x}\right)$ (2) $\frac{1}{4}\log_e\left(\frac{1-x}{1+x}\right)$

(3) $\frac{1}{4}(\log_8 e)\log_e\left(\frac{1+x}{1-x}\right)$ (4) $\frac{1}{4}\log_e\left(\frac{1+x}{1-x}\right)$

FN0190

EXERCISE (JA)

1. If functions f (x) and g (x) are defined on $\mathbb{R} \to \mathbb{R}$ such that

$f(x)=\begin{cases}0, & x\in \text{rational}\\ x, & x\in \text{irrational}\end{cases}$, $g(x)=\begin{cases}0, & x\in \text{irrational}\\ x, & x\in \text{rational}\end{cases}$, then (f – g) (x) is -

(A) one-one and onto (B) neither one-one nor onto

(C) one-one but not onto (D) onto but not one-one **[JEE 2005 (Scr.)]**

FN0191

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2. Let f(x) = x^2 and g(x) = sinx for all $x \in \mathbb{R}$. Then the set of all x satisfying $(f \circ g \circ g \circ f)(x) = (g \circ g \circ f)(x)$, where $(f \circ g)(x) = f(g(x))$, is- **[JEE 2011, 3, (–1)]**

(A) $\pm\sqrt{n\pi}, n \in \{0,1,2,....\}$

(B) $\pm\sqrt{n\pi}, n \in \{1,2,...\}$

(C) $\frac{\pi}{2} + 2n\pi, n \in \{....,-2,-1,0,1,2,.....\}$

(D) $2n\pi, n \in \{....,-2,-1,0,1,2,.....\}$

FN0192

3. The function $f : [0,3] \to [1,29]$, defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$, is : **[JEE 2012, 3, (–1)]**

(A) one-one and onto

(B) onto but not one-one

(C) one-one but not onto

(D) neither one-one nor onto

FN0193

4. Let $f : (-1,1) \to \mathbb{R}$ be such that $f(\cos 4\theta) = \frac{2}{2 - \sec^2\theta}$ for $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Then the value(s) of $f\left(\frac{1}{3}\right)$ is (are) - **[JEE 2012, 4]**

(A) $1 - \sqrt{\frac{3}{2}}$

(B) $1 + \sqrt{\frac{3}{2}}$

(C) $1 - \sqrt{\frac{2}{3}}$

(D) $1 + \sqrt{\frac{2}{3}}$

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FN0194

5. Let $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ be given by $f(x) = (\log(\sec x + \tan x))^3$ Then :-

(A) $f(x)$ is an odd function

(B) $f(x)$ is a one-one function

(C) $f(x)$ is an onto function

(D) $f(x)$ is an even function

[JEE(Advanced)-2014, 3]

FN0195

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ANSWER KEY

FUNCTION

EXERCISE (O-1)

1. (i) $-\infty < x \le 0$ (ii) $x \in \mathbb{R}$ (iii) $x \in \mathbb{R} - \{-1,0,1\}$

(iv) $-\infty < x < 0$ & $4 < x < \infty$ (v) $-\infty < x \le 1$ and $3 \le x < \infty$ (vi) $-\infty < x < 1$ and $2 < x < \infty$

(vii) $-1 \le x \le 1$ (viii) $0 < x < 1$ and $1 < x < \infty$. (ix) $-2 \le x < 0$ and $0 < x < 1$

(x) $\frac{3}{2} < x < 2$ and $2 < x < \infty$. (xi) $-1 < x < 0$ and $1 < x < 2$; $2 < x < \infty$

(xii) $2k\pi < x < (2k+1)\pi$, where k is an integer. (xiii) $4 \le x \le 6$ (xiv) $2 < x < 3$

2. (i) $\mathbb{R} - \{1\}$ (ii) $\mathbb{R} - \{0\}$ (iii) $(0,4/3]$ (iv) $[1/3, 3]$ (v) $[1,\infty)$ (vi) $\mathbb{R}$

(vii) $[0,\infty)$ (viii) $[3,7]$ (ix) $[-1,1]$ (x) $[2,\infty)$ (xi) $[1/3,1)$ (xii) $\left[\frac{1}{11}, \frac{1}{5}\right]$

3. C **4.** B **5.** C **6.** B **7.** C

8. (a) $y = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 < x \le 2 \end{cases}$ (b) $y = \begin{cases} 2, & 0 \le x < 1 \text{ and } 2 \le x < 3 \\ 0, & 1 \le x < 2 \text{ and } 3 \le x \le 4 \end{cases}$

9. C **10.** C **11.** D **12.** D **13.** A **14.** C **15.** A

16. (a) -2, (b) -3, 1, (c) $[-3, 3]$, $[-2, 3]$, (d) $[-1, 3]$, (e) B, (f) A,D

17.

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18. (a) $-2, 4$, (b) $-2, 2$, (c) $-3, 4$, (d) $[0, 4]$ (e) $[-4, 4], [-2, 3]$ (f) $[-4, 3]$, $[0.5, 4]$

19. (a) $\left[\frac{4}{3}, \frac{5}{3}\right] \cup \{3\}$ (b) $[-2, -1) \cup [1, 3]$ (c) $[-3, -2] \cup [-1, 1]$ (d) $\left(\frac{5}{3}, 3\right) - \{2\}$

20. D **21.** D **22.** C **23.** D **24.** B **25.** D **26.** B

27. C **28.** B **29.** D **30.** C **31.** B,C **32.** A,B,C,D

33. A,B,C **34.** A,B,C **35.** A,B,C **36.** A,C **37.** A,B,C

38. (A)→(P,R,T); (B)→(Q,R); (C)→(Q,S); (D)→(P,R,T) **39.** A **40.** C

EXERCISE (O-2)

1. B **2.** B **3.** A **4.** C **5.** C **6.** D **7.** D

8. A **9.** B **10.** A **11.** B **12.** B **13.** D **14.** C

15. D **16.** B **17.** C **18.** A **19.** A,C **20.** B,C,D **21.** A,C

22. C,D **23.** B,C,D **24.** A,B,C **25.** A,D **26.** A,B,C **27.** A,B,C,D

28. A,C **29.** B,C **30.** B,C **31.** B **32.** C **33.** C

34. (A)→(P,S); (B)→(R); (C)→(Q,S); (D)→(T) **35.** (A)→(P,R,T); (B)→(Q,R,T); (C)→(Q,R,T) ; (D)→(P,R,T)

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EXERCISE (S-1)

1. **(i)** $\left[-\frac{5\pi}{4}, -\frac{3\pi}{4}\right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$ **(ii)** $\left(-4, -\frac{1}{2}\right) \cup (2, \infty)$ **(iii)** $(-\infty, -3]$

(iv) $(-\infty, -1) \cup [0, \infty)$ **(v)** $(3 - 2\pi < x < 3 - \pi) \cup (3 < x \le 4)$

(vi) $\left(0, \frac{1}{100}\right) \cup \left(\frac{1}{100}, \frac{1}{\sqrt{10}}\right)$

(vii) $(-3, -1] \cup \{0\} \cup [1, 3)$ **(viii)** $\{4\} \cup [5, \infty)$ **(ix)** $(1, 2) \cup (2, 5/2)$ **(x)** $x \in \{4, 5\}$

2. **(i)** $D : x \in \mathbb{R}$ $\mathbb{R} : [0, 2]$

(ii) $D = \mathbb{R}$; range $[-1, 1]$

(iii) $D : \{x \mid x \in \mathbb{R}; x \ne -3; x \ne 2\}$ $\mathbb{R} : \{f(x) \mid f(x) \in \mathbb{R}, f(x) \ne 1/5; f(x) \ne 1\}$

(iv) $D : \mathbb{R}, \mathbb{R} : (-1, 1)$

(v) $D : -1 \le x \le 2$ $\mathbb{R} : \left[\sqrt{3}, \sqrt{6}\right]$

(vi) $D : [-4, \infty) - \{5\}; \mathbb{R} : \left(0, \frac{1}{6}\right) \cup \left(\frac{1}{6}, \frac{1}{3}\right]$

3. **(i)** **(a)** $y = \log(10 - 10^x), -\infty < x < 1$

(b) $y = x/3$ when $-\infty < x < 0$ & $y = x$ when $0 \le x < +\infty$

(ii) **(a)** $2K\pi \le x \le 2K\pi + \pi$ where $K \in I$ **(b)** $[-3/2, -1]$

(iii) **(a)** Range : $[-1/3, 3]$, Domain = $[4, 7]$; **(b)** Range $[-1, 9]$ and domain $[11, 14]$

4. 6 **5.** 401 **6.** 9 **7.** **(b)** (i), (iii) are identical

8. (a) $-3/4$; (b) 64; (c) 30, (d) 5050

9. **(a)** neither surjective nor injective **(b)** surjective but not injective

(c) neither injective nor surjective

10. **(a)** domain is $x \ge 0$; range $[-1, 1]$; **(b)** domain $2k\pi \le x \le 2k\pi + \pi$; range $[0, 1]$

(c) domain $x \in \mathbb{R}$; range $[-\sin 1, \sin 1]$ **(d)** domain is $0 \le x \le 1$; range is $[0, 1]$

12. **(a)** $\frac{(x+3)^{10}}{(x+3)^{10}+1}$, domain is $\mathbb{R}$, $\frac{1024}{1025}$; **(b)** $g(x) = \begin{cases} \frac{1}{x^2} & \text{if } 0 < x \le 1 \\ x^2 & \text{if } x > 1 \end{cases}$

13. $(gof)(x) = \begin{cases} x & \text{if } x \le 0 \\ -x^2 & \text{if } 0 < x < 1 \\ 1 - x^2 & \text{if } x \ge 1 \end{cases}$; $(fog)(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 1 + x & \text{if } 0 \le x < 1 \\ x & \text{if } x \ge 1 \end{cases}$

14. **(a)** $\frac{e^x - e^{-x}}{2}$; **(b)** $\frac{\log_2 x}{\log_2 x - 1}$; **(c)** $\frac{1}{2}\log\frac{1+x}{1-x}$

15. $x = 10$; $f^{-1}(x) = 10^{\log_2(x-8)}$ **16.** 6 **17.** $f^{-1}(1) = y$

18. (a) odd, (b) even, (c) neither odd nor even, (d) odd, (e) neither odd nor even,

(f) even, (g) even, (h) even

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19. **(a) (i)** Period of fog is π, period of gof is 2π **(ii)** range of fog is $[-1, 1]$, range of gof is $[-\tan 1, \tan 1]$
(b) 0.86

20. 9 **21.** 15 **22.** 2 **23.** 4024

24. (A) $\rightarrow$ (Q); (B) $\rightarrow$ (R); (C) $\rightarrow$ (P); (D) $\rightarrow$ (S) **25.** (A) S; (B) R; (C) P; (D) Q **26.** 6

EXERCISE (S-2)

1. **(i)** $(0, 1/4) \cup (3/4, 1) \cup \{x : x \in N, x \geq 2\}$ **(ii)** $R - \left\{-\frac{1}{2}, 0\right\}$

2. **(a) 727, (b) 3** **3.** 6016 **4.** $f(x) = 2x^2$

5. $fog(x) = \begin{cases} -(1+x), & -1 \leq x \leq 0 \\ x-1, & 0 < x \leq 2 \end{cases}$; $gof(x) = \begin{cases} x+1, & 0 \leq x < 1 \\ 3-x, & 1 \leq x \leq 2 \\ x-1, & 2 < x \leq 3 \\ 5-x, & 3 < x \leq 4 \end{cases}$

$fof(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 4-x, & 3 \leq x \leq 4 \end{cases}$; $gog(x) = \begin{cases} -x, & -1 \leq x \leq 0 \\ x, & 0 < x \leq 2 \\ 4-x, & 2 < x \leq 3 \end{cases}$

6. 21

7. $x = 0$ या 5/3 **8.** 1002.5 **9.** 5049 **10.** 20

11. 122 **12.** D **13.** A **14.** C **15.** $\{-1, 1\}$

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EXERCISE (JM)

1. 1 **2.** 3 **3.** 1 **4.** 4 **5.** 4 **6.** 2 **7.** 1 **8.** 4

9. 2 **10.** 1 **11.** 1 **12.** 2 **13.** 3 **14.** 4 **15.** 3 **16.** 2

17. 2 **18.** 4 **19.** 2 **20.** 3

EXERCISE (JA)

1. A **2.** A **3.** B

4. (zero marks to all) Reason : (as we are getting two f images for one x in the domain) **5.** A,B,C

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INVERSE TRIGONOMETRIC FUNCTION

1. INTRODUCTION :

The inverse trigonometric functions, denoted by $\sin^{-1}x$ or (arc sinx), $\cos^{-1}x$ etc., denote the angles whose sine, cosine etc, is equal to x. The angles are usually the numerically smallest angles, except in the case of $\cot^{-1}x$ and if positive & negative angles have same numerical value, the positive angle has been chosen.

It is worthwhile noting that the functions sinx, cosx etc are in general not invertible. Their inverse is defined by choosing an appropriate domain & co-domain so that they become invertible. For this reason the chosen value is usually the simplest and easy to remember.

2. DOMAIN & RANGE OF INVERSE TRIGONOMETRIC FUNCTIONS :

S.No	$f(x)$	Domain	Range
(1)	$\sin^{-1}x$	$\lvert x\rvert \le 1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(2)	$\cos^{-1}x$	$\lvert x\rvert \le 1$	$[0, \pi]$
(3)	$\tan^{-1}x$	$x \in \mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(4)	$\sec^{-1}x$	$\lvert x\rvert \ge 1$	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$ or $\left[0, \frac{\pi}{2}\right)\cup\left(\frac{\pi}{2}, \pi\right]$
(5)	$\text{cosec}^{-1}x$	$\lvert x\rvert \ge 1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]-\{0\}$
(6)	$\cot^{-1}x$	$x \in \mathbb{R}$	$(0, \pi)$

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3. GRAPH OF INVERSE TRIGONOMETRIC FUNCTIONS :

(a) $f : \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ $\qquad$ $f^{-1} : [-1, 1] \to [-\pi/2, \pi/2]$

$f(x) = \sin x$ $\qquad$ $f^{-1}(x) = \sin^{-1}(x)$

(Taking image of sin x about y = x to get $\sin^{-1}x$)

($y = \sin^{-1}x$)

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(b) $f : [0, \pi] \to [-1, 1]$

$f(x) = \cos x$

(Taking image of cos x about y = x)

$f^{-1} : [-1, 1] \to [0, \pi]$

$f^{-1}(x) = \cos^{-1} x$

$(y = \cos^{-1}x)$

(c) $f : (-\pi/2, \pi/2) \to \mathbb{R}$

$f(x) = \tan x$

(Taking image of tan x about y = x)

$f^{-1} : \mathbb{R} \to (-\pi/2, \pi/2)$

$f^{-1}(x) = \tan^{-1} x$

$(y = \tan^{-1}x)$

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(d) $f : (0, \pi) \to \mathbb{R}$

$f(x) = \cot x$

(Taking image of cot x about y = x)

$f^{-1} : \mathbb{R} \to (0, \pi)$

$f^{-1}(x) = \cot^{-1} x$

$(y = \cot^{-1}x)$

(e) $f : [0, \pi/2) \cup (\pi/2, \pi] \to (-\infty, -1] \cup [1, \infty)$

$f(x) = \sec x$

$f^{-1} : (-\infty, -1] \cup [1, \infty) \to [0, \pi/2) \cup (\pi/2, \pi]$

$f^{-1}(x) = \sec^{-1} x$

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(f) $f : [-\pi/2, 0) \cup (0, \pi/2] \to (-\infty, -1] \cup [1, \infty)$

$f(x) = \text{cosec } x$

$f^{-1} : (-\infty, -1] \cup [1, \infty) \to [-\pi/2, 0) \cup (0, \pi/2]$

$f^{-1}(x) = \text{cosec}^{-1} x$

From the above discussions following IMPORTANT points can be concluded :

(i) All the inverse trigonometric functions represent an angle.

(ii) If $x > 0$, then all six inverse trigonometric functions viz $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\sec^{-1} x$, $\text{cosec}^{-1} x$, $\cot^{-1} x$ represent an acute angle.

(iii) If $x < 0$, then $\sin^{-1} x$, $\tan^{-1} x$ & $\text{cosec}^{-1} x$ represent an angle from $-\pi/2$ to 0 (IVth quadrant)

(iv) If $x < 0$, then $\cos^{-1} x$, $\cot^{-1} x$ & $\sec^{-1} x$ represent an obtuse angle. (IInd quadrant)

(v) IIIrd quadrant is never used in range of inverse trigonometric function.

Illustration 1 : The value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$ is equal to

(A) $\frac{\pi}{4}$ (B) $\frac{5\pi}{12}$ (C) $\frac{3\pi}{4}$ (D) $\frac{13\pi}{12}$

Solution : $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6} = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$ **Ans.(C)**

Illustration 2 : If $\sum_{i=1}^{2n} \cos^{-1} x_i = 0$ then find the value of $\sum_{i=1}^{2n} x_i$

Solution : We know, $0 \le \cos^{-1} x \le \pi$

Hence, each value $\cos^{-1}x_1, \cos^{-1}x_2, \cos^{-1}x_3, \ldots\ldots, \cos^{-1}x_{2n}$ are non-negative their sum is zero only when each value is zero.

i.e., $\cos^{-1}x_i = 0$ for all i

$\Rightarrow$ $x_i = 1$ for all i

$\therefore$ $\sum_{i=1}^{2n} x_i = x_1 + x_2 + x_3 + \ldots\ldots + x_{2n} = \underbrace{\{1+1+1\ldots\ldots\ldots+1\}}_{2n \text{ times}} = 2n$ {using (i)}

$\Rightarrow$ $\sum_{i=1}^{2n} x_i = 2n$ **Ans.**

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Do yourself - 1 :

(i) If α, β are roots of the equation $6x^2 + 11x + 3 = 0$, then

(A) both $\cos^{-1}\alpha$ and $\cos^{-1}\beta$ are real (B) both $\text{cosec}^{-1}\alpha$ and $\text{cosec}^{-1}\beta$ are real

(C) both $\cot^{-1}\alpha$ and $\cot^{-1}\beta$ are real (D) none of these

(ii) If $\sin^{-1}x + \sin^{-1}y = \pi$ and $x = ky$, then find the value of $39^{2k} + 5^k$.

3. PROPERTIES OF INVERSE CIRCULAR FUNCTIONS :

P-1 (i) $y = \sin(\sin^{-1}x) = x$

$x \in [-1,1], y \in [-1,1]$

(ii) $y = \cos(\cos^{-1}x) = x$

$x \in [-1,1], y \in [-1,1]$

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(iii) $y = \tan(\tan^{-1}x) = x$

$x \in \mathbb{R}, y \in \mathbb{R}$

(iv) $y = \cot(\cot^{-1}x) = x$,

$x \in \mathbb{R}; y \in \mathbb{R}$

(v) $y = \text{cosec}(\text{cosec}^{-1}x) = x$,

$|x| \geq 1, |y| \geq 1$

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(vi) $y = \sec(\sec^{-1} x) = x$

$|x| \geq 1$; $|y| \geq 1$

Note : All the above functions are aperiodic.

Illustration 3 : Evaluate the following :

(i) $\sin(\cos^{-1}3/5)$ (ii) $\cos(\tan^{-1} 3/4)$ (iii) $\sin\left(\frac{\pi}{2} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$

(iv) $\tan\left\{2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right\}$

Solution : (i) Let $\cos^{-1} 3/5 = \theta$. Then,

$\cos\theta = 3/5 \Rightarrow \sin\theta = 4/5$

$\therefore \sin(\cos^{-1} 3/5) = \sin\theta = 4/5$

(ii) Let $\tan^{-1} 3/4 = \theta$. Then,

$\tan\theta = 3/4$

$\Rightarrow \cos\theta = \frac{4}{5}$ $\left\{\because \text{ as } \cos^2\theta = \frac{1}{1+\tan^2\theta}\right\}$

$\therefore \cos(\tan^{-1} 3/4) = \cos\theta = 4/5$

(iii) $\sin\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{-1}{2}\right)\right) = \sin\left(\frac{\pi}{2} - \left(-\frac{\pi}{6}\right)\right) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

(iv) Let $\tan^{-1}\left(\frac{1}{5}\right) = \theta \Rightarrow \tan\theta = \frac{1}{5}$

$$\tan\left(2\theta - \frac{\pi}{4}\right) = \frac{\tan(2\theta) - 1}{1 + \tan 2\theta} \text{ and } \tan 2\theta = \frac{2 \times \frac{1}{5}}{1 - \frac{1}{25}} = \frac{5}{12} \quad (\because \tan\theta = \frac{1}{5})$$

$$\Rightarrow \tan\left(2\theta - \frac{\pi}{4}\right) = \frac{\frac{5}{12} - 1}{1 + \frac{5}{12}} = -\frac{7}{17}$$ **Ans.**

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Illustration 4 : Prove that $\sec^2(\tan^{-1}2) + \text{cosec}^2(\cot^{-1} 3) = 15$

Solution : We have,

$\sec^2 (\tan^{-1}2) + \text{cosec}^2 (\cot^{-1}3)$

$$=\left\{\sec\left(\tan^{-1}2\right)\right\}^2+\left\{\text{cosec}\left(\cot^{-1}3\right)\right\}^2=\left\{\sec\left(\tan^{-1}\frac{2}{1}\right)\right\}^2+\left\{\text{cosec}\left(\cot^{-1}\frac{3}{1}\right)\right\}^2$$

$$=\left\{\sec\left(\sec^{-1}\sqrt{5}\right)\right\}^2+\left\{\text{cosec}\left(\text{cosec}^{-1}\sqrt{10}\right)\right\}^2=\left(\sqrt{5}\right)^2+\left(\sqrt{10}\right)^2=15$$

Do yourself - 2 :

Evaluate the following :

(i) $\tan\left(\cos^{-1}\left(\frac{8}{17}\right)\right)$ **(ii)** $\sin\left(\frac{1}{2}\cos^{-1}\left(\frac{4}{5}\right)\right)$ **(iii)** $\sin\left(\frac{\pi}{3}-\sin^{-1}\left(\frac{-1}{2}\right)\right)$ **(iv)** $\sin\left(\cos^{-1}\frac{3}{5}\right)$

P-2 (i) $y = \sin^{-1} (\sin x)$, $x \in \mathbb{R}$, $y \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ periodic with period 2π and it is an odd function.

$$\sin^{-1}(\sin x)=\begin{cases}-\pi-x & ,-\pi\le x\le-\frac{\pi}{2}\\ x & ,\ -\frac{\pi}{2}<x<\frac{\pi}{2}\\ \pi-x & ,\ \frac{\pi}{2}\le x\le\pi\end{cases}$$

(ii) $y = \cos^{-1} (\cos x)$, $x \in \mathbb{R}$, $y \in [0,\pi]$, periodic with period 2π and it is an even function.

$$\cos^{-1}(\cos x)=\begin{cases}-x & ,\ -\pi\le x\le 0\\ x & ,\ 0<x\le\pi\end{cases}$$

(iii) $y = \tan^{-1} (\tan x)$

$x \in \mathbb{R}-\left\{(2n-1)\frac{\pi}{2}, n\in I\right\}$; $y\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, periodic with period π and it is an odd function.

$$\tan^{-1}(\tan x)=\begin{cases}x+\pi & ,\ -\frac{3\pi}{2}<x<-\frac{\pi}{2}\\ x & ,\ -\frac{\pi}{2}<x<\frac{\pi}{2}\\ x-\pi & ,\ \frac{\pi}{2}<x<\frac{3\pi}{2}\end{cases}$$

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(iv) $y = \cot^{-1}(\cot x)$, $x \in \mathbb{R} - \{n\pi, n \in I\}$, $y \in (0, \pi)$, periodic with period π and neither even nor odd function.

$$\cot^{-1}(\cot x) = \begin{cases} x + \pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \\ x - \pi, & \pi < x < 2\pi \end{cases}$$

(v) $y = \text{cosec}^{-1}(\text{cosec } x)$, $x \in \mathbb{R} - \{n\pi, n \in I\}$ $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, is periodic with period 2π and it is an odd function.

(vi) $y = \sec^{-1}(\sec x)$, y is periodic with period 2π and it is an even function.

$x \in \mathbb{R} - \left\{(2n-1)\frac{\pi}{2} n \in I\right\}$, $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

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Illustration 5 : The value of $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\cos(7\pi/6))$ is -

(A) $5\pi/6$ (B) $\pi/2$ (C) $3\pi/2$ (D) none of these

Solution : $\sin^{-1}\left(-\sqrt{3}/2\right) = -\sin^{-1}\left(\sqrt{3}/2\right) = -\pi/3$

and $\cos^{-1}(\cos(7\pi/6) = \cos^{-1}\cos(2\pi - 5\pi/6) = \cos^{-1}\cos(5\pi/6) = 5\pi/6$

Hence $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\cos 7\pi/6) = -\frac{\pi}{3} + \frac{5\pi}{6} = \frac{\pi}{2}$ **Ans.(B)**

Illustration 6 : Evaluate the following :

(i) $\sin^{-1}(\sin 10)$ (ii) $\tan^{-1}(\tan(-6))$ (iii) $\cot^{-1}(\cot 4)$

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Solution : (i) We know that $\sin^{-1}(\sin\theta) = \theta$, if $-\pi/2 \le \theta \le \pi/2$

Here, $\theta = 10$ radians which does not lie between $-\pi/2$ and $\pi/2$

But, $3\pi - \theta$ i.e., $3\pi - 10$ lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

Also, $\sin(3\pi - 10) = \sin 10$

$\therefore \quad \sin^{-1}(\sin 10) = \sin^{-1}(\sin(3\pi - 10)) = (3\pi - 10)$

(ii) We know that,

$\tan^{-1}(\tan\theta) = \theta$, if $-\pi/2 < \theta < \pi/2$. Here, $\theta = -6$, radians which does not lie between $-\pi/2$ and $\pi/2$. We find that $2\pi - 6$ lies between $-\pi/2$ and $\pi/2$ such that;

$\tan(2\pi - 6) = -\tan 6 = \tan(-6)$

$\therefore \quad \tan^{-1}(\tan(-6)) = \tan^{-1}(\tan(2\pi - 6)) = (2\pi - 6)$

(iii) $\cot^{-1}(\cot 4) = \cot^{-1}(\cot(\pi + (4 - \pi))) = \cot^{-1}(\cot(4 - \pi)) = (4 - \pi)$

Ans.

Illustration 7 : Find the number of solutions of (x, y) which satisfy $|y| = \cos x$ and $y = \sin^{-1}(\sin x)$, where $|x| \le 3\pi$.

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Solution : Graphs of $y = \sin^{-1}(\sin x)$ and $|y| = \cos x$ meet exactly six times in $[-3\pi, 3\pi]$.

Do yourself - 3 :

Evaluate the following :

(i) $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ **(ii)** $\tan^{-1}\left(\tan\left(\frac{7\pi}{6}\right)\right)$ **(iii)** $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$

P-3 (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ $\quad |x| \le 1$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$ $\quad x \in \mathbb{R}$

(iii) $\text{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$ $\quad |x| \ge 1$

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P-4 (i) $\sin^{-1}(-x) = -\sin^{-1} x$, $|x| \le 1$

(ii) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$, $|x| \ge 1$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x$, $x \in \mathbb{R}$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1} x$, $x \in \mathbb{R}$

(v) $\cos^{-1}(-x) = \pi - \cos^{-1} x$, $|x| \le 1$

(vi) $\sec^{-1}(-x) = \pi - \sec^{-1} x$, $|x| \ge 1$

P-5 (i) $\text{cosec}^{-1} x = \sin^{-1}\frac{1}{x}$; $|x| \ge 1$

(ii) $\sec^{-1} x = \cos^{-1}\frac{1}{x}$; $|x| \ge 1$

(iii) $\cot^{-1} x = \begin{cases} \tan^{-1}\frac{1}{x} & ; \ x > 0 \\ \pi + \tan^{-1}\frac{1}{x} & ; \ x < 0 \end{cases}$

Illustration 8 : Find value of x if $\cos^{-1}(-x) + \tan^{-1}(-x) - 2\sin^{-1}(x) + \sec^{-1}\left(-\frac{1}{x}\right) = \frac{\pi}{4}$ for $|x| \le 1$.

Solution : $\pi - \cos^{-1}(x) - \tan^{-1}(x) - 2\sin^{-1}(x) + \cos^{-1}(-x) = \frac{\pi}{4}$

$\pi - \cos^{-1}(x) - \tan^{-1}(x) - 2\sin^{-1}(x) + \pi - \cos^{-1}(x) = \frac{\pi}{4}$

$2\pi - 2(\sin^{-1}x + \cos^{-1}x) - \frac{\pi}{4} = \tan^{-1}x$

$2\pi - \pi - \frac{\pi}{4} = \tan^{-1}x \Rightarrow \tan^{-1}x = \frac{3\pi}{4}$ Hence no soluton **Ans.**

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Do yourself - 4 :

(i) Prove the following :

(a) $\cos^{-1}\left(\frac{5}{13}\right) = \tan^{-1}\left(\frac{12}{5}\right)$ (b) $\sin^{-1}\left(-\frac{4}{5}\right) = \tan^{-1}\left(-\frac{4}{3}\right) = \cos^{-1}\left(-\frac{3}{5}\right) - \pi$

(ii) Find the value of $\sin(\tan^{-1}a + \tan^{-1}\frac{1}{a})$; $a \ne 0$

P-6 (i) (a) $\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1}\frac{x+y}{1-xy} & \text{where } x > 0, y > 0 \ \& \ xy < 1 \\ \pi + \tan^{-1}\frac{x+y}{1-xy} & \text{where } x > 0, y > 0 \ \& \ xy > 1 \\ \frac{\pi}{2}, & \text{where } x > 0, y > 0 \ \& \ xy = 1 \end{cases}$

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(b) $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \dfrac{x-y}{1+xy}$ where $x > 0, y > 0$

(c) $\tan^{-1}x + \tan^{-1}y + \tan^{-1} z = \tan^{-1} \left[\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right]$ if $x > 0, y > 0, z > 0$ & $xy + yz + zx < 1$

(ii) (a) $\sin^{-1}x + \sin^{-1}y = \begin{cases} \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}] & \text{where } x > 0, y > 0 \ \& \ (x^2 + y^2) \le 1 \\ \pi - \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}] & \text{where } x > 0, y > 0 \ \& \ x^2 + y^2 > 1 \end{cases}$

(b) $\sin^{-1} x - \sin^{-1} y = \sin^{-1} [x\sqrt{1-y^2} - y\sqrt{1-x^2}]$ where $x > 0, y > 0$

(iii) (a) $\cos^{-1} x + \cos^{-1} y = \cos^{-1} [xy - \sqrt{1-x^2}\sqrt{1-y^2}]$ where $x > 0, y > 0$

(b) $\cos^{-1} x - \cos^{-1} y = \begin{cases} \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) ; & x < y, \quad x, y > 0 \\ -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right); & x > y, \quad x, y > 0 \end{cases}$

Note : In the above results x & y are taken positive. In case if these are given as negative, we first apply **P-4** and then use above results.

Illustration 9 : Prove that

(i) $\tan^{-1}\dfrac{1}{7} + \tan^{-1}\dfrac{1}{13} = \tan^{-1}\dfrac{2}{9}$ (ii) $\tan^{-1}\dfrac{1}{5} + \tan^{-1}\dfrac{1}{7} + \tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{8} = \dfrac{\pi}{4}$

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Solution : (i) L.H.S. $= \tan^{-1}\dfrac{1}{7} + \tan^{-1}\dfrac{1}{13}$

$$= \tan^{-1}\left\{\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\times\frac{1}{13}}\right\} \qquad \left\{\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right); \text{ if } xy < 1\right\}$$

$$= \tan^{-1}\left(\frac{20}{90}\right) = \tan^{-1}\left(\frac{2}{9}\right) = \text{R.H.S.}$$

(ii) $\left(\tan^{-1}\dfrac{1}{5} + \tan^{-1}\dfrac{1}{7}\right) + \left(\tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{8}\right)$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right) = \tan^{-1}\left(\frac{6}{17}\right) + \tan^{-1}\left(\frac{11}{23}\right)$$

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right) = \tan^{-1}\left(\frac{325}{325}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$ **Ans.**

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Illustration 10 : Prove that $\sin^{-1}\frac{12}{13}+\cot^{-1}\frac{4}{3}+\tan^{-1}\frac{63}{16}=\pi$

Solution : We have,

$$\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}$$

$$=\tan^{-1}\frac{12}{5}+\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{63}{16} \qquad \left[\because \sin^{-1}\frac{12}{13}=\tan^{-1}\frac{12}{5} \text{ and } \cos^{-1}\frac{4}{5}=\tan^{-1}\frac{3}{4}\right]$$

$$=\pi+\tan^{-1}\left\{\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}}\right\}+\tan^{-1}\frac{63}{16} \qquad \left[\because \tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{if } xy>1\right]$$

$$=\pi+\tan^{-1}\left(\frac{63}{-16}\right)+\tan^{-1}\left(\frac{63}{16}\right)$$

$$=\pi-\tan^{-1}\frac{63}{16}+\tan^{-1}\frac{63}{16} \qquad \left[\because \tan^{-1}(-x)=-\tan^{-1}x\right]$$

$$=\pi$$

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Illustration 11 : Prove that : $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{56}{65}$

Solution : We have, L.H.S. = $\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\tan^{-1}\left(\frac{5}{12}\right)+\tan^{-1}\left(\frac{3}{4}\right)$

$$\left[\because \cos^{-1}\left(\frac{12}{13}\right)=\tan^{-1}\left(\frac{5}{12}\right) \;\&\; \sin^{-1}\left(\frac{3}{5}\right)=\tan^{-1}\left(\frac{3}{4}\right)\right]$$

$$\text{L.H.S.} = \tan^{-1}\left(\frac{\frac{5}{12}+\frac{3}{4}}{1-\frac{5}{12}\times\frac{3}{4}}\right)=\tan^{-1}\left(\frac{56}{33}\right)$$

$$\text{R.H.S.} = \sin^{-1}\left(\frac{56}{65}\right)=\tan^{-1}\left(\frac{56}{33}\right)$$

L.H.S. = R.H.S. Hence Proved

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Do yourself - 5 :

Prove the following :

(i) $\sin^{-1}\left(\frac{3}{5}\right)+\sin^{-1}\left(\frac{8}{17}\right)=\cos^{-1}\left(\frac{36}{85}\right)$

(ii) $\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{8}{19}\right)=\frac{\pi}{4}$

(iii) $\tan^{-1}\left(\frac{2}{11}\right)+\tan^{-1}\left(\frac{7}{24}\right)=\tan^{-1}\left(\frac{1}{2}\right)$

4. SIMPLIFIED INVERSE TRIGONOMETRIC FUNCTIONS :

(a) $y=f(x)=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\begin{cases}2\tan^{-1}x & \text{if } |x|\le 1\\ \pi-2\tan^{-1}x & \text{if } x>1\\ -(\pi+2\tan^{-1}x) & \text{if } x<-1\end{cases}$

(b) $y=f(x)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=\begin{cases}2\tan^{-1}x & \text{if } x\ge 0\\ -2\tan^{-1}x & \text{if } x<0\end{cases}$

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(c) $y=f(x)=\tan^{-1}\frac{2x}{1-x^2}=\begin{cases}2\tan^{-1}x & \text{if } |x|<1\\ \pi+2\tan^{-1}x & \text{if } x<-1\\ -(\pi-2\tan^{-1}x) & \text{if } x>1\end{cases}$

(d) $y=f(x)=\sin^{-1}(3x-4x^3)$

$$=\begin{cases}-(\pi+3\sin^{-1}x) & \text{if } -1\le x\le -\frac{1}{2}\\ 3\sin^{-1}x & \text{if } -\frac{1}{2}\le x\le \frac{1}{2}\\ \pi-3\sin^{-1}x & \text{if } \frac{1}{2}\le x\le 1\end{cases}$$

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(e) $y = f(x) = \cos^{-1}(4x^3 - 3x)$

$$= \begin{cases} 3\cos^{-1}x - 2\pi & \text{if} \quad -1 \le x \le -\frac{1}{2} \\ 2\pi - 3\cos^{-1}x & \text{if} \quad -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1}x & \text{if} \quad \frac{1}{2} \le x \le 1 \end{cases}$$

(f) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \begin{cases} -(\pi + 2\sin^{-1}x) & -1 \le x \le -\frac{1}{\sqrt{2}} \\ 2\sin^{-1}x & -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x & \frac{1}{\sqrt{2}} \le x \le 1 \end{cases}$

(g) $\cos^{-1}(2x^2 - 1) = \begin{cases} 2\cos^{-1}x & 0 \le x \le 1 \\ 2\pi - 2\cos^{-1}x & -1 \le x \le 0 \end{cases}$

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Illustration 12 : Prove that : $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

Solution : We have, $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$

$$= \tan^{-1}\left\{\frac{2 \times \frac{1}{2}}{1 - \left(\frac{1}{2}\right)^2}\right\} + \tan^{-1}\frac{1}{7} \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right), \text{if } -1 < x < 1\right]$$

$$= \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{7} = \tan^{-1}\left\{\frac{\frac{4}{3} + \frac{1}{7}}{1 - \frac{4}{3} \times \frac{1}{7}}\right\} = \tan^{-1}\frac{31}{17}$$

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Illustration 13 : Prove that $\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x\in[0,1]$

Solution : We have, $\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\cos^{-1}\left\{\frac{1-\left(\sqrt{x}\right)^2}{1+\left(\sqrt{x}\right)^2}\right\}=\frac{1}{2}\times 2\tan^{-1}\sqrt{x}=\tan^{-1}\sqrt{x}.$

Alter : Putting $\sqrt{x}=\tan\theta$, we have $\Rightarrow \theta\in\left[0,\frac{\pi}{4}\right]$

$$\text{RHS}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\frac{1}{2}\cos^{-1}(\cos 2\theta)=\theta \quad \because\left(2\theta\in\left[0,\frac{\pi}{2}\right]\right)$$

$$=\tan^{-1}\sqrt{x}=\text{LHS}$$

Illustration 14 : Prove that : (i) $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=\frac{\pi}{4}$

(ii) $2\tan^{-1}\frac{1}{5}+\sec^{-1}\frac{5\sqrt{2}}{7}+2\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

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Solution : (i) $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=2\left\{2\tan^{-1}\frac{1}{5}\right\}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}$

$$=2\left\{\tan^{-1}\frac{2\times 1/5}{1-(1/5)^2}\right\}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99} \qquad \left[\begin{array}{c}\because 2\tan^{-1}x\\ =\tan^{-1}\frac{2x}{1-x^2}, \text{if } |x|<1\end{array}\right]$$

$$=2\tan^{-1}\frac{5}{12}-\left\{\tan^{-1}\frac{1}{70}-\tan^{-1}\frac{1}{99}\right\}=\tan^{-1}\left\{\frac{2\times 5/12}{1-(5/12)^2}\right\}-\tan^{-1}.\left\{\frac{\frac{1}{70}-\frac{1}{99}}{1+\frac{1}{70}\times\frac{1}{99}}\right\}$$

$$=\tan^{-1}\frac{120}{119}-\tan^{-1}\frac{29}{6931}=\tan^{-1}\frac{120}{119}-\tan^{-1}\frac{1}{239}=\tan^{-1}\left\{\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\times\frac{1}{239}}\right\}=\tan^{-1}1=\frac{\pi}{4}$$

(ii) $2\tan^{-1}\frac{1}{5}+\sec^{-1}\frac{5\sqrt{2}}{7}+2\tan^{-1}\frac{1}{8}=2\left\{\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}\right\}+\sec^{-1}\frac{5\sqrt{2}}{7}$

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$$= 2\tan^{-1}\left\{\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\right\} + \tan^{-1}\sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2 - 1} \qquad \left[\because \sec^{-1}x = \tan^{-1}\sqrt{x^2-1}\right]$$

$$= 2\tan^{-1}\frac{13}{39} + \tan^{-1}\frac{1}{7} = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left\{\frac{2\times 1/3}{1-(1/3)^2}\right\} + \tan^{-1}\frac{1}{7} \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2},\ \text{if}\,|x|<1\right]$$

$$= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7} = \tan^{-1}\left\{\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right\} = \tan^{-1}1 = \frac{\pi}{4}$$

Do yourself - 6 :

Prove the following results :

(i) $2\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \tan^{-1}\left(\frac{4}{7}\right)$ **(ii)** $2\sin^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{17}{31}\right) = \frac{\pi}{4}$

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6. EQUATIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS :

Illustration 15 : The equation $2\cos^{-1}x + \sin^{-1}x = \frac{11\pi}{6}$ has

(A) no solution (B) only one solution (C) two solutions (D) three solutions

Solution : Given equation is $2\cos^{-1}x + \sin^{-1}x = \frac{11\pi}{6}$

$\Rightarrow \cos^{-1}x + (\cos^{-1}x + \sin^{-1}x) = \frac{11\pi}{6} \quad \Rightarrow \cos^{-1}x + \frac{\pi}{2} = \frac{11\pi}{6} \Rightarrow \cos^{-1}x = 4\pi/3$

which is not possible as $\cos^{-1}x \in [0, \pi]$ **Ans.(A)**

Illustration 16 : If $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = 5\pi^2/8$, then x is equal to-

(A) –1 (B) 0 (C) 1 (D) none of these

Solution : The given equation can be written as $(\tan^{-1}x + \cot^{-1}x)^2 - 2\tan^{-1}x\cot^{-1}x = 5\pi^2/8$

Since $\tan^{-1}x + \cot^{-1}x = \pi/2$ we have

$(\pi/2)^2 - 2\tan^{-1}x(\pi/2 - \tan^{-1}x) = 5\pi^2/8$

$\Rightarrow 2(\tan^{-1}x)^2 - 2(\pi/2)\tan^{-1}x - 3\pi^2/8 = 0 \Rightarrow \tan^{-1}x = -\pi/4 \Rightarrow x = -1$ **Ans. (A)**

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Illustration 17 : Solve the equation : $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$

Solution : $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$

taking tangent on both sides

$$\Rightarrow \quad \tan\left(\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)\right)=1 \Rightarrow \frac{\tan\left(\tan^{-1}\left(\frac{x-1}{x-2}\right)\right)+\tan\left(\tan^{-1}\left(\frac{x+1}{x+2}\right)\right)}{1-\tan\left(\tan^{-1}\left(\frac{x-1}{x-2}\right)\right)\tan\left(\tan^{-1}\left(\frac{x+1}{x+2}\right)\right)}=1$$

$$\Rightarrow \quad \frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}\cdot\frac{x+1}{x+2}}=1 \Rightarrow \frac{(x-1)(x+2)+(x-2)(x+1)}{x^2-4-(x^2-1)}=1 \Rightarrow 2x^2-4=-3 \Rightarrow x=\pm\frac{1}{\sqrt{2}}$$

Now verify $x=\frac{1}{\sqrt{2}}$

$$=\tan^{-1}\left(\frac{\frac{1}{\sqrt{2}}-1}{\frac{1}{\sqrt{2}}-2}\right)+\tan^{-1}\left(\frac{\frac{1}{\sqrt{2}}+1}{\frac{1}{\sqrt{2}}+2}\right)=\tan^{-1}\left(\frac{\sqrt{2}-1}{2\sqrt{2}-1}\right)+\tan^{-1}\left(\frac{\sqrt{2}+1}{2\sqrt{2}+1}\right)$$

$$=\tan^{-1}\left(\frac{(2\sqrt{2}+1)(\sqrt{2}-1)+(2\sqrt{2}-1)(\sqrt{2}+1)}{(2\sqrt{2}-1)(2\sqrt{2}+1)-(\sqrt{2}-1)(\sqrt{2}+1)}\right)=\tan^{-1}\left(\frac{6}{6}\right)=\tan^{-1}(1)=\frac{\pi}{4}$$

$x=-\frac{1}{\sqrt{2}}$

$$=\tan^{-1}\left(\frac{-\frac{1}{\sqrt{2}}-1}{-\frac{1}{\sqrt{2}}-2}\right)+\tan^{-1}\left(\frac{-\frac{1}{\sqrt{2}}+1}{-\frac{1}{\sqrt{2}}+2}\right)=\tan^{-1}\left(\frac{\sqrt{2}+1}{2\sqrt{2}+1}\right)+\tan^{-1}\left(\frac{\sqrt{2}-1}{2\sqrt{2}-1}\right)\{\text{same as above}\}$$

$=\tan^{-1}(1)=\frac{\pi}{4}$

$\therefore \quad x=\pm\frac{1}{\sqrt{2}}$ are solutions **Ans.**

Illustration 18 : Solve the equation : $2\tan^{-1}(2x+1)=\cos^{-1}x$.

Solution : Here, $2\tan^{-1}(2x+1)=\cos^{-1}x$

or $\cos(2\tan^{-1}(2x+1))=x$ $\left\{\text{We know } \cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}\right\}$

$\therefore \quad \frac{1-(2x+1)^2}{1+(2x+1)^2}=x \Rightarrow (1-2x-1)(1+2x+1)=x(4x^2+4x+2)$

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$\Rightarrow \quad -2x \,.\, 2(x+1) = 2x(2x^2+2x+1) \quad \Rightarrow \quad 2x(2x^2+2x+1+2x+2) = 0$

$\Rightarrow \quad 2x(2x^2+4x+3) = 0$

$\Rightarrow \quad x = 0 \quad \text{or} \quad 2x^2+4x+3 = 0 \quad \{\text{No solution}\}$

Verify $\quad x = 0$

$2\tan^{-1}(1) = \cos^{-1}(1) \quad \Rightarrow \quad \frac{\pi}{2} = \frac{\pi}{2}$

$\therefore \quad x = 0$ is only the solution **Ans.**

Do yourself - 7 :

Solve the following equation for x :

(i) $\sin\left[\sin^{-1}\left(\frac{1}{5}\right)+\cos^{-1}x\right] = 1$ **(ii)** $\cos^{-1}x + \sin^{-1}\frac{x}{2} = \frac{\pi}{6}$

7. INEQUATIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTION :

Illustration 19 : Find the complete solution set of $\sin^{-1}(\sin 5) > x^2 - 4x$.

Solution : $\sin^{-1}(\sin 5) > x^2 - 4x \quad \Rightarrow \quad \sin^{-1}[\sin(5-2\pi)] > x^2 - 4x$

$\Rightarrow \quad x^2 - 4x < 5 - 2\pi \quad \Rightarrow \quad x^2 - 4x + (2\pi - 5) < 0$

$\Rightarrow \quad 2-\sqrt{9-2\pi} < x < 2+\sqrt{9-2\pi} \quad \Rightarrow \quad x \in (2-\sqrt{9-2\pi},\ 2+\sqrt{9-2\pi})$

Ans.

Illustration 20 : Find the complete solution set of $[\cot^{-1}x]^2 - 6[\cot^{-1}x] + 9 \le 0$, where [.] denotes the greatest integer function.

Solution : $[\cot^{-1}x]^2 - 6[\cot^{-1}x] + 9 \le 0$

$\Rightarrow \quad ([\cot^{-1}x] - 3)^2 \le 0 \quad \Rightarrow \quad [\cot^{-1}x] = 3 \quad \Rightarrow \quad 3 \le \cot^{-1}x < 4 \quad \Rightarrow \quad x \in (-\infty, \cot 3]$

Illustration 21 : If $\cot^{-1}\frac{n}{\pi} > \frac{\pi}{6}$, $n \in N$, then the maximum value of n is -

(A) 1 (B) 5 (C) 9 (D) none of these

Solution : $\cot^{-1}\frac{n}{\pi} > \frac{\pi}{6}$

$\Rightarrow \quad \cot\left(\cot^{-1}\left(\frac{n}{\pi}\right)\right) < \cot\left(\frac{\pi}{6}\right) \Rightarrow \frac{n}{\pi} < \sqrt{3}$

$\Rightarrow \quad n < \pi\sqrt{3} \quad \Rightarrow \quad n < 5.5$ (approx)

$\Rightarrow \quad n = 5 \quad \because (n \in N)$ **Ans. (B)**

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Do yourself - 8 :

(i) Solve the inequality $\tan^{-1}x > \cot^{-1}x$.

(ii) Complete solution set of inequation $(\cos^{-1}x)^2 - (\sin^{-1}x)^2 > 0$, is

(A) $\left[0, \frac{1}{\sqrt{2}}\right)$ (B) $\left[-1, \frac{1}{\sqrt{2}}\right)$ (C) $(-1, \sqrt{2})$ (D) none of these

8. SUMMATION OF SERIES :

Illustration 22 : Prove that :

$$\tan^{-1}\left(\frac{c_1x-y}{c_1y+x}\right)+\tan^{-1}\left(\frac{c_2-c_1}{1+c_2c_1}\right)+\tan^{-1}\left(\frac{c_3-c_2}{1+c_3c_2}\right)+\ldots+\tan^{-1}\left(\frac{c_n-c_{n-1}}{1+c_nc_{n-1}}\right)+\tan^{-1}\left(\frac{1}{c_n}\right)=\tan^{-1}\left(\frac{x}{y}\right)$$

Solution : L.H.S.

$$\tan^{-1}\left(\frac{c_1x-y}{c_1y+x}\right)+\tan^{-1}\left(\frac{c_2-c_1}{1+c_2c_1}\right)+\tan^{-1}\left(\frac{c_3-c_2}{1+c_3c_2}\right)+\ldots+\tan^{-1}\left(\frac{c_n-c_{n-1}}{1+c_nc_{n-1}}\right)+\tan^{-1}\left(\frac{1}{c_n}\right)$$

$$=\tan^{-1}\left(\frac{\frac{x}{y}-\frac{1}{c_1}}{1+\frac{x}{y}.\frac{1}{c_1}}\right)+\left(\tan^{-1}c_2-\tan^{-1}c_1\right)+\left(\tan^{-1}c_3-\tan^{-1}c_2\right)+\ldots+\left(\tan^{-1}c_n-\tan^{-1}c_{n-1}\right)+\tan^{-1}\left(\frac{1}{c_n}\right)$$

$$=\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{1}{c_1}\right)-\tan^{-1}c_1+\tan^{-1}c_n+\tan^{-1}\left(\frac{1}{c_n}\right)$$

$$=\tan^{-1}\left(\frac{x}{y}\right)-\left(\cot^{-1}c_1+\tan^{-1}c_1\right)+\left(\tan^{-1}c_n+\cot^{-1}c_n\right)$$

$$=\tan^{-1}\left(\frac{x}{y}\right)-\frac{\pi}{2}+\frac{\pi}{2}=\tan^{-1}\left(\frac{x}{y}\right)=\text{R.H.S.}$$

Do yourself - 9 :

(i) Evaluate : $\sum_{r=1}^{\infty}\tan^{-1}\left(\frac{2}{1+(2r+1)(2r-1)}\right)$

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Miscellaneous Illustrations :

Illustration 23 : If $\tan^{-1} y = 4\tan^{-1} x, \left(|x| < \tan\frac{\pi}{8}\right)$, find y as an algebraic function of x and hence prove that $\tan\frac{\pi}{8}$ is a root of the equation $x^4 - 6x^2 + 1 = 0$.

Solution : We have $\tan^{-1} y = 4\tan^{-1} x$

$$\Rightarrow \quad \tan^{-1} y = 2\tan^{-1}\frac{2x}{1-x^2} \qquad (\text{as } |x| < 1)$$

$$= \tan^{-1}\frac{\frac{4x}{1-x^2}}{1-\frac{4x^2}{(1-x^2)^2}} = \tan^{-1}\frac{4x(1-x^2)}{x^4-6x^2+1} \qquad \left(\text{as}\left|\frac{2x}{1-x^2}\right| < 1\right)$$

$$\Rightarrow \quad y = \frac{4x(1-x^2)}{x^4-6x^2+1}$$

If $x = \tan\frac{\pi}{8}$

$\Rightarrow \quad \tan^{-1} y = 4\tan^{-1} x = \frac{\pi}{2} \Rightarrow$ y is not defined $\Rightarrow x^4 - 6x^2 + 1 = 0$ **Ans.**

Illustration 24 : If $A = 2\tan^{-1}\left(2\sqrt{2}-1\right)$ and $B = 3\sin^{-1}(1/3) + \sin^{-1}(3/5)$, then show A > B.

Solution : We have, $A = 2\tan^{-1}\left(2\sqrt{2}-1\right) = 2\tan^{-1}(1.828)$

$$\Rightarrow \quad A > 2\tan^{-1}\left(\sqrt{3}\right) \quad \Rightarrow \quad A > \frac{2\pi}{3} \qquad \text{..... (i)}$$

also we have, $\sin^{-1}\left(\frac{1}{3}\right) < \sin^{-1}\left(\frac{1}{2}\right) \Rightarrow \sin^{-1}\left(\frac{1}{3}\right) < \frac{\pi}{6}$

$$\Rightarrow \quad 3\sin^{-1}\left(\frac{1}{3}\right) < \frac{\pi}{2}$$

also, $3\sin^{-1}\left(\frac{1}{3}\right) = \sin^{-1}\left(3.\frac{1}{3} - 4\left(\frac{1}{3}\right)^3\right) = \sin^{-1}\left(\frac{23}{27}\right) = \sin^{-1}(0.852)$

$$\Rightarrow \quad 3\sin^{-1}(1/3) < \sin^{-1}\left(\sqrt{3}/2\right) \Rightarrow 3\sin^{-1}(1/3) < \pi/3$$

also, $\sin^{-1}(3/5) = \sin^{-1}(0.6) < \sin^{-1}\left(\sqrt{3}/2\right) \Rightarrow \sin^{-1}(3/5) < \pi/3$

Hence, $B = 3\sin^{-1}(1/3) + \sin^{-1}(3/5) < \frac{2\pi}{3}$ ..... (ii)

From (i) and (ii), we have A > B.

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Illustration 25 : If $\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right)$ then find the sum of all possible values of $\tan\theta$.

Solution : $\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right) \Rightarrow \theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{6\tan\theta}{9+\tan^2\theta}\right)$

$$\Rightarrow \theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left[\frac{2\left(\frac{1}{3}\tan\theta\right)}{1+\left(\frac{1}{3}\tan\theta\right)^2}\right] \Rightarrow \theta = \tan^{-1}(2\tan^2\theta) - \frac{2}{2}\tan^{-1}\left(\frac{1}{3}\tan\theta\right)$$

$$\Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \tan^{-1}\left(\frac{1}{3}\tan\theta\right) \qquad \text{..... (i)}$$

taking tangent on both sides

$$\Rightarrow \quad \tan\theta = \frac{6\tan^2\theta - \tan\theta}{3+2\tan^3\theta} \Rightarrow \quad 2\tan^4\theta - 6\tan^2\theta + 4\tan\theta = 0$$

$$\Rightarrow \quad 2\tan\theta(\tan^3\theta - 3\tan\theta + 2) = 0 \Rightarrow \quad 2\tan\theta(\tan\theta - 1)^2(\tan\theta + 2) = 0$$

$\Rightarrow \quad \tan\theta = 0, 1, -2$ which satisfy equation (i)

$\therefore \quad$ sum $= 0 + 1 - 2 = -1$ **Ans.**

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ANSWERS FOR DO YOURSELF

1 :	**(i)**	C	**(ii)**	1526				
2 :	**(i)**	$\frac{15}{8}$	**(ii)**	$\frac{1}{\sqrt{10}}$	**(iii)**	1	**(iv)**	$\frac{4}{5}$
3 :	**(i)**	$\frac{\pi}{6}$	**(ii)**	$\frac{\pi}{6}$	**(iii)**	$\frac{\pi}{6}$		
4 :	**(ii)**	$\begin{cases} 1, & \text{if } a > 0 \\ -1, & \text{if } a < 0 \end{cases}$						
7 :	**(i)**	$\frac{1}{5}$	**(ii)**	1	**(iii)**	$\sqrt{3}$		
8 :	**(i)**	$(1, \infty)$	**(ii)**	B				
9 :	**(i)**	$\pi/4$						

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EXERCISE (O-1)

Straight Objective Type

1. The domain of the function $\sin^{-1}\left(\log_2\left(\frac{x}{3}\right)\right)$ is-

(A) $\left[\frac{1}{2},3\right]$ (B) $\left[\frac{1}{2},3\right]$ (C) $\left[\frac{3}{2},6\right]$ (D) $\left[\frac{1}{2},2\right]$

IT0001

2. Domain of the function $f(x) = \log_e \cos^{-1}\{\sqrt{x}\}$ is, where $\{.\}$ represents fractional part function -

(A) $x \in \mathbb{R}$ (B) $x \in [0, \infty)$ (C) $x \in (0, \infty)$ (D) $x \in \mathbb{R} - \{x \mid x \in I\}$

IT0002

3. The value of $\tan^2(\sec^{-1}3) + \cot^2(\text{cosec}^{-1}4)$ is -

(A) 9 (B) 16 (C) 25 (D) 23

IT0003

4. $\tan^{-1}\left(1-x^2-\frac{1}{x^2}\right)+\sin^{-1}\left(x^2+\frac{1}{x^2}-1\right)$ (where $x \neq 0$) is equal to

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$ (C) $\frac{3\pi}{4}$ (D) π

IT0004

5. If $2 \le a < 3$, then the value of $\cos^{-1}\cos[a]+\text{cosec}^{-1}\text{cosec}[a]+\cot^{-1}\cot[a]$, (where [.] denotes greatest integer less than equal to x) is equal to

(A) $2 - \pi$ (B) $2 + \pi$ (C) π (D) 6

IT0005

6. If $x > 0$ $\cos^{-1}\left(\frac{12}{x}\right)=\frac{\pi}{2}-\cos^{-1}\left(\frac{16}{x}\right)$ then x is -

(A) 12 (B) 16 (C) 20 (D) 320

IT0006

7. Number of integral ordered pairs (a,b) for which $\sin^{-1}(1+b+b^2+....\infty)+\cos^{-1}\left(a-\frac{a^2}{3}+\frac{a^3}{9}-.....\infty\right)=\frac{\pi}{2}$ is-

(A) 0 (B) 4 (C) 9 (D) Infinitely many

IT0007

8. If $\cos^{-1}(2x^2 - 1) = 2\pi - 2\cos^{-1}x$, then -

(A) $x \in [-1, 0]$ (B) $x \in [0, 1]$ (C) $x \in \left[0, \frac{1}{\sqrt{2}}\right]$ (D) $x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

IT0008

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9. $\lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\frac{2r+1}{r^4+2r^3+r^2+1}$ is equal to -

(A) $\frac{\pi}{4}$ (B) $\frac{3\pi}{4}$ (C) $\frac{\pi}{2}$ (D) $-\frac{\pi}{8}$

IT0009

Multiple Correct Answer Type

10. If $\sin\left(2\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)\right)+\cos\left(2\tan^{-1}\left(\frac{1}{3}\right)\right)=\frac{p}{q}$, where p & q are relatively prime then digit at units place of $(p-q)^{2k+1}$, $k \in N$, can be -

(A) 1 (B) 3 (C) 7 (D) 9

IT0010

11. Let $f(x)=\sin^{-1}(\tan x)+\cos^{-1}(\cot x)$ then

(A) $f(x)=\frac{\pi}{2}$ wherever defined

(B) domain of $f(x)$ is $x=n\pi\pm\frac{\pi}{4}$, $n \in I$

(C) period of $f(x)$ is $\frac{\pi}{2}$

(D) $f(x)$ is many one function

IT0011

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12. Which of the following is/are correct ?

(A) $\cot^{-1}(x)=\tan^{-1}\left(\frac{1}{x}\right)\forall\ x \in R-\{0\}$

(B) If $f: R \to R$ such that $f(x)=\text{sgn}(e^x)$ then $f(x)$ is an into function.

(C) If $f: R^+ \to R$ such that $f(x)=\sin x+x$ then $f(x)$ is an odd function.

(D) If $f: R \to R$ such that $f(x)=\frac{e^x}{e^{[x]}}$ then $f(x)$ is a periodic function .

(where [.] represents greatest integer function)

IT0012

13. Consider the function $f(x)=e^x$ and $g(x)=\sin^{-1}x$, then which of the following is/are necessarily true.

(A) Domain of gof = Domain of f

(B) Range of $gof \subset$ Range of g

(C) Domain of gof is $(-\infty, 0]$

(D) Range of gof is $\left[-\frac{\pi}{2}, 0\right)$

IT0013

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14. Let $f(x) = e^{x^3 - x^2 + x}$ be an invertible function such that $f^{-1} = g$, then -

(A) $g(e) = 0$ (B) Domain of 'g' is $\mathbb{R}^+$

(C) Range of 'g' is $\mathbb{R}$ (D) $f(g(e)) = e$

IT0014

15. Value of $3\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$ is greater than

(A) $\frac{\pi}{2}$ (B) $\frac{2\pi}{3}$ (C) $\frac{3\pi}{4}$ (D) $\frac{5\pi}{6}$

IT0015

EXERCISE (O-2)

Straight Objective Type

1. The range of the function $f(x) = \sin^{-1}(\log_2(-x^2 + 2x + 3))$ is -

(A) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (B) $\left[-\frac{\pi}{2}, 0\right]$ (C) $\left[0, \frac{\pi}{2}\right]$ (D) $[-1, 1]$

IT0016

2. Range of $f(x) = \cot^{-1}(\log_e(1 - x^2))$ is -

(A) $(0,\pi)$ (B) $\left(0, \frac{\pi}{2}\right)$ (C) $\left[\frac{\pi}{2}, \pi\right)$ (D) $\left(0, \frac{\pi}{2}\right]$

IT0017

3. The value of $\sin^{-1}\left\{\cot\left(\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}} + \cos^{-1}\frac{\sqrt{12}}{4} + \sec^{-1}\sqrt{2}\right)\right\}$ is-

(A) 5 (B) 6 (C) 0 (D) 10

IT0018

4. Number of solution(s) of the equation $\cos^{-1}\sqrt{x} - \sin^{-1}\sqrt{x-1} + \cos^{-1}\sqrt{1-x} - \sin^{-1}\frac{1}{\sqrt{x}} = \frac{\pi}{2}$ is -

(A) 0 (B) 1 (C) 2 (D) 4

IT0019

5. $\sum_{r=0}^{\infty} \tan^{-1}\left(\frac{r((r+1)!)}{(r+1) + ((r+1)!)^2}\right)$ is equal to -

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$ (C) $\cot^{-1}3$ (D) $\tan^{-1}2$

IT0020

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Multiple Correct Answer Type

6. Let $f(x)=\sqrt{\cos^{-1}\sqrt{1-x^2}-\sin^{-1}x}$ then which of the following statement/s is/are correct -

(A) Domain of $f(x)$ is $[-1,1]$
(B) Domain of $f(x)$ is $[0,1]$
(C) Range of $f(x)$ is $\{0\}$
(D) Range of $f(x)$ is $\left[0,\sqrt{\pi}\right]$

IT0021

7. If $\alpha=2\tan^{-1}(\sqrt{3-2\sqrt{2}})+\sin^{-1}\left(\frac{1}{\sqrt{6}-\sqrt{2}}\right)$, $\beta=\cot^{-1}\left(\sqrt{3}-2\right)+\frac{1}{8}\sec^{-1}(-2)$ & $\gamma=\tan^{-1}\frac{1}{\sqrt{2}}+\cos^{-1}\frac{1}{\sqrt{3}}$, then

(A) $\alpha=\beta$
(B) $\alpha+\beta=3\gamma$
(C) $4(\beta-\gamma)=\alpha$
(D) $\beta=\gamma$

IT0022

8. If α is only real root of the equation $x^3+(\cos 1)x^2+(\sin 1)x+1=0$, then $\left(\tan^{-1}\alpha+\tan^{-1}\frac{1}{\alpha}\right)$ **cannot** be equal to-

(A) 0
(B) $\frac{\pi}{2}$
(C) $-\frac{\pi}{2}$
(D) π

IT0023

9. Let $f(x)=\begin{cases} x^2-4 & \text{if } |x|\le 3 \\ 5\,\text{sgn}|x-3| & \text{if } |x|>3 \end{cases}$ and $g(x)=2\tan^{-1}(e^x)-\frac{\pi}{2}$ for all $x\in\mathbb{R}$, then which of the following is wrong ?

(where sgn(x) denotes signum function of x)

(A) $f(x)$ is an even function
(B) $gof(x)$ is an even function
(C) $g(x)$ is an odd function
(D) $fof(x)$ is an odd function

IT0024

Linked Comprehension Type

Paragraph for Question 10 to 12

Consider a continuous function such that each image has atmost three preimage & atleast one image has exactly three preimages. This type of function is to be called as three-one function.

On the basis of above information, answer the following questions :

10. Which of the following function is a three-one function ?

(A) $|\ell n|x||$
(B) $e^{|x|}$
(C) x^3+3x^2-7x+6
(D) $\cos(\cos^{-1}x)$

IT0025

11. If $f(x)=\sin^{-1}(\sin x)$ is a three-one function, then possible interval of x is -

(A) $[-\pi,\pi]$
(B) $\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]$
(C) $(-2\pi,0]$
(D) $\left[\frac{-3\pi}{2},\frac{\pi}{2}\right]$

IT0025

12. If $f(x)$ is a three-one function such that $f(a)=f(b)$ (where $a\ne b$), then number of maximum possible values of b is -

(A) 1
(B) 2
(C) 3
(D) 4

IT0025

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Subjective Type Questions

13. If $\sin^{-1}\sin\left(\frac{10\pi}{3}\right)+\cos^{-1}\cos\frac{22\pi}{3}+\tan^{-1}\tan 10=a\pi+b$, then $\left|\frac{3ab}{80}\right|$ is equal to

IT0026

14. Number of integral solutions of the equation $2\sin^{-1}\sqrt{x^2-x+1}+\cos^{-1}\sqrt{x^2-x}=\frac{3\pi}{2}$ is

IT0027

15. The product of all real values of x satisfying the equation

$$\sin^{-1}\cos\left(\frac{2x^2+10|x|+4}{x^2+5|x|+3}\right)=\cot\left(\cot^{-1}\left(\frac{2-18|x|}{9|x|}\right)\right)+\frac{\pi}{2} \text{ is}$$

IT0028

16. Least integral value of x for which inequality $\sin^{-1}\left(\sin\left(\frac{2e^x+3}{e^x+1}\right)\right)>\pi-\frac{5}{2}$ holds, is

IT0029

EXERCISE (S-1)

1. Given is a partial graph of an even periodic function f whose period is 8.

If [*] denotes greatest integer function then find the value of the expression.

$$f(-3)+2|f(-1)|+\left[f\left(\frac{7}{8}\right)\right]+f(0)+\arccos(f(-2))+f(-7)+f(20)$$

IT0030

2. (a) Find the following :

(i) $\tan\left[\cos^{-1}\frac{1}{2}+\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)\right]$ (ii) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

(iii) $\cos\left(\tan^{-1}\frac{3}{4}\right)$ (iv) $\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$

IT0031

(b) Find the following :

(i) $\sin\left[\frac{\pi}{2}-\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)\right]$ (ii) $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]$

(iii) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ (iv) $\sin\left(\frac{1}{4}\arcsin\frac{\sqrt{63}}{8}\right)$

IT0032

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3. Find the domain of definition the following functions.
(Read the symbols [*] and { * } as greatest integers and fractional part functions respectively)

(i) $f(x) = \arccos\dfrac{2x}{1+x}$ **IT0033**

(ii) $f(x) = \sqrt{\cos(\sin x)} + \sin^{-1}\dfrac{1+x^2}{2x}$ **IT0034**

(iii) $f(x) = \sin^{-1}\left(\dfrac{x-3}{2}\right) - \log_{10}(4-x)$ **IT0035**

(iv) $f(x) = \sin^{-1}(2x + x^2)$ **IT0036**

(v) $f(x) = \dfrac{\sqrt{1-\sin x}}{\log_5(1-4x^2)} + \cos^{-1}(1-\{x\})$, where $\{x\}$ is the fractional part of x. **IT0037**

(vi) $f(x) = \sqrt{3-x} + \cos^{-1}\left(\dfrac{3-2x}{5}\right) + \log_6(2|x|-3) + \sin^{-1}(\log_2 x)$ **IT0038**

(vii) $f(x) = e^{\sin^{-1}\left(\frac{x}{2}\right)} + \tan^{-1}\left[\dfrac{x}{2}-1\right] + \ell n\left(\sqrt{x-[x]}\right)$ **IT0039**

4. Identify the pair(s) of functions which are identical. Also plot the graphs in each case.

(a) $y = \tan(\cos^{-1} x);\ y = \dfrac{\sqrt{1-x^2}}{x}$ (b) $y = \tan(\cot^{-1} x);\ y = \dfrac{1}{x}$

(c) $y = \sin(\arctan x);\ y = \dfrac{x}{\sqrt{1+x^2}}$ (d) $y = \cos(\arctan x);\ y = \sin(\operatorname{arccot} x)$

IT0040

5. Let $y = \sin^{-1}(\sin 8) - \tan^{-1}(\tan 10) + \cos^{-1}(\cos 12) - \sec^{-1}(\sec 9) + \cot^{-1}(\cot 6) - \operatorname{cosec}^{-1}(\operatorname{cosec} 7)$. If y simplifies to $a\pi + b$, then find $(a-b)$.

IT0041

6. Show that : $\sin^{-1}\left(\sin\dfrac{33\pi}{7}\right) + \cos^{-1}\left(\cos\dfrac{46\pi}{7}\right) + \tan^{-1}\left(-\tan\dfrac{13\pi}{8}\right) + \cot^{-1}\left(\cot\left(-\dfrac{19\pi}{8}\right)\right) = \dfrac{13\pi}{7}$

IT0042

7. Prove that :(a) $2\cos^{-1}\dfrac{3}{\sqrt{13}} + \cot^{-1}\dfrac{16}{63} + \dfrac{1}{2}\cos^{-1}\dfrac{7}{25} = \pi$ (b) $\arccos\sqrt{\dfrac{2}{3}} - \arccos\dfrac{\sqrt{6}+1}{2\sqrt{3}} = \dfrac{\pi}{6}$

IT0043

8. If α and β are the roots of the equation $x^2 + 5x - 49 = 0$, then find the value of $\cot(\cot^{-1}\alpha + \cot^{-1}\beta)$.

IT0044

9. If $a > b > c > 0$, then find the value of : $\cot^{-1}\left(\dfrac{1+ab}{a-b}\right) + \cot^{-1}\left(\dfrac{1+bc}{b-c}\right) + \cot^{-1}\left(\dfrac{1+ca}{c-a}\right)$.

IT0045

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10. Find all values of k for which there is a triangle whose angles have measure $\tan^{-1}\left(\frac{1}{2}\right), \tan^{-1}\left(\frac{1}{2}+k\right)$ and $\tan^{-1}\left(\frac{1}{2}+2k\right)$.

IT0046

11. Find the simplest value of

(a) $f(x) = \text{arc cos}x + \text{arc cos}\left(\frac{x}{2}+\frac{1}{2}\sqrt{3-3x^2}\right),\ x \in \left(\frac{1}{2}, 1\right)$

IT0047

(b) $f(x) = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right),\ x \in \mathbb{R}-\{0\}$

IT0048

12. Prove that the identities :

(a) $\sin^{-1}\cos(\sin^{-1}x) + \cos^{-1}\sin(\cos^{-1}x) = \frac{\pi}{2}, |x| \le 1$

IT0049

(b) $\tan(\tan^{-1}x + \tan^{-1}y + \tan^{-1}z) = \cot(\cot^{-1}x + \cot^{-1}y + \cot^{-1}z)$

IT0050

13. (a) Solve the inequality : $(\text{arc sec}x)^2 - 6(\text{arc sec}x) + 8 > 0$

IT0051

(b) If $\sin^2x + \sin^2y < 1$; $x, y \in R$ then prove that $\sin^{-1}(\tan x \,.\, \tan y) \in (-\pi/2, \pi/2)$.

IT0052

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14. Solve the following :

(a) $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}$ **IT0053**

(b) $\tan^{-1}\frac{1}{1+2x} + \tan^{-1}\frac{1}{1+4x} = \tan^{-1}\frac{2}{x^2}$ **IT0054**

(c) $\tan^{-1}(x-1) + \tan^{-1}(x) + \tan^{-1}(x+1) = \tan^{-1}(3x)$ **IT0055**

(d) $3\cos^{-1}x = \sin^{-1}\left(\sqrt{1-x^2}(4x^2-1)\right)$ **IT0056**

(e) $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$ & $\cos^{-1}x - \cos^{-1}y = \frac{\pi}{3}$ **IT0057**

15. Find the sum of the series :

(a) $\cot^{-1}7 + \cot^{-1}13 + \cot^{-1}21 + \cot^{-1}31 + ...$ to n terms. **IT0058**

(b) $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{2}{9} + ... + \tan^{-1}\frac{2^{n-1}}{1+2^{2n-1}} + ... \infty$ **IT0059**

(c) $\tan^{-1}\frac{1}{x^2+x+1} + \tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7} + \tan^{-1}\frac{1}{x^2+7x+13}$ to n terms.

IT0060

(d) $\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{65}} + \sin^{-1}\frac{1}{\sqrt{325}} + ... + \sin^{-1}\frac{1}{\sqrt{4n^4+1}} + ...\infty$ terms. **IT0061**

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EXERCISE (S-2)

1. If $\alpha = 2\arctan\left(\frac{1+x}{1-x}\right)$ & $\beta = \arcsin\left(\frac{1-x^2}{1+x^2}\right)$ for $0 < x < 1$, then prove that $\alpha + \beta = \pi$, what the value of $\alpha + \beta$ will be if $x > 1$.

IT0062

2. Solve the following :

(a) $\tan^{-1}\frac{x-1}{x+1} + \tan^{-1}\frac{2x-1}{2x+1} = \tan^{-1}\frac{23}{36}$ **IT0063**

(b) $2\tan^{-1}x = \cos^{-1}\frac{1-a^2}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2}$ $(a > 0,\ b > 0)$ **IT0064**

3. If α and β are the roots of the equation $x^2 - 4x + 1 = 0$ $(\alpha > \beta)$ then find the value of

$$f(\alpha, \beta) = \frac{\beta^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right) + \frac{\alpha^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right).$$

IT0065

4. Find all the positive integral solutions of, $\tan^{-1}x + \cos^{-1}\frac{y}{\sqrt{1+y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$.

IT0066

5. Prove that the equation, $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = \alpha\pi^3$ has no roots for $\alpha < \frac{1}{32}$ and $\alpha > \frac{7}{8}$.

IT0067

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6. Solve the following inequalities :

(a) $\operatorname{arc\,cot}^2 x - 5\operatorname{arc\,cot} x + 6 > 0$ **IT0068**

(b) $\arcsin x > \arccos x$ **IT0069**

(c) $\tan^2(\arcsin x) > 1$ **IT0070**

7. Solve the following system of inequalities :

$4\arctan^2 x - 8\arctan x + 3 < 0$ & $4\operatorname{arc\,cot} x - \operatorname{arc\,cot}^2 x - 3 \geq 0$

IT0071

8. If the sum $\sum_{n=1}^{10}\sum_{m=1}^{10}\tan^{-1}\left(\frac{m}{n}\right) = k\pi$, find the value of k.

IT0072

9. Show that the roots r, s and t of the cubic $x(x-2)(3x-7) = 2$, are real and positive. Also compute the value of $\tan^{-1}(r) + \tan^{-1}(s) + \tan^{-1}(t)$.

IT0073

10. Solve for x : $\sin^{-1}\left(\sin\left(\frac{2x^2+4}{1+x^2}\right)\right) < \pi - 3$.

IT0074

11. Find the set of values of 'a' for which the equation $2\cos^{-1}x = a + a^2(\cos^{-1}x)^{-1}$ posses a solution.

IT0075

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EXERCISE (JM)

1. Let $f : (-1, 1) \to B$, be a function defined by $f(x) = \tan^{-1}\frac{2x}{1-x^2}$, then f is both one-one and onto when B is the interval- **[AIEEE-2005]**

(1) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (2) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (3) $\left(0, \frac{\pi}{2}\right)$ (4) $\left[0, \frac{\pi}{2}\right)$

IT0076

2. If $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2$ is equal to - **[AIEEE-2005]**

(1) $2\sin 2\alpha$ (2) 4 (3) $4\sin^2\alpha$ (4) $-4\sin^2\alpha$

IT0077

3. If $\sin^{-1}\left(\frac{x}{5}\right) + \text{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$, then a value of x is- **[AIEEE-2007]**

(1) 1 (2) 3 (3) 4 (4) 5

IT0078

4. The value of $\cot\left(\text{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right)$ is equal to- **[AIEEE-2008]**

(1) $\frac{6}{17}$ (2) $\frac{3}{17}$ (3) $\frac{4}{17}$ (4) $\frac{5}{17}$

IT0079

5. If x, y, z are in A.P. and $\tan^{-1}x$, $\tan^{-1}y$ and $\tan^{-1}z$ are also in A.P., then **[JEE (Main)-2013]**

(1) x = y = z (2) 2x = 3y = 6z (3) 6x = 3y = 2z (4) 6x = 4y = 3z

IT0080

6. Let $\tan^{-1}y = \tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $|x| < \frac{1}{\sqrt{3}}$. Then a value of y is : **[JEE (Main)-2015]**

(1) $\frac{3x-x^3}{1+3x^2}$ (2) $\frac{3x+x^3}{1+3x^2}$ (3) $\frac{3x-x^3}{1-3x^2}$ (4) $\frac{3x+x^3}{1-3x^2}$

IT0081

7. If $\cos^{-1}\left(\frac{2}{3x}\right) + \cos^{-1}\left(\frac{3}{4x}\right) = \frac{\pi}{2}\left(x > \frac{3}{4}\right)$, then x is equal to : **[JEE(Main)-2019]**

(1) $\frac{\sqrt{145}}{12}$ (2) $\frac{\sqrt{145}}{10}$ (3) $\frac{\sqrt{146}}{12}$ (4) $\frac{\sqrt{145}}{11}$

IT0082

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8. If $x = \sin^{-1}(\sin 10)$ and $y = \cos^{-1}(\cos 10)$, then $y - x$ is equal to: **[JEE(Main)-2019]**

(1) π (2) 7π (3) 0 (4) 10

IT0083

9. The value of $\cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)\right)$ is : **[JEE(Main)-2019]**

(1) $\frac{22}{23}$ (2) $\frac{23}{22}$ (3) $\frac{21}{19}$ (4) $\frac{19}{21}$

IT0084

10. All x satisfying the inequality $(\cot^{-1} x)^2 - 7(\cot^{-1} x) + 10 > 0$, lie in the interval :-

[JEE(Main)-2019]

(1) $(-\infty, \cot 5) \cup (\cot 4, \cot 2)$ (2) $(\cot 5, \cot 4)$

(3) $(\cot 2, \infty)$ (4) $(-\infty, \cot 5) \cup (\cot 2, \infty)$

IT0085

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11. Considering only the principal values of inverse functions, the set $A = \left\{x \geq 0 : \tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}\right\}$

[JEE(Main)-2019]

(1) is an empty set (2) Contains more than two elements

(3) Contains two elements (4) is a singleton

IT0086

12. If $\alpha = \cos^{-1}\left(\frac{3}{5}\right), \beta = \tan^{-1}\left(\frac{1}{3}\right)$, where $0 < \alpha, \beta < \frac{\pi}{2}$, then $\alpha - \beta$ is equal to : **[JEE(Main)-2019]**

(1) $\sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$ (2) $\tan^{-1}\left(\frac{9}{14}\right)$ (3) $\cos^{-1}\left(\frac{9}{5\sqrt{10}}\right)$ (4) $\tan^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

IT0087

13. If $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, where $-1 \leq x \leq 1, -2 \leq y \leq 2, x \leq \frac{y}{2}$, then for all x, y, $4x^2 - 4xy\cos\alpha + y^2$ is equal to **[JEE(Main)-2019]**

(1) $4\sin^2\alpha - 2x^2y^2$ (2) $4\cos^2\alpha + 2x^2y^2$ (3) $4\sin^2\alpha$ (4) $2\sin^2\alpha$

IT0088

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14. The value of $\sin^{-1}\left(\frac{12}{13}\right)-\sin^{-1}\left(\frac{3}{5}\right)$ is equal to : **[JEE(Main)-2019]**

(1) $\pi-\sin^{-1}\left(\frac{63}{65}\right)$ (2) $\pi-\cos^{-1}\left(\frac{33}{65}\right)$ (3) $\frac{\pi}{2}-\sin^{-1}\left(\frac{56}{65}\right)$ (4) $\frac{\pi}{2}-\cos^{-1}\left(\frac{9}{65}\right)$

IT0089

EXERCISE (JA)

1. Let (x, y) be such that $\sin^{-1}(ax) + \cos^{-1}(y) + \cos^{-1}(bxy) = \frac{\pi}{2}$ **[JEE 2007, 6]**

Match the statements in column-I with statements in column-II and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrixgiven in the ORS.

Column-I	Column-II
(A) If a = 1 and b = 0, then (x, y)	(p) lies on the circle $x^2 + y^2 = 1$
(B) If a = 1 and b = 1, then (x, y)	(q) lies on $(x^2 - 1)(y^2 - 1) = 0$
(C) If a = 1 and b = 2, then (x, y)	(r) lies on y = x
(D) If a = 2 and b = 2, then (x, y)	(s) lies on $(4x^2 - 1)(y^2 - 1) = 0$

IT0090

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2. If 0 < x< 1, then $\sqrt{1+x^2}\,[\{x\cos(\cot^{-1}x) + \sin(\cot^{-1}x)\}^2 - 1]^{1/2} =$ **[JEE 2008, 3]**

(A) $\frac{x}{\sqrt{1+x^2}}$ (B) x (C) $x\sqrt{1+x^2}$ (D) $\sqrt{1+x^2}$

IT0091

3. The value of $\cot\left(\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right)$ is **[JEE(Advanced) 2013, 2]**

(A) $\frac{23}{25}$ (B) $\frac{25}{23}$ (C) $\frac{23}{24}$ (D) $\frac{24}{23}$

IT0092

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4. Match List-I with List-II and select the correct answer using the code given below the lists.

List-I		List-II
P.	$\left(\frac{1}{y^2}\left(\frac{\cos\left(\tan^{-1}y\right)+y\sin\left(\tan^{-1}y\right)}{\cot\left(\sin^{-1}y\right)+\tan\left(\sin^{-1}y\right)}\right)^2+y^4\right)^{1/2}$ takes value	1. $\frac{1}{2}\sqrt{\frac{5}{3}}$
Q.	If cosx + cosy + cosz = 0 = sinx + siny + sinz, then possible value of $\cos\frac{x-y}{2}$ is	2. $\sqrt{2}$
R.	If $\cos\left(\frac{\pi}{4}-x\right)$cos2x + sinx sin2x secx = cosx sin2x secx + $\cos\left(\frac{\pi}{4}+x\right)$ cos2x, then possible value of secx is	3. $\frac{1}{2}$
S.	If $\cot\left(\sin^{-1}\sqrt{1-x^2}\right)=\sin\left(\tan^{-1}\left(x\sqrt{6}\right)\right), x\neq 0$, then possible value of x is	4. 1

Codes :

	P	Q	R	S
(A)	4	3	1	2
(B)	4	3	2	1
(C)	3	4	2	1
(D)	3	4	1	2

[JEE(Advanced) 2013, 3, (–1)]

IT0093

5. Let $f:[0,4\pi]\to[0,\pi]$ be defined by $f(x)=\cos^{-1}(\cos x)$. The number of points $x\in[0,4\pi]$ satisfying the equation $f(x)=\frac{10-x}{10}$ is **[JEE(Advanced)-2014, 3]**

IT0094

6. If $\alpha=3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta=3\cos^{-1}\left(\frac{4}{9}\right)$ where the inverse trigonometric functions take only the principal values, then the correct option(s) is(are) **[JEE(Advanced)-2015, 4]**

(A) $\cos\beta>0$ (B) $\sin\beta<0$ (C) $\cos(\alpha+\beta)>0$ (D) $\cos\alpha<0$

IT0095

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7. The number of real solutions of the equation

$$\sin^{-1}\left(\sum_{i=1}^{\infty}x^{i+1}-x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^{i}\right)=\frac{\pi}{2}-\cos^{-1}\left(\sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^{i}-\sum_{i=1}^{\infty}(-x)^{i}\right)$$ lying in the interval $\left(-\frac{1}{2},\frac{1}{2}\right)$ is ____

(Here, the inverse trigonometric functions $\sin^{-1}x$ and $\cos^{-1}x$ assume value in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and $[0, \pi]$, respectively.)

[JEE(Advanced)-2018]
IT0096

8. Let $E_1 = \left\{x \in \mathbb{R} : x \neq 1 \text{ and } \frac{x}{x-1} > 0\right\}$ and $E_2 = \left\{x \in E_1 : \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right) \text{ is a real number}\right\}$.

$\left(\text{Here, the inverse trigonometric function } \sin^{-1}x \text{ assumes values in } \left[-\frac{\pi}{2},\frac{\pi}{2}\right].\right)$

Let $f : E_1 \to \mathbb{R}$ be the function defined by $f(x) = \log_e\left(\frac{x}{x-1}\right)$

and $g : E_2 \to \mathbb{R}$ be the function defined by $g(x) = \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right)$.

LIST-I	LIST-II
P. The range of f is	**1.** $\left(-\infty,\frac{1}{1-e}\right]\cup\left[\frac{e}{e-1},\infty\right)$
Q. The range of g contains	**2.** $(0, 1)$
R. The domain of f contains	**3.** $\left[-\frac{1}{2},\frac{1}{2}\right]$
S. The domain of g is	**4.** $(-\infty,0)\cup(0,\infty)$
	5. $\left(-\infty,\frac{e}{e-1}\right]$
	6. $(-\infty,0)\cup\left(\frac{1}{2},\frac{e}{e-1}\right]$

The correct option is : **[JEE(Advanced)-2018]**

(A) **P → 4; Q → 2; R → 1; S → 1**
(B) **P → 3; Q → 3; R → 6; S → 5**
(C) **P → 4; Q → 2; R → 1; S → 6**
(D) **P → 4; Q → 3; R → 6; S → 5**

IT0097

9. The value of $\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)\right)$ in the interval $\left[-\frac{\pi}{4},\frac{3\pi}{4}\right]$ equals

[JEE(Advanced)-2019, 3(0)]
IT0098

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ANSWER KEY

INVERSE TRIGONOMETRIC FUNCTION

EXERCISE (O-1)

1.	C	**2.**	B	**3.**	D	**4.**	B	**5.**	B	**6.**	C	**7.**	A
8.	A	**9.**	A	**10.**	B,C	**11.**	A,B,C,D			**12.**	B,D	**13.**	B,C
14.	B,C,D	**15.**	A,B										

EXERCISE (O-2)

1.	A	**2.**	C	**3.**	C	**4.**	A	**5.**	B	**6.**	A,D	**7.**	A,C
8.	A,B,D	**9.**	D	**10.**	C	**11.**	A	**12.**	B	**13.**	1	**14.**	2
15.	9	**16.**	1										

EXERCISE (S-1)

1. 5

2. **(a)** (i) $\frac{1}{\sqrt{3}}$, (ii) $\frac{5\pi}{6}$, (iii) $\frac{4}{5}$, (iv) $\frac{17}{6}$; **(b)** (i) $\frac{1}{2}$, (ii) -1, (iii) $-\frac{\pi}{4}$, (iv) $\frac{\sqrt{2}}{4}$

3. **(i)** $-1/3 \le x \le 1$ **(ii)** $\{1, -1\}$ **(iii)** $1 \le x < 4$ **(iv)** $[-(1+\sqrt{2}), (\sqrt{2}-1)]$

(v) $x \in (-1/2, 1/2), x \ne 0$ **(vi)** $(3/2, 2]$ **(vii)** $(-2, 2) - \{-1, 0, 1\}$

4. (a)

(b)

(c)

and (d)

all are identical.

5. 53 **8.** 10 **9.** π **10.** $k = \frac{11}{4}$ **11.** **(a)** $\frac{\pi}{3}$; **(b)** $\frac{1}{2}\tan^{-1}x$

13. **(a)** $(-\infty, \sec 2) \cup [1, \infty)$

14. **(a)** $x = \frac{1}{2}\sqrt{\frac{3}{7}}$; **(b)** $x = 3$; **(c)** $x = 0, \frac{1}{2}, -\frac{1}{2}$; **(d)** $\left[\frac{\sqrt{3}}{2}, 1\right]$; **(e)** $x = \frac{1}{2}, y = 1$

15. **(a)** $\text{arccot}\left[\frac{2n+5}{n}\right]$, **(b)** $\frac{\pi}{4}$, **(c)** $\text{arc}\tan(x+n) - \text{arc}\tan x$, **(d)** $\frac{\pi}{4}$

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EXERCISE (S-2)

1. $-\pi$ **2.** **(a)** $x = \frac{4}{3}$; **(b)** $x = \frac{a-b}{1+ab}$; **3.** 56 **4.** $x = 1; y = 2$ & $x = 2;\ y = 7$

6. **(a)** $(\cot 2, \infty) \cup (-\infty, \cot 3)$ **(b)** $\left(\frac{\sqrt{2}}{2}, 1\right]$ **(c)** $\left(\frac{\sqrt{2}}{2}, 1\right) \cup \left(-1, -\frac{\sqrt{2}}{2}\right)$

7. $\left(\tan\frac{1}{2}, \cot 1\right]$ **8.** $k = 25$ **9.** $\frac{3\pi}{4}$ **10.** $x \in (-1, 1)$

11. $a \in [-2\pi, \pi] - \{0\}$

EXERCISE (JM)

1. 2 **2.** 3 **3.** 2 **4.** 1 **5.** 1 **6.** 3 **7.** 1 **8.** 1
9. 3 **10.** 3 **11.** 4 **12.** 1 **13.** 3 **14.** 3

EXERCISE (JA)

1. (A) P; (B) Q; (C) P; (D) S **2.** C **3.** B **4.** B **5.** 3
6. B,C,D **7.** 2 **8.** A **9.** 0.00

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COMPLEX NUMBERS

1. DEFINITION :

Complex numbers are defined as expressions of the form $a + ib$ where $a, b \in R$ & $i = \sqrt{-1}$. It is denoted by z i.e. $z = a + ib$. 'a' is called real part of z (Re z) and 'b' is called imaginary part of z (Im z).

Every Complex Number Can Be Regarded As

- Purely real if $b = 0$
- Imaginary if $b \neq 0$
  - Purely imaginary if $a = 0$

Note :

(i) The set R of real numbers is a proper subset of the Complex Numbers. Hence the Complex Number system is $N \subset W \subset I \subset Q \subset R \subset C$.

(ii) Zero is both purely real as well as purely imaginary but not imaginary.

(iii) $i = \sqrt{-1}$ is called the imaginary unit. Also $i^2 = -1$; $i^3 = -i$; $i^4 = 1$ etc.

In general $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$, where $n \in I$

(iv) $\sqrt{a}\sqrt{b} = \sqrt{ab}$ only if atleast one of either a or b is non-negative.

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***Illustration* 1 :** The value of $i^{57} + 1/i^{125}$ is :-

(A) 0 (B) –2i (C) 2i (D) 2

Solution : $i^{57} + 1/i^{125} = i^{56}.\ i + \dfrac{1}{i^{124}.i}$

$$= \left(i^4\right)^{14} i + \frac{1}{\left(i^4\right)^{31} i}$$

$$= i + \frac{1}{i} = i + \frac{i}{i^2} = i - i = 0$$

Ans. (A)

2. ARGAND DIAGRAM :

Master Argand had done a systematic study on complex numbers and represented every complex number $z = x + iy$ as a set of ordered pair (x, y) on a plane called complex plane (Argand Diagram) containing two perpendicular axes. Horizontal axis is known as Real axis & vertical axis is known as Imaginary axis.

All complex numbers lying on the real axis are called as purely real and those lying on imaginary axis as purely imaginary.

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3. ALGEBRAIC OPERATIONS :

Fundamental operations with complex numbers :

(a) Addition $(a + bi) + (c + di) = (a + c) + (b + d)i$

(b) Subtraction $(a + bi) - (c + di) = (a - c) + (b - d)i$

(c) Multiplication $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

(d) Division $\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i$

Note :

(i) The algebraic operations on complex numbers are similar to those on real numbers treating i as a polynomial.

(ii) Inequalities in complex numbers (non-real) are not defined. There is no validity if we say that complex number (non-real) is positive or negative.

e.g. $z > 0$, $4 + 2i < 2 + 4i$ are meaningless.

(iii) In real numbers, if $a^2 + b^2 = 0$, then $a = 0 = b$ but in complex numbers, $z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.

Illustration 2 : $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ will be purely imaginary, if $\theta =$

(A) $2n\pi \pm \frac{\pi}{3}, n \in I$ (B) $n\pi + \frac{\pi}{3}, n \in I$ (C) $n\pi \pm \frac{\pi}{3}, n \in I$ (D) none of these

Solution : $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ will be purely imaginary, if the real part vanishes, i.e.,

$$\frac{(3+2i\sin\theta)}{(1-2i\sin\theta)} \times \frac{(1+2i\sin\theta)}{(1+2i\sin\theta)} = \frac{\left(3-4\sin^2\theta\right)+i\left(8\sin\theta\right)}{\left(1+4\sin^2\theta\right)}$$

$$\frac{3-4\sin^2\theta}{1+4\sin^2\theta} = 0 \quad \Rightarrow \quad 3 - 4\sin^2\theta = 0 \text{ (only if } \theta \text{ be real)}$$

$$\Rightarrow \sin^2\theta = \left(\frac{\sqrt{3}}{2}\right)^2 = \left(\sin\frac{\pi}{3}\right)^2$$

$$\Rightarrow \theta = n\pi \pm \frac{\pi}{3}, n \in I$$

Ans. (C)

Do yourself - 1 :

(i) Determine least positive value of n for which $\left(\frac{1+i}{1-i}\right)^n = 1$

(ii) Find the value of the sum $\sum_{n=1}^{5}(i^n + i^{n+2})$, where $i = \sqrt{-1}$.

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4. EQUALITY IN COMPLEX NUMBER :

Two complex numbers $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$ are equal if and only if their real & imaginary parts are respectively equal.

Illustration 3 : The values of x and y satisfying the equation $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$ are

(A) x = –1, y =3 (B) x = 3, y = –1 (C) x = 0, y = 1 (D) x = 1, y = 0

Solution : $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i \Rightarrow (4+2i)\,x+(9-7i)\,y-3i-3=10i$

Equating real and imaginary parts, we get $2x - 7y = 13$ and $4x + 9y = 3$.

Hence x = 3 and y = –1. **Ans.(B)**

Illustration 4 : Find the square root of 7 + 24 i.

Solution : Let $\sqrt{7+24i} = a + ib$

Squaring $a^2 - b^2 + 2iab = 7 + 24i$

Compare real & imaginary parts $a^2 - b^2 = 7$ & $2ab = 24$

By solving these two equations

We get $a = \pm 4$, $b = \pm 3$

$\sqrt{7+24i} = \pm(4 + 3i)$

Illustration 5 : If $x = -5+2\sqrt{-4}$, find the value of $x^4 + 9x^3 + 35x^2 - x + 4$.

Solution : We have , $x = -5 + 2\sqrt{-4}$

$\Rightarrow x + 5 = 4i \quad \Rightarrow (x + 5)^2 = 16i^2$

$\Rightarrow x^2 + 10x + 25 = -16 \quad \Rightarrow x^2 + 10x + 41 = 0$

Now,

$x^4 + 9x^3 + 35x^2 - x + 4$

$\Rightarrow x^2(x^2 + 10x + 41) - x(x^2 + 10x + 41) + 4(x^2 + 10x + 41) - 160$

$\Rightarrow x^2(0) - x(0) + 4(0) - 160 \quad \Rightarrow \quad -160$ **Ans.**

Do yourself - 2 :

(i) Find the value of $x^3 + 7x^2 - x + 16$, where x = 1 + 2i.

(ii) If $a + ib = \frac{c+i}{c-i}$, where c is a real number, then prove that : $a^2 + b^2 = 1$ and $\frac{b}{a}=\frac{2c}{c^2-1}$.

(iii) Find square root of –15 – 8i

5. THREE IMPORTANT TERMS : CONJUGATE/MODULUS/ARGUMENT :

(a) CONJUGATE COMPLEX :

If $z = a + ib$ then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by $\bar{z}$. i.e. $\bar{z} = a - ib$.

Note that :

(i) $z + \bar{z} = 2\,\mathrm{Re}(z)$

(ii) $z - \bar{z} = 2i\,\mathrm{Im}(z)$

(iii) $z\bar{z} = a^2 + b^2$, which is purely real

(iv) If z is purely real, then $z - \bar{z} = 0$

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(v) If z is purely imaginary, then $z + \bar{z} = 0$

(vi) If z lies in the 1st quadrant, then $\bar{z}$ lies in the 4th quadrant and $-\bar{z}$ lies in the 2nd quadrant.

(b) Modulus :

If P denotes complex number z = x + iy, then the length OP is called modulus of complex number z. It is denoted by |z|.

$$OP = |z| = \sqrt{x^2 + y^2}$$

Geometrically $|z|$ represents the distance of point P from origin. ($|z| \geq 0$)

Note : Unlike real numbers, $|z| = \begin{cases} z & \text{if } z > 0 \\ -z & \text{if } z < 0 \end{cases}$

is not correct.

(c) Argument or Amplitude :

If P denotes complex number z = x + iy and if OP makes an angle θ with real axis, then θ is called one of the arguments of z.

$\theta = \tan^{-1}\dfrac{y}{x}$ (angle made by OP with positive real axis)

Note :

(i) Argument of a complex number is a many valued function. If θ is the argument of a complex number, then $2n\pi + \theta$; $n \in I$ will also be the argument of that complex number. Any two arguments of a complex number differ by $2n\pi$.

(ii) The unique value of θ such that $-\pi < \theta \leq \pi$ is called **Amplitude (principal value of the argument).**

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(iii) Principal argument of a complex number z = x + iy can be found out using method given below :

(a) Find $\theta = \tan^{-1}\left|\dfrac{y}{x}\right|$ such that $\theta \in \left(0, \dfrac{\pi}{2}\right)$.

(b) Use given figure to find out the principal argument according as the point lies in respective quadrant.

(iv) Unless otherwise stated, amp z implies principal value of the argument.

(v) The unique value of $\theta = \tan^{-1}\dfrac{y}{x}$ such that $0 < \theta \leq 2\pi$ is called **least positive argument.**

(vi) If z = 0, arg(z) is not defined

(vii) If z is real & negative, arg(z) = π.

(viii) If z is real & positive, arg(z) = 0

(ix) If $\theta = \dfrac{\pi}{2}$, z lies on the positive side of imaginary axis.

(x) If $\theta = -\dfrac{\pi}{2}$, z lies on the negative side of imaginary axis.

By specifying the modulus & argument a complex number is defined completely. Argument impart direction & modulus impart distance from origin.

For the complex number 0 + 0i the argument is not defined and this is the only complex number which is given by its modulus only.

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Illustration 6 : Find the modulus, argument, principal value of argument, least positive argument of complex numbers (a) $1+i\sqrt{3}$ (b) $-1+i\sqrt{3}$ (c) $1-i\sqrt{3}$ (d) $-1-i\sqrt{3}$

Solution : (a) For $z = 1+i\sqrt{3}$

$|z| = \sqrt{1^2+(\sqrt{3})^2} = 2$

$\arg(z) = 2n\pi + \frac{\pi}{3}, \quad n \in I$

Least positive argument is $\frac{\pi}{3}$

If the point is lying in first or second quadrant then amp(z) is taken in anticlockwise direction.

In this case amp(z) = $\frac{\pi}{3}$

(b) For $z = -1 + i\sqrt{3}$

$|z| = 2$

$\arg(z) = 2n\pi + \frac{2\pi}{3}, \quad n \in I$

Least positive argument = $\frac{2\pi}{3}$

amp(z) = $\frac{2\pi}{3}$

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(c) For $z = 1 - i\sqrt{3}$

$|z| = 2$

$\arg(z) = 2n\pi - \frac{\pi}{3}, \; n \in I$

Least positive argument = $\frac{5\pi}{3}$

If the point lies in third or fourth quadrant then consider amp(z) in clockwise direction.

In this case amp(z) = $-\frac{\pi}{3}$

(d) For $z = -1 - i\sqrt{3}$

$|z| = 2$

$\arg(z) = 2n\pi - \frac{2\pi}{3}, \quad n \in I$

Least positive argument = $\frac{4\pi}{3}$

amp(z) = $-\frac{2\pi}{3}$

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Illustration 7 : Find modulus and argument for $z = 1 - \sin\alpha + i\cos\alpha$, $\alpha \in (0, 2\pi)$

Solution : $|z| = \sqrt{(1-\sin\alpha)^2 + (\cos\alpha)^2} = \sqrt{2 - 2\sin\alpha} = \sqrt{2}\left|\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right|$

Case (i) For $\alpha \in \left(0, \frac{\pi}{2}\right)$, z will lie in I quadrant.

$$\text{amp}(z) = \tan^{-1}\frac{\cos\alpha}{1-\sin\alpha} \Rightarrow \text{amp}(z) = \tan^{-1}\frac{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}{\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)^2} = \tan^{-1}\frac{\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}}$$

$$\Rightarrow \arg z = \tan^{-1}\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$$

Since $\frac{\pi}{4} + \frac{\alpha}{2} \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$$\therefore \text{amp}(z) = \left(\frac{\pi}{4} + \frac{\alpha}{2}\right), \ |z| = \sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)$$

Case (ii) at $\alpha = \frac{\pi}{2}$: $z = 0 + 0i$

$|z| = 0$

amp (z) is not defined.

Case (iii) For $\alpha \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, z will lie in IV quadrant

so $\text{amp}(z) = -\tan^{-1}\tan\left(\frac{\alpha}{2} + \frac{\pi}{4}\right)$

Since $\frac{\alpha}{2} + \frac{\pi}{4} \in \left(\frac{\pi}{2}, \pi\right)$

$$\therefore \text{amp}(z) = -\left(\frac{\alpha}{2} + \frac{\pi}{4} - \pi\right) = \frac{3\pi}{4} - \frac{\alpha}{2}, \ |z| = \sqrt{2}\left(\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\right)$$

Case (iv) at $\alpha = \frac{3\pi}{2}$: $z = 2 + 0i$

$|z| = 2$

$\text{amp}(z) = 0$

Case (v) For $\alpha \in \left(\frac{3\pi}{2}, 2\pi\right)$

z will lie in I quadrant

$\arg(z) = \tan^{-1}\tan\left(\frac{\alpha}{2} + \frac{\pi}{4}\right)$

Since $\frac{\alpha}{2} + \frac{\pi}{4} \in \left(\pi, \frac{5\pi}{4}\right)$

$$\therefore \arg z = \frac{\alpha}{2} + \frac{\pi}{4} - \pi = \frac{\alpha}{2} - \frac{3\pi}{4}, \ |z| = \sqrt{2}\left(\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\right)$$

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Do yourself - 3 :

Find the modulus and amplitude of following complex numbers :

(i) $-2+2\sqrt{3}i$ **(ii)** $-\sqrt{3}-i$ **(iii)** $-2i$ **(iv)** $\frac{1+2i}{1-3i}$ **(v)** $\frac{2+6\sqrt{3}i}{5+\sqrt{3}i}$

6. REPRESENTATION OF A COMPLEX NUMBER IN VARIOUS FORMS :

(a) Cartesian Form (Geometrical Representation) :

Every complex number $z = x + i\,y$ can be represented by a point on the cartesian plane known as complex plane by the ordered pair (x , y). There exists a one-one correspondence between the points of the plane and the members of the set of complex numbers.

For $z = x + iy$; $|z| = \sqrt{x^2+y^2}$; $\bar{z} = x - iy$ and $\theta = \tan^{-1}\frac{y}{x}$

Note :

(i) Distance between the two complex numbers z_1 & z_2 is given by $|z_1 - z_2|$.

(ii) $|z - z_0| = r$, represents a circle, whose centre is z_0 and radius is r.

Illustration 8 : Find the locus of :

(a) $|z-1|^2 + |z+1|^2 = 4$ (b) $Re(z^2) = 0$

Solution : (a) Let $z = x + iy$

$\Rightarrow$ $(|x + iy - 1|)^2 + (|x + iy + 1|)^2 = 4$

$\Rightarrow$ $(x-1)^2 + y^2 + (x+1)^2 + y^2 = 4$

$\Rightarrow$ $x^2 - 2x + 1 + y^2 + x^2 + 2x + 1 + y^2 = 4 \Rightarrow x^2 + y^2 = 1$

Above represents a circle on complex plane with center at origin and radius unity.

(b) Let $z = x + iy$

$\Rightarrow$ $z^2 = x^2 - y^2 + 2xyi$

$\therefore$ $Re(z^2) = 0$

$\Rightarrow$ $x^2 - y^2 = 0 \Rightarrow y = \pm x$

Thus $Re(z^2) = 0$ represents a pair of straight lines passing through origin.

Illustration 9 : If z is a complex number such that $z^2 = (\bar{z})^2$, then

(A) z is purely real (B) z is purely imaginary

(C) either z is purely real or purely imaginary (D) none of these

Solution : Let $z = x + iy$, then its conjugate $\bar{z} = x - iy$

Given that $z^2 = (\bar{z})^2 \Rightarrow x^2 - y^2 + 2ixy = x^2 - y^2 - 2ixy \Rightarrow 4ixy = 0$

If $x \neq 0$ then $y = 0$ and if $y \neq 0$ then $x = 0$. **Ans. (C)**

Illustration 10 : Among the complex number z which satisfies $|z - 25i| \le 15$, find the complex numbers z having

(a) least positive argument (b) maximum positive argument

(c) least modulus (d) maximum modulus

Solution : The complex numbers z satisfying the condition

$|z - 25i| \le 15$

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are represented by the points inside and on the circle of radius 15 and centre at the point C(0, 25).

The complex number having least positive argument and maximum positive arguments in this region are the points of contact of tangents drawn from origin to the circle

Here θ = least positive argument

and ϕ = maximum positive argument

$\therefore$ In $\Delta OCP, OP = \sqrt{(OC)^2 - (CP)^2} = \sqrt{(25)^2 - (15)^2} = 20$

and $\sin\theta = \frac{OP}{OC} = \frac{20}{25} = \frac{4}{5}$

$\therefore$ $\tan\theta = \frac{4}{3} \Rightarrow \theta = \tan^{-1}\left(\frac{4}{3}\right)$

Thus, complex number at P has modulus 20 and argument $\theta = \tan^{-1}\left(\frac{4}{3}\right)$

$\therefore$ $z_p = 20(\cos\theta + i\sin\theta) = 20\left(\frac{3}{5} + i\frac{4}{5}\right)$

$\therefore$ $z_p = 12 + 16i$

Similarly $z_Q = -12 + 16i$

From the figure, E is the point with least modulus and D is the point with maximum modulus.

Hence, $z_E = \overrightarrow{OE} = \overrightarrow{OC} - \overrightarrow{EC} = 25i - 15i = 10i$

and $z_D = \overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = 25i + 15i = 40i$

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Do yourself - 4 :

(i) Find the distance between two complex numbers $z_1 = 2 + 3i$ & $z_2 = 7 - 9i$ on the complex plane

(ii) Find the locus of $|z - 2 - 3i| = 1$.

(iii) If z is a complex number, then $z^2 + \bar{z}^2 = 2$ represents -

(A) a circle (B) a straight line (C) a hyperbola (D) an ellipse

(c) Trigonometric / Polar Representation :

$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$; $\arg z = \theta$; $\bar{z} = r(\cos\theta - i\sin\theta)$

Note : $\cos\theta + i\sin\theta$ is also written as CiS θ.

Euler's formula :

The formula $e^{ix} = \cos x + i\sin x$ is called Euler's formula.

It was introduced by Euler in 1748, and is used as a method of expressing complex numbers.

Also $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ & $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$ are known as Euler's identities.

(d) Exponential Representation :

Let z be a complex number such that $|z| = r$ & $\arg z = \theta$, then $z = r.e^{i\theta}$

Illustration 11 : Express the following complex numbers in polar and exponential form :

(i) $\frac{1+3i}{1-2i}$ (ii) $\frac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$

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Solution : (i) Let $z = \frac{1+3i}{1-2i} = \frac{1+3i}{1-2i} \times \frac{1+2i}{1+2i} = -1+i$

$|z| = \sqrt{(-1)^2+1^2} = \sqrt{2}$

$\tan\alpha = \left|\frac{1}{-1}\right| = 1 = \tan\frac{\pi}{4} \Rightarrow \alpha = \frac{\pi}{4}$

$\because$ Re(z) < 0 and Im(z) > 0 $\Rightarrow$ z lies in second quadrant.

$\therefore$ $\theta = \arg(z) = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Hence Polar form is $z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

and exponential form is $z = \sqrt{2}\, e^{\frac{3\pi}{4}i}$

(ii) Let $z = \frac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}} = \frac{i-1}{\frac{1}{2} + \frac{i\sqrt{3}}{2}} = \frac{2(i-1)}{(1+i\sqrt{3})}$

$\Rightarrow$ $z = \frac{2(i-1)}{(1+i\sqrt{3})} \times \frac{(1-i\sqrt{3})}{(1-i\sqrt{3})} \Rightarrow z = \left(\frac{\sqrt{3}-1}{2}\right) + i\left(\frac{\sqrt{3}+1}{2}\right)$

$\because$ Re(z) > 0 and Im(z) > 0 $\Rightarrow$ z lies in first quadrant.

$\therefore$ $|z| = \sqrt{\left(\frac{\sqrt{3}-1}{2}\right)^2 + \left(\frac{\sqrt{3}+1}{2}\right)^2} = \sqrt{\frac{2(3+1)}{4}} = \sqrt{2}$.

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$\tan\theta = \left|\frac{\sqrt{3}+1}{\sqrt{3}-1}\right| = \tan\frac{5\pi}{12} \Rightarrow \alpha = \frac{5\pi}{12}$

Hence Polar form is $z = \sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$

and exponential form is $z = \sqrt{2}e^{\frac{5\pi}{12}i}$

Illustration 12 : If $x_n = \cos\left(\frac{\pi}{2^n}\right) + i\sin\left(\frac{\pi}{2^n}\right)$, then $x_1x_2x_3.....\infty$ is equal to -

(A) –1 (B) 1 (C) 0 (D) ∞

Solution : $x_n = \cos\left(\frac{\pi}{2^n}\right) + i\sin\left(\frac{\pi}{2^n}\right) = 1 \times e^{i\frac{\pi}{2^n}}$

$x_1x_2x_3.....\infty$

$= e^{i\frac{\pi}{2^1}}.e^{i\frac{\pi}{2^2}} - - - e^{i\frac{\pi}{2^n}} = e^{i\left(\frac{\pi}{2}+\frac{\pi}{2^2}+---+\frac{\pi}{2^n}\right)}$

$= \cos\left(\frac{\pi}{2}+\frac{\pi}{2^2}+\frac{\pi}{2^3}+.....\right) + i\sin\left(\frac{\pi}{2}+\frac{\pi}{2^2}+\frac{\pi}{2^3}+.....\right) = -1$

$\left(\text{as } \frac{\pi}{2}+\frac{\pi}{2^2}+\frac{\pi}{2^3}+..... = \frac{\pi/2}{1-1/2} = \pi\right)$

Ans. (A)

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Do yourself - 5 :

Express the following complex number in polar form and exponential form :

(i) $-2+2i$ **(ii)** $-1-\sqrt{3}i$ **(iii)** $\dfrac{(1+7i)}{(2-i)^2}$ **(iv)** $(1-\cos\theta+i\sin\theta),\ \theta\in(0,\pi)$

7. IMPORTANT PROPERTIES OF CONJUGATE :

(a) $z+\bar{z}=2\,\text{Re}(z)$ **(b)** $z-\bar{z}=2i\,\text{Im}(z)$ **(c)** $\overline{(\bar{z})}=z$

(d) $\overline{z_1+z_2}=\bar{z}_1+\bar{z}_2$ **(e)** $\overline{z_1-z_2}=\bar{z}_1-\bar{z}_2$

(f) $\overline{z_1 z_2}=\bar{z}_1.\bar{z}_2$. In general $\overline{z_1z_2.....z_n}=\bar{z}_1.\bar{z}_2.....\bar{z}_n$

(g) $\overline{\left(\dfrac{z_1}{z_2}\right)}=\dfrac{\bar{z}_1}{\bar{z}_2}$; $z_2\neq 0$ **(h)** If $f(\alpha+i\beta)=x+iy\Rightarrow f(\alpha-i\beta)=x-iy$

8. IMPORTANT PROPERTIES OF MODULUS :

(a) $|z|\geq 0$ **(b)** $|z|\geq \text{Re}(z)$ **(c)** $|z|\geq \text{Im}(z)$

(d) $|z|=|\bar{z}|=|-z|=|-\bar{z}|$ **(e)** $z\bar{z}=|z|^2$

(f) $|z_1 z_2|=|z_1|.|z_2|$. In general $|z_1z_2.....z_n|=|z_1|.|z_2|.....|z_n|$

(g) $\left|\dfrac{z_1}{z_2}\right|=\dfrac{|z_1|}{|z_2|}$, $z_2\neq 0$

(h) $|z^n|=|z|^n,\ n\in I$

(i) $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2\,\text{Re}(z_1\bar{z}_2)$

(j) $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\alpha-\beta)$, where α,β are $\arg(z_1)$, $\arg(z_2)$ respectively.

(k) $|z_1+z_2|^2+|z_1-z_2|^2=2\left[|z_1|^2+|z_2|^2\right]$

(l) $\big||z_1|-|z_2|\big|\leq|z_1+z_2|\leq|z_1|+|z_2|$ **[Triangle Inequality]**

(m) $\big||z_1|-|z_2|\big|\leq|z_1-z_2|\leq|z_1|+|z_2|$ **[Triangle Inequality]**

9. IMPORTANT PROPERTIES OF AMPLITUDE :

(a) $\text{amp}(z_1.z_2)=\text{amp}\ z_1+\text{amp}\ z_2+2k\pi$; $k\in I$

(b) $\text{amp}\left(\dfrac{z_1}{z_2}\right)=\text{amp}\ z_1-\text{amp}\ z_2+2k\pi$; $k\in I$

(c) $\text{amp}(z^n)=n\,\text{amp}(z)+2k\pi$; $n,k\in I$
where proper value of k must be chosen so that RHS lies in $(-\pi,\pi]$.

Illustration 13 : Find amp z and |z| if $z=\left[\dfrac{(3+4i)(1+i)(1+\sqrt{3}i)}{(1-i)(4-3i)(2i)}\right]^2$.

Solution : $\text{amp}\ z=2\left[\text{amp}(3+4i)+\text{amp}(1+i)+\text{amp}(1+\sqrt{3}i)-\text{amp}(1-i)-\text{amp}(4-3i)-\text{amp}(2i)\right]+2k\pi$
where $k\in I$ and k chosen so that amp z lies in $(-\pi,\pi]$.

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$$\Rightarrow \quad \text{amp}\, z = 2\left[\tan^{-1}\frac{4}{3}+\frac{\pi}{4}+\frac{\pi}{3}-\left(-\frac{\pi}{4}\right)-\tan^{-1}\left(-\frac{3}{4}\right)-\frac{\pi}{2}\right]+2k\pi$$

$$\Rightarrow \quad \text{amp } z = 2\left[\tan^{-1}\frac{4}{3}+\cot^{-1}\frac{4}{3}+\frac{\pi}{3}\right]+2k\pi \Rightarrow \text{amp}\, z = 2\left[\frac{\pi}{2}+\frac{\pi}{3}\right]+2k\pi$$

$$\Rightarrow \quad \text{amp } z = -\frac{\pi}{3} \quad [\text{at } k=-1]$$ **Ans.**

Also,

$$|z| = \left|\frac{(3+4i)(1+i)\left(1+\sqrt{3}i\right)}{(1-i)(4-3i)(2i)}\right|^2 \Rightarrow \quad |z| = \left(\frac{|3+4i||1+i|\left|1+\sqrt{3}i\right|}{|1-i||4-3i||2i|}\right)^2$$

$$\Rightarrow \quad |z| = \left(\frac{5\times\sqrt{2}\times 2}{\sqrt{2}\times 5\times 2}\right)^2 = 1$$ **Ans.**

Aliter

$$z = \left[\frac{(3+4i)(1+i)\left(1+\sqrt{3}i\right)}{(1-i)(4-3i)(2i)}\right]^2 \Rightarrow z = \left[-\frac{\sqrt{3}+i}{2}\right]^2 \Rightarrow z = \frac{2-2\sqrt{3}i}{4} = \frac{1}{2}-\frac{\sqrt{3}i}{2}$$

Hence $|z| = 1$, $\text{amp}(z) = -\frac{\pi}{3}$.

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Illustration 14 : If $\left|\frac{z-i}{z+i}\right| = 1$, then locus of z is -

(A) x-axis (B) y-axis (C) x = 1 (D) y = 1

Solution : We have, $\left|\frac{z-i}{z+i}\right| = 1 \Rightarrow \left|\frac{x+i(y-1)}{x+i(y+1)}\right| = 1$

$$\Rightarrow \frac{|x+i(y-1)|^2}{|x+i(y+1)|^2} = 1 \Rightarrow x^2+(y-1)^2 = x^2+(y+1)^2 \Rightarrow 4y = 0; y = 0$$

which is x-axis **Ans. (A)**

Illustration 15 : If $|z_1+z_2|^2 = |z_1|^2+|z_2|^2$ then $\left(\frac{z_1}{z_2}\right)$ is -

(A) zero or purely imaginary (B) purely imaginary
(C) purely real (D) none of these

Solution : Here let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $|z_1| = r_1$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, $|z_2| = r_2$

$$\therefore \quad |(z_1+z_2)|^2 = \left|(r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)\right|^2$$

$$= r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1-\theta_2) = |z_1|^2 + |z_2|^2 \text{ if } \cos(\theta_1-\theta_2) = 0$$

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$\therefore \quad \theta_1 - \theta_2 = \pm\frac{\pi}{2}$

$\Rightarrow \text{amp}(z_1) - \text{amp}(z_2) = \pm\frac{\pi}{2} \Rightarrow \text{amp}\left(\frac{z_1}{z_2}\right) = \pm\frac{\pi}{2} \Rightarrow \frac{z_1}{z_2}$ is purely imaginary **Ans. (B)**

Illustration 16 : z_1 and z_2 are two complex numbers such that $\frac{z_1 - 2z_2}{2 - z_1\overline{z_2}}$ is unimodular (whose modulus is one), while z_2 is not unimodular. Find $|z_1|$.

Solution : Here $\left|\frac{z_1 - 2z_2}{2 - z_1\overline{z_2}}\right| = 1 \Rightarrow \frac{|z_1 - 2z_2|}{|2 - z_1\overline{z_2}|} = 1$

$\Rightarrow \quad |z_1 - 2z_2| = |2 - z_1\overline{z_2}| \Rightarrow \quad |z_1 - 2z_2|^2 = |2 - z_1\overline{z_2}|^2$

$\Rightarrow \quad (z_1 - 2z_2)(\overline{z_1 - 2z_2}) = (2 - z_1\overline{z_2})(\overline{2 - z_1\overline{z_2}})$

$\Rightarrow \quad (z_1 - 2z_2)(\overline{z_1} - 2\overline{z_2}) = (2 - z_1\overline{z_2})(2 - \overline{z_1}z_2)$

$\Rightarrow \quad z_1\overline{z_1} - 2z_1\overline{z_2} - 2z_2\overline{z_1} + 4z_2\overline{z_2} = 4 - 2\overline{z_1}z_2 - 2z_1\overline{z_2} + z_1\overline{z_1}z_2\overline{z_2}$

$\Rightarrow \quad |z_1|^2 + 4|z_2|^2 = 4 + |z_1|^2|z_2|^2 \Rightarrow \quad |z_1|^2 - |z_1|^2|z_2|^2 + 4|z_2|^2 - 4 = 0$

$\Rightarrow \quad (|z_1|^2 - 4)(1 - |z_2|^2) = 0$

But $|z_2| \neq 1$ (given)

$\therefore \quad |z_1|^2 = 4$

Hence, $\quad |z_1| = 2.$

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Illustration 17 : The locus of the complex number z in argand plane satisfying the inequality

$\log_{1/2}\left(\frac{|z-1|+4}{3|z-1|-2}\right) > 1 \left(\text{where } |z-1| \neq \frac{2}{3}\right)$ is -

(A) a circle (B) interior of a circle (C) exterior of a circle (D) none of these

Solution : We have, $\log_{1/2}\left(\frac{|z-1|+4}{3|z-1|-2}\right) > 1 = \log_{1/2}\left(\frac{1}{2}\right)$

$\Rightarrow \frac{|z-1|+4}{3|z-1|-2} < \frac{1}{2}$ $\quad [\because \log_a x \text{ is a decreasing function if } a < 1]$

$\Rightarrow \quad 2|z-1| + 8 < 3|z-1| - 2 \quad \text{as} \quad |z-1| > 2/3$

$\Rightarrow \quad |z-1| > 10$

which is exterior of a circle. **Ans. (C)**

Illustration 18 : If $\left|z - \frac{4}{z}\right| = 2$, then the greatest value of $|z|$ is -

(A) $1 + \sqrt{2}$ (B) $2 + \sqrt{2}$ (C) $\sqrt{3} + 1$ (D) $\sqrt{5} + 1$

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Solution : We have $|z| = \left|z - \frac{4}{z} + \frac{4}{z}\right| \le \left|z - \frac{4}{z}\right| + \frac{4}{|z|} = 2 + \frac{4}{|z|}$

$\Rightarrow \quad |z|^2 \le 2|z| + 4 \quad \Rightarrow \quad (|z| - 1)^2 \le 5$

$\Rightarrow \quad |z| - 1 \le \sqrt{5} \quad \Rightarrow \quad |z| \le \sqrt{5} + 1$

Therefore, the greatest value of $|z|$ is $\sqrt{5} + 1$. **Ans. (D)**

Illustration 19 : Shaded region is given by -

(A) $|z + 2| \ge 6,\ 0 \le \arg(z) \le \frac{\pi}{6}$

(B) $|z + 2| \le 6,\ 0 \le \arg(z + 2) \le \frac{\pi}{3}$

(C) $|z + 2| \ge 6,\ 0 \le \arg(z + 2) \le \frac{\pi}{3}$

(D) None of these

Solution : Note that AB = 6 and $1 + 3\sqrt{3}i = -2 + 3 + 3\sqrt{3}i = -2 + 6\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -2 + 6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$\therefore \quad \angle BAC = \frac{\pi}{3}$

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Thus, shaded region is given by $|z + 2| \ge 6$ and $0 \le \arg(z + 2) \le \frac{\pi}{3}$ **Ans. (C)**

Do yourself - 6 :

(i) The inequality $|z - 4| < |z - 2|$ represents region given by -

(A) $\text{Re}(z) > 0$ (B) $\text{Re}(z) < 0$ (C) $\text{Re}(z) > 3$ (D) none

(ii) If $z = re^{i\theta}$, then the value of $|e^{iz}|$ is equal to -

(A) $e^{-r\cos\theta}$ (B) $e^{r\cos\theta}$ (C) $e^{r\sin\theta}$ (D) $e^{-r\sin\theta}$

10. SECTION FORMULA AND COORDINATES OF ORTHOCENTRE, CENTROID, CIRCUMCENTRE, INCENTRE OF A TRIANGLE :

If z_1 & z_2 are two complex numbers then the complex number $z = \frac{nz_1 + mz_2}{m + n}$ divides the join of z_1 & z_2 in the ratio m : n.

Note :

(i) If a , b , c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$; where $a + b + c = 0$ and a,b,c are not all simultaneously zero, then the complex numbers z_1, z_2 & z_3 are collinear.

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(ii) If the vertices A, B, C of a triangle represent the complex numbers z_1, z_2, z_3 respectively, then :

- Centroid of the $\Delta ABC = \dfrac{z_1 + z_2 + z_3}{3}$
- Orthocentre of the $\Delta ABC =$

$$\frac{(a \sec A)z_1 + (b \sec B)z_2 + (c \sec C)z_3}{a \sec A + b \sec B + c \sec C} \quad \text{or} \quad \frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\tan A + \tan B + \tan C}$$

- Incentre of the $\Delta ABC = \dfrac{(az_1 + bz_2 + cz_3)}{(a+b+c)}$
- Circumcentre of the $\Delta ABC = \dfrac{(z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C)}{(\sin 2A + \sin 2B + \sin 2C)}$

11. VECTORIAL REPRESENTATION OF A COMPLEX NUMBER :

(a) In complex number every point can be represented in terms of position vector. If the point P represents the complex number z then, $\vec{OP} = z$ & $|\vec{OP}| = |z|$.

(b) If $P(z_1)$ & $Q(z_2)$ be two complex numbers on argand plane then $\vec{PQ}$ represents complex number $z_2 - z_1$.

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Note :

(i) If $\vec{OP} = z = re^{i\theta}$ then $\vec{OQ} = z_1 = re^{i(\theta+\phi)} = z.\,e^{i\phi}$. If $\vec{OP}$ and $\vec{OQ}$ are of unequal magnitude then $\hat{OQ} = \hat{OP}\, e^{i\phi}$ i.e. $\dfrac{z_1}{|z_1|} = \dfrac{z}{|z|} e^{i\phi}$

(ii) In general, if z_1, z_2, z_3 be the three vertices of ΔABC then

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i\alpha}. \text{ Here } \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \alpha.$$

(iii) Note that the locus of z satisfying $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \alpha$ is:

Case (a) $0 < \alpha < \pi/2$

Locus is major arc of circle as shown excluding z_1 & z_2

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Case (b) $\frac{\pi}{2} < \alpha < \pi$

Locus is minor arc of circle as shown excluding z_1 & z_2

(iv) If A, B, C & D are four points representing the complex numbers z_1, z_2, z_3 & z_4 then AB || CD if $\frac{z_4 - z_3}{z_2 - z_1}$ is purely real ;

AB ⊥ CD if $\frac{z_4 - z_3}{z_2 - z_1}$ is purely imaginary.

(v) If z_1, z_2, z_3 are the vertices of an equilateral triangle where z_0 is its circumcentre then

(1) $z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$ (2) $z_1^2 + z_2^2 + z_3^2 = 3 z_0^2$

Illustration 20 : Complex numbers z_1, z_2, z_3 are the vertices A, B, C respectively of an isosceles right angled triangle with right angle at C. Show that $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

Solution : In the isosceles triangle ABC, AC = BC and BC⊥AC. It means that AC is rotated through angle π/2 to occupy the position BC.

Hence we have, $\frac{z_2 - z_3}{z_1 - z_3} = e^{+i\pi/2} = +i \Rightarrow z_2 - z_3 = +i(z_1 - z_3)$

$\Rightarrow z_2^2 + z_3^2 - 2z_2z_3 = -\left(z_1^2 + z_3^2 - 2z_1z_3\right)$

$\Rightarrow z_1^2 + z_2^2 - 2z_1z_2 = 2z_1z_3 + 2z_2z_3 - 2z_1z_2 - 2z_3^2 = 2\left(z_1 - z_3\right)\left(z_3 - z_2\right)$

$\Rightarrow \left(z_1 - z_2\right)^2 = 2\left(z_1 - z_3\right)\left(z_3 - z_2\right)$

Illustration 21: If the vertices of a square ABCD are z_1, z_2, z_3 & z_4 then find z_3 & z_4 in terms of z_1 & z_2.

Solution : Using vector rotation at angle A

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i\frac{\pi}{4}}$$

$\because |z_3 - z_1| = AC$ and $|z_2 - z_1| = AB$

Also $AC = \sqrt{2}\ AB$

$\therefore |z_3 - z_1| = \sqrt{2}|z_2 - z_1|$

$$\Rightarrow \frac{z_3 - z_1}{z_2 - z_1} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$\Rightarrow z_3 - z_1 = (z_2 - z_1)(1 + i)$

$\Rightarrow z_3 = z_1 + (z_2 - z_1)(1 + i)$

Similarly $z_4 = z_2 + (1 + i)(z_1 - z_2)$

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Illustration 22 : Plot the region represented by $\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$ in the Argand plane.

Solution : Let us take $\arg\left(\frac{z+1}{z-1}\right) = \frac{2\pi}{3}$, clearly z lies on the minor arc of the circle passing through (1, 0) and (–1, 0). Similarly, $\arg\left(\frac{z+1}{z-1}\right) = \frac{2\pi}{3}$ means that 'z' is lying on the major arc of the circle passing through (1, 0) and (–1, 0). Now if we take any point in the region included between two arcs say $P_1(z_1)$ we get $\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$

Thus $\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$ represents the shaded region (excluding points (1, 0) and (–1, 0)).

Do yourself - 7 :

(i) A complex number z = 3 + 4i is rotated about another fixed complex number $z_1 = 1 + 2i$ in anticlockwise direction by 45° angle. Find the complex number represented by new position of z in argand plane.

(ii) If A, B, C are three points in argand plane representing the complex number z_1, z_2, z_3 such that $z_1 = \frac{\lambda z_2 + z_3}{\lambda + 1}$, where $\lambda \in R$, then find the distance of point A from the line joining points B and C.

(iii) If $A(z_1), B(z_2), C(z_3)$ are vertices of ΔABC in which $\angle ABC = \frac{\pi}{4}$ and $\frac{AB}{BC} = \sqrt{2}$, then find z_2 in terms of z_1 and z_3.

(iv) If a & b are real numbers between 0 and 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle then a and b are equal to :-

(A) a = b = 1/2 (B) $a = b = 2 - \sqrt{3}$ (C) $a = b = -2 + \sqrt{3}$ (D) $a = b = \sqrt{2} - 1$

(v) If $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$, find locus of z.

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12. DE'MOIVRE'S THEOREM :

The value of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$ if 'n' is integer & it is one of the values of $(\cos\theta + i\sin\theta)^n$ if n is a rational number of the form p/q, where p & q are co-prime.

Note : Continued product of the roots of a complex quantity should be determined by using theory of equations.

Illustration 23: If $\cos\alpha + \cos\beta + \cos\gamma = 0$ and also $\sin\alpha + \sin\beta + \sin\gamma = 0$, then prove that

(a) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

(b) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

(c) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

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Solution : Let $z_1 = \cos\alpha + i\sin\alpha$, $z_2 = \cos\beta + i\sin\beta$ & $z_3 = \cos\gamma + i\sin\gamma$.

$\therefore$ $z_1 + z_2 + z_3 = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 + i\,.\,0 = 0$ ..... (i)

(a) Also $\dfrac{1}{z_1} = (\cos\alpha + i\sin\alpha)^{-1} = \cos\alpha - i\sin\alpha$

$\dfrac{1}{z_2} = \cos\beta - i\sin\beta,\ \dfrac{1}{z_3} = \cos\gamma - i\sin\gamma$

$\therefore$ $\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3} = (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$ ..... (ii)

$= 0 - i\,.\,0 = 0$

Now $z_1^2 + z_2^2 + z_3^2 = (z_1 + z_2 + z_3)^2 - 2(z_1z_2 + z_2z_3 + z_3z_1)$

$= 0 - 2z_1z_2z_3\left(\dfrac{1}{z_3} + \dfrac{1}{z_1} + \dfrac{1}{z_2}\right) = 0 - 2z_1z_2z_3\,.\,0 = 0$ {using (i) and (ii)}

or $(\cos\alpha + i\sin\alpha)^2 + (\cos\beta + i\sin\beta)^2 + (\cos\gamma + i\sin\gamma)^2 = 0$

or $\cos2\alpha + i\sin2\alpha + \cos2\beta + i\sin2\beta + \cos2\gamma + i\sin2\gamma = 0 + i.0$

Equating real and imaginary parts on both sides,

$\cos2\alpha + \cos2\beta + \cos2\gamma = 0$ and $\sin2\alpha + \sin2\beta + \sin2\gamma = 0$

(b) If $z_1 + z_2 + z_3 = 0$ then $z_1^3 + z_2^3 + z_3^3 = 3z_1z_2z_3$

$\therefore$ $(\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$

$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$

or $\cos3\alpha + i\sin3\alpha + \cos3\beta + i\sin3\beta + \cos3\gamma + i\sin3\gamma$

$= 3\{\cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)\}$

Equating imaginary parts on both sides, $\sin3\alpha + \sin3\beta + \sin3\gamma = 3\sin(\alpha + \beta + \gamma)$

(c) Equating real parts on both sides, $\cos3\alpha + \cos3\beta + \cos3\gamma = 3\cos(\alpha + \beta + \gamma)$

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Do yourself - 8 :

(i) If $z_r = \cos\dfrac{2r\pi}{5} + i\sin\dfrac{2r\pi}{5}$, $r = 0, 1, 3, 4, \ldots\ldots$, then $z_1z_2z_3z_4z_5$ is equal to -

(A) –1 (B) 0 (C) 1 (D) none of these

(ii) If $(x - 1)^4 - 16 = 0$, then the sum of non-real complex values of x is -

(A) 2 (B) 0 (C) 4 (D) none of these

(iii) If $(\sqrt{3} - i)^n = 2^n$, $n \in Z$, then n is a multiple of -

(A) 6 (B) 10 (C) 9 (D) 12

13. CUBE ROOT OF UNITY :

(a) The cube roots of unity are 1, $\dfrac{-1 + i\sqrt{3}}{2}(\omega)$, $\dfrac{-1 - i\sqrt{3}}{2}(\omega^2)$.

(b) If ω is one of the imaginary cube roots of unity then $1 + \omega + \omega^2 = 0$. In general $1 + \omega^r + \omega^{2r} = 0$; where $r \in I$ but is not the multiple of 3 & $1 + \omega^r + \omega^{2r} = 3$ if $r = 3\lambda$; $\lambda \in I$

(c) In polar form the cube roots of unity are :

$1 = \cos 0 + i\sin 0$; $\omega = \cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}$, $\omega^2 = \cos\dfrac{4\pi}{3} + i\sin\dfrac{4\pi}{3}$

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(d) The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.

(e) The following factorisation should be remembered :
(a, b, c $\in$ R & ω is the cube root of unity)
$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$; $x^2 + x + 1 = (x - \omega)(x - \omega^2)$;
$a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$;
$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

Illustration 24 : If α & β are imaginary cube roots of unity then $\alpha^n + \beta^n$ is equal to (where $n \in I$) -

(A) $2\cos\frac{2n\pi}{3}$ (B) $\cos\frac{2n\pi}{3}$ (C) $2i\sin\frac{2n\pi}{3}$ (D) $i\sin\frac{2n\pi}{3}$

Solution : $\alpha = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$;

$\beta = \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}$

$$\alpha^n + \beta^n = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^n + \left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)^n$$

$$= \left(\cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}\right) + \left(\cos\frac{2n\pi}{3} - i\sin\left(\frac{2n\pi}{3}\right)\right) = 2\cos\left(\frac{2n\pi}{3}\right)$$ **Ans. (A)**

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Illustration 25 : If α, β, γ are roots of $x^3 - 3x^2 + 3x + 7 = 0$ (and ω is imaginary cube root of unity), then find the value of $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1}$.

Solution : We have $x^3 - 3x^2 + 3x + 7 = 0$

$\therefore$ $(x - 1)^3 + 8 = 0$

$\therefore$ $(x - 1)^3 = (-2)^3$

$\Rightarrow$ $\left(\frac{x-1}{-2}\right)^3 = 1$ $\Rightarrow$ $\frac{x-1}{-2} = (1)^{1/3} = 1, \omega, \omega^2$ (cube roots of unity)

$\therefore$ $x = -1, 1 - 2\omega, 1 - 2\omega^2$

Here $\alpha = -1, \beta = 1 - 2\omega, \gamma = 1 - 2\omega^2$

$\therefore$ $\alpha - 1 = -2, \beta - 1 = -2\omega, \gamma - 1 = -2\omega^2$

Then $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1} = \left(\frac{-2}{-2\omega}\right) + \left(\frac{-2\omega}{-2\omega^2}\right) + \left(\frac{-2\omega^2}{-2}\right) = \frac{1}{\omega} + \frac{1}{\omega} + \omega^2 = \omega^2 + \omega^2 + \omega^2$

Therefore $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1} = 3\omega^2$. **Ans.**

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Do yourself - 9 :

(i) If ω is an imaginary cube root of unity, then $(1 + \omega - \omega^2)^2$ equals : -

(A) ω (B) -4ω (C) ω^2 (D) 4ω

(ii) If ω is a non real cube root of unity, then the expression $(1-\omega)(1-\omega^2)(1+\omega^4)(1+\omega^8)$ is equal to :-

(A) 0 (B) 3 (C) 1 (D) 2

14. n^{th} ROOTS OF UNITY :

If $1, \alpha_1, \alpha_2, \alpha_3 .... \alpha_{n-1}$ are the n, n^{th} root of unity then :

(a) They are in G.P. with common ratio $e^{i(2\pi/n)}$

(b) Their arguments are in A.P. with common difference $\frac{2\pi}{n}$

(c) The points represented by n, n^{th} roots of unity are located at the vertices of a regular polygon of n sides inscribed in a unit circle having center at origin, one vertex being on positive real axis.

(d) $1^p + \alpha_1^p + \alpha_2^p + ... + \alpha_{n-1}^p = 0$ if p is not an integral multiple of n

$= n$ if p is an integral multiple of n

(e) $(1-\alpha_1)(1-\alpha_2)..... (1-\alpha_{n-1}) = n$

(f) $(1+\alpha_1)(1+\alpha_2)..... (1+\alpha_{n-1}) = 0$ if n is even and

$= 1$ if n is odd.

(g) $1.\alpha_1.\alpha_2.\alpha_3..... \alpha_{n-1} = 1$ or -1 according as n is odd or even.

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Illustration 26: Find the value $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - \cos\frac{2\pi k}{7}\right)$

Solution : $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7}\right) - \sum_{k=1}^{6}\left(\cos\frac{2\pi k}{7}\right) = \sum_{k=1}^{6}\sin\frac{2\pi k}{7} - \sum_{k=0}^{6}\cos\frac{2\pi k}{7} + 1$

$= \sum_{k=0}^{6}$ (Sum of imaginary part of seven seventh roots of unity)

$-\sum_{k=0}^{6}$ (Sum of real part of seven seventh roots of unity) $+1 = 0 - 0 + 1 = 1$

15. THE SUM OF THE FOLLOWING SERIES SHOULD BE REMEMBERED :

(a) $\cos\theta + \cos 2\theta + \cos 3\theta + ... + \cos n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \cos\left(\frac{n+1}{2}\right)\theta.$

(b) $\sin\theta + \sin 2\theta + \sin 3\theta + ... + \sin n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \sin\left(\frac{n+1}{2}\right)\theta.$

Note : If $\theta = (2\pi/n)$ then the sum of the above series vanishes.

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16. STRAIGHT LINES & CIRCLES IN TERMS OF COMPLEX NUMBERS :

(a) $\text{amp}(z-\alpha)=\theta$ is a ray emanating from the complex

point α and inclined at an angle θ to the x–axis.

(b) $|z-a|=|z-b|$ is the perpendicular bisector of the segment joining a & b.

(c) The equation of a line joining z_1 & z_2 is given by ;

$z=z_1+t(z_1-z_2)$ where t is a parameter.

(d) $z=z_1(1+it)$ where t is a real parameter, is a line through the point z_1 & perpendicular to z_1.

(e) The equation of a line passing through z_1 & z_2 can be expressed in the determinant form as

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix}=0.$$ This is also the condition for three complex numbers to be collinear.

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(f) Complex equation of a straight line through two given points z_1 & z_2 can be written as $z(\bar{z}_1-\bar{z}_2)-\bar{z}(z_1-z_2)+(z_1\bar{z}_2-\bar{z}_1z_2)=0$, which on manipulating takes the form as $\bar{\alpha}z+\alpha\bar{z}+r=0$ where r is real and α is a non zero complex constant.

(g) The equation of circle having centre z_0 & radius ρ is :

$|z-z_0|=\rho$ or $z\bar{z}-z_0\bar{z}-\bar{z}_0z+\bar{z}_0z_0-\rho^2=0$ which is of the form

$z\bar{z}+\bar{\alpha}z+\alpha\bar{z}+r=0$, r is real centre $=-\alpha$ & radius $=\sqrt{\alpha\bar{\alpha}-r}$.

Circle will be real if $\alpha\bar{\alpha}-r\ge 0$.

(h) $\arg\left(\dfrac{z-z_2}{z-z_1}\right)=\pm\dfrac{\pi}{2}$ or $(z-z_1)(\bar{z}-\bar{z}_2)+(z-z_2)(\bar{z}-\bar{z}_1)=0$

this equation represents the circle described on the line segment joining z_1 & z_2 as diameter.

(i) Condition for four given points z_1, z_2, z_3 & z_4 to be concyclic is, the number $\dfrac{z_3-z_1}{z_3-z_2}\cdot\dfrac{z_4-z_2}{z_4-z_1}$ is real. Hence the equation of a circle through 3 non collinear points z_1, z_2 & z_3 can be taken as

$$\frac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)} \text{ is real } \Rightarrow \frac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)}=\frac{(\bar{z}-\bar{z}_2)(\bar{z}_3-\bar{z}_1)}{(\bar{z}-\bar{z}_1)(\bar{z}_3-\bar{z}_2)}$$

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Miscellaneous Illustration :

Illustration 27 : If z is a point on the Argand plane such that $|z-1| = 1$, then $\frac{z-2}{z}$ is equal to -

(A) tan (arg z) (B) cot (arg z) (C) i tan (arg z) (D) none of these

Solution : Since $|z-1| = 1$,

$\therefore$ let $z-1 = \cos\theta + i\sin\theta$

Then, $z-2 = \cos\theta + i\sin\theta - 1$

$$= -2\sin^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2i\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \text{ .... (i)}$$

and $z = 1 + \cos\theta + i\sin\theta$

$$= 2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \text{ .... (ii)}$$

From (i) and (ii), we get $\frac{z-2}{z} = i\tan\frac{\theta}{2} = i\tan(\arg z)\left(\because \arg z = \frac{\theta}{2} \text{ from (ii)}\right)$ **Ans. (C)**

Illustration 28 : Let a be a complex number such that $|a| < 1$ and $z_1, z_2, \ldots\ldots, z_n$ be the vertices of a polygon such that $z_k = 1 + a + a^2 + \ldots. a^k$, then show that vertices of the polygon lie within the circle $\left|z - \frac{1}{1-a}\right| = \frac{1}{|1-a|}$.

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Solution : We have, $z_k = 1 + a + a^2 + \ldots.. + a^k = \frac{1-a^{k+1}}{1-a}$

$$\Rightarrow \quad z_k - \frac{1}{1-a} = \frac{-a^{k+1}}{1-a} \quad \Rightarrow \quad \left|z_k - \frac{1}{1-a}\right| = \frac{|a|^{k+1}}{|1-a|} < \frac{1}{|1-a|} \qquad (\because |a| < 1)$$

$\therefore$ Vertices of the polygon $z_1, z_2, \ldots.., z_n$ lie within the circle $\left|z - \frac{1}{1-a}\right| = \frac{1}{|1-a|}$

Illustration 29 : If z_1 and z_2 are two complex numbers and $C > 0$, then prove that

$$|z_1 + z_2|^2 \le (1 + C)\,|z_1|^2 + (1 + C^{-1})|z_2|^2$$

Solution : We have to prove that : $|z_1 + z_2|^2 \le (1 + C)\,|z_1|^2 + (1 + C^{-1})|z_2|^2$

i.e. $|z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2 \le (1+C)\,|z_1|^2 + (1 + C^{-1})|z_2|^2$

or $z_1\bar{z}_2 + \bar{z}_1 z_2 \le C|z_1|^2 + C^{-1}|z_2|^2$

or $C|z_1|^2 + \frac{1}{C}|z_2|^2 - z_1\bar{z}_2 - \bar{z}_1 z_2 \ge 0$ (using $\text{Re}\,(z_1\bar{z}_2) \le |z_1\bar{z}_2|$)

or $\left(\sqrt{C}\,|z_1| - \frac{1}{\sqrt{C}}|z_2|\right)^2 \ge 0$ which is always true.

Illustration 30 : If $\theta \in [\pi/6, \pi/3]$, i = 1, 2, 3, 4, 5 and $z^4\cos\theta_1 + z^3\cos\theta_2 + z^2\cos\theta_3 + z\cos\theta_4 + \cos\theta_5 = 2\sqrt{3}$, then show that $|z| > \frac{3}{4}$

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Solution : Given that $\cos\theta_1.z^4+\cos\theta_2.z^3+\cos\theta_3.z^2+\cos\theta_4.z+\cos\theta_5=2\sqrt{3}$

or $\left|\cos\theta_1.z^4+\cos\theta_2.z^3+\cos\theta_3.z^2+\cos\theta_4.z+\cos\theta_5\right|=2\sqrt{3}$

$$2\sqrt{3}\le\left|\cos\theta_1.z^4\right|+\left|\cos\theta_2.z^3\right|+\left|\cos\theta_3.z^2\right|+\left|\cos\theta_4.z\right|+\left|\cos\theta_5\right|$$

$\because$ $\theta_i\in[\pi/6,\ \pi/3]$

$\therefore$ $\frac{1}{2}\le\cos\theta_i\le\frac{\sqrt{3}}{2}$

$$2\sqrt{3}\le\frac{\sqrt{3}}{2}|z|^4+\frac{\sqrt{3}}{2}|z|^3+\frac{\sqrt{3}}{2}|z|^2+\frac{\sqrt{3}}{2}|z|+\frac{\sqrt{3}}{2}$$

$\Rightarrow 3\le|z|^4+|z|^3+|z|^2+|z|$ $\Rightarrow 3<|z|+|z|^2+|z|^3+|z|^4+|z|^5+.....\infty$

$\Rightarrow 3<\frac{|z|}{1-|z|}$ $\Rightarrow 3-3|z|<|z|$ $\Rightarrow 4|z|>3$

$\therefore$ $|z|>\frac{3}{4}$

Illustration 31 : If z_1, z_2, z_3 are complex numbers such that $\frac{2}{z_1}=\frac{1}{z_2}+\frac{1}{z_3}$, show that the points represented by z_1, z_2, z_3 lie on a circle passing through the origin.

Solution : We have, $\frac{2}{z_1}=\frac{1}{z_2}+\frac{1}{z_3}\Rightarrow\frac{1}{z_1}-\frac{1}{z_2}=\frac{1}{z_3}-\frac{1}{z_1}\Rightarrow\frac{z_2-z_1}{z_1z_2}=\frac{z_1-z_3}{z_1z_3}$

$\Rightarrow\frac{z_2-z_1}{z_3-z_1}=\frac{-z_2}{z_3}$ $\Rightarrow\arg\left(\frac{z_2-z_1}{z_3-z_1}\right)=\arg\left(\frac{-z_2}{z_3}\right)$

$\arg\left(\frac{z_2-z_1}{z_3-z_1}\right)=\pi+\arg\left(\frac{z_2}{z_3}\right)\Rightarrow$ or $\beta=\pi-\arg\frac{z_3}{z_2}=\pi-\alpha=\alpha+\beta=\pi$

Thus the sum of a pair of opposite angle of a quadrilateral is 180°. Hence, the points 0, z_1, z_2 and z_3 are the vertices of a cyclic quadrilateral i.e. lie on a circle.

Illustration 32 : Two given points P & Q are the reflection points w.r.t. a given straight line if the given line is the right bisector of the segment PQ. Prove that the two points denoted by the complex numbers z_1 & z_2 will be the reflection points for the straight line $\bar{\alpha}z+\alpha\bar{z}+r=0$ if and only if; $\bar{\alpha}z_1+\alpha\bar{z}_2+r=0$, where r is real and α is non zero complex constant.

Solution : Let $P(z_1)$ is the reflection point of $Q(z_2)$ then the perpendicular bisector of z_1 & z_2 must be the line

$\bar{\alpha}z+\alpha\bar{z}+r=0$ ..... (i)

Now perpendicular bisector of z_1 & z_2 is, $|z-z_1|=|z-z_2|$

or $(z-z_1)(\bar{z}-\bar{z}_1)=(z-z_2)(\bar{z}-\bar{z}_2)$

$-z\bar{z}_1-z_1\bar{z}+z_1\bar{z}_1=-z\bar{z}_2-z_2\bar{z}+z_2\bar{z}_2$ ($z\bar{z}$ cancels on either side)

or $(\bar{z}_2-\bar{z}_1)z+(z_2-z_1)\bar{z}+z_1\bar{z}_1-z_2\bar{z}_2=0$ ..... (ii)

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Comparing (i) & (ii) $\dfrac{\bar{\alpha}}{\bar{z}_2-\bar{z}_1}=\dfrac{\alpha}{z_2-z_1}=\dfrac{r}{z_1\bar{z}_1-z_2\bar{z}_2}=\lambda$

$\therefore \quad \bar{\alpha}=\lambda(\bar{z}_2-\bar{z}_1)$ ..... (iii) $\qquad \alpha=\lambda(z_2-z_1)$ ..... (iv)

$r=\lambda(z_1\bar{z}_1-z_2\bar{z}_2)$ ..... (v)

Multiplying (iii) by z_1; (iv) by $\bar{z}_2$ and adding

$\bar{\alpha}z_1+\alpha\bar{z}_2+r=0$

Note that we could also multiply (iii) by z_2 & (iv) by $\bar{z}_1$ & add to get the same result.

Hence $\bar{\alpha}z_1+\alpha\bar{z}_2+r=0$

Again, let $\bar{\alpha}z_1+\alpha\bar{z}_2+r=0$ is true w.r.t. the line $\bar{\alpha}z+\alpha\bar{z}+r=0$.

Subtracting $\bar{\alpha}(z-z_1)+\alpha(\bar{z}-\bar{z}_2)=0$

or $\quad |(z-z_1)||\bar{\alpha}|=|\alpha||(\bar{z}-\bar{z}_2)| \quad$ or $\quad |z-z_1|=|\overline{z-z_2}|=|z-z_2|$

Hence 'z' lies on the perpendicular bisector of joins of z_1 & z_2.

ANSWERS FOR DO YOURSELF

1 : **(i)** $n=4$ **(ii)** 0

2 : **(i)** $-17+24i$ **(iii)** $\pm(1-4i)$

3 : **(i)** $|z|=4;\ \text{amp}(z)=\dfrac{2\pi}{3}$ **(ii)** $|z|=2;\ \text{amp}(z)=-\dfrac{5\pi}{6}$ **(iii)** $|z|=2;\ \text{amp}(z)=-\dfrac{\pi}{2}$

(iv) $|z|=\dfrac{1}{\sqrt{2}};\ \text{amp}(z)=\dfrac{3\pi}{4}$ **(v)** $|z|=2;\ \text{amp}(z)=\dfrac{\pi}{3}$

4 : **(i)** 13 units **(ii)** locus is a circle on complex plane with center at (2,3) and radius 1 unit. **(iii)** C

5 : **(i)** $2\sqrt{2}\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right);\ 2\sqrt{2}e^{i\left(\frac{3\pi}{4}\right)}$ **(ii)** $2\left(\cos\dfrac{4\pi}{3}+i\sin\dfrac{4\pi}{3}\right);\ 2e^{i\left(\frac{4\pi}{3}\right)}$

(iii) $\sqrt{2}\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right);\sqrt{2}e^{i\left(\frac{3\pi}{4}\right)}$ **(iv)** $2\sin\left(\dfrac{\theta}{2}\right)\left(\cos\left(\dfrac{\pi}{2}-\dfrac{\theta}{2}\right)+i\sin\left(\dfrac{\pi}{2}-\dfrac{\theta}{2}\right)\right);\ 2\sin\left(\dfrac{\theta}{2}\right)e^{i\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}$

6 : **(i)** C **(ii)** D

7 : **(i)** $1+(2+2\sqrt{2})i$ **(ii)** 0 **(iii)** $z_2=z_3+i(z_1-z_3)$ **(iv)** B

(v) Locus is all the points on the major arc of circle as shown excluding points 1 & –1.

8 : **(i)** C **(ii)** A **(iii)** D

9 : **(i)** D **(ii)** B

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ELEMENTARY EXERCISE

1. Simplify and express the result in the form of a + b*i*

(a) $\left(\frac{1+2i}{2+i}\right)^2$ **CN0001**

(b) $-i(9+6i)(2-i)^{-1}$ **CN0002**

(c) $\left(\frac{4i^3-i}{2i+1}\right)^2$ **CN0003**

(d) $\frac{(2+i)^2}{2-i}-\frac{(2-i)^2}{2+i}$ **CN0004**

(e) A square $P_1P_2P_3P_4$ is drawn in the complex plane with P_1 at (1, 0) and P_3 at (3, 0). Let P_n denotes the point (x_n, y_n) n = 1, 2, 3, 4. Find the numerical value of the product of complex numbers $(x_1 + i\,y_1)(x_2 + i\,y_2)(x_3 + i\,y_3)(x_4 + i\,y_4)$. **CN0005**

2. Given that x, y $\in$ R, solve :

(a) $(x + 2y) + i\,(2x - 3y) = 5 - 4i$ **CN0006**

(b) $(x + iy) + (7 - 5i) = 9 + 4i$ **CN0007**

(c) $x^2 - y^2 - i\,(2x + y) = 2i$ **CN0008**

3. Find the square root of :

(a) $9 + 40\,i$ **CN0009**

(b) $-11 - 60\,i$ **CN0010**

(c) $50\,i$ **CN0011**

4. (a) If $f(x) = x^4 + 9x^3 + 35x^2 - x + 4$, find $f(-5 + 4i)$ **CN0012**

(b) If $g(x) = x^4 - x^3 + x^2 + 3x - 5$, find $g(2 + 3i)$ **CN0013**

5. Solve the following equations over C and express the result in the form a + ib, a, b $\in$ R.

(a) $ix^2 - 3x - 2i = 0$ **CN0014**

(b) $2\,(1 + i)\,x^2 - 4\,(2 - i)\,x - 5 - 3\,i = 0$ **CN0015**

6. Locate the points representing the complex number z on the Argand plane :

(a) $|z + 1 - 2i| = \sqrt{7}$ **CN0016**

(b) $|z - 1|^2 + |z + 1|^2 = 4$ **CN0017**

(c) $\left|\frac{z-3}{z+3}\right| = 3$ **CN0018**

(d) $|z - 3| = |z - 6|$ **CN0019**

7. If a & b are real numbers between 0 & 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ & $z_3 = 0$ form an equilateral triangle, then find the values of 'a' and 'b'. **CN0020**

8. Let $z_1 = 1 + i$ and $z_2 = -1 - i$. Find $z_3 \in C$ such that triangle $z_1z_2z_3$ is equilateral. **CN0021**

9. For what real values of x & y are the numbers $-3 + ix^2y$ & $x^2 + y + 4i$ conjugate complex ? **CN0022**

10. If $(x + iy)^{1/3} = a + bi$, then prove that $4\,(a^2 - b^2) = \frac{x}{a} + \frac{y}{b}$. **CN0023**

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11. (a) Prove the identity, $|1-z_1\bar{z}_2|^2-|z_1-z_2|^2=\left(1-|z_1|^2\right)\left(1-|z_2|^2\right)$ **CN0124**

(b) Prove the identity, $|1+z_1\bar{z}_2|^2+|z_1-z_2|^2=\left(1+|z_1|^2\right)\left(1+|z_2|^2\right)$ **CN0125**

(c) For any two complex numbers, prove that $|z_1+z_2|^2+|z_1-z_2|^2=2\left[|z_1|^2+|z_2|^2\right]$. Also give the geometrical interpretation of this identity. **C** **CN0026**

12. Find the Cartesian equation of the locus of 'z' in the complex plane satisfying, $|z-4|+|z+4|=16$. **CN0027**

Paragraph for question nos. 13 to 15

Consider a complex number $w=\dfrac{z-i}{2z+1}$, where $z=x+iy$ and $x, y \in R$.

13. If the complex number w is purely imaginary then locus of z is -

(A) a straight line

(B) a circle with centre $\left(-\dfrac{1}{4},\dfrac{1}{2}\right)$ and radius $\dfrac{\sqrt{5}}{4}$.

(C) a circle with centre $\left(\dfrac{1}{4},-\dfrac{1}{2}\right)$ and passing through origin.

(D) neither a circle nor a straight line.

CN0028

14. If the complex number w is purely real then locus of z is

(A) a straight line passing through origin

(B) a straight line with gradient 3 and y intercept (–1)

(C) a straight line with gradient 2 and y intercept 1.

(D) none

CN0028

15. If $|w| = 1$ then the locus of P(z) is

(A) a point circle (B) an imaginary circle (C) a real circle (D) not a circle.

CN0028

EXERCISE (O-1)

1. If $z+z^3=0$ then which of the following must be true on the complex plane?

(A) $Re(z) < 0$ (B) $Re(z) = 0$ (C) $Im(z) = 0$ (D) $z^4 = 1$

CN0029

2. Number of integral values of n for which the quantity $(n+i)^4$ where $i^2=-1$, is an integer is

(A) 1 (B) 2 (C) 3 (D) 4

CN0030

3. Let $i=\sqrt{-1}$. The product of the real part of the roots of $z^2-z=5-5i$ is

(A) – 25 (B) – 6 (C) – 5 (D) 25

CN0031

4. There is only one way to choose real numbers M and N such that when the polynomial $5x^4+4x^3+3x^2+Mx+N$ is divided by the polynomial x^2+1, the remainder is 0. If M and N assume these unique values, then M – N is

(A) – 6 (B) – 2 (C) 6 (D) 2

CN0032

5. The complex number z satisfying $z+|z|=1+7i$ then the value of $|z|^2$ equals

(A) 625 (B) 169 (C) 49 (D) 25

CN0033

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6. Number of values of z (real or complex) simultaneously satisfying the system of equations $1 + z + z^2 + z^3 + ... + z^{17} = 0$ and $1 + z + z^2 + z^3 + ... + z^{13} = 0$ is

(A) 1 (B) 2 (C) 3 (D) 4

CN0034

7. If $\frac{x-3}{3+i} + \frac{y-3}{3-i} = i$ where $x, y \in R$ then

(A) $x = 2$ & $y = -8$ (B) $x = -2$ & $y = 8$ (C) $x = -2$ & $y = -6$ (D) $x = 2$ & $y = 8$

CN0035

8. Number of complex numbers z satisfying $z^3 = \bar{z}$ is

(A) 1 (B) 2 (C) 4 (D) 5

CN0036

9. Let $z = 9 + bi$ where b is non zero real and $i^2 = -1$. If the imaginary part of z^2 and z^3 are equal, then b^2 equals

(A) 261 (B) 225 (C) 125 (D) 361

CN0037

10. The value of sum $\sum_{n=1}^{13}(i^n + i^{n+1})$, where $i = \sqrt{-1}$, equals

(A) i (B) i – 1 (C) –1 (D) 0

CN0038

11. The diagram shows several numbers in the complex plane. The circle is the unit circle centered at the origin. One of these numbers is the reciprocal of F, which is

(A) A (B) B

(C) C (D) D

CN0039

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12. If $z = x + iy$ & $\omega = \frac{1 - iz}{z - i}$ then $|\omega| = 1$ implies that, in the complex plane

(A) z lies on the imaginary axis (B) z lies on the real axis

(C) z lies on the unit circle (D) none

CN0040

13. On the complex plane locus of a point z satisfying the inequality $2 \le |z - 1| < 3$ denotes

(A) region between the concentric circles of radii 3 and 1 centered at (1, 0)

(B) region between the concentric circles of radii 3 and 2 centered at (1, 0) excluding the inner and outer boundaries.

(C) region between the concentric circles of radii 3 and 2 centered at (1, 0) including the inner and outer boundaries.

(D) region between the concentric circles of radii 3 and 2 centered at (1, 0) including the inner boundary and excluding the outer boundary.

CN0041

14. The complex number z satisfies $z + |z| = 2 + 8i$. The value of $|z|$ is

(A) 10 (B) 13 (C) 17 (D) 23

CN0042

15. Let $Z_1 = (8 + i)\sin\theta + (7 + 4i)\cos\theta$ and $Z_2 = (1 + 8i)\sin\theta + (4 + 7i)\cos\theta$ are two complex numbers. If $Z_1 \cdot Z_2 = a + ib$ where $a, b \in R$ then the largest value of $(a + b)$ $\forall$ $\theta \in R$, is

(A) 75 (B) 100 (C) 125 (D) 130

CN0043

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16. The locus of z, for $\arg z = -\pi/3$ is

(A) same as the locus of z for $\arg z = 2\pi/3$

(B) same as the locus of z for $\arg z = \pi/3$

(C) the part of the straight line $\sqrt{3}x + y = 0$ with $(y < 0, x > 0)$

(D) the part of the straight line $\sqrt{3}x + y = 0$ with $(y > 0, x < 0)$

CN0044

17. If z_1 & $\bar{z}_1$ represent adjacent vertices of a regular polygon of n sides with centre at the origin & if $\frac{\text{Im}\, z_1}{\text{Re}\, z_1} = \sqrt{2} - 1$ then the value of n is equal to :

(A) 8 (B) 12 (C) 16 (D) 24

CN0045

18. All real numbers x which satisfy the inequality $\left|1 + 4i - 2^{-x}\right| \le 5$ where $i = \sqrt{-1}$, $x \in R$ are

(A) $[-2, \infty)$ (B) $(-\infty, 2]$ (C) $[0, \infty)$ (D) $[-2, 0]$

CN0046

19. For $Z_1 = \sqrt[6]{\frac{1-i}{1+i\sqrt{3}}}$; $Z_2 = \sqrt[6]{\frac{1-i}{\sqrt{3}+i}}$; $Z_3 = \sqrt[6]{\frac{1+i}{\sqrt{3}-i}}$ which of the following holds good?

(A) $\sum |Z_1|^2 = \frac{3}{2}$ (B) $|Z_1|^4 + |Z_2|^4 = |Z_3|^{-8}$

(C) $\sum |Z_1|^3 + |Z_2|^3 = |Z_3|^{-6}$ (D) $|Z_1|^4 + |Z_2|^4 = |Z_3|^8$

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CN0047

20. Number of real or purely imaginary solution of the equation, $z^3 + iz - 1 = 0$ is :

(A) zero (B) one (C) two (D) three

CN0048

21. A point 'z' moves on the curve $|z - 4 - 3i| = 2$ in an argand plane. The maximum and minimum values of $|z|$ are

(A) 2, 1 (B) 6, 5 (C) 4, 3 (D) 7, 3

CN0049

22. If z is a complex number satisfying the equation $|z+i| + |z-i| = 8$, on the complex plane then maximum value of $|z|$ is

(A) 2 (B) 4 (C) 6 (D) 8

CN0050

23. Let z_r $(1 \le r \le 4)$ be complex numbers such that $|z_r| = \sqrt{r+1}$
and $|30 z_1 + 20 z_2 + 15 z_3 + 12 z_4| = k\,|z_1 z_2 z_3 + z_2 z_3 z_4 + z_3 z_4 z_1 + z_4 z_1 z_2|$.
Then the value of k equals

(A) $|z_1 z_2 z_3|$ (B) $|z_2 z_3 z_4|$ (C) $|z_3 z_4 z_1|$ (D) $|z_4 z_1 z_2|$

CN0051

24. Let Z be a complex number satisfying the equation $(Z^3 + 3)^2 = -16$ then $|Z|$ has the value equal to

(A) $5^{1/2}$ (B) $5^{1/3}$ (C) $5^{2/3}$ (D) 5

CN0052

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25. If z_1, z_2, z_3 are 3 distinct complex numbers such that $\frac{3}{|z_2 - z_3|} = \frac{4}{|z_3 - z_1|} = \frac{5}{|z_1 - z_2|}$, then the value of $\frac{9}{z_2 - z_3} + \frac{16}{z_3 - z_1} + \frac{25}{z_1 - z_2}$ equals

(A) 0 (B) 3 (C) 4 (D) 5

CN0053

26. The area of the triangle whose vertices are the roots $z^3 + iz^2 + 2i = 0$ is

(A) 2 (B) $\frac{3}{2}\sqrt{7}$ (C) $\frac{3}{4}\sqrt{7}$ (D) $\sqrt{7}$

CN0054

27. Consider two complex numbers α and β as

$\alpha = \left(\frac{a+bi}{a-bi}\right)^2 + \left(\frac{a-bi}{a+bi}\right)^2$, where a, b $\in$ R and $\beta = \frac{z-1}{z+1}$, $z \neq \pm 1$ where $|z| = 1$, then

(A) Both α and β are purely real
(B) Both α and β are purely imaginary
(C) α is purely real and β is purely imaginary
(D) β is purely real and α is purely imaginary

CN0055

28. Let Z is complex satisfying the equation $z^2 - (3+i)z + m + 2i = 0$, where $m \in R$. Suppose the equation has a real root. The additive inverse of non real root, is

(A) $1 - i$ (B) $1 + i$ (C) $-1 - i$ (D) -2

CN0056

29. The minimum value of $|z - 1 + 2i| + |4i - 3 - z|$ is

(A) $\sqrt{5}$ (B) 5 (C) $2\sqrt{13}$ (D) $\sqrt{15}$

CN0057

30. If $i = \sqrt{-1}$, then $4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}$ is equal to

(A) $1 - i\sqrt{3}$ (B) $-1 + i\sqrt{3}$ (C) $i\sqrt{3}$ (D) $-i\sqrt{3}$

CN0058

31. Let C_1 and C_2 are concentric circles of radius 1 and 8/3 respectively having centre at (3, 0) on the argand plane. If the complex number z satisfies the inequality, $\log_{1/3}\left(\frac{|z-3|^2 + 2}{11|z-3| - 2}\right) > 1$ then :

(A) z lies outside C_1 but inside C_2
(B) z lies inside of both C_1 and C_2
(C) z lies outside both of C_1 and C_2
(D) none of these

CN0059

32. Identify the incorrect statement.

(A) no non zero complex number z satisfies the equation, $\bar{z} = -4z$
(B) $\bar{z} = z$ implies that z is purely real
(C) $\bar{z} = -z$ implies that z is purely imaginary
(D) if z_1, z_2 are the roots of the quadratic equation $az^2 + bz + c = 0$ such that $\text{Im}(z_1 z_2) \neq 0$ then a, b, c must be real numbers .

CN0060

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33. The equation of the radical axis of the two circles represented by the equations, $|z-2| = 3$ and $|z-2-3i| = 4$ on the complex plane is :

(A) $3y + 1 = 0$ (B) $3y - 1 = 0$ (C) $2y - 1 = 0$ (D) none

CN0061

34. $z_1 = \frac{a}{1-i}$; $z_2 = \frac{b}{2+i}$; $z_3 = a - bi$ for $a, b \in R$

if $z_1 - z_2 = 1$ then the centroid of the triangle formed by the points z_1, z_2, z_3 in the argand's plane is given by

(A) $\frac{1}{9}(1+7i)$ (B) $\frac{1}{3}(1+7i)$ (C) $\frac{1}{3}(1-3i)$ (D) $\frac{1}{9}(1-3i)$

CN0062

35. Consider the equation $10z^2 - 3iz - k = 0$, where z is a complex variable and $i^2 = -1$. Which of the following statements is True?

(A) For all real positive numbers k, both roots are pure imaginary.
(B) For negative real numbers k, both roots are pure imaginary.
(C) For all pure imaginary numbers k, both roots are real and irrational.
(D) For all complex numbers k, neither root is real.

CN0063

36. Number of complex numbers z such that $|z| = 1$ and $\left|\frac{z}{\bar{z}} + \frac{\bar{z}}{z}\right| = 1$ is

(A) 4 (B) 6 (C) 8 (D) more than 8

CN0064

37. If z is a complex number satisfying the equation $|z-(1+i)|^2 = 2$ and $\omega = \frac{2}{z}$, then the locus traced by 'ω' in the complex plane is

(A) $x - y - 1 = 0$ (B) $x + y - 1 = 0$ (C) $x - y + 1 = 0$ (D) $x + y + 1 = 0$

CN0065

38. If P and Q are respectively by the complex numbers z_1 and z_2 such that $\left|\frac{1}{z_1} + \frac{1}{z_2}\right| = \left|\frac{1}{z_1} - \frac{1}{z_2}\right|$, then the circumcentre of ΔOPQ (where O is the origin) is

(A) $\frac{z_1 - z_2}{2}$ (B) $\frac{z_1 + z_2}{2}$ (C) $\frac{z_1 + z_2}{3}$ (D) $z_1 + z_2$

CN0066

39. If z_1 & z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then Arg z_1 – Arg z_2 is equal to

(A) $-\pi$ (B) $-\pi/2$ (C) 0 (D) $\pi/2$

CN0067

40. A particle starts from a point $z_0 = 1 + i$, where $i = \sqrt{-1}$. It moves horizontally away from origin by 2 units and then vertically away from origin by 3 units to reach a point z_1. From z_1 particle moves $\sqrt{5}$ units in the direction of $2\hat{i} + \hat{j}$ and then it moves through an angle of $\text{cosec}^{-1}\sqrt{2}$ in anticlockwise direction of a circle with centre at origin to reach a point z_2. The arg z_2 is given by

(A) $\sec^{-1}2$ (B) $\cot^{-1}0$ (C) $\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)$ (D) $\cos^{-1}\left(\frac{-1}{2}\right)$

CN0068

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41. Consider $az^2 + bz + c = 0$, where a, b, c $\in$ R and $4ac > b^2$.

(i) If z_1 and z_2 are the roots of the equation given above, then which one of the following complex numbers is purely real?

(A) $z_1\bar{z}_2$ (B) $\bar{z}_1 z_2$ (C) $z_1 - z_2$ (D) $(z_1 - z_2)i$

(ii) In the argand's plane, if A is the point representing z_1, B is the point representing z_2 & $z = \frac{\overrightarrow{OA}}{\overrightarrow{OB}}$ then

(A) z is purely real (B) z is purely imaginary
(C) $|z| = 1$ (D) ΔAOB is a scalene triangle.

CN0069

42. If the complex number z satisfies the condition $|z| \geq 3$, then the least value of $\left|z + \frac{1}{z}\right|$ is equal to :

(A) 5/3 (B) 8/3 (C) 11/3 (D) none of these

CN0070

43. Given $z_p = \cos\left(\frac{\pi}{2^P}\right) + i\sin\left(\frac{\pi}{2^P}\right)$, then $\lim_{n\to\infty}(z_1 z_2 z_3 \ldots z_n) =$

(A) 1 (B) -1 (C) i (D) $-i$

CN0071

44. The maximum & minimum values of $|z+1|$ when $|z+3| \leq 3$ are :

(A) (5, 0) (B) (6, 0) (C) (7, 1) (D) (5, 1)

CN0072

45. If $|z| = 1$ and $|\omega - 1| = 1$ where $z, \omega \in C$, then the largest set of values of $|2z - 1|^2 + |2\omega - 1|^2$ equals

(A) [1, 9] (B) [2, 6] (C) [2, 12] (D) [2, 18]

CN0073

46. If $\text{Arg}(z + a) = \frac{\pi}{6}$ and $\text{Arg}(z - a) = \frac{2\pi}{3}$; $a \in R^+$, then

(A) z is independent of a (B) $|a| = |z + a|$ (C) $z = a\,\text{Cis}\,\frac{\pi}{6}$ (D) $z = a\,\text{Cis}\,\frac{\pi}{3}$

CN0074

47. If z_1, z_2, z_3 are the vertices of the ΔABC on the complex plane which are also the roots of the equation, $z^3 - 3\alpha z^2 + 3\beta z + x = 0$, then the condition for the ΔABC to be equilateral triangle is

(A) $\alpha^2 = \beta$ (B) $\alpha = \beta^2$ (C) $\alpha^2 = 3\beta$ (D) $\alpha = 3\beta^2$

CN0075

48. The locus represented by the equation, $|z-1| + |z+1| = 2$ is :

(A) an ellipse with focii (1, 0) ; (–1, 0)
(B) one of the family of circles passing through the points of intersection of the circles $|z-1| = 1$ & $|z+1| = 1$
(C) the radical axis of the circles $|z-1| = 1$ and $|z+1| = 1$
(D) the portion of the real axis between the points (1, 0) ; (–1, 0) including both.

CN0076

Paragraph for question nos. 49 to 51

Let A, B, C be three sets of complex numbers as defined below.

$A = \{z : |z+1| \leq 2 + \text{Re}(z)\}$, $B = \{z : |z-1| \geq 1\}$ and $C = \left\{z : \left|\frac{z-1}{z+1}\right| \geq 1\right\}$

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49. The number of point(s) having integral coordinates in the region $A \cap B \cap C$ is
(A) 4 (B) 5 (C) 6 (D) 10

CN0077

50. The area of region bounded by $A \cap B \cap C$ is
(A) $2\sqrt{3}$ (B) $\sqrt{3}$ (C) $4\sqrt{3}$ (D) 2

CN0077

51. The real part of the complex number in the region $A \cap B \cap C$ and having maximum amplitude is
(A) -1 (B) $\frac{-3}{2}$ (C) $\frac{1}{2}$ (D) -2

CN0077

[MATCH THE COLUMN]

52. Match the equation in z, in **Column-I** with the corresponding values of arg(z) in **Column-II**.

Column-I (equations in z)	**Column-II** (principal value of arg (z))
(A) $z^2 - z + 1 = 0$	(P) $-2\pi/3$
(B) $z^2 + z + 1 = 0$	(Q) $-\pi/3$
(C) $2z^2 + 1 + i\sqrt{3} = 0$	(R) $\pi/3$
(D) $2z^2 + 1 - i\sqrt{3} = 0$	(S) $2\pi/3$

CN0078

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EXERCISE (O-2)

1. Let z_1 & z_2 be non zero complex numbers satisfying the equation, $z_1^2 - 2z_1z_2 + 2z_2^2 = 0$. The geometrical nature of the triangle whose vertices are the origin and the points representing z_1 & z_2 is :
(A) an isosceles right angled triangle (B) a right angled triangle which is not isosceles
(C) an equilateral triangle (D) an isosceles triangle which is not right angled.

CN0079

2. Let P denotes a complex number z on the Argand's plane, and Q denotes a complex number $\sqrt{2|z|^2}\ \text{CiS}\left(\frac{\pi}{4}+\theta\right)$ where $\theta = \text{amp } z$. If 'O' is the origin, then the Δ OPQ is :
(A) isosceles but not right angled (B) right angled but not isosceles
(C) right isosceles (D) equilateral.

CN0080

3. If $z = \frac{\pi}{4}(1+i)^4\left(\frac{1-\sqrt{\pi}\,i}{\sqrt{\pi}+i} + \frac{\sqrt{\pi}-i}{1+\sqrt{\pi}\,i}\right)$, then $\left(\frac{|z|}{\text{amp } z}\right)$ equals
(A) 1 (B) π (C) 3π (D) 4

CN0081

4. z is a complex number such that $z + \frac{1}{z} = 2\cos 3°$, then the value of $z^{2000} + \frac{1}{z^{2000}} + 1$ is equal to
(A) 0 (B) -1 (C) $\sqrt{3}+1$ (D) $1-\sqrt{3}$

CN0082

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5. The complex number ω satisfying the equation $\omega^3 = 8i$ and lying in the second quadrant on the complex plane is

(A) $-\sqrt{3}+i$ (B) $-\frac{\sqrt{3}}{2}+\frac{1}{2}i$ (C) $-2\sqrt{3}+i$ (D) $-\sqrt{3}+2i$

CN0083

6. If $z^4 + 1 = \sqrt{3}\,i$

(A) z^3 is purely real

(B) z represents the vertices of a square of side $2^{1/4}$

(C) z^9 is purely imaginary

(D) z represents the vertices of a square of side $2^{3/4}$.

CN0084

7. Let z is a complex number satisfying the equation $Z^6 + Z^3 + 1 = 0$. If this equation has a root $re^{i\theta}$ with $90° < \theta < 180°$ then the value of 'θ' is

(A) 100° (B) 110° (C) 160° (D) 170°

CN0085

8. If A and B be two complex numbers satisfying $\frac{A}{B}+\frac{B}{A}=1$. Then the two points represented by A and B and the origin form the vertices of

(A) an equilateral triangle (B) an isosceles triangle which is not equilateral

(C) an isosceles triangle which is not right angled (D) a right angled triangle

CN0086

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9. If $1, \alpha_1, \alpha_2 \ldots\ldots, \alpha_{2008}$ are $(2009)^{th}$ roots of unity, then the value of $\sum_{r=1}^{2008} r(\alpha_r + \alpha_{2009-r})$ equals

(A) 2009 (B) 2008 (C) 0 (D) – 2009

CN0087

10. If $x = \frac{1+\sqrt{3}\,i}{2}$ then the value of the expression, $y = x^4 - x^2 + 6x - 4$, equals

(A) $-1 + 2\sqrt{3}\,i$ (B) $2 - 2\sqrt{3}\,i$ (C) $2 + 2\sqrt{3}\,i$ (D) none

CN0088

11. (a) If $w(\neq 1)$ is a cube root of unity and $(1+w)^7 = A + Bw$, then A & B are respectively the numbers

(A) 0, 1 (B) 1, 1 (C) 1, 0 (D) –1, 1

(b) If $(w \neq 1)$ is a cube root of unity then $\begin{vmatrix} 1 & 1+i+w^2 & w^2 \\ 1-i & -1 & w^2-1 \\ -i & -i+w-1 & -1 \end{vmatrix} =$

(A) 0 (B) 1 (C) i (D) w

CN0089

12. If $z_n = \cos\frac{\pi}{(2n+1)(2n+3)} + i\sin\frac{\pi}{(2n+1)(2n+3)}$, then $\lim_{n\to\infty}(z_1 . z_2 . z_3 . \ldots\ldots z_n) =$

(A) $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ (B) $\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$ (C) $\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$ (D) $\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$

CN0090

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13. The straight line $(1+2i)z+(2i-1)\bar{z}=10i$ on the complex plane, has intercept on the imaginary axis equal to

(A) 5 (B) 5/2 (C) – 5/2 (D) – 5

CN0091

14. If $\cos\theta + i\sin\theta$ is a root of the equation $x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$ then the value of $\sum_{r=1}^{n} a_r \cos r\theta$ equals (where all coefficient are real)

(A) 0 (B) 1 (C) –1 (D) none

CN0092

15. Intercept made by the circle $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + r = 0$ on the real axis on complex plane, is

(A) $\sqrt{(\alpha+\bar{\alpha})-r}$ (B) $\sqrt{(\alpha+\bar{\alpha})^2-2r}$ (C) $\sqrt{(\alpha+\bar{\alpha})^2+r}$ (D) $\sqrt{(\alpha+\bar{\alpha})^2-4r}$

CN0093

16. If Z_r ; $r = 1, 2, 3,..., 50$ are the roots of the equation $\sum_{r=0}^{50} (Z)^r = 0$, then the value of $\sum_{r=1}^{50} \frac{1}{Z_r - 1}$ is

(A) – 85 (B) – 25 (C) 25 (D) 75

CN0094

17. All roots of the equation, $(1+z)^6 + z^6 = 0$:

(A) lie on a unit circle with centre at the origin

(B) lie on a unit circle with centre at (–1, 0)

(C) lie on the vertices of a regular polygon with centre at the origin

(D) are collinear

CN0095

18. It is given that complex numbers z_1 and z_2 satisfy $|z_1| = 2$ and $|z_2| = 3$. If the included angle of their corresponding vectors is 60° then $\left|\frac{z_1 + z_2}{z_1 - z_2}\right|$ can be expressed as $\frac{\sqrt{N}}{7}$ where N is natural number then N equals

(A) 126 (B) 119 (C) 133 (D) 19

CN0096

One ore more than one is/are correct :

19. If z is a complex number which simultaneously satisfies the equations $3|z - 12| = 5|z - 8i|$ and $|z - 4| = |z - 8|$ then the Im(z) can be

(A) 15 (B) 16 (C) 17 (D) 8

CN0097

20. Let z_1, z_2, z_3 be non-zero complex numbers satisfying the equation $z^4 = iz$.
Which of the following statement(s) is/are correct?

(A) The complex number having least positive argument is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

(B) $\sum_{k=1}^{3} \text{Amp}(z_k) = \frac{\pi}{2}$

(C) Centroid of the triangle formed by z_1, z_2 and z_3 is $\left(\frac{1}{\sqrt{3}}, \frac{-1}{3}\right)$

(D) Area of triangle formed by z_1, z_2 and z_3 is $\frac{3\sqrt{3}}{2}$

CN0098

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21. If $z \in C$, which of the following relation(s) represents a circle on an Argand diagram?

(A) $|z-1|+|z+1|=3$
(B) $(z-3+i)(\bar{z}-3-i)=5$
(C) $3|z-2+i|=7$
(D) $|z-3|=2$

CN0099

22. The locii of a point P(z) in the complex plane satisfying the $\left|z+\frac{1}{z}\right|=2$ are two circles C_1 and C_2. These circles

(A) have centres on real axis.
(B) cut each other orthogonally.
(C) are congruent
(D) have exactly two common tangents.

CN0100

23. Let A and B be two distinct points denoting the complex numbers α and β respectively. A complex number z lies between A and B where $z \neq \alpha$, $z \neq \beta$. Which of the following relation(s) hold good?

(A) $|\alpha - z| + |z - \beta| = |\alpha - \beta|$
(B) $\exists$ a positive real number 't' such that $z = (1-t)\alpha + t\beta$
(C) $\begin{vmatrix} z-\alpha & \bar{z}-\bar{\alpha} \\ \beta-\alpha & \bar{\beta}-\bar{\alpha} \end{vmatrix} = 0$
(D) $\begin{vmatrix} z & \bar{z} & 1 \\ \alpha & \bar{\alpha} & 1 \\ \beta & \bar{\beta} & 1 \end{vmatrix} = 0$

CN0101

24. Let z_1, z_2, z_3 are the coordinates of the vertices of the triangle $A_1A_2A_3$. Which of the following statements are equivalent.

(A) $A_1A_2A_3$ is an equilateral triangle.
(B) $(z_1+\omega z_2+\omega^2 z_3)(z_1+\omega^2 z_2+\omega z_3)=0$, where ω is the cube root of unity.
(C) $\frac{z_2-z_1}{z_3-z_2} = \frac{z_3-z_2}{z_1-z_3}$
(D) $\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_2 & z_3 & z_1 \end{vmatrix} = 0$

CN0102

25. Let tangents at $A(z_1)$ and $B(z_2)$ are drawn to the circle $|z| = 2$. Then which of the following is/are **CORRECT** ?

(A) The equation of tangent at A is given by $\frac{z}{z_1}+\frac{\bar{z}}{\bar{z}_1}=2$.
(B) If tangents at $A(z_1)$ and $B(z_2)$ intersect at $P(z_p)$, then $z_p = \frac{2z_1z_2}{z_1+z_2}$.
(C) Slope of tangent at $A(z_1)$ is $\frac{1}{i}\left(\frac{z_1+\bar{z}_1}{z_1-\bar{z}_1}\right)$
(D) If points $A(z_1)$ and $B(z_2)$ on the circle $|z|=2$ are such that $z_1+z_2=0$, then tangents intersect at $\frac{\pi}{2}$.

CN0103

26. Equation of a straight line on the complex plane passing through a point P denoting the complex number α and perpendicular to the vector $\overrightarrow{OP}$ where 'O' is the origin can be written as

(A) $\text{Im}\left(\frac{z-\alpha}{\alpha}\right)=0$
(B) $\text{Re}\left(\frac{z-\alpha}{\alpha}\right)=0$
(C) $\text{Re}(\bar{\alpha} z)=0$
(D) $\bar{\alpha} z + \alpha\bar{z} - 2|\alpha|^2 = 0$

CN0104

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27. If $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_{n-1}$ are the imaginary n^{th} roots of unity then the product $\prod_{r=1}^{n-1}(i-\alpha_r)$ (where $i=\sqrt{-1}$) can take the value equal to

(A) 0 (B) 1 (C) i (D) (1 + i)

CN0105

28. If the expression $(1+ir)^3$ is of the form of $s(1+i)$ for some real 's' where 'r' is also real and $i=\sqrt{-1}$, then the value of 'r' can be

(A) $\cot\frac{\pi}{8}$ (B) $\sec\pi$ (C) $\tan\frac{\pi}{12}$ (D) $\tan\frac{5\pi}{12}$

CN0106

29. Let point z moves on $|z-1|=1$ such that minimum & maximum value of $|z-2\sqrt{6}i|$ are m & M respectively, then-

(A) $m+M=10$ (B) $m^2+M^2=52$

(C) $m+M=8$ (D) $m^2+M^2=16$

CN0107

30. Locus of all points z in argand plane which satisfy $|z^2+1|=|z^2-1|$ is -

(A) same as $Re(z)=0$ (B) same as $Re(z^2)=0$

(C) a pair of straight lines (D) a circle with unit radius

CN0108

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EXERCISE (S-1)

1. Find the modulus, argument and the principal argument of the complex numbers.

(i) $6(\cos 310° - i\sin 310°)$ (ii) $-2(\cos 30° + i\sin 30°)$ (iii) $\frac{2+i}{4i+(1+i)^2}$

CN0109

2. (a) Let z is complex number satisfying the equation, $z^2-(3+i)z+m+2i=0$, where $m \in R$. Suppose the equation has a real root, then find the value of *m*. **CN0110**

(b) a, b, c are real numbers in the polynomial, $P(z)=2z^4+az^3+bz^2+cz+3$
If two roots of the equation $P(z)=0$ are 2 and i, then find the value of 'a'.

CN0111

3. Find the real values of x & y for which $z_1=9y^2-4-10ix$ and $z_2=8y^2-20i$ are conjugate complex of each other.

CN0112

4. (a) Solve the following equation $z^2-(3-2i)z=(5i-5)$ expressing your answer in the form of $(a+ib)$. **CN0113**

(b) If $(1-i)$ is a root of the equation $z^3-2(2-i)z^2+(4-5i)z-1+3i=0$, then find the other two roots.

CN0114

5. (a) If $iz^3+z^2-z+i=0$, then find $|z|$. **CN0115**

(b) Let z_1 and z_2 be two complex numbers such that $\left|\frac{z_1-2z_2}{2-z_1\bar{z}_2}\right|=1$ and $|z_2|\neq 1$, find $|z_1|$. **CN0116**

(c) Find the minimum value of the expression $E=|z|^2+|z-3|^2+|z-6i|^2$ (where $z=x+iy, x,y \in R$)

CN0117

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6. Show that the product,

$$\left[1+\left(\frac{1+i}{2}\right)\right]\left[1+\left(\frac{1+i}{2}\right)^2\right]\left[1+\left(\frac{1+i}{2}\right)^{2^2}\right].....\left[1+\left(\frac{1+i}{2}\right)^{2^n}\right]$$ is equal to $\left(1-\frac{1}{2^{2^n}}\right)(1+i)$ where $n \geq 2$.

CN0118

7. Let z be a complex number such that $z \in c\backslash R$ and $\frac{1+z+z^2}{1-z+z^2} \in R$, then prove that $|z|=1$.

CN0119

8. Let $z=(0, 1) \in C$. Express $\sum_{k=0}^{n} z^k$ in terms of the positive integer n.

CN0120

9. Among the complex numbers z satisfying the condition $\left|z+3-\sqrt{3}i\right| = \sqrt{3}$, find the number having the least positive argument.

CN0121

10. If A, B and C are the angles of a triangle $D = \begin{vmatrix} e^{-2iA} & e^{iC} & e^{iB} \\ e^{iC} & e^{-2iB} & e^{iA} \\ e^{iB} & e^{iA} & e^{-2iC} \end{vmatrix}$, where $i = \sqrt{-1}$, then find the value of D.

CN0122

11. Dividing f(z) by $z-i$, we get the remainder i and dividing it by $z+i$, we get the remainder $1+i$. Find the remainder upon the division of f(z) by z^2+1.

CN0123

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12. (a) Find all non–zero complex numbers z satisfying $\bar{z} = i z^2$. **CN0124**

(b) If the complex numbers $z_1, z_2, ...z_n$ lie on the unit circle $|z| = 1$ then show that

$$|z_1 + z_2 + ...+z_n| = |z_1^{-1} + z_2^{-1} + ...+z_n^{-1}|.$$

CN0125

13. Let $|z| = 2$ and $w = \frac{z+1}{z-1}$ where $z, w \in C$ (where C is the set of complex numbers). If M and n respectively be the greatest and least modulus of w, then find the value of $(2010m + M)$.

CN0126

14. If ω is the imaginary cube root of unity, then find the number of pairs of integers (a, b) such the $|a\omega + b| = 1$.

CN0127

EXERCISE (S-2)

1. Find the sum of the series $1(2-\omega)(2-\omega^2) + 2(3-\omega)(3-\omega^2) ..... (n-1)(n-\omega)(n-\omega^2)$ where ω is one of the imaginary cube root of unity.

CN0128

2. Resolve z^5+1 into linear & quadratic factors with real coefficients. Deduce that : $4.\sin\frac{\pi}{10}.\cos\frac{\pi}{5} = 1$.

CN0129

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3. Prove that, with regard to the quadratic equation $z^2 + (p + ip')z + q + iq' = 0$ where p, p', q, q' are all real.

(i) if the equation has one real root then $q'^2 - pp'q' + qp'^2 = 0$.

(ii) if the equation has two equal roots then $p^2 - p'^2 = 4q$ & $pp' = 2q'$.

State whether these equal roots are real or complex.

CN0130

4. (a) Let $z = x + iy$ be a complex number, where x and y are real numbers. Let A and B be the sets defined by **CN0131**

$A = \{z \mid |z| \le 2\}$ and $B = \{z \mid (1 - i)z + (1 + i)\bar{z} \ge 4\}$. Find the area of the region $A \cap B$.

(b) If z is a non-real complex number, then find the minimum value of $\dfrac{\text{Im}\, z^5}{\text{Im}^5\, z}$. **CN0132**

5. Interpret the following locii in $z \in C$.

(a) $1 < |z - 2i| < 3$ **CN0133**

(b) $\text{Re}\left(\dfrac{z + 2i}{iz + 2}\right) \le 4 \quad (z \ne 2i)$ **CN0134**

(c) $\text{Arg}(z + i) - \text{Arg}(z - i) = \pi/2$ **CN0135**

(d) $\text{Arg}(z - a) = \pi/3$ where $a = 3 + 4i$. **CN0136**

6. If the equation $(z + 1)^7 + z^7 = 0$ has roots $z_1, z_2, \ldots z_7$, find the value of

(a) $\sum_{r=1}^{7} \text{Re}(z_r)$ and **CN0137**

(b) $\sum_{r=1}^{7} \text{Im}(z_r)$ **CN0138**

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7. If the expression $z^5 - 32$ can be factorised into linear and quadratic factors over real coefficients as $(z^5 - 32) = (z - 2)(z^2 - pz + 4)(z^2 - qz + 4)$ then find the value of $(p^2 + 2p)$.

CN0139

8. Let $z_i (i = 1, 2, 3, 4)$ represent the vertices of a square all of which lie on the sides of the triangle with vertices (0, 0), (2, 1) and (3, 0). If z_1 and z_2 are purely real, then area of triangle formed by z_3, z_4 and origin is $\dfrac{m}{n}$ (where m and n are in their lowest form). Find the value of $(m + n)$.

CN0140

9. Let $f(x) = ax^3 + bx^2 + cx + d$ be a cubic polynomial with real coefficients satisfying $f(i) = 0$ and $f(1 + i) = 5$. Find the value of $a^2 + b^2 + c^2 + d^2$. (where $i = \sqrt{-1}$)

CN0141

10. A particle starts to travel from a point P on the curve $C_1 : |z - 3 - 4i| = 5$, where $|z|$ is maximum. From P, the particle moves through an angle $\tan^{-1}\dfrac{3}{4}$ in anticlockwise direction on $|z - 3 - 4i| = 5$ and reaches at point Q. From Q, it comes down parallel to imaginary axis by 2 units and reaches at point R. Find the complex number corresponding to point R in the Argand plane.

CN0142

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11. Evaluate : $\sum_{p=1}^{32}(3p+2)\left(\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11}-i\cos\frac{2q\pi}{11}\right)\right)^{p}$.

CN0143

12. If the biquadratic $x^4+ax^3+bx^2+cx+d=0$ $(a, b, c, d \in R)$ has 4 non real roots, two with sum $3+4i$ and the other two with product $13+i$. Find the value of 'b'.

CN0144

EXERCISE (JM)

1. If $z \neq 1$ and $\frac{z^2}{z-1}$ is real, then the point represented by the complex number z lies : **[AIEEE-2012]**

(1) on the imaginary axis.
(2) either on the real axis or on a circle passing through the origin.
(3) on a circle with centre at the origin.
(4) either on the real axis or on a circle not passing through the origin.

CN0145

2. If z is a complex number of unit modulus and argument θ, then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ equals **[JEE (Main)-2013]**

(1) $-\theta$ (2) $\frac{\pi}{2}-\theta$ (3) θ (4) $\pi-\theta$ ∴

CN0146

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3. If z is a complex number such that $|z| \geq 2$, then the minimum value of $\left|z+\frac{1}{2}\right|$: **[JEE(Main)-2014]**

(1) is equal to $\frac{5}{2}$
(2) lies in the interval (1, 2)
(3) is strictly greater than $\frac{5}{2}$
(4) is strictly greater than $\frac{3}{2}$ but less than $\frac{5}{2}$

CN0147

4. A complex number z is said to be unimodular if $|z| = 1$. Suppose z_1 and z_2 are complex numbers such that $\frac{z_1-2z_2}{2-z_1\bar{z}_2}$ is unimodular and z_2 is not unimodular. Then the point z_1 lies on a : **[JEE(Main)-2015]**

(1) circle of radius 2
(2) circle of radius $\sqrt{2}$
(3) straight line parallel to x-axis
(4) straight line parallel to y-axis

CN0148

5. A value of θ for which $\frac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary, is : **[JEE(Main)-2016]**

(1) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{6}$ (4) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$

CN0149

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6. Let ω be a complex number such that 2ω + 1 = z where $z = \sqrt{-3}$. If $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$, then k is equal to :- **[JEE(Main)-2017]**

(1) 1 (2) –z (3) z (4) –1

CN0150

7. If α, β ∈ C are the distinct roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{101} + \beta^{107}$ is equal to- **[JEE(Main)-2018]**

(1) 0 (2) 1 (3) 2 (4) –1

CN0151

8. Let z be a complex number such that |z| + z = 3 + i (where $i = \sqrt{-1}$). Then |z| is equal to :- **[JEE(Main)-2019]**

(1) $\frac{5}{4}$ (2) $\frac{\sqrt{41}}{4}$ (3) $\frac{\sqrt{34}}{3}$ (4) $\frac{5}{3}$

CN0152

9. If $\frac{z-\alpha}{z+\alpha}(\alpha \in R)$ is a purely imaginary number and |z| = 2, then a value of α is : **[JEE(Main)-2019]**

(1) 1 (2) 2 (3) $\sqrt{2}$ (4) $\frac{1}{2}$

CN0153

10. Let Z_1 and Z_2 be two complex numbers satisfying $|Z_1| = 9$ and $|Z_2 - 3 - 4i| = 4$. Then the minimum value of $|Z_1 - Z_2|$ is : **[JEE(Main)-2019]**

(1) 0 (2) 1 (3) $\sqrt{2}$ (4) 2

CN0154

11. If $z = \frac{\sqrt{3}}{2} + \frac{i}{2}\left(i = \sqrt{-1}\right)$, then $\left(1 + iz + z^5 + iz^8\right)^9$ is equal to **[JEE(Main)-2019]**

(1) –1 (2) 1 (3) 0 (4) $(-1+2i)^9$

CN0155

12. Let z∈C be such that |z| < 1. If $\omega = \frac{5+3z}{5(1-z)}$, then :- **[JEE(Main)-2019]**

(1) 5Im(ω) < 1 (2) 4Im(ω) > 5 (3) 5Re(ω) > 1 (4) 5Re(ω) > 4

CN0156

13. If z and w are two complex numbers such that |zw| = 1 and arg(z) – arg(w) = $\frac{\pi}{2}$, then : **[JEE(Main)-2019]**

(1) $\bar{z}w = i$ (2) $\bar{z}w = -i$ (3) $z\bar{w} = \frac{1-i}{\sqrt{2}}$ (4) $z\bar{w} = \frac{-1+i}{\sqrt{2}}$

CN0157

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14. The equation $|z-i| = |z-1|$, $i = \sqrt{-1}$, represents: **[JEE(Main)-2019]**

(1) the line through the origin with slope –1.
(2) a circle of radius 1.
(3) a circle of radius $\frac{1}{2}$.
(4) the line through the origin with slope 1.

CN0158

15. If $\text{Re}\left(\frac{z-1}{2z+i}\right) = 1$, where $z = x + iy$, then the point (x,y) lies on a : **[JEE(Main)-2020]**

(1) circle whose centre is at $\left(-\frac{1}{2}, -\frac{3}{2}\right)$
(2) circle whose diameter is $\frac{\sqrt{5}}{2}$
(3) straight line whose slope is $\frac{3}{2}$
(4) straight line whose slope is $-\frac{2}{3}$

CN0159

16. Let $\alpha = \frac{-1+i\sqrt{3}}{2}$. If $a = (1+\alpha)\sum_{k=0}^{100}\alpha^{2k}$ and $b = \sum_{k=0}^{100}\alpha^{3k}$, then a and b are the roots of the quadratic equation: **[JEE(Main)-2020]**

(1) $x^2 - 102x + 101 = 0$
(2) $x^2 + 101x + 100 = 0$
(3) $x^2 - 101x + 100 = 0$
(4) $x^2 + 102x + 101 = 0$

CN0160

17. If the equation, $x^2 + bx + 45 = 0$ $(b \in R)$ has conjugate complex roots and they satisfy $|z+1| = 2\sqrt{10}$, then **[JEE(Main)-2020]**

(1) $b^2 - b = 42$
(2) $b^2 + b = 12$
(3) $b^2 + b = 72$
(4) $b^2 - b = 30$

CN0161

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18. If z be a complex number satisfying $|\text{Re}(z)| + |\text{Im}(z)| = 4$, then $|z|$ cannot be **[JEE(Main)-2020]**

(1) $\sqrt{\frac{17}{2}}$
(2) $\sqrt{10}$
(3) $\sqrt{8}$
(4) $\sqrt{7}$

CN0162

19. Let z be complex number such that $\left|\frac{z-i}{z+2i}\right| = 1$ and $|z| = \frac{5}{2}$. Then the value of $|z+3i|$ is : **[JEE(Main)-2020]**

(1) $\sqrt{10}$
(2) $2\sqrt{3}$
(3) $\frac{7}{2}$
(4) $\frac{15}{4}$

CN0163

EXERCISE (JA)

1. Let z_1 and z_2 be two distinct complex numbers and let $z = (1-t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If Arg(w) denotes the principal argument of a nonzero complex number w, then **[JEE 2010, 3M]**

(A) $|z - z_1| + |z - z_2| = |z_1 - z_2|$
(B) $\text{Arg}(z - z_1) = \text{Arg}(z - z_2)$
(C) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$
(D) $\text{Arg}(z - z_1) = \text{Arg}(z_2 - z_1)$

CN0164

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2. Let ω be the complex number $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}=0$ is equal to **[JEE 2010, 3M]**

CN0165

3. Match the statements in **Column-I** with those in **Column-II.**
[**Note :** Here z takes values in the complex plane and Im z and Re z denote, respectively, the imaginary part and the real part of z.]

	Column I		**Column II**
(A)	The set of points z satisfying $\lvert z-i\lvert z\rvert\rvert=\lvert z+i\lvert z\rvert\rvert$ is contained in or equal to	(p)	an ellipse with eccentricity $\frac{4}{5}$
(B)	The set of points z satisfying $\lvert z+4\rvert+\lvert z-4\rvert=10$ is contained in or equal to	(q)	the set of points z satisfying Im z = 0
(C)	If $\lvert w\rvert=2$, then the set of points $z=w-\frac{1}{w}$ is contained in or equal to	(t)	the set of points z satisfying $\lvert \text{Im } z\rvert\le 1$
(D)	If $\lvert w\rvert=1$, then the set of points $z=w+\frac{1}{w}$ is contained in or equal to	(s)	the set of points z satisfying $\lvert \text{Re } z\rvert\le 2$
		(t)	the set of points z satisfying $\lvert z\rvert\le 3$

[JEE 10, 3+3+8]
CN0166

4. **Comprehension (3 questions together)**
Let a,b and c be three real numbers satisfying

$$\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad ...(E)$$

(i) If the point P(a,b,c), with reference to (E), lies on the plane 2x + y + z = 1, then the value of 7a + b + c is
(A) 0 (B) 12 (C) 7 (D) 6

(ii) Let ω be a solution of $x^3 - 1 = 0$ with Im(ω) > 0. If a = 2 with b and c satisfying (E), then the value of $\frac{3}{\omega^a}+\frac{1}{\omega^b}+\frac{3}{\omega^c}$ is equal to -
(A) –2 (B) 2 (C) 3 (D) –3

(iii) Let b = 6, with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^n$ is - **[JEE 2011, 3+3+3]**

(A) 6 (B) 7 (C) $\frac{6}{7}$ (D) ∞ **CN0167**

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5. If z is any complex number satisfying $|z - 3 - 2i| \le 2$, then the minimum value of $|2z - 6 + 5i|$ is **[JEE 2011, 4M]**

CN0168

6. Let $\omega = e^{i\pi/3}$, and a, b, c, x, y, z be non-zero complex numbers such that

$a + b + c = x$

$a + b\omega + c\omega^2 = y$

$a + b\omega^2 + c\omega = z.$

Then the value of $\dfrac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is **[JEE 2011, 4M]**

CN0169

7. Match the statements given in **Column I** with the values given in **Column II**

Column I	**Column II**
(A) If $\vec{a} = \hat{j} + \sqrt{3}\hat{k}$, $\vec{b} = -\hat{j} + \sqrt{3}\hat{k}$ and $\vec{c} = 2\sqrt{3}\hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is	(p) $\dfrac{\pi}{6}$
(B) If $\int_a^b (f(x) - 3x)dx = a^2 - b^2$, then the value of $f\left(\dfrac{\pi}{6}\right)$ is	(q) $\dfrac{2\pi}{3}$
(C) The value of $\dfrac{\pi^2}{\ell n3}\int_{7/6}^{5/6} \sec(\pi x)dx$ is	(r) $\dfrac{\pi}{3}$
(D) The maximum value of $\left\lvert \text{Arg}\left(\dfrac{1}{1-z}\right)\right\rvert$ for $\lvert z\rvert = 1, z \ne 1$ is given by	(s) π
	(t) $\dfrac{\pi}{2}$

[JEE 2011, 2+2+2+2M]

CN0170

8. Match the statements given in **Column I** with the intervals/union of intervals given in **Column II**

Column I	**Column II**
(A) The set $\left\{\text{Re}\left(\dfrac{2iz}{1-z^2}\right) : z \text{ is a complex number, } \lvert z\rvert=1, z \ne \pm 1\right\}$ is	(p) $(-\infty, -1) \cup (1, \infty)$
(B) The domain of the function $f(x) = \sin^{-1}\left(\dfrac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ is	(q) $(-\infty, 0) \cup (0, \infty)$
(C) If $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set	(r) $[2, \infty)$
$\left\{f(\theta) : 0 \le \theta < \dfrac{\pi}{2}\right\}$ is	(s) $(-\infty, -1] \cup [1, \infty)$
(D) If $f(x) = x^{3/2}(3x - 10), x \ge 0$, then $f(x)$ is increasing in(t)	$(-\infty, 0] \cup [2, \infty)$

[JEE 2011, 2+2+2+2M]

CN0171

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9. Let z be a complex number such that the imaginary part of z is nonzero and $a = z^2 + z + 1$ is real. Then a **cannot** take the value - **[JEE 2012, 3M, –1M]**

(A) –1 (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

CN0172

10. Let complex numbers α and $\frac{1}{\bar{\alpha}}$ lie on circles $(x-x_0)^2+(y-y_0)^2=r^2$ and $(x-x_0)^2+(y-y_0)^2=4r^2$ respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha| =$

[JEE(Advanced) 2013, 2M]

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{7}}$ (D) $\frac{1}{3}$

CN0173

11. Let ω be a complex cube root of unity with $\omega \neq 1$ and $P = [p_{ij}]$ be a $n \times n$ matrix with $p_{ij} = \omega^{i+j}$. Then $P^2 \neq 0$, when n = **[JEE(Advanced) 2013, 3, (–1)]**

(A) 57 (B) 55 (C) 58 (D) 56

CN0174

12. Let $w = \frac{\sqrt{3}+i}{2}$ and $P = \{w^n : n = 1, 2, 3, ...\}$. Further $H_1 = \left\{z \in C : \text{Re}\, z > \frac{1}{2}\right\}$ and $H_2 = \left\{z \in C : \text{Re}\, z < \frac{-1}{2}\right\}$, where C is the set of all complex numbers. If $z_1 \in P \cap H_1$, $z_2 \in P \cap H_2$ and O represents the origin, then $\angle z_1Oz_2 =$ **[JEE-Advanced 2013, 4, (–1)]**

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{6}$ (C) $\frac{2\pi}{3}$ (D) $\frac{5\pi}{6}$ **CN0175**

Paragraph for Question 13 and 14

Let $S = S_1 \cap S_2 \cap S_3$, where $S_1 = \{z \in C : |z| < 4\}$, $S_2 = \left\{z \in C : \text{Im}\left[\frac{z-1+\sqrt{3}i}{1-\sqrt{3}i}\right] > 0\right\}$ and $S_3 = \{z \in C : \text{Re}\, z > 0\}$.

13. $\min\limits_{z \in S} |1-3i-z| =$ **[JEE(Advanced) 2013, 3, (–1)]**

(A) $\frac{2-\sqrt{3}}{2}$ (B) $\frac{2+\sqrt{3}}{2}$ (C) $\frac{3-\sqrt{3}}{2}$ (D) $\frac{3+\sqrt{3}}{2}$

CN0176

14. Area of S = **[JEE(Advanced) 2013, 3, (–1)]**

(A) $\frac{10\pi}{3}$ (B) $\frac{20\pi}{3}$ (C) $\frac{16\pi}{3}$ (D) $\frac{32\pi}{3}$

CN0176

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15. The quadratic equation p(x) = 0 with real coefficients has purely imaginary roots. Then the equation p(p(x)) = 0 has **[JEE(Advanced) 2014, 3(–1)]**

(A) only purely imaginary roots
(B) all real roots
(C) two real and two purely imaginary roots
(D) neither real nor purely imaginary roots.

CN0177

16. Let $z_k = \cos\left(\frac{2k\pi}{10}\right) + i\sin\left(\frac{2k\pi}{10}\right)$; k = 1, 2, ... 9.

	List-I		**List-II**
P.	For each z_k there exists a z_j such that $z_k \cdot z_j = 1$	1.	True
Q.	There exists a $k \in \{1, 2, ..., 9\}$ such that $z_1 \cdot z = z_k$ has no solution z in the set of complex numbers.	2.	False
R.	$\frac{\lvert 1-z_1\rvert \lvert 1-z_2\rvert \ldots \lvert 1-z_9\rvert}{10}$ equals	3.	1
S.	$1 - \sum_{k=1}^{9} \cos\left(\frac{2k\pi}{10}\right)$ equals	4.	2

Codes :

	P	Q	R	S
(A)	1	2	4	3
(B)	2	1	3	4
(C)	1	2	3	4
(D)	2	1	4	3

[JEE(Advanced) 2014, 3(–1)]

CN0178

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17.

	Column-I		**Column-II**
(A)	In $\mathbb{R}^2$, if the magnitude of the projection vector of the vector $\alpha\hat{i} + \beta\hat{j}$ on $\sqrt{3}\hat{i} + \hat{j}$ is $\sqrt{3}$ and if $\alpha = 2 + \sqrt{3}\beta$, then possible value(s) of $\lvert\alpha\rvert$ is (are)	(P)	1
(B)	Let a and b be real numbers such that the function $f(x) = \begin{cases} -3ax^2 - 2, & x < 1 \\ bx + a^2, & x \geq 1 \end{cases}$ is differentiable for all $x \in \mathbb{R}$. Then possible value(s) of a is (are)	(Q)	2
(C)	Let $\omega \neq 1$ be a complex cube root of unity. If $(3 - 3\omega + 2\omega^2)^{4n+3} + (2 + 3\omega - 3\omega^2)^{4n+3} + (-3 + 2\omega + 3\omega^2)^{4n+3} = 0$ then possible value(s) of n is (are)	(R)	3

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(D) Let the harmonic mean of two positive real number a and b be 4, If q is a positive real number such that a, 5, q, b is an arithmetic progression, then the value(s) of $|q - a|$ is (are)

(S) 4

(T) 5

[JEE 2015, 8(Each 2M, –1M)]

CN0179

18. For any integer k, let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i\sin\left(\frac{k\pi}{7}\right)$, where $i = \sqrt{-1}$. The value of the expression $\dfrac{\sum_{k=1}^{12}|\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1} - \alpha_{4k-2}|}$ is

[JEE 2015, 4M, –0M]

CN0180

19. Let $z = \frac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and r, s $\in$ {1, 2, 3}. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is

[JEE(Advanced)-2016, 3(0)]

CN0181

20. Let a,b $\in \mathbb{R}$ and $a^2 + b^2 \neq 0$. Suppose $S = \left\{z \in \mathbb{C} : z = \frac{1}{a+ibt}, t \in \mathbb{R}, t \neq 0\right\}$, where $i = \sqrt{-1}$. If z = x + iy and z $\in$ S, then (x,y) lies on

(A) the circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for a > 0, b $\neq$ 0

(B) the circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for a < 0, b $\neq$ 0

(C) the x-axis for a $\neq$ 0, b = 0

(D) the y-axis for a = 0, b $\neq$ 0

[JEE(Advanced)-2016, 4(–2)]

CN0182

21. Let a, b, x and y be real numbers such that a – b = 1 and y $\neq$ 0. If the complex number $z = x + iy$ satisfies $\text{Im}\left(\frac{az+b}{z+1}\right) = y$, then which of the following is(are) possible value(s) of x ?

[JEE(Advanced)-2017, 4(–2)]

(A) $-1-\sqrt{1-y^2}$ (B) $1+\sqrt{1+y^2}$ (C) $1-\sqrt{1+y^2}$ (D) $-1+\sqrt{1-y^2}$

CN0183

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22. For a non-zero complex number z, let arg(z) denotes the principal argument with $-\pi < \arg(z) \le \pi$. Then, which of the following statement(s) is (are) **FALSE** ? **[JEE(Advanced)-2018, 4(–2)]**

(A) $\arg(-1 - i) = \frac{\pi}{4}$, where $i = \sqrt{-1}$

(B) The function $f : \mathbb{R} \to (-\pi, \pi]$, defined by $f(t) = \arg(-1 + it)$ for all $t \in \mathbb{R}$, is continuous at all points of $\mathbb{R}$, where $i = \sqrt{-1}$

(C) For any two non-zero complex numbers z_1 and z_2, $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$ is an integer multiple of 2π

(D) For any three given distinct complex numbers z_1, z_2 and z_3, the locus of the point z satisfying the condition $\arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right) = \pi$, lies on a straight line

CN0184

23. Let s, t, r be the non-zero complex numbers and L be the set of solutions $z = x + iy$ $\left(x, y \in \mathbb{R}, i = \sqrt{-1}\right)$ of the equation $sz + t\bar{z} + r = 0$, where $\bar{z} = x - iy$. Then, which of the following statement(s) is (are) TRUE ?

[JEE(Advanced)-2018, 4(–2)]

(A) If L has exactly one element, then $|s| \ne |t|$

(B) If $|s| = |t|$, then L has infinitely many elements

(C) The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2

(D) If L has more than one element, then L has infinitely many elements

CN0185

24. Let S be the set of all complex numbers z satisfying $|z - 2 + i| \ge \sqrt{5}$. If the complex number z_0 is such that $\frac{1}{|z_0 - 1|}$ is the maximum of the set $\left\{\frac{1}{|z-1|} : z \in S\right\}$, then the principal argument of $\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2i}$ is

[JEE(Advanced)-2019, 3(–1)]

(1) $\frac{\pi}{4}$ (2) $-\frac{\pi}{2}$ (3) $\frac{3\pi}{4}$ (4) $\frac{\pi}{2}$

CN0186

25. Let $\omega \ne 1$ be a cube root of unity. Then the minimum of the set

$$\{|a + b\omega + c\omega^2|^2 : a, b, c \text{ distinct non-zero integers}\}$$

equals ____ **[JEE(Advanced)-2019, 3(0)]**

CN0187

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ANSWER KEY

ELEMENTARY EXERCISE

1. **(a)** $\frac{7}{25}+\frac{24}{25}i$; **(b)** $\frac{21}{5}-\frac{12}{5}i$; **(c)** $3+4i$; **(d)** $\frac{22}{5}i$; **(e)** 15

2. **(a)** $x=1, y=2$; **(b)** (2,9); **(c)** $(-2,2)$ or $\left(-\frac{2}{3},-\frac{2}{3}\right)$ **3.** **(a)** $\pm(5+4i)$; **(b)** $\pm(5-6i)$; **(c)** $\pm 5(1+i)$

4. **(a)** -160 ; **(b)** $-(77+108i)$ **5.** **(a)** $-i, -2i$ **(b)** $\frac{3-5i}{2}$ or $-\frac{1+i}{2}$

6. **(a)** on a circle of radius $\sqrt{7}$ with centre $(-1, 2)$; **(b)** on a unit circle with centre at origin
(c) on a circle with centre $(-15/4, 0)$ & radius 9/4; **(d)** a straight line.

7. $a = b = 2-\sqrt{3}$ **8.** $z_3=\sqrt{3}(1-i)$ and $z'_3=\sqrt{3}(-1+i)$

9. $x = 1, y = -4$ or $x = -1, y = -4$ **12.** $\frac{x^2}{64}+\frac{y^2}{48}=1$ **13.** B **14.** C **15.** C

EXERCISE (O-1)

1. B	**2.** C	**3.** B	**4.** C	**5.** A	**6.** A	**7.** B	**8.** D
9. B	**10.** B	**11.** C	**12.** B	**13.** D	**14.** C	**15.** C	**16.** C
17. A	**18.** A	**19.** B	**20.** A	**21.** D	**22.** B	**23.** D	**24.** B
25. A	**26.** A	**27.** C	**28.** C	**29.** C	**30.** C	**31.** A	**32.** D
33. B	**34.** A	**35.** B	**36.** C	**37.** A	**38.** B	**39.** C	**40.** B
41. (i) D; (ii) C		**42.** B	**43.** B	**44.** A	**45.** D	**46.** D	**47.** A

48. D **49.** B **50.** A **51.** B **52.** (A) Q,R; (B) P,S; (C) Q,S; (D) P,R

EXERCISE (O-2)

1. A	**2.** C	**3.** D	**4.** A	**5.** A	**6.** D	**7.** C	**8.** A
9. D	**10.** A	**11.** **(a)** B; **(b)** A		**12.** B	**13.** A	**14.** C	**15.** D
16. B	**17.** D	**18.** C	**19.** C,D	**20.** A,B	**21.** B,C,D	**22.** B,C,D	
23. A,B,C,D	**24.** A,B,C,D		**25.** A,B,C	**26.** B,D	**27.** A,B,C,D	**28.** B,C,D	
29. A,B	**30.** B,C						

EXERCISE (S-1)

1. **(i)** Modulus = 6, Arg $=2k\pi+\frac{5\pi}{18}$ $(k \in I)$, Principal Arg $=\frac{5\pi}{18}$

(ii) Modulus = 2, Arg $=2k\pi+\frac{7\pi}{6}$ $(k \in I)$, Principal Arg $=-\frac{5\pi}{6}$

(iii) Modulus $=\frac{\sqrt{5}}{6}$, Arg $=2k\pi-\tan^{-1}2$ $(k \in I)$, Principal Arg $=-\tan^{-1}2$

2. **(a)** 2, **(b)** $-11/2$ **3.** $[(-2, 2); (-2, -2)]$ **4.** **(a)** $z=(2+i)$ or $(1-3i)$; **(b)** $z=1$ or $2-i$

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5. **(a)** 1, **(b)** 2 **(c)** 30 **8.** $\begin{cases}(1,\ 0) & \text{for } n = 4k \\ (1,\ 1) & \text{for } n = 4k+1 \\ (0,\ 1) & \text{for } n = 4k+2 \\ (0,\ 0) & \text{for } n = 4k+3\end{cases}$ **9.** $-\frac{3}{2} + \frac{3\sqrt{3}}{2}i$ **10.** -4

11. $\frac{iz}{2} + \frac{1}{2} + i$ **12.** **(a)** $\frac{\sqrt{3}}{2} - \frac{i}{2}, -\frac{\sqrt{3}}{2} - \frac{i}{2}, i$ **13.** 673 **14.** 6

EXERCISE (S-2)

1. $\left[\frac{n(n+1)}{2}\right]^2 - n$ **2.** $(Z + 1)(Z^2 - 2Z\cos 36° + 1)(Z^2 - 2Z\cos 108° + 1)$ **4.** **(a)** $\pi - 2$, **(b)** -4

5. **(a)** The region between the concentric circles with centre at $(0, 2)$ & radii 1 & 3 units

(b) region outside or on the circle with centre $\frac{1}{2} + 2i$ and radius $\frac{1}{2}$.

(c) semicircle (in the 1st & 4th quadrant) $x^2 + y^2 = 1$

(d) a ray emanating from the point $(3 + 4i)$ directed away from the origin & having equation $\sqrt{3}x - y + 4 - 3\sqrt{3} = 0$

6. (a) $-\frac{7}{2}$, (b) zero **7.** 4 **8.** 41 **9.** 26 **10.** $(3+7i)$ **11.** $48(1-i)$ **12.** 51

EXERCISE (JM)

1. 2 **2.** 3 **3.** 2 **4.** 1 **5.** 1 **6.** 2 **7.** 2 **8.** 4
9. 2 **10.** 1 **11.** 1 **12.** 3 **13.** 2 **14.** 4 **15.** 2 **16.** 1
17. 4 **18.** 4 **19.** 3

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EXERCISE (JA)

1. A,C,D **2.** 1 **3.** (A) → (q,r), (B) → (p), (C) → (p,s,t), (D) → (q,r,s,t) **4.** **(i)** D, **(ii)** A, **(iii)** B

5. 5 **6.** Bonus **7.** (A) → (q); (B) → (p) or (p,q,r,s,t); (C) → (s); (D) → (t)

8. (A) → (s); (B) → (t); (C) → (r); (D) → (r) **9.** D **10.** C **11.** B,C,D

12. C,D **13.** C **14.** B **15.** D **16.** C

17. (A) → (P,Q); (B) → (P,Q); (C) → (P,Q,S,T); (D) → (Q,T) **18.** 4 **19.** 1 **20.** A,C,D

21. A,D **22.** A,B,D **23.** A,C,D **24.** 2 **25.** 3.00

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LIMIT

1. INTRODUCTION :

The concept of limit of a function is one of the fundamental ideas that distinguishes calculus from algebra and trigonometry. We use limits to describe the way a function f varies. Some functions vary continuously; small changes in x produce only small changes in f(x). Other functions can have values that jump or vary erratically. We also use limits to define tangent lines to graphs of functions. This geometric application leads at once to the important concept of derivative of a function.

2. DEFINITION :

Let f(x) be defined on an open interval about 'a' except possibly at 'a' itself. If f(x) gets arbitrarily close to L (a finite number) for all x sufficiently close to 'a' we say that f(x) approaches the limit L as x approaches 'a' and we write $\lim\limits_{x\to a} f(x) = L$ and say "the limit of f(x), as x approaches a, equals L".

This implies if we can make the value of f(x) arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

3. LEFT HAND LIMIT AND RIGHT HAND LIMIT OF A FUNCTION :

The value to which f(x) approaches, as x tends to 'a' from the left hand side ($x \to a^-$) is called left hand limit of f(x) at x = a. Symbolically, LHL = $\lim\limits_{x\to a^-} f(x) = \lim\limits_{h\to 0} f(a-h)$.

The value to which f(x) approaches, as x tends to 'a' from the right hand side ($x \to a^+$) is called right hand limit of f(x) at x = a. Symbolically, RHL = $\lim\limits_{x\to a^+} f(x) = \lim\limits_{h\to 0} f(a+h)$.

Limit of a function f(x) is said to exist as, $x \to$ a when $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x)$ = Finite quantity.

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Example :

Graph of y = f(x)

Fig. 1

$\lim\limits_{x\to -1^+} f(x) = \lim\limits_{h\to 0} f(-1+h) = f(-1^+) = -1$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{h\to 0} f(0-h) = f(0^-) = 0$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0} f(0+h) = f(0^+) = 0$

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{h\to 0} f(1-h) = f(1^-) = -1$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{h\to 0} f(1+h) = f(1^+) = 0$

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{h\to 0} f(2-h) = f(2^-) = 1$

$\lim\limits_{x\to 0} f(x) = 0$ and $\lim\limits_{x\to 1} f(x)$ does not exist.

Important note :

In $\lim\limits_{x\to a} f(x)$, $x \to a$ necessarily implies $x \neq a$. That is while evaluating limit at x = a, we are not concerned with the value of the function at x = a. In fact the function may or may not be defined at x = a.

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Also it is necessary to note that if f(x) is defined only on one side of **'x = a'**, one sided limits are good enough to establish the existence of limits, & if f(x) is defined on either side of **'a'** both sided limits are to be considered.

As in $\lim\limits_{x\to 1} \cos^{-1} x = 0$, though f(x) is not defined for x >1, even in it's immediate vicinity.

Illustration 1 : Consider the adjacent graph of y = f (x)

Find the following :

(a) $\lim\limits_{x\to 0^-} f(x)$ (b) $\lim\limits_{x\to 0^+} f(x)$ (c) $\lim\limits_{x\to 1^-} f(x)$

(d) $\lim\limits_{x\to 1^+} f(x)$ (e) $\lim\limits_{x\to 2^-} f(x)$ (f) $\lim\limits_{x\to 2^+} f(x)$

(g) $\lim\limits_{x\to 3^-} f(x)$ (h) $\lim\limits_{x\to 3^+} f(x)$ (i) $\lim\limits_{x\to 4^-} f(x)$

(j) $\lim\limits_{x\to 4^+} f(x)$ (k) $\lim\limits_{x\to \infty} f(x) = 2$ (l) $\lim\limits_{x\to 6^-} f(x) = -\infty$

Solution : (a) As $x \to 0^-$: limit does not exist (the function is not defined to the left of x = 0)

(b) As $x \to 0^+$: $f(x) \to -1 \Rightarrow \lim\limits_{x\to 0^+} f(x) = -1$. (c) As $x \to 1^-$: $f(x) \to 1 \Rightarrow \lim\limits_{x\to 1^-} f(x) = 1$.

(d) As $x \to 1^+$: $f(x) \to 2 \Rightarrow \lim\limits_{x\to 1^+} f(x) = 2$. (e) As $x \to 2^-$: $f(x) \to 3 \Rightarrow \lim\limits_{x\to 2^-} f(x) = 3$.

(f) As $x \to 2^+$: $f(x) \to 3 \Rightarrow \lim\limits_{x\to 2^-} f(x) = 3$. (g) As $x \to 3^-$: $f(x) \to 2 \Rightarrow \lim\limits_{x\to 3^-} f(x) = 2$.

(h) As $x \to 3^+$: $f(x) \to 3 \Rightarrow \lim\limits_{x\to 3^+} f(x) = 3$. (i) As $x \to 4^-$: $f(x) \to 4 \Rightarrow \lim\limits_{x\to 4^-} f(x) = 4$.

(j) As $x \to 4^+$: $f(x) \to 4 \Rightarrow \lim\limits_{x\to 4^+} f(x) = 4$. (k) As $x \to \infty$: $f(x) \to 2 \Rightarrow \lim\limits_{x\to \infty} f(x) = 2$.

(l) As $x \to 6^-$,$f(x) \to -\infty \Rightarrow \lim\limits_{x\to 6^-} f(x) = -\infty$ limit does not exist because it is not finite.

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Do yourself - 1 :

(i) Which of the following statements about the function y = f(x) graphed here are true, and which are false ?

(a) $\lim\limits_{x\to -1^+} f(x) = 1$ (b) $\lim\limits_{x\to 2} f(x)$ does not exist

(c) $\lim\limits_{x\to 2} f(x) = 2$ (d) $\lim\limits_{x\to 1^-} f(x) = 2$

(e) $\lim\limits_{x\to 1} f(x)$ does not exist (f) $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^-} f(x)$

(g) $\lim\limits_{x\to c} f(x)$ exists at every $c \in (-1, 1)$

(h) $\lim\limits_{x\to c} f(x)$ exists at every $c \in (1, 3)$

(i) $\lim\limits_{x\to 1^-} f(x) = 0$ (j) $\lim\limits_{x\to 3^+} f(x)$ does not exist.

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4. FUNDAMENTAL THEOREMS ON LIMITS :

Let $\lim\limits_{x\to a} f(x) = l$ & $\lim\limits_{x\to a} g(x) = m$. If l & m exist finitely then :

(a) Sum rule : $\lim\limits_{x\to a}\{f(x)+g(x)\} = l+m$ **(b)** Difference rule : $\lim\limits_{x\to a}\{f(x)-g(x)\} = l-m$

(c) Product rule : $\lim\limits_{x\to a} f(x).g(x) = l.m$ **(d)** Quotient rule : $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{l}{m}$, provided $m \neq 0$

(e) Constant multiple rule : $\lim\limits_{x\to a} kf(x) = k\lim\limits_{x\to a} f(x)$; where k is constant.

(f) Power rule : If m and n are integers then $\lim\limits_{x\to a}[f(x)]^{m/n} = l^{m/n}$ provided $l^{m/n}$ is a real number.

(g) $\lim\limits_{x\to a} f[g(x)] = f\left(\lim\limits_{x\to a} g(x)\right) = f(m)$; provided f(x) is continuous at x = m.

For example : $\lim\limits_{x\to a} \ell n(g(x)) = \ell n[\lim\limits_{x\to a} g(x)]$

$= \ell n\,(m)$; provided ℓnx is continuous at x = m, m = $\lim\limits_{x\to a} g(x)$.

5. INDETERMINATE FORMS :

$\dfrac{0}{0}, \dfrac{\infty}{\infty}, \infty-\infty, 0\times\infty, 1^{\infty}, 0^{0}, \infty^{0}$

Initially we will deal with first five forms only and the other two forms will come up after we have gone through differentiation.

Note : (i) Here 0,1 are not exact, infact both are approaching to their corresponding values.

(ii) We cannot plot ∞ on the paper. Infinity (∞) is a symbol & not a number It does not obey the laws of elementary algebra,

(a) $\infty+\infty \to \infty$ (b) $\infty\times\infty \to \infty$ (c) $\infty^{\infty} \to \infty$ (d) $0^{\infty} \to 0$

6. GENERAL METHODS TO BE USED TO EVALUATE LIMITS :

(a) Factorization :

Important factors :

(i) $x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + \ldots\ldots\ldots + a^{n-1})$, $n \in N$

(ii) $x^n + a^n = (x+a)(x^{n-1} - ax^{n-2} + \ldots\ldots\ldots + a^{n-1})$, n is an odd natural number.

Note : $\lim\limits_{x\to a}\dfrac{x^n - a^n}{x-a} = na^{n-1}$

Illustration 2 : Evaluate : $\lim\limits_{x\to 2}\left[\dfrac{1}{x-2} - \dfrac{2(2x-3)}{x^3-3x^2+2x}\right]$

Solution : We have

$$\lim_{x\to 2}\left[\frac{1}{x-2} - \frac{2(2x-3)}{x^3-3x^2+2x}\right] = \lim_{x\to 2}\left[\frac{1}{x-2} - \frac{2(2x-3)}{x(x-1)(x-2)}\right] = \lim_{x\to 2}\left[\frac{x(x-1)-2(2x-3)}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{x^2-5x+6}{x(x-1)(x-2)}\right] = \lim_{x\to 2}\left[\frac{(x-2)(x-3)}{x(x-1)(x-2)}\right] = \lim_{x\to 2}\left[\frac{x-3}{x(x-1)}\right] = -\frac{1}{2}$$

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Do yourself - 2 :

(i) Evaluate : $\lim\limits_{x\to1}\dfrac{x-1}{2x^2-7x+5}$

(b) Rationalization or double rationalization :

Illustration 3 : Evaluate : $\lim\limits_{x\to1}\dfrac{4-\sqrt{15x+1}}{2-\sqrt{3x+1}}$

Solution :

$$\lim_{x\to1}\frac{4-\sqrt{15x+1}}{2-\sqrt{3x+1}}=\lim_{x\to1}\frac{(4-\sqrt{15x+1})(2+\sqrt{3x+1})(4+\sqrt{15x+1})}{(2-\sqrt{3x+1})(4+\sqrt{15x+1})(2+\sqrt{3x+1})}$$

$$=\lim_{x\to1}\frac{(15-15x)}{(3-3x)}\times\frac{2+\sqrt{3x+1}}{4+\sqrt{15x+1}}=\frac{5}{2}$$

Illustration 4 : Evaluate : $\lim\limits_{x\to1}\left(\dfrac{\sqrt{x^2+8}-\sqrt{10-x^2}}{\sqrt{x^2+3}-\sqrt{5-x^2}}\right)$

Solution : This is of the form $\dfrac{3-3}{2-2}=\dfrac{0}{0}$ if we put x = 1

To eliminate the $\dfrac{0}{0}$ factor, multiply by the conjugate of numerator and the conjugate of the denominator

$$\therefore\ \text{Limit}=\lim_{x\to1}\left(\sqrt{x^2+8}-\sqrt{10-x^2}\right)\frac{(\sqrt{x^2+8}+\sqrt{10-x^2})}{(\sqrt{x^2+8}+\sqrt{10-x^2})}\times\frac{(\sqrt{x^2+3}+\sqrt{5-x^2})}{(\sqrt{x^2+3}+\sqrt{5-x^2})(\sqrt{x^2+3}-\sqrt{5-x^2})}$$

$$=\lim_{x\to1}\frac{\sqrt{x^2+3}+\sqrt{5-x^2}}{\sqrt{x^2+8}+\sqrt{10-x^2}}\times\frac{(x^2+8)-(10-x^2)}{(x^2+3)-(5-x^2)}=\lim_{x\to1}\left(\frac{\sqrt{x^2+3}+\sqrt{5-x^2}}{\sqrt{x^2+8}+\sqrt{10-x^2}}\right)\times1=\frac{2+2}{3+3}=\frac{2}{3}$$

Do yourself - 3 :

(i) Evaluate : $\lim\limits_{x\to0}\dfrac{\sqrt{p+x}-\sqrt{p-x}}{\sqrt{q+x}-\sqrt{q-x}}$

(ii) Evaluate : $\lim\limits_{x\to a}\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$, $a\neq0$

(iii) If $G(x)=-\sqrt{25-x^2}$, then find the $\lim\limits_{x\to1}\left(\dfrac{G(x)-G(1)}{x-1}\right)$

(c) Limit when $x\to\infty$:

(i) Divide by greatest power of x in numerator and denominator.

(ii) Put x = 1/y and apply $y\to0$

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Illustration 5 : Evaluate : $\lim\limits_{x\to\infty}\dfrac{x^2+x+1}{3x^2+2x-5}$

Solution : $\lim\limits_{x\to\infty}\dfrac{x^2+x+1}{3x^2+2x-5}$, $\left(\dfrac{\infty}{\infty}\text{ form}\right)$

Put $x=\dfrac{1}{y}$

$$\text{Limit}=\lim_{y\to 0}\frac{1+y+y^2}{3+2y-5y^2}=\frac{1}{3}$$

Illustration 6 : If $\lim\limits_{x\to\infty}\left(\dfrac{x^3+1}{x^2+1}-(ax+b)\right)=2$, then

(A) a = 1, b = 1 (B) a = 1, b = 2 (C) a = 1, b = –2 (D) none of these

Solution : $\lim\limits_{x\to\infty}\left(\dfrac{x^3+1}{x^2+1}-(ax+b)\right)=2\Rightarrow\lim\limits_{x\to\infty}\dfrac{x^3(1-a)-bx^2-ax+(1-b)}{x^2+1}=2$

$$\Rightarrow\lim_{x\to\infty}\frac{x(1-a)-b-\frac{a}{x}+\frac{(1-b)}{x^2}}{1+\frac{1}{x^2}}=2\Rightarrow 1-a=0,-b=2\Rightarrow a=1,b=-2$$ **Ans. (C)**

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Do yourself - 4 :

(i) Evaluate : $\lim\limits_{n\to\infty}\dfrac{\lfloor n+2+\lfloor n+1}{\lfloor n+2-\lfloor n+1}$ **(ii)** Evaluate : $\lim\limits_{n\to\infty}(n-\sqrt{n^2+n})$

(d) Squeeze play theorem (Sandwich theorem) :

Statement : If $f(x)\le g(x)\le h(x)$; $\forall$ x in the neighbourhood at x = a and

$\lim\limits_{x\to a}f(x)=\ell=\lim\limits_{x\to a}h(x)$ then $\lim\limits_{x\to a}g(x)=\ell$,

Ex.1 $\lim\limits_{x\to 0}x^2\sin\dfrac{1}{x}=0$,

$\because\ \sin\left(\dfrac{1}{x}\right)$ lies between –1 & 1

$\Rightarrow -x^2\le x^2\sin\dfrac{1}{x}\le x^2$

$\Rightarrow\lim\limits_{x\to 0}x^2\sin\dfrac{1}{x}=0$ as $\lim\limits_{x\to 0}(-x^2)=\lim\limits_{x\to 0}x^2=0$

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Ex.2 $\lim\limits_{x\to 0} x\sin\frac{1}{x} = 0$

$\because \quad \sin\left(\frac{1}{x}\right)$ lies between –1 & 1

$\Rightarrow -x \le x\sin\frac{1}{x} \le x$

$\Rightarrow \underset{x\to 0}{\text{Lim}}\, x\sin\frac{1}{x} = 0$ as $\underset{x\to 0}{\text{Lim}}(-x) = \underset{x\to 0}{\text{Lim}}\, x = 0$

Illustration 7 : Evaluate : $\lim\limits_{n\to\infty}\frac{[x]+[2x]+[3x]+.....[nx]}{n^2}$ (Where [.] denotes the greatest integer function.)

Solution : We know that $x - 1 < [x] \le x$

$\Rightarrow x + 2x + ...nx - n < \sum\limits_{r=1}^{n}[rx] \le x + 2x + ...+ nx$

$\Rightarrow \frac{xn}{2}(n+1) - n < \sum\limits_{r=1}^{n}[rx] \le \frac{x.n(n+1)}{2} \Rightarrow \frac{x}{2}\left(1+\frac{1}{n}\right) - \frac{1}{n} < \frac{1}{n^2}\sum\limits_{r=1}^{n}[rx] \le \frac{x}{2}\left(1+\frac{1}{n}\right)$

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Now, $\lim\limits_{n\to\infty}\frac{x}{2}\left(1+\frac{1}{n}\right) = \frac{x}{2}$ and $\lim\limits_{n\to\infty}\frac{x}{2}\left(1+\frac{1}{n}\right) - \frac{1}{n} = \frac{x}{2}$

Thus, $\lim\limits_{n\to\infty}\frac{[x]+[2x]+.....+[nx]}{n^2} = \frac{x}{2}$

---

7. LIMIT OF TRIGONOMETRIC FUNCTIONS :

$\underset{x\to 0}{\text{Lim}}\frac{\sin x}{x} = 1 = \underset{x\to 0}{\text{Lim}}\frac{\tan x}{x} = \underset{x\to 0}{\text{Lim}}\frac{\tan^{-1} x}{x} = \underset{x\to 0}{\text{Lim}}\frac{\sin^{-1} x}{x}$ [where x is measured in radians]

(a) If $\underset{x\to a}{\text{Lim}}\, f(x) = 0$, then $\underset{x\to a}{\text{Lim}}\frac{\sin f(x)}{f(x)} = 1$, e.g. $\underset{x\to 1}{\text{Lim}}\frac{\sin(\ell nx)}{(\ell nx)} = 1$

Illustration 8 : Evaluate : $\lim\limits_{x\to 0}\frac{x^3\cot x}{1-\cos x}$

Solution : $\lim\limits_{x\to 0}\frac{x^3\cos x}{\sin x(1-\cos x)} = \lim\limits_{x\to 0}\frac{x^3\cos x(1+\cos x)}{\sin x.\sin^2 x} = \lim\limits_{x\to 0}\frac{x^3}{\sin^3 x}.\cos x(1 + \cos x) = 2$

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Illustration 9 : Evaluate : $\lim\limits_{x\to 0}\dfrac{(2+x)\sin(2+x)-2\sin 2}{x}$

Solution : $\lim\limits_{x\to 0}\dfrac{2(\sin(2+x)-\sin 2)+x\sin(2+x)}{x}=\lim\limits_{x\to 0}\left(\dfrac{2.2.\cos\left(2+\dfrac{x}{2}\right)\sin\dfrac{x}{2}}{x}+\sin(2+x)\right)$

$$=\lim_{x\to 0}\frac{2\cos\left(2+\frac{x}{2}\right)\sin\frac{x}{2}}{\frac{x}{2}}+\lim_{x\to 0}\sin(2+x)=2\cos 2+\sin 2$$

Illustration 10 : Evaluate : $\lim\limits_{n\to\infty}\dfrac{\sin\dfrac{a}{n}}{\tan\dfrac{b}{n+1}}$

Solution : As $n\to\infty$, $\dfrac{1}{n}\to 0$ and $\dfrac{a}{n}$ also tends to zero

$\sin\dfrac{a}{n}$ should be written as $\dfrac{\sin\frac{a}{n}}{\frac{a}{n}}$ so that it looks like $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}$

The given limit $=\lim\limits_{n\to\infty}\left(\dfrac{\sin\frac{a}{n}}{\frac{a}{n}}\right)\left(\dfrac{\frac{b}{n+1}}{\tan\frac{b}{n+1}}\right).\dfrac{a(n+1)}{n.b}$

$$=\lim_{n\to\infty}\left(\frac{\sin\frac{a}{n}}{\frac{a}{n}}\right)\left(\frac{\frac{b}{n+1}}{\tan\frac{b}{n+1}}\right).\frac{a}{b}\left(1+\frac{1}{n}\right)=1\times 1\times\frac{a}{b}\times 1=\frac{a}{b}$$

Do yourself - 5 :

(i) Evaluate :

(a) $\lim\limits_{x\to 0}\dfrac{\sin\alpha x}{\tan\beta x}$ **(b)** $\lim\limits_{x\to y}\dfrac{\sin^2 x-\sin^2 y}{x^2-y^2}$ **(c)** $\lim\limits_{h\to 0}\dfrac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$

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8. LIMIT OF EXPONENTIAL FUNCTIONS :

(a) $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\ell n\, a (a>0)$ In particular $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1$.

In general if $\lim\limits_{x\to a} f(x)=0$,then $\lim\limits_{x\to a}\dfrac{a^{f(x)}-1}{f(x)}=\ell na,\ a>0$

Illustration 11 : Evaluate : $\lim\limits_{x\to 0}\dfrac{e^{\tan x}-e^x}{\tan x-x}$

Solution : $\lim\limits_{x\to 0}\dfrac{e^{\tan x}-e^x}{\tan x-x}=\lim\limits_{x\to 0}\dfrac{e^x\times e^{(\tan x-x)}-e^x}{\tan x-x}$

$=\lim\limits_{x\to 0}\dfrac{e^x(e^{\tan x-x}-1)}{\tan x-x}=\lim\limits_{x\to 0\, y\to 0}\dfrac{e^x(e^y-1)}{y}$ where $y=\tan x-x$ and $\lim\limits_{y\to 0}\dfrac{e^y-1}{y}=1$

$=e^0\times 1$ [as $x\to 0$, $\tan x-x\to 0$]

$=1\times 1=1$

Do yourself - 6 :

(i) Evaluate : $\lim\limits_{x\to a}\dfrac{e^x-e^a}{x-a}$ **(ii)** Evaluate : $\lim\limits_{x\to 0}\dfrac{2^x-1}{(1+x)^{1/2}-1}$

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(b) (i) $\lim\limits_{x\to 0}(1+x)^{1/x}=e=\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x$ (Note : The base and exponent depends on the same variable.) **In general**, if $\lim\limits_{x\to a} f(x)=0$, then $\lim\limits_{x\to a}(1+f(x))^{1/f(x)}=e$

(ii) $\lim\limits_{x\to 0}\dfrac{\ell n(1+x)}{x}=1$

(iii) If $\lim\limits_{x\to a} f(x)=1$ and $\lim\limits_{x\to a}\phi(x)=\infty$, then ; $\lim\limits_{x\to a}[f(x)]^{\phi(x)}=e^k$

where $k=\lim\limits_{x\to a}\phi(x)\,[f(x)-1]$

Illustration 12 : Evaluate $\lim\limits_{x\to 1}(\log_3 3x)^{\log_x 3}$

Solution : $\lim\limits_{x\to 1}(\log_3 3x)^{\log_x 3}=\lim\limits_{x\to 1}(\log_3 3+\log_3 x)^{\log_x 3}$

$=\lim\limits_{x\to 1}(1+\log_3 x)^{1/\log_3 x}=e$ $\because\ \log_b a=\dfrac{1}{\log_a b}$

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Illustration 13 : Evaluate : $\lim\limits_{x\to 0}\dfrac{x\ell n(1+2\tan x)}{1-\cos x}$

Solution : $\lim\limits_{x\to 0}\dfrac{x\ell n(1+2\tan x)}{1-\cos x}=\lim\limits_{x\to 0}\dfrac{x\ell n(1+2\tan x)}{\dfrac{1-\cos x}{x^2}.x^2}.\dfrac{2\tan x}{2\tan x}=4$

Illustration 14 : Evaluate : $\lim\limits_{x\to\infty}\left(\dfrac{2x^2-1}{2x^2+3}\right)^{4x^2+2}$

Solution : Since it is in the form of 1^∞

$$\lim_{x\to\infty}\left(\frac{2x^2-1}{2x^2+3}\right)^{4x^2+2}=e^{\lim\limits_{x\to\infty}\left(\frac{2x^2-1-2x^2-3}{2x^2+3}\right)(4x^2+2)}=e^{-8}$$

Do yourself - 7 :

(i) Evaluate : $\lim\limits_{x\to\infty}x\{\ell n(x+a)-\ell nx\}$

(ii) Evaluate : $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{pn+q}$

(iii) Evaluate : $\lim\limits_{x\to 0}\left(1+\tan^2\sqrt{x}\right)^{\frac{1}{2x}}$

(iv) Evaluate : $\lim\limits_{x\to\infty}\left(\dfrac{x+6}{x+1}\right)^{x+4}$

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(c) If $\lim\limits_{x\to a} f(x)=A>0$ & $\lim\limits_{x\to a}\phi(x)=B$, then $\lim\limits_{x\to a}[f(x)]^{\phi(x)}=e^{B\ln A}=A^B$

Illustration 15 : Evaluate : $\lim\limits_{x\to\infty}\left(\dfrac{7x^2+1}{5x^2-1}\right)^{\frac{x^5}{1-x^3}}$

Solution : Here $f(x)=\dfrac{7x^2+1}{5x^2-1}$, $\phi(x)=\dfrac{x^5}{1-x^3}=\dfrac{x^2.x^3}{1-x^3}=\dfrac{x^2}{\dfrac{1}{x^3}-1}$

$\therefore\quad \lim\limits_{x\to\infty}f(x)=\dfrac{7}{5}$ & $\lim\limits_{x\to\infty}\phi(x)\to-\infty$

$\Rightarrow\quad \lim\limits_{x\to\infty}(f(x))^{\phi(x)}=\left(\dfrac{7}{5}\right)^{-\infty}=0$

Do yourself - 8 :

(i) Evaluate : $\lim\limits_{x\to\infty}\left(\dfrac{1+5x^2}{1+3x^2}\right)^{-x^2}$

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9. LIMIT USING SERIES EXPANSION :

Expansion of function like binomial expansion, exponential & logarithmic expansion, expansion of sinx, cosx, tanx should be remembered by heart which are given below :

(a) $a^x = 1 + \frac{x\,\ell na}{1!} + \frac{x^2 \ell n^2 a}{2!} + \frac{x^3 \ell n^3 a}{3!} + ..., x \in \mathbb{R}, a > 0, a \neq 1$

(b) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ..., x \in \mathbb{R}$

(c) $\ell n\left(1+x\right) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$ for $-1 < x \leq 1$

(d) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ..., x \in \mathbb{R}$

(e) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ..., x \in \mathbb{R}$

(f) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + ..., -\frac{\pi}{2} < x < \frac{\pi}{2}$

(g) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ..., x \in (-1,1)$

(h) $\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + ..., x \in (-1,1)$

(i) $\sec^{-1} x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + ..., x \in (-\infty, -1) \cup (1, \infty)$

(j) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ..., n \in \mathbb{R}, x \in (-1, 1)$

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Illustration 16 : $\lim\limits_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$

Solution : $\lim\limits_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} \Rightarrow \lim\limits_{x\to 0} \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + ...\right) - 2x}{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} .....\right)}$

$$\Rightarrow \lim_{x\to 0} \frac{2.\frac{x^3}{6} + 2.\frac{x^5}{5!} + ...}{\frac{x^3}{6} + \frac{x^5}{5!} .....} \Rightarrow \lim_{x\to 0} \frac{x^3\left(\frac{1}{3} + \frac{1}{60}x^2 + ...\right)}{x^3\left(\frac{1}{6} + \frac{1}{120}x^2 + ...\right)} = \frac{1/3}{1/6} = 2$$

Do yourself - 9 :

(i) Evaluate : $\lim\limits_{x\to 0} \frac{x - \sin x}{\sin(x^3)}$

(ii) Evaluate : $\lim\limits_{x\to 0} \frac{x - \tan^{-1} x}{x^3}$

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Miscellaneous Illustrations :

Illustration 17 : Evaluate $\lim_{x\to 0} \sin\frac{\pi}{x}$.

Solution : Again the function $f(x) = \sin(\pi/x)$ is undefined at 0. Evaluating the function for some small values of x, we get $f(1) = \sin\pi = 0$, $f\left(\frac{1}{2}\right) = \sin 2\pi = 0$,

$f(0.1) = \sin 10\pi = 0$, $f(0.01) = \sin 100\pi = 0$.

On the basis of this information we might be tempted to guess that $\lim_{x\to 0} \sin\frac{\pi}{x} = 0$ but this time our guess is wrong. Note that although $f(1/n) = \sin n\pi = 0$ for any integer n, it is also true that $f(x) = 1$ for infinitely many values of x that approach 0. [In fact, $\sin(\pi/x) = 1$ when $\frac{\pi}{x} = \frac{\pi}{2} + 2n\pi$ and solving for x, we get $x = 2/(4n + 1)$]. The graph of f is given in following figure

The dashed line indicate that the values of $\sin(\pi/x)$ oscillate between 1 and –1 infinitely often as x approaches 0. Since the values of f(x) do not approach a fixed number as x approaches 0,

$\Rightarrow$ $\lim_{x\to 0} \sin\frac{\pi}{x}$ does not exist.

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ANSWERS FOR DO YOURSELF

1 :	**(i)**	(a) T	**(b)** F	**(c)** F	**(d)** T	**(e)** T	**(f)** T	**(g)** T	**(h)** T	**(i)** F	**(j)** T
2 :	**(i)**	$-\frac{1}{3}$									
3 :	**(i)**	$\frac{\sqrt{q}}{\sqrt{p}}$	**(ii)**	$\frac{2}{3\sqrt{3}}$	**(iii)**	$\frac{1}{\sqrt{24}}$					
4 :	**(i)**	1	**(ii)**	$-\frac{1}{2}$							
5 :	**(i)**	**(a)** $\frac{\alpha}{\beta}$	**(b)**	$\frac{\sin 2y}{2y}$		**(c)**	$2a\sin a + a^2\cos a$				
6 :	**(i)**	e^a	**(ii)**	$2\ln 2$							
7 :	**(i)**	a	**(ii)**	e^p	**(iii)**	$e^{\frac{1}{2}}$	**(iv)**	e^5			
8 :	**(i)**	0									
9.	**(i)**	$\frac{1}{6}$	**(ii)**	$\frac{1}{3}$							

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EXERCISE (O-1)

[SINGLE CORRECT CHOICE TYPE]

1. $\lim_{x\to 1}\left(\frac{1}{1-x}-\frac{3}{1-x^3}\right)$ is equal to

(A) –1 (B) 0 (C) 1 (D) D.N.E.

LT0001

2. $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{2x}$ is equal to

(A) 0 (B) 1 (C) $\frac{1}{2}$ (D) $\frac{1}{4}$

LT0002

3. $\lim_{x\to 2}\frac{\sqrt{1+\sqrt{2+x}}-\sqrt{3}}{x-2}$ is equal to

(A) $\frac{1}{\sqrt{3}}$ (B) $\sqrt{3}$ (C) $\frac{1}{4\sqrt{3}}$ (D) $\frac{1}{8\sqrt{3}}$

LT0003

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4. $\lim_{x\to 1}\frac{\sqrt[n]{x}-1}{\sqrt[m]{x}-1}$ (m and n integers) is equal to

(A) 0 (B) 1 (C) $\frac{m}{n}$ (D) $\frac{n}{m}$

LT0004

5. If $\lim_{x\to a}\frac{2x-\sqrt{x^2+3a^2}}{\sqrt{x+a}-\sqrt{2a}}=\sqrt{2}$ (where $a\in\mathbb{R}^+$), then a is equal to -

(A) $\frac{1}{3}$ (B) $\frac{1}{2\sqrt{2}}$ (C) $\frac{1}{3\sqrt{2}}$ (D) $\frac{1}{9}$

LT0005

6. $\lim_{x\to 0}\frac{\ell n(\sin 3x)}{\ell n(\sin x)}$ is equal to

(A) 0 (B) 1 (C) 2 (D) Non existent

LT0006

7. $\lim_{x\to 0}\frac{\sqrt[3]{1+x^2}-\sqrt[4]{1-2x}}{x+x^2}$ is equal to

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) 1 (D) D.N.E.

LT0007

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8. $\lim_{x\to 1}\frac{\sqrt[3]{7+x^3}-\sqrt{3+x^2}}{x-1}$ is equal to

(A) $\frac{1}{4}$ (B) $\frac{1}{6}$ (C) $-\frac{1}{4}$ (D) $-\frac{1}{6}$

LT0008

9. $\lim_{n\to\infty}\frac{(n+1)^4-(n-1)^4}{(n+1)^4+(n-1)^4}$ is equal to

(A) –1 (B) 0 (C) 1 (D) D.N.E.

LT0009

10. $\lim_{x\to\infty}\frac{(x+1)^{10}+(x+2)^{10}+....+(x+100)^{10}}{x^{10}+10^{10}}$ is equal to

(A) 1 (B) 100 (C) 200 (D) 10

LT0010

11. $\lim_{x\to-\infty}\left(\sqrt{x^2-2x-1}-\sqrt{x^2-7x+3}\right)$ is equal to

(A) $-\frac{5}{2}$ (B) $\frac{5}{2}$ (C) 0 (D) D.N.E

LT0011

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12. If $\lim_{n\to\infty}\left(\sqrt{2n^2+n}-\lambda\sqrt{2n^2-n}\right)=\frac{1}{\sqrt{2}}$ (where λ is a real number), then-

(A) $\lambda = 1$ (B) $\lambda = -1$ (C) $\lambda = \pm 1$ (D) $\lambda \in (-\infty, 1)$

LT0012

13. Let $U_n = \frac{n!}{(n+2)!}$ where $n \in \mathbb{N}$. If $S_n = \sum_{n=1}^{n} U_n$ then $\lim_{n\to\infty} S_n$ equals

(A) 2 (B) 1 (C) 1/2 (D) Non existent

LT0013

14. For $n \in \mathbb{N}$, let $a_n = \sum_{k=1}^{n} 2k$ and $b_n = \sum_{k=1}^{n}(2k-1)$. Then $\lim_{n\to\infty}\left(\sqrt{a_n}-\sqrt{b_n}\right)$ is equal to-

(A) 1 (B) $\frac{1}{2}$ (C) 0 (D) 2

LT0014

15. Let $P_n = \prod_{k=2}^{n}\left(1-\frac{1}{{}^{k+1}C_2}\right)$. If $\lim_{n\to\infty} P_n$ can be expressed as lowest rational in the form $\frac{a}{b}$, then value of (a + b) is

(A) 4 (B) 8 (C) 10 (D) 12

LT0015

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16. $\lim_{x\to -1}\frac{\cos 2-\cos 2x}{x^2-|x|}$ is equal to

(A) 0 (B) cos2 (C) 2sin2 (D) sin1

LT0016

17. $\lim_{x\to 0}\left(\left[\frac{-5\sin x}{x}\right]+\left[\frac{6\sin x}{x}\right]\right)$ (where [.] denotes greatest integer function) is equal to -

(A) 0 (B) –12 (C) 1 (D) 2

LT0017

18. Let $f(x)=\left[\frac{\sin x}{x}\right]+\left[\frac{2\sin 2x}{x}\right]+....+\left[\frac{10\sin 10x}{x}\right]$ (where [y] is the largest integer ≤ y). The value of $\lim_{x\to 0} f(x)$ equals

(A) 55 (B) 164 (C) 165 (D) 375

LT0018

19. Let $f(x)=\frac{\sin\{x\}}{x^2+ax+b}$. If $f(5^+)$ & $f(3^+)$ exists finitely and are not zero, then the value of (a + b) is (where {.} represents fractional part function) -

(A) 7 (B) 10 (C) 11 (D) 20

LT0019

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20. $\lim_{x\to 0}\frac{|\cos(\sin(3x))|-1}{x^2}$ equals

(A) $\frac{-9}{2}$ (B) $\frac{-3}{2}$ (C) $\frac{3}{2}$ (D) $\frac{9}{2}$

LT0020

21. Let a = min{x^2 + 2x + 3, x ∈ ℝ} and b = $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}$. Then value of $\sum_{r=0}^{n} a^r.b^{n-r}$ is :

(A) $\frac{2^{n+1}-1}{3.2^n}$ (B) $\frac{2^{n+1}+1}{3.2^n}$ (C) $\frac{4^{n+1}-1}{3.2^n}$ (D) N.O.T.

LT0021

22. Let BC is diameter of a circle centred at O. Point A is a variable point, moving on the circumference of circle. If BC = 1 unit, then $\lim_{A\to B}\frac{BM}{(\text{Area of sector OAB})^2}$ is equal to -

(A) 1 (B) 2
(C) 4 (D) 16

LT0022

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23. $\lim_{x\to\infty}\left(\frac{x^2-2x+1}{x^2-4x+2}\right)^x$ is equal to

(A) 1 (B) e (C) $\frac{1}{e^2}$ (D) e^2

LT0023

24. $\lim_{x\to 0}(1+\sin x)^{\cos x}$ is equal to

(A) 0 (B) e (C) 1 (D) $\frac{1}{e}$

LT0024

25. $\lim_{x\to 0}(\cos x + a\sin bx)^{\frac{1}{x}}$ is equal to

(A) e^a (B) e^{ab} (C) e^b (D) $e^{a/b}$

LT0025

26. $\lim_{x\to 0}\left(\tan\left(\frac{\pi}{4}+x\right)\right)^{1/x}$ is equal to

(A) e^{-2} (B) $\frac{1}{e}$ (C) e (D) e^2

LT0026

27. $\lim_{n\to\infty}(4^n+5^n)^{1/n}$ is equal to

(A) 5 (B) 4 (C) 0 (D) D.N.E.

LT0027

28. $\lim_{x\to\infty}\left(\frac{1^{1/x}+2^{1/x}+3^{1/x}+.....+n^{1/x}}{n}\right)^{nx}$, $n\in\mathbb{N}$ is equal to

(A) n! (B) 1 (C) $\frac{1}{n!}$ (D) 0

LT0028

29. If $\lim_{x\to\lambda}\left(2-\frac{\lambda}{x}\right)^{\lambda\tan\left(\frac{\pi x}{2\lambda}\right)}=\frac{1}{e}$, then λ is equal to -

(A) $-\pi$ (B) π (C) $\frac{\pi}{2}$ (D) $-\frac{2}{\pi}$

LT0029

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30. If $\lim_{x\to 0}(1 + ax + bx^2)^{2/x} = e^3$, then

(A) $a = \frac{3}{2}$ and $b \in \mathbb{R}$
(B) $a = \frac{3}{2}$ and $b \in \mathbb{R}^+$
(C) a = 0 and b = 1
(D) a = 1 and b = 0

LT0030

31. If f(x) is a polynomial of least degree, such that $\lim_{x\to 0}\left(1+\frac{f(x)+x^2}{x^2}\right)^{1/x} = e^2$, then f(2) is -

(A) 2
(B) 8
(C) 10
(D) 12

LT0031

32. Let $f(x) = \frac{\tan x}{x}$, then the value of $\lim_{x\to 0}\left(\left[f(x)\right]+x^2\right)^{\frac{1}{\{f(x)\}}}$ is equal to (where [.], {.} denotes greatest integer function and fractional part functions respectively) -

(A) e^{-3}
(B) e^3
(C) e^2
(D) non-existent

LT0032

33. $\lim_{n\to\infty}\frac{e^n}{\left(1+\frac{1}{n}\right)^{n^2}}$ equals -

(A) 1
(B) $\frac{1}{2}$
(C) e
(D) $\sqrt{e}$

LT0033

34. If f(x) is odd linear polynomial with f(1) = 1, then $\lim_{x\to 0}\frac{2^{f(\tan x)} - 2^{f(\sin x)}}{x^2 f(\sin x)}$ is

(A) 1
(B) $\ell n2$
(C) $\frac{1}{2}\ell n2$
(D) cos2

LT0034

35. $\lim_{x\to 0}\frac{x(1+a\cos x) - b\sin x}{x^3} = 1$ then

(A) a = –5/2
(B) a = –3/2, b = –1/2
(C) a = –3/2, b = –5/2
(D) a = –5/2, b = –3/2

LT0035

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[MULTIPLE CORRECT CHOICE TYPE]

36. Consider following statements and identify correct options

(i) $\lim\limits_{x\to4}\left(\frac{2x}{x-4}-\frac{8}{x-4}\right)=\lim\limits_{x\to4}\frac{2x}{x-4}-\lim\limits_{x\to4}\frac{8}{x-4}$

(ii) $\lim\limits_{x\to1}\frac{x^2+6x-7}{x^2+5x-6}=\frac{\lim\limits_{x\to1}(x^2+6x-7)}{\lim\limits_{x\to1}(x^2+5x-6)}$

(iii) $\lim\limits_{x\to1}\frac{x-3}{x^2+2x-4}=\frac{\lim\limits_{x\to1}(x-3)}{\lim\limits_{x\to1}(x^2+2x-4)}$

(iv) If $\lim\limits_{x\to5}f(x)=2$ and $\lim\limits_{x\to5}g(x)=0$, then $\lim\limits_{x\to5}\frac{f(x)}{g(x)}$ does not exist.

(v) If $\lim\limits_{x\to5}f(x)=0$ and $\lim\limits_{x\to5}g(x)=2$, then $\lim\limits_{x\to5}\frac{f(x)}{g(x)}$ does not exist.

(A) Only one is true.
(B) Only two are true.
(C) Only three are false.
(D) Only two are false.

LT0036

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37. Which of the following limits equal to $\frac{1}{2}$

(A) $\lim\limits_{n\to\infty}\left(\frac{1}{1.3}+\frac{1}{3.5}+...+\frac{1}{(2n-1)(2n+1)}\right)$

(B) $\lim\limits_{x\to\infty}\left[\frac{3x^2}{2x+1}-\frac{(2x-1)(3x^2+x+2)}{4x^2}\right]$

(C) $\lim\limits_{n\to\infty}\frac{1}{n^2}(1+2+3+.....+n)$

(D) $\lim\limits_{n\to\infty}\frac{(n+2)!+(n+1)!}{(n+2)!-(n+1)!}$

LT0037

38. Let $f(x)=\begin{cases}\sin x\,;\text{where } x=\text{integer}\\ 0\ ;\ \text{otherwise}\end{cases}$: $g(x)=\begin{cases}x^2+1 & ;\ x\neq0,2\\ 4 & ;\ x=0\\ 5 & ;\ x=2\end{cases}$, then

(A) $\lim\limits_{x\to0}g(f(x))=4$
(B) $\lim\limits_{x\to0}f(g(x))=0$
(C) $\lim\limits_{x\to1}f(g(x))=0$
(D) $\lim\limits_{x\to1}g(f(x))=5$

LT0038

39. If $f(x)=\begin{cases}x^2 & \text{if } x \text{ is rational}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$, then

(A) $\lim\limits_{x\to0}f(x)=0$
(B) $\lim\limits_{x\to0}f(x)$ does not exist
(C) $\lim\limits_{x\to2}f(x)=4$
(D) $\lim\limits_{x\to2}f(x)$ does not exist

LT0039

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40. Let $f(\beta)=\lim\limits_{\alpha\to\beta}\dfrac{\sin^2\alpha-\sin^2\beta}{\alpha^2-\beta^2}$, then $f\left(\dfrac{\pi}{4}\right)$ is greater than-

(A) $\lim\limits_{x\to0}\dfrac{1-\cos^3 x}{x\sin 2x}$

(B) $\lim\limits_{x\to\pi/2}\dfrac{\cot x-\cos x}{(\pi-2x)^3}$

(C) $\lim\limits_{x\to\infty}\left(\cos\sqrt{x+1}-\cos\sqrt{x}\right)$

(D) $\lim\limits_{x\to a}\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$ where $a>0$

LT0040

41. If $\dfrac{\sin x+ae^x+be^{-x}+c\ln(1+x)}{x^3}$ has a finite limit L as $x\to0$, then

(A) $a=-\dfrac{1}{2}$ (B) $b=\dfrac{1}{2}$ (C) $c=0$ (D) $L=-\dfrac{1}{3}$

LT0041

42. Let $\ell=\lim\limits_{x\to\infty}\dfrac{a^x-a^{-x}}{a^x+a^{-x}}(a>0)$, then

(A) $\ell=1\ \forall\ a>0$ (B) $\ell=-1\ \forall\ a\in(0,1)$ (C) $\ell=0$, if $a=1$ (D) $\ell=1\ \forall\ a>1$

LT0042

[MATCH THE COLUMN TYPE]

43. For the function g(t) whose graph is given, match the entries of column-I to column-II

Column-I	Column-II
(A) $\lim\limits_{t\to0^+}g(t)+\lim\limits_{t\to2^-}g(t)$	(P) $\lim\limits_{t\to2^+}g(t)$
(B) $\lim\limits_{t\to0^-}g(t)+g(2)$	(Q) does not exist
(C) $\lim\limits_{t\to0}g(t)$	(R) 0
(D) $\lim\limits_{t\to2}g(t)$	(S) $\lim\limits_{t\to4}g(t)$

LT0043

44.

Column-I	Column-II	
(A) $\lim\limits_{n\to\infty}n\sin\left(\dfrac{\pi}{4n}\right)\cos\left(\dfrac{\pi}{4n}\right)$ is equal to	(P) 0	**LT0044**
(B) $\lim\limits_{x\to0}\dfrac{\sin x^\circ}{x}$ is equal to	(Q) $\dfrac{1}{2}$	**LT0045**
(C) $\lim\limits_{x\to0}\left(\dfrac{1}{\sin x}-\dfrac{1}{\tan x}\right)$ is equal to	(R) $\dfrac{\pi}{4}$	**LT0046**
(D) $\lim\limits_{x\to\pi/2}\dfrac{1+\cos 2x}{(\pi-2x)^2}$ is equal to	(S) $\dfrac{\pi}{180}$	**LT0047**

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45. **Column-I** **Column-II**

(A) $\lim\limits_{x\to\infty}\frac{a^x}{a^x+1}$ $(a > 0)$ can be equal to (P) $\lim\limits_{x\to\infty} x(e^{1/x}-1)$

LT0048

(B) $\lim\limits_{x\to 2}\frac{\sin(e^{x-2}-1)}{\log(x-1)}$ is equal to (Q) $\lim\limits_{x\to 0}\frac{a^x+b^x+c^x-3}{x}(a,b,c>0\,\&\,abc=1)$

LT0049

(C) $\lim\limits_{x\to e}\frac{(\ell n x-1)e}{x-e}$ is equal to (R) $\lim\limits_{x\to 0}\frac{e^{4x}-e^{3x}}{x}$

LT0050

(D) $\lim\limits_{x\to 0}\frac{x(5^x-1)}{(1-\cos x)4\ell n5}$ is equal to (S) $\frac{1}{2}$

LT0051

(T) 0

EXERCISE (O-2)

[SINGLE CORRECT CHOICE TYPE]

1. $\lim\limits_{h\to 0}\frac{\sin(a+3h)-3\sin(a+2h)+3\sin(a+h)-\sin a}{h^3}$ is equal to

(A) cosa (B) –cosa (C) sina (D) sina cosa

LT0052

2. $\lim\limits_{x\to\frac{\pi}{2}}\tan^2 x(\sqrt{2\sin^2 x+3\sin x+4}-\sqrt{\sin^2 x+6\sin x+2})$ is equal to

(A) $\frac{3}{4}$ (B) $\frac{1}{6}$ (C) $\frac{1}{12}$ (D) $\frac{5}{12}$

LT0053

3. $\lim\limits_{x\to\infty} x\left(\arctan\frac{x+1}{x+2}-\arctan\frac{x}{x+2}\right)$ is equal to

(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) 1 (D) D.N.E

LT0054

4. $\lim\limits_{h\to 0}\frac{\tan(a+2h)-2\tan(a+h)+\tan a}{h^2}$ is equal to

(A) tana (B) $\tan^2 a$ (C) seca (D) $2(\sec^2 a)(\tan a)$

LT0055

5. $\lim\limits_{x\to 0}\left(2^{x-1}+\frac{1}{2}\right)^{1/x}$ equals

(A) $\sqrt{2}$ (B) $\frac{1}{2}\ell n2$ (C) $\ell n2$ (D) 2

LT0056

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6. If $\lim\limits_{x\to 0}\left(\cos x + a^3 \sin\left(b^6 x\right)\right)^{\frac{1}{x}} = e^{512}$, then the value of ab^2 is equal to

(A) –512 (B) 512 (C) 8 (D) $8\sqrt{8}$

LT0057

7. The value of $\lim\limits_{x\to 0}\dfrac{\sin\left(\sqrt[3]{x}\right)\ell n(1+3x)}{\left(\tan^{-1}\sqrt{x}\right)^2\left(e^{5\left(\sqrt[3]{x}\right)}-1\right)}$ is equal to

(A) $\frac{1}{5}$ (B) $\frac{3}{5}$ (C) $\frac{2}{5}$ (D) $\frac{4}{5}$

LT0058

8. The figure shows an isosceles triangle ABC with $\angle B = \angle C$. The bisector of angle B intersects the side AC at the point P. Suppose that BC remains fixed but the altitude AM approaches 0, so that A→M (mid-point of BC). Limiting value of BP, is

(A) $\frac{a}{3}$ (B) $\frac{a}{2}$ (C) $\frac{2a}{3}$ (D) $\frac{3a}{4}$

where a is fixed side BC.

LT0059

9. The value of $\lim\limits_{x\to 2}\dfrac{\sec^x\theta - \tan^x\theta - 1}{x-2}$ is equal to

(A) $\sec^2\theta.\ \ell n\sec\theta + \tan^2\theta.\ \ell n\tan\theta$ (B) $\sec^2\theta.\ \ell n\tan\theta + \tan^2\theta.\ \ell n\sec\theta$

(C) $\sec^2\theta.\ \ell n\tan\theta - \tan^2\theta.\ \ell n\sec\theta$ (D) $\sec^2\theta.\ \ell n\sec\theta - \tan^2\theta.\ \ell n\tan\theta$

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LT0060

10. Consider the function $f(x) = \begin{cases} 1-x, & 0 \le x \le 1 \\ x+2, & 1 < x < 2 \\ 4-x, & 2 \le x \le 4 \end{cases}$. Let $\lim\limits_{x\to 1} f(f(x)) = \ell$ and $\lim\limits_{x\to 2} f(f(x)) = m$ then which one of the following hold good ?

(A) ℓ exists but m does not. (B) m exists but ℓ does not.

(C) Both ℓ and m exist (D) Neither ℓ nor m exist.

LT0061

11. If $f(x) = e^x$, then $\lim\limits_{x\to 0} f(f(x))^{\frac{1}{\{f(x)\}}}$ is equal to (where {x} denotes fractional part of x).

(A) f(1) (B) f(0) (C) 0 (D) does not exist

LT0062

12. Let $f(x)$ be a quadratic function such that $f(0) = f(1) = 0$ & $f(2) = 1$, then $\lim\limits_{x\to 0}\dfrac{\cos\left(\frac{\pi}{2}\cos^2 x\right)}{f^2(x)}$ is equal to

(A) $\frac{\pi}{2}$ (B) π (C) 2π (D) 4π

LT0063

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[MULTIPLE CORRECT CHOICE TYPE]

13. If $\ell = \lim\limits_{x\to a}\dfrac{\sqrt{3x^2+a^2}-\sqrt{x^2+3a^2}}{(x-a)}$ then -

(A) $\ell = 1\ \forall\ a \in \mathbb{R}$ (B) $\ell = 1\ \forall\ a > 0$ (C) $\ell = -1\ \forall\ a < 0$ (D) ℓ = D.N.E. if a = 0

LT0064

14. Which of the following limits vanish ?

(A) $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}$ (B) $\lim\limits_{x\to\infty}\dfrac{\arctan x}{x}$ (C) $\lim\limits_{x\to\infty}\dfrac{x+\sin x}{x+\cos x}$ (D) $\lim\limits_{x\to 1}\dfrac{\arcsin x}{\tan\frac{\pi x}{2}}$

LT0065

15. Which of the following statement are true for the function f defined for $-1 \le x \le 3$ in the figure shown.

(A) $\lim\limits_{x\to -1^+} f(x) = 1$

(B) $\lim\limits_{x\to 2} f(x)$ does not exist

(C) $\lim\limits_{x\to 1^-} f(x) = 2$

(D) $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^-} f(x)$

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LT0066

16. Let $f(x) = x + \sqrt{x^2+2x}$ and $g(x) = \sqrt{x^2+2x} - x$, then

(A) $\lim\limits_{x\to\infty} g(x) = 1$ (B) $\lim\limits_{x\to\infty} f(x) = 1$ (C) $\lim\limits_{x\to-\infty} f(x) = -1$ (D) $\lim\limits_{x\to-\infty} g(x) = -1$

LT0067

17. If $A = \lim\limits_{x\to 0}\dfrac{\sin^{-1}(\sin x)}{\cos^{-1}(\cos x)}$ and $B = \lim\limits_{x\to 0}\dfrac{[|x|]}{x}$, then (where [.] denotes greatest integer function)-

(A) A = 1 (B) A does not exist

(C) B = 0 (D) B = 1

LT0068

18. Which of the following limit tends to unity ?

(A) $\lim\limits_{x\to 0}\dfrac{1-\cos x+2\sin x-\sin^3 x-x^2+3x^4}{\tan^3 x-6\sin^2 x+x-5x^3}$ (B) $\lim\limits_{x\to\infty}\dfrac{x}{[x]}$

(C) $\lim\limits_{x\to\infty}\dfrac{1}{\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)}$ (D) $\lim\limits_{x\to\infty}\left(\dfrac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}\right)$

LT0069

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19. Which of the following limits does not exist ?

(A) $\lim\limits_{x\to 1^+}([x])^{\frac{1}{x-1}}$

(B) $\lim\limits_{x\to 3}\dfrac{(x^2-9-\sqrt{x^2-6x+9})}{|x-1|-2}$

(C) $\lim\limits_{x\to 0^+}(x)^{\ell nx}$

(D) $\lim\limits_{x\to 0^+}\left(\dfrac{1-\cos(\sin^2 x)}{x^2}\right)^{\frac{\ell n(1-2x^2)}{\sin^2 x}}$

(where [.] represents greatest integer function)

LT0070

20. The value(s) of 'n' for which $\lim\limits_{x\to 1}\dfrac{e^{x-1}-x}{(x-1)^n}$ exists is/are -

(A) 1 (B) 2 (C) 3 (D) 4

LT0071

21. Let $f:\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)\to\mathbb{R},\ f(x)=\begin{cases}\lim\limits_{n\to\infty}\left(\dfrac{(\tan x)^{2n}+x^2}{\sin^2 x+(\tan x)^{2n}}\right) & ;\ x\neq 0\\ 1 & ;\ x=0\end{cases}$, $n\in\mathbb{N}$. Which of the following holds good ?

(A) $f\left(-\dfrac{\pi^-}{4}\right)=f\left(\dfrac{\pi^+}{4}\right)$

(B) $f\left(-\dfrac{\pi^-}{4}\right)=f\left(-\dfrac{\pi^+}{4}\right)$

(C) $f\left(\dfrac{\pi^-}{4}\right)=f\left(\dfrac{\pi^+}{4}\right)$

(D) $f(0^+)=f(0)=f(0^-)$

LT0072

22. Let $f(x)=\begin{bmatrix}\dfrac{\tan^2\{x\}}{x^2-[x]^2} & \text{for } x>0\\ 1 & \text{for } x=0\\ \sqrt{\{x\}\cot\{x\}} & \text{for } x<0\end{bmatrix}$ where [x] is the step up function and {x} is the fractional part function of x, then-

(A) $\lim\limits_{x\to 0^+}f(x)=1$

(B) $\lim\limits_{x\to 0^-}f(x)=1$

(C) $\cot^{-1}\left(\lim\limits_{x\to 0^-}f(x)\right)^2=1$

(D) None

LT0073

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23. $\lim_{x\to c} f(x)$ does not exist when (where [x] is the step up function, {x} is the fractional part function of x & sgn(x) denotes signum function), then-

(A) $f(x) = [[x]] - [2x - 1]$; $c = 3$

(B) $f(x) = [x] - x$, $c = 1$

(C) $f(x) = \{x\}^2 - \{-x\}^2$, $c = 0$

(D) $f(x) = \dfrac{\tan(\text{sgn } x)}{\text{sgn } x}, c = 0$

LT0074

24. Which of the following limits does not exist ?

(A) $\lim_{x\to\infty} \text{cosec}^{-1}\left(\dfrac{x}{x+7}\right)$

(B) $\lim_{x\to 1} \sec^{-1}(\sin^{-1} x)$

(C) $\lim_{x\to 0^+} x^{\frac{1}{x}}$

(D) $\lim_{x\to 0}\left(\tan\left(\dfrac{\pi}{8}+x\right)\right)^{\cot x}$

LT0075

25. Which of the following statement(s) is (are) INCORRECT ?

(A) If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both does not exist then $\lim_{x\to c} f(x)\, g(x)$ also does not exist.

(B) If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both does not exist then $\lim_{x\to c} f(g(x))$ also does not exist.

(C) If $\lim_{x\to c} f(x)$ exists and $\lim_{x\to c} g(x)$ does not exist then $\lim_{x\to c} g(f(x))$ does not exist.

(D) If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist then $\lim_{x\to c} f(g(x))$ and $\lim_{x\to c} g(f(x))$ also exist.

LT0076

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EXERCISE (S-1)

1. $\lim_{x\to 1} \dfrac{x^2 - x.\ln x + \ln x - 1}{x-1}$ **LT0077**

2. $\lim_{x\to 1} \dfrac{\left[\sum_{K=1}^{100} x^k\right] - 100}{x-1}$ **LT0078**

3. $\lim_{x\to \frac{\pi}{4}} \dfrac{1-\tan x}{1-\sqrt{2}\sin x}$ **LT0079**

4. $\lim_{x\to 0} \dfrac{8}{x^8}\left[1-\cos\dfrac{x^2}{2}-\cos\dfrac{x^2}{4}+\cos\dfrac{x^2}{2}\cos\dfrac{x^2}{4}\right]$ **LT0080**

5. $\lim_{\theta\to\frac{\pi}{4}} \dfrac{\sqrt{2}-\cos\theta-\sin\theta}{(4\theta-\pi)^2}$ **LT0081**

6. $\lim_{h\to 0} \dfrac{\sin\left(\frac{\pi}{3}+4h\right)-4\sin\left(\frac{\pi}{3}+3h\right)+6\sin\left(\frac{\pi}{3}+2h\right)-4\sin\left(\frac{\pi}{3}+h\right)+\sin\frac{\pi}{3}}{h^4}$ **LT0082**

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7. $\lim\limits_{x\to\infty} x^2\left(\sqrt{\frac{x+2}{x}}-\sqrt[3]{\frac{x+3}{x}}\right)$ **LT0083**

8. $\lim\limits_{x\to-\infty}\frac{(3x^4+2x^2)\sin\frac{1}{x}+|x|^3+5}{|x|^3+|x|^2+|x|+1}$ **LT0084**

9. If $\ell=\lim\limits_{n\to\infty}\sum\limits_{r=2}^{n}\left((r+1)\sin\frac{\pi}{r+1}-r\sin\frac{\pi}{r}\right)$ then find $\{\ell\}$. (where { } denotes the fractional part function)

LT0085

10. Find a & b if : (i) $\lim\limits_{x\to\infty}\left[\frac{x^2+1}{x+1}-ax-b\right]=0$ (ii) $\lim\limits_{x\to-\infty}\left[\sqrt{x^2-x+1}-ax-b\right]=0$

LT0086

11. $\lim\limits_{x\to0}[\ell n\,(1+\sin^2 x).\cot(\ell n^2\,(1+x))]$ **LT0087**

12. $\lim\limits_{x\to0}\frac{27^x-9^x-3^x+1}{\sqrt{2}-\sqrt{1+\cos x}}$ **LT0088**

13. (a) $\lim\limits_{x\to0}\tan^{-1}\frac{a}{x^2}$, where $a\in\mathbb{R}$; (b) Plot the graph of the function $f(x)=\lim\limits_{t\to0}\left(\frac{2x}{\pi}\tan^{-1}\frac{x}{t^2}\right)$

LT0089

14. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences such that

(i) $a_n+b_n+c_n=2n+1$; **(ii)** $a_nb_n+b_nc_n+c_na_n=2n-1$;

(iii) $a_nb_nc_n=-1$; **(iv)** $a_n<b_n<c_n$

Then find the value of $\lim\limits_{n\to\infty}(na_n)$.

LT0090

15. Let $f(x)=ax^3+bx^2+cx+d$ and $g(x)=x^2+x-2$.

If $\lim\limits_{x\to1}\frac{f(x)}{g(x)}=1$ and $\lim\limits_{x\to-2}\frac{f(x)}{g(x)}=4$, then find the value of $\frac{c^2+d^2}{a^2+b^2}$. **LT0091**

16. $\lim\limits_{x\to\infty}\left[\frac{2x^2+3}{2x^2+5}\right]^{8x^2+3}$ **LT0092**

17. $\lim\limits_{x\to\infty}\left(\frac{x+c}{x-c}\right)^x=4$ then find c **LT0093**

18. $\lim\limits_{x\to1}\left(\tan\frac{\pi x}{4}\right)^{\tan\frac{\pi x}{2}}$ **LT0094**

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19. $\lim\limits_{x\to 0}\left(\dfrac{x-1+\cos x}{x}\right)^{\frac{1}{x}}$ **LT0095**

20. If $n \in \mathbb{N}$ and $a_n = 2^2 + 4^2 + 6^2 + ... + (2n)^2$ and $b_n = 1^2 + 3^2 + 5^2 + ...+ (2n-1)^2$.

Find the value $\lim\limits_{n\to\infty}\dfrac{\sqrt{a_n}-\sqrt{b_n}}{\sqrt{n}}$.

LT0096

EXERCISE (S-2)

1. $\lim\limits_{n\to\infty}\left(\dfrac{\sqrt{n^2+n}-1}{n}\right)^{2\sqrt{n^2+n}-1}$ **LT0097**

2. $\lim\limits_{x\to\infty}\left(\dfrac{a_1^{\frac{1}{x}}+a_2^{\frac{1}{x}}+a_3^{\frac{1}{x}}+.....+a_n^{\frac{1}{x}}}{n}\right)^{nx}$, $n \in \mathbb{N}$, where $a_1, a_2, a_3,....a_n > 0$ **LT0098**

3. $\lim\limits_{x\to 0}\left[\dfrac{(1+x)^{1/x}}{e}\right]^{1/x}$ **LT0099**

4. If $\lim\limits_{x\to\infty}\dfrac{a(2x^3-x^2)+b(x^3+5x^2-1)-c(3x^3+x^2)}{a(5x^4-x)-bx^4+c(4x^4+1)+2x^2+5x} = 1$, then the value of $(a + b + c)$ can be expressed in the lowest form as $\dfrac{p}{q}$. Find the value of $(p + q)$.

LT0100

5. $\lim\limits_{x\to 0}\left[\dfrac{\ell n(1+x)^{1+x}}{x^2}-\dfrac{1}{x}\right]$ **LT0101**

6. Let $L = \prod\limits_{n=3}^{\infty}\left(1-\dfrac{4}{n^2}\right)$; $M = \prod\limits_{n=2}^{\infty}\left(\dfrac{n^3-1}{n^3+1}\right)$ and $N = \prod\limits_{n=1}^{\infty}\dfrac{(1+n^{-1})^2}{1+2n^{-1}}$, then find the value of $L^{-1}+M^{-1}+N^{-1}$.

LT0102

7. A circular arc of radius 1 subtends an angle of x radians, $0 < x < \dfrac{\pi}{2}$ as shown in the figure. The point C is the intersection of the two tangent lines at A & B. Let T(x) be the area of triangle ABC & let S(x) be the area of the shaded region. Compute :

(a) T(x) (b) S(x) & (c) the limit of $\dfrac{T(x)}{S(x)}$ as $x \to 0$.

LT0103

8. Let $f(x) = \lim\limits_{n\to\infty}\sum\limits_{n=1}^{n}3^{n-1}\sin^3\dfrac{x}{3^n}$ and $g(x) = x - 4f(x)$. Evaluate $\lim\limits_{x\to 0}(1+g(x))^{\cot x}$.

LT0104

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9. If $f(n, \theta) = \prod_{r=1}^{n}\left(1-\tan^2\frac{\theta}{2^r}\right)$, then compute $\lim_{n\to\infty} f(n, \theta)$

LT0105

10. Evaluate $\lim_{x\to\infty}\left(\frac{x}{e} - x\left(\frac{x}{x+1}\right)^x\right)$

LT0106

11. f(x) is the function such that $\lim_{x\to 0}\frac{f(x)}{x} = 1$. If $\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{(f(x))^3} = 1$, then find the value of a and b.

LT0107

12. Through a point A on a circle, a chord AP is drawn & on the tangent at A a point T is taken such that AT = AP. If TP produced meet the diameter through A at Q, prove that the limiting value of AQ when P moves upto A is double the diameter of the circle.

LT0108

13. At the end points A, B of the fixed segment of length L, lines are drawn meeting in C and making angles θ and 2θ respectively with the given segment. Let D be the foot of the altitude CD and let x represents the length of AD. Find the value of x as θ tends to zero i.e. $\lim_{\theta\to 0} x$.

LT0109

14. Let $f(x) = \lim_{n\to\infty}\frac{2x^{2n}\sin\frac{1}{x} + x}{1+x^{2n}}$, $n \in \mathbb{N}$, then find

(a) $\lim_{x\to\infty} xf(x)$, (b) $\lim_{x\to 1} f(x)$, (c) $\lim_{x\to 0} f(x)$, (d) $\lim_{x\to -\infty} f(x)$

LT0110

15. Using Sandwich theorem, evaluate

(a) $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots\ldots + \frac{1}{\sqrt{n^2+2n}}\right)$

LT0111

(b) $\lim_{n\to\infty}\frac{1}{1+n^2} + \frac{2}{2+n^2} + \ldots\ldots + \frac{n}{n+n^2}$

LT0112

EXERCISE (JM)

1. Let f : R → R be a positive increasing function with $\lim_{x\to\infty}\frac{f(3x)}{f(x)} = 1$. Then $\lim_{x\to\infty}\frac{f(2x)}{f(x)} =$

[AIEEE-2010]

(1) 1 (2) $\frac{2}{3}$ (3) $\frac{3}{2}$ (4) 3

LT0113

2. $\lim_{x\to 2}\left(\frac{\sqrt{1-\cos\{2(x-2)\}}}{x-2}\right)$ **[AIEEE-2011]**

(1) equals $-\sqrt{2}$ (2) equals $\frac{1}{\sqrt{2}}$ (3) does not exist (4) equals $\sqrt{2}$

LT0114

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3. Let $f : R \to [0, \infty)$ be such that $\lim_{x\to5} f(x)$ exists and $\lim_{x\to5}\frac{(f(x))^2-9}{\sqrt{|x-5|}}=0$. Then $\lim_{x\to5} f(x)$ equal - **[AIEEE-2011]**

(1) 3 (2) 0 (3) 1 (4) 2

LT0115

4. $\lim_{x\to0}\frac{\sin(\pi\cos^2 x)}{x^2}$ is equal to : **[JEE Mains Offline-2014]**

(1) $\frac{\pi}{2}$ (2) 1 (3) $-\pi$ (4) π

LT0116

5. If $\lim_{x\to2}\frac{\tan(x-2)\{x^2+(k-2)x-2k\}}{x^2-4x+4}=5$ then k is equal to **[JEE Mains Online-2014]**

(1) 3 (2) 1 (3) 0 (4) 2

LT0117

6. Let $p=\lim_{x\to0+}\left(1+\tan^2\sqrt{x}\right)^{\frac{1}{2x}}$ then log p is equal to - **[JEE(Main)-2016]**

(1) $\frac{1}{4}$ (2) 2 (3) 1 (4) $\frac{1}{2}$

LT0118

7. $\lim_{x\to\frac{\pi}{2}}\frac{\cot x-\cos x}{(\pi-2x)^3}$ equals :- **[JEE(Main)-2017]**

(1) $\frac{1}{4}$ (2) $\frac{1}{24}$ (3) $\frac{1}{16}$ (4) $\frac{1}{8}$

LT0119

8. For each $t \in R$, let [t] be the greatest integer less than or equal to t. Then

$\lim_{x\to0+} x\left(\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+....+\left[\frac{15}{x}\right]\right)$ **[JEE(Main)-2018]**

(1) is equal to 15. (2) is equal to 120.
(3) does not exist (in R). (4) is equal to 0.

LT0120

9. $\lim_{y\to0}\frac{\sqrt{1+\sqrt{1+y^4}}-\sqrt{2}}{y^4}$ **[JEE(Main)-2019]**

(1) exists and equals $\frac{1}{4\sqrt{2}}$ (2) does not exist

(3) exists and equals $\frac{1}{2\sqrt{2}}$ (4) exists and equals $\frac{1}{2\sqrt{2}\left(\sqrt{2}+1\right)}$

LT0121

10. For each $x \in R$, let [x] be the greatest integer less than or equal to x.

Then $\lim_{x\to0^-}\frac{x([x]+|x|)\sin[x]}{|x|}$ is equal to **[JEE(Main)-2019]**

(1) $-\sin1$ (2) 0 (3) 1 (4) $\sin1$

LT0122

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11. For each t ∈ R, let [t] be the greatest integer less than or equal to t.

Then, $\lim_{x\to 1+}\frac{(1-|x|+\sin|1-x|)\sin\left(\frac{\pi}{2}[1-x]\right)}{|1-x|[1-x]}$ **[JEE(Main)-2019]**

(1) equals –1 (2) equals 1 (3) does not exist (4) equals 0

LT0123

12. Let [x] denote the greatest integer less than or equal to x.

Then $\lim_{x\to 0}\frac{\tan(\pi\sin^2 x)+\left(|x|-\sin(x[x])\right)^2}{x^2}$ **[JEE(Main)-2019]**

(1) equals π (2) equals 0 (3) equals π + 1 (4) does not exist

LT0124

13. $\lim_{x\to\pi/4}\frac{\cot^3 x-\tan x}{\cos\left(x+\pi/4\right)}$ is : **[JEE(Main)-2019]**

(1) 4 (2) $8\sqrt{2}$ (3) 8 (4) $4\sqrt{2}$

LT0125

14. $\lim_{x\to 1-}\frac{\sqrt{\pi}-\sqrt{2\sin^{-1}x}}{\sqrt{1-x}}$ equal to : **[JEE(Main)-2019]**

(1) $\frac{1}{\sqrt{2\pi}}$ (2) $\sqrt{\frac{\pi}{2}}$ (3) $\sqrt{\frac{2}{\pi}}$ (4) $\sqrt{\pi}$

LT0126

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15. $\lim_{x\to 0}\frac{\sin^2 x}{\sqrt{2}-\sqrt{1+\cos x}}$ equals : **[JEE(Main)-2019]**

(1) $2\sqrt{2}$ (2) $4\sqrt{2}$ (3) $\sqrt{2}$ (4) 4

LT0127

16. Let f : R → R be a differentiable function satisfying $f'(3) + f'(2) = 0$.

Then $\lim_{x\to 0}\left(\frac{1+f(3+x)-f(3)}{1+f(2-x)-f(2)}\right)^{\frac{1}{x}}$ is equal to **[JEE(Main)-2019]**

(1) e^2 (2) e (3) e^{-1} (4) 1

LT0128

17. If $\lim_{x\to 1}\frac{x^4-1}{x-1}=\lim_{x\to k}\frac{x^3-k^3}{x^2-k^2}$, then k is : **[JEE(Main)-2019]**

(1) $\frac{3}{8}$ (2) $\frac{3}{2}$ (3) $\frac{4}{3}$ (4) $\frac{8}{3}$

LT0129

18. If $\lim_{x\to 1}\frac{x^2-ax+b}{x-1}=5$, then a + b is equal to :- **[JEE(Main)-2019]**

(1) –7 (2) – 4 (3) 5 (4) 1

LT0130

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EXERCISE (JA)

1. Let $L = \lim_{x\to 0} \dfrac{a-\sqrt{a^2-x^2}-\dfrac{x^2}{4}}{x^4}$, $a > 0$. If L is finite, then - **[JEE 2009, 4]**

(A) $a = 2$ (B) $a = 1$ (C) $L = \dfrac{1}{64}$ (D) $L = \dfrac{1}{32}$

LT0131

2. If $\lim_{x\to 0}\left[1+x\ell n(1+b^2)\right]^{\frac{1}{x}} = 2b\sin^2\theta, b > 0$ and $\theta \in (-\pi,\pi]$, then the value of θ is-

[JEE 2011, 3M, –1M]

(A) $\pm\dfrac{\pi}{4}$ (B) $\pm\dfrac{\pi}{3}$ (C) $\pm\dfrac{\pi}{6}$ (D) $\pm\dfrac{\pi}{2}$

LT0132

3. If $\lim_{x\to\infty}\left(\dfrac{x^2+x+1}{x+1}-ax-b\right) = 4$, then - **[JEE 2012, 3M, –1M]**

(A) $a = 1, b = 4$ (B) $a = 1, b = -4$
(C) $a = 2, b = -3$ (D) $a = 2, b = 3$

LT0133

4. Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation $\left(\sqrt[3]{1+a}-1\right)x^2+\left(\sqrt{1+a}-1\right)x+\left(\sqrt[6]{1+a}-1\right)=0$ where $a > -1$. Then $\lim_{a\to 0^+}\alpha(a)$ and $\lim_{a\to 0^+}\beta(a)$ are **[JEE 2012, 3M, –1M]**

(A) $-\dfrac{5}{2}$ and 1 (B) $-\dfrac{1}{2}$ and –1 (C) $-\dfrac{7}{2}$ and 2 (D) $-\dfrac{9}{2}$ and 3

LT0134

5. The largest value of the non-negative integer a for which $\lim_{x\to 1}\left\{\dfrac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}} = \dfrac{1}{4}$ is

[JEE(Advanced)-2014, 3]
LT0135

6. Let m and n be two positive integers greater than 1. If $\lim_{\alpha\to 0}\left(\dfrac{e^{\cos(\alpha^n)}-e}{\alpha^m}\right) = -\left(\dfrac{e}{2}\right)$ then the value of $\dfrac{m}{n}$ is **[JEE 2015, 4M, –0M]**

LT0136

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7. Let $\alpha, \beta \in \mathbb{R}$ be such that $\lim_{x\to 0}\frac{x^2\sin(\beta x)}{\alpha x-\sin x}=1$. Then $6(\alpha+\beta)$ equals

[JEE(Advanced)-2016, 3(0)]

LT0137

8. Let $f(x)=\frac{1-x(1+|1-x|)}{|1-x|}\cos\left(\frac{1}{1-x}\right)$ for $x \neq 1$. Then **[JEE(Advanced)-2017, 4]**

(A) $\lim_{x\to 1^+} f(x)$ does not exist

(B) $\lim_{x\to 1^-} f(x)$ does not exist

(C) $\lim_{x\to 1^-} f(x) = 0$

(D) $\lim_{x\to 1^+} f(x) = 0$

LT0138

9. For any positive integer n, define $f_n : (0, \infty) \to \mathbb{R}$ as

$$f_n(x)=\sum_{j=1}^{n}\tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right) \text{ for all } x \in (0, \infty).$$

(Here, the inverse trigonometric function $\tan^{-1}x$ assume values in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.)

Then, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018]**

(A) $\sum_{j=1}^{5}\tan^2\left(f_j(0)\right)=55$

(B) $\sum_{j=1}^{10}\left(1+f_j'(0)\right)\sec^2\left(f_j(0)\right)=10$

(C) For any fixed positive integer n, $\lim_{x\to\infty}\tan\left(f_n(x)\right)=\frac{1}{n}$

(D) For any fixed positive integer n, $\lim_{x\to\infty}\sec^2\left(f_n(x)\right)=1$

LT0139

10. Let $F : \mathbb{R} \to \mathbb{R}$ be a function. We say that f has

PROPERTY 1 if $\lim_{h\to 0}\frac{f(h)-f(0)}{\sqrt{|h|}}$ exists and is finite, and

PROPERTY 2 if $\lim_{h\to 0}\frac{f(h)-f(0)}{h^2}$ exists and is finite.

Then which of the following options is/are correct ? **[JEE(Advanced)-2019, 4(–1)]**

(1) $f(x) = x|x|$ has PROPERTY 2

(2) $f(x) = x^{2/3}$ has PROPERTY 1

(3) $f(x) = \sin x$ has PROPERTY 2

(4) $f(x) = |x|$ has PROPERTY 1

LT0140

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ANSWER KEY

LIMIT

EXERCISE (O-1)

1. A	**2.** C	**3.** D	**4.** C	**5.** D	**6.** B	**7.** B	**8.** C
9. B	**10.** B	**11.** A	**12.** A	**13.** C	**14.** B	**15.** A	**16.** C
17. A	**18.** D	**19.** A	**20.** A	**21.** C	**22.** D	**23.** D	**24.** C
25. B	**26.** D	**27.** A	**28.** A	**29.** C	**30.** A	**31.** D	**32.** B
33. D	**34.** C	**35.** D	**36.** B,C	**37.** A,C	**38.** A,B,C	**39.** A,D	

40. B,C,D **41.** A,B,C,D **42.** B,C,D **43.** (A)→(P,R); (B)→(P,R); (C)→(Q); (D)→(Q)

44. (A)→(R);(B)→(S);(C)→(P);(D)→(Q) **45.** (A)→(P,Q,R,S,T);(B)→(P,R);(C)→(P,R); (D)→(S)

EXERCISE (O-2)

1. B	**2.** C	**3.** A	**4.** D	**5.** A	**6.** C	**7.** B	**8.** C
9. D	**10.** A	**11.** D	**12.** C	**13.** B,C,D	**14.** A,B,D	**15.** A,C,D	**16.** A,C
17. B,C	**18.** B,D	**19.** B,C,D	**20.** A,B	**21.** A,D	**22.** A,C	**23.** B,C	**24.** A,D
25. A,B,C,D							

EXERCISE (S-1)

1. 2 **2.** 5050 **3.** 2 **4.** $\frac{1}{32}$ **5.** $\frac{1}{16\sqrt{2}}$ **6.** $\frac{\sqrt{3}}{2}$ **7.** 1/2 **8.** –2

9. $\pi - 3$ **10.** **(i)** a = 1, b = –1 **(ii)** a = –1, b = $\frac{1}{2}$ **11.** 1 **12.** $8\sqrt{2}(\ell n3)^2$

13. **(a)** $\pi/2$ if a > 0 ; 0 if a = 0 and $-\pi/2$ if a < 0 ; **(b)** f(x) = | x | **14.** –1/2 **15.** 16

16. e^{-8} **17.** c = ln2 **18.** e^{-1} **19.** $e^{-1/2}$ **20.** $\frac{\sqrt{3}}{2}$

EXERCISE (S-2)

1. e^{-1} **2.** $(a_1.a_2.a_3.....a_n)$ **3.** $e^{-\frac{1}{2}}$ **4.** 167 **5.** 1/2 **6.** 8

7. $T(x) = \frac{1}{2}\tan^2\frac{x}{2}.\sin x$ or $\tan\frac{x}{2} - \frac{\sin x}{2}$, $S(x) = \frac{1}{2}x - \frac{1}{2}\sin x$, limit $= \frac{3}{2}$ **8.** g(x) = sinx and $\ell = e$

9. $\frac{\theta}{\tan\theta}$ **10.** $-\frac{1}{2e}$ **11.** a = –5/2, b = –3/2 **13.** $\frac{2L}{3}$

14. **(a)** 2, **(b)** D.N.E., **(c)** 0, **(d)** 0 **15.** **(a)** 2 ; **(b)** 1/2

EXERCISE (JM)

1. 1	**2.** 3	**3.** 1	**4.** 4	**5.** 1	**6.** 4	**7.** 3	**8.** 2	**9.** 1	**10.** 1
11. 4	**12.** 4	**13.** 3	**14.** 3	**15.** 2	**16.** 4	**17.** 4	**18.** 1		

EXERCISE (JA)

1. A,C **2.** D **3.** B **4.** B **5.** 0 **6.** 2 **7.** 7 **8.** A,C **9.** D

10. 2,4

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CONTINUITY

1. CONTINUOUS FUNCTIONS :

A function for which a small change in the independent variable causes only a small change and not a sudden jump in the dependent variable are called continuous functions. Naively, we may say that a function is continuous at a fixed point if we can draw the graph of the function around that point without lifting the pen from the plane of the paper.

A function $f(x)$ is said to be continuous at x = a, if $\lim\limits_{x\to a} f(x)$ exists and is equal to $f(a)$. Symbolically $f(x)$ is continuous at x = a if $\lim\limits_{h\to 0} f(a-h) = \lim\limits_{h\to 0} f(a+h) = f(a) =$ finite quantity.

i.e. $LHL|_{x=a} = RHL|_{x=a} =$ value of $f(x)|_{x=a} =$ finite quantity. $(h > 0)$

figure (1) figure (2) figure (3) figure (4) figure (5) figure (6)

In figure (1) and (2) $f(x)$ is continuous at x = a and x = 0 respectively and in figure (3) to (6) $f(x)$ is discontinuous at x = a.

Note 1 : Continuity of a function must be discussed only at points which are in the domain of the function.

Note 2 : If x = a is an isolated point of domain then $f(x)$ is always considered to be continuous at x = a.

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Illustration 1 : If $f(x) = \begin{cases} \sin\frac{\pi x}{2}, & x < 1 \\ [x] & x \ge 1 \end{cases}$ then find whether f(x) is continuous or not at x = 1, where [ ] denotes greatest integer function.

Solution : $f(x) = \begin{cases} \sin\frac{\pi x}{2}, & x < 1 \\ [x] \quad , & x \ge 1 \end{cases}$

For continuity at x = 1, we determine, f(1), $\lim\limits_{x\to 1^-} f(x)$ and $\lim\limits_{x\to 1^+} f(x)$.

Now, f(1) = [1] = 1

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sin\frac{\pi x}{2} = \sin\frac{\pi}{2} = 1 \text{ and } \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} [x] = 1$$

so $f(1) = \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x)$

$\therefore$ f(x) is continuous at x = 1

Illustration 2 : Let $f(x) = \begin{cases} \dfrac{a(1 - x\sin x) + b\cos x + 5}{x^2} & x < 0 \\ 3 & x = 0 \\ \left(1 + \left(\dfrac{cx + dx^3}{x^2}\right)\right)^{\frac{1}{x}} & x > 0 \end{cases}$

If f is continuous at x = 0, then find out the values of a, b, c and d.

Solution : Since f(x) is continuous at x = 0, so at x = 0, both left and right limits must exist and both must be equal to 3.

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Now $\lim\limits_{x\to 0^-} \dfrac{a(1-x\sin x)+b\cos x+5}{x^2} = \lim\limits_{x\to 0^-} \dfrac{(a+b+5)+\left(-a-\frac{b}{2}\right)x^2+\ldots}{x^2} = 3$

(By the expansions of sinx and cosx)

If $\lim\limits_{x\to 0^-} f(x)$ exists then $a + b + 5 = 0$ and $-a - \dfrac{b}{2} = 3 \Rightarrow a = -1$ and $b = -4$

since $\lim\limits_{x\to 0^+}\left(1+\left(\dfrac{cx+dx^3}{x^2}\right)\right)^{\frac{1}{x}}$ exists $\Rightarrow \lim\limits_{x\to 0^+}\dfrac{cx+dx^3}{x^2} = 0 \Rightarrow c = 0$

Now $\lim\limits_{x\to 0^+}(1+dx)^{\frac{1}{x}} = \lim\limits_{x\to 0^+}\left[(1+dx)^{\frac{1}{dx}}\right]^d = e^d$

So $e^d = 3 \Rightarrow d = \ell n\ 3$,

Hence $a = -1$, $b = -4$, $c = 0$ and $d = \ell n\ 3$.

Do yourself - 1 :

(i) If $f(x) = \begin{cases} \cos x; x \ge 0 \\ x+k;\ x<0 \end{cases}$ find the value of k if f (x) is continuous at x = 0.

(ii) If $f(x) = \begin{cases} \dfrac{|x+2|}{\tan^{-1}(x+2)} & ;\ x \ne -2 \\ 2 & ;\ x = -2 \end{cases}$ then discuss the continuity of f (x) at x=–2

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2. CONTINUITY OF THE FUNCTION IN AN INTERVAL :

(a) A function is said to be continuous in (a,b) if f is continuous at each & every point belonging to (a, b).

(b) A function is said to be continuous in a closed interval [a,b] if :

(i) f is continuous in the open interval (a,b)

(ii) f is right continuous at 'a' i.e. $\lim\limits_{x\to a^+} f(x) = f(a) =$ a finite quantity

(iii) f is left continuous at 'b' i.e. $\lim\limits_{x\to b^-} f(x) = f(b) =$ a finite quantity

Note :

(i) All polynomials, trigonometrical functions, exponential & logarithmic functions are continuous in their domains.

(ii) If f (x) & g(x) are two functions that are continuous at x = c then the function defined by :

$F_1(x) = f(x) \pm g(x); F_2(x) = K\,f(x)$, where K is any real number ; $F_3(x) = f(x).g(x)$ are also continuous at x = c.

Further, if g(c) is not zero, then $F_4(x) = \dfrac{f(x)}{g(x)}$ is also continuous at x = c.

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Illustration 3 : Discuss the continuity of $f(x) = \begin{cases} |x+1| & , \quad x < -2 \\ 2x+3 & , \quad -2 \le x < 0 \\ x^2+3 & , \quad 0 \le x < 3 \\ x^3-15 & , \quad x \ge 3 \end{cases}$

Solution : We write f(x) as $f(x) = \begin{cases} -x-1 & , \quad x < -2 \\ 2x+3 & , \quad -2 \le x < 0 \\ x^2+3 & , \quad 0 \le x < 3 \\ x^3-15 & , \quad x \ge 3 \end{cases}$

As we can see, f(x) is defined as a polynomial function in each of intervals $(-\infty, -2)$, $(-2, 0)$, $(0, 3)$ and $(3, \infty)$. Therefore, it is continuous in each of these four open intervals. Thus we check the continuity at $x = -2, 0, 3$.

At the point $x = -2$

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} (-x - 1) = +2 - 1 = 1$$

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (2x + 3) = 2.(-2) + 3 = -1$$

Therefore, $\lim_{x \to -2} f(x)$ does not exist and hence f(x) is discontinuous at $x = -2$.

At the point $x = 0$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x + 3) = 3$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x^2 + 3) = 3$$

$f(0) = 0^2 + 3 = 3$

Therefore f(x) is continuous at $x = 0$.

At the point $x = 3$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 + 3) = 3^2 + 3 = 12$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x^3 - 15) = 3^3 - 15 = 12$$

$f(3) = 3^3 - 15 = 12$

Therefore, f(x) is continuous at $x = 3$.

We find that f(x) is continuous at all points in $\mathbb{R}$ except at $x = -2$

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Do yourself -2 :

(i) If $f(x)=\begin{cases} \dfrac{x^2}{a} & ; \quad 0 \le x < 1 \\ -1 & ; \quad 1 \le x < \sqrt{2} \\ \dfrac{2b^2-4b}{x^2} & ; \quad \sqrt{2} \le x < \infty \end{cases}$ then find the value of a & b if $f(x)$ is continuous in $[0,\infty)$

(ii) Discuss the continuity of $f(x)=\begin{cases} |x-3| & ; \quad 0 \le x < 1 \\ \sin x & ; \quad 1 \le x \le \dfrac{\pi}{2} \\ \log_{\frac{\pi}{2}} x & ; \quad \dfrac{\pi}{2} < x < 3 \end{cases}$ in $[0,3)$

3. THE INTERMEDIATE VALUE THEOREM :

Suppose f(x) is continuous on an interval I, and a and b are any two points of I. Then if y_0 is a number between f(a) and f(b), there exists a number c between a and b such that $f(c) = y_0$

The function f, being continuous on [a,b] takes on every value between f(a) and f(b)

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Note that a function f which is continuous in [a,b] possesses the following properties :

(i) If f(a) & f(b) posses opposite signs, then there exists atleast one root of the equation $f(x) = 0$ in the open interval (a,b).

(ii) If K is any real number between f(a) & f(b), then there exists atleast one root of the equation $f(x) = K$ in the open interval (a,b).

Note : In above cases the number of roots is always odd.

Illustration 4 : Show that the function, $f(x) = (x-a)^2(x-b)^2 + x$, takes the value $\dfrac{a+b}{2}$ for some $x_0 \in (a, b)$

Solution : $f(x) = (x-a)^2(x-b)^2 + x$

$f(a) = a$

$f(b) = b$

& $\dfrac{a+b}{2} \in (f(a), f(b))$

$\therefore$ By intermediate value theorem, there is atleast one $x_0 \in (a, b)$ such that $f(x_0) = \dfrac{a+b}{2}$.

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Illustration 5 : Let $f : [0, 1] \xrightarrow{\text{onto}} [0, 1]$ be a continuous function, then prove that $f(x) = x$ for atleast one $x \in [0, 1]$

Solution : Consider $g(x) = f(x) - x$

$g(0) = f(0) - 0 = f(0) \geq 0$ $\{\because \quad 0 \leq f(x) \leq 1\}$

$g(1) = f(1) - 1 \leq 0$

$\Rightarrow$ $g(0) \,.\, g(1) \leq 0$

$\Rightarrow$ $g(x) = 0$ has atleast one root in $[0, 1]$

$\Rightarrow$ $f(x) = x$ for atleast one $x \in [0, 1]$

Do yourself -3 :

(i) If $f(x)$ is continuous in [a,b] such that $f(c) = \dfrac{2f(a)+3f(b)}{5}$, then prove that $c \in (a,b)$

4. SOME IMPORTANT POINTS :

(a) If f(x) is continuous & g(x) is discontinuous at x = a then the product function $\phi(x) = f(x).g(x)$ will **not necessarily be discontinuous at x = a**, e.g.

$$f(x) = x \;\&\; g(x) = \begin{cases} \sin\dfrac{\pi}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

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f(x) is continuous at x = 0 & g(x) is discontinuous at x = 0, but f(x).g(x) is continuous at x = 0.

(b) If f (x) and g (x) both are discontinuous at x = a then the product function $\phi(x) = f(x).g(x)$ **is not necessarily be discontinuous at x = a** , e.g.

$$f(x) = -g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

f(x) & g(x) both are discontinuous at x = 0 but the product function f.g(x) is still continuous at x = 0

(c) If f (x) and g (x) both are discontinuous at x = a then $f(x) \pm g(x)$ is not necessarily be discontinuous at x = a

(d) A continuous function whose domain is closed must have a range also in closed interval.

(e) If f is continuous at x = a & g is continuous at x = f (a) then the composite g[f(x)] is continuous at x = a. eg. $f(x) = \dfrac{x \sin x}{x^2+2}$ & $g(x) = |x|$ are continuous at x =0, hence the composite $(gof)(x) = \left|\dfrac{x \sin x}{x^2+2}\right|$ will also be continuous at x = 0

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Illustration 6: If $f(x) = \frac{x+1}{x-1}$ and $g(x) = \frac{1}{x-2}$, then discuss the continuity of f(x), g(x) and fog (x) in its domain.

Solution : $f(x) = \frac{x+1}{x-1}$

f(x) is a rational function it must be continuous in its domain and f is not defined at x = 1.

$g(x) = \frac{1}{x-2}$

g(x) is also a rational function. It must be continuous in its domain and g is not defined at x = 2.

Consider g(x) = 1

$\frac{1}{x-2} = 1 \quad \Rightarrow \quad x = 3$

$\therefore$ fog(x) is continuous in its domain : $\mathbb{R} - \{2, 3\}$

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Do yourself -4 :

(i) Let $f(x) = [x]$ & $g(x) = \text{sgn}(x)$ (where [.] denotes greatest integer function) , then discuss the continuity of $f(x) \pm g(x)$, $f(x).g(x)$ & $\frac{f(x)}{g(x)}$ at x= 0.

(ii) If $f(x) = \sin|x|$ & $g(x) = \tan|x|$ then discuss the continuity of $f(x) \pm g(x)$; $\frac{f(x)}{g(x)}$ & $f(x)\,g(x)$

5. CONTINUITY OVER COUNTABLE SET :

There are functions which are continuous over a countable set and else where discontinuous.

Illustration 7: If $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \notin \mathbb{Q} \end{cases}$, find the points where $f(x)$ is continuous

Solution : Let x = a be the point at which $f(x)$ is continuous.

$$\Rightarrow \lim_{\substack{x \to a \\ \text{through rational}}} f(x) = \lim_{\substack{x \to a \\ \text{through irrational}}} f(x)$$

$\Rightarrow$ a = –a

$\Rightarrow$ a = 0 $\Rightarrow$ function is continuous at x = 0.

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Do yourself -5 :

(i) If $g(x)=\begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$, then find the points where function is continuous.

(ii) If $f(x)=\begin{cases} x^2 & ; \ x \in \mathbb{Q} \\ 1-x^2 & ; \ x \notin \mathbb{Q} \end{cases}$, then find the points where function is continuous.

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ANSWERS FOR DO YOURSELF

1. **(i)** 1 **(ii)** discontinuous at $x = -2$

2. **(i)** $a=-1$ & $b=1$ **(ii)** Discontinuous at $x = 1$ & continuous at $x = \frac{\pi}{2}$

4. **(i)** All are discontinuous at $x = 0$.

(ii) $f(x)\,g(x)$ & $f(x) \pm g(x)$ are continuous in $\mathbb{R} - \{x : x = (2n+1)\frac{\pi}{2}; n \in \mathbb{Z}\}$

$\frac{f(x)}{g(x)}$ is continuous in $\mathbb{R} - \{x : x = \frac{n\pi}{2}; n \in \mathbb{Z}\}$

5. **(i)** $x = 0$ **(ii)** $x = \pm\frac{1}{\sqrt{2}}$

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EXERCISE (O-1)

[SINGLE CORRECT CHOICE TYPE]

1. Let $f(x)=\begin{cases} ax+1 & \text{if} \quad x<1 \\ 3 & \text{if} \quad x=1 \\ bx^2+1 & \text{if} \quad x>1 \end{cases}$. If $f(x)$ is continuous at x = 1 then (a – b) is equal to-

(A) 0 (B) 1 (C) 2 (D) 4

CY0001

2. For the function $f(x)=\begin{cases} \dfrac{1}{x+2^{\left(\frac{1}{x-2}\right)}}, x\neq 2 \\ k, \quad x=2 \end{cases}$ which of the following holds ?

(A) k = 1/2 and f is continuous at x = 2
(B) k ≠ 0, 1/2 and f is continuous at x = 2
(C) f can not be continuous at x = 2
(D) k = 0 and f is continuous at x = 2.

CY0002

3. The function $f(x)=\dfrac{4-x^2}{4x-x^3}$, is-

(A) discontinuous at only one point in its domain.
(B) discontinuous at two points in its domain.
(C) discontinuous at three points in its domain.
(D) continuous everywhere in its domain.

CY0003

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4. If $f(x)=\begin{cases} -4\sin x+\cos x & \text{for} \quad x\le -\dfrac{\pi}{2} \\ a\sin x+b & \text{for} \quad -\dfrac{\pi}{2}<x<\dfrac{\pi}{2} \\ \cos x+2 & \text{for} \quad x\ge \dfrac{\pi}{2} \end{cases}$ is continuous then :

(A) a = –1, b = 3
(B) a = 1, b = –3
(C) a = 1, b = 3
(D) a = –1, b = –3

CY0004

5. The function $f(x)=\begin{cases} \dfrac{1}{4}(3x^2+1) & -\infty<x\le 1 \\ 5-4x & 1<x<4 \\ 4-x & 4\le x<\infty \end{cases}$ is -

(A) continuous at x = 1 & x = 4
(B) continuous at x = 1, discontinuous at x= 4
(C) continuous at x = 4, discontinuous at x = 1
(D) discontinuous at x = 1 & x = 4

CY0005

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6. If $f(x)=\dfrac{x^2-bx+25}{x^2-7x+10}$ for $x \neq 5$ and f is continuous at $x = 5$, then $f(5)$ has the value equal to-

(A) 0 (B) 5 (C) 10 (D) 25

CY0006

7. If $f(x)=\dfrac{x-e^x+\cos 2x}{x^2}, x \neq 0$ is continuous at $x = 0$, then -

(A) $f(0)=\dfrac{5}{2}$ (B) $[f(0)] = -2$ (C) $\{f(0)\} = -0.5$ (D) $[f(0)].\{f(0)\} = -1.5$

where [.] and {.} denotes greatest integer and fractional part function

CY0007

8. $y = f(x)$ is a continuous function such that its graph passes through (a,0). Then $\lim\limits_{x\to a}\dfrac{\log_e(1+3f(x))}{2f(x)}$ is-

(A) 1 (B) 0 (C) $\dfrac{3}{2}$ (D) $\dfrac{2}{3}$

CY0008

9. In [1,3], the function $[x^2 + 1]$,[.] denoting the greatest integer function, is continuous -

(A) for all x

(B) for all x except at nine points

(C) for all x except at seven points

(D) for all x except at eight points

CY0009

10. Number of points of discontinuity of $f(x) = [2x^3 - 5]$ in [1,2), is equal to-

(where [x] denotes greatest integer less than or equal to x)

(A) 14 (B) 13 (C) 10 (D) 8

CY0010

11. Given $f(x)=\begin{cases}|x+1| & \text{if} \quad x<-2\\ 2x+3 & \text{if} \quad -2\le x<0\\ x^2+3 & \text{if} \quad 0\le x<3\\ x^3-15 & \text{if} \quad x\ge 3\end{cases}$. Then number of point(s) of discontinuity of $f(x)$ is-

(A) 0 (B) 1 (C) 2 (D) 3

CY0011

12. If $f(x)$ is continuous and $f\left(\dfrac{9}{2}\right)=\dfrac{2}{9}$, then the value of $\lim\limits_{x\to 0} f\left(\dfrac{1-\cos 3x}{x^2}\right)$ is-

(A) $\dfrac{2}{9}$ (B) $\dfrac{9}{2}$ (C) 0 (D) data insufficient

CY0012

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13. f is a continuous function on the real line. Given that $x^2 + (f(x) - 2)x - \sqrt{3} \cdot f(x) + 2\sqrt{3} - 3 = 0$. Then the value of $f(\sqrt{3})$

(A) can not be determined
(B) is $2(1 - \sqrt{3})$
(C) is zero
(D) is $\frac{2(\sqrt{3}-2)}{\sqrt{3}}$

CY0013

14. The function $f(x) = [x]^2 - [x^2]$ (where [y] is the greatest integer less than or equal to y), is discontinuous at :

(A) all integers
(B) all integers except 0 & 1
(C) all integers except 0
(D) all integers except 1

CY0014

EXERCISE (O-2)

[SINGLE CORRECT CHOICE TYPE]

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function $\forall\ x \in \mathbb{R}$ and $f(x) = 5\ \forall\ x \in$ irrational. Then the value of $f(3)$ is -

(A) 1
(B) 2
(C) 5
(D) cannot determine

CY0015

2. If $f(x) = \frac{1}{(x-1)(x-2)}$ and $g(x) = \frac{1}{x^2}$, then set of points in domain of $fog(x)$ at which $fog(x)$ is discontinuous.

(A) $\left\{-1, 0, 1, \frac{1}{\sqrt{2}}\right\}$
(B) ϕ
(C) $\{0, 1\}$
(D) $\left\{0, 1, \frac{1}{\sqrt{2}}\right\}$

CY0016

3. The function $f(x) = [x].\cos\frac{2x-1}{2}\pi$, where [•] denotes the greatest integer function, is discontinuous at :-

(A) all x
(B) all integer points
(C) no x
(D) x which is not an integer

CY0017

[MULTIPLE CORRECT CHOICE TYPE]

4. Which of the following function(s) is/are discontinuous at x = 0 ?

(A) $f(x) = \sin\frac{\pi}{2x}$, $x \neq 0$ and $f(0) = 1$
(B) $g(x) = x\sin\left(\frac{\pi}{x}\right)$, $x \neq 0$ and $g(0) = \pi$
(C) $h(x) = \frac{|x|}{x}$, $x \neq 0$ and $h(0) = 1$
(D) $k(x) = \frac{1}{1+e^{\cot x}}$, $x \neq 0$ and $k(0) = 0$.

CY0018

5. A function $f(x)$ is defined as $f(x) = \frac{A\sin x + \sin 2x}{x^3}, (x \neq 0)$. If the function is continuous at x = 0, then -

(A) A = –2
(B) $f(0) = -1$
(C) A = 1
(D) $f(0) = 1$

CY0019

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6. Which of the following function(s) can't be defined at x = 0 to make it continuous at x = 0 ?

(A) $f(x)=\dfrac{1}{1+2^{\frac{1}{x}}}$ (B) $f(x)=\tan^{-1}\dfrac{1}{x}$ (C) $f(x)=\dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$ (D) $f(x)=\dfrac{1}{\ell n|x|}$

CY0020

7. Which of the following function(s) can be defined continuously at x = 0 ?

(A) $f(x)=\dfrac{1}{1+2^{\cot x}}$ (B) $f(x)=\cos\left(\dfrac{|\sin x|}{x}\right)$

(C) $f(x)= x\sin\dfrac{\pi}{x}$ (D) $f(x)=\dfrac{1}{\ell n|x|}$

CY0021

8. Which of the following function(s) can be defined continuously at x = 0 ?

(A) $f(x)=\dfrac{1-\sec^2 2x}{4x^2}$ (B) $g(x)=\dfrac{\csc x-1}{x\csc x}$ (where csc x = cosec x)

(C) $h(x)=\dfrac{\sin 5x}{x}$ (D) $l(x)=(1+2x^2)^{\frac{1}{x^2}}$

CY0022

9. If f is defined on an interval [a, b]. Which of the following statement(s) is/are INCORRECT ?

(A) If f(a) and f(b), have opposite sign, then there must be a point $c \in (a, b)$ such that f(c) = 0.

(B) If f is continuous on [a, b], f(a) < 0 and f(b) > 0, then there must be a point $c \in (a, b)$ such that f(c)=0.

(C) If f is continuous on [a, b] and there is a point c in (a, b) such that f(c) = 0, then f(a) and f(b) have opposite sign.

(D) If f has no zeroes on [a, b], then f(a) and f(b) have the same sign.

CY0023

10. Which of the following functions can be defined at indicated point so that resulting function is continuous -

(A) $f(x)=\dfrac{x^2-2x-8}{x+2}$ at x = –2 (B) $f(x)=\dfrac{x-7}{|x-7|}$ at x = 7

(C) $f(x)=\dfrac{x^3+64}{x+4}$ at x = –4 (D) $f(x)=\dfrac{3-\sqrt{x}}{9-x}$ at x = 9

CY0024

11. In which of the following cases the given equations has atleast one root in the indicated interval ?

(A) x – cos x = 0 in (0, π/2)

(B) x + sinx = 1 in (0, π/6)

(C) $\dfrac{a}{x-1}+\dfrac{b}{x-3}=0$, a, b > 0 in (1, 3)

(D) f(x) – g(x) = 0 in (a, b) where f and g are continuous on [a, b] and f(a) > g(a) and f(b) < g(b).

CY0025

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[MATRIX TYPE]

12.

Column-I	Column-II
(A) $\lim\limits_{x\to 1}\dfrac{x^3-1}{\ell nx}$ is	(P) 2
(B) $\lim\limits_{x\to 0}\dfrac{x(\cos x-\cos 2x)}{2\sin x-\sin 2x}$ is	(Q) 3
(C) $\lim\limits_{x\to 0}\dfrac{\tan x\sqrt{\tan x}-\sin x\sqrt{\sin x}}{x^3.\sqrt{x}}$ is	(R) $\dfrac{3}{2}$
(D) If $f(x) = \cos(x\cos\frac{1}{x})$ and $g(x) = \dfrac{\ell n(\sec^2 x)}{x\sin x}$ are both continuous at x = 0 then $f(0) + g(0)$ equals	(S) $\dfrac{3}{4}$

CY0026

13. Match the function in Column-I with its behaviour at x = 0 in column-II, where [.] denotes greatest integer function & sgn(x) denotes signum function.

Column-I	Column-II
(A) $f(x) = [x][1 + x]$	(P) LHL exist at x = 0
(B) $f(x) = [-x][1 + x]$	(Q) RHL exist at x = 0
(C) $f(x) = (\text{sgn}(x))[2 - x][1 + \lvert x\rvert]$	(R) Continuous at x = 0
(D) $f(x) = [\cos x]$	(S) $\lim\limits_{x\to 0} f(x)$ exists but function is discontinuous at x = 0
	(T) $\lim\limits_{x\to 0} f(x)$ does not exist

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CY0027

EXERCISE (S-1)

1. If the function $f(x) = \dfrac{3x^2+ax+a+3}{x^2+x-2}$, $(x \neq -2)$ is continuous at x = – 2. Find f(–2).

CY0028

2. Find all possible values of a and b so that f(x) is continuous for all $x \in \mathbb{R}$ if

$$f(x) = \begin{cases} |ax+3| & \text{if } x \le -1 \\ |3x+a| & \text{if } -1 < x \le 0 \\ \dfrac{b\sin 2x}{x} - 2b & \text{if } 0 < x < \pi \\ \cos^2 x - 3 & \text{if } x \ge \pi \end{cases}$$

CY0029

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3. The function $f(x) = \begin{cases} \left(\frac{6}{5}\right)^{\frac{\tan 6x}{\tan 5x}} & \text{if } 0 < x < \frac{\pi}{2} \\ b+2 & \text{if } x = \frac{\pi}{2} \\ (1+|\cos x|)^{\left(\frac{a|\tan x|}{b}\right)} & \text{if } \frac{\pi}{2} < x < \pi \end{cases}$

Determine the values of 'a' & 'b', if f is continuous at $x = \pi/2$.

CY0030

4. Suppose that $f(x) = x^3 - 3x^2 - 4x + 12$ and $h(x) = \begin{cases} \frac{f(x)}{x-3} & , x \neq 3 \\ K & , x = 3 \end{cases}$ then

(a) find all zeroes of f(x).

(b) find the value of K that makes h continuous at x = 3.

(c) using the value of K found in (b), determine whether h is an even function.

CY0031

5. Let $f(x) = \begin{cases} \frac{1-\sin \pi x}{1+\cos 2\pi x}, & x < \frac{1}{2} \\ p, & x = \frac{1}{2} \\ \frac{\sqrt{2x-1}}{\sqrt{4+\sqrt{2x-1}}-2}, & x > \frac{1}{2} \end{cases}$. Determine the value of p, if possible, so that the function is continuous at x = 1/2.

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CY0032

6. Given the function $g(x) = \sqrt{6-2x}$ and $h(x) = 2x^2 - 3x + a$. Then

(a) evaluate h (g(2))

(b) If $f(x) = \begin{cases} g(x), & x \le 1 \\ h(x), & x > 1 \end{cases}$, find 'a' so that f is continuous.

CY0033

7. Let $f(x) = \begin{cases} 1+x, & 0 \le x \le 2 \\ 3-x, & 2 < x \le 3 \end{cases}$. Determine the form of $g(x) = f[f(x)]$ & hence find the point of discontinuity of g, if any.

CY0034

8. Let $f(x) = \begin{cases} \frac{\ell n \cos x}{\sqrt[4]{1+x^2}-1} & \text{if } x > 0 \\ \frac{e^{\sin 4x}-1}{\ell n(1+\tan 2x)} & \text{if } x < 0 \end{cases}$

Is it possible to define f(0) to make the function continuous at x = 0. If yes what is the value of f(0).

CY0035

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9. Determine a & b so that f is continuous at $x = \frac{\pi}{2}$ where $f(x) = \begin{cases} \frac{1-\sin^3 x}{3\cos^2 x} & \text{if } x < \frac{\pi}{2} \\ a & \text{if } x = \frac{\pi}{2} \\ \frac{b(1-\sin x)}{(\pi-2x)^2} & \text{if } x > \frac{\pi}{2} \end{cases}$

CY0036

10. Determine the values of a, b & c for which the function $f(x) = \begin{cases} \frac{\sin(a+1)x+\sin x}{x} & \text{for } x < 0 \\ c & \text{for } x = 0 \\ \frac{(x+bx^2)^{1/2}-x^{1/2}}{bx^{3/2}} & \text{for } x > 0 \end{cases}$ is continuous at x = 0.

CY0037

EXERCISE (S-2)

1. If $f(x) = \frac{\sin 3x + A\sin 2x + B\sin x}{x^5}$ $(x \neq 0)$ is cont. at x = 0. Find A & B. Also find f(0).

CY0038

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2. Let $f(x) = \begin{cases} \frac{\left(\frac{\pi}{2}-\sin^{-1}\left(1-\{x\}^2\right)\right)\sin^{-1}(1-\{x\})}{\sqrt{2}\left(\{x\}-\{x\}^3\right)} & \text{for } x \neq 0 \\ \frac{\pi}{2} & \text{for } x = 0 \end{cases}$ where {x} is the fractional part of x.

Consider another function g(x) ; such that

$g(x) = \begin{cases} f(x) & \text{for } x \geq 0 \\ 2\sqrt{2}f(x) & \text{for } x < 0 \end{cases}$

Discuss the continuity of the functions f(x) & g(x) at x = 0.

CY0039

3. If f(x) = x + {–x} + [x], where [x] is the integral part & {x} is the fractional part of x. Discuss the continuity of f in [–2, 2].

CY0040

4. Find the locus of (a, b) for which the function $f(x) = \begin{cases} ax-b & \text{for } x \leq 1 \\ 3x & \text{for } 1 < x < 2 \\ bx^2-a & \text{for } x \geq 2 \end{cases}$ is continuous at x = 1 but discontinuous at x = 2.

CY0041

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5. A function $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = \lim\limits_{n\to\infty} \dfrac{ax^2 + bx + c + e^{nx}}{1 + c\cdot e^{nx}}$ where f is continuous on $\mathbb{R}$. Find the values of a, b and c.

CY0042

6. Let $f(x) = \begin{cases} (\sin x + \cos x)^{\operatorname{cosec} x} & ; \quad \dfrac{-\pi}{2} < x < 0 \\ a & ; \quad x = 0 \\ \dfrac{e^{1/x} + e^{2/x} + e^{3/|x|}}{ae^{2/x} + be^{3/|x|}} & ; \quad 0 < x < \dfrac{\pi}{2} \end{cases}$

If f(x) is continuous at x = 0, find the value of $(a^2 + b^2)$.

CY0043

7. Given $f(x) = \sum\limits_{r=1}^{n} \tan\left(\dfrac{x}{2^r}\right)\sec\left(\dfrac{x}{2^{r-1}}\right)$; r, n $\in$ N

$$g(x) = \lim_{n\to\infty} \frac{\ell n\left(f(x) + \tan\dfrac{x}{2^n}\right) - \left(f(x) + \tan\dfrac{x}{2^n}\right)^n \cdot \left[\sin\left(\tan\dfrac{x}{2}\right)\right]}{1 + \left(f(x) + \tan\dfrac{x}{2^n}\right)^n}$$

$= k$ for $x = \dfrac{\pi}{4}$ and the domain of g(x) is $(0, \pi/2)$.

where [ ] denotes the greatest integer function.

Find the value of k, if possible, so that g(x) is continuous at $x = \pi/4$. Also state the points of discontinuity of g(x) in $(0, \pi/4)$, if any.

CY0044

8. Let $f(x) = x^3 - x^2 - 3x - 1$ and $h(x) = \dfrac{f(x)}{g(x)}$, where h is a rational function such that

(a) Domain of h(x) is $\mathbb{R} - \{-1\}$

(b) $\lim\limits_{x\to\infty} h(x) = \infty$ and (c) $\lim\limits_{x\to -1} h(x) = \dfrac{1}{2}$.

Find $\lim\limits_{x\to 0} (3h(x) + f(x) - 2g(x))$

CY0045

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9. (a) Let f be a real valued continuous function on $\mathbb{R}$ and satisfying $f(-x) - f(x) = 0 \ \forall \in \mathbb{R}$. If $f(-5) = 5$, $f(-2) = 4$, $f(3) = -2$ and $f(0) = 0$ then find the minimum number of zero's of the equation $f(x) = 0$.

CY0046

(b) Find the number of points of discontinuity of the function $f(x) = [5x] + \{3x\}$ in $[0, 5]$ where $[y]$ and $\{y\}$ denote largest integer less than or equal to y and fractional part of y respectively.

CY0047

10. (a) If $g : [a, b] \to [a, b]$ is continuous & onto function, then show that there is some $c \in [a, b]$ such that $g(c) = c$.

CY0048

(b) Let f be continuous on the interval $[0, 1]$ to $\mathbb{R}$ such that $f(0) = f(1)$. Prove that there exists a point c in $\left[0, \frac{1}{2}\right]$ such that $f(c) = f\left(c + \frac{1}{2}\right)$

CY0049

Paragraph 11 to 13 :

In case $\lim\limits_{x \to a} f(x)$ exists but is not equal to f(a) (where f(a) is defined) then the function is said to have a removable discontinuity or discontinuity of the first kind. In this case we can redefine the function such that $\lim\limits_{x \to a} f(x) = f(a)$ & make it continuous at $x = a$.

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Similarly,

In case $\lim\limits_{x \to a} f(x)$ does not exist and $f(a)$ is defined then it is not possible to make the function continuous by redefining it at $x = a$. Such a discontinuity is known as non-removable discontinuity or discontinuity of the 2nd kind.

11. Let $f(x) = \begin{cases} x - 1 & , \quad x < 0 \\ 1/4 & , \quad x = 0 \\ x^2 - 1 & , \quad x > 0 \end{cases}$. Discuss the continuity, and if discontinuous remove the discontinuity by redefining the function at $x = 0$ (if possible).

CY0050

12. Let $f(x) = \begin{cases} \sin\frac{\pi}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ discuss the continuity at $x = 0$ and if it is discontinuous then identify the type of discontinuity.

CY0050

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13. Let $f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1} & ; \text{ when } x \neq 0 \\ 0, & ; \text{ when } x = 0 \end{cases}$ discuss the continuity at $x = 0$ and if it is discontinuous then identify the type of discontinuity.

CY0050

EXERCISE (JM)

1. The function $f : \mathbb{R}/\{0\} \to \mathbb{R}$ given by $f(x) = \dfrac{1}{x} - \dfrac{2}{e^{2x}-1}$ can be made continuous at $x = 0$ by defining f(0) as-

[AIEEE 2007]

(1) 2 (2) –1 (3) 0 (4) 1

CY0051

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function defined by $f(x) = \dfrac{1}{e^x + 2e^{-x}}$

Statement–1 : $f(c) = \dfrac{1}{3}$, for some $c \in \mathbb{R}$.

Statement–2 : $0 < f(x) \leq \dfrac{1}{2\sqrt{2}}$, for all $x \in \mathbb{R}$. **[AIEEE-2010]**

(1) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.

(2) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for statement–1.

(3) Statement–1 is true, Statement–2 is false.

(4) Statement–1 is false, Statement–2 is true.

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CY0052

3. The values of p and q for which the function $f(x) = \begin{cases} \dfrac{\sin(p+1)x + \sin x}{x} & , \ x < 0 \\ q & , \ x = 0 \\ \dfrac{\sqrt{x+x^2} - \sqrt{x}}{x^{\frac{3}{2}}} & , \ x > 0 \end{cases}$ is continuous for all x in $\mathbb{R}$, are :-

[AIEEE 2011]

(1) $p = -\dfrac{3}{2}, q = \dfrac{1}{2}$ (2) $p = \dfrac{1}{2}, q = \dfrac{3}{2}$ (3) $p = \dfrac{1}{2}, q = -\dfrac{3}{2}$ (4) $p = \dfrac{5}{2}, q = \dfrac{1}{2}$

CY0053

4. Define F(x) as the product of two real functions $f_1(x) = x, x \in \mathbb{R}$,

and $f_2(x) = \begin{cases} \sin\dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ as follows : $F(x) = \begin{cases} f_1(x).f_2(x) & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ **[AIEEE 2011]**

Statement-1 : F(x) is continuous on $\mathbb{R}$.

Statement-2 : $f_1(x)$ and $f_2(x)$ are continuous on $\mathbb{R}$.

(1) Statemen-1 is false, statement-2 is true.

(2) Statemen-1 is true, statement-2 is true; Statement-2 is correct explanation for statement1.

(3) Statement-1 is true, statement-2 is true, statement-2 is not a correct explanation for statement1

(4) Statement-1 is true, statement-2 is false

CY0054

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5. If f(x) is continuous and f(9/2) = 2/9, then $\lim_{x\to 0} f\left(\frac{1-\cos 3x}{x^2}\right)$ is equal to: **[JEE Mains Offline-2014]**

(1) 9/2 (2) 0 (3) 2/9 (4) 8/9

CY0055

6. If the function $f(x)=\begin{cases}\frac{\sqrt{2+\cos x}-1}{(\pi-x)^2}, & x\neq\pi\\ k, & x=\pi\end{cases}$ is continuous at x = π, then k equals:-

[JEE Mains Offline-2014]

(1) $\frac{1}{4}$ (2) $\frac{1}{2}$ (3) 2 (4) 0

CY0056

7. Let f : R → R be a function defined as : f(x) = $\begin{cases}5, & \text{if } x\le 1\\ a+bx, & \text{if } 1<x<3\\ b+5x, & \text{if } 3\le x<5\\ 30, & \text{if } x\ge 5\end{cases}$. Then, f is :

[JEE(Main)-2019]

(1) continuous if a = 5 and b = 5
(2) continuous if a = –5 and b = 10
(3) continuous if a = 0 and b = 5
(4) not continuous for any values of a and b

CY0057

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8. Let f : [–1,3] → R be defined as $f(x)=\begin{cases}|x|+[x], & -1\le x<1\\ x+|x|, & 1\le x<2\\ x+[x], & 2\le x\le 3\end{cases}$,

where [t] denotes the greatest integer less than or equal to t. Then, f is discontinuous at:

[JEE(Main)-2019]

(1) four or more points
(2) only one point
(3) only two points
(4) only three points

CY0058

9. If the function f defined on $\left(\frac{\pi}{6},\frac{\pi}{3}\right)$ by $f(x)=\begin{cases}\frac{\sqrt{2}\cos x-1}{\cot x-1}, & x\neq\frac{\pi}{4}\\ k, & x=\frac{\pi}{4}\end{cases}$ is continuous, then k is equal to

[JEE(Main)-2019]

(1) $\frac{1}{2}$ (2) 1 (3) $\frac{1}{\sqrt{2}}$ (4) 2

CY0059

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10. If $f(x)=[x]-\left[\frac{x}{4}\right], x\in R$, where [x] denotes the greatest integer function, then :

[JEE(Main)-2019]

(1) Both $\lim_{x\to 4-} f(x)$ and $\lim_{x\to 4+} f(x)$ exist but are not equal

(2) $\lim_{x\to 4-} f(x)$ exists but $\lim_{x\to 4+} f(x)$ does not exist

(3) $\lim_{x\to 4+} f(x)$ exists but $\lim_{x\to 4-} f(x)$ does not exist

(4) f is continuous at x = 4

CY0060

11. If the function $f(x)=\begin{cases} a|\pi-x|+1, x\le 5 \\ b|x-\pi|+3, x>5 \end{cases}$ is continuous at x = 5, then the value of a – b is :-

[JEE(Main)-2019]

(1) $\frac{2}{5-\pi}$ (2) $\frac{2}{\pi-5}$ (3) $\frac{2}{\pi+5}$ (4) $\frac{-2}{\pi+5}$

CY0061

12. If $f(x)=\begin{cases} \frac{\sin(p+1)+\sin x}{x}, & x<0 \\ q, & x=0 \\ \frac{\sqrt{x+x^2}-\sqrt{x}}{x^{3/2}}, & x>0 \end{cases}$ is continuous at x = 0, then the ordered pair (p,q) is equal to :

[JEE(Main)-2019]

(1) $\left(\frac{5}{2},\frac{1}{2}\right)$ (2) $\left(-\frac{3}{2},-\frac{1}{2}\right)$ (3) $\left(-\frac{1}{2},\frac{3}{2}\right)$ (4) $\left(-\frac{3}{2},\frac{1}{2}\right)$

CY0062

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EXERCISE (JA)

1. Discuss the continuity of the function $f(x)=\begin{cases} \frac{e^{1/(x-1)}-2}{e^{1/(x-1)}+2}, & x\ne 1 \\ 1, & x=1 \end{cases}$

at x = 1. **[REE 2001 (Mains), 3 out of 100]**

CY0063

2. For every integer n, let a_n and b_n be real numbers. Let function $f:\mathbb{R}\to\mathbb{R}$ be given by

$f(x)=\begin{cases} a_n+\sin\pi x, & \text{for } x\in[2n,2n+1] \\ b_n+\cos\pi x, & \text{for } x\in(2n-1,2n) \end{cases}$, for all integers n.

If f is continuous, then which of the following holds(s) for all n ? **[JEE 2012, 4]**

(A) $a_{n-1}-b_{n-1}=0$ (B) $a_n-b_n=1$ (C) $a_n-b_{n+1}=1$ (D) $a_{n-1}-b_n=-1$

CY0064

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3. For every pair of continuous function $f, g : [0, 1] \to \mathbb{R}$ such that

$$\max\{f(x) : x \in [0, 1]\} = \max\{g(x) : x \in [0,1]\},$$

the correct statement(s) is(are) :

(A) $(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0,1]$

(B) $(f(c))^2 + f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0,1]$

(C) $(f(c))^2 + 3f(c) = (g(c))^2 + g(c)$ for some $c \in [0,1]$

(D) $(f(c))^2 = (g(c))^2$ for some $c \in [0,1]$ **[JEE(Advanced)-2014, 3]**

CY0065

4. Let [x] be the greatest integer less than or equal to x. Then, at which of the following point(s) the function $f(x) = x\cos(\pi(x + [x]))$ is discontinuous ? **[JEE(Advanced)-2017, 4(–2)]**

(A) $x = -1$ (B) $x = 0$ (C) $x = 2$ (D) $x = 1$

CY0066

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ANSWER KEY

CONTINUITY

EXERCISE (O-1)

1. A	**2.** C	**3.** D	**4.** A	**5.** B	**6.** A	**7.** D
8. C	**9.** D	**10.** B	**11.** B	**12.** A	**13.** B	**14.** D

EXERCISE (O-2)

1. C **2.** B **3.** C **4.** A,B,C,D **5.** A,B **6.** A,B,C **7.** B,C,D

8. A,C,D **9.** A, C, D **10.** A, C, D **11.** A, B, C, D

12. (A)→(Q); (B)→(R); (C)→(S); (D)→(P)

13. (A)→(P,Q,R); (B)→(P,Q,T); (C)→(P,Q,T); (D)→(P,Q,S)

EXERCISE (S-1)

1. –1 **2.** a = 0, b = 1 **3.** a = 0 ; b = –1 **4.** (a) –2, 2, 3; (b) K = 5; (c) even

5. P not possible. **6.** (a) $4-3\sqrt{2}$ +a, (b) a = 3

7. g(x) = 2 + x for $0 \le x \le 1$, 2 – x for $1 < x \le 2$, 4 – x for $2 < x \le 3$, g is discontinuous at x = 1 & x = 2

8. $f(0^+) = -2$; $f(0^-) = 2$ hence f(0) not possible to define **9.** a = 1/2, b = 4

10. a = –3/2, b ≠ 0, c = 1/2

EXERCISE (S-2)

1. A = –4, B = 5, f(0) = 1

2. $f(0^+) = \frac{\pi}{2}$; $f(0^-) = \frac{\pi}{4\sqrt{2}}$ ⇒ f is discont. at x = 0 ; $g(0^+) = g(0^-) = g(0) = \pi/2$ ⇒ g is cont. at x = 0

3. discontinuous at all integral values in [–2, 2]

4. locus (a, b) → x, y is y = x – 3 excluding the points where y = 3 intersects it.

5. c = 1, a, b ∈ ℝ **6.** $e^2 + e^{-2}$

7. k = 0 ; $g(x) = \begin{cases} \ell n(\tan x) & \text{if } 0 < x < \frac{\pi}{4} \\ 0 & \text{if } \frac{\pi}{4} \le x < \frac{\pi}{2} \end{cases}$. Hence g(x) is continuous everywhere.

8. g(x) = 4 (x + 1) and limit $= -\frac{39}{4}$ **9.** **(a)** 5 **(b)** 30

11. f(x) has removable discontinuity and f(x) could be made continuous by taking f(0) = – 1

12. f(x) has non removable type discontinuity at x = 0

13. f(x) has non removable type discontinuity at x = 0

EXERCISE (JM)

1. 4	**2.** 1	**3.** 1	**4.** 4	**5.** 3	**6.** 1	**7.** 4
8. 4	**9.** 1	**10.** 4	**11.** 1	**12.** 4		

EXERCISE (JA)

1. Discontinuous at x = 1 ; $f(1^+) = 1$ and $f(1^-) = -1$ **2.** B,D **3.** A,D **4.** A,C,D

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DIFFERENTIABILITY

1. MEANING OF DERIVATIVE :

The instantaneous rate of change of a function with respect to the dependent variable is called derivative. Let 'f' be a given function of one variable and let Δx denote a number (positive or negative) to be added to the number x. Let Δf denote the corresponding change of 'f' then $\Delta f = f(x+\Delta x) - f(x)$

$$\Rightarrow \frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

If $\Delta f/\Delta x$ approaches a limit as Δx approaches zero, this limit is the derivative of 'f' at the point x. The derivative of a function 'f' is a function ; this function is denoted by symbols such as

$$f'(x),\ \frac{df}{dx}, \frac{d}{dx}f(x) \text{ or } \frac{df(x)}{dx}$$

$$\Rightarrow \quad \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

The derivative evaluated at a point a, is written, $f'(a),\ \left.\frac{df(x)}{dx}\right|_{x=a}, f'(x)_{x=a}$, etc.

2. EXISTENCE OF DERIVATIVE AT x = a :

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(a) Right hand derivative :

The right hand derivative of f(x) at x = a denoted by Rf'(a) is defined as :

$$Rf'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}, \text{ provided the limit exists \& is finite. } (h > 0)$$

(b) Left hand derivative :

The left hand derivative of f(x) at x = a denoted by Lf'(a) is defined as :

$$Lf'(a) = \lim_{h \to 0} \frac{f(a-h)-f(a)}{-h}, \text{ provided the limit exists \& is finite. } (h > 0)$$

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Hence f(x) is said to be **derivable or differentiable at x = a.** $\boxed{\text{If } Rf'(a) = Lf'(a) = \text{finite quantity}}$

and it is denoted by f'(a); where $f'(a) = Lf'(a) = Rf'(a)$ & it is called derivative or differential coefficient of f(x) at x = a.

3. DIFFERENTIABILITY & CONTINUITY :

Theorem : If a function f(x) is derivable at x = a, then f(x) is continuous at x = a.

Proof : $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ exists.

Also $f(a+h)-f(a) = \dfrac{f(a+h)-f(a)}{h}.h \quad [h \neq 0]$

$\therefore \quad \lim\limits_{h\to 0} [f(a+h)-f(a)] = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}.h = f'(a).0 = 0$

$\Rightarrow \quad \lim\limits_{h\to 0}[f(a+h)-f(a)] = 0 \Rightarrow \lim\limits_{h\to 0} f(a+h) = f(a) \Rightarrow f(x)$ is continuous at x = a.

Note :

(i) $\boxed{\text{Differentiable} \Rightarrow \text{Continuous ; Continuity} \not\Rightarrow \text{Differentiable ; Not Differentiable} \not\Rightarrow \text{Not Continuous But Not Continuous} \Rightarrow \text{Not Differentiable}}$

(ii) All polynomial, rational, trigonometric, logarithmic and exponential function are continuous and differentiable in their domains.

(iii) If f(x) & g(x) are differentiable at x = a, then the functions f(x) + g(x), f(x) – g(x), f(x). g(x) will also be differentiable at x = a & if $g(a) \neq 0$ then the function f(x)/g(x) will also be differentiable at x = a.

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Illustration 1 : Let $f(x) = \begin{cases} \text{sgn}(x) + x; & -\infty < x < 0 \\ -1 + \sin x; & 0 \le x < \dfrac{\pi}{2} \\ \cos x; & \dfrac{\pi}{2} \le x < \infty \end{cases}$. Discuss the continuity & differentiability at $x = 0$ & $\dfrac{\pi}{2}$.

Solution : $f(x) = \begin{cases} -1 + x; & -\infty < x < 0 \\ -1 + \sin x; & 0 \le x < \dfrac{\pi}{2} \\ \cos x; & \dfrac{\pi}{2} \le x < \infty \end{cases}$

To check the differentiability at x = 0

$$\text{LHD} = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{-1+0-h-(-1)}{-h} = 1$$

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$$\text{RHD} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{-1+\sinh+1}{h} = 1$$

$\because$ LHD = RHD

$\therefore$ Differentiable at x = 0.

$\Rightarrow$ Continuous at x = 0.

To check the continuity at $x = \frac{\pi}{2}$

LHL $\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^-}(-1+\sin x) = 0$

RHL $\lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{x\to\frac{\pi}{2}^+}\cos x = 0$

$\because$ LHL = RHL = $f\left(\frac{\pi}{2}\right) = 0$

$\therefore$ Continuous at $x = \frac{\pi}{2}$.

To check the differentiability at $x = \frac{\pi}{2}$

$$\text{LHD} = \lim_{h\to 0}\frac{f\left(\frac{\pi}{2}-h\right)-f\left(\frac{\pi}{2}\right)}{-h} = \lim_{h\to 0}\frac{-1+\cosh-0}{-h} = 0$$

$$\text{RHD} = \lim_{h\to 0}\frac{f\left(\frac{\pi}{2}+h\right)-f\left(\frac{\pi}{2}\right)}{h} = \lim_{h\to 0}\frac{-\sinh-0}{h} = -1$$

$\because$ LHD $\neq$ RHD

$\therefore$ not differentiable at $x = \frac{\pi}{2}$.

Illustration 2 : If $f(x) = \begin{cases} A+Bx^2 & x<1 \\ 3Ax-B+2 & x\geq 1\end{cases}$

then find A and B so that f(x) become differentiable at x = 1.

Solution : $Rf'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{3A(1+h)-B+2-3A+B-2}{h} = \lim_{h\to 0}\frac{3Ah}{h} = 3A$

$$Lf'(1) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{A+B(1-h)^2-3A+B-2}{-h} = \lim_{h\to 0}\frac{(-2A+2B-2)+Bh^2-2Bh}{-h}$$

hence for this limit to be defined

$-2A + 2B - 2 = 0$

$B = A + 1$

$Lf'(1) = \lim_{h\to 0} -(Bh - 2B) = 2B$

$\therefore$ $Lf'(1) = Rf'(1)$

$3A = 2B = 2(A + 1)$

$A = 2,\ B = 3$

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Illustration 3 : $f(x)=\begin{cases}[\cos \pi x] & x \le 1 \\ 2\{x\}-1 & x > 1\end{cases}$ comment on the derivability at x =1, where [ ] denotes greatest integer function & { } denotes fractional part function.

Solution :

$$Lf'(1) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{[\cos(\pi-\pi h)]+1}{-h} = \lim_{h\to 0}\frac{-1+1}{-h} = 0$$

$$Rf'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{2\{1+h\}-1+1}{h} = \lim_{h\to 0}\frac{2h}{h} = 2$$

Hence f(x) is not differentiable at x = 1.

Do yourself -1 :

(i) A function is defined as follows :

$f(x)=\begin{cases}x^3 \ ; & x^2 < 1 \\ x \ ; & x^2 \ge 1\end{cases}$ discuss the continuity and differentiability at x = 1.

(ii) If $f(x)=\begin{cases}ax^3+b, & \text{for } 0 \le x \le 1 \\ 2\cos \pi x + \tan^{-1} x, & \text{for } 1 < x \le 2\end{cases}$ be the differentiable function in [0, 2], then find a and b. (where [.] denotes the greatest integer function)

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4. IMPORTANT NOTE :

(a) Let $Rf'(a) = p$ & $Lf'(a) = q$ where p & q are finite then :

(i) $p = q \Rightarrow$ f is differentiable at x = a $\Rightarrow$ f is continuous at x = a

(ii) $p \ne q \Rightarrow$ f is not differentiable at x = a, but f is continuous at x = a.

Illustration 4 : Determine the values of x for which the following functions fails to be continuous or differentiable $f(x)=\begin{cases}(1-x), & x < 1 \\ (1-x)(2-x), & 1 \le x \le 2 \\ (3-x), & x > 2\end{cases}$, Justify your answer.

Solution : By the given definition it is clear that the function f is continuous and differentiable at all points except possibily at x = 1 and x = 2.

Check the differentiability at x = 1

$$q = LHD = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{1-(1-h)-0}{-h} = -1$$

$$p = RHD = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{\{1-(1+h)\}\{2-(1+h)\}-0}{h} = -1$$

$\because$ q = p $\therefore$ Differentiable at x = 1. $\Rightarrow$ Continuous at x = 1.

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Check the differentiability at $x = 2$

$$q = \text{LHD} = \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h} = \lim_{h\to 0}\frac{(1-2+h)(2-2+h)-0}{-h} = 1 = \text{finite}$$

$$p = \text{RHD} = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{(3-2-h)-0}{h} \to \infty \quad \text{(not finite)}$$

$\because \quad q \neq p \qquad \therefore \quad$ not differentiable at $x = 2$.

Now we have to check the continuity at $x = 2$

$$\text{LHL} = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(1-x)(2-x) = \lim_{h\to 0}(1-(2-h))(2-(2-h)) = 0$$

$$\text{RHL} = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(3-x) = \lim_{h\to 0}(3-(2+h)) = 1$$

$\therefore \quad \text{LHL} \neq \text{RHL}$

$\Rightarrow \quad$ not continuous at $x = 2$.

Do yourself - 2 :

(i) Let $f(x) = (x-1)|x-1|$. Discuss the continuity and differentiability of $f(x)$ at $x = 1$.

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(b) Vertical tangent :

(i) If $y = f(x)$ is continuous at $x = a$ and $\lim_{x\to a}|f'(x)|$ approaches to ∞, then $y = f(x)$ has a vertical tangent at $x = a$. If a function has vertical tangent at $x = a$ then it is non differentiable at $x = a$.

e.g. (1) $f(x) = x^{1/3}$ has vertical tangent at $x = 0$

since $f'_+(0) \to \infty$ and $f'_-(0) \to \infty$ hence $f(x)$ is not differentiable at $x = 0$

(2) $g(x) = x^{2/3}$ have vertical tangent at $x = 0$

since $g'_+(0) \to \infty$ and $g'_-(0) \to -\infty$ hence $g(x)$ is not differentiable at $x = 0$.

(c) Geometrical interpretation of differentiability :

(i) If the function $y = f(x)$ is differentiable at $x = a$, then a unique non vertical tangent can be drawn to the curve $y = f(x)$ at the point $P(a, f(a))$ & $f'(a)$ represent the slope of the tangent at point P.

(ii) If a function $f(x)$ does not have a unique tangent (p & q are finite but unequal), then f is continuous at $x = a$, it geometrically implies a corner at $x = a$.

e.g. $f(x) = |x|$ is continuous but not differentiable at $x = 0$ & there is corner at $x = 0$.

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(does not have unique tangent, corner at x = 0) $\begin{cases} p = 1 \\ q = -1 \end{cases}$

(iii) If one of p & q tends to ∞ and other tends to $-\infty$, then their will be a cusp at x = a. Where p = $Rf'(a)$ and q = $Lf'(a)$

e.g. (1) $f(x) = |x|^{1/3}$ is continuous but not differentiable at x = 0 & there is cusp at x = 0.

(has a vertical tangent, cusp at x = 0) $\begin{cases} p \to +\infty \\ q \to -\infty \end{cases}$

(2) $f(x) = x^{1/3}$ is continuous but not differentiable at x = 0 because $Rf'(0) \to \infty$ and $Lf'(0) \to \infty$.

$\begin{cases} p \to +\infty \\ q \to +\infty \end{cases}$

(have a unique vertical tangent but does not have corner)

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Note : $\boxed{\text{corner/cusp/vertical tangent} \Rightarrow \text{non differentiable}}$

$\boxed{\text{non differentiable} \not\Rightarrow \text{corner/cusp/vertical tangent}}$

Illustration 5 : If $f(x) = \begin{cases} x-3 & x<0 \\ x^2-3x+2 & x \ge 0 \end{cases}$. Draw the graph of the function & discuss the continuity and differentiability of $f(|x|)$ and $|f(x)|$.

Solution : $f(|x|) = \begin{cases} |x|-3; & |x|<0 \to \text{not possible} \\ |x|^2-3|x|+2; & |x| \ge 0 \end{cases}$

$f(|x|) = \begin{cases} x^2+3x+2, & x<0 \\ x^2-3x+2, & x \ge 0 \end{cases}$

at x = 0

$$q = LHD = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{h^2-3h+2-2}{-h} = 3$$

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$$p = RHD = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2-3h+2-2}{h} = -3$$

$\because$ $q \neq p$

$\therefore$ not differentiable at x = 0. but p & q are both are finite

$\Rightarrow$ continuous at x = 0

$$\text{Now, } |f(x)| = \begin{cases} 3-x & , \quad x<0 \\ (x^2-3x+2), & 0\le x<1 \\ -(x^2-3x+2) & , \quad 1\le x\le 2 \\ (x^2-3x+2), & x>2 \end{cases}$$

To check differentiability at x = 0

$$q = LHD = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{3+h-2}{-h} = \lim_{h\to 0}\frac{(1+h)}{-h} \to -\infty$$

$$p = RHD = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2-3h+2-2}{h} = -3$$

$\Rightarrow$ not differentiable at x = 0.

Now to check continuity at x = 0

$$LHL = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 3-x = 3$$

$$RHL = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2-3x+2 = 2$$

$\Rightarrow$ not continuous at x = 0.

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To check differentiability at x = 1

$$q = LHD = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0}\frac{(1-h)^2-3(1-h)+2-0}{-h} = \lim_{h\to 0}\frac{h^2+h}{-h} = -1$$

$$p = RHD = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{-(h^2+2h+1-3+3h+2)-0}{h} = \lim_{h\to 0}\frac{-(h^2-h)}{h} = 1$$

$\Rightarrow$ not differentiable at x = 1.

but $|f(x)|$ is continous at x = 1, because $p \neq q$ and both are finite.

To check differentiability at x = 2

$$q = LHD = \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$$

$$= \lim_{h\to 0}\frac{-(4+h^2-4h-6+3h+2)-0}{-h} = \lim_{h\to 0}\frac{h^2-h}{h} = -1$$

$$p = RHD = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{(h^2+4h+4-6-3h+2)-0}{h} = \lim_{h\to 0}\frac{(h^2+h)}{h} = 1$$

$\Rightarrow$ not differentiable at x = 2.

but $|f(x)|$ is continous at x = 2, because $p \neq q$ and both are finite.

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Do yourself - 3 :

(i) Let $f(x)=\begin{cases}-4 & ;\ -4<x<0\\ x^2-4; & 0\le x<4\end{cases}$

Discuss the continuity and differentiablity of $g(x)=|f(x)|$.

(ii) Let $f(x)=\min\{|x-1|, |x+1|, 1\}$. Find the number of points where it is not differentiable.

5. DIFFERENTIABILITY OVER AN INTERVAL :

(a) f(x) is said to be differentiable over an open interval (a, b) if it is differentiable at each & every point of the open interval (a, b).

(b) f(x) is said to be differentiable over the closed interval [a, b] if :

(i) f(x) is differentiable in (a, b) &

(ii) for the points a and b, $f'(a^+)$ & $f'(b^-)$ exist.

Illustration 6 : If $f(x)=\begin{cases}e^{-|x|}, & -5<x<0\\ -e^{-|x-1|}+e^{-1}+1, & 0\le x<2\\ e^{-|x-2|}, & 2\le x<4\end{cases}$

Discuss the continuity and differentiability of f(x) in the interval (–5, 4).

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Solution : $f(x)=\begin{cases}e^{+x} & -5<x<0\\ -e^{x-1}+e^{-1}+1 & 0\le x\le 1\\ -e^{-x+1}+e^{-1}+1 & 1<x<2\\ e^{-x+2} & 2\le x<4\end{cases}$

Check the differentiability at x = 0

$$\text{LHD}=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{e^{-h}-1}{-h}=1$$

$$\text{RHD}=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{-e^{h-1}+e^{-1}+1-1}{h}=-e^{-1}$$

$\because$ LHD $\neq$ RHD

$\therefore$ Not differentiable at x = 0, but continuous at x = 0 since LHD and RHD both are finite.

Check the differentiability at x = 1

$$\text{LHD}=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}=\lim_{h\to 0}\frac{-e^{1-h-1}+e^{-1}+1-e^{-1}}{-h}=-1$$

$$\text{RHD}=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{-e^{1-h-1}+e^{-1}+1-e^{-1}}{h}=1$$

$\because$ LHD $\neq$ RHD

$\therefore$ Not differentiable at x = 1, but continuous at x = 1 since LHD and RHD both are finite.

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Check the differentiability at x = 2

$$\text{LHD} = \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h} = \lim_{h\to 0}\frac{-e^{-2+h+1}+e^{-1}+1-1}{-h} = \lim_{h\to 0}\frac{-e^{-1}(e^{h}-1)}{-h} = e^{-1}$$

$$\text{RHD} = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{e^{-h}-1}{h} = -1$$

$\because$ LHD $\neq$ RHD

$\therefore$ Not differentiable at x = 2, but continuous at x = 2 since LHD & RHD both are finite.

Note :

(i) If f(x) is differentiable at x = a & g(x) is not differentiable at x =a, then the product function F(x)=f(x).g(x) can still be differentiable at x = a.

e.g. Consider f(x) = x & g(x) = |x|. f is differentiable at x = 0 & g is non-differentiable at x = 0, but f(x).g(x) is still differentiable at x = 0.

(ii) If f(x) & g(x) both are not differentiable at x = a then the product function; F(x)=f(x).g(x) can still be differentiable at x = a.

e.g. Consider f(x) = |x| & g(x) = –|x|. f & g are both non differentiable at x = 0, but f(x).g(x) still differentiable at x = 0.

(iii) If f(x) & g(x) both are non-differentiable at x=a then the sum function F(x)=f(x)+g(x) may be a differentiable function.

e.g. f(x)=|x| & g(x)= –|x|. f & g are both non differentiable at x = 0, but (f+g)(x) still differentiable at x = 0.

(iv) If f(x) is differentiable at x = a $\not\Rightarrow$ f'(x) is continuous at x = a.

$$\text{e.g. } f(x)=\begin{cases} x^2\sin\dfrac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x = 0\end{cases}$$

Do yourself - 4 :

(i) Let f(x) = max {sinx, 1/2}, where $0 \le x \le \frac{5\pi}{2}$. Find the number of points where it is not differentiable.

(ii) Let $f(x)=\begin{cases}[x] & ;\ 0<x\le 2\\ 2x-2 & ;2<x<3\end{cases}$, where [.] denotes greatest integer function.

(a) Find that points at which continuity and differentiability should be checked.

(b) Discuss the continuity & differentiability of f(x) in the interval (0, 3).

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6. DETERMINATION OF FUNCTION WHICH SATISFYING THE GIVEN FUNCTIONAL EQUATION :

Illustration 7 : Let $f(x+y) = f(x) + f(y) - 2xy - 1$ for all x and y. If $f'(0)$ exists and $f'(0) = -\sin\alpha$, then find $f\{f'(0)\}$.

Solution :

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\{f(x) + f(h) - 2xh - 1\} - f(x)}{h} \qquad \text{(Using the given relation)}$$

$$= \lim_{h\to 0} -2x + \lim_{h\to 0} \frac{f(h) - 1}{h} = -2x + \lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

[Putting $x = 0 = y$ in the given relation we find $f(0) = f(0) + f(0) + 0 - 1 \Rightarrow f(0) = 1$]

$\therefore \quad f'(x) = -2x + f'(0) = -2x - \sin\alpha$

$\Rightarrow \quad f(x) = -x^2 - (\sin\alpha).\, x + c$

$f(0) = -0 - 0 + c \Rightarrow \quad c = 1$

$\therefore \quad f(x) = -x^2 - (\sin\alpha).\, x + 1$

So, $\quad f\{f'(0)\} = f(-\sin\alpha) = -\sin^2\alpha + \sin^2\alpha + 1$

$\therefore \quad f\{f'(0)\} = 1.$

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Do yourself - 5 :

(i) A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation $f(x+y) = f(x).f(y)$ for all $x, y \in \mathbb{R}$, $f(x) \neq 0$. Suppose that the function is differentiable everywhere and $f'(0) = 2$. Prove that $f'(x) = 2f(x)$.

<u>Miscellaneous Illustrations</u> :

Illustration 8 : Discuss the continuity and differentiability of the function $y = f(x)$ defined parametrically ; $x = 2t - |t-1|$ and $y = 2t^2 + t|t|$.

Solution : Here $x = 2t - |t-1|$ and $y = 2t^2 + t|t|$.

Now when $t < 0$;

$$x = 2t - \{-(t-1)\} = 3t - 1 \quad \text{and} \quad y = 2t^2 - t^2 = t^2 \quad \Rightarrow \quad y = \frac{1}{9}(x+1)^2$$

when $0 \le t < 1$

$$x = 2t - (-(t-1)) = 3t - 1 \quad \text{and} \quad y = 2t^2 + t^2 = 3t^2 \quad \Rightarrow \quad y = \frac{1}{3}(x+1)^2$$

when $t \ge 1$;

$$x = 2t - (t-1) = t + 1 \quad \text{and} \quad y = 2t^2 + t^2 = 3t^2 \quad \Rightarrow \quad y = 3(x-1)^2$$

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Thus, $y = f(x) = \begin{cases} \frac{1}{9}(x+1)^2, & x < -1 \\ \frac{1}{3}(x+1)^2, & -1 \le x < 2 \\ 3(x-1)^2, & x \ge 2 \end{cases}$

We have to check differentiability at $x = -1$ and 2.

Differentiability at $x = -1$;

$$\text{LHD} = f'_-(-1) = \lim_{h\to 0} \frac{f(-1-h)-f(-1)}{-h} = \lim_{h\to 0} \frac{\frac{1}{9}(-1-h+1)^2 - 0}{-h} = 0$$

$$\text{RHD} = f'_+(-1) = \lim_{h\to 0} \frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0} \frac{\frac{1}{3}(-1+h+1)^2 - 0}{-h} = 0$$

Hence f(x) is differentiable at $x = -1$.

$\Rightarrow$ continuous at $x = -1$.

To check differentiability at $x = 2$;

$$\text{LHD} = f'_-(2) = \lim_{h\to 0} \frac{\frac{1}{3}(2-h+1)^2 - 3}{-h} = 2 \ \& \ \text{RHD} = f'_+(2) = \lim_{h\to 0} \frac{3(2+h-1)^2 - 3}{h} = 6$$

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Hence f(x) is not differentiable at $x = 2$.

But continuous at $x = 2$, because LHD & RHD both are finite.

$\therefore$ f(x) is continuous for all x and differentiable for all x, except $x = 2$.

ANSWERS FOR DO YOURSELF

1: **(i)** Continuous but not differentiable at x = 1 **(ii)** $a = \frac{1}{6}, b = \frac{\pi}{4} - \frac{13}{6}$

2: **(i)** Continuous & differentiable at x = 1

3: **(i)** Continuous everywhere but not differentiable at x = 2 only **(ii)** 5

4: **(i)** 3 **(ii)** **(a)** 1 & 2 **(b)** Not continuous at x = 1 & 2 and not differentiable at x = 1 & 2.

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EXERCISE (O-1)

[SINGLE CORRECT CHOICE TYPE]

1. Let $f(x) = [\tan^2 x]$, (where [.] denotes greatest integer function). Then -

(A) $\lim_{x\to 0} f(x)$ does not exist
(B) $f(x)$ is continuous at x = 0.
(C) $f(x)$ is not differentiable at x = 0
(D) $f'(0) = 1$

DY0001

2. The number of points where $f(x) = [\sin x + \cos x]$ (where [] denotes the greatest integer function), $x \in (0,2\pi)$ is not continuous is -

(A) 3 (B) 4 (C) 5 (D) 6

DY0002

3. If 6,8 and 12 are l^{th}, m^{th} and n^{th} terms of an A.P. and $f(x) = nx^2 + 2lx - 2m$, then the equation $f(x) = 0$ has -

(A) a root between 0 and 1
(B) both roots imaginary.
(C) both roots negative.
(D) both roots greater than 1.

DY0003

4. Let f be differentiable at x = 0 and $f'(0) = 1$. Then $\lim_{h\to 0}\frac{f(h)-f(-2h)}{h} =$

(A) 3 (B) 2 (C) 1 (D) –1

DY0004

5. Let $g(x) = \begin{cases} 3x^2 - 4\sqrt{x} + 1 & \text{for } x < 1 \\ ax + b & \text{for } x \ge 1 \end{cases}$.

If g(x) is continuous and differentiable at x = 1 then -

(A) a = b = 4
(B) a = b = – 4
(C) a = 4 and b = – 4
(D) a = – 4 and b = 4

DY0005

6. If $f(x)\, f(y) + 2 = f(x) + f(y) + f(xy)$ and $f(1) = 2$, $f'(1) = 2$ then sgn $f(x)$ is equal to (where sgn denotes signum function) -

(A) 0 (B) 1 (C) –1 (D) 4

DY0006

7. The function $g(x) = \begin{cases} x + b, & x < 0 \\ \cos x, & x \ge 0 \end{cases}$ can be made differentiable at x = 0 -

(A) if b is equal to zero
(B) if b is not equal to zero
(C) if b takes any real value
(D) for no value of b

DY0007

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8. Which one of the following functions is continuous everywhere in its domain but has atleast one point where it is not differentiable ?

(A) $f(x) = x^{1/3}$ (B) $f(x) = \frac{|x|}{x}$ (C) $f(x) = e^{-x}$ (D) $f(x) = \tan x$

DY0008

9. If the right hand derivative of $f(x) = [x] \tan \pi x$ at $x = 7$ is $k\pi$, then k is equal to ([y] denotes greatest integer $\le y$)

(A) 6 (B) 7 (C) –7 (D) 49

DY0009

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous onto function satisfying $f(x) + f(-x) = 0$, $\forall x \in \mathbb{R}$. If $f(-3) = 2$ and $f(5) = 4$ in $[-5,5]$, then the equation $f(x) = 0$ has -

(A) exactly three real roots (B) exactly two real roots

(C) atleast five real roots (D) atleast three real roots

DY0010

11. Let $f(x) = \begin{cases} \lim\limits_{n\to\infty} \dfrac{ax(x-1)\left(\cot\frac{\pi x}{4}\right)^n + \left(px^2+2\right)}{\left(\cot\frac{\pi x}{4}\right)^n + 1}, & x \in (0,1)\cup(1,2) \\ 0, & x = 1 \end{cases}$

If $f(x)$ is differentiable for all $x \in (0,2)$ then $(a^2 + p^2)$ equals -

(A) 18 (B) 20 (C) 22 (D) 24

DY0011

12. If $2x + 3|y| = 4y$, then y as a function of x i.e. $y = f(x)$, is -

(A) discontinuous at one point

(B) non differentiable at one point

(C) discontinuous & non differentiable at same point

(D) continuous & differentiable everywhere

DY0012

13. If $f(x) = (x^5 + 1)\,|x^2 - 4x - 5| + \sin|x| + \cos(|x - 1|)$, then $f(x)$ is not differentiable at -

(A) 2 points (B) 3 points (C) 4 points (D) zero points

DY0013

14. Let $f(x) = \begin{cases} x^3 + 2x^2 & x \in \mathbb{Q} \\ -x^3 + 2x^2 + ax & x \notin \mathbb{Q} \end{cases}$, then the integral value of 'a' so that $f(x)$ is differentiable at $x = 1$, is

(A) 2 (B) 6 (C) 7 (D) not possible

DY0014

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15. Let $\mathbb{R}$ be the set of real numbers and $f : \mathbb{R} \to \mathbb{R}$, be a differentiable function such that $|f(x) - f(y)| \le |x - y|^3 \ \forall\ x,y \in \mathbb{R}$. If $f(10) = 100$, then the value of $f(20)$ is equal to -

(A) 0 (B) 10 (C) 20 (D) 100

DY0015

16. For what triplets of real numbers (a, b, c) with $a \ne 0$ the function

$$f(x) = \begin{cases} x & x \le 1 \\ ax^2 + bx + c & \text{otherwise} \end{cases}$$ is differentiable for all real x ?

(A) $\{(a, 1 - 2a, a) \mid a \in \mathbb{R}, a \ne 0\}$ (B) $\{(a, 1 - 2a, c) \mid a, c \in \mathbb{R}, a \ne 0\}$

(C) $\{(a, b, c) \mid a, b, c \in \mathbb{R}, a + b + c = 1\}$ (D) $\{(a, 1 - 2a, 0) \mid a \in \mathbb{R}, a \ne 0\}$

DY0016

17. Number of points of non-differentiability of the function

$g(x) = [x^2]\{\cos^2 4x\} + \{x^2\}[\cos^2 4x] + x^2 \sin^2 4x + [x^2][\cos^2 4x] + \{x^2\}\{\cos^2 4x\}$ in $(-50, 50)$ where [x] and {x} denotes the greatest integer function and fractional part function of x respectively, is equal to :-

(A) 98 (B) 99 (C) 100 (D) 0

DY0017

18. Let $f(x) = [n + p \sin x]$, $x \in (0, \pi)$, $n \in \mathbb{Z}$ and p is a prime number. The number of points where f(x) is not differentiable is :-

(A) $p - 1$ (B) $p + 1$ (C) $2p + 1$ (D) $2p - 1$

DY0018

Here [x] denotes greatest integer function.

19. The function $f(x) = (x^2 - 1)\,|\,x^2 - 3x + 2\,| + \cos(|x|)$ is NOT differentiable at :

(A) –1 (B) 0 (C) 1 (D) 2

DY0019

EXERCISE (O-2)

[MULTIPLE CORRECT CHOICE TYPE]

1. If $f(x) = x\left(\sqrt{x} - \sqrt{x+1}\right)$, then -

(A) $Rf'(0)$ exist (B) $L\ f'(0)$ exist but $R\ f'(0)$ does not exist

(C) $\lim\limits_{x \to 0^+} f(x)$ exist (D) $f(x)$ is differentiable at x = 0.

DY0020

2. The function $f(x) = \begin{cases} |x-3|, x \ge 1 \\ \left(\frac{x^2}{4}\right) - \left(\frac{3x}{2}\right) + \left(\frac{13}{4}\right), x < 1 \end{cases}$ is -

(A) continuous at x = 1 (B) differentiable at x = 1

(C) continuous at x = 3 (D) differentiable at x = 3

DY0021

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3. Select the correct statements -

(A) The function f defined by $f(x)=\begin{cases}2x^2+3 & \text{for } x\le 1\\ 3x+2 & \text{for } x>1\end{cases}$ is neither differentiable nor continuous at x = 1.

(B) The function $f(x) = x^2|x|$ is twice differentiable at x = 0.

(C) If f is continuous at x = 5 and $f(5) = 2$ then $\lim\limits_{x\to 2} f(4x^2-11)$ exists

(D) If $\lim\limits_{x\to a}\left(f(x)+g(x)\right)=2$ and $\lim\limits_{x\to a}\left(f(x)-g(x)\right)=1$ then $\lim\limits_{x\to a} f(x).g(x)$ need not exist.

DY0022

4. If $f(x) = \text{sgn}(x^5)$, then which of the following is/are **false** (where sgn denotes signum function) -

(A) $f'_+(0) = 1$

(B) $f'_-(0) = -1$

(C) f is continuous but not differentiable at x = 0

(D) f is discontinuous at x = 0

DY0023

5. Graph of $f(x)$ is shown in adjacent figure, then

(A) $f(x)$ has non removable discontinuity at two points

(B) $f(x)$ is non differentiable at three points in its domain

(C) $\lim\limits_{x\to 1} f(f(x)) = 1$

(D) Number of points of discontinuity = number of points of non-differentiability

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DY0024

6. Let S denotes the set of all points where $\sqrt[5]{x^2|x|^3}-\sqrt[3]{x^2|x|}-1$ is not differentiable then S is a subset of -

(A) {0,1} (B) {0,1,–1} (C) {0,1} (D) {0}

DY0025

7. Which of the following statements is/are correct ?

(A) There exist a function $f:[0,1]\to\mathbb{R}$ which is discontinuous at every point in [0,1] & $|f(x)|$ is continuous at every point in [0,1]

(B) Let $F(x) = f(x).\,g(x)$. If $f(x)$ is differentiable at x = a, $f(a) = 0$ and g(x) is continuous at x = a then F(x) is always differentiable at x = a.

(C) If $Rf'(a) = 2$ & $Lf'(a) = 3$, then $f(x)$ is non-differentiable at x = a but will be always continuous at x = a

(D) If $f(a)$ and $f(b)$ possess opposite signs then there must exist at least one solution of the equation $f(x) = 0$ in (a,b) provided f is continuous on [a,b]

DY0026

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8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = |f(x)|$ for all x. Then which of the following is/are not always true-

(A) If f is continuous then g is also continuous
(B) If f is one-one then g is also one-one
(C) If f is onto then g is also onto
(D) If f is differentiable then g is also differentiable

DY0027

9. The function $\phi(x) = [|x| - \sin|x|]$ (where [.] denotes greater integer function) is -

(A) derivable at x = 0
(B) continuous at x = 0
(C) $\lim\limits_{x\to 0}\phi(x)$ does not exists
(D) continuous and derivable at x = 0

DY0028

10. Let $f(x) = \begin{cases} x^2\cos\frac{1}{x}, & x<0 \\ 0, & x=0 \\ x^2\sin\frac{1}{x}, & x>0 \end{cases}$, then which of the following is (are) correct ?

(A) f(x) is continuous but not differentiable at x = 0
(B) f(x) is continuous and differentiable at x = 0
(C) f'(x) is continuous but not differentiable at x = 0
(D) f'(x) is discontinuous at x = 0

DY0029

[MATCH THE COLUMN]

11.

	Column - I		Column - II
(A)	If f(x) is derivable at x = 3 & f'(3) = 2, then $\underset{h\to 0}{\text{Limit}}\frac{f(3+h^2)-f(3-h^2)}{2h^2}$ equals	(P)	0
(B)	Let f(x) be a function satisfying the condition f(–x) = f(x) for all real x. If f'(0) exists, then its value is equal to	(Q)	1
(C)	For the function $f(x) = \begin{cases} \frac{x}{1+e^{1/x}}, x\neq 0 \\ 0, \ x=0 \end{cases}$ the derivative from the left Lf '(0) equals	(R)	2
(D)	The number of points at which the function f(x) = max. {a – x, a + x, b}, –∞ < x < ∞, 0 < a < b cannot be differentiable is	(S)	3

DY0030

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EXERCISE (S-1)

1. Discuss the continuity & differentiability of the function $f(x) = \sin x + \sin|x|$, $x \in \mathbb{R}$. Draw a rough sketch of the graph of f(x).

DY0031

2. Examine the continuity and differentiability of $f(x) = |x| + |x-1| + |x-2|$ $x \in \mathbb{R}$. Also draw the graph of f(x).

DY0032

3. If the function f(x) de7fined as $f(x) = \begin{cases} -\dfrac{x^2}{2} & \text{for } x \le 0 \\ x^n \sin\dfrac{1}{x} & \text{for } x > 0 \end{cases}$ is continuous but not derivable at x = 0 then find the range of n.

DY0033

4. A function f is defined as follows : $f(x) = \begin{cases} 1 & \text{for } -\infty < x < 0 \\ 1+|\sin x| & \text{for } 0 \le x < \dfrac{\pi}{2} \\ 2+\left(x-\dfrac{\pi}{2}\right)^2 & \text{for } \dfrac{\pi}{2} \le x < +\infty \end{cases}$

Discuss the continuity & differentiability at x = 0 & x = π/2.

DY0034

5. Examine the origin for continuity & derivability in the case of the function f defined by $f(x) = x\tan^{-1}(1/x)$, $x \ne 0$ and $f(0) = 0$.

DY0035

6. Let f(0) = 0 and f'(0) = 1. For a positive integer k, show that

$$\lim_{x\to 0}\frac{1}{x}\left(f(x)+f\left(\frac{x}{2}\right)+.....+f\left(\frac{x}{k}\right)\right)=1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{k}$$

DY0036

7. Let $f(x) = xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}$; $x \ne 0$, f(0) = 0, test the continuity & differentiability at x = 0

DY0037

8. If $f(x) = |x-1|.([x]-[-x])$, then find $Rf'(1)$ & $Lf'(1)$ where [x] denotes greatest integer function.

DY0038

9. If $f(x) = \begin{cases} ax^2 - b & \text{if } |x| < 1 \\ -\dfrac{1}{|x|} & \text{if } |x| \ge 1 \end{cases}$ is derivable at x = 1. Find the values of a & b.

DY0039

10. Let $g(x) = \begin{cases} a\sqrt{x+2}, & 0 < x < 2 \\ bx+2, & 2 \le x < 5 \end{cases}$. If g(x) is derivable on (0, 5), then find (2a + b).

DY0040

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EXERCISE (S-2)

1. Let f(x) be defined in the interval [–2, 2] such that $f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ x-1, & 0 < x \le 2 \end{cases}$ & $g(x) = f(|x|) + |f(x)|$. Test the differentiability of g(x) in (–2, 2).

DY0041

2. Discuss the continuity & the derivability in [0, 2] of $f(x) = \begin{cases} |2x-3|[x] & \text{for} \quad x \ge 1 \\ \sin\frac{\pi x}{2} & \text{for} \quad x < 1 \end{cases}$

where [.] denotes the greatest integer function

DY0042

3. Examine the function, $f(x) = x \cdot \dfrac{a^{1/x} - a^{-1/x}}{a^{1/x} + a^{-1/x}}$, $x \ne 0$ $(a > 0)$ and $f(0) = 0$ for continuity and existence of the derivative at the origin.

DY0043

4. For any real number x, let [x] denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval [–3,3] by $f(x) = \begin{cases} -x-[-x] & \text{if} \quad [x] \text{ is even} \\ x-[x] & \text{if} \quad [x] \text{ is odd} \end{cases}$

If L denotes the number of point of discontinuity and M denotes the number of points of non-derivability of f(x), then find (L + M).

DY0044

5. $f(x) = \begin{cases} 1-x & , \quad (0 \le x \le 1) \\ x+2 & , \quad (1 < x < 2) \\ 4-x & , \quad (2 \le x \le 4) \end{cases}$. Discuss the continuity & differentiability of y = f(f(x)) for $0 \le x \le 4$.

DY0045

6. A derivable function $f : \mathbb{R}^+ \to \mathbb{R}$ satisfies the condition $f(x) - f(y) \ge \ell n\,(x/y) + x - y$ for every $x, y \in \mathbb{R}^+$. If g denotes the derivative of f then compute the value of the sum $\sum_{n=1}^{100} g\left(\frac{1}{n}\right)$.

DY0046

7. If $\lim_{x \to 0} \dfrac{1-\cos\left(1-\cos\frac{x}{2}\right)}{2^m x^n}$ is equal to the left hand derivative of $e^{-|x|}$ at x = 0, then find the value of (n – 10m)

DY0047

8. If f is a differentiable function such that $f\left(\dfrac{x+y}{3}\right) = \dfrac{f(x)+f(y)+f(0)}{3}$, $\forall$ x,y $\in \mathbb{R}$ and $f'(0) = 2$, find $f(x)$

DY0048

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9. If $\lim_{x\to 0}\frac{f(3-\sin x)-f(3+x)}{x}=8$, then $|f'(3)|$ is

DY0049

10. Let $f(x)$ be a differentiable function such that $2f(x + y) + f(x - y) = 3f(x) + 3f(y) + 2xy$ $\forall$ x, y $\in \mathbb{R}$ & $f'(0) = 0$, then $f(10) + f'(10)$ is equal to

DY0050

EXERCISE (JM)

1. Let $f(x) = x|x|$ and $g(x) = \sin x$. **[AIEEE-2009]**

Statement–1 : gof is differentiable at x = 0 and its derivative is continuous at that point.

Statement–2 : gof is twice differentiable at x = 0.

(1) Statement–1 is true, Statement–2 is false.

(2) Statement–1 is false, Statement–2 is true.

(3) Statement–1 is true, Statement–2 is true; Statement–2 is a correct explanation for Statement–1.

(4) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for statement–1.

DY0051

2. If function f(x) is differentiable at x = a then $\lim_{x\to a}\frac{x^2f(a)-a^2f(x)}{x-a}$ **[AIEEE-2011]**

(1) $2a\,f(a) + a^2 f'(a)$ (2) $-a^2 f'(a)$ (3) $af(a) - a^2 f'(a)$ (4) $2af(a) - a^2 f'(a)$

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DY0052

3. Consider the function,

$f(x) = |x - 2| + |x - 5|$, $x \in R$.

Statement–1 : $f'(4) = 0$.

Statement–2 : f is continuous in [2, 5], differentiable in (2, 5) and f(2) = f(5). **[AIEEE 2012]**

(1) Statement–1 is true, Statement–2 is false.

(2) Statement–1 is false, Statement–2 is true.

(3) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement1.

(4) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for Statement1.

DY0053

4. Let $f(x) = x|x|$, $g(x) = \sin x$ and $h(x) = (gof)(x)$. Then **[On-line 2014]**

(1) $h'(x)$ is differentiable at x = 0

(2) $h'(x)$ is continuous at x = 0 but is not differentiable at x = 0

(3) h(x) is differentiable at x = 0 but $h'(x)$ is not continuous at x = 0

(4) h(x) is not differentiable at x = 0

DY0054

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5. Let f, g: R → R be two functions defined by $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$, and g(x) = xf(x) :-

Statement I : f is a continuous function at x = 0. **[On-line 2014]**

Statement II : g is a differentiable function at x = 0.

(1) Statement I is false and statement II is true

(2) Statement I is true and statement II is false

(3) Both statement I and II are true

(4) Both statements I and II are false

DY0055

6. Let f: R → R be a function such that $|f(x)| \le x^2$, for all $x \in R$. Then, at x = 0 , f is:

(1) Neither continuous nor differentiable **[On-line 2014]**

(2) differentiable but not continuous

(3) continuous as well as differentiable

(4) continuous but not differentiable

DY0056

7. For $x \in R$, $f(x) = |\log 2 - \sin x|$ and g(x) = f(f(x)), then : **[JEE(Main)-2016]**

(1) g is differentiable at x = 0 and g'(0) = –sin(log2)

(2) g is not differentiable at x = 0

(3) g'(0) = cos(log2)

(4) g'(0) = –cos(log2)

DY0057

8. Let $S = \{t \in R : f(x) = |x - \pi|\cdot(e^{|x|} - 1)\sin|x|$ is not differentiable at t}. Then the set S is equal to: **[JEE(Main)-2018]**

(1) {0} (2) {π} (3) {0, π} (4) ϕ (an empty set)

DY0058

9. Let f be a differentiable function from R to R such that $|f(x) - f(y)| \le 2|x - y|^{\frac{3}{2}}$, for all x, y ∈ R.

If $f(0) = 1$ then $\int_0^1 f^2(x)dx$ is equal to **[JEE(Main)-2019]**

(1) 0 (2) $\frac{1}{2}$ (3) 2 (4) 1

DY0059

10. Let $f(x) = \begin{cases} \max\{|x|, x^2\}, & |x| \le 2 \\ 8 - 2|x|, & 2 < |x| \le 4 \end{cases}$. Let S be the set of points in the interval (–4,4) at which f is not differentiable. Then S : **[JEE(Main)-2019]**

(1) is an empty set (2) equals {–2, –1, 1, 2}

(3) equals {–2, –1, 0, 1, 2} (4) equals {–2, 2}

DY0060

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11. Let f : (–1,1)→R be a function defined by $f(x) = \max\left\{-|x|, -\sqrt{1-x^2}\right\}$. If K be the set of all points at which f is not differentiable, then K has exactly : **[JEE(Main)-2019]**

(1) Three elements (2) One element (3) Five elements (4) Two elements

DY0061

12. Let K be the set of all real values of x where the function $f(x) = \sin|x| - |x| + 2(x - \pi)\cos|x|$ is not differentiable. Then the set K is equal to :- **[JEE(Main)-2019]**

(1) $\{\pi\}$ (2) $\{0\}$ (3) ϕ (an empty set) (4) $\{0, \pi\}$

DY0062

13. Let S be the set of all points in $(-\pi,\pi)$ at which the function, $f(x) = \min\{\sin x, \cos x\}$ is not differentiable. Then S is a subset of which of the following? **[JEE(Main)-2019]**

(1) $\left\{-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{4}\right\}$ (2) $\left\{-\frac{3\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$ (3) $\left\{-\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}\right\}$ (4) $\left\{-\frac{\pi}{4}, 0, \frac{\pi}{4}\right\}$

DY0063

14. Let $f(x) = 15 - |x - 10|$; $x \in R$. Then the set of all values of x, at which the function, $g(x) = f(f(x))$ is not differentiable, is : **[JEE(Main)-2019]**

(1) {5,10,15,20} (2) {10,15} (3) {5,10,15} (4) {10}

DY0064

15. Let $f : R \to R$ be differentiable at $c \in R$ and $f(c) = 0$. If $g(x) = |f(x)|$, then at x = c, g is : **[JEE(Main)-2019]**

(1) differentiable if $f'(c) = 0$ (2) not differentiable

(3) differentiable if $f'(c) \neq 0$ (4) not differentiable if $f'(c) = 0$

DY0065

EXERCISE (JA)

1. Let $f : R \to R$ be a function such that

$$f(x + y) = f(x) + f(y), \forall x, y \in R.$$

If $f(x)$ is differentiable at x = 0, then

(A) $f(x)$ is differentiable only in a finite interval containing zero

(B) $f(x)$ is continuous $\forall x \in R$

(C) $f'(x)$ is constant $\forall x \in R$

(D) $f(x)$ is differentiable except at finitely many points **[JEE 2011, 4M]**

DY0066

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2. If $f(x)=\begin{cases}-x-\frac{\pi}{2} & , \quad x\le -\frac{\pi}{2}\\ -\cos x & , \quad -\frac{\pi}{2}<x\le 0\\ x-1 & , \quad 0<x\le 1\\ \ell nx & , \quad x>1\end{cases}$ then - **[JEE 2011, 4M]**

(A) $f(x)$ is continuous at $x=-\frac{\pi}{2}$

(B) $f(x)$ is not differentiable at x = 0

(C) $f(x)$ is differentiable at x = 1

(D) $f(x)$ is differentiable at $x=-\frac{3}{2}$

DY0067

3. Let $f(x)=\begin{cases}x^2\left|\cos\frac{\pi}{x}\right| & , \quad x\ne 0\\ 0 & , \quad x=0\end{cases}$, $x\in\mathbb{R}$, then f is - **[JEE 2012, 3M, –1M]**

(A) differentiable both at x = 0 and at x = 2

(B) differentiable at x = 0 but not differentiable at x = 2

(C) not differentiable at x = 0 but differentiable at x = 2

(D) differentiable neither at x = 0 nor at x = 2

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DY0068

4. Let $f_1:\mathbb{R}\to\mathbb{R}$, $f_2:[0,\infty)\to\mathbb{R}$, $f_3:\mathbb{R}\to\mathbb{R}$ and $f_4:\mathbb{R}\to[0,\infty)$ be defined by

$f_1(x)=\begin{cases}|x| & \text{if} \quad x<0,\\ e^x & \text{if} \quad x\ge 0;\end{cases}$

$f_2(x)=x^2$;

$f_3(x)=\begin{cases}\sin x & \text{if} \quad x<0,\\ x & \text{if} \quad x\ge 0\end{cases}$

and $f_4(x)=\begin{cases}f_2(f_1(x)) & \text{if} \quad x<0,\\ f_2(f_1(x))-1 & \text{if} \quad x\ge 0.\end{cases}$

List-I		List-II	
P.	f_4 is	1.	onto but not one-one
Q.	f_3 is	2.	neither continuous nor one-one
R.	$f_2 o f_1$ is	3.	differentiable but not one-one
S.	f_2 is	4.	continuous and one-one

Codes :

	P	Q	R	S
(A)	3	1	4	2
(B)	1	3	4	2
(C)	3	1	2	4
(D)	1	3	2	4

[JEE(Advanced)-2014, 3(–1)]

DY0069

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5. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$. Define

$$h : \mathbb{R} \to \mathbb{R} \text{ by } h(x) = \begin{cases} \max\{f(x), g(x)\} & \text{if } x \le 0, \\ \min\{f(x), g(x)\} & \text{if } x > 0. \end{cases}$$

The number of points at which h(x) is not differentiable is **[JEE(Advanced)-2014, 3]**

DY0070

6. Let a, b $\in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = a\cos(|x^3 - x|) + b|x|\sin(|x^3 + x|)$. Then f is -

(A) differentiable at x = 0 if a = 0 and b = 1

(B) differentiable at x = 1 if a = 1 and b = 0

(C) **NOT** differentiable at x = 0 if a = 1 and b = 0

(D) **NOT** differentiable at x = 1 if a = 1 and b = 1 **[JEE(Advanced)-2016, 4(–2)]**

DY0071

7. Let $f : \left[-\frac{1}{2}, 2\right] \to \mathbb{R}$ and $g : \left[-\frac{1}{2}, 2\right] \to \mathbb{R}$ be function defined by $f(x) = [x^2 - 3]$ and $g(x) = |x|\, f(x) + |4x - 7|\, f(x)$, where [y] denotes the greatest integer less than or equal to y for y $\in \mathbb{R}$. Then

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(A) f is discontinuous exactly at three points in $\left[-\frac{1}{2}, 2\right]$

(B) f is discontinuous exactly at four points in $\left[-\frac{1}{2}, 2\right]$

(C) g is NOT differentiable exactly at four points in $\left(-\frac{1}{2}, 2\right)$

(D) g is NOT differentiable exactly at five points in $\left(-\frac{1}{2}, 2\right)$ **[JEE(Advanced)-2016, 4(–2)]**

DY0072

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f(0) = 1$ and satisfying the equation

$$f(x + y) = f(x)f'(y) + f'(x)f(y) \text{ for all } x, y \in \mathbb{R}.$$

Then, then value of $\log_e(f(4))$ is _____ . **[JEE(Advanced)-2018]**

DY0073

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9. Let $f_1 : \mathbb{R} \to \mathbb{R}$, $f_2 : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, $f_3 : \left(-1, e^{\frac{\pi}{2}} - 2\right) \to \mathbb{R}$ and $f_4 : \mathbb{R} \to \mathbb{R}$ be functions defined by

(i) $f_1(x) = \sin\left(\sqrt{1 - e^{-x^2}}\right)$

(ii) $f_2(x) = \begin{cases} \frac{|\sin x|}{\tan^{-1} x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$, where the inverse trigonometric function $\tan^{-1}x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

(iii) $f_3(x) = [\sin(\log_e(x + 2)]$, where for $t \in \mathbb{R}$, $[t]$ denotes the greatest integer less than or equal to t,

(iv) $f_4(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

List-I		List-II	
P.	The function f_1 is	**1.**	**NOT** continuous at $x = 0$
Q.	The function f_2 is	**2.**	continuous at $x = 0$ and **NOT** differentiable at $x = 0$
R.	The function f_3 is	**3.**	differentiable at $x = 0$ and its derivative is **NOT** continuous at $x = 0$
S.	The function f_4 is	**4.**	differentiable at $x = 0$ and its derivative is continuous at $x = 0$

The correct option is :

(A) **P → 2; Q → 3, R → 1; S → 4**

(B) **P → 4; Q → 1; R → 2; S → 3**

(C) **P → 4; Q → 2, R → 1; S → 3**

(D) **P → 2; Q → 1; R → 4; S → 3** **[JEE(Advanced)-2018]**

DY0074

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ANSWER KEY

DIFFERENTIABILITY

EXERCISE (O-1)

1. B	**2.** C	**3.** A	**4.** A	**5.** C	**6.** B	**7.** D
8. A	**9.** B	**10.** D	**11.** B	**12.** B	**13.** A	**14.** D
15. D	**16.** A	**17.** D	**18.** D	**19.** D		

EXERCISE (O-2)

1. A,C,D	**2.** A,B,C	**3.** B,C	**4.** A,B,C	**5.** B,C	**6.** A.B,C,D	**7.** A,B,C,D
8. B,C,D	**9.** A,B,D	**10.** B,D	**11.** (A) R, (B) P, (C) Q, (D) R			

EXERCISE (S-1)

1. f(x) is conti. but not derivable at $x = 0$

2. conti. $\forall\, x \in \mathbb{R}$, not diff. at $x = 0, 1$ & 2

3. $0 < n \le 1$

4. conti. but not diff. at $x = 0$; diff. & conti. at $x = \pi/2$

5. conti. but not diff. at $x = 0$

7. f is cont. but not diff. at $x = 0$

8. $Rf'(1) = 3,\ Lf'(1) = -1$

9. $a = 1/2,\ b = 3/2$

10. 3

EXERCISE (S-2)

1. not derivable at $x = 0$ & $x = 1$

2. f is conti. at $x = 1, 3/2$ & disconti. at $x = 2$, f is not diff. at $x = 1, 3/2, 2$

3. If $a \in (0, 1)$ $Rf'(0) = -1$; $Lf'(0) = 1 \Rightarrow$ continuous but not derivable

$a = 1$; $f(x) = 0$ which is constant $\Rightarrow$ continuous and derivable

If $a > 1$, $Lf'(0) = -1$; $Rf'(0) = 1 \Rightarrow$ continuous but not derivable

4. 8 **5.** f is conti. but not diff. at $x = 1$, disconti. at $x = 2$ & $x = 3$. cont. & diff. at all other points

6. 5150 **7.** 74 **8.** $f(x) = 2x + c$ **9.** 4 **10.** 120

EXERCISE (JM)

1. 1	**2.** 4	**3.** 4	**4.** 2	**5.** 3	**6.** 3	**7.** 3	**8.** 4
9. 4	**10.** 3	**11.** 1	**12.** 3	**13.** 1	**14.** 3	**15.** 1	

EXERCISE (JA)

1. B,C	**2.** A,B,C,D	**3.** B	**4.** D	**5.** 3	**6.** A,B	**7.** B,C	**8.** 2
9. D							

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METHOD OF DIFFERENTIATION

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JEE (Main/Advanced) Syllabus

JEE (Main) Syllabus :

Differentiation of the sum, difference, product and quotient of two functions. Differentiation of trigonometric, inverse trigonometric, logarithmic, exponential, composite and implicit functions; derivatives of order upto two.

JEE (Advanced) Syllabus :

Derivative of a function, derivative of the sum, difference, product and quotient of two functions, chain rule, derivatives of polynomial, rational, trigonometric, inverse trigonometric, exponential and logarithmic functions. Derivatives of implicit functions, derivatives up to order two. L'Hospital rule of evaluation of limits of functions.

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METHODS OF DIFFERENTIATION

The process of calculating derivative is called differentiation.

1. DERIVATIVE OF f(x) FROM THE FIRST PRINCIPLE :

Obtaining the derivative using the definition $\lim\limits_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim\limits_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}=f'(x)=\frac{dy}{dx}$ is called calculating derivative using first principle or ab initio or delta method.

Note : $\frac{dy}{dx}$ can also be represented as y_1 or y' or Dy or $f'(x)$. $\frac{dy}{dx}$ represents instantaneous rate of change of y w.r.t. x.

Illustration 1 : Differentiate each of following functions by first principle :

(i) $f(x) = \tan x$ (ii) $f(x) = e^{\sin x}$

Solution : (i) $f'(x) = \lim\limits_{h\to 0}\frac{\tan(x+h)-\tan x}{h} = \lim\limits_{h\to 0}\frac{\tan(x+h-x)[1+\tan x\tan(x+h)]}{h}$

$= \lim\limits_{h\to 0}\frac{\tan h}{h}.(1+\tan^2 x) = \sec^2 x.$ **Ans.**

(ii) $f'(x) = \lim\limits_{h\to 0}\frac{e^{\sin(x+h)}-e^{\sin x}}{h} = \lim\limits_{h\to 0} e^{\sin x}\frac{[e^{\sin(x+h)-\sin x}-1]}{\sin(x+h)-\sin x}\left(\frac{\sin(x+h)-\sin x}{h}\right)$

$= e^{\sin x}\lim\limits_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = e^{\sin x}\cos x$ **Ans.**

Do yourself -1 :

(i) Differentiate each of following functions by first principle:

(a) $f(x) = \ell nx$ (b) $f(x) = \frac{1}{x}$

2. DERIVATIVE OF STANDARD FUNCTIONS :

	f(x)	f'(x)		f(x)	f'(x)
(i)	x^n	nx^{n-1}	(ii)	e^x	e^x
(iii)	a^x	$a^x\ell na,\ a>0$	(iv)	ℓnx	$1/x$
(v)	$\log_a x$	$(1/x)\log_a e,\ a>0,\ a\neq 1$	(vi)	$\sin x$	$\cos x$
(vii)	$\cos x$	$-\sin x$	(viii)	$\tan x$	$\sec^2 x$
(ix)	$\sec x$	$\sec x\tan x$	(x)	$\operatorname{cosec} x$	$-\operatorname{cosec} x\,.\cot x$
(xi)	$\cot x$	$-\operatorname{cosec}^2 x$	(xii)	constant	0
(xiii)	$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}},\ -1<x<1$	(xiv)	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}},\ -1<x<1$
(xv)	$\tan^{-1} x$	$\frac{1}{1+x^2},\ x\in R$	(xvi)	$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}},\ \lvert x\rvert>1$
(xvii)	$\operatorname{cosec}^{-1} x$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}},\ \lvert x\rvert>1$	(xviii)	$\cot^{-1} x$	$\frac{-1}{1+x^2},\ x\in R$

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3. FUNDAMENTAL THEOREMS :

If f and g are derivable functions of x, then,

(a) $\frac{d}{dx}(f \pm g) = \frac{df}{dx} \pm \frac{dg}{dx}$ (b) $\frac{d}{dx}(cf) = c\frac{df}{dx}$, where c is any constant

(c) $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$ known as **"PRODUCT RULE"**

(d) $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\left(\frac{df}{dx}\right) - f\left(\frac{dg}{dx}\right)}{g^2}$ where $g \neq 0$ known as **"QUOTIENT RULE"**

(e) If y = f(u) & u = g (x) then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ known as **"CHAIN RULE"**

Note : In general if y = f(u) then $\frac{dy}{dx} = f'(u).\frac{du}{dx}$.

Illustration 2 : If $y = e^x \tan x + x\log_e x$, find $\frac{dy}{dx}$.

Solution : $y = e^x.\tan x + x \cdot \log_e x$

On differentiating we get,

$$\frac{dy}{dx} = e^x \cdot \tan x + e^x \cdot \sec^2 x + 1 \cdot \log x + x \cdot \frac{1}{x}$$

Hence, $\frac{dy}{dx} = e^x(\tan x + \sec^2 x) + (\log x + 1)$ **Ans.**

Illustration 3 : If $y = \frac{\log x}{x} + e^x \sin 2x + \log_5 x$, find $\frac{dy}{dx}$.

Solution : On differentiating we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\log x}{x}\right) + \frac{d}{dx}(e^x \sin 2x) + \frac{d}{dx}(\log_5 x) = \frac{\frac{1}{x}\cdot x - \log x \,.\, 1}{x^2} + e^x \sin 2x + 2e^x \,.\, \cos 2x + \frac{1}{x\log_e 5}$$

Hence, $\frac{dy}{dx} = \left(\frac{1-\log x}{x^2}\right) + e^x(\sin 2x + 2\cos 2x) + \frac{1}{x\log_e 5}$ **Ans.**

Illustration 4 : If $y = \log_e(\tan^{-1}\sqrt{1+x^2})$, find $\frac{dy}{dx}$.

Solution : $y = \log_e(\tan^{-1}\sqrt{1+x^2})$

On differentiating we get,

$$= \frac{1}{\tan^{-1}\sqrt{1+x^2}} \cdot \frac{1}{1+(\sqrt{1+x^2})^2} \cdot \frac{1}{2\sqrt{1+x^2}} \cdot 2x$$

$$= \frac{x}{\left(\tan^{-1}\sqrt{1+x^2}\right)\left\{1+\left(\sqrt{1+x^2}\right)^2\right\}\sqrt{1+x^2}} = \frac{x}{\left(\tan^{-1}\sqrt{1+x^2}\right)(2+x^2)\sqrt{1+x^2}}$$ **Ans.**

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Do yourself -2 :

(i) Find $\frac{dy}{dx}$ if -

(a) $y = (x + 1)(x + 2)(x + 3)$ (b) $y = e^{5x}\tan(x^2 + 2)$

4. LOGARITHMIC DIFFERENTIATION :

To find the derivative of a function :

(a) which is the product or quotient of a number of functions or

(b) of the form $[f(x)]^{g(x)}$ where f & g are both derivable functions.

It is convenient to take the logarithm of the function first & then differentiate.

Illustration 5 : If $y = (\sin x)^{\ell n\, x}$, find $\frac{dy}{dx}$

Solution : $\ell n\, y = \ell n\, x.\ \ell n\,(\sin x)$

On differentiating we get,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ \ell n\,(\sin x) + \ell n\, x.\ \frac{\cos x}{\sin x} \Rightarrow \frac{dy}{dx} = (\sin x)^{\ell n\, x}\left[\frac{\ell n(\sin x)}{x} + \cot x\ \ell n\, x\right]$$ **Ans.**

Illustration 6 : If $x = \exp\left(\tan^{-1}\left(\frac{y - x^2}{x^2}\right)\right)$, then $\frac{dy}{dx}$ equals -

(A) $x\,[1 + \tan(\log x) + \sec^2 x]$ (B) $2x\,[1 + \tan(\log x)] + \sec^2 x$

(C) $2x\,[1 + \tan(\log x)] + \sec x$ (D) $2x + x[1 + \tan(\log x)]^2$

Solution : Taking log on both sides, we get

$$\log x = \tan^{-1}\left(\frac{y - x^2}{x^2}\right) \Rightarrow \tan(\log x) = (y - x^2)/x^2$$

$$\Rightarrow y = x^2 + x^2\tan(\log x)$$

On differentiating, we get

$$\therefore \frac{dy}{dx} = 2x + 2x\tan(\log x) + x\sec^2(\log x) \Rightarrow 2x\,[1 + \tan(\log x)] + x\sec^2(\log x)$$

$$= 2x + x[1 + \tan(\log x)]^2$$ **Ans. (D)**

Illustration 7 : If $y = \frac{x^{1/2}(1-2x)^{2/3}}{(2-3x)^{3/4}(3-4x)^{4/5}}$ find $\frac{dy}{dx}$

Solution : $\ell n\, y = \frac{1}{2}\ \ell n\, x + \frac{2}{3}\ \ell n\,(1 - 2x) - \frac{3}{4}\ \ell n\,(2 - 3x) - \frac{4}{5}\ \ell n\,(3 - 4x)$

On differentiating we get,

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{2x} - \frac{4}{3(1-2x)} + \frac{9}{4(2-3x)} + \frac{16}{5(3-4x)}$$

$$\frac{dy}{dx} = y\left(\frac{1}{2x} - \frac{4}{3(1-2x)} + \frac{9}{4(2-3x)} + \frac{16}{5(3-4x)}\right)$$ **Ans.**

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Do yourself -3 :

(i) Find $\frac{dy}{dx}$ if $y = x^x$ **(ii)** Find $\frac{dy}{dx}$ if $y = e^x.e^{x^2}.e^{x^3}.e^{x^4}$

5. PARAMETRIC DIFFERENTIATION :

If $y = f(\theta)$ & $x = g(\theta)$ where θ is a parameter, then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)}{g'(\theta)}$

Illustration 8: If y = a cos t and x = a(t – sint) find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{2}$

Solution : $\frac{dy}{dx} = \frac{-a\sin t}{a(1-\cos t)} \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = -1$ **Ans.**

Illustration 9 : Prove that the function represented parametrically by the equations. $x = \frac{1+t}{t^3}; y = \frac{3}{2t^2} + \frac{2}{t}$ satisfies the relationship : $x(y')^3 = 1 + y'$ (where $y' = \frac{dy}{dx}$)

Solution : Here $x = \frac{1+t}{t^3} = \frac{1}{t^3} + \frac{1}{t^2}$

Differentiating w.r. to t

$$\frac{dx}{dt} = -\frac{3}{t^4} - \frac{2}{t^3}$$

$$y = \frac{3}{2t^2} + \frac{2}{t}$$

Differentiating w.r. to t

$$\frac{dy}{dt} = -\frac{3}{t^3} - \frac{2}{t^2}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = t = y'$$

Since $x = \frac{1+t}{t^3} \Rightarrow x = \frac{1+y'}{(y')^3}$ or $x(y')^3 = 1 + y'$

Ans.

6. DERIVATIVE OF A FUNCTION W.R.T. ANOTHER FUNCTION :

Let y= f (x) ; z = g (x) then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$

Illustration 10: Differentiate $\log_e(\tan x)$ with respect to $\sin^{-1}(e^x)$.

Solution : $\frac{d(\log_e \tan x)}{d(\sin^{-1}(e^x))} = \frac{\frac{d}{dx}(\log_e \tan x)}{\frac{d}{dx}\sin^{-1}(e^x)} = \frac{\cot x.\sec^2 x}{e^x.1/\sqrt{1-e^{2x}}} = \frac{e^{-x}\sqrt{1-e^{2x}}}{\sin x \cos x}$ **Ans.**

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Do yourself-4 :

(i) Find $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$ if $y = \cos^4 t$ & $x = \sin^4 t$.

(ii) Find the slope of the tangent at a point P(t) on the curve $x = at^2$, $y=2at$.

(iii) Differentiate $x^{\ell nx}$ with respect to ℓnx.

7. DIFFERENTIATION OF IMPLICIT FUNCTIONS : $\phi(x, y) = 0$

(a) To find dy /dx of implicit functions, we differentiate each term w.r.t. x regarding y as a function of x & then collect terms with dy/dx together on one side.

(b) Also $\frac{dy}{dx} = -\frac{\frac{\partial\phi}{\partial x}}{\frac{\partial\phi}{\partial y}}$, where $\frac{\partial\phi}{\partial x}$ = partial derivative of $\phi(x, y)$ w.r.t. x taking y as a constant and

$\frac{\partial\phi}{\partial y}$ = partial derivative of $\phi(x, y)$ w.r.t. y taking x as a constant.

(c) In the case of implicit functions, generally, both x & y are present in answers of dy/dx.

Illustration 11 : If $x^y + y^x = 2$, then find $\frac{dy}{dx}$.

Solution : Let $u = x^y$ and $v = y^x$

$u + v = 2 \Rightarrow \frac{du}{dx} + \frac{dv}{dx} = 0$

Now $u = x^y$ and $v = y^x$

$\Rightarrow \ell n\, u = y\, \ell nx$ and $\ell n\, v = x\, \ell n\, y$

$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{y}{x} + \ell nx\frac{dy}{dx}$ and $\frac{1}{v}\frac{dv}{dx} = \ell n\, y + \frac{x}{y}\frac{dy}{dx}$

$\Rightarrow \frac{du}{dx} = x^y\left(\frac{y}{x} + \ell nx\frac{dy}{dx}\right)$ and $\frac{dv}{dx} = y^x\left(\ell n\, y + \frac{x}{y}\frac{dy}{dx}\right)$

$\Rightarrow x^y\left(\frac{y}{x} + \ell n\, x\frac{dy}{dx}\right) + y^x\left(\ell ny + \frac{x}{y}\frac{dy}{dx}\right) = 0 \Rightarrow \frac{dy}{dx} = -\frac{\left(y^x\ell ny + x^y.\frac{y}{x}\right)}{\left(x^y\ell nx + y^x.\frac{x}{y}\right)}$ **Ans.**

Aliter :

$\phi(x,y) = x^y + y^x - 2 = 0$

$\frac{dy}{dx} = \frac{-\partial\phi/\partial x}{\partial\phi/\partial y} = \frac{yx^{y-1} + y^x\ell ny}{x^y\ell nx + xy^{x-1}}$

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Illustration 12 : If $y = \cfrac{\sin x}{1+\cfrac{\cos x}{1+\cfrac{\sin x}{1+\cos x.....}}}$, prove that $\dfrac{dy}{dx} = \dfrac{(1+y)\cos x + y\sin x}{1+2y+\cos x-\sin x}$.

Solution : Given function is $y = \cfrac{\sin x}{1+\cfrac{\cos x}{1+y}} = \dfrac{(1+y)\sin x}{1+y+\cos x}$

or $y + y^2 + y\cos x = (1 + y)\sin x$ .....(i)

Differentiate both sides with respect to x,

$$\frac{dy}{dx} + 2y\frac{dy}{dx} + \frac{dy}{dx}\cos x - y\sin x = (1 + y)\cos x + \frac{dy}{dx}\sin x$$

$$\frac{dy}{dx}(1 + 2y + \cos x - \sin x) = (1 + y)\cos x + y\sin x$$

or $\dfrac{dy}{dx} = \dfrac{(1+y)\cos x + y\sin x}{1+2y+\cos x-\sin x}$ **Ans.**

Aliter :

From (i) $\phi(x,y) = (1 + y)\sin x - y - y^2 - y\cos x = 0$

$$\frac{dy}{dx} = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y} = -\frac{(1+y)\cos x + y\sin x}{\sin x - 1 - 2y - \cos x} = \frac{(1+y)\cos x + y\sin x}{1+2y+\cos x-\sin x}$$

Do yourself -5 :

(i) Find $\dfrac{dy}{dx}$, if $x + y = \sin(x - y)$

(ii) If $x^2 + xe^y + y = 0$, find y', also find the value of y' at point (0,0).

8. DIFFERENTIATION BY TRIGONOMETRIC TRANSFORMATION :

Some Standard Substitutions :

Expression	Substitution
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = \tan\theta$ or $a\cot\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $a\operatorname{cosec}\theta$
$\sqrt{\dfrac{a+x}{a-x}}$ or $\sqrt{\dfrac{a-x}{a+x}}$	$x = a\cos\theta$ or $a\cos 2\theta$
$\sqrt{2ax - x^2}$	$x = a(1 - \cos\theta)$

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Illustration 13: If $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ then find

(i) $f'(2)$ (ii) $f'\left(\frac{1}{2}\right)$ (iii) $f'(1)$

Solution : $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $\Rightarrow$ $y = \sin^{-1}(\sin 2\theta)$

$$y = \begin{cases} \pi - 2\theta & \frac{\pi}{2} < 2\theta < \pi \\ 2\theta & \frac{-\pi}{2} \le 2\theta \le \frac{\pi}{2} \\ -(\pi + 2\theta) & -\pi < 2\theta < -\frac{\pi}{2} \end{cases} \Rightarrow f(x) = \begin{cases} \pi - 2\tan^{-1} x & x > 1 \\ 2\tan^{-1} x & -1 \le x \le 1 \\ -(\pi + 2\tan^{-1} x) & x < -1 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} -\frac{2}{1+x^2} & x > 1 \\ \frac{2}{1+x^2} & -1 < x < 1 \\ \frac{-2}{1+x^2} & x < -1 \end{cases}$$

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(i) $f'(2) = -\frac{2}{5}$ (ii) $f'\left(\frac{1}{2}\right) = \frac{8}{5}$ (iii) $f'(1^+) = -1$ and $f'(1^-) = +1 \Rightarrow f'(1)$ does not exist **Ans.**

Illustration 14 : $\frac{d}{dx}\left\{\sin^2\left(\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right)\right\} =$

(A) $-\frac{1}{2}$ (B) 0 (C) $\frac{1}{2}$ (D) -1

Solution : Let $y = \sin^2\left(\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right)$. Put $x = \cos 2\theta$ $\theta \in \left(0, \frac{\pi}{2}\right]$

$\therefore$ $y = \sin^2 \cot^{-1}\left(\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}\right) = \sin^2 \cot^{-1}(\cot\theta)$

$\therefore$ $y = \sin^2\theta = \frac{1-\cos 2\theta}{2} = \frac{1-x}{2} = \frac{1}{2} - \frac{x}{2}$

$\therefore$ $\frac{dy}{dx} = -\frac{1}{2}$. **Ans (A)**

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Illustration 15 : Obtain differential coefficient of $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ with respect to $\cos^{-1}\sqrt{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}$

Solution : Assume $u = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$, $v = \cos^{-1}\sqrt{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}$

The function needs simplification before differentiation Let $x = \tan\theta$; $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\therefore \quad u = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

$$v = \cos^{-1}\sqrt{\frac{1+\sec\theta}{2\sec\theta}} = \cos^{-1}\sqrt{\frac{1+\cos\theta}{2}} = \cos^{-1}\left(\cos\frac{\theta}{2}\right) = \frac{\theta}{2} \Rightarrow u = v$$

$$\therefore \quad \frac{du}{dv} = 1.$$ **Ans.**

Do yourself : 6

(i) If $y = \cos^{-1}(4x^3 - 3x)$, then find :

(a) $f'\left(-\frac{\sqrt{3}}{2}\right)$, (b) $f'(0)$, (c) $f'\left(\frac{\sqrt{3}}{2}\right)$.

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9. DERIVATIVE OF A FUNCTION AND ITS INVERSE FUNCTION :

If g is inverse of f, then

(a) $g\{f(x)\} = x$, $g'\{f(x)\}f'(x)=1$ (b) $f\{g(x)\} = x$, $f'\{g(x)\}g'(x) = 1$

Illustration 16 : If g is inverse of f and $f'(x) = \frac{1}{1+x^n}$, then $g'(x)$ equals :-

(A) $1 + x^n$ (B) $1 + [f(x)]^n$ (C) $1 + [g(x)]^n$ (D) none of these

Solution : Since g is the inverse of f. Therefore

$f(g(x)) = x$ for all x

$$\Rightarrow \frac{d}{dx}f(g(x)) = 1 \text{ for all } x$$

$$\Rightarrow f'(g(x))\, g'(x) = 1 \quad \Rightarrow \quad g'(x) = \frac{1}{f'(g(x))} = 1 + (g(x))^n$$ **Ans. (C)**

Do yourself -7 :

(i) If g is inverse of f and $f(x) = 2x + \sin x$; then $g'(x)$ equals:

(A) $-\frac{3}{x^2} + \frac{1}{\sqrt{1-x^2}}$ (B) $2 + \sin^{-1}x$ (C) $2 + \cos g(x)$ (D) $\frac{1}{2+\cos(g(x))}$

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10. HIGHER ORDER DERIVATIVES :

Let a function y = f (x) be defined on an interval (a, b). If f (x) is differentiable function, then its derivative f '(x) [or (dy/dx) or y'] is called the first derivative of y w.r.t. x. If f '(x) is again differentiable function on (a, b), then its derivative f "(x) [or d^2y/dx^2 or y"] is called second derivative of y w.r.t. x.

Similarly, the 3rd order derivative of y w.r.to x, if it exists, is defined by $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ and denoted by f "'(x) or y"' and so on.

Note : If x = f(θ) and y = g(θ) where 'θ' is a parameter then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ & $\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\Big/\frac{dx}{d\theta}$

In general $\frac{d^ny}{dx^n} = \frac{d}{d\theta}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)\Big/\frac{dx}{d\theta}$.

11. ANALYSIS AND GRAPHS OF SOME INVERSE TRIGONOMETRIC FUNCTIONS :

(a) $y = f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & |x| \le 1 \\ \pi - 2\tan^{-1}x & x > 1 \\ -(\pi + 2\tan^{-1}x) & x < -1 \end{cases}$

Important points :

(i) Domain is $x \in R$ & range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) f is continuous for all x but not differentiable at $x = 1, -1$

(iii) $\frac{dy}{dx} = \begin{cases} \frac{2}{1+x^2} & \text{for } |x| < 1 \\ \text{non existent} & \text{for } |x| = 1 \\ \frac{-2}{1+x^2} & \text{for } |x| > 1 \end{cases}$

(iv) Increasing in [–1, 1] & Decreasing in (–∞, –1], [1, ∞)

(b) Consider $y = f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$

Important points :

(i) Domain is $x \in R$ & range is [0, π)

(ii) Continuous for all x but not differentiable at x = 0

(iii) $\frac{dy}{dx} = \begin{cases} \frac{2}{1+x^2} & \text{for } x > 0 \\ \text{non existent} & \text{for } x = 0 \\ -\frac{2}{1+x^2} & \text{for } x < 0 \end{cases}$

(iv) Increasing in [0, ∞) & Decreasing in (–∞, 0]

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(c) $y = f(x) = \tan^{-1}\dfrac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & |x|<1 \\ \pi + 2\tan^{-1}x & x<-1 \\ -(\pi - 2\tan^{-1}x) & x>1 \end{cases}$

Important points :

(i) Domain is $R-\{1, -1\}$ & range is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(ii) $\dfrac{dy}{dx} = \begin{cases} \dfrac{2}{1+x^2} & |x| \neq 1 \\ \text{non existent} & |x| = 1 \end{cases}$

(iii) It is bounded for all x

(d) $y = f(x) = \sin^{-1}(3x - 4x^3) = \begin{cases} -(\pi + 3\sin^{-1}x) & \text{if } -1 \le x < -\dfrac{1}{2} \\ 3\sin^{-1}x & \text{if } -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ \pi - 3\sin^{-1}x & \text{if } \dfrac{1}{2} < x \le 1 \end{cases}$

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Important points :

(i) Domain is $x \in [-1, 1]$ & range is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

(ii) Continuous everywhere in its domain

(iii) Not derivable at $x = -\dfrac{1}{2}, \dfrac{1}{2}, 1, -1$

(iv) $\dfrac{dy}{dx} = \begin{cases} \dfrac{3}{\sqrt{1-x^2}} & \text{if } x \in (-\dfrac{1}{2}, \dfrac{1}{2}) \\ -\dfrac{3}{\sqrt{1-x^2}} & \text{if } x \in (-1, -\dfrac{1}{2}) \cup (\dfrac{1}{2}, 1) \end{cases}$

(v) Increasing in $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$ and Decreasing in $\left[-1, -\dfrac{1}{2}\right], \left[\dfrac{1}{2}, 1\right]$

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(e) $y = f(x) = \cos^{-1}(4x^3 - 3x) = \begin{cases} 3\cos^{-1}x - 2\pi & \text{if } -1 \le x < -\frac{1}{2} \\ 2\pi - 3\cos^{-1}x & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1}x & \text{if } \frac{1}{2} < x \le 1 \end{cases}$

Important points :

(i) Domain is $x \in [-1, 1]$ & range is $[0, \pi]$

(ii) Continuous everywhere in its domain

(iii) Not derivable at $x = -\frac{1}{2}, \frac{1}{2}, 1, -1$

(iv) $\frac{dy}{dx} = \begin{cases} \frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \\ -\frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \end{cases}$

(v) Increasing in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ &

Decreasing in $\left[-1, -\frac{1}{2}\right], \left[\frac{1}{2}, 1\right]$

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GENERAL NOTE :

Concavity is decided by the sign of 2nd derivative as :

$\frac{d^2y}{dx^2} \ge 0 \Rightarrow$ Concave upwards ; $\frac{d^2y}{dx^2} \le 0 \Rightarrow$ Concave downwards

---

Illustration 17: Find the interval for which $f(x) = x^3 + x + 1$ is

(i) concave upwards (ii) concave downwards.

Solution : $f(x) = x^3 + x + 1$

$f'(x) = 3x^2 + 1$

$f''(x) = 6x$

(i) $f''(x) = 6x \ge 0 \Rightarrow$ Concave upwards

$\Rightarrow x \in [0, \infty)$

(ii) $f''(x) = 6x \le 0 \Rightarrow$ Concave downwards

$\Rightarrow x \in (-\infty, 0]$ **Ans.**

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Illustration 18: If x = a (t + sin t) and y = a(1 – cos t), find $\frac{d^2y}{dx^2}$.

Solution : Here x = a (t + sin t) and y = a (1–cos t)

Differentiating both sides w.r.t. t, we get :

$$\frac{dx}{dt} = a(1 + \cos t) \quad \text{and} \quad \frac{dy}{dt} = a(\sin t)$$

$$\therefore \quad \frac{dy}{dx} = \frac{a\sin t}{a(1+\cos t)} = \frac{2\sin\frac{t}{2}.\cos\frac{t}{2}}{2\cos^2\frac{t}{2}} = \tan\left(\frac{t}{2}\right)$$

Again differentiating both sides, we get,

$$\frac{d^2y}{dx^2} = \sec^2\left(\frac{t}{2}\right)\cdot\frac{1}{2}\cdot\frac{dt}{dx} = \frac{1}{2}\sec^2(t/2)\cdot\frac{1}{a(1+\cos t)} = \frac{1}{2a}\cdot\frac{\sec^2\left(\frac{t}{2}\right)}{2\left(\cos^2\frac{t}{2}\right)}$$

Hence, $\frac{d^2y}{dx^2} = \frac{1}{4a}\cdot\sec^4\left(\frac{t}{2}\right)$ **Ans.**

Illustration 19: y = f(x) and x = g(y) are inverse functions of each other then express g'(y) and g''(y) in terms of derivative of f(x).

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Solution : $\frac{dy}{dx} = f'(x)$ and $\frac{dx}{dy} = g'(y)$

$$\Rightarrow \quad g'(y) = \frac{1}{f'(x)} \qquad \text{.....(i)}$$

Again differentiating w.r.t. to y

$$g''(y) = \frac{d}{dy}\left(\frac{1}{f'(x)}\right) = \frac{d}{dx}\left(\frac{1}{f'(x)}\right)\cdot\frac{dx}{dy} = -\frac{f''(x)}{(f'(x))^2}\cdot\left(\frac{1}{f'(x)}\right)$$

$$\Rightarrow \quad g''(y) = -\frac{f''(x)}{(f'(x))^3} \qquad \text{.....(ii)}$$

Which can also be remembered as $\frac{d^2x}{dy^2} = -\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}$ **Ans.**

Do yourself : 8

(i) If $y = xe^{x^2}$ then find y''.

(ii) Find y'' at x = π/4, if y = x tan x.

(iii) Prove that the function $y = e^x \sin x$ satisfies the relationship y'' – 2y' + 2y = 0.

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12. DIFFERENTIATION OF DETERMINANTS :

If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where f, g, h. l, m, n, u, v, w are differentiable functions of x then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

Note : Sometimes it is batter to expand the determinant first & then differentiate.

Illustration 20: If $f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$, find f'(x).

Solution : Here, $f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$

On differentiating, we get,

$$\Rightarrow f'(x) = \begin{vmatrix} \frac{d}{dx}(x) & \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ \frac{d}{dx}(1) & \frac{d}{dx}(2x) & \frac{d}{dx}(3x^2) \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ \frac{d}{dx}(0) & \frac{d}{dx}(2) & \frac{d}{dx}(6x) \end{vmatrix}$$

or $$f'(x) = \begin{vmatrix} 1 & 2x & 3x^2 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 0 & 2 & 6x \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix}$$

As we know if any two rows or columns are equal, then value of determinant is zero.

$$= 0 + 0 + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix} \qquad \therefore \quad f'(x) = 6\,(2x^2 - x^2)$$

Therefore, $f'(x) = 6x^2$ **Ans.**

Do yourself : 9

(i) If $f(x) = \begin{vmatrix} e^x & x^2 \\ \ell nx & \sin x \end{vmatrix}$, then find $f'(1)$. **(ii)** If $f(x) = \begin{vmatrix} 2x & x^2 & x^3 \\ x^2+2x & 1 & 3x+1 \\ 2x & 1-3x^2 & 5x \end{vmatrix}$, then find $f'(1)$.

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13. L' HÔPITAL'S RULE :

(a) This rule is applicable for the indeterminate forms of the type $\frac{0}{0}, \frac{\infty}{\infty}$. If the function f(x) and g(x) are differentiable in certain neighbourhood of the point 'a', except, may be, at the point 'a' itself and $g'(x) \neq 0$, and if

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \quad \text{or} \quad \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty,$$

then $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists (L' Hôpital's rule). The point 'a' may be either finite or improper ($+\infty$ or $-\infty$).

(b) Indeterminate forms of the type $0.\infty$ or $\infty - \infty$ are reduced to forms of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by algebraic transformations.

(c) Indeterminate forms of the type $1^\infty, \infty^0$ or 0^0 are reduced to forms of the type $0 \times \infty$ by taking logarithms or by the transformation $[f(x)]^{\phi(x)} = e^{\phi(x).\ell nf(x)}$.

Illustration 21: Evaluate $\lim_{x\to 0} |x|^{\sin x}$

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Solution : $\lim_{x\to 0} |x|^{\sin x} = \lim_{x\to 0} e^{\sin x \log_e |x|} = e^{\lim_{x\to 0} \frac{\log_e |x|}{\operatorname{cosec} x}}$

$= e^{\lim_{x\to 0} \frac{1/x}{-\operatorname{cosec} x \cot x}}$ (applying L'Hôpital's rule)

$= e^{\lim_{x\to 0} -\frac{\sin^2 x}{x \cos x}} = e^{\lim_{x\to 0} -\left(\frac{\sin x}{x}\right)^2 \cdot \left(\frac{x}{\cos x}\right)} = e^{-(1)^2 \cdot (0)} = e^0 = 1$ **Ans.**

Illustration 22: Solve $\lim_{x\to 0^+} \log_{\sin x} \sin 2x$.

Solution : Here $\lim_{x\to 0^+} \log_{\sin x} \sin 2x$

$= \lim_{x\to 0^+} \frac{\log \sin 2x}{\log \sin x}$ $\left(\frac{-\infty}{-\infty} \text{form}\right)$

$= \lim_{x\to 0^+} \frac{\frac{1}{\sin 2x} \cdot 2\cos 2x}{\frac{1}{\sin x} \cdot \cos x}$ {applying L'Hôpital's rule}

$= \lim_{x\to 0^+} \frac{\left(\frac{(2x)}{\sin(2x)}\right)\cos 2x}{\left(\frac{x}{\sin x}\right)\cos x} = \lim_{x\to 0^+} \frac{\cos 2x}{\cos x} = 1$ **Ans.**

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Illustration 23: Evaluate $\lim_{n\to\infty}\left(\frac{e^n}{\pi}\right)^{1/n}$.

Solution : Here, $A = \lim_{n\to\infty}\left(\frac{e^n}{\pi}\right)^{1/n}$ (∞^0 form)

$$\therefore \quad \log A = \lim_{n\to\infty}\frac{1}{n}\log\left(\frac{e^n}{\pi}\right) = \lim_{n\to\infty}\frac{n\log e - \log\pi}{n}\left(\frac{\infty}{\infty}\text{form}\right)$$

$$= \lim_{n\to\infty}\frac{\log e - 0}{1} \quad \{\text{applying L'Hôpital's rule}\}$$

$\log A = 1 \Rightarrow A = e^1$ or $\lim_{n\to\infty}\left(\frac{e^n}{\pi}\right)^{1/n} = e$ **Ans.**

Do yourself : 10

(i) Using L'Hôpital's rule find (a) $\lim_{x\to 0}\frac{\tan x - x}{x^3}$ (b) $\lim_{x\to 0}\frac{e^x - x - 1}{x^2}$

(ii) Using L'Hôpital's rule verify that : (a) $\lim_{x\to 0}\frac{\sin x - \tan x}{x^3} = -\frac{1}{2}$ (b) $\lim_{x\to 0}\frac{\ell n(1+x)}{x} = 1$

Miscellaneous Illustrations :

Illustration 24: Find second order derivative of $y = \sin x$ with respect to $z = e^x$.

Solution :

$$\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{\cos x}{e^x}$$

$$\Rightarrow \quad \frac{d^2y}{dz^2} = \frac{d}{dx}\left(\frac{\cos x}{e^x}\right)\cdot\frac{dx}{dz} = \frac{-e^x\sin x - \cos x e^x}{\left(e^x\right)^2}\cdot\frac{1}{e^x}$$

$$\Rightarrow \quad \frac{d^2y}{dz^2} = -\frac{(\sin x + \cos x)}{e^{2x}}$$ **Ans.**

Illustration 25: Let a function f satisfies $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}\ \forall\, x, y \in R$ and $f'(0) = a$, $f(0) = b$, then find $f(x)$ hence find $f''(x)$.

Solution : $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$

Diff. w.r.t. 'x'

$f'\left(\frac{x+y}{2}\right)\cdot\frac{1}{2} = \frac{1}{2}f'(x)$ $\left(\because\ x \,\&\, y \text{ are independent to each other, } \therefore \frac{dy}{dx} = 0\right)$

$f'\left(\frac{x+y}{2}\right) = f'(x)$

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Let x = 0 & y = x $\quad f'\left(\frac{x}{2}\right) = f'(0) = a$

$\Rightarrow \quad f'(x) = a$

On integrating, we get $f(x) = ax + b \quad (\because f(0) = b)$

$\Rightarrow \quad f''(x) = 0$

Illustration 26: Prove that $\frac{1}{2^2}\sec^2\frac{x}{2} + \frac{1}{2^4}\sec^2\frac{x}{2^2} + \frac{1}{2^6}\sec^2\frac{x}{2^3} + ...\infty = \text{cosec}^2 x - \frac{1}{x^2}$

Solution : Let $\cos\frac{x}{2}.\cos\frac{x}{2^2}.\cos\frac{x}{2^3}.....\infty$

$$= \lim_{n\to\infty}\cos\frac{x}{2}.\cos\frac{x}{2^2}.\cos\frac{x}{2^3}.....\cos\frac{x}{2^n}$$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{\sin x}{2^n\sin\frac{x}{2^n}} = \frac{\sin x}{x} \quad \Rightarrow \quad \cos\frac{x}{2}.\cot\frac{x}{2^2}\cos\frac{x}{2^3}.....\infty = \frac{\sin x}{x}$$

$$\Rightarrow \quad \ell n\left(\cos\frac{x}{2}\right) + \ell n\left(\cos\frac{x}{2^2}\right) + \ell n\left(\cos\frac{x}{2^3}\right) + ...\infty = \ell n\sin x - \ell nx$$

Diff. w.r.t. x

$$-\left(\frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2} + ...\infty\right) = \cot x - \frac{1}{x}$$

Diff. w.r.t. x again

$$\frac{1}{2^2}\sec^2\frac{x}{2} + \frac{1}{2^4}\sec^2\frac{x}{2^2} + \frac{1}{2^6}\sec^2\frac{x}{2^3} + ...\infty = \text{cosec}^2 x - \frac{1}{x^2}$$

Hence proved

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ANSWERS FOR DO YOURSELF

1 :	**(i)**	**(a)** $\frac{1}{x}$	**(b)** $-\frac{1}{x^2}$	
2 :	**(i)**	**(a)** $3x^2 + 12x + 11$	**(b)** $5e^{5x}\tan(x^2+2) + 2xe^{5x}\sec^2(x^2+2)$	
3 :	**(i)**	$x^x(\ell nx + 1)$	**(ii)** $y(1 + 2x + 3x^2 + 4x^3)$	
4.	**(i)**	-1	**(ii)** $\frac{1}{t}$	**(iii)** $2(x^{\ell nx})(\ell nx)$
5 :	**(i)**	$\frac{\cos(x-y)-1}{\cos(x-y)+1}$	**(ii)** $y' = -\left(\frac{2x+e^y}{xe^y+1}\right), -1$	
6 :	**(i)**	**(a)** -6	**(b)** 3	**(c)** -6
7 :	**(i)**	D		
8 :	**(i)**	$y'' = 4y + 2xy'$	**(ii)** $\pi + 4$	
9 :	**(i)**	$e(\sin 1 + \cos 1) - 1$	**(ii)** 9	
10.	**(i)**	**(a)** $\frac{1}{3}$	**(b)** $\frac{1}{2}$	

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EXERCISE (O-1)

1. If $y = \frac{1}{1+x^{\beta-\alpha}+x^{\gamma-\alpha}} + \frac{1}{1+x^{\alpha-\beta}+x^{\gamma-\beta}} + \frac{1}{1+x^{\alpha-\gamma}+x^{\beta-\gamma}}$, then $\frac{dy}{dx}$ is equal to-

(A) 0
(B) 1
(C) $(\alpha + \beta + \gamma)x^{\alpha+\beta+\gamma-1}$
(D) $\alpha\beta\gamma$

DF0001

2. If f(x) = |cosx|, then $f'\left(\frac{3\pi}{4}\right)$ is equal to -

(A) $-\frac{1}{\sqrt{2}}$
(B) $\frac{1}{\sqrt{2}}$
(C) 1
(D) –1

DF0002

3. $\frac{d}{dx}(e^x \sin\sqrt{3}\,x)$ equals-

(A) $e^x \sin(\sqrt{3}\,x + \pi/3)$
(B) $2e^x \sin(\sqrt{3}\,x + \pi/3)$
(C) $\frac{1}{2} e^x \sin(\sqrt{3}\,x + \pi/3)$
(D) $\frac{1}{2} e^x \sin(\sqrt{3}\,x - \pi/3)$

DF0003

4. $\frac{d}{dx}\left(\ell n \sin\sqrt{x}\right)$ is equal to-

(A) $\frac{\tan\sqrt{x}}{2\sqrt{x}}$
(B) $\frac{\cot\sqrt{x}}{\sqrt{x}}$
(C) $\frac{\cot\sqrt{x}}{2x}$
(D) $\frac{\cot\sqrt{x}}{2\sqrt{x}}$

DF0004

5. If $y = \sqrt{\frac{1-x}{1+x}}$, then $\frac{dy}{dx}$ equals-

(A) $\frac{y}{1-x^2}$
(B) $\frac{y}{x^2-1}$
(C) $\frac{y}{1+x^2}$
(D) $\frac{y}{y^2-1}$

DF0005

6. If $y = \ell n\left\{\frac{x+\sqrt{(a^2+x^2)}}{a}\right\}$, then the value of $\frac{dy}{dx}$ is-

(A) $\sqrt{a^2-x^2}$
(B) $a\sqrt{a^2+x^2}$
(C) $\frac{1}{\sqrt{a^2+x^2}}$
(D) $x\sqrt{a^2+x^2}$

DF0006

7. If $x = y\,\ell n(xy)$, then $\frac{dx}{dy}$ equals-

(A) $\frac{y(x-y)}{x(x+y)}$
(B) $\frac{x(x+y)}{y(x-y)}$
(C) $\frac{y(x+y)}{x(x-y)}$
(D) $\frac{x(x-y)}{y(x+y)}$

DF0007

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8. If $(\cos x)^y = (\sin y)^x$, then $\frac{dy}{dx}$ equals-

(A) $\frac{\log\sin y - y\tan x}{\log\cos x + x\cot y}$ (B) $\frac{\log\sin y + y\tan x}{\log\cos x - x\cot y}$

(C) $\frac{\log\sin y + y\tan x}{\log\cos x + x\cot y}$ (D) $\frac{\log\sin y + y\tan x}{\log\cos y - y\cot x}$

DF0008

9. If $2^x + 2^y = 2^{x+y}$, then $\frac{dy}{dx}$ is equal to-

(A) $\frac{2^x + 2^y}{2^x - 2^y}$ (B) $\frac{2^x + 2^y}{1 + 2^{x+y}}$ (C) $2^{x-y}\left(\frac{2^y - 1}{1 - 2^x}\right)$ (D) $\frac{2^x + y - 2^x}{2^y}$

DF0009

10. If $x = a(t - \sin t)$, $y = a(1 + \cos t)$, then $\frac{dy}{dx}$ equals-

(A) $-\tan\frac{t}{2}$ (B) $\cot\frac{t}{2}$ (C) $-\cot\frac{t}{2}$ (D) $\tan\frac{t}{2}$

DF0010

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11. The differential coefficient of $\sec^{-1}\left(\frac{1}{2x^2 - 1}\right)$ w.r.t. $\sqrt{1 - x^2}$ is-

(A) $1/x^2$ (B) $2/x^3$ (C) $x/2$ (D) $2/x$

DF0011

12. If $x^3 - y^3 + 3xy^2 - 3x^2y + 1 = 0$, then at (0, 1) $\frac{dy}{dx}$ equals-

(A) 1 (B) –1 (C) 2 (D) 0

DF0012

13. $\frac{d}{d\theta}\left(\tan^{-1}\left(\frac{1 - \cos\theta}{\sin\theta}\right)\right)$ equals, if $-\pi < \theta < \pi$ -

(A) 1/2 (B) 1 (C) $\sec\theta$ (D) $\operatorname{cosec}\theta$

DF0013

14. $\frac{d}{dx}\cot^{-1}\left(\frac{1 + x}{1 - x}\right)$ is equal to, if $x > -1$

(A) $\frac{1}{1 + x^2}$ (B) $\frac{1}{1 - x^2}$ (C) $-\frac{1}{1 + x^2}$ (D) $\frac{-1}{1 - x^2}$

DF0014

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15. If $y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x)$, then $\frac{dy}{dx}$ is equal to-

(A) 1 (B) 0 (C) –1 (D) –2

DF0015

16. $\frac{d}{dx}\left(\tan^{-1}\left(\frac{\sqrt{x}-x}{1+x^{3/2}}\right)\right)$ equals- $(x \geq 0)$

(A) $\frac{1}{2\sqrt{x}(1+x)} - \frac{1}{1+x^2}$ (B) $\frac{1}{2\sqrt{x}(1+x)} + \frac{1}{1+x^2}$

(C) $\frac{1}{1+x} - \frac{1}{1+x^2}$ (D) $\frac{1}{1+x} + \frac{1}{1+x^2}$

DF0016

17. If g is the inverse of f and $f'(x) = \frac{1}{1+x^3}$ then g'(x) is equal to-

(A) $1 + [g(x)]^3$ (B) $\frac{-1}{2(1+x^2)}$

(C) $\frac{1}{2(1+x^2)}$ (D) $\frac{1}{1+[g(x)]^3}$

DF0017

18. If $x^2 + y^2 = 1$, then-

(A) $yy'' - 2(y')^2 + 1 = 0$ (B) $yy'' + (y')^2 + 1 = 0$

(C) $yy'' + (y')^2 - 1 = 0$ (D) $yy'' + 2(y')^2 + 1 = 0$

DF0018

19. Let f be a function defined for all $x \in R$. If f is differentiable and $f(x^3) = x^5$ for all $x \in R$ ($x \neq 0$), then the value of f'(27) is-

(A) 15 (B) 45 (C) 0 (D) 35

DF0019

20. If $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$ are given, then -

(A) $\Delta_1 = 3(\Delta_2)^2$ (B) $\frac{d}{dx}\Delta_1 = 3\Delta_2$

(C) $\frac{d}{dx}\Delta_1 = 3(\Delta_2)^2$ (D) $\Delta_1 = 3(\Delta_2)^{3/2}$

DF0020

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EXERCISE (O-2)

1. Let u(x) and v(x) are differentiable functions such that $\frac{u(x)}{v(x)} = 7$. If $\frac{u'(x)}{v'(x)} = p$ and $\left(\frac{u(x)}{v(x)}\right)' = q$, then $\frac{p+q}{p-q}$ has the value equal to -

(A) 1 (B) 0 (C) 7 (D) –7

DF0021

2. Let f (x) be a polynomial function of second degree. If f (1) = f (–1) and a, b, c are in A.P., then f '(a), f '(b) and f '(c) are in

(A) G.P. (B) H.P. (C) A.G.P. (D) A.P.

DF0022

3. Suppose the function $f(x) - f(2x)$ has the derivative 5 at x = 1 and derivative 7 at x = 2. The derivative of the function $f(x) - f(4x)$ at x = 1, has the value equal to

(A) 19 (B) 9 (C) 17 (D) 14

DF0023

4. If $f(x) = \begin{cases} x\tan^{-1}x + \sec^{-1}(1/x), & x \in (-1,1) - \{0\} \\ \pi/2, & \text{if } x = 0 \end{cases}$, then $f'(0)$ is -

(A) equal to –1 (B) equal to 0 (C) equal to 1 (D) non existent

DF0024

5. Given: $f(x) = 4x^3 - 6x^2 \cos 2a + 3x \sin 2a \,.\, \sin 6a + \sqrt{\ell n\left(2a - a^2\right)}$ then

(A) f(x) is not defined at x = 1/2 (B) f′ (1/2) < 0

(C) f′(x) is not defined at x = 1/2 (D) f′ (1/2) > 0

DF0025

6. If $x = t^3 + t + 5$ & $y = \sin t$, then $\frac{d^2y}{dx^2} =$

(A) $-\frac{(3t^2+1)\sin t + 6t\cos t}{(3t^2+1)^3}$ (B) $\frac{(3t^2+1)\sin t + 6t\cos t}{(3t^2+1)^2}$

(C) $-\frac{(3t^2+1)\sin t + 6t\cos t}{(3t^2+1)^2}$ (D) $\frac{\cos t}{3t^2+1}$

DF0026

7. If $f(x) = \frac{a + \sqrt{a^2 - x^2} + x}{\sqrt{a^2 - x^2} + a - x}$ where a > 0 and x < a, then $f'(0)$ has the value equal to -

(A) $\sqrt{a}$ (B) a (C) $\frac{1}{\sqrt{a}}$ (D) $\frac{1}{a}$

DF0027

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8. Suppose that f (0) = 0 and f '(0) = 2, and let g (x) = $f\left(-x+f\left(f(x)\right)\right)$. The value of g ' (0) is equal to

(A) 0 (B) 1 (C) 6 (D) 8

DF0028

9. If f is differentiable in (0, 6) & f '(4) = 5, then $\lim\limits_{x\to 2}\dfrac{f(4)-f(x^2)}{2-x}=$

(A) 5 (B) 5/4 (C) 10 (D) 20

DF0029

10. If $y = x + e^x$ then $\dfrac{d^2x}{dy^2}$ is :

(A) e^x (B) $-\dfrac{e^x}{(1+e^x)^3}$ (C) $-\dfrac{e^x}{(1+e^x)^2}$ (D) $\dfrac{-1}{(1+e^x)^3}$

DF0030

11. If f is twice differentiable such that $f''(x) = -f(x),\ f'(x) = g(x)$

$h'(x) = [f(x)]^2 + [g(x)]^2$ and

$h(0) = 2,\ h(1) = 4$

then the equation $y = h(x)$ represents :

(A) a curve of degree 2 (B) a curve passing through the origin

(C) a straight line with slope 2 (D) a straight line with y intercept equal to -2.

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DF0031

12. If $y = (A + Bx)\,e^{mx} + (m-1)^{-2}\,e^x$ then $\dfrac{d^2y}{dx^2} - 2m\dfrac{dy}{dx} + m^2y$ is equal to -

(A) e^x (B) e^{mx} (C) e^{-mx} (D) $e^{(1-m)x}$

DF0032

13. If $\dfrac{d^2x}{dy^2}\left(\dfrac{dy}{dx}\right)^3 + \dfrac{d^2y}{dx^2} = K$ then the value of K is equal to

(A) 1 (B) –1 (C) 2 (D) 0

DF0033

14. Let $f(x)$ be differentiable at x = h, then $\lim\limits_{x\to h}\dfrac{(x+h)f(x)-2hf(h)}{x-h}$ is equal to -

(A) $f(h) + 2hf'(h)$ (B) $2f(h) + hf'(h)$ (C) $hf(h) + 2f'(h)$ (D) $hf(h) - 2f'(h)$

DF0034

15. If $y = at^2 + 2bt + c$ and $t = ax^2 + 2bx + c$, then $\dfrac{d^3y}{dx^3}$ equals

(A) $24\,a^2\,(at + b)$ (B) $24\,a\,(ax + b)^2$

(C) $24\,a\,(at + b)^2$ (D) $24\,a^2\,(ax + b)$

DF0035

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16. Let $f(x) = x + \sin x$. Suppose g denotes the inverse function of f. The value of $g'\left(\frac{\pi}{4}+\frac{1}{\sqrt{2}}\right)$ has the value equal to

(A) $\sqrt{2}-1$ (B) $\frac{\sqrt{2}+1}{\sqrt{2}}$

(C) $2-\sqrt{2}$ (D) $\sqrt{2}+1$

DF0036

Multiple objective type :

17. If $f(x) = (2x - 3\pi)^5 + \frac{4}{3}x + \cos x$ and g is the inverse function of f, then

(A) $g'(2\pi) = \frac{7}{3}$ (B) $g'(2\pi) = \frac{3}{7}$ (C) $g''(2\pi) = \frac{7}{3}$ (D) $g''(2\pi) = 0$

DF0037

18. If $f(x) = x.|x|$, then its derivative is :

(A) 2x (B) –2x (C) 2|x| (D) 2xsgnx

DF0038

19. If $y = \tan x \tan 2x \tan 3x$, $(\sin 12x \neq 0)$ then $\frac{dy}{dx}$ has the value equal to

(A) $3\sec^2 3x \tan x \tan 2x + \sec^2 x \tan 2x \tan 3x + 2\sec^2 2x \tan 3x \tan x$

(B) $2y(\operatorname{cosec} 2x + 2\operatorname{cosec} 4x + 3\operatorname{cosec} 6x)$

(C) $3\sec^2 3x - 2\sec^2 2x - \sec^2 x$

(D) $\sec^2 x + 2\sec^2 2x + 3\sec^2 3x$

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DF0039

20. Which of the following statements are true ?

(A) If $xe^{xy} = y + \sin^2 x$, then at x = 0, (dy/dx) = 1.

(B) If $f(x) \equiv a_0x^{2m+1} + a_1x^{2m} + a_3x^{2m-1} + \ldots + a_{2m+1} = 0$ $(a_0 \neq 0)$ is a polynomial equation with rational coefficients then the equation $f'(x) = 0$ must have a real root. $(m \in N)$.

(C) If $(x - r)$ is a factor of the polynomial $f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0$ repeated m times where $1 \leq m \leq n$ then r is a root of the equation $f'(x) = 0$ repeated (m–1) times.

(D) If $y = \sin^{-1}(\cos \sin^{-1}x) + \cos^{-1}(\sin \cos^{-1}x)$ then $\frac{dy}{dx}$ is dependent on x.

DF0040

21. Let $f(x) = \frac{\sqrt{x - 2\sqrt{x-1}}}{\sqrt{x-1}-1}.x$ then

(A) $f'(10) = 1$ (B) $f'(3/2) = -1$ (C) domain of $f(x)$ is $x \geq 1$ (D) none

DF0041

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22. $\lim\limits_{x\to\pi}\dfrac{x^{\pi}-\pi^{x}}{x^{x}-\pi^{\pi}}$ is equal to -

(A) $\log_{\pi e}\left(\dfrac{e}{\pi}\right)$

(B) $\log_{\pi e}\left(\dfrac{\pi}{e}\right)$

(C) $\tan(\cot^{-1}(\ell n\pi)-\cot^{-1}(1))$

(D) $\tan(\tan^{-1}(1)-\tan^{-1}(\ell n\pi))$

DF0042

23. Let P(x) be the polynomial $x^3 + ax^2 + bx + c$, where a,b,c $\in$ R. If P(–3) = P(2) = 0 and P'(–3)<0, which of the following is a possible value of 'c' ?

(A) –27 (B) –18 (C) –6 (D) –3

DF0043

24. Two functions f & g have first & second derivatives at x = 0 & satisfy the relations,

$f(0)=\dfrac{2}{g(0)}$, $f'(0) = 2g'(0) = 4g(0)$, $g''(0) = 5$ $f''(0) = 6f(0) = 3$ then -

(A) if $h(x)=\dfrac{f(x)}{g(x)}$ then $h'(0)=\dfrac{15}{4}$

(B) if $k(x) = f(x).\ g(x)\sin x$ then $k'(0) = 2$

(C) $\lim\limits_{x\to 0}\dfrac{g'(x)}{f'(x)}=\dfrac{1}{2}$

(D) none

DF0044

25. If $y = x^{(\ell nx)^{\ell n(\ell nx)}}$, then $\dfrac{dy}{dx}$ is equal to :

(A) $\dfrac{y}{x}\left(\ell n x^{\ell n x-1}+2\ell n x\ \ell n\left(\ell n x\right)\right)$

(B) $\dfrac{y}{x}(\ln x)^{\ln(\ln x)}(2\ln(\ln x)+1)$

(C) $\dfrac{y}{x\ \ell n x}\left((\ln x)^2+2\ln(\ln x)\right)$

(D) $\dfrac{y\ \ell n y}{x\ \ell n x}(2\ln(\ln x)+1)$

DF0045

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EXERCISE (S-1)

1. Let f, g and h are differentiable functions. If f(0) = 1 ; g(0) = 2 ; h(0) = 3 and the derivatives of their pair wise products at x = 0 are (f g)' (0) = 6 ; (g h)' (0) = 4 and (h f)' (0) = 5 then compute the value of (fgh)'(0).

DF0046

2. **(a)** If $y = (\cos x)^{\ell nx} + (\ell nx)^x$ find $\frac{dy}{dx}$

DF0047

(b) If $y = e^{x^{e^x}} + e^{x^{x^e}} + x^{e^{e^x}}$. Find $\frac{dy}{dx}$.

DF0048

3. If $y = \frac{x^2}{2} + \frac{1}{2}x\sqrt{x^2+1} + \ell n\sqrt{x+\sqrt{x^2+1}}$ prove that 2y = xy' + ℓn y'. Where ' denotes the derivative.

DF0049

4. If $y = \ln\left(x^{e^x.a^y}\right)^{y^x}$ find $\frac{dy}{dx}$.

DF0050

5. If $y = 1 + \frac{x_1}{x-x_1} + \frac{x_2.x}{(x-x_1)(x-x_2)} + \frac{x_3.x^2}{(x-x_1)(x-x_2)(x-x_3)} + ...$ upto (n + 1) terms then prove that $\frac{dy}{dx} = \frac{y}{x}\left[\frac{x_1}{x_1-x} + \frac{x_2}{x_2-x} + \frac{x_3}{x_3-x} + ... + \frac{x_n}{x_n-x}\right]$

DF0051

6. If x = cosec θ – sin θ ; y = $\text{cosec}^n\theta - \sin^n\theta$, then show that $(x^2+4)\left(\frac{dy}{dx}\right)^2 - n^2(y^2+4) = 0$.

DF0052

7. If a curve is represented parametrically by the equations

$$x = \sin\left(t+\frac{7\pi}{12}\right) + \sin\left(t-\frac{\pi}{12}\right) + \sin\left(t+\frac{3\pi}{12}\right),$$

$$y = \cos\left(t+\frac{7\pi}{12}\right) + \cos\left(t-\frac{\pi}{12}\right) + \cos\left(t+\frac{3\pi}{12}\right)$$

then find the value of $\frac{d}{dt}\left(\frac{x}{y}-\frac{y}{x}\right)$ at $t = \frac{\pi}{8}$.

DF0053

8. If $x = \frac{1+\ell nt}{t^2}$ and $y = \frac{3+2\ell nt}{t}$. Show that $y\frac{dy}{dx} = 2x\left(\frac{dy}{dx}\right)^2 + 1$.

DF0054

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9. Differentiate $\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$ w.r.t. $\sqrt{1-x^4}$.

DF0055

10. Let g(x) be a polynomial, of degree one & f(x) be defined by $f(x) = \begin{cases} g(x), & x \le 0 \\ \left(\dfrac{1+x}{2+x}\right)^{1/x}, & x > 0 \end{cases}$

Find the continuous function f(x) satisfying $f'(1) = f(-1)$.

DF0056

11. If $\sqrt{1-x^6}+\sqrt{1-y^6} = a^3.(x^3-y^3)$, prove that $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}\sqrt{\dfrac{1-y^6}{1-x^6}}$.

DF0057

12. If $y = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ldots\ldots\ldots\ldots}}}$, prove that $\dfrac{dy}{dx} = \cfrac{1}{2 - \cfrac{x}{x + \cfrac{1}{x + \cfrac{1}{x + \ldots\ldots\ldots\ldots\ldots\ldots}}}}$

DF0058

13. Let $f(x) = x + \dfrac{1}{2x+}\dfrac{1}{2x+}\dfrac{1}{2x+}\ldots\ldots\ldots\infty$. Compute the value of $f(100)\cdot f'(100)$.

DF0059

14. Find the derivative with respect to x of the function :

$(\log_{\cos x}\sin x)(\log_{\sin x}\cos x)^{-1} + \arcsin\dfrac{2x}{1+x^2}$ at $x = \dfrac{\pi}{4}$.

DF0060

15. Suppose $f(x) = \tan(\sin^{-1}(2x))$

(a) Find the domain and range of f.

(b) Express f(x) as an algebraic function of x.

(c) Find $f'(1/4)$

DF0061

16. (a) Let $f(x) = x^2 - 4x - 3$, $x > 2$ and let g be the inverse of f. Find the value of g' where f(x) = 2.

DF0062

(b) Let $f : R \to R$ be defined as $f(x) = x^3 + 3x^2 + 6x - 5 + 4e^{2x}$ and $g(x) = f^{-1}(x)$, then find $g'(-1)$.

DF0063

(c) Suppose f^{-1} is the inverse function of a differentiable function f and let $G(x) = \dfrac{1}{f^{-1}(x)}$.

If $f(3) = 2$ and $f'(3) = \dfrac{1}{9}$, find $G'(2)$.

DF0064

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17. If $y = \tan^{-1}\frac{1}{x^2+x+1} + \tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7} + \tan^{-1}\frac{1}{x^2+7x+13} + ...$ to n terms. Find dy/dx, expressing your answer in 2 terms.

DF0065

18. If $y = \tan^{-1}\frac{u}{\sqrt{1-u^2}}$ & $x = \sec^{-1}\frac{1}{2u^2-1}$, $u \in \left(0,\frac{1}{\sqrt{2}}\right)\cup\left(\frac{1}{\sqrt{2}},1\right)$ prove that $2\frac{dy}{dx}+1=0$.

DF0066

19. If $y = \cot^{-1}\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$, find $\frac{dy}{dx}$ if $x \in \left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right)$.

DF0067

20. If $f(x) = \begin{vmatrix}(x-a)^4 & (x-a)^3 & 1\\ (x-b)^4 & (x-b)^3 & 1\\ (x-c)^4 & (x-c)^3 & 1\end{vmatrix}$, then $f'(x) = \lambda.\begin{vmatrix}(x-a)^4 & (x-a)^2 & 1\\ (x-b)^4 & (x-b)^2 & 1\\ (x-c)^4 & (x-c)^2 & 1\end{vmatrix}$. Find the value of λ.

DF0068

21. **(a)** If $y = y(x)$ and it follows the relation $e^{xy} + y\cos x = 2$, then find **(i)** $y'(0)$ and **(ii)** $y''(0)$.

DF0069

(b) A twice differentiable function f(x) is defined for all real numbers and satisfies the following conditions

$f(0) = 2$; $f'(0) = -5$ and $f''(0) = 3$.

The function g(x) is defined by $g(x) = e^{ax} + f(x)\ \forall\ x \in R$, where 'a' is any constant. If $g'(0) + g''(0) = 0$. Find the value(s) of 'a'.

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DF0070

22. If $x = 2\cos t - \cos 2t$ & $y = 2\sin t - \sin 2t$, find the value of (d^2y/dx^2) when $t = (\pi/2)$.

DF0071

23. Find the value of the expression $y^3\frac{d^2y}{dx^2}$ on the ellipse $3x^2 + 4y^2 = 12$.

DF0072

24. If $f(x) = \begin{vmatrix}\cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2)\\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2)\\ \sin 2x & 0 & \sin 2x^2\end{vmatrix}$ then find $f'(x)$.

DF0073

25. Let P(x) be a polynomial of degree 4 such that $P(1) = P(3) = P(5) = P'(7) = 0$. If the real number $x \neq 1, 3, 5$ is such that $P(x) = 0$ can be expressed as $x = p/q$ where 'p' and 'q' are relatively prime, then $(p + q)$ equals.

DF0074

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EXERCISE (S-2)

1. Find a polynomial function f(x) such that f(2x) = f '(x) f "(x).

DF0075

2. If $2x = y^{\frac{1}{5}} + y^{-\frac{1}{5}}$ then $(x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = ky$, then find the value of 'k'.

DF0076

3. Let y = x sin kx. Find the possible value of k for which the differential equation $\frac{d^2y}{dx^2} + y = 2k\cos kx$ holds true for all $x \in R$.

DF0077

4. Prove that if $|a_1 \sin x + a_2 \sin 2x + \ldots\ldots\ldots + a_n \sin nx| \le |\sin x|$ for $x \in R$, then
$|a_1 + 2a_2 + 3a_3 + \ldots\ldots\ldots + na_n| \le 1$

DF0078

5. The function f: R → R satisfies $f(x^2) \cdot f''(x) = f'(x) \cdot f'(x^2)$ for all real x. Given that f(1) = 1 and f '"(1) = 8, compute the value of f '(1) + f "(1).

DF0079

6. Show that $R = \frac{\left[1+(dy/dx)^2\right]^{\frac{3}{2}}}{d^2y/dx^2}$ can be reduced to the from $R^{2/3} = \frac{1}{(d^2y/dx^2)^{2/3}} + \frac{1}{(d^2x/dy^2)^{2/3}}$.

DF0080

7. Let $f(x) = \frac{\sin x}{x}$ if $x \ne 0$ and f(0) = 1. Define the function f '(x) for all x and find f "(0) if it exist.

DF0081

8. Suppose f and g are two functions such that f, g : R → R,

$f(x) = \ln\left(1+\sqrt{1+x^2}\right)$ and $g(x) = \ln\left(x+\sqrt{1+x^2}\right)$

then find the value of $x\, e^{g(x)}\left(f\left(\frac{1}{x}\right)\right)' + g'(x)$ at x = 1.

DF0082

9. Let $\frac{f(x+y)-f(x)}{2} = \frac{f(y)-a}{2} + xy$ for all real x and y. If f(x) is differentiable and f '(0) exists for all real permissible values of 'a' and is equal to $\sqrt{5a-1-a^2}$. Prove that f(x) is positive for all real x.

DF0083

10. If $y = \tan^{-1}\frac{x}{1+\sqrt{1-x^2}} + \sin\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)$, then find $\frac{dy}{dx}$ for $x \in (-1, 1)$.

DF0084

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11. If $f : R \to R$ is a function such that $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$ for all $x \in R$, then prove that $f(2) = f(1) - f(0)$.

DF0085

12. If α be a repeated root of a quadratic equation $f(x) = 0$ & $A(x)$, $B(x)$, $C(x)$ be the polynomials of degree 3, 4 & 5 respectively, then show that $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is divisible by $f(x)$, where dash denotes the derivative.

DF0086

EXERCISE (S-3)

Evaluate the following limits using L'Hospital's Rule or otherwise (Q.No.1 to 5) :

1. $\lim\limits_{x \to 0} \left[\dfrac{1}{x \sin^{-1} x} - \dfrac{1 - x^2}{x^2} \right]$

DF0087

2. $\lim\limits_{x \to 0} \dfrac{x + \ell n\left(\sqrt{x^2 + 1} - x\right)}{x^3}$

DF0088

3. $\lim\limits_{x \to 0} \left[\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x} \right]$

DF0089

4. $\lim\limits_{x \to 0} \dfrac{1 + \sin x - \cos x + \ell n(1 - x)}{x \cdot \tan^2 x}$

DF0090

5. $\lim\limits_{x \to 0} \dfrac{x^{6000} - (\sin x)^{6000}}{x^2 . (\sin x)^{6000}}$

DF0091

6. If $\lim\limits_{x \to 0} \dfrac{1 - \cos x . \cos 2x . \cos 3x ..... \cos nx}{x^2}$ has the value equal to 253, find the value of n (where $n \in N$).

DF0092

7. Given a real valued function f(x) as follows :

$f(x) = \dfrac{x^2 + 2\cos x - 2}{x^4}$ for $x < 0$; $f(0) = \dfrac{1}{12}$ & $f(x) = \dfrac{\sin x - ln(e^x \cos x)}{6x^2}$ for $x > 0$. Test the continuity and differentiability of f(x) at x = 0.

DF0093

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8. Let $a_1 > a_2 > a_3$ ..... $a_n > 1$; $p_1 > p_2 > p_3$ ..... $> p_n > 0$; such that $p_1 + p_2 + p_3 + ... + p_n = 1$. Also $F(x) = \left(p_1a_1^x + p_2a_2^x + ... + p_na_n^x\right)^{1/x}$. Compute

(a) $\lim\limits_{x\to 0^+} F(x)$ (b) $\lim\limits_{x\to\infty} F(x)$ (c) $\lim\limits_{x\to -\infty} F(x)$

DF0094

9. If x_1, x_1, x_2, x_3, ...x_{n-1} be n zero's of the polynomial $P(x) = x^n + \alpha x + \beta$, where $x_i \neq x_j$ $\forall$ i & j = 1, 2, 3, ... (n – 1).
Prove that the value of $Q(x) = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)...(x_1 - x_{n-1})$, is equal to ${}^nC_2 x_1^{n-2}$.

DF0095

10. **Column - I** contains function defined on R and **Column-II** contains their properties. Match them :-

Column - I		**Column - II**	
(A)	$\lim\limits_{n\to\infty}\left(\dfrac{1+\tan\frac{\pi}{2n}}{1+\sin\frac{\pi}{3n}}\right)^n$ equal	(P)	e
(B)	$\lim\limits_{x\to 0^+}\dfrac{1}{(1+\operatorname{cosec}x)^{\frac{1}{\ell n(\sin x)}}}$ equals	(Q)	e^2
(C)	$\lim\limits_{x\to 0}\left(\dfrac{2}{\pi}\cos^{-1}x\right)^{1/x}$ equals	(R)	$e^{-2/\pi}$
		(S)	$e^{\pi/6}$

DF0096

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EXERCISE (JM)

1. Let f : (–1, 1) → R be a differentiable function with f(0) = – 1 and f'(0) = 1. Let $g(x) = [f(2f(x) + 2)]^2$. Then g'(0) :- **[AIEEE-2010]**

(1) 4 (2) –4 (3) 0 (4) –2

DF0097

2. $\dfrac{d^2x}{dy^2}$ equals :- **[AIEEE-2011]**

(1) $\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-2}$ (2) $-\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-3}$ (3) $\left(\dfrac{d^2y}{dx^2}\right)^{-1}$ (4) $-\left(\dfrac{d^2y}{dx^2}\right)^{-1}\left(\dfrac{dy}{dx}\right)^{-3}$

DF0098

3. If $y = \sec(\tan^{-1}x)$, then $\dfrac{dy}{dx}$ at x = 1 is equal to : **[JEE(Main)-2013]**

(1) $\dfrac{1}{\sqrt{2}}$ (2) $\dfrac{1}{2}$ (3) 1 (4) $\sqrt{2}$

DF0099

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4. If g is the inverse of a function f and $f'(x) = \frac{1}{1+x^5}$, then g'(x) is equal to : **[JEE(Main)-2014]**

(1) $1 + x^5$ (2) $5x^4$ (3) $\frac{1}{1+\{g(x)\}^5}$ (4) $1+\{g(x)\}^5$

DF0100

5. If for $x \in \left(0, \frac{1}{4}\right)$, the derivative of $\tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ is $\sqrt{x} \cdot g(x)$, then g(x) equals :- **[JEE(Main)-2017]**

(1) $\frac{3}{1+9x^3}$ (2) $\frac{9}{1+9x^3}$ (3) $\frac{3x\sqrt{x}}{1-9x^3}$ (4) $\frac{3x}{1-9x^3}$

DF0101

6. If x = 3 tan t and y = 3 sec t, then the value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$, is: **[JEE(Main)-2019]**

(1) $\frac{3}{2\sqrt{2}}$ (2) $\frac{1}{3\sqrt{2}}$ (3) $\frac{1}{6}$ (4) $\frac{1}{6\sqrt{2}}$

DF0102

7. Let $f(x) = \begin{cases} -1, & -2 \le x < 0 \\ x^2 - 1, & 0 \le x \le 2 \end{cases}$ and $g(x) = |f(x)| + f(|x|)$. Then, in the interval (–2, 2), g is :- **[JEE(Main)-2019]**

(1) differentiable at all points (2) not differentiable at two points
(3) not continuous (4) not differentiable at one point

DF0103

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8. For x >1, if $(2x)^{2y} = 4e^{2x-2y}$, then $(1+\log_e 2x)^2 \frac{dy}{dx}$ is equal to : **[JEE(Main)-2019]**

(1) $\log_e 2x$ (2) $\frac{x\log_e 2x + \log_e 2}{x}$ (3) $x\log_e 2x$ (4) $\frac{x\log_e 2x - \log_e 2}{x}$

DF0104

9. If $f(1) = 1$, $f'(1) = 3$, then the derivative of $f(f(f(x))) + (f(x))^2$ at x = 1 is : **[JEE(Main)-2019]**

(1) 12 (2) 33 (3) 9 (4) 15

DF0105

10. If $e^y + xy = e$, the ordered pair $\left(\frac{dy}{dx}, \frac{d^2y}{dx^2}\right)$ at x = 0 is equal to : **[JEE(Main)-2019]**

(1) $\left(-\frac{1}{e}, \frac{1}{e^2}\right)$ (2) $\left(\frac{1}{e}, \frac{1}{e^2}\right)$ (3) $\left(\frac{1}{e}, -\frac{1}{e^2}\right)$ (4) $\left(-\frac{1}{e}, -\frac{1}{e^2}\right)$

DF0106

11. The derivative of $\tan^{-1}\left(\frac{\sin x - \cos x}{\sin x + \cos x}\right)$, with respect to $\frac{x}{2}$, where $\left(x \in \left(0, \frac{\pi}{2}\right)\right)$ is : **[JEE(Main)-2019]**

(1) $\frac{1}{2}$ (2) $\frac{2}{3}$ (3) 1 (4) 2

DF0107

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EXERCISE (JA)

1. Let g(x) = ℓn f(x) where f(x) is a twice differentiable positive function on (0, ∞) such that f(x + 1) = x f(x). Then for N = 1, 2, 3,.....,

$g''\left(N+\frac{1}{2}\right)-g''\left(\frac{1}{2}\right)=$ **[JEE 2008, 3]**

(A) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+.....\frac{1}{(2N-1)^2}\right\}$

(B) $4\left\{1+\frac{1}{9}+\frac{1}{25}+.....\frac{1}{(2N-1)^2}\right\}$

(C) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+.....\frac{1}{(2N+1)^2}\right\}$

(D) $4\left\{1+\frac{1}{9}+\frac{1}{25}+.....\frac{1}{(2N+1)^2}\right\}$

DF0108

2. Let f and g be real valued functions defined on interval (–1, 1) such that g''(x) is continuous, g(0) $\neq$ 0, g'(0) = 0, g''(0) $\neq$ 0, and f(x) = g(x) sin x.

Statement-1 : $\lim\limits_{x\to 0}$ [g(x) cot x – g(0)cosecx] = f ''(0)

and

Statement-2 : f '(0) = g(0)

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation of statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true. **[JEE 2008, 3]**

DF0109

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3. If the function f(x) = $x^3+e^{\frac{x}{2}}$ and g(x) = f^{-1}(x), then the value of g'(1) is **[JEE 2009, 4]**

DF0110

4. Let $f(\theta)=\sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\frac{\pi}{4}<\theta<\frac{\pi}{4}$. Then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is

[JEE 2011, 4]

DF0111

5. The slope of the tangent to the curve $(y-x^5)^2 = x(1+x^2)^2$ at the point (1,3) is

[JEE(Advanced)-2014, 3]

DF0112

6. Let $f:\mathbb{R}\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$ and $h:\mathbb{R}\to\mathbb{R}$ be differentiable functions such that $f(x) = x^3 + 3x + 2$, $g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x\in\mathbb{R}$. Then- **[JEE(Advanced)-2016, 4(–2)]**

(A) $g'(2)=\frac{1}{15}$ (B) h'(1) = 666 (C) h(0) = 16 (D) h(g(3)) = 36

DF0113

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ANSWER KEY

EXERCISE (O-1)

1. A	**2.** B	**3.** B	**4.** D	**5.** B	**6.** C	**7.** B	**8.** B
9. C	**10.** C	**11.** D	**12.** A	**13.** A	**14.** C	**15.** D	**16.** A
17. A	**18.** B	**19.** A	**20.** B				

EXERCISE (O-2)

1. A	**2.** D	**3.** A	**4.** A	**5.** D	**6.** A	**7.** D	**8.** C
9. D	**10.** B	**11.** C	**12.** A	**13.** D	**14.** A	**15.** D	**16.** C
17. B,D	**18.** C,D	**19.** A,B,C	**20.** A,C	**21.** A,B	**22.** A,C,D	**23.** A	
24. A,B,C	**25.** B,D						

EXERCISE (S-1)

1. 16

2. (a) $Dy = (\cos x)^{\ell nx}\left[\frac{\ell n(\cos x)}{x} - \tan x\ \ell nx\right] + (\ell nx)^x\left[\frac{1}{1nx} + \ell n(\ell nx)\right]$;

(b) $\frac{dy}{dx} = e^{x^{e^x}} . x^{e^x}\left[\frac{e^x}{x} + e^x \ell nx\right] + e^{x^{x^e}}\ x^{e-1}\ x^{x^e}\,[1 + e\ell nx] + x^{e^{e^x}}\ e^{e^x}\left[\frac{1}{x} + e^x \ell nx\right]$

4. $\frac{y}{x}.\frac{x\,\ell nx + x\,\ell nx.\ell ny + 1}{\ell n\,x(1 - x - y\,\ell na)}$ **7.** 8 **9.** $\frac{1+\sqrt{1-x^4}}{x^6}$ **10.** $f(x) = \begin{cases} -\frac{2}{3}\left[\frac{1}{6} + \ell n\frac{3}{2}\right]x & \text{if } x \le 0 \\ \left(\frac{1+x}{2+x}\right)^{1/x} & \text{if } x > 0 \end{cases}$

13. 100 **14.** $\frac{32}{16+\pi^2} - \frac{8}{\ell n2}$ **15.** **(a)** $\left(-\frac{1}{2}, \frac{1}{2}\right)$, $(-\infty, \infty)$; **(b)** $f(x) = \frac{2x}{\sqrt{1-4x^2}}$; **(c)** $\frac{16\sqrt{3}}{9}$

16. (a) 1/6; (b) $\frac{1}{14}$; (c) –1 **17.** $\frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$ **19.** $\frac{1}{2}$ or $-\frac{1}{2}$ **20.** 3

21. (a) (i) $y'(0) = -1$; (ii) $y''(0) = 2$; (b) $a = 1, -2$ **22.** $\frac{-3}{2}$ **23.** $\frac{-9}{4}$ **24.** $2(1 + 2x).\cos 2(x + x^2)$ **25.** 100

EXERCISE (S-2)

1. $\frac{4x^3}{9}$ **2.** 25 **3.** k = 1, –1 or 0 **5.** 6 **7.** $f'(x) = \begin{cases} \frac{x\cos x - \sin x}{x^2} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$; $f''(0) = -\frac{1}{3}$

8. Zero **10.** $\frac{1-2x}{2\sqrt{1-x^2}}$

EXERCISE (S-3)

1. $\frac{5}{6}$ **2.** $\frac{1}{6}$ **3.** $-\frac{1}{3}$ **4.** $-\frac{1}{2}$ **5.** 1000 **6.** 11

7. f is cont. but not derivable at x = 0 **8.** **(a)** $a_1^{p_1}.a_2^{p_2}.....a_n^{p_n}$; **(b)** a_1; **(c)** a_n **10.** (A) S; (B) P ; (C) R

EXERCISE (JM)

1. 2	**2.** 2	**3.** 1	**4.** 4	**5.** 2	**6.** 4	**7.** 4	**8.** 4
9. 2	**10.** 1	**11.** 4					

EXERCISE (JA)

1. A	**2.** A	**3.** 2	**4.** 1	**5.** 8	**6.** B,C

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Important Notes

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INDEFINITE INTEGRATION

If f & F are function of x such that F' (x) = f(x) then the function F is called a **PRIMITIVE OR ANTIDERIVATIVE OR INTEGRAL** of f(x) w.r.t. x and is written symbolically as

$\int f(x)\,dx = F(x) + C \Leftrightarrow \frac{d}{dx}\{F(x) + C\} = f(x)$, where C is called the **constant of integration.**

1. GEOMETRICAL INTERPRETATION OF INDEFINITE INTEGRAL :

$\int f(x)dx = F(x) + C = y(\text{say})$, represents a family of curves. The different values of c will correspond to different members of this family and these members can be obtained by shifting any one of the curves parallel to itself. This is the geometrical interpretation of indefinite integral.

Let f(x) = 2x. Then $\int f(x)dx = x^2 + C$. For different values of C, we get different integrals. But these integrals are very similar geometrically.

Thus, $y = x^2 + C$, where C is arbitrary constant, represents a family of integrals. By assigning different values to C, we get different members of the family. These together constitute the indefinite integral. In this case, each integral represents a parabola with its axis along y-axis.

If the line x = a intersects the parabolas $y = x^2$, $y = x^2 + 1$, $y = x^2 + 2$, $y = x^2 - 1$, $y = x^2 - 2$ at $P_0, P_1, P_2, P_{-1}, P_{-2}$ etc., respectively, then $\frac{dy}{dx}$ at these points equals 2a. This indicates that the tangents to the curves at these points are parallel. Thus,

$\int 2x\,dx = x^2 + C = f(x) + C$ **(say), implies that the tangents to all the curves f(x) + C, C ∈ R, at the points of intersection of the curves by the line x = a, (a ∈ R) , are parallel.**

2. STANDARD FORMULAE :

(i) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C;\ n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\ell n|\,ax+b\,| + C$

(iii) $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C$

(iv) $\int a^{px+q}dx = \frac{1}{p}\frac{a^{px+q}}{\ell n\,a} + C,\ (a > 0)$

(v) $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + C$

(vi) $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$

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(vii) $\int \tan(ax+b)dx = \frac{1}{a}\ell n|\sec(ax+b)|+C$ **(viii)** $\int \cot(ax+b)dx = \frac{1}{a}\ell n|\sin(ax+b)|+C$

(ix) $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b)+C$

(x) $\int \operatorname{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b)+C$

(xi) $\int \operatorname{cosec}(ax+b).\cot(ax+b)dx = -\frac{1}{a}\operatorname{cosec}(ax+b)+C$

(xii) $\int \sec(ax+b).\tan(ax+b)dx = \frac{1}{a}\sec(ax+b)+C$

(xiii) $\int \sec x\,dx = \ell n|\sec x+\tan x|+C = \ell n\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|+C$

(xiv) $\int \operatorname{cosec} x\,dx = \ell n|\operatorname{cosec} x-\cot x|+C = \ell n\left|\tan\frac{x}{2}\right|+C = -\ell n|\operatorname{cosec} x+\cot x|+C$

(xv) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+C$ **(xvi)** $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+C$

(xvii) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}+C$ **(xviii)** $\int \frac{dx}{\sqrt{x^2+a^2}} = \ell n\left[x+\sqrt{x^2+a^2}\right]+C$

(xix) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ell n\left[x+\sqrt{x^2-a^2}\right]+C$ **(xx)** $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ell n\left|\frac{a+x}{a-x}\right|+C$

(xxi) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ell n\left|\frac{x-a}{x+a}\right|+C$ **(xxii)** $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

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(xxiii) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ell n\left(x+\sqrt{x^2+a^2}\right)+C$

(xxiv) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ell n\left(x+\sqrt{x^2-a^2}\right)+C$

(xxv) $\int e^{ax}.\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+C = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)+C$

(xxvi) $\int e^{ax}.\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+C = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx-\tan^{-1}\frac{b}{a}\right)+C$

3. TECHNIQUES OF INTEGRATION :

(a) Substitution or change of independent variable :

If $\phi(x)$ is a continuous differentiable function, then to evaluate integrals of the form $\int f(\phi(x))\phi'(x)dx$, we substitute $\phi(x)=t$ and $\phi'(x)dx=dt$.

Hence $I=\int f(\phi(x))\phi'(x)dx$ reduces to $\int f(t)dt$.

(i) Fundamental deductions of method of substitution :

$\int [f(x)]^n f'(x)dx$ OR $\int \frac{f'(x)}{[f(x)]^n}dx$ put $f(x)=t$ & proceed.

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Illustration 1 : Evaluate $\int \frac{\cos^3 x}{\sin^2 x + \sin x} dx$

Solution : $I = \int \frac{(1-\sin^2 x)\cos x}{\sin x(1+\sin x)} dx = \int \frac{1-\sin x}{\sin x} \cos x \, dx$

Put sinx = t $\Rightarrow$ cosx dx = dt

$\Rightarrow \; I = \int \frac{1-t}{t} dt = \ell n|t| - t + C = \ell n|\sin x| - \sin x + C$ **Ans.**

Illustration 2 : Evaluate $\int \frac{(x^2-1)dx}{(x^4+3x^2+1)\tan^{-1}\left(x+\frac{1}{x}\right)}$

Solution : The given integral can be written as

$$I = \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left[\left(x+\frac{1}{x}\right)^2+1\right]\tan^{-1}\left(x+\frac{1}{x}\right)}$$

Let $\left(x+\frac{1}{x}\right) = t$. Differentiating we get $\left(1-\frac{1}{x^2}\right)dx = dt$

Hence $I = \int \frac{dt}{(t^2+1)\tan^{-1} t}$

Now make one more substitution $\tan^{-1}t = u$. Then $\frac{dt}{t^2+1} = du$ and $I = \int \frac{du}{u} = \ell n|u| + C$

Returning to t, and then to x, we have

$I = \ell n|\tan^{-1} t| + C = \ell n\left|\tan^{-1}\left(x+\frac{1}{x}\right)\right| + C$ **Ans.**

Do yourself -1 :

(i) Evaluate : $\int \frac{x^2}{9+16x^6} dx$

(ii) Evaluate : $\int \cos^3 x \, dx$

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(ii) Standard substitutions :

$\int \frac{dx}{\sqrt{a^2+x^2}}$ or $\int \sqrt{a^2+x^2}dx$; put $x = a\tan\theta$ or $x = a\cot\theta$

$\int \frac{dx}{\sqrt{a^2-x^2}}$ or $\int \sqrt{a^2-x^2}dx$; put $x = a\sin\theta$ or $x = a\cos\theta$

$\int \frac{dx}{\sqrt{x^2-a^2}}$ or $\int \sqrt{x^2-a^2}dx$; put $x = a\sec\theta$ or $x = a\,\text{cosec}\,\theta$

$\int \sqrt{\frac{a-x}{a+x}}dx$; put $x = a\cos 2\theta$

$\int \sqrt{\frac{x-\alpha}{\beta-x}}dx$ or $\int \sqrt{(x-\alpha)(\beta-x)}$; put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$\int \sqrt{\frac{x-\alpha}{x-\beta}}dx$ or $\int \sqrt{(x-\alpha)(x-\beta)}$; put $x = \alpha\sec^2\theta - \beta\tan^2\theta$

$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$; put $x-\alpha = t^2$ or $x-\beta = t^2$.

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Illustration 3 : Evaluate $\int \frac{dx}{\sqrt{(x-a)(b-x)}}$

Solution : Put $x = a\cos^2\theta + b\sin^2\theta$, the given integral becomes

$$I = \int \frac{2(b-a)\sin\theta\cos\theta d\theta}{\left\{(a\cos^2\theta + b\sin^2\theta - a)(b - a\cos^2\theta - b\sin^2\theta\right\}^{\frac{1}{2}}}$$

$$= \int \frac{2(b-a)\sin\theta\cos\theta d\theta}{(b-a)\sin\theta\cos\theta} = \left(\frac{b-a}{b-a}\right)\int 2d\theta = 2\theta + C = 2\sin^{-1}\sqrt{\left(\frac{x-a}{b-a}\right)} + C$$ **Ans.**

Illustration 4 : Evaluate $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}.\frac{1}{x}dx$

Solution : Put $x = \cos^2\theta \Rightarrow dx = -2\sin\theta\cos\theta\, d\theta$

$$\Rightarrow \quad I = \int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}.\frac{1}{\cos^2\theta}(-2\sin\theta\cos\theta)d\theta = -\int 2\tan\frac{\theta}{2}\tan\theta\, d\theta$$

$$= -4\int \frac{\sin^2(\theta/2)}{\cos\theta}d\theta = -2\int \frac{1-\cos\theta}{\cos\theta}d\theta = -2\ell n\,|\sec\theta + \tan\theta| + 2\theta + C$$

$$= -2\ell n\left|\frac{1+\sqrt{1-x}}{\sqrt{x}}\right| + 2\cos^{-1}\sqrt{x} + C$$

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Do yourself -2 :

(i) Evaluate : $\int \sqrt{\frac{x-3}{2-x}}dx$ **(ii)** Evaluate : $\int \frac{dx}{x\sqrt{x^2+4}}$

(b) **Integration by part :** $\int u.v\,dx = u\int v\,dx - \int\left[\frac{du}{dx}.\int v\,dx\right]dx$ where u & v are differentiable functions and are commonly designated as first & second function respectively.

Note : While using integration by parts, choose u & v such that

(i) $\int v dx$ & (ii) $\int\left[\frac{du}{dx}.\int v\,dx\right]dx$ are simple to integrate.

This is generally obtained by choosing first function as the function which comes first in the word **ILATE**, where; I-Inverse function, L-Logarithmic function, A-Algebraic function, T-Trigonometric function & E-Exponential function.

Illustration 5 : Evaluate : $\int \cos\sqrt{x}\,dx$

Solution : Consider $I = \int \cos\sqrt{x}dx$

Let $\sqrt{x} = t$ then $\frac{1}{2\sqrt{x}}dx = dt$

i.e. $dx = 2\sqrt{x}dt$ or $dx = 2t\,dt$

so $I = \int \cos t.2tdt$

Taking t as first function, integrate it by part

$$\Rightarrow \quad I = 2\left[t\int \cos tdt - \int\left\{\frac{dt}{dt}\int \cos tdt\right\}dt\right]$$

$$I = 2\left[t\sin t - \int 1.\sin tdt\right] = 2[t\sin t + \cos t] + C$$

$$I = 2\left[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right] + C$$ **Ans.**

Illustration 6 : Evaluate : $\int \frac{x}{1+\sin x}dx$

Solution : Let $I = \int \frac{x}{1+\sin x}dx = \int \frac{x(1-\sin x)}{(1+\sin x)(1-\sin x)}dx$

$$= \int \frac{x(1-\sin x)}{1-\sin^2 x}dx = \int \frac{x(1-\sin x)}{\cos^2 x}dx = \int x\sec^2 xdx - \int x\sec x\tan xdx$$

$$= \left[x\int \sec^2 xdx - \int\left\{\frac{dx}{dx}\int \sec^2 xdx\right\}dx\right]$$

$$-\left[x\int \sec x\tan xdx - \int\left\{\frac{dx}{dx}\int \sec x\tan xdx\right\}dx\right]$$

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$$=\left[x\tan x-\int\tan x dx\right]-\left[x\sec x-\int\sec x dx\right]$$

$$=[x\tan x-\ell n|\sec x|]-[x\sec x-\ell n|\sec x+\tan x|]+C$$

$$=x(\tan x-\sec x)+\ell n\left|\frac{(\sec x+\tan x)}{\sec x}\right|+C$$

$$=\frac{-x(1-\sin x)}{\cos x}+\ell n|1+\sin x|+C$$ **Ans.**

Do yourself -3 :

(i) Evaluate : $\int xe^x dx$ **(ii)** Evaluate : $\int x^3\sin(x^2)dx$

Two classic integrands :

(i) $\int e^x[f(x)+f'(x)]dx=e^x.f(x)+C$

Illustration 7 : Evaluate $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$

Solution : $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx=\int e^x\frac{(1-2x+x^2)}{(1+x^2)^2}dx$

$$=\int e^x\left(\frac{1}{(1+x^2)}-\frac{2x}{(1+x^2)^2}\right)dx=\frac{e^x}{1+x^2}+C$$ **Ans.**

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Illustration 8 : The value of $\int e^x\left(\frac{x^4+2}{(1+x^2)^{5/2}}\right)dx$ is equal to -

(A) $\frac{e^x(x+1)}{(1+x^2)^{3/2}}+C$ (B) $\frac{e^x(1-x+x^2)}{(1+x^2)^{3/2}}+C$ (C) $\frac{e^x(1-x)}{(1+x^2)^{3/2}}+C$ (D) none of these

Solution : Let $I=\int e^x\left(\frac{x^4+2}{(1+x^2)^{5/2}}\right)dx=\int e^x\left(\frac{1}{(1+x^2)^{1/2}}+\frac{1-2x^2}{(1+x^2)^{5/2}}\right).dx$

$$=\int e^x\left(\frac{1}{(1+x^2)^{1/2}}-\frac{x}{(1+x^2)^{3/2}}+\frac{x}{(1+x^2)^{3/2}}+\frac{1-2x^2}{(1+x^2)^{5/2}}\right)dx$$

$$=\frac{e^x}{(1+x^2)^{1/2}}+\frac{xe^x}{(1+x^2)^{3/2}}+C=\frac{e^x\{1+x^2+x\}}{(1+x^2)^{3/2}}+C$$ **Ans. (D)**

Do yourself - 4 :

(i) Evaluate : $\int e^x\left(\tan^{-1}x+\frac{1}{1+x^2}\right)dx$ **(ii)** Evaluate : $\int xe^{x^2}\left(\sin x^2+\cos x^2\right)dx$

(ii) $\int[f(x)+xf'(x)]dx=xf(x)+C$

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Illustration 9 : Evaluate $\int \frac{x+\sin x}{1+\cos x}dx$

Solution : $I=\int \frac{x+\sin x}{1+\cos x}dx=\int\left(\frac{x+\sin x}{2\cos^2\frac{x}{2}}\right)dx=\int\left(x\frac{1}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right)dx=x\tan\frac{x}{2}+C$ **Ans.**

Do yourself -5 :

(i) Evaluate : $\int\left(\tan(e^x)+xe^x\sec^2(e^x)\right)dx$ **(ii)** Evaluate : $\int(\ell nx+1)dx$

(c) Integration of trigonometric functions :

(i) $\int \sin^m x\cos^n x dx$

Case-I : When m & n $\in$ natural numbers.

* If one of them is odd, then substitute for the term of even power.
* If both are odd, substitute either of the term.
* If both are even, use trigonometric identities to convert integrand into cosines of multiple angles.

Case-II : m + n is a negative even integer.

* In this case the best substitution is tanx = t.

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Illustration 10 : Evaluate $\int \sin^3 x\cos^5 x\,dx$

Solution : Put cos x = t; – sin x dx = dt.

so that $I=-\int(1-t^2).t^5dt$

$$=\int(t^7-t^5)dt=\frac{t^8}{8}-\frac{t^6}{6}=\frac{\cos^8 x}{8}-\frac{\cos^6 x}{6}+C$$

Alternate :

Put sin x = t; cos x dx = dt

so that $I=\int t^3(1-t^2)^2dt=\int\left(t^3-2t^5+t^7\right)dt$

$$=\frac{\sin^4 x}{4}-\frac{2\sin^6 x}{6}+\frac{\sin^8 x}{8}+C$$

Note : This problem can also be handled by successive reduction or by trigonometric identities.

Illustration 11 : Evaluate $\int \sin^2 x\cos^4 x\,dx$

Solution : $\int \sin^2 x\cos^4 x dx=\int\left(\frac{1-\cos 2x}{2}\right)\left(\frac{\cos 2x+1}{2}\right)^2dx$

$$=\int\frac{1}{8}(1-\cos 2x)\left(\cos^2 2x+2\cos 2x+1\right)dx$$

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$$=\frac{1}{8}\int\left(\cos^2 2x+2\cos 2x+1-\cos^3 2x-2\cos^2 2x-\cos 2x\right)dx$$

$$=\frac{1}{8}\int\left(-\cos^3 2x-\cos^2 2x+\cos 2x+1\right)dx$$

$$=-\frac{1}{8}\int\left(\frac{\cos 6x+3\cos 2x}{4}+\frac{1+\cos 4x}{2}-\cos 2x-1\right)dx$$

$$=-\frac{1}{32}\left[\frac{\sin 6x}{6}+\frac{3\sin 2x}{2}\right]-\frac{1}{16}x-\frac{\sin 4x}{64}+\frac{\sin 2x}{16}+\frac{x}{8}+C$$

$$=-\frac{\sin 6x}{192}-\frac{\sin 4x}{64}+\frac{1}{64}\sin 2x+\frac{x}{16}+C$$

Illustration 12 : Evaluate $\int\frac{\sqrt{\sin x}}{\cos^{9/2}x}dx$

Solution : Let $I=\int\frac{\sin^{1/2}x}{\cos^{9/2}x}dx=\int\frac{dx}{\sin^{-1/2}x\cos^{9/2}x}$

Here $m+n=\frac{1}{2}-\frac{9}{2}=-4$ (negative even integer).

Divide Numerator & Denominator by $\cos^4 x$.

$$I=\int\sqrt{\tan x}\sec^4 x\,dx=\int\sqrt{\tan x}(1+\tan^2 x)\sec^2 x\,dx$$

$$=\int\sqrt{t}(1+t^2)dt \quad \text{(using } \tan x=t\text{)}$$

$$=\frac{2}{3}t^{3/2}+\frac{2}{7}t^{7/2}+C=\frac{2}{3}\tan^{3/2}x+\frac{2}{7}\tan^{7/2}x+C$$

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Do yourself -6 :

(i) Evaluate : $\int\frac{\sin^2 x}{\cos^4 x}dx$ **(ii)** Evaluate : $\int\frac{\sqrt{\sin x}dx}{\cos^{5/2}x}$ **(iii)** Evaluate : $\int\sin^2 x\cos^5 x\,dx$

(ii) $\int\frac{dx}{a+b\sin^2 x}$ OR $\int\frac{dx}{a+b\cos^2 x}$ OR $\int\frac{dx}{a\sin^2 x+b\sin x\cos x+c\cos^2 x}$

Divide N^r & D^r by $\cos^2 x$ & put $\tan x=t$.

Illustration 13 : Evaluate : $\int\frac{dx}{2+\sin^2 x}$

Solution : Divide numerator and denominator by $\cos^2 x$

$$I=\int\frac{\sec^2 x\,dx}{2\sec^2 x+\tan^2 x}=\int\frac{\sec^2 x\,dx}{2+3\tan^2 x}$$

Let $\sqrt{3}\tan x=t$ $\therefore$ $\sqrt{3}\sec^2 x\,dx=dt$

So $I=\frac{1}{\sqrt{3}}\int\frac{dt}{2+t^2}=\frac{1}{\sqrt{3}}.\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}+C=\frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{\sqrt{3}\tan x}{\sqrt{2}}\right)+C$ **Ans.**

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Illustration 14 : Evaluate : $\int \frac{dx}{(2\sin x + 3\cos x)^2}$

Solution : Divide numerator and denominator by $\cos^2 x$

$$\therefore \quad I = \int \frac{\sec^2 x dx}{(2\tan x + 3)^2}$$

Let $2\tan x + 3 = t$, $\quad \therefore \; 2\sec^2 x dx = dt$

$$I = \frac{1}{2}\int \frac{dt}{t^2} = -\frac{1}{2t} + C = -\frac{1}{2(2\tan x + 3)} + C$$ **Ans.**

Do yourself -7 :

(i) Evaluate : $\int \frac{dx}{1 + 4\sin^2 x}$ **(ii)** Evaluate : $\int \frac{dx}{3\sin^2 x + \sin x \cos x + 1}$

(iii) $\int \frac{dx}{a + b\sin x}$ OR $\int \frac{dx}{a + b\cos x}$ OR $\int \frac{dx}{a + b\sin x + c\cos x}$

Convert sines & cosines into their respective tangents of half the angles

& put $\tan\frac{x}{2} = t$

In this case $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $x = 2\tan^{-1}t$; $dx = \frac{2dt}{1+t^2}$

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Illustration 15 : Evaluate : $\int \frac{dx}{3\sin x + 4\cos x}$

Solution :

$$I = \int \frac{dx}{3\sin x + 4\cos x} = \int \frac{dx}{3\left\{\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right\} + 4\left\{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right\}} = \int \frac{\sec^2\frac{x}{2}dx}{4 + 6\tan\frac{x}{2} - 4\tan^2\frac{x}{2}}$$

let $\tan\frac{x}{2} = t$, $\quad \therefore \; \frac{1}{2}\sec^2\frac{x}{2}dx = dt$

$$\text{so} \quad I = \int \frac{2dt}{4 + 6t - 4t^2} = \frac{1}{2}\int \frac{dt}{1 - \left(t^2 - \frac{3}{2}t\right)} = \frac{1}{2}\int \frac{dt}{\frac{25}{16} - \left(t - \frac{3}{4}\right)^2}$$

$$= \frac{1}{2}.\frac{1}{2\left(\frac{5}{4}\right)}\ell n\left|\frac{\frac{5}{4} + \left(t - \frac{3}{4}\right)}{\frac{5}{4} - \left(t - \frac{3}{4}\right)}\right| + C = \frac{1}{5}\ell n\left|\frac{1 + 2\tan\frac{x}{2}}{4 - 2\tan\frac{x}{2}}\right| + C$$ **Ans.**

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Do yourself-8 :

(i) Evaluate : $\int \frac{dx}{3+\sin x}$ **(ii)** Evaluate : $\int \frac{dx}{1+4\sin x+3\cos x}$

(iv) $\int \frac{a\cos x+b\sin x+c}{p\cos x+q\sin x+r}dx$

Express Numerator $(N^r) = \ell(D^r) + m\frac{d}{dx}(D^r) + n$ & proceed.

Illustration 16 : Evaluate : $\int \frac{2+3\cos\theta}{\sin\theta+2\cos\theta+3}d\theta$

Solution : Write the Numerator = ℓ(denominator) + m(d.c. of denominator) + n

$\Rightarrow 2 + 3\cos\theta = \ell(\sin\theta+2\cos\theta+3) + m(\cos\theta - 2\sin\theta) + n.$

Comparing the coefficients of $\sin\theta$, $\cos\theta$ and constant terms,

we get $3\ell + n = 2$, $2\ell + m = 3$, $\ell - 2m = 0 \Rightarrow \ell = 6/5$, $m = 3/5$ and $n = -8/5$

Hence $I = \int \frac{6}{5}d\theta + \frac{3}{5}\int \frac{\cos\theta-2\sin\theta}{\sin\theta+2\cos\theta+3}d\theta - \frac{8}{5}\int \frac{d\theta}{\sin\theta+2\cos\theta+3}$

$= \frac{6}{5}\theta + \frac{3}{5}\ell n|\sin\theta+2\cos\theta+3| - \frac{8}{5}I_3$ where $I_3 = \int \frac{d\theta}{\sin\theta+2\cos\theta+3}$

In I_3, put $\tan\frac{\theta}{2} = t \Rightarrow \sec^2\frac{\theta}{2}d\theta = 2dt$

$I_3 = 2\int \frac{dt}{t^2+2t+5} = 2\int \frac{dt}{(t+1)^2+2^2} = 2.\frac{1}{2}\tan^{-1}\left(\frac{t+1}{2}\right) = \tan^{-1}\left(\frac{\tan\theta/2+1}{2}\right)$

Hence $I = \frac{6\theta}{5} + \frac{3}{5}\ell n|\sin\theta+2\cos\theta+3| - \frac{8}{5}\tan^{-1}\left(\frac{\tan\theta/2+1}{2}\right) + C$ **Ans.**

Do yourself -9 :

(i) Evaluate : $\int \frac{\sin x}{\sin x+\cos x}dx$ **(ii)** Evaluate : $\int \frac{3\sin x+2\cos x}{3\cos x+2\sin x}dx$

(d) Integration of rational function :

(i) Rational function is defined as the ratio of two polynomials in the form $\frac{P(x)}{Q(x)}$, where P(x) and Q(x) are polynomials in x and $Q(x) \neq 0$. If the degree of P(x) is less than the degree of Q(x), then the rational function is called proper, otherwise, it is called improper. The improper rational function can be reduced to the proper rational functions by long division process. Thus, if $\frac{P(x)}{Q(x)}$ is improper, then $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$, where T(x) is a polynomial in x and $\frac{P_1(x)}{Q(x)}$ is proper rational function. It is always possible to write the integrand as a sum of simpler rational functions by a method called partial fraction decomposition. After this, the integration can be carried out easily using the already known methods.

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S. No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
2.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
3.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ where x^2+bx+c cannot be factorised further	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

Illustration 17 : Evaluate : $\int \frac{x}{(x-2)(x+5)}dx$

Solution : $\frac{x}{(x-2)(x+5)}=\frac{A}{x-2}+\frac{B}{x+5}$

or $x = A(x + 5) + B(x - 2)$.

by comparing the coefficients, we get

A = 2/7 and B = 5/7 so that

$$\int \frac{x}{(x-2)(x+5)}dx=\frac{2}{7}\int\frac{dx}{x-2}+\frac{5}{7}\int\frac{dx}{x+5}=\frac{2}{7}\ell n|(x-2)|+\frac{5}{7}\ell n|(x+5)|+C$$ **Ans.**

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Illustration 18 : Evaluate $\int \frac{x^4}{(x+2)(x^2+1)}dx$

Solution : $\frac{x^4}{(x+2)(x^2+1)}=(x-2)+\frac{3x^2+4}{(x+2)(x^2+1)}$

Now, $\frac{3x^2+4}{(x+2)(x^2+1)}=\frac{16}{5(x+2)}+\frac{-\frac{1}{5}x+\frac{2}{5}}{x^2+1}$

So, $\frac{x^4}{(x+2)(x^2+1)}=x-2+\frac{16}{5(x+2)}+\frac{-\frac{1}{5}x+\frac{2}{5}}{x^2+1}$

Now, $\int\left((x-2)+\frac{16}{5(x+2)}+\frac{-\frac{1}{5}x+\frac{2}{5}}{x^2+1}\right)dx$

$$=\frac{x^2}{2}-2x+\frac{2}{5}\tan^{-1}x+\frac{16}{5}\ell n|x+2|-\frac{1}{10}\ell n(x^2+1)+C$$ **Ans.**

Do yourself - 10 :

(i) Evaluate : $\int \frac{3x+2}{(x+1)(x+3)}dx$ **(ii)** Evaluate : $\int \frac{x^2-1}{(x+1)(x+2)^2}dx$

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(ii) $\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\,dx$

Express $ax^2 + bx + c$ in the form of perfect square & then apply the standard results.

(iii) $\int \frac{px+q}{ax^2+bx+c}dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Express $px + q = \ell$ (differential coefficient of denominator) + m.

Illustration 19 : Evaluate $\int \frac{dx}{2x^2+x-1}$

Solution :

$$I = \int \frac{dx}{2x^2+x-1} = \frac{1}{2}\int \frac{dx}{x^2+\frac{x}{2}-\frac{1}{2}} = \frac{1}{2}\int \frac{dx}{x^2+\frac{x}{2}+\frac{1}{16}-\frac{1}{16}-\frac{1}{2}}$$

$$= \frac{1}{2}\int \frac{dx}{(x+1/4)^2-9/16} = \frac{1}{2}\int \frac{dx}{(x+1/4)^2-(3/4)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{2(3/4)}\log\left|\frac{x+1/4-3/4}{x+1/4+3/4}\right| + C \qquad \left\{\text{using, } \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C\right\}$$

$$= \frac{1}{3}\log\left|\frac{x-1/2}{x+1}\right| + C = \frac{1}{3}\log\left|\frac{2x-1}{2(x+1)}\right| + C$$ **Ans.**

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Illustration 20 : Evaluate $\int \frac{3x+2}{4x^2+4x+5}dx$

Solution : Express $3x + 2 = \ell$(d.c. of $4x^2 + 4x + 5$) + m

or, $3x + 2 = \ell(8x + 4) + m$

Comparing the coefficients, we get

$8\ell = 3$ and $4\ell + m = 2 \Rightarrow \ell = 3/8$ and $m = 2 - 4\ell = 1/2$

$$\Rightarrow I = \frac{3}{8}\int \frac{8x+4}{4x^2+4x+5}dx + \frac{1}{2}\int \frac{dx}{4x^2+4x+5}$$

$$= \frac{3}{8}\log\left|4x^2+4x+5\right| + \frac{1}{8}\int \frac{dx}{x^2+x+\frac{5}{4}}$$

$$= \frac{3}{8}\log\left|4x^2+4x+5\right| + \frac{1}{8}\tan^{-1}\left(x+\frac{1}{2}\right) + C$$ **Ans.**

Do yourself -11 :

(i) Evaluate : $\int \frac{dx}{x^2+x+1}$

(ii) Evaluate : $\int \frac{5x+4}{\sqrt{x^2+4x+1}}dx$

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(iv) Integrals of the form $\int \frac{x^2+1}{x^4+Kx^2+1}dx$ OR $\int \frac{x^2-1}{x^4+Kx^2+1}dx$, where K is any constant.

Divide N^r & D^r by x^2 & proceed.

Note : Sometimes it is useful to write the integral as a sum of two related integrals, which can be evaluated by making suitable substitutions e.g.

* $\int \frac{2x^2}{x^4+1}dx = \int \frac{x^2+1}{x^4+1}dx + \int \frac{x^2-1}{x^4+1}dx$ * $\int \frac{2}{x^4+1}dx = \int \frac{x^2+1}{x^4+1}dx - \int \frac{x^2-1}{x^4+1}dx$

These integrals can be called as **Algebric Twins**.

Illustration 21 : Evaluate : $\int \frac{4}{\sin^4 x+\cos^4 x}dx$

Solution : $I = 4\int \frac{1}{\sin^4 x+\cos^4 x}dx = 4\int \frac{\sec^4 x}{1+\tan^4 x}dx$

$$= 4\int \frac{(\tan^2 x+1)\sec^2 x}{(\tan^4 x+1)}dx$$

Now, put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\Rightarrow I = 4\int \frac{1+t^2}{1+t^4}dt = 4\int \frac{1/t^2+1}{t^2+1/t^2}dt$$

Now, put $t - 1/t = z \Rightarrow \left(1+\frac{1}{t^2}\right)dt = dz$

$$\Rightarrow I = 4\int \frac{dz}{z^2+2} = \frac{4}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}} + C = 2\sqrt{2}\tan^{-1}\frac{t-1/t}{\sqrt{2}} + C$$

$$= 2\sqrt{2}\tan^{-1}\left(\frac{\tan x - 1/\tan x}{\sqrt{2}}\right) + C$$ **Ans.**

Illustration 22 : Evaluate : $\int \frac{1}{x^4+5x^2+1}dx$

Solution : $I = \frac{1}{2}\int \frac{2}{x^4+5x^2+1}dx$

$$\Rightarrow \quad I = \frac{1}{2}\int \frac{1+x^2}{x^4+5x^2+1}dx + \frac{1}{2}\int \frac{1-x^2}{x^4+5x^2+1}dx$$

$$= \frac{1}{2}\int \frac{1+1/x^2}{x^2+5+1/x^2}dx - \frac{1}{2}\int \frac{1-1/x^2}{x^2+5+1/x^2}dx$$

{dividing N^r and D^r by x^2}

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$$=\frac{1}{2}\int\frac{(1+1/x^2)}{(x-1/x)^2+7}dx-\frac{1}{2}\int\frac{(1-1/x^2)dx}{(x+1/x)^2+3}$$

$$=\frac{1}{2}\int\frac{dt}{t^2+(\sqrt{7})^2}-\frac{1}{2}\int\frac{du}{u^2+(\sqrt{3})^2}$$

where $t = x-\frac{1}{x}$ and $u = x+\frac{1}{x}$

$$I=\frac{1}{2}.\frac{1}{\sqrt{7}}\left(\tan^{-1}\frac{t}{\sqrt{7}}\right)-\frac{1}{2}.\frac{1}{\sqrt{3}}\left(\tan^{-1}\frac{u}{\sqrt{3}}\right)+C$$

$$=\frac{1}{2}\left[\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{7}}\right)-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+1/x}{\sqrt{3}}\right)\right]+C$$ **Ans.**

Do yourself -12 :

(i) Evaluate : $\int\frac{x^2+1}{x^4-x^2+1}dx$ **(ii)** Evaluate : $\int\frac{1}{1+x^4}dx$

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(e) Integration of Irrational functions :

(i) $\int\frac{dx}{(ax+b)\sqrt{px+q}}$ & $\int\frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$; put $px+q=t^2$

(ii) $\int\frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$, put $ax+b=\frac{1}{t}$; $\int\frac{dx}{(ax^2+b)\sqrt{px^2+q}}$, put $x=\frac{1}{t}$

Illustration 23 : Evaluate $\int\frac{x+2}{(x^2+3x+3)\sqrt{x+1}}.dx$

Solution : Let, $I=\int\frac{x+2}{(x^2+3x+3)\sqrt{x+1}}.dx$ Put $x+1=t^2 \Rightarrow dx=2tdt$

$$\therefore\quad I=\int\frac{(t^2-1)+2}{\{(t^2-1)^2+3(t^2-1)+3\}\sqrt{t^2}}.(2t)dt=2\int\frac{t^2+1}{t^4+t^2+1}dt=2\int\frac{1+1/t^2}{t^2+1+1/t^2}dt$$

$$=2\int\frac{1+1/t^2}{(t-1/t)^2+(\sqrt{3})^2}.dt=2\int\frac{du}{u^2+(\sqrt{3})^2}\qquad\left\{\text{where } u=t-\frac{1}{t}\right\}$$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)+C=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{3}t}\right)+C=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3(x+1)}}\right)+C$$ **Ans.**

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Illustration 24 : Evaluate $\int \frac{dx}{(x-1)\sqrt{x^2+x+1}}$

Solution : Let, $I = \int \frac{dx}{(x-1)\sqrt{x^2+x+1}}$ put $x - 1 = \frac{1}{t}$ $\Rightarrow$ $dx = -1/t^2\, dt$

$$I = \int \frac{-1/t^2 dt}{1/t\sqrt{\left(\frac{1}{t}+1\right)^2+\left(\frac{1}{t}+1\right)+1}} = -\int \frac{dt}{\sqrt{3t^2+3t+1}}$$

$$= -\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2+1/12}} = -\frac{1}{\sqrt{3}}\log\left|(t+1/2)+\sqrt{(t+1/2)^2+1/12}\right|+C$$

$$= -\frac{1}{\sqrt{3}}\log\left|\left(\frac{1}{x-1}+\frac{1}{2}\right)+\sqrt{\frac{12\left(\frac{1}{x-1}+\frac{1}{2}\right)^2+1}{12}}\right|+C$$ **Ans.**

Illustration 25 : Evaluate : $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Solution : Let, $I = \int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$ Put $x = \frac{1}{t}$, So that $dx = \frac{-1}{t^2}dt$

$$\therefore \quad I = \int \frac{-1/t^2 dt}{(1+1/t^2)\sqrt{1-1/t^2}} = -\int \frac{t dt}{(t^2+1)\sqrt{t^2-1}}$$

Again let, $t^2 = u$. So that $2t\, dt = du$.

$= \frac{-1}{2}\int \frac{du}{(u+1)\sqrt{u-1}}$ which reduces to the form $\int \frac{dx}{P\sqrt{Q}}$ where both P and Q are linear

so that we put $u - 1 = z^2$ so that $du = 2z\, dz$

$$\therefore \quad I = -\frac{1}{2}\int \frac{2zdz}{(z^2+1+1)\sqrt{z^2}} = -\int \frac{dz}{(z^2+2)}$$

$$I = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z}{\sqrt{2}}\right)+C$$

$$I = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{u-1}}{\sqrt{2}}\right)+C = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{t^2-1}}{\sqrt{2}}\right)+C$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right)+C$$ **Ans.**

Do yourself -13 :

(i) Evaluate : $\int \frac{x}{(x-3)\sqrt{x+1}}dx$ **(ii)** Evaluate : $\int \frac{dx}{x^2\sqrt{1+x^2}}$

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(f) Manipulating integrands :

(i) $\int \frac{dx}{x(x^n+1)}$, $n \in N$, take x^n common & put $1 + x^{-n} = t$.

(ii) $\int \frac{dx}{x^2(x^n+1)^{(n-1)/n}}$, $n \in N$, take x^n common & put $1 + x^{-n} = t^n$

(iii) $\int \frac{dx}{x^n(1+x^n)^{1/n}}$, take x^n common and put $1 + x^{-n} = t^n$.

Illustration 26 : Evaluate : $\int \frac{dx}{x^n(1+x^n)^{1/n}}$

Solution : Let $I = \int \frac{dx}{x^n(1+x^n)^{1/n}} = \int \frac{dx}{x^{n+1}\left(1+\frac{1}{x^n}\right)^{1/n}}$

Put $1+\frac{1}{x^n} = t^n$, then $\frac{1}{x^{n+1}}dx = -t^{n-1}dt$

$$I = -\int \frac{t^{n-1}dt}{t} = -\int t^{n-2}dt = -\frac{t^{n-1}}{n-1} + C = \frac{-1}{n-1}\left(1+\frac{1}{x^n}\right)^{\frac{n-1}{n}} + C$$ **Ans.**

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Do yourself -14 :

(i) Evaluate : $\int \frac{dx}{x(x^2+1)}$

(ii) Evaluate : $\int \frac{dx}{x^2(x^3+1)^{2/3}}$

(iii) Evaluate : $\int \frac{dx}{x^3(x^3+1)^{1/3}}$

<u>Miscellaneous Illustrations</u> :

Illustration 27 : Evaluate $\int \frac{\cos^4 x dx}{\sin^3 x\{\sin^5 x + \cos^5 x\}^{\frac{3}{5}}}$

Solution : $I = \int \frac{\cos^4 x}{\sin^3 x\{\sin^5 x + \cos^5 x\}^{\frac{3}{5}}}dx = \int \frac{\cos^4 x}{\sin^6 x\{1+\cot^5 x\}^{\frac{3}{5}}}dx = \int \frac{\cot^4 x \operatorname{cosec}^2 x dx}{(1+\cot^5 x)^{3/5}}$

Put $1 + \cot^5 x = t$

$5\cot^4 x \operatorname{cosec}^2 x dx = -dt$

$$= -\frac{1}{5}\int \frac{dt}{t^{3/5}} = -\frac{1}{2}t^{2/5} + C = -\frac{1}{2}(1+\cot^5 x)^{2/5} + C$$ **Ans.**

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Illustration 28 : $\int \frac{dx}{\cos^6 x + \sin^6 x}$ is equal to -

(A) $\ell n|\tan x - \cot x| + C$ (B) $\ell n|\cot x - \tan x| + C$

(C) $\tan^{-1}(\tan x - \cot x) + C$ (D) $\tan^{-1}(-2\cot 2x) + C$

Solution : Let $I = \int \frac{dx}{\cos^6 x + \sin^6 x} = \int \frac{\sec^6 x}{1+\tan^6 x} dx = \int \frac{(1+\tan^2 x)^2 \sec^2 x dx}{1+\tan^6 x}$

If $\tan x = p$, then $\sec^2 x\, dx = dp$

$$\Rightarrow \quad I = \int \frac{(1+p^2)^2 dp}{1+p^6} = \int \frac{(1+p^2)}{p^4 - p^2 + 1} dp = \int \frac{p^2\left(1+\frac{1}{p^2}\right)}{p^2\left(p^2 + \frac{1}{p^2} - 1\right)} dp$$

$$= \int \frac{dk}{k^2+1} = \tan^{-1}(k) + C \qquad \left(\text{where } p - \frac{1}{p} = k, \left(1+\frac{1}{p^2}\right)dp = dk\right)$$

$$= \tan^{-1}\left(p - \frac{1}{p}\right) + C = \tan^{-1}(\tan x - \cot x) + C = \tan^{-1}(-2\cot 2x) + C$$ **Ans. (C,D)**

Illustration 29 : Evaluate : $\int \frac{2\sin 2x - \cos x}{6 - \cos^2 x - 4\sin x} dx$

Solution : $I = \int \frac{2\sin 2x - \cos x}{6 - \cos^2 x - 4\sin x} dx = \int \frac{(4\sin x - 1)\cos x}{6 - (1 - \sin^2 x) - 4\sin x} dx = \int \frac{(4\sin x - 1)\cos x}{\sin^2 x - 4\sin x + 5} dx$

Put $\sin x = t$, so that $\cos x\, dx = dt$.

$$\therefore \quad I = \int \frac{(4t-1)dt}{(t^2 - 4t + 5)} \qquad \text{..... (i)}$$

Now, let $(4t - 1) = \lambda(2t - 4) + \mu$

Comparing coefficients of like powers of t, we get

$2\lambda = 4, -4\lambda + \mu = -1$ ..... (ii)

$\lambda = 2, \mu = 7$

$$\therefore \quad I = \int \frac{2(2t-4)+7}{t^2 - 4t + 5} dt \qquad \{\text{using (i) and (ii)}\}$$

$$= 2\int \frac{2t-4}{t^2-4t+5} dt + 7\int \frac{dt}{t^2-4t+5} = 2\log|t^2 - 4t + 5| + 7\int \frac{dt}{t^2 - 4t + 4 - 4 + 5}$$

$$= 2\log|t^2 - 4t + 5| + 7\int \frac{dt}{(t-2)^2 + (1)^2} = 2\log|t^2 - 4t + 5| + 7.\tan^{-1}(t-2) + C$$

$= 2\log|\sin^2 x - 4\sin x + 5| + 7\tan^{-1}(\sin x - 2) + C.$ **Ans.**

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Illustration 30 : The value of $\int\sqrt{\frac{3-x}{3+x}}.\sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-x}\right)dx$, is equal to -

(A) $\frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2+2\sqrt{9-x^2}.\cos^{-1}\left(\frac{x}{3}\right)+2x\right\}+C$

(B) $\frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2+2\sqrt{9-x^2}.\sin^{-1}\left(\frac{x}{3}\right)+2x\right\}+C$

(C) $\frac{1}{4}\left\{-3\left(\sin^{-1}\left(\frac{x}{3}\right)\right)^2+2\sqrt{9-x^2}.\sin^{-1}\left(\frac{x}{3}\right)+2x\right\}+C$

(D) none of these

Solution : Here, $I=\int\sqrt{\frac{3-x}{3+x}}.\sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-x}\right)dx$

Put $x = 3\cos 2\theta \quad\Rightarrow\quad dx = -6\sin 2\theta d\theta$

$$=\int\sqrt{\frac{3-3\cos 2\theta}{3+3\cos 2\theta}}.\sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-3\cos 2\theta}\right)(-6\sin 2\theta)d\theta$$

$$=\int\frac{\sin\theta}{\cos\theta}.\sin^{-1}(\sin\theta).(-6\sin 2\theta)d\theta=-6\int\theta.(2\sin^2\theta)d\theta$$

$$=-6\int\theta(1-\cos 2\theta)d\theta=-6\left\{\frac{\theta^2}{2}-\int\theta\cos 2\theta\, d\theta\right\}+C$$

$$=-6\left\{\frac{\theta^2}{2}-\left(\theta\frac{\sin 2\theta}{2}-\int 1.\left(\frac{\sin 2\theta}{2}\right)d\theta\right)\right\}+C=-3\theta^2+6\left\{\theta\frac{\sin 2\theta}{2}+\frac{\cos 2\theta}{4}\right\}+C$$

$$=\frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2+2\sqrt{9-x^2}.\cos^{-1}\left(\frac{x}{3}\right)+2x\right\}+C$$ **Ans. (A)**

Illustration 31 : Evaluate : $\int\frac{\tan\left(\frac{\pi}{4}-x\right)}{\cos^2 x\sqrt{\tan^3 x+\tan^2 x+\tan x}}dx$

Solution : $I=\int\frac{\tan\left(\frac{\pi}{4}-x\right)}{\cos^2 x\sqrt{\tan^3 x+\tan^2 x+\tan x}}dx=\int\frac{(1-\tan^2 x)dx}{(1+\tan x)^2\cos^2 x\sqrt{\tan^3 x+\tan^2 x+\tan x}}$

$$I=\int\frac{-\left(1-\frac{1}{\tan^2 x}\right)\sec^2 x dx}{\left(\tan x+2+\frac{1}{\tan x}\right)\sqrt{\tan x+1+\frac{1}{\tan x}}}$$

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$$\text{let, } y=\sqrt{\tan x+1+\frac{1}{\tan x}} \Rightarrow 2y\,dy=\left(\sec^2 x-\frac{1}{\tan^2 x}.\sec^2 x\right)dx$$

$$\therefore \quad I=\int\frac{-2ydy}{(y^2+1).y}=-2\int\frac{dy}{1+y^2}$$

$$=-2\tan^{-1}y+c=-2\tan^{-1}\left(\sqrt{\tan x+1+\frac{1}{\tan x}}\right)+C$$ **Ans.**

ANSWERS FOR DO YOURSELF

1 : **(i)** $\frac{1}{36}\tan^{-1}\left(\frac{4x^3}{3}\right)+C$ **(ii)** $\sin x-\frac{1}{3}\sin^3 x+C$

2 : **(i)** $\sqrt{(x-2)(3-x)}-\sin^{-1}\sqrt{3-x}+C$ **(ii)** $\frac{1}{2}\ell n\left[\frac{\sqrt{x^2+4}-2}{x}\right]+C$

3 : **(i)** xe^x-e^x+C **(ii)** $\frac{1}{2}\left[-x^2\cos x^2+\sin x^2\right]+C$

4 : **(i)** $e^x\tan^{-1}x+C$ **(ii)** $\frac{1}{2}e^{x^2}\sin(x^2)+C$

5 : **(i)** $x\tan(e^x)+C$ **(ii)** $x\ell nx+C$

6 : **(i)** $\frac{1}{3}\tan^3 x+C$ **(ii)** $\frac{2}{3}\tan^{3/2}x+C$ **(iii)** $\frac{1}{3}\sin^3 x-\frac{2}{5}\sin^5 x+\frac{1}{7}\sin^7 x+C$

7 : **(i)** $\frac{1}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\tan x\right)+C$ **(ii)** $\frac{2}{\sqrt{15}}\tan^{-1}\left(\frac{8\tan x+1}{\sqrt{15}}\right)+C$

8 : **(i)** $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3\tan x/2+1}{2\sqrt{2}}\right)+C$ **(ii)** $\frac{1}{2\sqrt{6}}\ell n\left|\frac{\sqrt{6}+\tan x/2-2}{\sqrt{6}-\tan x/2+2}\right|+C$

9 : **(i)** $\frac{1}{2}x-\frac{1}{2}\ell n|\sin x+\cos x|+C$ **(ii)** $\frac{12}{13}x-\frac{5}{13}\ell n|3\cos x+2\sin x|+C$

10 : **(i)** $-\frac{1}{2}\ell n|x+1|+\frac{7}{2}\ell n|x+3|+C$ **(ii)** $\ell n|x+2|+\frac{3}{x+2}+C$

11 : **(i)** $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$ **(ii)** $5\sqrt{x^2+4x+1}-6\ell n\left[(x+2)+\sqrt{x^2+4x+1}\right]+C$

12 : **(i)** $\tan^{-1}\left(\frac{x^2-1}{x}\right)+C$ **(ii)** $\frac{1}{2\sqrt{2}}\left[\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right)-\frac{1}{2}\ell n\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right]$

13 : **(i)** $2\sqrt{x+1}+\frac{3}{2}\ell n\left|\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right|+C$ **(ii)** $-\frac{1}{x}\sqrt{1+x^2}+C$

14 : **(i)** $-\frac{1}{2}\ell n\left(\frac{x^2+1}{x^2}\right)+C$ **(ii)** $-\left(1+\frac{1}{x^3}\right)^{1/3}+C$ **(iii)** $-\frac{1}{2}\left(1+\frac{1}{x^3}\right)^{2/3}+C$

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EXERCISE (O-1)

1. Let g (x) be an antiderivative for f(x). Then $ln\left(1+(g(x))^2\right)$ is an antiderivative for

(A) $\frac{2f(x)g(x)}{1+(f(x))^2}$ (B) $\frac{2f(x)g(x)}{1+(g(x))^2}$ (C) $\frac{2f(x)}{1+(f(x))^2}$ (D) none

II0001

2. A function y = f (x) satisfies $f''(x) = -\frac{1}{x^2} - \pi^2 \sin(\pi x)$; $f'(2) = \pi+\frac{1}{2}$ and $f(1) = 0$. The value of $f\left(\frac{1}{2}\right)$ is

(A) $ln\,2$ (B) 1 (C) $\frac{\pi}{2} - ln\,2$ (D) $1 - ln\,2$

II0002

3. Consider $f(x) = \frac{x^2}{1+x^3}$; $g(t) = \int f(t)dt$. If g(1) = 0 then g(x) equals-

(A) $\frac{1}{3}\ell n(1+x^3)$ (B) $\frac{1}{3}\ell n\left(\frac{1+x^3}{2}\right)$ (C) $\frac{1}{2}\ell n\left(\frac{1+x^3}{3}\right)$ (D) $\frac{1}{3}\ell n\left(\frac{1+x^3}{3}\right)$

II0003

4. $\int \frac{x^2(1-ln\,x)}{ln^4x - x^4}dx$ equals

(A) $\frac{1}{2}ln\left(\frac{x}{ln\,x}\right)-\frac{1}{4}ln\left(ln^2x-x^2\right)+C$ (B) $\frac{1}{4}ln\left(\frac{ln\,x-x}{ln\,x+x}\right)-\frac{1}{2}\tan^{-1}\left(\frac{ln\,x}{x}\right)+C$

(C) $\frac{1}{4}ln\left(\frac{ln\,x+x}{ln\,x-x}\right)+\frac{1}{2}\tan^{-1}\left(\frac{ln\,x}{x}\right)+C$ (D) $\frac{1}{4}\left(ln\left(\frac{ln\,x-x}{ln\,x+x}\right)+\tan^{-1}\left(\frac{ln\,x}{x}\right)\right)+C$

II0004

5. If $\int \frac{(2x+3)dx}{x(x+1)(x+2)(x+3)+1} = C - \frac{1}{f(x)}$, where f(x) is of the form of $ax^2 + bx + c$ then (a + b + c) equals

(A) 4 (B) 5 (C) 6 (D) none

II0005

6. $\int \frac{x^3}{(2x^2+1)^3}dx$ is equal to-

(A) $\frac{1}{4}\left(2+\frac{1}{x^2}\right)^{-2}+C$ (B) $-\frac{1}{4}\left(2+\frac{1}{x^2}\right)^{-2}+C$

(C) $\frac{1}{2}\left(2+\frac{1}{x^2}\right)^{-2}+C$ (D) $\frac{1}{4}\left(2+\frac{1}{x^2}\right)^{2}+C$

(where 'C' is integration constant)

II0006

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7. Primitive of $\frac{3x^4-1}{(x^4+x+1)^2}$ w.r.t. x is -

(A) $\frac{x}{x^4+x+1}+c$ (B) $-\frac{x}{x^4+x+1}+c$ (C) $\frac{x+1}{x^4+x+1}+c$ (D) $-\frac{x+1}{x^4+x+1}+c$

II0007

8. If $\int \frac{\cos x-\sin x+1-x}{e^x+\sin x+x}dx = ln(f(x)) + g(x) + C$ where C is the constant of integration and $f(x)$ is positive, then $f(x) + g(x)$ has the value equal to

(A) $e^x + \sin x + 2x$ (B) $e^x + \sin x$ (C) $e^x - \sin x$ (D) $e^x + \sin x + x$

II0008

9. $\int \frac{dx}{\sqrt[3]{x^{5/2}(x+1)^{7/2}}}$

(A) $-\left(\frac{x+1}{x}\right)^{1/6}+C$ (B) $6\left(\frac{x+1}{x}\right)^{-1/6}+C$ (C) $\left(\frac{x}{x+1}\right)^{5/6}+C$ (D) $-\left(\frac{x}{x+1}\right)^{5/6}+C$

II0009

10. $\int \frac{1-x^7}{x(1+x^7)}dx$ equals -

(A) $ln\,x + \frac{2}{7}\,ln\,(1+x^7) + c$ (B) $ln\,x - \frac{2}{7}\,ln\,(1-x^7) + c$

(C) $ln\,x - \frac{2}{7}\,ln\,(1+x^7) + c$ (D) $ln\,x + \frac{2}{7}\,ln\,(1-x^7) + c$

II0010

11. Integral of $\sqrt{1+2\cot x(\cot x+\operatorname{cosec}x)}$ w.r.t. x is

(A) $2\ln\cos\frac{x}{2}+c$ (B) $2\ln\sin\frac{x}{2}+c$

(C) $\frac{1}{2}\ln\cos\frac{x}{2}+c$ (D) $\ln\sin x - \ln(\operatorname{cosec}x - \cot x) + c$

II0011

12. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\left(x+\sqrt{x}\right)dx$

(A) $2e^{\sqrt{x}}\left[x-\sqrt{x}+1\right]+C$ (B) $e^{\sqrt{x}}\left[x-2\sqrt{x}+1\right]$

(C) $e^{\sqrt{x}}\left(x+\sqrt{x}\right)+C$ (D) $e^{\sqrt{x}}\left(x+\sqrt{x}+1\right)+C$

II0012

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13. Let $f(x)=\frac{2\sin^2 x-1}{\cos x}+\frac{\cos x(2\sin x+1)}{1+\sin x}$ then $\int e^x(f(x)+f'(x))dx$ where c is the constant of integration)

(A) $e^x \tan x + c$ (B) $e^x \cot x + c$ (C) $e^x \operatorname{cosec}^2 x + c$ (D) $e^x \sec^2 x + c$

II0013

14. $\int e^x\left(\frac{x^2-3}{(x-1)^2}\right)dx$ is equal to -

(A) $e^x\frac{(x+3)}{(x-1)}+C$ (B) $e^x\left(\frac{x-3}{x-1}\right)+C$ (C) $e^x\left(\frac{x+1}{x-1}\right)+C$ (D) $e^x\left(\frac{1}{x-1}\right)^2+C$

(where C is constant of integration)

II0014

15. $\int x\,.\frac{ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}dx$ equals -

(A) $\sqrt{1+x^2}\ ln\left(x+\sqrt{1+x^2}\right)-x+c$ (B) $\frac{x}{2}\,.\,ln^2\left(x+\sqrt{1+x^2}\right)-\frac{x}{\sqrt{1+x^2}}+c$

(C) $\frac{x}{2}\,.\,ln^2\left(x+\sqrt{1+x^2}\right)+\frac{x}{\sqrt{1+x^2}}+c$ (D) $\sqrt{1+x^2}\,ln\left(x+\sqrt{1+x^2}\right)+x+c$

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II0015

EXERCISE (O-2)

Paragraph for Question Nos. 1 to 3

A curve is represented parametrically by the equations $x = e^t \cos t$ and $y = e^t \sin t$ where t is a parameter. Then

1. The relation between the parameter 't' and the angle α between the tangent to the given curve and the x-axis is given by, 't' equals

(A) $\frac{\pi}{2}-\alpha$ (B) $\frac{\pi}{4}+\alpha$ (C) $\alpha-\frac{\pi}{4}$ (D) $\frac{\pi}{4}-\alpha$

II0016

2. The value of $\frac{d^2y}{dx^2}$ at the point where t = 0 is

(A) 1 (B) 2 (C) – 2 (D) 3

II0016

3. If $F(t)=\int(x+y)\,dt$ then the value of $F\left(\frac{\pi}{2}\right)-F(0)$ is

(A) 1 (B) – 1 (C) $e^{\pi/2}$ (D) 0

II0016

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4. The evaluation of $\int \frac{p\,x^{p+2q-1} - q\,x^{q-1}}{x^{2p+2q} + 2x^{p+q} + 1}dx$ is

(A) $-\frac{x^p}{x^{p+q}+1} + C$

(B) $\frac{x^q}{x^{p+q}+1} + C$

(C) $-\frac{x^q}{x^{p+q}+1} + C$

(D) $\frac{x^p}{x^{p+q}+1} + C$

II0017

5. The integral $\int \sqrt{\cot x}\, e^{\sqrt{\sin x}} \sqrt{\cos x}\, dx$ equals

(A) $\frac{\sqrt{\tan x}\, e^{\sqrt{\sin x}}}{\sqrt{\cos x}} + C$

(B) $2e^{\sqrt{\sin x}} + C$

(C) $-\frac{1}{2}e^{\sqrt{\sin x}} + C$

(D) $\frac{\sqrt{\cot x}\, e^{\sqrt{\sin x}}}{2\sqrt{\cos x}} + C$

II0018

6. $\int \frac{\cot^{-1}(e^x)}{e^x}dx$ is equal to -

(A) $\frac{1}{2}\ell n\left(e^{2x}+1\right) - \frac{\cot^{-1}(e^x)}{e^x} + x + c$

(B) $\frac{1}{2}\ell n\left(e^{2x}+1\right) + \frac{\cot^{-1}(e^x)}{e^x} + x + c$

(C) $\frac{1}{2}\ell n\left(e^{2x}+1\right) - \frac{\cot^{-1}(e^x)}{e^x} - x + c$

(D) $\frac{1}{2}\ell n\left(e^{2x}+1\right) + \frac{\cot^{-1}(e^x)}{e^x} - x + c$

II0019

7. $\int \left(\sin(101x)\cdot\sin^{99} x\right)dx$ equals

(A) $\frac{\sin(100x)(\sin x)^{100}}{100} + C$

(B) $\frac{\cos(100x)(\sin x)^{100}}{100} + C$

(C) $\frac{\cos(100x)(\cos x)^{100}}{100} + C$

(D) $\frac{\sin(100x)(\sin x)^{101}}{101} + C$

II0020

8. $\int \sec^2\theta(\sec\theta + \tan\theta)^2 d\theta$

(A) $\frac{(\sec\theta+\tan\theta)}{2}[2 + \tan\theta(\sec\theta + \tan\theta)] + C$

(B) $\frac{(\sec\theta+\tan\theta)}{3}[2 + 4\tan\theta\,(\sec\theta + \tan\theta)] + C$

(C) $\frac{(\sec\theta+\tan\theta)}{3}[2 + \tan\theta(\sec\theta + \tan\theta)] + C$

(D) $\frac{3(\sec\theta+\tan\theta)}{2}[2 + \tan\theta(\sec\theta + \tan\theta)] + C$

II0021

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Multiple Correct :

9. Which one of the following is **FALSE** ?

(A) $x.\int \frac{dx}{x} = x\,ln|x| + C$

(B) $x.\int \frac{dx}{x} = x\,ln|x| + Cx$

(C) $\frac{1}{\cos x}.\int \cos x\ dx = \tan x + C$

(D) $\frac{1}{\cos x}.\int \cos x\ dx = x + C$

II0022

10. If $I_n = \int (\sin x)^n dx\ \ n \in N$, then $5I_4 - 6I_6$ is equal to-

(A) $\sin x.(\cos x)^5 + C$

(B) $\cos x.(\sin x)^5 + C$

(C) $\frac{\sin 2x}{8}[\cos^2 2x + 1 - 2\cos 2x] + C$

(D) $\frac{\sin 2x}{8}[\cos^2 2x + 1 + 2\cos 2x] + C$

II0023

11. Let $f(x) = \sin^3 x + \sin^3\left(x + \frac{2\pi}{3}\right) + \sin^3\left(x + \frac{4\pi}{3}\right)$ then the primitive of $f(x)$ w.r.t. x is

(A) $-\frac{3\sin 3x}{4} + C$

(B) $\frac{1}{2}\cos^2\left(\frac{3x}{2}\right) + C$

(C) $\frac{\sin 3x}{4} + C$

(D) $\frac{\cos 3x}{4} + C$

where C is an arbitrary constant.

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II0024

12. Suppose $J = \int \frac{\sin^2 x + \sin x}{1 + \sin x + \cos x} dx$ and $K = \int \frac{\cos^2 x + \cos x}{1 + \sin x + \cos x} dx$. If C is an arbitrary constant of integration then which of the following is/are correct?

(A) $J = \frac{1}{2}(x - \sin x + \cos x) + C$

(B) $J = K - (\sin x + \cos x) + C$

(C) $J = x - K + C$

(D) $K = \frac{1}{2}(x - \sin x + \cos x) + C$

II0025

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Match the column :

13. **Column-I** **Column-II**

(A) Let $f(x) = \int x^{\sin x}(1 + x\cos x \cdot \ln x + \sin x)dx$ and $f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$ then the value of $f(\pi)$ is (P) rational

II0026

(B) Let $g(x) = \int \frac{1+2\cos x}{(\cos x+2)^2}dx$ and $g(0) = 0$ then the value of $g\left(\frac{\pi}{2}\right)$ is (Q) irrational

II0027

(C) Let $k(x) = \int \frac{(x^2+1)\,dx}{\sqrt[3]{x^3+3x+6}}$ and $k(-1) = \frac{1}{\sqrt[3]{2}}$ then the value of $k(-2)$ is (R) integral

(S) prime

II0028

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14. **Column-I** **Column-II**

(A) $\int \frac{x^2(x^6+x^5-1)dx}{(2x^6+3x^5+2)^2}$ (P) $\frac{1}{6}\left(\frac{1}{x^{-3}}+\frac{3}{x^{-2}}\right)^{\frac{1}{2}}+C$

II0029

(B) $\int \frac{(x^5+x^4+x^2)dx}{\sqrt{4x^7+5x^6+10x^4}}$ (Q) $\frac{1}{2}(1+x^{-2}+x^{-5})^{-2}+C$

II0030

(C) $\int \frac{(2x^{12}+5x^9)dx}{(x^5+x^3+1)^3}$ (R) $\frac{-1}{6}(2x^3+3x^2+2x^{-3})^{-1}+C$

II0031

(where C is the constant of integration.) (S) $x\left(\frac{x^3}{25}+\frac{x^2}{20}+\frac{1}{10}\right)^{\frac{1}{2}}+C$

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EXERCISE (S-1)

1. $\int\left[\sin\alpha\sin(x-\alpha)+\sin^2\left(\frac{x}{2}-\alpha\right)\right]dx$ **II0032**

2. $\int\frac{dx}{\cot\frac{x}{2}.\cot\frac{x}{3}.\cot\frac{x}{6}}$ **II0033**

3. Let $\int\frac{f'(x)g(x)-g'(x)f(x)}{(f(x)+g(x))\sqrt{f(x)g(x)-g^2(x)}}dx=\sqrt{m}\tan^{-1}\left(\sqrt{\frac{f(x)-g(x)}{ng(x)}}\right)+C$, **II0034**

where $m,n\in N$ and 'C' is constant of integration $(g(x)>0)$. Find the value of (m^2+n^2).

4. $\int\frac{dx}{(x-\alpha)\sqrt{(x-\alpha)(x-\beta)}}$ **II0035**

5. $\int\frac{l\,n\left(l\,n\left(\frac{1+x}{1-x}\right)\right)}{1-x^2}dx$ **II0036**

6. $\int\frac{\tan 2\theta}{\sqrt{\cos^6\theta+\sin^6\theta}}d\theta$ **II0037**

7. $\int\frac{(ax^2-b)\ \ dx}{x\sqrt{c^2x^2-(ax^2+b)^2}}$ **II0038**

8. $\int\left[\frac{\sqrt{x^2+1}\ \left[\ell n(x^2+1)-2\ell n\,x\right]}{x^4}\right]dx$ **II0039**

9. $\int\frac{(\sqrt{x}+1)(x^2-\sqrt{x})}{x\sqrt{x}+x+\sqrt{x}}dx$ **II0040**

10. A function g defined for all positive real numbers, satisfies $g'(x^2)=x^3$ for all $x>0$ and $g(1)=1$. Compute g(4).

II0041

11. $\int\sqrt{\frac{\operatorname{cosec}x-\cot x}{\operatorname{cosec}x+\cot x}}.\frac{\sec x}{\sqrt{1+2\sec x}}dx$ **II0042**

12. $\int\frac{(\sqrt{x}+1)dx}{\sqrt{x}\left(\sqrt[3]{x}+1\right)}$ **II0043**

13. $\int\frac{x^5+3x^4-x^3+8x^2-x+8}{x^2+1}dx$ **II0044**

14. $\int\sin^{-1}\sqrt{\frac{x}{a+x}}dx$ **II0045**

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15. $\int \frac{e^{\cos x}(x\sin^3 x+\cos x)}{\sin^2 x}dx$ **II0046**

16. $\int \frac{5x^4+4x^5}{(x^5+x+1)^2}dx$ **II0047**

17. $\int (\sin x)^{-11/3}(\cos x)^{-1/3}dx$ **II0048**

18. $\int \frac{dx}{\sin x+\sec x}$ **II0049**

19. If the value $\int \frac{1-(\cot x)^{2008}}{\tan x+(\cot x)^{2009}}dx=\frac{1}{k}l\,n\,|\sin^k x+\cos^k x|+C$, then find k. **II0050**

20. Suppose $\int \frac{1-7\cos^2 x}{\sin^7 x\cos^2 x}dx=\frac{g(x)}{\sin^7 x}+C$, where C is arbitrary constant of integration. Then find the value of $g'(0)+g''\left(\frac{\pi}{4}\right)$

II0051

21. $\int \frac{4x^5-7x^4+8x^3-2x^2+4x-7}{x^2(x^2+1)^2}dx$ **II0052**

22. $\int \frac{\left(e^{\sqrt{x}}-e^{-\sqrt{x}}\right)\cos\left(e^{\sqrt{x}}+e^{-\sqrt{x}}+\frac{\pi}{4}\right)+\left(e^{\sqrt{x}}+e^{-\sqrt{x}}\right)\cos\left(e^{\sqrt{x}}-e^{-\sqrt{x}}+\frac{\pi}{4}\right)}{\sqrt{x}}dx$ **II0053**

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23. $\int \frac{3x^2+1}{(x^2-1)^3}dx$ **II0054**

24. $\int \frac{x^2+3}{x^6(x^2+1)}dx$ **II0055**

25. $\int \frac{x^2+x}{(e^x+x+1)^2}dx$ **II0056**

26. $\int \left[\left(\frac{x}{e}\right)^x+\left(\frac{e}{x}\right)^x\right]\ell n\,x\,dx$ **II0057**

27. $\int \frac{x\ell nx}{(x^2-1)^{3/2}}.dx$ **II0058**

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EXERCISE (S-2)

1. $\int \frac{\tan(\ln x)\tan\left(\ln\frac{x}{2}\right)\tan(\ln 2)}{x}dx$ **II0059**

2. $\int \sqrt{\frac{\sin(x-a)}{\sin(x+a)}}dx$ **II0060**

3. $\int \frac{\cos x - \sin x}{7-9\sin 2x}dx$ **II0061**

4. $\int \cos 2\theta . \ell n \frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}d\theta$ **II0062**

5. $\int \frac{\cos^2 x}{1+\tan x}dx$ **II0063**

6. $\int \frac{(1+x^2)dx}{1-2x^2\cos\alpha+x^4}\ \alpha\in(0,\pi)$ **II0064**

7. $\int \frac{dx}{\sqrt{\sin^3 x\sin(x+\alpha)}}$ **II0065**

8. $\int \frac{dx}{\left(x+\sqrt{x(1+x)}\right)^2}$ **II0066**

9. $\int \frac{e^x\left(2-x^2\right)}{(1-x)\sqrt{1-x^2}}dx$ **II0067**

10. $\int \frac{x}{(7x-10-x^2)^{3/2}}dx$ **II0068**

11. Let f(x) is a quadratic function such that f(0) = 1 and $\int \frac{f(x)dx}{x^2(x+1)^3}$ is a rational function, find the value of f '(0)

II0069

Match of Column :

12. $I_1 = \int \tan x\tan(ax+b)dx$ and $I_2 = \int \cot x\cot(ax+b)dx$

	Column-I		Column-II
(A)	value of I_1 for a = 1 is	(P)	$x-\cot b\, ln\frac{\cos(x-b)}{\cos x}+C$
(B)	value of I_2 for a = 1 is	(Q)	$\cot b\, ln\frac{\sin x}{\sin(x+b)}-x+C$
(C)	value of I_1 for a = –1 is	(R)	$\cot b\, ln\left(\frac{\cos x}{\cos(x+b)}\right)-x+C$
(D)	value of I_2 for a =–1 is	(S)	$x+\cot b\, ln\left(\frac{\sin x}{\sin(b-x)}\right)+C$

II0070

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EXERCISE (JM)

1. If the integral $\int \frac{5\tan x}{\tan x - 2}dx = x + a\ln|\sin x - 2\cos x| + k$ then a is equal to : **[AIEEE-2012]**

(1) 2 (2) –1 (3) –2 (4) 1

II0071

2. If $\int f(x)dx = \Psi(x)$, then $\int x^5 f(x^3)dx$ is equal to : **[JEE-MAIN-2013]**

(1) $\frac{1}{3}\left[x^3\Psi(x^3) - \int x^2\Psi(x^3)dx\right] + C$

(2) $\frac{1}{3}x^3\Psi(x^3) - 3\int x^3\Psi(x^3)dx + C$

(3) $\frac{1}{3}x^3\Psi(x^3) - \int x^2\Psi(x^3)dx + C$

(4) $\frac{1}{3}\left[x^3\Psi(x^3) - \int x^3\Psi(x^3)dx\right] + C$

II0072

3. The integral $\int \left(1 + x - \frac{1}{x}\right)e^{x+\frac{1}{x}}dx$ is equal to : **[JEE-MAIN-2014]**

(1) $(x-1)e^{x+\frac{1}{x}} + c$

(2) $xe^{x+\frac{1}{x}} + c$

(3) $(x+1)e^{x+\frac{1}{x}} + c$

(4) $-xe^{x+\frac{1}{x}} + c$

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II0073

4. The integral $\int \frac{dx}{x^2(x^4+1)^{\frac{3}{4}}}$ equals : **[JEE-MAIN-2015]**

(1) $-(x^4+1)^{\frac{1}{4}} + c$

(2) $-\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}} + c$

(3) $\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}} + c$

(4) $(x^4+1)^{\frac{1}{4}} + c$

II0074

5. The integral $\int \frac{2x^{12}+5x^9}{(x^5+x^3+1)^3}dx$ is equal to :- **[JEE-MAIN 2016]**

(1) $\frac{-x^{10}}{2(x^5+x^3+1)^2} + C$

(2) $\frac{-x^5}{(x^5+x^3+1)^2} + C$

(3) $\frac{x^{10}}{2(x^5+x^3+1)^2} + C$

(4) $\frac{x^5}{2(x^5+x^3+1)^2} + C$

where C is an arbitrary constant.

II0075

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6. Let $I_n = \int \tan^n x\,dx, (n>1)$. $I_4 + I_6 = a\tan^5 x + bx^5 + C$, where C is a constant of integration, then the ordered pair (a, b) is equal to :- **[JEE-MAIN 2017]**

(1) $\left(-\frac{1}{5}, 0\right)$ (2) $\left(-\frac{1}{5}, 1\right)$ (3) $\left(\frac{1}{5}, 0\right)$ (4) $\left(\frac{1}{5}, -1\right)$

II0076

7. The integral $\int \frac{\sin^2 x\cos^2 x}{(\sin^5 x + \cos^3 x\sin^2 x + \sin^3 x\cos^2 x + \cos^5 x)^2}dx$ is equal to **[JEE-MAIN 2018]**

(1) $\frac{-1}{3(1+\tan^3 x)} + C$ (2) $\frac{1}{1+\cot^3 x} + C$

(3) $\frac{-1}{1+\cot^3 x} + C$ (4) $\frac{1}{3(1+\tan^3 x)} + C$

(where C is a constant of integration)

II0077

8. For $x^2 \neq n\pi + 1$, $n \in N$ (the set of natural numbers), the integral $\int x\sqrt{\frac{2\sin(x^2-1) - \sin 2(x^2-1)}{2\sin(x^2-1) + \sin 2(x^2-1)}}dx$ is equal to :
(where c is a constant of integration)

(1) $\log_e\left|\sec\left(\frac{x^2-1}{2}\right)\right| + c$ (2) $\log_e\left|\frac{1}{2}\sec^2(x^2-1)\right| + c$

(3) $\frac{1}{2}\log_e\left|\sec^2\left(\frac{x^2-1}{2}\right)\right| + c$ (4) $\frac{1}{2}\log_e\left|\sec(x^2-1)\right| + c$ **[JEE-MAIN 2019]**

II0078

9. If $\int \frac{\sqrt{1-x^2}}{x^4}dx = A(x)\left(\sqrt{1-x^2}\right)^m + C$, for a suitable chosen integer m and a function A(x), where C is a constant of integration then $(A(x))^m$ equals : **[JEE-MAIN 2019]**

(1) $\frac{-1}{3x^3}$ (2) $\frac{-1}{27x^9}$ (3) $\frac{1}{9x^4}$ (4) $\frac{1}{27x^6}$

II0079

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10. The integral $\int \cos(\log_e x)dx$ is equal to : (where C is a constant of integration) **[JEE-MAIN 2019]**

(1) $\frac{x}{2}[\sin(\log_e x)-\cos(\log_e x)]+C$ (2) $\frac{x}{2}[\cos(\log_e x)+\sin(\log_e x)]+C$

(3) $x[\cos(\log_e x)+\sin(\log_e x)]+C$ (4) $x[\cos(\log_e x)-\sin(\log_e x)]+C$

II0080

11. If $\int \frac{\cos x\, dx}{\sin^3 x\left(1+\sin^6 x\right)^{2/3}} = f(x)\left(1+\sin^6 x\right)^{1/\lambda}+c$

where c is a constant of integration, then $\lambda f\left(\frac{\pi}{3}\right)$ is equal to **[JEE-MAIN 2020]**

(1) –2 (2) $-\frac{9}{8}$ (3) 2 (4) $\frac{9}{8}$

II0081

12. If $\int \frac{d\theta}{\cos^2\theta(\tan 2\theta+\sec 2\theta)} = \lambda\tan\theta + 2\log_e|f(\theta)| + C$ where C is a constant of integration, then the ordered pair $(\lambda, f(\theta))$ is equal to : **[JEE-MAIN 2020]**

(1) $(-1, 1 + \tan\theta)$ (2) $(-1, 1 - \tan\theta)$ (3) $(1, 1 - \tan\theta)$ (4) $(1, 1 + \tan\theta)$

II0082

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13. The integral $\int \frac{dx}{(x+4)^{\frac{8}{7}}(x-3)^{\frac{6}{7}}}$ is equal to :

(where C is a constant of integration) **[JEE-MAIN 2020]**

(1) $\left(\frac{x-3}{x+4}\right)^{\frac{1}{7}}+C$ (2) $-\left(\frac{x-3}{x+4}\right)^{-\frac{1}{7}}+C$ (3) $\frac{1}{2}\left(\frac{x-3}{x+4}\right)^{\frac{3}{7}}+C$ (4) $-\frac{1}{13}\left(\frac{x-3}{x+4}\right)^{-\frac{13}{7}}+C$

II0083

EXERCISE (JA)

1. $\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}dx$ is equal to - **[JEE 2006, (3M, –1M)]**

(A) $\frac{\sqrt{2x^4-2x^2+1}}{x^2}+c$ (B) $\frac{\sqrt{2x^4-2x^2+1}}{x^3}+c$ (C) $\frac{\sqrt{2x^4-2x^2+1}}{x}+c$ (D) $\frac{\sqrt{2x^4-2x^2+1}}{2x^2}+c$

II0084

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2. Let $f(x) = \frac{x}{(1+x^n)^{1/n}}$ for $n \geq 2$ and $g(x) = \underbrace{(f \circ f \circ ... \circ f)}_{f \text{ occurs } n \text{ times}}(x)$. Then $\int x^{n-2} g(x)dx$ equals.

(A) $\frac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}} + K$

(B) $\frac{1}{n-1}(1+nx^n)^{1-\frac{1}{n}} + K$ **[JEE 2007, 3]**

(C) $\frac{1}{n(n+1)}(1+nx^n)^{1+\frac{1}{n}} + K$

(D) $\frac{1}{n+1}(1+nx^n)^{1+\frac{1}{n}} + K$

II0085

3. Let F(x) be an indefinite integral of $\sin^2 x$. **[JEE 2007, 3]**

Statement-1 : The function F(x) satisfies $F(x + \pi) = F(x)$ for all real x.

because

Statement-2 : $\sin^2(x + \pi) = \sin^2 x$ for all real x.

(A) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1.

(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1.

(C) Statement-1 is True, Statement-2 is False.

(D) Statement-1 is False, Statement-2 is True.

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II0086

4. Let $I = \int \frac{e^x}{e^{4x}+e^{2x}+1}dx$, $J = \int \frac{e^{-x}}{e^{-4x}+e^{-2x}+1}dx$. Then, for an arbitrary constant c, the value of J – I equals **[JEE 2008, 3 (–1)]**

(A) $\frac{1}{2}\ell n\left(\frac{e^{4x}-e^{2x}+1}{e^{4x}+e^{2x}+1}\right)+c$

(B) $\frac{1}{2}\ell n\left(\frac{e^{2x}+e^{x}+1}{e^{2x}-e^{x}+1}\right)+c$

(C) $\frac{1}{2}\ell n\left(\frac{e^{2x}-e^{x}+1}{e^{2x}+e^{x}+1}\right)+c$

(D) $\frac{1}{2}\ell n\left(\frac{e^{4x}+e^{2x}+1}{e^{4x}-e^{2x}+1}\right)+c$

II0087

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5. The integral $\int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}} dx$ equals (for some arbitrary constant K) **[JEE 2012, 3M, –1M]**

(A) $-\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(B) $\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(C) $-\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(D) $\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

II0088

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ANSWER KEY

INDEFINITE INTEGRATION

EXERCISE (O-1)

1. B	**2.** D	**3.** B	**4.** B	**5.** B	**6.** A	**7.** B	**8.** B
9. B	**10.** C	**11.** B	**12.** A	**13.** A	**14.** C	**15.** A	

EXERCISE (O-2)

1. C	**2.** B	**3.** C	**4.** C	**5.** B	**6.** C	**7.** A	**8.** C
9. A,C,D	**10.** B,C	**11.** B,D	**12.** B,C	**13.** (A) Q ; (B) P; (C) P,R,S			

14. (A) R ; (B) S; (C) Q

EXERCISE (S-1)

1. $\frac{1}{2}(x-\sin x)+C$ **2.** $2ln\left(\sec\frac{x}{2}\right)-3ln\left(\sec\frac{x}{3}\right)-6ln\left(\sec\frac{x}{6}\right)+C$ **3.** 8

4. $\frac{-2}{\alpha-\beta}\sqrt{\frac{x-\beta}{x-\alpha}}+C$ **5.** $\frac{1}{2}\left[ln\left(\frac{1+x}{1-x}\right).ln\left(ln\frac{1+x}{1-x}\right)-ln\left(\frac{1+x}{1-x}\right)\right]+C$

6. $ln\left(\frac{1+\sqrt{1+3\cos^2 2\theta}}{\cos 2\theta}\right)+C$ **7.** $\sin^{-1}\left(\frac{ax^2+b}{cx}\right)+k$ **8.** $\frac{(x^2+1)\sqrt{x^2+1}}{9x^3}\left[2-3ln\left(1+\frac{1}{x^2}\right)\right]$

9. $\frac{x^2}{2}-x+C$ **10.** $\frac{67}{5}$ **11.** $\sin^{-1}\left(\frac{1}{2}\sec^2\frac{x}{2}\right)+C$

12. $6\left[\frac{t^4}{4}-\frac{t^2}{2}+t+\frac{1}{2}ln(1+t^2)-\tan^{-1}t\right]+C$ where $t=x^{1/6}$

13. $\frac{x^4}{4}+x^3-x^2+5x+\frac{1}{2}ln(x^2+1)+3\tan^{-1}x+C$ **14.** $(a+x)\arctan\sqrt{\frac{x}{a}}-\sqrt{ax}+C$

15. $C-e^{\cos x}(x+\operatorname{cosec} x)$ **16.** $C-\frac{x+1}{x^5+x+1}$ or $C+\frac{x^5}{x^5+x+1}$ **17.** $-\frac{3(1+4\tan^2 x)}{8(\tan x)^{8/3}}+C$

18. $\frac{1}{2\sqrt{3}}ln\frac{\sqrt{3}+\sin x-\cos x}{\sqrt{3}-\sin x+\cos x}+\arctan(\sin x+\cos x)+C$ **19.** 2010

20. 5 **21.** $4\ln x+\frac{7}{x}+6\tan^{-1}(x)+\frac{6x}{1+x^2}+C$ **22.** $2\sqrt{2}\left(\cos(e^{-\sqrt{x}})\right)\left(\sin(e^{\sqrt{x}})+\cos(e^{\sqrt{x}})\right)+C$

23. $C-\frac{x}{(x^2-1)^2}$ **24.** $C-\frac{2}{x}+\frac{2}{3x^3}-\frac{3}{5x^5}-2\tan^{-1}x$ **25.** $C-ln(1+(x+1)e^{-x})-\frac{1}{1+(x+1)e^{-x}}$

26. $\left(\frac{x}{e}\right)^x-\left(\frac{e}{x}\right)^x+C$ **27.** $\operatorname{arcsec} x-\frac{ln\,x}{\sqrt{x^2-1}}+C$

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EXERCISE (S-2)

1. $l\text{n}\left(\dfrac{\sec(l\text{n}\,x)}{\sec(l\text{n}\,x/2)x^{\tan(l\text{n}\,2)}}\right)+C$ **2.** $\cos a.\arccos\left(\dfrac{\cos x}{\cos a}\right)-\sin a.l\text{n}\left(\sin x+\sqrt{\sin^2 x-\sin^2 a}\right)+C$

3. $\dfrac{1}{24}\ell\text{n}\left|\dfrac{(4+3\sin x+3\cos x)}{(4-3\sin x-3\cos x)}\right|+C$ **4.** $\dfrac{1}{2}(\sin 2\theta)l\text{n}\left(\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)-\dfrac{1}{2}l\text{n}(\sec 2\theta)+C$

5. $\dfrac{1}{4}ln(\cos x+\sin x)+\dfrac{x}{2}+\dfrac{1}{8}(\sin 2x+\cos 2x)+C$ **6.** $\dfrac{1}{2}\left(\text{cosec}\dfrac{\alpha}{2}\right)\tan^{-1}\left(\left(\dfrac{x^2-1}{2x}\right)\text{cosec}\dfrac{\alpha}{2}\right)$

7. $C-\dfrac{2}{\sin\alpha}\sqrt{\dfrac{\sin(x+\alpha)}{\sin x}}$ **8.** $2l\text{n}\dfrac{t}{2t+1}+\dfrac{1}{2t+1}+C$, when $t=x+\sqrt{x^2+x}$

9. $e^x\sqrt{\dfrac{1+x}{1-x}}+c$ **10.** $\dfrac{2(7x-20)}{9\sqrt{7x-10-x^2}}+C$ **11.** 3 **12.** (A) R ; (B) Q; (C) P ; (D) S

EXERCISE (JM)

1. 1 **2.** 3 **3.** 2 **4.** 2 **5.** 3 **6.** 3 **7.** 1 **8.** 1 or 3
9. 2 **10.** 2 **11.** 1 **12.** 1 **13.** 1

EXERCISE (JA)

1. D **2.** A **3.** D **4.** C **5.** C

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DEFINITE INTEGRATION

A definite integral is denoted by $\int_a^b f(x)dx$ which represent the algebraic area bounded by the curve y = f(x), the ordinates x = a, x = b and the x axis.

1. THE FUNDAMENTAL THEOREM OF CALCULUS :

The Fundamental Theorem of Calculus is appropriately named because it establishes a connection between the two branches of calculus : differential calculus and integral calculus. Differential calculus arose from the tangent problem, whereas integral calculus arose from a seemingly unrelated problem, the area problem. Newton's teacher at Cambridge, Isaac Barrow (1630-1677), discovered that these two problems are actually closely related. In fact, he realized that differentiation and integration are inverse processes. The Fundamental Theorem of Calculus gives the precise inverse relationship between the derivative and the integral. It was Newton and Leibnitz who exploited this relationship and used it to develop calculus into a systematic mathematical method. In particular, they saw that the Fundamental Theorem enabled them to compute areas and integrals very easily without having to compute them as limits of sums.

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The Fundamental Theorem of Calculus, Part 1

If f is continuous on [a, b], then the function g defined by

$$g(x) = \int_a^x f(t)dt \quad a \le x \le b$$

is continuous on [a, b] and differentiable on (a, b), and g'(x) = f(x).

The Fundamental Theorem of Calculus, Part 2

If f is continuous on [a, b], then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that F '= f.

Note : If $\int_a^b f(x)dx = 0 \Rightarrow$ then the equation f(x) = 0 has atleast one root lying in (a, b) provided f is a continuous function in (a,b).

2. PROPERTIES OF DEFINITE INTEGRAL :

(a) $\int_a^b f(x)dx = \int_a^b f(t)\,dt$ provided f is same

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(b) $\int_a^b f(x)dx = -\int_b^a f(x)\,dx$

(c) $\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where c may lie inside or outside the interval [a,b]. This property is to be used when f is piecewise continuous in (a, b).

Illustration 1 : If $f(x) = \begin{cases} x^2, & 0 < x < 2 \\ 3x-4, & 2 \le x < 3 \end{cases}$ then evaluate $\int_0^3 f(x)dx$

Solution : $\int_0^3 f(x)dx = \int_0^2 f(x)dx + \int_2^3 f(x)dx = \int_0^2 x^2dx + \int_2^3 (3x-4)dx$

$= \left(\frac{x^3}{3}\right)_0^2 + \left(\frac{3x^2}{2} - 4x\right)_2^3 = \frac{8}{3} + \frac{27}{2} - 12 - 6 + 8 = 37/6$ **Ans.**

Illustration 2 : If $f(x) = \begin{cases} 3[x] - 5\frac{|x|}{x}, & x \ne 0 \\ 2, & x = 0 \end{cases}$ then $\int_{-3/2}^{2} f(x)dx$ is equal to ([.] denotes the greatest integer function)

(A) $-\frac{11}{2}$ (B) $-\frac{7}{2}$ (C) -6 (D) $-\frac{17}{2}$

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Solution : $3[x] - 5\frac{|x|}{x} = 3[x] - 5$, if $x > 0$

$= 3[x] + 5$, if $x < 0$

$\Rightarrow \int_{-3/2}^{2} f(x)dx = \int_{-3/2}^{-1} (-1)dx + \int_{-1}^{0} (2)dx + \int_{0}^{1} (-5)dx + \int_{1}^{2} (-2)dx$

$= -1\left(-1 + \frac{3}{2}\right) + 2(1) + 1(-5) + (-2) = -\frac{1}{2} + 2 - 5 - 2 = -\frac{11}{2}$ **Ans. (A)**

Illustration 3 : The value of $\int_1^2 (x^{[x^2]} + [x^2]^x)dx$, where [.] denotes the greatest integer function, is equal to -

(A) $\frac{5}{4} + \sqrt{3} + (2^{\sqrt{3}} - 2^{\sqrt{2}}) + \frac{1}{\log 3}(9 - 3^{\sqrt{3}})$

(B) $\frac{5}{4} + \sqrt{3} + \frac{\sqrt{2}}{3} + \frac{1}{\log 2}(2^{\sqrt{3}} - 2^{\sqrt{2}}) + \frac{1}{\log 3}(9 - 3^{\sqrt{3}})$

(C) $\frac{5}{4} + \frac{\sqrt{2}}{3} + \frac{1}{\log 2}(2^{\sqrt{3}} - 2^{\sqrt{2}}) + \frac{1}{\log 3}(9 - 3^{\sqrt{3}})$

(D) none of these

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Solution : We have, $I = \int_1^2 (x^{[x^2]} + [x^2]^x)dx = \int_1^{\sqrt{2}} (x+1)dx + \int_{\sqrt{2}}^{\sqrt{3}} (x^2 + 2^x)dx + \int_{\sqrt{3}}^{2} (x^3 + 3^x)dx$

$$= \left(\frac{x^2}{2} + x\right)_1^{\sqrt{2}} + \left(\frac{x^3}{3} + \frac{2^x}{\log 2}\right)_{\sqrt{2}}^{\sqrt{3}} + \left(\frac{x^4}{4} + \frac{3^x}{\log 3}\right)_{\sqrt{3}}^{2}$$

$$= \frac{5}{4} + \sqrt{3} + \frac{\sqrt{2}}{3} + \frac{1}{\log 2}(2^{\sqrt{3}} - 2^{\sqrt{2}}) + \frac{1}{\log 3}(3^2 - 3^{\sqrt{3}})$$ **Ans. (B)**

Illustration 4 : Evaluate : $\int_{-10}^{20} [\cot^{-1} x]dx$. Here [.] is the greatest integer function.

Solution : $I = \int_{-10}^{20} [\cot^{-1} x]dx$, we know $\cot^{-1} x \in (0, \pi)\ \forall\ x \in R$

$$\text{Thus } [\cot^{-1} x] = \begin{cases} 3, & x \in (-\infty, \cot 3) \\ 2, & x \in (\cot 3, \cot 2) \\ 1, & x \in (\cot 2, \cot 1) \\ 0 & x \in (\cot 1, \infty) \end{cases}$$

Hence $I = \int_{-10}^{\cot 3} 3dx + \int_{\cot 3}^{\cot 2} 2dx + \int_{\cot 2}^{\cot 1} 1dx + \int_{\cot 1}^{20} 0dx = 30 + \cot 1 + \cot 2 + \cot 3$ **Ans.**

Do yourself -1 :
Evaluate :

(i) $\int_0^3 |x^2 - x - 2|dx$

(ii) $\int_0^4 \{x\}dx$, where {.} denotes fractional part of x.

(iii) $\int_0^{\pi/2} |\sin x - \cos x|dx$

(iv) If $f(x) = \begin{cases} 2 & 0 \le x \le 1 \\ x + [x] & 1 \le x < 3 \end{cases}$, where [.] denotes the greatest integer function. Evaluate $\int_0^2 f(x)dx$

(d) $\int_{-a}^{a} f(x)\,dx = \int_0^a [f(x) + f(-x)]dx = \begin{cases} 0 & ;\text{if } f(x) \text{ is an odd function} \\ 2\int_0^a f(x)dx & ;\text{if } f(x) \text{ is an even function} \end{cases}$

Illustration 5 : Evaluate $\int_{-1/2}^{1/2} \cos x\, \ell n\left(\frac{1+x}{1-x}\right)dx$

Solution : $f(-x) = \cos(-x)\, \ell n\left(\frac{1-x}{1+x}\right) = -\cos\, \ell n\left(\frac{1+x}{1-x}\right) = -f(x)$

$\Rightarrow$ f(x) is odd

Hence, the value of the given integral = 0. **Ans.**

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Illustration 6 : If $f(x) = \begin{vmatrix} \cos x & e^{x^2} & 2x\cos^2 x/2 \\ x^2 & \sec x & \sin x + x^3 \\ 1 & 2 & x + \tan x \end{vmatrix}$, then the value of $\int_{-\pi/2}^{\pi/2} (x^2+1)(f(x)+f''(x))dx$

(A) 1 (B) –1 (C) 2 (D) none of these

Solution : As, $f(x) = \begin{vmatrix} \cos x & e^{x^2} & 2x\cos^2 x/2 \\ x^2 & \sec x & \sin x + x^3 \\ 1 & 2 & x + \tan x \end{vmatrix}$

$\Rightarrow$ $f(-x) = -f(x)$ $\Rightarrow$ $f(x)$ is odd

$\Rightarrow$ $f'(x)$ is even $\Rightarrow$ $f''(x)$ is odd

Thus, $f(x) + f''(x)$ is odd function let,

$\phi(x) = (x^2 + 1).\{f(x) + f''(x)\}$

$\Rightarrow$ $\phi(-x) = -\phi(x)$

i.e. $\phi(x)$ is odd

$\therefore$ $\int_{-\pi/2}^{\pi/2} \phi(x)dx = 0$ **Ans. (D)**

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Do yourself -2 :

Evaluate :

(i) $\int_{-\pi/2}^{\pi/2} (x^2 \sin^3 x + \cos x)dx$ **(ii)** $\int_{-\pi/2}^{\pi/2} \ell n\left[2\left(\frac{4-\sin\theta}{4+\sin\theta}\right)\right]d\theta$

(e) $\int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx$, In particular $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx$

Illustration 7 : If f, g, h be continuous functions on [0, a] such that $f(a-x) = -f(x)$, $g(a-x) = g(x)$ and $3h(x) - 4h(a-x) = 5$, then prove that $\int_0^a f(x)g(x)h(x)dx = 0$

Solution : $I = \int_0^a f(x)g(x)h(x)dx = \int_0^a f(a-x)g(a-x)h(a-x)dx = -\int_0^a f(x)g(x)h(a-x)dx$

$7I = 3I + 4I$

$= \int_0^a f(x)g(x)\{3h(x) - 4h(a-x)\}dx = 5\int_0^a f(x)g(x)dx = 0$

(since $f(a-x)\,g(a-x) = -f(x)g(x)$)

$\Rightarrow$ $I = 0$ **Ans.**

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Illustration 8 : Evaluate $\int_{-\pi}^{\pi} \frac{x \sin x}{e^x + 1} dx$

Solution : $I = \int_{-\pi}^{0} \frac{x \sin x}{e^x + 1} dx + \int_{0}^{\pi} \frac{x \sin x}{e^x + 1} dx = I_1 + I_2$

where $I_1 = \int_{-\pi}^{0} \frac{x \sin x}{e^x + 1} dx$

Put $x = -t \Rightarrow dx = -dt$

$$\Rightarrow \quad I_1 = \int_{\pi}^{0} \frac{(-t)\sin(-t)(-dt)}{e^{-t}+1} = \int_{0}^{\pi} \frac{t \sin t\, dt}{e^{-t}+1} = \int_{0}^{\pi} \frac{e^t t \sin t\, dt}{e^t + 1} = \int_{0}^{\pi} \frac{e^x x \sin x\, dx}{e^x + 1}$$

Hence $I = I_1 + I_2 = \int_{0}^{\pi} \frac{e^x x \sin x}{e^x + 1} dx + \int_{0}^{\pi} \frac{x \sin x}{e^x + 1} dx$

$$I = \int_{0}^{\pi} x \sin x dx = \int_{0}^{\pi} (\pi - x)\sin(\pi - x) dx = \pi \int_{0}^{\pi} \sin x dx - I$$

$$\Rightarrow \quad 2I = \pi \int_{0}^{\pi} \sin x dx = \pi |-\cos x|_0^{\pi} = 2\pi \Rightarrow I = \pi$$ **Ans.**

Illustration 9 : Evaluate $\int_{0}^{2} \frac{dx}{(17 + 8x - 4x^2)[e^{6(1-x)} + 1]}$

Solution : Let $I = \int_{0}^{2} \frac{dx}{(17 + 8x - 4x^2)[e^{6(1-x)} + 1]}$

Also $I = \int_{0}^{2} \frac{dx}{(17 + 8x - 4x^2)[e^{-6(1-x)} + 1]}$ $\left[\because \int_{0}^{a} f(x)dx = \int_{0}^{a} f(a-x)dx\right]$

Adding, we get

$$2I = \int_{0}^{2} \frac{1}{17 + 8x - 4x^2}\left(\frac{1}{e^{6(1-x)} + 1} + \frac{1}{e^{-6(1-x)} + 1}\right) dx$$

$$= \int_{0}^{2} \frac{1}{17 + 8x - 4x^2} dx = -\frac{1}{4}\int_{0}^{2} \frac{dx}{x^2 - 2x - 17/4}$$

$$= -\frac{1}{4}\int_{0}^{2} \frac{dx}{(x-1)^2 - 21/4} = -\frac{1}{4} \times \frac{1}{2 \times \frac{\sqrt{21}}{2}} \left[\log\left|\frac{x - 1 - \frac{\sqrt{21}}{2}}{x - 1 + \frac{\sqrt{21}}{2}}\right|\right]_0^2$$

$$= -\frac{1}{4\sqrt{21}}\left[\log\left|\frac{2x - 2 - \sqrt{21}}{2x - 2 + \sqrt{21}}\right|\right]_0^2 \quad \Rightarrow \quad I = -\frac{1}{8\sqrt{21}}\left[\log\left|\frac{2 - \sqrt{21}}{2 + \sqrt{21}}\right| - \log\left|\frac{2 + \sqrt{21}}{\sqrt{21} - 2}\right|\right]$$

$$= -\frac{1}{4\sqrt{21}}\left[\log\left|\frac{\sqrt{21} - 2}{2 + \sqrt{21}}\right|\right]$$ **Ans.**

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Illustration 10 : $\int_0^1 \cot^{-1}(1-x+x^2)dx$ equals -

(A) $\frac{\pi}{2}+\log 2$ (B) $\frac{\pi}{2}-\log 2$ (C) $\pi - \log 2$ (D) none of these

Solution : $I=\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx=\int_0^1 \tan^{-1}\left(\frac{x+(1-x)}{1-x(1-x)}\right)dx$

$=\int_0^1 [\tan^{-1}x+\tan^{-1}(1-x)]\ dx=\int_0^1 \tan^{-1}x\ dx+\int_0^1 \tan^{-1}(1-x)\ dx$

$=2\int_0^1 \tan^{-1}x\ dx=2\left[x\ \tan^{-1}x-\frac{1}{2}\log(1+x^2)\right]_0^1=2\frac{\pi}{4}-\log 2=\frac{\pi}{2}-\log 2$ **Ans. (B)**

Illustration 11 : $\int_0^{\pi/2}\frac{a\sin x+b\cos x}{\sin x+\cos x}dx$

Solution : $I=\int_0^{\pi/2}\frac{a\sin x+b\cos x}{\sin x+\cos x}dx$ .....(i)

$I=\int_0^{\pi/2}\frac{a\sin(\pi/2-x)+b\cos(\pi/2-x)}{\sin(\pi/2-x)+\cos(\pi/2-x)}dx=\int_0^{\pi/2}\frac{a\cos x+b\sin x}{\sin x+\cos x}dx$ .....(ii)

$\therefore\ 2I=\int_0^{\pi/2}\frac{(a+b)(\sin x+\cos x)}{\sin x+\cos x}dx=\int_0^{\pi/2}(a+b)dx=(a+b)\pi/2 \Rightarrow I=(a+b)\pi/4$ **Ans.**

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Illustration 12 : $\int_0^{\pi/2}\frac{2^{\sin x}}{2^{\sin x}+2^{\cos x}}dx$ equals -

(A) 2 (B) π (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$

Solution : $I=\int_0^{\pi/2}\frac{2^{\sin x}}{2^{\sin x}+2^{\cos x}}dx=\int_0^{\pi/2}\frac{2^{\sin(\pi/2-x)}}{2^{\sin x(\pi/2-x)}+2^{\cos(\pi/2-x)}}dx=\int_0^{\pi/2}\frac{2^{\cos x}}{2^{\cos x}+2^{\sin x}}dx$

$2I=\int_0^{\pi/2}dx=\frac{\pi}{2} \quad\Rightarrow\quad I=\frac{\pi}{4}$ **Ans. (C)**

Do yourself - 3 :

Evaluate :

(i) $\int_1^5\frac{\sqrt{x}}{\sqrt{6-x}+\sqrt{x}}dx$ **(ii)** $\int_{\pi/6}^{\pi/3}\frac{dx}{1+\tan^5 x}$

(f) $\int_0^{2a}f(x)dx=\int_0^a f(x)dx+\int_0^a f(2a-x)dx=\begin{cases}2\int_0^a f(x)dx\ ;\ \text{if}\ f(2a-x)=f(x)\\ 0\ ;\ \text{if}\ f(2a-x)=-f(x)\end{cases}$

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Illustration 13 : Evaluate $\int_0^{\pi} \frac{xdx}{1+\cos^2 x}$

Solution : Let $I = \int_0^{\pi} \frac{xdx}{1+\cos^2 x} = \int_0^{\pi} \frac{(\pi - x)dx}{1+\cos^2(\pi - x)} = \int_0^{\pi} \frac{\pi dx}{1+\cos^2 x} - I$

$$\Rightarrow 2I = \int_0^{\pi} \frac{\pi dx}{1+\cos^2 x} = 2\pi \int_0^{\pi/2} \frac{dx}{1+\cos^2 x} = 2\pi \int_0^{\pi/2} \frac{\sec^2 x dx}{2+\tan^2 x}$$

Let tan x = t so that for $x \to 0$, $t \to 0$ and for $x \to \pi/2$, $t \to \infty$. Hence we can write,

$$I = \pi \int_0^{\infty} \frac{dt}{2+t^2} = \pi \frac{1}{\sqrt{2}} \left[\tan^{-1} \frac{t}{\sqrt{2}} \right]_0^{\infty} = \frac{\pi^2}{2\sqrt{2}}$$ **Ans.**

Illustration 14 : Prove that $\int_0^{\pi/2} \log(\sin x)dx = \int_0^{\pi/2} \log(\cos x)dx = -\frac{\pi}{2}\log 2$

Solution : Let $I = \int_0^{\pi/2} \log(\sin x)dx$ ..... (i)

then $I = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right)dx = \int_0^{\pi/2} \log(\cos x)dx$ ..... (ii)

adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \log \sin x\, dx + \int_0^{\pi/2} \log \cos x\, dx = \int_0^{\pi/2} (\log \sin x + \log \cos x)dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \log(\sin x \cos x)dx = \int_0^{\pi/2} \log\left(\frac{2\sin x \cos x}{2}\right)dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)dx = \int_0^{\pi/2} \log(\sin 2x)dx - \int_0^{\pi/2} (\log 2)dx = \int_0^{\pi/2} \log \sin 2x . dx - (\log 2)(x)_0^{\pi/2}$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \log(\sin 2x)dx - \frac{\pi}{2}\log 2 \quad \text{..... (iii)}$$

Let $I_1 = \int_0^{\pi/2} \log(\sin 2x)dx$, putting 2x = t, we get

$$I_1 = \int_0^{\pi} \log(\sin t)\frac{dt}{2} = \frac{1}{2}\int_0^{\pi} \log(\sin t)dt = \frac{1}{2}.2\int_0^{\pi/2} \log(\sin t)dt$$

$$I_1 = \int_0^{\pi/2} \log(\sin x)dx$$

$\therefore$ (iii) becomes ; $2I = I - \frac{\pi}{2}\log 2$

Hence $\int_0^{\pi/2} \log \sin x\, dx = -\frac{\pi}{2}\log 2$ **Ans.**

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Illustration 15 : $\int_0^{\pi/2} (2\log\sin x - \log\sin 2x)dx$ equals -

(A) $\pi \log 2$ (B) $-\pi \log 2$ (C) $(\pi/2) \log 2$ (D) $-(\pi/2) \log 2$

Solution : $I = \int_0^{\pi/2} (2\log\sin x - \log 2\sin x\cos x)dx = \int_0^{\pi/2} (2\log\sin x - \log 2 - \log\sin x - \log\cos x)dx$

$= \int_0^{\pi/2} \log\sin x dx - \int_0^{\pi/2} \log 2 dx - \int_0^{\pi/2} \log\cos x dx = -(\pi/2)\log 2$ **Ans. (D)**

Do yourself -4 :

Evaluate :

(i) $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{(1+e^x)(1+x^2)}$ **(ii)** $\int_0^{\pi/2} \ell n\left(\sin^2 x\cos x\right)dx$

(iii) $\int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x\cos x}dx$ **(iv)** $\int_{-\pi/2}^{\pi/2} \sqrt{\cos x - \cos^3 x}\, dx$

(g) $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx$, $(n \in I)$; where 'T' is the period of the function i.e. $f(T + x) = f(x)$

Note that : $\int_x^{T+x} f(t)dt$ will be independent of x and equal to $\int_0^T f(t)dt$

(h) $\int_{a+nT}^{b+nT} f(x)dx = \int_a^b f(x)dx$ where f(x) is periodic with period T & $n \in I$.

(i) $\int_{mT}^{nT} f(x)dx = (n-m)\int_0^T f(x)dx$, $(n, m \in I)$ if f(x) is periodic with period 'T'.

Illustration 16 : Evaluate $\int_0^{4\pi} |\cos x|\, dx$

Solution : Note that $|\cos x|$ is a periodic function with period π. Hence the given integral.

$$I = 4\int_0^{\pi} |\cos x|\, dx = 4\left[\int_0^{\frac{\pi}{2}} \cos x dx - \int_{\frac{\pi}{2}}^{\pi} \cos x dx\right] = 4\left[[\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi}\right] = 4[1+1] = 8$$ **Ans.**

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Illustration 17 : Evaluate $\int_0^{16\pi/3} |\sin x|\,dx$

Solution : $\int_0^{16\pi/3} |\sin x|\,dx = \int_0^{5\pi} |\sin x|\,dx + \int_{5\pi}^{5\pi+\pi/3} |\sin x|\,dx = 5\int_0^{\pi} |\sin x|\,dx + \int_0^{\pi/3} |\sin x|\,dx$

$= 5[-\cos x]_0^{\pi} + [-\cos x]_0^{\pi/3} = 10 + \left(-\frac{1}{2}+1\right) = \frac{21}{2}$ **Ans.**

Illustration 18 : Evaluate : $\int_0^{2n\pi} [\sin x + \cos x]dx$. Here [.] is the greatest integer function.

Solution : Let $I = \int_0^{2n\pi} [\sin x + \cos x]dx = n\int_0^{2\pi} [\sin x + \cos x]dx$

($\because$ [sinx + cosx] is periodic function with period 2π]

$$[\sin x + \cos x] = \begin{cases} 1, & 0 \le x \le \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \le x \le \frac{3\pi}{4} \\ -1, & \frac{3\pi}{4} < x \le \pi \\ -2, & \pi < x \le \frac{3\pi}{2} \\ -1, & \frac{3\pi}{2} < x \le \frac{7\pi}{4} \\ 0, & \frac{7\pi}{4} < x \le 2\pi \end{cases}$$

Hence $I = n\left[\int_0^{\pi/2} 1dx + \int_{\pi/2}^{3\pi/4} 0dx + \int_{3\pi/4}^{\pi} -1dx + \int_{\pi}^{3\pi/2} -2dx + \int_{3\pi/2}^{7\pi/4} -1dx + \int_{7\pi/4}^{2\pi} 0dx\right]$

$I = n\left[\frac{\pi}{2} + 0 - \pi + \frac{3\pi}{4} - 3\pi + 2\pi - \frac{7\pi}{4} + \frac{3\pi}{2} + 0\right] = -n\pi$ **Ans.**

Do yourself -5 :

Evaluate :

(i) $\int_{-1.5}^{10} \{2x\}dx$, where {.} denotes fractional part of x. **(ii)** $\int_{20\pi+\frac{\pi}{6}}^{20\pi+\frac{\pi}{3}} (\sin x + \cos x)dx$

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3. WALLI'S FORMULA :

If m, n $\in$ N & m, n $\geq$ 2, then

(a) $$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{(n-1)(n-3).....(1 \text{ or } 2)}{n(n-2).....(1 \text{ or } 2)}K$$

$$\text{where } K = \begin{cases} \pi/2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

(b) $$\sin^n x.\cos^m x\,dx = \frac{[(n-1)(n-3)(n-5)....1 \text{ or } 2][(m-1)(m-3)....1 \text{ or } 2]}{(m+n)(m+n-2)(m+n-4)....1 \text{ or } 2}K$$

$$\text{Where } K = \begin{cases} \frac{\pi}{2} & \text{if both m and n are even} \\ 1 & \text{otherwise} \end{cases}$$

Illustration 19 : $\int_{-\pi/2}^{\pi/2} \sin^4 x \cos^6 x \, dx =$

(A) $\frac{3\pi}{64}$ (B) $\frac{3\pi}{572}$ (C) $\frac{3\pi}{256}$ (D) $\frac{3\pi}{128}$

Solution : $I = \int_{-\pi/2}^{\pi/2} \sin^4 x \cos^6 x \, dx = 2\int_0^{\pi/2} \sin^4 x \cos^6 x.dx = 2\frac{(3.1)(5.3.1)}{10.8.6.4.2}.\frac{\pi}{2} = \frac{3\pi}{256}$ **Ans. (C)**

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4. DERIVATIVE OF ANTIDERIVATIVE FUNCTION (Newton-Leibnitz Formula) :

If h(x) & g(x) are differentiable functions of x then, $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f[h(x)].h'(x) - f[g(x)].g'(x)$

Illustration 20 : Find the points of maxima/minima of $\int_0^{x^2} \frac{t^2-5t+4}{2+e^t}dt$

Solution : Let $f(x) = \int_0^{x^2} \frac{t^2-5t+4}{2+e^t}dt$

$$f'(x) = \frac{x^4-5x^2+4}{2+e^{x^2}}2x - 0 = \frac{(x-1)(x+1)(x-2)(x+2)2x}{2+e^{x^2}}$$

Graph of f'(x)

From the wavy curve, it is clear that f'(x) changes its sign at x = ± 2, ±1, 0 and hence the points of maxima are –1, 1 and of the minima are –2, 0, 2.

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Illustration 21 : Evaluate $\frac{d}{dt}\int_{t^2}^{t^3}\frac{1}{\log x}dx$

Solution : $\frac{d}{dt}\int_{t^2}^{t^3}\frac{1}{\log x}dx = \frac{1}{\log t^3}.\frac{d}{dt}(t^3) - \frac{1}{\log t^2}.\frac{d}{dt}(t^2) = \frac{3t^2}{3\log t} - \frac{2t}{2\log t} = \frac{t(t-1)}{\log t}$ **Ans.**

Do yourself - 6 :

(i) If $f(x) = \int_{1/x}^{\sqrt{x}} \sin t\, dt$, then find $f'(1)$.

(ii) $\int_{\pi/3}^{x}\sqrt{3-\sin^2 t}\,dt + \int_0^y \cos t\,dt = 0$, then evaluate $\frac{dy}{dx}$.

5. DEFINITE INTEGRAL AS LIMIT OF A SUM :

An alternative way of describing $\int_a^b f(x)dx$ is that the definite integral $\int_a^b f(x)dx$ is a limiting case of the summation of an infinite series, provided $f(x)$ is continuous on [a,b]

i.e. $\int_a^b f(x)dx = \lim_{n\to\infty} h\sum_{r=0}^{n-1} f(a+rh)$ where $h = \frac{b-a}{n}$. The converse is also true i.e., if we have an infinite series of the above form, it can be expressed as a definite integral.

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Step I : Express the given series in the form $\sum \frac{1}{n} f\left(\frac{r}{n}\right)$

Step II : Then the limit is its sum when $n \to \infty$, i.e. $\lim_{n\to\infty}\frac{1}{n} f\left(\frac{r}{n}\right)$

Step III : Replace $\frac{r}{n}$ by x and $\frac{1}{n}$ by dx and $\lim_{n\to\infty}\sum$ by the sign of $\int$

Step IV : The lower and the upper limit of integration are the limiting values of $\frac{r}{n}$ for the first and the last term of r respectively.

Illustration 22 : Evaluate $\lim_{n\to\infty}\left(\frac{1}{2n+1}+\frac{1}{2n+2}+.....+\frac{1}{6n}\right)$

Solution : Let $S_n = \frac{1}{2n+1}+\frac{1}{2n+2}+.....+\frac{1}{6n} = \sum_{r=1}^{4n}\frac{1}{2n+r} = \sum_{r=1}^{4n}\frac{1}{n}.\frac{1}{2+\left(\frac{r}{n}\right)}$

$\Rightarrow \quad S = \lim_{n\to\infty} S_n = \int_0^4 \frac{dx}{2+x} = [\ell n|2+x|]_0^4 = \ell n6 - \ell n2 = \ell n3$ **Ans.**

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Illustration 23 : Evaluate $\lim\limits_{n\to\infty}\left[\frac{\sqrt{n}}{(3+4\sqrt{n})^2}+\frac{\sqrt{n}}{\sqrt{2}(3\sqrt{2}+4\sqrt{n})^2}+\frac{\sqrt{n}}{\sqrt{3}(3\sqrt{3}+4\sqrt{n})^2}+.....+\frac{1}{49n}\right]$

Solution : Let $p = \lim\limits_{n\to\infty}\left[\frac{\sqrt{n}}{(3+4\sqrt{n})^2}+\frac{\sqrt{n}}{\sqrt{2}(3\sqrt{2}+4\sqrt{n})^2}+.....+\frac{\sqrt{n}}{\sqrt{n}(3\sqrt{n}+4\sqrt{n})^2}\right]$

Analyzing the expression with the view of increasing integral value we get the expression in terms of r as

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{n}}{\sqrt{r}(3\sqrt{r}+4\sqrt{n})^2} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n\sqrt{\frac{r}{n}}\left(3\sqrt{\frac{r}{n}}+4\right)^2} = \int_0^1\frac{dx}{\sqrt{x}(3\sqrt{x}+4)^2}$$

Put $3\sqrt{x}+4 = t, \quad \therefore \quad \frac{3}{2\sqrt{x}}dx = dt$

Hence $p = \frac{2}{3}\int_4^7\frac{dt}{t^2} = \frac{2}{3}\left[-\frac{1}{t}\right]_4^7 = \frac{2}{3}\left(-\frac{1}{7}+\frac{1}{4}\right) = \frac{1}{14}$ **Ans.**

Do yourself - 7 :

Evaluate :

(i) $\lim\limits_{n\to\infty}\left[\frac{1}{n+2.1}+\frac{1}{n+2.2}+\frac{1}{n+2.3}.....\frac{1}{3n}\right]$ **(ii)** $\lim\limits_{n\to\infty}\sum\limits_{r=0}^{n-1}\frac{1}{\sqrt{n^2-r^2}}$

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6. ESTIMATION OF DEFINITE INTEGRAL :

(a) If f(x) is continuous in [a, b] and it's range in this interval is [m, M], then

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

Illustration 24 : Prove that $4 \le \int_1^3\sqrt{3+x^3}dx \le 2\sqrt{30}$

Solution : Since the function $f(x) = \sqrt{3+x^3}$ increases monotonically on the interval [1, 3], m = 2, M = $\sqrt{30}$, b – a =2.

Hence, $2.2 \le \int_1^3\sqrt{3+x^3}dx \le 2\sqrt{30} \Rightarrow 4 \le \int_1^3\sqrt{3+x^3}dx \le 2\sqrt{30}$ **Ans.**

(b) If $f(x) \le \phi(x)$ for $a \le x \le b$ then $\int_a^b f(x)dx \le \int_a^b \phi(x)dx$

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Illustration 25 : Prove that $\frac{\pi}{6} \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \frac{\pi}{4\sqrt{2}}$

Solution : Since $4 - x^2 \ge 4 - x^2 - x^3 \ge 4 - 2x^2 > 0 \ \forall \ x \in [0, 1]$

$$\sqrt{4-x^2} \ge \sqrt{4-x^2-x^3} \ge \sqrt{4-2x^2} > 0 \ \forall \ x \in [0, 1]$$

$$\Rightarrow \quad 0 < \frac{1}{\sqrt{4-x^2}} \le \frac{1}{\sqrt{4-x^2-x^3}} \le \frac{1}{\sqrt{4-2x^2}} \ \forall \ x \in [0 \ 1]$$

$$\Rightarrow \quad \int_0^1 \frac{dx}{\sqrt{4-x^2}} \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \int_0^1 \frac{dx}{\sqrt{4-2x^2}} \ \forall x \in [0, 1]$$

$$\Rightarrow \quad \left[\sin^{-1}\frac{x}{2}\right]_0^1 \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{x}{\sqrt{2}}\right]_0^1 \Rightarrow \frac{\pi}{6} \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \frac{\pi}{4\sqrt{2}}$$ **Ans.**

(c) $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$.

Illustration 26 : Prove that $\left|\int_{10}^{19} \frac{\sin x}{1+x^8}dx\right| < 10^{-7}$

Solution : To find $I = \left|\int_{10}^{19} \frac{\sin x}{1+x^8}dx\right| \le \int_{10}^{19}\left|\frac{\sin x}{1+x^8}\right|dx$ ..... (i)

Since $|\sin x| \le 1$ for $x \ge 10$

The inequality $\left|\frac{\sin x}{1+x^8}\right| \le \frac{1}{|1+x^8|}$ ..... (ii)

also, $10 \le x \le 19$

$\Rightarrow$ $1 + x^8 > 10^8$

$\Rightarrow$ $\frac{1}{1+x^8} < \frac{1}{10^8}$ or $\frac{1}{|1+x^8|} < 10^{-8}$ ..... (iii)

from (ii) and (iii) ;

$$\left|\frac{\sin x}{1+x^8}\right| < 10^{-8}$$

$$\left|\int_{10}^{19} \frac{\sin x}{1+x^8}dx\right| < \int_{10}^{19} 10^{-8}dx$$

$$\therefore \quad \left|\int_{10}^{19} \frac{\sin x}{1+x^8}dx\right| < (19-10).10^{-8} < 10^{-7}$$ **Ans.**

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Illustration 27 : If f (x) is integrable function such that $|f(x) - f(y)| \le |x^2 - y^2|$, $\forall$ x, y $\in$ [a,b] then prove that $\left|\int_a^b \frac{f(x)-f(a)}{x+a}dx\right| \le \frac{(a-b)^2}{2}$.

Solution : Given, $\left|\int_a^b \frac{f(x)-f(a)}{x+a}dx\right| \le \int_a^b \left|\frac{f(x)-f(a)}{x+a}\right|dx$

$$\le \int_a^b \left|\frac{x^2-a^2}{x+a}\right|dx = \int_a^b |x-a|\,dx = \int_a^b (x-a)dx = \frac{(a-b)^2}{2}$$

(d) If $f(x) \ge 0$ on the interval [a,b], then $\int_a^b f(x)dx \ge 0$.

Illustration 28: If f (x) is a continuous function such that $f(x) \ge 0$ $\forall$ x $\in$ [2,10] and $\int_4^8 f(x)\,dx = 0$, then find f (6).

Solution : f (x) is above the x-axis or on the x-axis for all x $\in$ [2,10]. If f(x) is greater than zero for any sub interval of [4,8], then $\int_4^8 f(x)dx$ must be greater than zero.

But $\int_4^8 f(x)dx = 0 \Rightarrow f(x) = 0$ $\forall$ x $\in$ [4,8]

$\Rightarrow f(6) = 0$.

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Do yourself - 8 :

(i) Prove that $4 \le \int_1^3 \sqrt{3+x^2}dx \le 4\sqrt{3}$

(ii) Prove that $\frac{\pi}{4} \le \int_0^{2\pi} \frac{dx}{5+3\sin x} \le \pi$.

(iii) Show that $\frac{3}{5}\left(2^{1/3}-1\right) \le \int_0^1 \frac{x^4}{(1+x^6)^{2/3}}dx \le 1$

<u>Miscellaneous Illustrations</u> **:**

Illustration 29 : Evaluate : $\int_0^\pi \frac{x^3\cos^4 x\sin^2 x}{(\pi^2-3\pi x+3x^2)}dx$

Solution : Let $I = \int_0^\pi \frac{x^3\cos^4 x\sin^2 x}{(\pi^2-3\pi x+3x^2)}dx$ ..... (i)

$$= \int_0^\pi \frac{(\pi-x)^3\cos^4(\pi-x)\sin^2(\pi-x)dx}{\pi^2-3\pi(\pi-x)+3(\pi-x)^2} \quad \text{(By. Prop.)}$$

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$$= \int_0^{\pi} \frac{(\pi^3 - x^3 - 3\pi^2 x + 3\pi x^2)\cos^4 x \sin^2 x}{(\pi^2 - 3\pi x + 3x^2)} dx \qquad \text{..... (ii)}$$

Adding (i) and (ii) we have

$$2I = \int_0^{\pi} \frac{(\pi^3 - 3\pi^2 x + 3\pi x^2)\cos^4 x \sin^2 x}{(\pi^2 - 3\pi x + 3x^2)} dx$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi} \cos^4 x \sin^2 x dx \quad \Rightarrow \quad 2I = 2\pi\int_0^{\pi/2} \cos^4 x \sin^2 x dx$$

$$\therefore \quad I = \pi\int_0^{\pi/2} \cos^4 x \sin^2 x\, dx$$

Using walli's formula, we get $I = \pi\frac{(3.1)(1)}{6.4.2}\frac{\pi}{2} = \frac{\pi^2}{32}$ **Ans.**

Illustration 30 : Let f be an injective function such that $f(x)\,f(y) + 2 = f(x) + f(y) + f(xy)$ for all non negative real x and y with f(0) = 1 and f'(1) = 2 find f(x) and show that $3\int f(x)dx - x(f(x)+2)$ is a constant.

Solution : We have f(x)f(y) + 2 = f(x) + f(y) + f(xy)

Putting x = 1 & y = 1

then f(1)f(1) + 2 = 3f(1)

we get f(1) = 1,2

$f(1) \neq 1$ ($\because$ f(0) = 1 & function is injective)

then f(1) = 2

Replacing y by $\frac{1}{x}$ in (1) then

$$f(x)f\left(\frac{1}{x}\right) + 2 = f(x) + f\left(\frac{1}{x}\right) + f(1) \quad \Rightarrow \quad f(x)f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

Hence f(x) is of the type

$f(x) = 1 \pm x^n$

$\because$ f(1) = 2

$\therefore$ $f(x) = 1 + x^n$

and $f'(x) = nx^{n-1} \Rightarrow f'(1) = n = 2$

$f(x) = 1 + x^2$

$$\therefore \quad 3\int f(x)dx - x(f(x)+2) = 3\int (1+x^2)dx - x(1+x^2+2)$$

$$= 3\left(x + \frac{x^3}{3}\right) - x(3+x^2) + c = c = \text{constant}$$

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Illustration 31 : Evaluate : $\int_{-1}^{1}[x[1+\sin \pi x]+1]dx$, [.] is the greatest integer function.

Solution : Let $I = \int_{-1}^{1}[x[1+\sin \pi x]+1]dx = \int_{-1}^{0}[x[1+\sin \pi x]+1]dx + \int_{0}^{1}[x[1+\sin \pi x]+1]dx$

Now $[1 + \sin\pi x] = 0$ if $-1 < x < 0$

$[1 + \sin\pi x] = 1$ if $0 < x < 1$

$\therefore \quad I = \int_{-1}^{0}1.dx + \int_{0}^{1}[x+1]dx = 1 + 1\int_{0}^{1}dx = 1 + 1 = 2.$ **Ans.**

Illustration 32 : Find the limit, when $n \to \infty$ of

$$\frac{1}{\sqrt{(2n-1^2)}}+\frac{1}{\sqrt{(4n-2^2)}}+\frac{1}{\sqrt{(6n-3^2)}}+.....+\frac{1}{n}$$

Solution : Let $P = \lim_{n\to\infty}\left[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\frac{1}{\sqrt{6n-3^2}}+....+\frac{1}{n}\right]$

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$$= \lim_{n\to\infty}\left[\frac{1}{\sqrt{1(2n)-1^2}}+\frac{1}{\sqrt{2(2n)-2^2}}+\frac{1}{\sqrt{3(2n)-3^2}}+....+\frac{1}{\sqrt{n(2n)-n^2}}\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{r(2n)-r^2}} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n.\sqrt{2\frac{r}{n}-\left(\frac{r}{n}\right)^2}} = \int_{0}^{1}\frac{dx}{\sqrt{(2x-x^2)}}$$

Put $x = t^2 \Rightarrow dx = 2t\,dt$

$$\therefore \quad P = \int_{0}^{1}\frac{2tdt}{t\sqrt{2-t^2}} = \left[2\sin^{-1}\left(\frac{t}{\sqrt{2}}\right)\right]_0^1 = 2\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 2\left(\frac{\pi}{4}\right)$$

Hence $P = \pi/2$. **Ans.**

Illustration 33 : If $f(x) = \begin{cases}1-|x|, & |x|\le 1\\ |x|-1, & |x|>1\end{cases}$, and $g(x) = f(x-1) + f(x+1)$. Find the value of $\int_{-3}^{5}g(x)\,dx$.

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Solution : Given,

$$f(x)=\begin{cases}-x-1, & x<-1\\ 1+x, & -1\le x<0\\ 1-x, & 0\le x\le 1\\ x-1, & x>1\end{cases}\quad;\quad f(x-1)=\begin{cases}-x, & x-1<-1 & \Rightarrow x<0\\ x, & -1\le x-1<0 & \Rightarrow 0\le x<1\\ 2-x, & 0\le x-1\le 1 & \Rightarrow 1\le x\le 2\\ x-2, & x-1>1 & \Rightarrow x>2\end{cases}$$

Similarly

$$f(x+1)=\begin{cases}-x-2, & x+1<-1 & \Rightarrow x<-2\\ x+2, & -1\le x+1<0 & \Rightarrow -2\le x<-1\\ -x, & 0\le x+1\le 1 & \Rightarrow -1\le x\le 0\\ x, & x+1>1 & \Rightarrow x>0\end{cases}$$

$$\Rightarrow\quad g(x)=f(x-1)+f(x+1)=\begin{cases}-2x-2 & x<-2\\ 2, & -2\le x<-1\\ -2x, & -1\le x\le 0\\ 2x, & 0<x<1\\ 2, & 1<x\le 2\\ 2x-2, & 2<x\end{cases}$$

Clearly g(x) is even,

Now $\int_{-3}^{5} g(x)dx = 2\int_{0}^{3} g(x)\,dx + \int_{3}^{5} g(x)\,dx$

$$= 2\left(\int_0^1 2x\,dx+\int_1^2 2dx+\int_2^3 (2x-2)\,dx\right)+\int_3^5 (2x-2)\,dx = 24$$

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ANSWERS FOR DO YOURSELF

1 :	**(i)**	$\frac{31}{6}$	**(ii)**	2	**(iii)**	$2(\sqrt{2}-1)$	**(iv)**	$\frac{9}{2}$
2 :	**(i)**	2	**(ii)**	$\pi\ell n2$				
3 :	**(i)**	2	**(ii)**	$\pi/12$				
4 :	**(i)**	$\frac{\pi}{3}$	**(ii)**	$-\left(\frac{3\pi}{2}\right)\ell n2$	**(iii)**	0	**(iv)**	$\frac{4}{3}$
5 :	**(i)**	$\frac{23}{4}$	**(ii)**	$\left(\sqrt{3}-1\right)$				
6 :	**(i)**	$\frac{3}{2}\sin 1$	**(ii)**	$\frac{dy}{dx}=\frac{-\sqrt{3-\sin^2 x}}{\cos y}$				
7 :	**(i)**	$\frac{1}{2}\ell n3$	**(ii)**	$\frac{\pi}{2}$				

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EXERCISE (O-1)

1. $\int_{\frac{1}{2}}^{3\frac{1}{2}} \left\{\frac{1}{2}(|x-3|+|1-x|-4)\right\} dx$ equals-

(A) $-\frac{3}{2}$ (B) $\frac{9}{8}$ (C) $\frac{1}{4}$ (D) $\frac{3}{2}$

Where {.} denotes the fraction part function.

DI0001

2. The value of $\int_{-1}^{1} \frac{dx}{\sqrt{|x|}}$ is -

(A) $\frac{1}{2}$ (B) 2 (C) 4 (D) undefined

DI0002

3. The value of the definite integral $\int_{2}^{4} \left(x(3-x)(4+x)(6-x)(10-x)+\sin x\right) dx$ equals-

(A) cos 2 + cos 4 (B) cos 2 – cos 4 (C) sin 2 + sin 4 (D) sin 2 – sin 4

DI0003

4. If $\frac{C_0}{1}+\frac{C_1}{2}+\frac{C_2}{3}=0$, where C_0, C_1, C_2 are all real, the equation $C_2x^2 + C_1x + C_0 = 0$ has :

(A) atleast one root is (0,1)

(B) one root is (1,2) & other is (3,4)

(C) one root is (–1,1) & the other is (–5,–2)

(D) both roots imaginary

DI0004

5. Value of the definite integral $\int_{-1/2}^{1/2} (\sin^{-1}(3x-4x^3) - \cos^{-1}(4x^3-3x))\, dx$

(A) 0 (B) $-\frac{\pi}{2}$ (C) $\frac{7\pi}{2}$ (D) $\frac{\pi}{2}$

DI0005

6. The value of the definite integral $\int_{0}^{\pi/2} \sin x \sin 2x \sin 3x\, dx$ is equal to :

(A) $\frac{1}{3}$ (B) $-\frac{2}{3}$ (C) $-\frac{1}{3}$ (D) $\frac{1}{6}$

DI0006

7. Variable x and y are related by equation $x = \int_{0}^{y} \frac{dt}{\sqrt{1+t^2}}$. The value of $\frac{d^2y}{dx^2}$ is equal to

(A) $\frac{y}{\sqrt{1+y^2}}$ (B) y (C) $\frac{2y}{\sqrt{1+y^2}}$ (D) 4y

DI0007

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8. The value of the defined integral $\int_0^{\pi/2} (\sin x + \cos x).\sqrt{\frac{e^x}{\sin x}}dx$ equals

(A) $2\sqrt{e^{\pi/2}}$ (B) $\sqrt{e^{\pi/2}}$ (C) $2\sqrt{e^{\pi/2}}.\cos 1$ (D) $\frac{1}{2}e^{\pi/4}$

DI0008

9. If $g(x) = \int_0^x \cos^4 t\,dt$, then $g(x + \pi)$ equals

(A) $g(x) + g(\pi)$ (B) $g(x) - g(\pi)$ (C) $g(x)\,g(\pi)$ (D) $[g(x)/g(\pi)]$

DI0009

10. $\int_0^1 \frac{\tan^{-1}x}{x}dx =$

(A) $\int_0^{\pi/4} \frac{\sin x}{x}dx$ (B) $\int_0^{\pi/2} \frac{x}{\sin x}dx$ (C) $\frac{1}{2}\int_0^{\pi/2} \frac{x}{\sin x}dx$ (D) $\frac{1}{2}\int_0^{\pi/4} \frac{x}{\sin x}dx$

DI0010

11. If $I = \int_0^{\pi/2} \ell n(\sin x)dx$ then $\int_{-\pi/4}^{\pi/4} \ell n(\sin x + \cos x)dx$

(A) $\frac{I}{2}$ (B) $\frac{I}{4}$ (C) $\frac{I}{\sqrt{2}}$ (D) I

DI0011

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12. The value of the definite integral $\int_1^e \left((x+1)e^x.\ell nx\right)dx$ is -

(A) e (B) e^{e+1} (C) $e^e(e-1)$ (D) $e^e(e-1) + e$

DI0012

13. The value of the definite integral $\int_{19}^{37} \left(\{x\}^2 + 3(\sin 2\pi x)\right)dx$ where { x } denotes the fractional part function.

(A) 0 (B) 6 (C) 9 (D) can't be determined

DI0013

14. If g (x) is the inverse of f (x) and f (x) has domain $x \in [1, 5]$, where f (1) = 2 and f (5) = 10 then the values of $\int_1^5 f(x)dx + \int_2^{10} g(y)dy$ equals -

(A) 48 (B) 64 (C) 71 (D) 52

DI0014

15. $\int_{-1}^1 \frac{x^3 + |x| + 1}{x^2 + 2|x| + 1}dx = a\ell n2 + b$ then -

(A) a = 2; b = 1 (B) a = 2; b = 0 (C) a = 3; b = –2 (D) a = 4; b = –1

DI0015

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16. Suppose that F(x) is an antiderivative of $f(x) = \frac{\sin x}{x}$, $x > 0$ then $\int_1^3 \frac{\sin 2x}{x}$ can be expressed as -

(A) $F(6) - F(2)$ (B) $\frac{1}{2}(F(6) - F(2))$ (C) $\frac{1}{2}(F(3) - F(1))$ (D) $2(F(6) - F(2))$

DI0016

17. Let f be a positive function. Let $I_1 = \int_{1-k}^{k} x f(x(1-x))dx$; $I_2 = \int_{1-k}^{k} f(x(1-x))dx$, where $2k - 1 > 0$.

Then $\frac{I_2}{I_1}$ is -

(A) k (B) 1/2 (C) 1 (D) 2

DI0017

18. $\int_0^\infty f\left(x + \frac{1}{x}\right) \cdot \frac{\ln x}{x} dx$

(A) is equal to zero (B) is equal to one (C) is equal to $\frac{1}{2}$ (D) can not be evaluated

DI0018

19. $\int_0^{\pi/2} \frac{\cos\theta - \sin\theta}{(1+\cos\theta)(1+\sin\theta)} d\theta$ equals -

(A) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (B) $\cos^{-1}(0)$ (C) $\cos^{-1}(1)$ (D) $\cos^{1}(-1)$

DI0019

20. Let $f(x)$ be a continuous function on [0,4] satisfying $f(x)\, f(4 - x) = 1$.

The value of the definite integral $\int_0^4 \frac{1}{1+f(x)} dx$ equals-

(A) 0 (B) 1 (C) 2 (D) 4

DI0020

21. If $g(x) = \int_1^x e^{t^2} dt$ then the value of $\int_3^{x^3} e^{t^2} dt$ equals

(A) $g(x^3) - g(3)$ (B) $g(x^3) + g(3)$ (C) $g(x^3) - 3$ (D) $g(x^3) - 3g(x)$

DI0021

22. If $\int_0^x f(t)\,dt = x + \int_x^1 t^2 \cdot f(t)dt + \frac{\pi}{4} - 1$, then the value of the integral $\int_{-1}^{1} f(x)\,dx$ is equal to

(A) 0 (B) $\pi/4$ (C) $\pi/2$ (D) π

DI0022

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23. $\lim_{x\to\infty}\left(x^3\int_{-1/x}^{1/x}\frac{ln(1+t^2)}{1+e^t}dt\right)$ equals

(A) 1/3 (B) 2/3 (C) 1 (D) 0

DI0023

24. $\lim_{x\to 0}\frac{\int_0^x \sin t^2 dt}{x(1-\cos x)}$ equals -

(A) $\frac{1}{3}$ (B) 2 (C) $\frac{1}{2}$ (D) $\frac{2}{3}$

DI0024

25. The true solution set of the inequality, $\sqrt{5x-6-x^2}+\left(\frac{\pi}{2}\int_0^x dz\right) > x\int_0^\pi \sin^2 x dx$ is :

(A) R (B) (1,6) (C) (–6,1) (D) (2,3)

DI0025

26. Let $u=\int_0^1\frac{ln(x+1)}{x^2+1}dx$ and $v=\int_0^{\pi/2} ln(\sin 2x)dx$ then -

(A) u = 4v (B) 4u + v = 0 (C) u + 4v = 0 (D) 2u + v = 0

DI0026

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27. $\lim_{h\to 0}\frac{\int_a^{x+h} ln^2 t dt-\int_a^x ln^2 t dt}{h}=$

(A) 0 (B) ln^2x (C) $\frac{2ln x}{x}$ (D) does not exist

DI0027

28. Let $I_1=\int_0^{\pi/2} e^{-x^2}\sin(x)dx; I_2=\int_0^{\pi/2} e^{-x^2}dx; I_3=\int_0^{\pi/2} e^{-x^2}(1+x)dx$

and consider the statements

I $I_1 < I_2$ **II** $I_2 < I_3$ **III** $I_1 = I_3$

Which of the following is(are) true ?

(A) I only (B) II only

(C) Neither I nor II nor III (D) Both I and II

DI0028

29. $\lim_{n\to\infty}\frac{1}{\sqrt{n}\sqrt{n+1}}+\frac{1}{\sqrt{n}\sqrt{n+2}}+.....+\frac{1}{\sqrt{n}\sqrt{4n}}$ is equal to -

(A) 2 (B) 4 (C) $2\left(\sqrt{2}-1\right)$ (D) $2\sqrt{2}-1$

DI0029

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30. If $f(x) = x\sin x^2$; $g(x) = x\cos x^2$ for $x \in [-1, 2]$

$A = \int_{-1}^{2} f(x)\,dx$; $B = \int_{-1}^{2} g(x)\,dx$, then

(A) $A > 0$; $B < 0$ (B) $A < 0$; $B > 0$ (C) $A > 0$; $B > 0$ (D) $A < 0$; $B < 0$

DI0030

EXERCISE (O-2)

1. The value of $\int_{\pi}^{2\pi} [2\cos x]dx$ where [.] represents the greatest integer function, is -

(A) $-\frac{5\pi}{6}$ (B) $-\frac{\pi}{2}$ (C) $-\pi$ (D) none

DI0031

2. Which one of the following functions is not continuous on $(0,\pi)$?

(A) $f(x) = \cot x$ (B) $g(x) = \int_0^x t\sin\frac{1}{t}\,dt$

(C) $h(x) = \begin{cases} 1 & 0 < x \le \frac{3\pi}{4} \\ 2\sin\frac{2}{9}x & \frac{3\pi}{4} < x < \pi \end{cases}$ (D) $l(x) = \begin{cases} x\sin x\ , & 0 < x \le \frac{\pi}{2} \\ \frac{\pi}{2}\sin(x+\pi)\ , & \frac{\pi}{2} < x < \pi \end{cases}$

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DI0032

3. $\int_0^{\pi/2} \sqrt{\frac{\sec x - \tan x}{\sec x + \tan x}}\,\frac{\operatorname{cosec} x}{\sqrt{1 + 2\operatorname{cosec} x}}\,dx$ has the value -

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $2\pi/3$

DI0033

4. Let f be a continuous function satisfying $f'(\ln x) = \begin{cases} 1 & \text{for } 0 < x \le 1 \\ x & \text{for } x > 1 \end{cases}$ and $f(0) = 0$ then $f(x)$ can be defined as

(A) $f(x) = \begin{cases} 1 & \text{if } x \le 0 \\ 1 - e^x & \text{if } x > 0 \end{cases}$ (B) $f(x) = \begin{cases} 1 & \text{if } x \le 0 \\ e^x - 1 & \text{if } x > 0 \end{cases}$

(C) $f(x) = \begin{cases} x & \text{if } x < 0 \\ e^x & \text{if } x > 0 \end{cases}$ (D) $f(x) = \begin{cases} x & \text{if } x \le 0 \\ e^x - 1 & \text{if } x > 0 \end{cases}$

DI0034

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5. The value of the definite integral $\int_0^{\pi/2} \sin|2x-\alpha|\,dx$ where $\alpha \in [0, \pi]$

(A) 1 (B) $\cos\alpha$ (C) $\frac{1+\cos\alpha}{2}$ (D) $\frac{1-\cos\alpha}{2}$

DI0035

6. Let $f(x) = \int_0^{g(x)} \frac{dt}{\sqrt{1+t^2}}$ where $g(x) = \int_0^{\cos x} (1+\sin t^2)\,dt$. Also $h(x) = e^{-|x|}$ and $l(x) = x^2 \sin\frac{1}{x}$ if $x \neq 0$ and $l(0) = 0$ then $f'\left(\frac{\pi}{2}\right)$ equals

(A) $l'(0)$ (B) $h'(0^-)$ (C) $h'(0^+)$ (D) $\lim_{x\to 0} \frac{1-\cos x}{x\sin x}$

DI0036

7. Suppose f is continuous and satisfies $f(x) + f(-x) = x^2$ then the integral $\int_{-1}^{1} f(x)dx$ has the value equal to

(A) $\frac{2}{3}$ (B) $\frac{1}{3}$ (C) $\frac{4}{3}$ (D) zero

DI0037

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8. If the value of the integral $\int_1^2 e^{x^2} dx$ is α, then the value of $\int_e^{e^4} \sqrt{\ell nx}\, dx$ is -

(A) $e^4 - e - \alpha$ (B) $2e^4 - e - \alpha$ (C) $2(e^4 - e) - \alpha$ (D) $2e^4 - 1 - \alpha$

DI0038

9. If $\int_0^1 \frac{e^t dt}{1+t} = A$ then the value of $\int_0^1 \frac{e^t dt}{(1+t)^2}$ is -

(A) $A + \frac{e}{2} - 1$ (B) $A - \frac{e}{2} + 1$ (C) $A - \frac{e}{2} - 1$ (D) $A + \frac{e}{2} + 1$

DI0039

10. The value of the definite integral $\int_0^{\pi/3} l\,n\left(1+\sqrt{3}\tan x\right)dx$ equals-

(A) $\frac{\pi}{3} l\,n\,2$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi^2}{6} l\,n\,2$ (D) $\frac{\pi}{2} l\,n\,2$

DI0040

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Paragraph for Question Nos. 11 to 13

Let the function f satisfies

$$f(x)\cdot f'(-x) = f(-x)\cdot f'(x) \text{ for all x and } f(0) = 3.$$

11. The value of $f(x)\cdot f(-x)$ for all x, is

(A) 4 (B) 9 (C) 12 (D) 16

DI0041

12. $\int_{-51}^{51} \frac{dx}{3+f(x)}$ has the value equal to

(A) 17 (B) 34 (C) 102 (D) 0

DI0041

13. Number of roots of $f(x) = 0$ in $[-2, 2]$ is

(A) 0 (B) 1 (C) 2 (D) 4

DI0041

14. $\lim_{k\to 0} \frac{1}{k}\int_0^k (1+\sin 2x)^{\frac{1}{x}} dx$ -

(A) 2 (B) 1 (C) e^2 (D) non existent

DI0042

15. The absolute value of $\frac{\int_0^{\pi/2} (x\cos x+1)e^{\sin x}dx}{\int_0^{\pi/2} (x\sin x-1)e^{\cos x}dx}$ is equal to -

(A) e (B) πe (C) e/2 (D) π/e

DI0043

16. For positive integers k = 1, 2, 3, ... n, let S_k denotes the area of ΔAOB_k (where 'O' is origin) such that that $\angle AOB_k = \frac{k\pi}{2n}$, $OA = 1$ and $OB_k = k$. The value of the $\lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n} S_k$ is

(A) $\frac{2}{\pi^2}$ (B) $\frac{4}{\pi^2}$ (C) $\frac{8}{\pi^2}$ (D) $\frac{1}{2\pi^2}$

DI0044

17. The true set of values of 'a' for which the inequality $\int_a^0 (3^{-2x} - 2.3^{-x})dx \ge 0$ is true is -

(A) $[0,1]$ (B) $(-\infty,-1]$ (C) $[0,\infty)$ (D) $(-\infty,-1] \cup [0,\infty)$

DI0045

18. The value of $\lim_{n\to\infty} \sum_{r=1}^{r=4n} \frac{\sqrt{n}}{\sqrt{r}\left(3\sqrt{r}+4\sqrt{n}\right)^2}$ is equal to

(A) $\frac{1}{35}$ (B) $\frac{1}{14}$ (C) $\frac{1}{10}$ (D) $\frac{1}{5}$

DI0046

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Multiple Correct :

19. $\int_0^\infty \frac{x}{(1+x)(1+x^2)}dx =$

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$

(C) is same as $\int_0^\infty \frac{dx}{(1+x)(1+x^2)}$ (D) cannot be evaluated

DI0047

20. Let $u = \int_0^\infty \frac{dx}{x^4+7x^2+1}$ & $v = \int_0^\infty \frac{x^2dx}{x^4+7x^2+1}$ then -

(A) $v > u$ (B) $6v = \pi$ (C) $3u + 2v = 5\pi/6$ (D) $u + v = \pi/3$

DI0048

21. Which of the following pair(s) of functions are the primitive of one and the same function ?

(A) $f(x) = ln\, ax$; $g(x) = lnx$

(B) $f(x) = 2\sin^2x$; $g(x) = -\cos 2x$

(C) $f(x) = (e^x + e^{-x})^2$; $g(x) = (e^x - e^{-x})^2$

(D) $f(x) = \frac{e^x\left(e^x+e^{-x}\right)}{2}; g(x) = \frac{e^x\left(e^x-e^{-x}\right)}{2}$

DI0049

22. Let $S_n = \sum_{k=1}^{n} \frac{k^2+n^2}{n^3}$ and $T_n = \sum_{k=0}^{n-1} \frac{k^2+n^2}{n^3}$ for n = 1,2,3,..... Then -

(A) $S_n < \frac{4}{3}$ (B) $T_n > \frac{4}{3}$ (C) $S_n > \frac{4}{3}$ (D) $T_n < \frac{4}{3}$

DI0050

23. Which of the following statement(s) is/are **TRUE ?**

(A) $\int_0^1 \ell nxdx = -1$

(B) $\lim_{n\to\infty} \frac{1}{n} ln\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right).....\left(1+\frac{n}{n}\right)\right) = 1 + 2ln\,2.$

(C) Let f be a continuous and non-negative function defined on [a.b].

If $\int_a^b f(x)dx = 0$ then $f(x) = 0\ \forall\ x \in [a,b]$

(D) Let f be a continuous function defined on [a ,b] such that $\int_a^b f(x)dx = 0$, then there exists atleast one $c \in (a,b)$ for which $f(c) = 0$.

DI0051

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24. Let $f(x)=\begin{cases} x+1, & 0\le x\le 1 \\ 2x^2-6x+6, & 1<x\le 2\end{cases}$ and $g(t)=\int_{t-1}^{t} f(x)dx$ for $t\in[1,2]$

Which of the following hold(s) good ?

(A) $f(x)$ is continuous and differentiable in [0,2]

(B) g'(t) vanishes for t = 3/2 and 2

(C) g(t) is maximum at t = 3/2

(D) g(t) is minimum at t = 1

DI0052

25. Let $F(x)=\int_{\sin x}^{\cos x} e^{(1+\arcsin t)^2}dt$ on $\left[0,\frac{\pi}{2}\right]$ then

(A) $F''(c)=0$ for all $c\in\left(0,\frac{\pi}{2}\right)$

(B) $F''(c)=0$ for some $c\in\left(0,\frac{\pi}{2}\right)$

(C) $F'(c)\ne 0$ for all $c\in\left(0,\frac{\pi}{2}\right)$

(D) $F(c)=0$ for some $c\in\left(0,\frac{\pi}{2}\right)$

DI0053

26. Let $f:\left[0,\frac{\pi}{2}\right]\to[0,1]$ be a differentiable function such that $f(0)=0$, $f\left(\frac{\pi}{2}\right)=1$, then

(A) $f'(\alpha)=\sqrt{1-f^2(\alpha)}$ for all $\alpha\in\left(0,\frac{\pi}{2}\right)$.

(B) $f'(\alpha)=\frac{2}{\pi}$ for all $\alpha\in\left(0,\frac{\pi}{2}\right)$.

(C) $f(\alpha)\,f'(\alpha)=\frac{1}{\pi}$ for atleast one $\alpha\in\left(0,\frac{\pi}{2}\right)$

(D) $f'(\alpha)=\frac{8\alpha}{\pi^2}$ for atleast one $\alpha\in\left(0,\frac{\pi}{2}\right)$.

DI0054

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EXERCISE (S-1)

1. Evaluate : $\int_0^1 \frac{\sin^{-1}\sqrt{x}}{\sqrt{x(1-x)}}dx$

DI0055

2. If $P=\int_0^\infty \frac{x^2}{1+x^4}dx; Q=\int_0^\infty \frac{x\,dx}{1+x^4}$ and $R=\int_0^\infty \frac{dx}{1+x^4}$, then prove that :

(a) $Q=\frac{\pi}{4}$, **(b)** $P=R$, **(c)** $P-\sqrt{2}Q+R=\frac{\pi}{2\sqrt{2}}$

DI0056

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3. Evaluate : $\int_0^{\pi/2} \frac{1+2\cos x}{(2+\cos x)^2}\,dx$

DI0057

4. Evaluate :

(i) $\int_0^1 e^{l n \tan^{-1} x}.\sin^{-1}(\cos x).dx$ **DI0058**

(ii) $\int_{1/3}^{3} \frac{\sin^{-1}\frac{x}{\sqrt{1+x^2}}}{x}dx$ **DI0059**

5. Evaluate : $\int_0^{\pi/2} \sqrt{\frac{1-\sin 2x}{1+\sin 2x}}dx$ **DI0060**

6. Let h(x) = (fog)(x) + K where K is any constant. If $\frac{d}{dx}(h(x)) = -\frac{\sin x}{\cos^2(\cos x)}$ then compute the value of j(0) where $j(x) = \int_{g(x)}^{f(x)} \frac{f(t)}{g(t)}dt$, where f and g are trigonometric functions.

DI0061

7. Let $u = \int_0^{\pi/4} \left(\frac{\cos x}{\sin x+\cos x}\right)^2 dx$ and $v = \int_0^{\pi/4} \left(\frac{\sin x+\cos x}{\cos x}\right)^2 dx$. Find the value of $\frac{v}{u}$.

DI0062

8. Evaluate : $\int_1^{\frac{1+\sqrt{5}}{2}} \frac{x^2+1}{x^4-x^2+1} l n\left(1+x-\frac{1}{x}\right)dx$

DI0063

9. If $\int_0^{\pi} \sqrt{(\cos x+\cos 2x+\cos 3x)^2+(\sin x+\sin 2x+\sin 3x)^2}dx$ has the value equal to $\left(\frac{\pi}{k}+\sqrt{w}\right)$ where k and w are positive integers, find the value of $(k^2 + w^2)$.

DI0064

10. Let $I_n = \int_{-n}^{n} \left(\{x+1\}.\{x^2+2\}+\{x^2+2\}\{x^3+4\}\right)dx$, where {.} denotes the fractional part of x. Find I_1.

DI0065

11. Evaluate : $\int_0^{\frac{l n 3}{2}} \frac{e^x+1}{e^{2x}+1}dx$ **DI0066**

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12. Let $I = \int_0^1 \frac{2+3x+4x^2}{2\sqrt{1+x+x^2}}dx$. Find the value of I^2. **DI0067**

13. Let f and g be function that are differentiable for all real numbers x and that have the following properties:

(i) $f'(x) = f(x) - g(x)$; (ii) $g'(x) = g(x) - f(x)$; (iii) $f(0) = 5$; (iv) $g(0) = 1$

(a) Prove that $f(x) + g(x) = 6$ for all x. **(b)** Find f(x) and g(x)

DI0068

14. Evaluate : $\int_0^{\pi/2} e^x \left\{\cos(\sin x)\cos^2\frac{x}{2} + \sin(\sin x)\sin^2\frac{x}{2}\right\}dx$

DI0069

15. Evaluate : $\int_1^e \left\{(1+x)e^x + (1-x)e^{-x}\right\} \ln x\, dx$

DI0070

16. $\int_1^2 \frac{(x^2-1)dx}{x^3.\sqrt{2x^4-2x^2+1}} = \frac{u}{v}$ where u and v are in their lowest form. Find the value of $\frac{(1000)u}{v}$.

DI0071

17. Evaluate : $\int_{-2}^{2} \frac{x^2-x}{\sqrt{x^2+4}}dx$

DI0072

18. Evaluate : $\int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^7+3x^6-10x^5-7x^3-12x^2+x+1}{x^2+2}dx$

DI0073

19. If a_1, a_2 and a_3 are the three values of a which satisfy the equation

$\int_0^{\pi/2} (\sin x + a\cos x)^3 dx - \frac{4a}{\pi-2}\int_0^{\pi/2} x\cos x\, dx = 2$ then find the value of $1000\left(a_1^2 + a_2^2 + a_3^2\right)$.

DI0074

20. Evaluate : $\int_0^1 \frac{\sin^{-1}\sqrt{x}}{x^2-x+1}dx$ **DI0075**

21. Evaluate : $\int_0^{\pi/4} \frac{x\,dx}{\cos x(\cos x + \sin x)}$ **DI0076**

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22. Evaluate : $\int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin\left(\frac{\pi}{4}+x\right)}dx$ **DI0077**

23. A continuous real function f satisfies f(2x) = 3 f(x) $\forall$ x $\in$ R

If $\int_0^1 f(x)dx = 1$, then compute the value of definite integral $\int_1^2 f(x)dx$ **DI0078**

24. Evaluate : $\int_0^3 \sqrt{\frac{x}{3-x}}\,dx$ **DI0079**

25. Evaluate : $\int_0^{\pi} \frac{(2x+3)\sin x}{(1+\cos^2 x)}dx$ **DI0080**

26. Evaluate : $\int_0^{\pi} \frac{(ax+b)\sec x\tan x}{4+\tan^2 x}dx (a, b > 0)$ **DI0081**

27. Evaluate : $\int_0^{a} \frac{l\,n(1+ax)}{1+x^2}dx, a \in N$ **DI0082**

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28. Evaluate : $\int_0^{\pi} \frac{x^2\sin 2x.\sin\left(\frac{\pi}{2}.\cos x\right)}{2x-\pi}dx$ **DI0083**

29. Show that $\int_0^{p+q\pi} |\cos x|\,dx = 2q + \sin p$ where $q \in N$ & $-\frac{\pi}{2} < p < \frac{\pi}{2}$. **DI0084**

30. Evaluate : $\int_{-1}^{1} \frac{(2x^{332} + x^{998} + 4x^{1668}.\sin x^{691})}{1+x^{666}}dx$ **DI0085**

31. Let f(x) be a function defined on R such that f '(x) = f '(3 – x) $\forall$ x $\in$ [0,3] with f(0) = –32 and f(3) = 46. Then find the value of $\int_0^3 f(x)dx$.

DI0086

32. Let $U_{10} = \int_0^{\frac{\pi}{2}} x\sin^{10} x\,dx$, then find the value of $\left(\frac{100U_{10}-1}{U_8}\right)$. **DI0087**

33. $\lim_{n\to\infty} n^2 \int_{-1/n}^{1/n} (2010\sin x + 2012\cos x)|x|\,dx$ **DI0088**

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34. If $\phi(x) = \cos x - \int_0^x (x-t)\phi(t)dt$. Then find the value of $\phi''(x) + \phi(x)$. **DI0089**

35. Evaluate : $\lim_{x\to+\infty} \frac{d}{dx} \int_{2\sin\frac{1}{x}}^{3\sqrt{x}} \frac{3t^4+1}{(t-3)(t^2+3)}dt$ **DI0090**

36. Find a positive real valued continuously differentiable functions f on the real line such that for all x

$$f^2(x) = \int_0^x \left((f(t))^2 + (f'(t))^2\right)dt + e^2$$

DI0091

37. **(a)** Let $g(x) = x^c . e^{2x}$ & let $f(x) = \int_0^x e^{2t}.(3t^2+1)^{1/2}dt$. For a certain value of 'c', the limit of $\frac{f'(x)}{g'(x)}$ as $x \to \infty$ is finite and non-zero. Determine the value of 'c' and the limit.

DI0092

(b) Find the constants 'a' $(a > 0)$ and 'b' such that $\lim_{x\to 0} \frac{\int_0^x \frac{t^2\,dt}{\sqrt{a+t}}}{bx - \sin x} = 1$.

DI0093

38. Evaluate

(a) $\lim_{n\to\infty} \left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)....\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$ **DI0094**

(b) $\lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{n+1} + \frac{2}{n+2} + ... + \frac{3n}{4n}\right]$ **DI0095**

39. Consider a function $f(n) = \frac{1}{1+n^2}$. Let $\alpha_n = \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$ and $\beta_n = \frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)$ for n = 1,2,3,.....

Also $\alpha = \lim_{n\to\infty} \alpha_n$ & $\beta = \lim_{n\to\infty} \beta_n$. Then prove **(a)** $\alpha_n < \beta_n$ **(b)** $\alpha = \beta$ **(c)** $\alpha_n < \frac{\pi}{4} < \beta_n$

DI0096

40. Prove the inequalities :

(a) $\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{\pi\sqrt{2}}{8}$ **DI0097**

(b) $2e^{-1/4} < \int_0^2 e^{x^2-x}dx < 2e^2$. **DI0098**

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EXERCISE (S-2)

1. Evaluate : $\int_0^{\sqrt{3}} \sin^{-1}\frac{2x}{1+x^2}dx$ **DI0099**

2. Evaluate : $\int_1^{16} \tan^{-1}\sqrt{\sqrt{x}-1}dx$ **DI0100**

3. Evaluate : $\int_{\frac{\sqrt{3a^2+b^2}}{2}}^{\sqrt{\frac{a^2+b^2}{2}}} \frac{x.dx}{\sqrt{(x^2-a^2)(b^2-x^2)}}$ **DI0101**

4. **(a)** Show that $\int_0^{\infty}\frac{dx}{x^2+2x\cos\theta+1} = 2\int_0^1\frac{dx}{x^2+2x\cos\theta+1}$ **DI0102**

(b) Evaluate : $f(\theta) = \int_0^{\infty}\frac{\tan^{-1}x}{x^2+2x\cos\theta+1}dx$, $\theta \in (0,\pi)$. **DI0103**

5. Evaluate : $\int_0^1 \frac{1-x}{1+x}\frac{dx}{\sqrt{x+x^2+x^3}}$ **DI0104**

6. Prove that :

(a) $\int_\alpha^\beta \sqrt{(x-\alpha)(\beta-x)}\,dx = \frac{(\beta-\alpha)^2\pi}{8}$ **DI0105**

(b) $\int_\alpha^\beta \sqrt{\frac{x-\alpha}{\beta-x}}dx = (\beta-\alpha)\frac{\pi}{2}$ **DI0106**

(c) $\int_\alpha^\beta \frac{dx}{x\sqrt{(x-\alpha)(\beta-x)}} = \frac{\pi}{\sqrt{\alpha\beta}}$ where $\alpha, \beta > 0$ **DI0107**

(d) $\int_\alpha^\beta \frac{x\,.\,dx}{\sqrt{(x-\alpha)(\beta-x)}} = (\alpha+\beta)\frac{\pi}{2}$, where $\alpha < \beta$ **DI0108**

7. **(a)** Let $\beta(n) = \int_0^{n\pi}\sqrt{1-\sin t}\,dt$. Find the value of $\beta(2) - \beta(1)$. **DI0109**

(b) Determine a positive integer $n \le 5$, such that $\int_0^1 e^x(x-1)^n dx = 16-6e$. **DI0110**

8. Suppose f is continuous, f (0) = 0, f (1) = 1, f ' (x) > 0 and $\int_0^1 f(x)dx = \frac{1}{3}$. Find the value of the definite integral $\int_0^1 f^{-1}(y)dy$.

DI0111

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9. Evaluate : $\int_0^1 \frac{2-x^2}{(1+x)\sqrt{1-x^2}}\,dx$ **DI0112**

10. Let $A = \int_{3/4}^{4/3} \frac{2x^2+x+1}{x^3+x^2+x+1}dx$, then find the value of e^A. **DI0113**

11. Evaluate : $\int_0^1 x^5\sqrt{\frac{1+x^2}{1-x^2}}\,dx$ **DI0114**

12. Evaluate : $\int_0^1 x\,(\tan^{-1}x)^2\,dx$ **DI0115**

13. Find the range of the function, $f(x) = \int_{-1}^{1}\frac{\sin x\,dt}{1-2t\cos x+t^2}$ **DI0116**

14. Let $y = f(x)$ be a quadratic function with $f'(2) = 1$. Find the value of the integral $\int_{2-\pi}^{2+\pi} f(x).\sin\left(\frac{x-2}{2}\right)dx$.

DI0117

15. If $y = x^{\int_1^x \ln t\,dt}$, find $\frac{dy}{dx}$ at $x = e$. **DI0118**

16. Let $f(x) = \begin{cases} -1 & \text{if } -2 \le x \le 0 \\ |x-1| & \text{if } 0 < x \le 2 \end{cases}$ and $g(x) = \int_{-2}^{x} f(t)dt$. Define $g(x)$ as a function of x and test the continuity and differentiability of $g(x)$ in $(-2,2)$.

DI0119

17. Let $I = \int_0^{\pi/2}\frac{\cos x+4}{3\sin x+4\cos x+25}dx$ and $J = \int_0^{\pi/2}\frac{\sin x+3}{3\sin x+4\cos x+25}dx$.

If $25I = a\pi + b\ln\frac{c}{d}$ where a,b,c and d $\in$N and $\frac{c}{d}$ is not a perfect square of a rational then find the value of $(a + b + c + d)$.

DI0120

18. Evaluate : $\int_0^{\pi/2}\tan^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]dx$ **DI0121**

19. Evaluate : $\int_0^{2\pi}\frac{dx}{2+\sin 2x}$ **DI0122**

20. Find the value of the definite integral $\int_0^{\pi}\left|\sqrt{2}\sin x+2\cos x\right|dx$. **DI0123**

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21. If $U_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x} dx$, then show that $U_1, U_2, U_3, \ldots\ldots, U_n$ constitute an AP.

Hence or otherwise find the value of U_n. **DI0124**

22. For $a \geq 2$, if the value of the definite integral $\int_0^{\infty} \frac{dx}{a^2 + (x - (1/x))^2}$ equals $\frac{\pi}{5050}$. Find the value of a.

DI0125

23. If $f(x) = x + \sin x$ and I denotes the value of integral $\int_{\pi}^{2\pi} \left(f^{-1}(x) + \sin x\right) dx$ then the value of $\left[\frac{2I}{3}\right]$

(where [.] denotes greatest integer function)

DI0126

24. A students forgot the product rule for differentiation and made the mistake of thinking that $(f.g)' = f'g'$. However he was lucky to get the correct answer. The function f that he used was $f(x) = e^{x^2}$.

If the domain of g(x) was the interval $\left(\frac{1}{2}, \infty\right)$ with $g(1) = e$. Find the value of g(5).

DI0127

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25. Determine a pair of number a and b for which $\int_0^1 \frac{ax + b}{(x^2 + 3x + 2)^2} dx = \frac{5}{2}$. **DI0128**

26. A curve C_1 is defined by : $\frac{dy}{dx} = e^x \cos x$ for $x \in [0, 2\pi]$ and passes through the origin. Prove that the roots of the function y = 0 (other than zero) occurs in the ranges $\frac{\pi}{2} < x < \pi$ and $\frac{3\pi}{2} < x < 2\pi$.

DI0129

27. Let $F(x) = \int_{-1}^{x} \sqrt{4 + t^2} dt$ and $G(x) = \int_{x}^{1} \sqrt{4 + t^2} dt$ then compute the value of (FG)' (0) where dash denotes the derivative.

DI0130

28. Comment upon the nature of roots of the quadratic equation $x^2 + 2x = k + \int_0^1 |t + k| dt$ depending on the value of $k \in R$.

DI0131

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29. Evaluate : $\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=0}^{n-1}\left[k\int_k^{k+1}\sqrt{(x-k)(k+1-x)}\,dx\right]$ **DI0132**

30. (a) $\lim_{n\to\infty}\left[\frac{n!}{n^n}\right]^{1/n}$ **DI0133**

(b) Let $P_n=\sqrt[n]{\frac{(3n)!}{(2n)!}}$ (n= 1,2,3.....), then find $\lim_{n\to\infty}\frac{P_n}{n}$. **DI0134**

31. Prove the inequalities :

(a) $\frac{1}{3}<\int_0^1 x^{(\sin x+\cos x)^2}dx<\frac{1}{2}$ **DI0135**

(b) $\frac{1}{2}\le\int_0^2\frac{dx}{2+x^2}\le\frac{5}{6}$ **DI0136**

32. Let $a+b=4$, where $a<2$ and let $g(x)$ be a differentiable function. If $\frac{dg}{dx}>0$ for all x, prove that $\int_0^a g(x)\,dx+\int_0^b g(x)\,dx$ increases as $(b-a)$ increases.

DI0137

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EXERCISE (JM)

1. The value of $\int_0^1\frac{8\log(1+x)}{1+x^2}dx$ is :- **[AIEEE-2011]**

(1) $\frac{\pi}{2}\log 2$ (2) log 2 (3) π log 2 (4) $\frac{\pi}{8}\log 2$

DI0138

2. Let [.] denote the greatest integer function then the value of $\int_0^{1.5}x[x^2]dx$ is :- **[AIEEE-2011]**

(1) $\frac{5}{4}$ (2) 0 (3) $\frac{3}{2}$ (4) $\frac{3}{4}$

DI0139

3. For $x\in\left(0,\frac{5\pi}{2}\right)$, define $f(x)=\int_0^x\sqrt{t}\sin t\,dt$. Then f has :- **[AIEEE-2011]**

(1) local minimum at π and local maximum at 2π

(2) local maximum at π and local minimum at 2π

(3) local maximum at π and 2π

(4) local minimum at π and 2π

DI0140

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4. If $g(x) = \int_0^x \cos 4t\, dt$, then $g(x + \pi)$ equals : **[AIEEE-2012]**

(1) $g(x) \cdot g(\pi)$ (2) $\frac{g(x)}{g(\pi)}$ (3) $g(x) + g(\pi)$ (4) $g(x) - g(\pi)$

DI0141

5. **Statement-I :** The value of the integral $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ is equal to $\frac{\pi}{6}$. **[JEE-MAIN-2013]**

Statement-II : $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$.

(1) Statement-I is true, Statement-II is true; Statement-II is a **correct** explanation for Statement-I.
(2) Statement-I is true, Statement-II is true; Statement-II is **not** a correct explanation for Statement-I.
(3) Statement-I is true, Statement-II is false.
(4) Statement-I is false, Statement-II is true.

DI0142

6. The intercepts on x-axis made by tangents to the curve, $y = \int_0^x |t|\, dt$, $x \in R$, which are parallel to the line $y = 2x$, are equal to **[JEE-MAIN 2013]**

(1) ± 1 (2) ± 2 (3) ± 3 (4) ± 4

DI0143

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7. The integral $\int_0^{\pi} \sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$ equals : **[JEE-MAIN-2014]**

(1) $\pi - 4$ (2) $\frac{2\pi}{3}-4-4\sqrt{3}$ (3) $4\sqrt{3}-4$ (4) $4\sqrt{3}-4-\frac{\pi}{3}$

DI0144

8. The integral $\int_2^4 \frac{\log x^2}{\log x^2 + \log(36-12x+x^2)}dx$ is equal to : **[JEE-MAIN-2015]**

(1) 1 (2) 6 (3) 2 (4) 4

DI0145

9. The integral $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1+\cos x}$ is equal to :- **[JEE-MAIN-2017]**

(1) –1 (2) –2 (3) 2 (4) 4

DI0146

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10. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+2^x}\,dx$ is : **[JEE-MAIN-2018]**

(1) $\frac{\pi}{2}$ (2) 4π (3) $\frac{\pi}{4}$ (4) $\frac{\pi}{8}$

DI0147

11. The value of $\int_{-\pi/2}^{\pi/2} \frac{dx}{[x]+[\sin x]+4}$, where [t] denotes the greatest integer less than or equal to t, is : **[JEE-MAIN 2019]**

(1) $\frac{1}{12}(7\pi+5)$ (2) $\frac{3}{10}(4\pi-3)$ (3) $\frac{1}{12}(7\pi-5)$ (4) $\frac{3}{20}(4\pi-3)$

DI0148

12. If $\int_0^x f(t)dt = x^2 + \int_x^1 t^2 f(t)dt$, then f'(1/2) is : **[JEE-MAIN 2019]**

(1) $\frac{6}{25}$ (2) $\frac{24}{25}$ (3) $\frac{18}{25}$ (4) $\frac{4}{5}$

DI0149

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13. Let $I = \int_a^b (x^4 - 2x^2)dx$. If I is minimum then the ordered pair (a, b) is : **[JEE-MAIN 2019]**

(1) $(-\sqrt{2}, 0)$ (2) $(-\sqrt{2}, \sqrt{2})$ (3) $(0, \sqrt{2})$ (4) $(\sqrt{2}, -\sqrt{2})$

DI0150

14. $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+....+\frac{1}{5n}\right)$ is equal to : **[JEE-MAIN 2019]**

(1) $\frac{\pi}{4}$ (2) $\tan^{-1}(2)$ (3) $\tan^{-1}(3)$ (4) $\frac{\pi}{2}$

DI0151

15. The value of α for which $4\alpha\int_{-1}^{2} e^{-\alpha|x|}dx = 5$, is : **[JEE-MAIN 2020]**

(1) $\log_e\left(\frac{3}{2}\right)$ (2) $\log_e\left(\frac{4}{3}\right)$ (3) $\log_e 2$ (4) $\log_e \sqrt{2}$

DI0152

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16. If f (a + b + 1 – x) = f(x), for all x, where a and b are fixed positive real numbers, then $\frac{1}{a+b}\int_a^b x(f(x)+f(x+1))dx$ is equal to : **[JEE-MAIN 2020]**

(1) $\int_{a+1}^{b+1} f(x)dx$ (2) $\int_{a+1}^{b+1} f(x+1)dx$ (3) $\int_{a-1}^{b-1} f(x+1)dx$ (4) $\int_{a-1}^{b-1} f(x)dx$

DI0153

17. If $I=\int_1^2 \frac{dx}{\sqrt{2x^3-9x^2+12x+4}}$, then : **[JEE-MAIN 2020]**

(1) $\frac{1}{9}<I^2<\frac{1}{8}$ (2) $\frac{1}{16}<I^2<\frac{1}{9}$ (3) $\frac{1}{6}<I^2<\frac{1}{2}$ (4) $\frac{1}{8}<I^2<\frac{1}{4}$

DI0154

18. The value of $\int_0^{2\pi} \frac{x\sin^8 x}{\sin^8 x+\cos^8 x}dx$ is equal to : **[JEE-MAIN 2020]**

(1) 2π (2) 4π (3) $2\pi^2$ (4) π^2

DI0155

EXERCISE (JA)

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1. The value of $\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x\sin x^2}{\sin x^2+\sin(\ln 6-x^2)}dx$ is **[JEE 2011, 3 (–1)]**

(A) $\frac{1}{4}\ln\frac{3}{2}$ (B) $\frac{1}{2}\ln\frac{3}{2}$ (C) $\ln\frac{3}{2}$ (D) $\frac{1}{6}\ln\frac{3}{2}$

DI0156

2. Let S be the area of the region enclosed by $y=e^{-x^2}$, y = 0, x = 0, and x = 1. Then - **[JEE 2012, 4]**

(A) $S\ge\frac{1}{e}$ (B) $S\ge 1-\frac{1}{e}$ (C) $S\le\frac{1}{4}\left(1+\frac{1}{\sqrt{e}}\right)$ (D) $S\le\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{e}}\left(1-\frac{1}{\sqrt{2}}\right)$

DI0157

3. The value of the integral $\int_{-\pi/2}^{\pi/2}\left(x^2+\ln\frac{\pi+x}{\pi-x}\right)\cos x dx$ is **[JEE 2012, 3, (–1)]**

(A) 0 (B) $\frac{\pi^2}{2}-4$ (C) $\frac{\pi^2}{2}+4$ (D) $\frac{\pi^2}{2}$

DI0158

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Paragraph for Question 4 and 5

Let $f(x) = (1-x)^2\sin^2 x + x^2$ for all $x \in \mathbb{R}$, and let $g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ell nt\right) f(t)dt$ for all $x \in (1,\infty)$.

4. Consider the statements :
P : There exists some $x \in \mathbb{R}$ such that $f(x) + 2x = 2(1 + x^2)$
Q : There exists some $x \in \mathbb{R}$ such that $2f(x) + 1 = 2x(1 + x)$
Then **[JEE 2012, 3M, –1M]**
(A) both **P** and **Q** are true
(B) **P** is true and **Q** is false
(C) **P** is false and **Q** is true
(D) both **P** and **Q** are false

DI0159

5. Which of the following is true ? **[JEE 2012, 3M, –1M]**
(A) g is increasing on $(1,\infty)$
(B) g is decreasing on $(1,\infty)$
(C) g is increasing on $(1,2)$ and decreasing on $(2,\infty)$
(D) g is decreasing on $(1,2)$ and increasing on $(2,\infty)$

DI0159

6. If $f(x) = \int_0^x e^{t^2}(t-2)(t-3)dt$ for all $x \in (0,\infty)$, then - **[JEE 2012, 4M]**
(A) f has a local maximum at $x = 2$
(B) f is decreasing on $(2,3)$
(C) there exists some $c \in (0,\infty)$ such that $f''(c) = 0$
(D) f has a local minimum at $x = 3$

DI0160

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7. For $a \in R$ (the set of all real numbers), $a \neq -1$. $\lim_{n\to\infty} \frac{(1^a + 2^a + ... + n^a)}{(n+1)^{a-1}[(na+1)+(na+2)+.....+(na+n)]} = \frac{1}{60}$
Then a = **[JEE(Advanced) 2013, 3, (–1)]**
(A) 5
(B) 7
(C) $\frac{-15}{2}$
(D) $\frac{-17}{2}$

DI0161

8. Let $f : [a, b] \to [1, \infty)$ be a continuous function and let $g : \mathbb{R} \to \mathbb{R}$ be defined as

$$g(x) = \begin{cases} 0 & \text{if } x < a \\ \int_a^x f(t)dt & \text{if } a \le x \le b \\ \int_a^b f(t)dt & \text{if } x > b \end{cases}$$

Then **[JEE(Advanced)-2014, 3]**
(A) g(x) is continuous but not differentiable at a
(B) g(x) is differentiable on $\mathbb{R}$
(C) g(x) is continuous but not differentiable at b
(D) g(x) is continuous and differentiable at either a or b but not both.

DI0162

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9. The value of $\int_0^1 4x^3\left\{\frac{d^2}{dx^2}\left(1-x^2\right)^5\right\}dx$ is **[JEE(Advanced)-2014, 3]**

DI0163

10. The following integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(2\operatorname{cosec} x)^{17}dx$ is equal to - **[JEE(Advanced)-2014, 3(–1)]**

(A) $\int_0^{\log(1+\sqrt{2})}2(e^u+e^{-u})^{16}du$ (B) $\int_0^{\log(1+\sqrt{2})}(e^u+e^{-u})^{17}du$

(C) $\int_0^{\log(1+\sqrt{2})}(e^u-e^{-u})^{17}du$ (D) $\int_0^{\log(1+\sqrt{2})}2(e^u-e^{-u})^{16}du$

DI0164

11. Let $f : [0, 2] \to \mathbb{R}$ be a function which is continuous on [0, 2] and is differentiable on (0, 2) with $f(0) = 1$. Let $F(x)=\int_0^{x^2}f(\sqrt{t})dt$ for $x \in [0, 2]$. If $F'(x) = f'(x)$ for all $x \in (0, 2)$, then F(2) equals -

(A) $e^2 - 1$ (B) $e^4 - 1$ (C) $e - 1$ (D) e^4

[JEE(Advanced)-2014, 3(–1)]

DI0165

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Paragraph For Questions 12 and 13

Given that for each $a \in (0,1)$, $\lim_{h\to 0^+}\int_h^{1-h}t^{-a}(1-t)^{a-1}dt$ exists. Let this limit be g(a). In addition, it is given that the function g(a) is differentiable on (0,1).

12. The value of $g\left(\frac{1}{2}\right)$ is - **[JEE(Advanced)-2014, 3(–1)]**

(A) π (B) 2π (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$

DI0166

13. The value of $g'\left(\frac{1}{2}\right)$ is- **[JEE(Advanced)-2014, 3(–1)]**

(A) $\frac{\pi}{2}$ (B) π (C) $-\frac{\pi}{2}$ (D) 0

DI0166

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14. Let $f : (0 , \infty) \to \mathbb{R}$ be given by $f(x) = \int_{\frac{1}{x}}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$. Then **[JEE(Advanced)-2014, 3]**

(A) $f(x)$ is monotonically increasing on $[1,\infty)$

(B) $f(x)$ is monotonically decreasing on $[0, 1)$

(C) $f(x) + f\left(\frac{1}{x}\right) = 0$, for all $x \in (0, \infty)$

(D) $f(2^x)$ is an odd function of x on $\mathbb{R}$

DI0167

15.

	List-I		**List-II**
P.	The number of polynomials f(x) with non-negative integer coefficients of degree ≤ 2, satisfying $f(0) = 0$ and $\int_0^1 f(x)dx = 1$, is	**1.**	8
Q.	The number of points in the interval $\left[-\sqrt{13}, \sqrt{13}\right]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is	**2.**	2
R.	$\int_{-2}^{2} \frac{3x^2}{(1+e^x)} dx$ equals	**3.**	4
S.	$\frac{\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos 2x \log\left(\frac{1+x}{1-x}\right) dx\right)}{\left(\int_{0}^{\frac{1}{2}} \cos 2x \log\left(\frac{1+x}{1-x}\right) dx\right)}$ equals	**4.**	0

DI0168

DI0169

DI0170

DI0171

Codes :

	P	Q	R	S
(A)	3	2	4	1
(B)	2	3	4	1
(C)	3	2	1	4
(D)	2	3	1	4

[JEE(Advanced)-2014, 3(–1)]

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16. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \begin{cases} [x] & , \quad x \le 2 \\ 0 & , \quad x > 2 \end{cases}$,

where [x] is the greatest integer less than or equal to x. If $I = \int_{-1}^{2} \frac{x f(x^2)}{2 + f(x+1)} dx$, then the value of (4I – 1) is **[JEE 2015, 4M, –0M]**

DI0172

17. If $\alpha = \int_0^1 \left(e^{9x + 3\tan^{-1}x}\right)\left(\frac{12 + 9x^2}{1 + x^2}\right) dx$, where $\tan^{-1}x$ takes only principal values, then the value of $\left(\log_e |1 + \alpha| - \frac{3\pi}{4}\right)$ is **[JEE 2015, 4M, –0M]**

DI0173

18. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous odd function, which vanishes exactly at one point and $f(1) = \frac{1}{2}$.

Suppose that $F(x) = \int_{-1}^{x} f(t)dt$ for all $x \in [-1,2]$ and $G(x) = \int_{-1}^{x} t\left|f(f(t))\right| dt$ for all $x \in [-1,2]$. If $\lim_{x \to 1} \frac{F(x)}{G(x)} = \frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is **[JEE 2015, 4M, –0M]**

DI0174

19. The option(s) with the values of a and L that satisfy the following equation is(are)

$$\frac{\int_0^{4\pi} e^t(\sin^6 at + \cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt} = L\ ?$$ **[JEE 2015, 4M, –0M]**

(A) $a = 2, L = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$

(B) $a = 2, L = \frac{e^{4\pi} + 1}{e^{\pi} + 1}$

(C) $a = 4, L = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$

(D) $a = 4, L = \frac{e^{4\pi} + 1}{e^{\pi} + 1}$

DI0175

20. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the correct expression(s) is(are) **[JEE 2015, 4M, –0M]**

(A) $\int_0^{\pi/4} x f(x)dx = \frac{1}{12}$

(B) $\int_0^{\pi/4} f(x)dx = 0$

(C) $\int_0^{\pi/4} x f(x)dx = \frac{1}{6}$

(D) $\int_0^{\pi/4} f(x)dx = 1$

DI0176

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21. Let $f'(x) = \frac{192x^3}{2+\sin^4 \pi x}$ for all $x \in \mathbb{R}$ with $f\left(\frac{1}{2}\right) = 0$. If $m \le \int_{1/2}^{1} f(x)dx \le M$, then the possible values of m and M are **[JEE 2015, 4M, –0M]**

(A) m = 13, M = 24

(B) $m = \frac{1}{4}, M = \frac{1}{2}$

(C) m = –11, M = 0

(D) m = 1, M = 12

DI0177

Paragraph For Questions 22 and 23

Let $F : \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function. Suppose that F(1) = 0, F(3) = –4, F'(x) < 0 for all $x \in (1/2, 3)$. Let $f(x) = xF(x)$ for all $x \in \mathbb{R}$.

22. The correct statement(s) is(are) **[JEE 2015, 4M, –0M]**

(A) $f'(1) < 0$

(B) $f(2) < 0$

(C) $f'(x) \ne 0$ for any $x \in (1, 3)$

(D) $f'(x) = 0$ for some $x \in (1, 3)$

DI0178

23. If $\int_1^3 x^2F'(x)dx = -12$ and $\int_1^3 x^3F''(x)dx = 40$, then the correct expression(s) is(are) **[JEE 2015, 4M, –0M]**

(A) $9f'(3) + f'(1) - 32 = 0$

(B) $\int_1^3 f(x)dx = 12$

(C) $9f'(3) - f'(1) + 32 = 0$

(D) $\int_1^3 f(x)dx = -12$

DI0178

24. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \cos x}{1+e^x} dx$ is equal to **[JEE(Advanced)-2016, 3(–1)]**

(A) $\frac{\pi^2}{4} - 2$

(B) $\frac{\pi^2}{4} + 2$

(C) $\pi^2 - e^{\frac{\pi}{2}}$

(D) $\pi^2 + e^{\frac{\pi}{2}}$

DI0179

25. The total number of distinct $x \in [0, 1]$ for which $\int_0^x \frac{t^2}{1+t^4} dt = 2x - 1$ is

[JEE(Advanced)-2016, 3(0)]

DI0180

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26. Let $f(x)=\lim_{n\to\infty}\left(\dfrac{n^n(x+n)\left(x+\frac{n}{2}\right).....\left(x+\frac{n}{n}\right)}{n!(x^2+n^2)\left(x^2+\frac{n^2}{4}\right).....\left(x^2+\frac{n^2}{n^2}\right)}\right)^{x/n}$, for all $x > 0$. Then

[JEE(Advanced)-2016, 4(–2)]

(A) $f\left(\frac{1}{2}\right)\ge f(1)$

(B) $f\left(\frac{1}{3}\right)\le f\left(\frac{2}{3}\right)$

(C) $f'(2)\le 0$

(D) $\dfrac{f'(3)}{f(3)}\ge\dfrac{f'(2)}{f(2)}$

DI0181

27. Let $f : R \to R$ be a differentiable function such that $f(0) = 0$, $f\left(\frac{\pi}{2}\right) = 3$ and $f'(0) = 1$. If

$$g(x) = \int_x^{\frac{\pi}{2}}\left[f'(t)\operatorname{cosec} t-\cot t\operatorname{cosec} t\, f(t)\right]dt$$

for $x \in \left(0,\frac{\pi}{2}\right]$, then $\lim_{x\to 0} g(x) =$ **[JEE(Advanced)-2017, 3]**

DI0182

28. If $I=\sum_{k=1}^{98}\int_k^{k+1}\dfrac{k+1}{x(x+1)}dx$, then **[JEE(Advanced)-2017, 4]**

(A) $I<\dfrac{49}{50}$

(B) $I < \log_e 99$

(C) $I>\dfrac{49}{50}$

(D) $I > \log_e 99$

DI0183

29. If $g(x) = \int_{\sin x}^{\sin(2x)}\sin^{-1}(t)\,dt$, then **[JEE(Advanced)-2017, 4]**

(A) $g'\left(\frac{\pi}{2}\right) = -2\pi$

(B) $g'\left(-\frac{\pi}{2}\right) = 2\pi$

(C) $g'\left(\frac{\pi}{2}\right) = 2\pi$

(D) $g'\left(-\frac{\pi}{2}\right) = -2\pi$

DI0184

30. Let $f:\mathbb{R}\to(0,1)$ be a continuous function. Then, which of the following function(s) has(have) the value zero at some point in the interval (0, 1)? **[JEE(Advanced)-2017]**

(A) $e^x-\int_0^x f(t)\sin t\,dt$

(B) $x^9 - f(x)$

(C) $f(x)+\int_0^{\frac{\pi}{2}} f(t)\sin t\,dt$

(D) $x-\int_0^{\frac{\pi}{2}-x} f(t)\cos t\,dt$

DI0185

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31. For each positive integer n, let

$$y_n = \frac{1}{n}(n+1)(n+2)...(n+n)^{1/n}$$

For $x \in \mathbb{R}$, let [x] be the greatest integer less than or equal to x. If $\lim_{n\to\infty} y_n = L$, then the value of [L] is _____ **[JEE(Advanced)-2018, 3(0)]**

DI0186

32. The value of the integral

$$\int_0^{\frac{1}{2}} \frac{1+\sqrt{3}}{\left((x+1)^2(1-x)^6\right)^{\frac{1}{4}}}dx$$

is _____ . **[JEE(Advanced)-2018, 3(0)]**

DI0187

33. Let $f : \mathbb{R} \to \mathbb{R}$ be given by f(x) = (x – 1) (x – 2) (x – 5). Define $F(x) = \int_0^x f(t)\,dt$, x > 0. Then which of the following options is/are correct ? **[JEE(Advanced)-2019, 4(–1)]**

(1) F has a local minimum at x = 1

(2) F has a local maximum at x = 2

(3) $F(x) \neq 0$ for all $x \in (0, 5)$

(4) F has two local maxima and one local minimum in $(0, \infty)$

DI0188

34. If $I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+e^{\sin x})(2-\cos 2x)}$ then $27I^2$ equals _____ **[JEE(Advanced)-2019, 3(0)]**

DI0189

35. For $a \in \mathbb{R}$, $|a| > 1$, let $\lim_{n\to\infty}\left(\frac{1+\sqrt[3]{2}+.....+\sqrt[3]{n}}{n^{7/3}\left(\frac{1}{(an+1)^2}+\frac{1}{(an+2)^2}+.....+\frac{1}{(an+n)^2}\right)}\right) = 54$. Then the possible value(s) of a is/are : **[JEE(Advanced)-2019, 4(–1)]**

(1) 8 (2) –9 (3) –6 (4) 7

DI0190

36. The value of the integral $\int_0^{\pi/2} \frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5}d\theta$ equals **[JEE(Advanced)-2019, 3(0)]**

DI0191

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ANSWER KEY

DEFINITE INTEGRATION

EXERCISE (O-1)

1. C	**2.** C	**3.** B	**4.** A	**5.** B	**6.** D	**7.** B	**8.** A
9. A	**10.** C	**11.** A	**12.** D	**13.** B	**14.** A	**15.** B	**16.** A
17. D	**18.** A	**19.** C	**20.** C	**21.** A	**22.** C	**23.** A	**24.** D
25. D	**26.** B	**27.** B	**28.** D	**29.** A	**30.** A		

EXERCISE (O-2)

1. B	**2.** D	**3.** C	**4.** D	**5.** A	**6.** C	**7.** B	**8.** B
9. B	**10.** A	**11.** B	**12.** A	**13.** A	**14.** C	**15.** A	**16.** A
17. D	**18.** C	**19.** A,C	**20.** B,C,D	**21.** A,B,C,D		**22.** C,D	**23.** A,C,D
24. B,C,D	**25.** B,C,D	**26.** C,D					

EXERCISE (S-1)

1. $\frac{\pi^2}{4}$ **3.** $\frac{1}{2}$ **4.** (i) $\frac{\pi^2}{8}-\frac{\pi}{4}(1+l\,n2)+\frac{1}{2}$; (ii) $\frac{\pi\, l n 3}{2}$ **5.** $l n2$ **6.** $1-\sec(1)$

7. 4 **8.** $\frac{\pi}{8}l\,n2$ **9.** 153 **10.** 2/3 **11.** $\frac{1}{2}\left[\frac{\pi}{6}+l\,n3-l\,n2\right]$ **12.** 3

13. **(b)** $f(x) = 3 + 2e^{2x}$; $g(x) = 3 - 2e^{2x}$ **14.** $\frac{1}{2}\left[e^{\pi/2}(\cos1+\sin1)-1\right]$

15. $e^{1+e} + e^{1-e} + e^{-e} - e^{e} + e - e^{-1}$ **16.** 125 **17.** $4\sqrt{2}-4l\,n(\sqrt{2}+1)$ **18.** $\frac{\pi}{2\sqrt{2}}-\frac{16\sqrt{2}}{5}$

19. 5250 **20.** $\frac{\pi^2}{6\sqrt{3}}$ **21.** $\frac{\pi}{8}l n2$ **22.** $\frac{\pi(a+b)}{2\sqrt{2}}$ **23.** 5 **24.** $\frac{3\pi}{2}$

25. $\frac{\pi(\pi+3)}{2}$ **26.** $\frac{(a\pi+2b)\pi}{3\sqrt{3}}$ **27.** $\tan^{-1}(a).l\,n\sqrt{1+a^2}$ **28.** $\frac{8}{\pi}$ **30.** $\frac{\pi+4}{666}$ **31.** 21

32. 90 **33.** 2012 **34.** $-\cos x$ **35.** 13.5 **36.** $f(x) = e^{x+1}$

37. **(a)** c = 1 and $\lim_{x\to\infty}$ will be $\frac{\sqrt{3}}{2}$; **(b)** a = 4 and b = 1 **38.** **(a)** $2\,e^{(1/2)(\pi-4)}$; **(b)** $3 - l n4$

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EXERCISE (S-2)

1. $\frac{\pi\sqrt{3}}{3}$ **2.** $\frac{16\pi}{3}-2\sqrt{3}$ **3.** $\frac{\pi}{12}$ **4.** **(b)** $\frac{\pi\theta}{4\sin\theta}$ **5.** $\frac{\pi}{3}$

7. **(a)** 4; **(b)** 3 **8.** 2/3 **9.** $\frac{\pi}{2}$ **10.** $\frac{16}{9}$ **11.** $\frac{3\pi+8}{24}$ **12.** $\frac{\pi}{4}\left(\frac{\pi}{4}-1\right)+\frac{1}{2}\ell n2$

13. $\left\{-\frac{\pi}{2},\frac{\pi}{2}\right\}$ **14.** I = 8 as $\int_0^{\pi/2} y\sin y\,dy = 1$ **15.** 1 + e

16. g(x) is cont. in(–2,2); g(x) is der. at x = 1 & not der. at x = 0. Note that : $g(x)=\begin{cases}-(x+2) & \text{for } -2\le x\le 0\\ -2+x-\frac{x^2}{2} & \text{for } 0<x<1\\ \frac{x^2}{2}-x-1 & \text{for } 1\le x\le 2\end{cases}$

17. 62 **18.** $\frac{3\pi^2}{16}$ **19.** $\frac{2\pi}{\sqrt{3}}$ **20.** $2\sqrt{6}$ **21.** $U_n=\frac{n\pi}{2}$ **22.** 2525

23. 9 **24.** $g(5)=3e^5$ **25.** $a=15,\ b=\frac{45}{2}$ **27.** 0

28. real & distinct $\forall\ k\in R$ **29.** $\frac{\pi}{16}$ **30.** **(a)** $\frac{1}{e}$; **(b)** $\frac{27}{4e}$

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EXERCISE (JM)

1. 3 **2.** 4 **3.** 2 **4.** 3,4 **5.** 4 **6.** 1 **7.** 4 **8.** 1

9. 3 **10.** 3 **11.** 4 **12.** 2 **13.** 2 **14.** 2 **15.** 3

16. 1 or 3 (Ans. 1 By NTA) **17.** 1 **18.** 4

EXERCISE (JA)

1. A **2.** A,B,D **3.** B **4.** C **5.** B **6.** A,B,C,D **7.** B **8.** A,C

9. 2 **10.** A **11.** B **12.** A **13.** D **14.** A,C,D **15.** D **16.** 0

17. 9 **18.** 7 **19.** A,C **20.** A,B **21.** D **22.** A,B,C **23.** C,D **24.** A

25. 1 **26.** B,C **27.** 2 **28.** B,C **29.** **Bonus** **(Reason : g'(π/2) = g'(–π/2) = 0)**

30. B,D **31.** 1 **32.** 2 **33.** 1,2,3 **34.** 4.00 **35.** 1,2 **36.** 0.50

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STATISTICS

MEASURES OF CENTRAL TENDENCY :

An average value or a central value of a distribution is the value of variable which is representative of the entire distribution, this representative value are called the measures of central tendency. Generally the following five measures of central tendency.

(a) Mathematical average
 (i) Arithmetic mean (ii) Geometric mean (iii) Harmonic mean

(b) Positional average
 (i) Median (ii) Mode

1. ARITHMETIC MEAN :

(i) For ungrouped dist. : If $x_1, x_2, ... x_n$ are n values of variate x_i then their A.M. $\overline{x}$ is defined as

$$\overline{x} = \frac{x_1 + x_2 + ... + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\Rightarrow \qquad \Sigma x_i = n\overline{x}$$

(ii) For ungrouped and grouped freq. dist. : If $x_1, x_2, ... x_n$ are values of variate with corresponding frequencies $f_1, f_2, ... f_n$ then their A.M. is given by

$$\overline{x} = \frac{f_1x_1 + f_2x_2 + ... + f_nx_n}{f_1 + f_2 + ... + f_n} = \frac{\sum_{i=1}^{n} f_ix_i}{N}, \text{ where } N = \sum_{i=1}^{n} f_i$$

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Ex.1 Find the A.M. of the following freq. dist.

x_i	5	8	11	14	17
f_i	4	5	6	10	20

Sol. Here $N = \Sigma f_i = 4 + 5 + 6 + 10 + 20 = 45$

$\Sigma f_ix_i = (5 \times 4) + (8 \times 5) + (11 \times 6) + (14 \times 10) + (17 \times 20) = 606$

$$\therefore \quad \overline{x} = \frac{\Sigma f_ix_i}{N} = \frac{606}{45} = 13.47$$

(iii) By short method : If the value of x_i are large, then the calculation of A.M. by using previous formula is quite tedious and time consuming. In such case we take deviation of variate from an arbitrary point a.

Let $\qquad d_i = x_i - a$

$$\therefore \qquad \overline{x} = a + \frac{\Sigma f_id_i}{N}, \quad \text{where a is assumed mean}$$

(iv) By step deviation method : Sometime during the application of short method of finding the A.M. If each deviation d_i are divisible by a common number h(let)

Let $\qquad u_i = \frac{d_i}{h} = \frac{x_i - a}{h}$

$$\therefore \qquad \overline{x} = a + \left(\frac{\Sigma f_iu_i}{N}\right)h$$

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Ex.2 Find the mean of the following freq. dist.

x_i	5	15	25	35	45	55
f_i	12	18	27	20	17	6

Sol. Let assumed mean a = 35, h = 10

here $N = \Sigma f_i = 100$, $u_i = \dfrac{(x_i - 35)}{10}$

$\therefore$ $\Sigma f_i u_i = (12 \times -3) + (18 \times -2) + (27 \times -1) + (20 \times 0) + (17 \times 1) + (6 \times 2) = -70$

$\therefore$ $\bar{x} = a + \left(\dfrac{\Sigma f_i u_i}{N}\right)h = 35 + \dfrac{(-70)}{100} \times 10 = 28$

(v) Weighted mean : If $w_1, w_2, \ldots\ldots w_n$ are the weights assigned to the values $x_1, x_2, \ldots\ldots x_n$ respectively then their weighted mean is defined as

$$\text{Weighted mean} = \frac{w_1x_1 + w_2x_2 + \ldots\ldots + w_nx_n}{w_1 + \ldots\ldots + w_n} = \frac{\sum_{i=1}^{n} w_ix_i}{\sum_{i=1}^{n} w_i}$$

Ex.3 Find the weighted mean of first n natural numbers when their weights are equal to their squares respectively

Sol. $$\text{Weighted Mean} = \frac{1.1^2 + 2.2^2 + \ldots + n.n^2}{1^2 + 2^2 + \ldots + n^2} = \frac{1^3 + 2^3 + \ldots\ldots + n^3}{1^2 + 2^2 + \ldots + n^2} = \frac{[n(n+1)/2]^2}{[n(n+1)(2n+1)/6]} = \frac{3n(n+1)}{2(2n+1)}$$

(vi) Combined mean : If $\bar{x}_1$ and $\bar{x}_2$ be the means of two groups having n_1 and n_2 terms respectively then the mean (combined mean) of their composite group is given by

$$\text{combined mean} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

If there are more than two groups then, $\text{combined mean} = \dfrac{n_1\bar{x}_1 + n_1\bar{x}_2 + n_3\bar{x}_3 + \ldots}{n_1 + n_2 + n_3 + \ldots}$

Ex.4 The mean income of a group of persons is Rs. 400 and another group of persons is Rs. 480. If the mean income of all the persons of these two groups is Rs. 430 then find the ratio of the number of persons in the groups.

Sol. Here $\bar{x}_1 = 400$, $\bar{x}_2 = 480$, $\bar{x} = 430$

$\because$ $\bar{x} = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} \Rightarrow 430 = \dfrac{400n_1 + 480n_2}{n_1 + n_2}$

$\Rightarrow \dfrac{n_1}{n_2} = \dfrac{5}{3}$

(vii) Properties of Arithmetic mean :

- Sum of deviations of variate from their A.M. is always zero i.e. $\Sigma(x_i - \bar{x}) = 0$, $\Sigma f_i(x_i - \bar{x}) = 0$
- Sum of square of deviations of variate from their A.M. is minimum i.e. $\Sigma(x_i - \bar{x})^2$ is minimum
- If $\bar{x}$ is the mean of variate x_i then A.M. of $(x_i + \lambda) = \bar{x} + \lambda$
  A.M. of $(\lambda x_i) = \lambda\bar{x}$
  A.M. of $(ax_i + b) = a\bar{x} + b$ (where λ, a, b are constant)
- A.M. is independent of change of assumed mean i.e. it is not effected by any change in assumed mean.

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2. MEDIAN :

The median of a series is the value of middle term of the series when the values are written in ascending order. Therefore median, divided an arranged series into two equal parts.

Formulae of median :

(i) For ungrouped distribution : Let n be the number of variate in a series then

$$\text{Median} = \begin{cases} \left(\frac{n+1}{2}\right)^{th} \text{term}, \text{ (when n is odd)} \\ \text{Mean of } \left(\frac{n}{2}\right)^{th} \text{ and } \left(\frac{n}{2}+1\right)^{th} \text{ terms, (when n is even)} \end{cases}$$

(ii) For ungrouped freq. dist. : First we prepare the cumulative frequency (c.f.) column and Find value of N then

$$\text{Median} = \begin{cases} \left(\frac{N+1}{2}\right)^{th} \text{term}, \text{ (when N is odd)} \\ \text{Mean of } \left(\frac{N}{2}\right)^{th} \text{ and } \left(\frac{N}{2}+1\right)^{th} \text{ terms, (when N is even)} \end{cases}$$

(iii) For grouped freq. dist : Prepare c.f. column and find value of $\frac{N}{2}$ then find the class which contain value of c.f. is equal or just greater to N/2, this is median class

$$\therefore \text{ Median} = \ell + \frac{\left(\frac{N}{2} - F\right)}{f} \times h$$

where ℓ — lower limit of median class

f — freq. of median class

F — c.f. of the class preceeding median class

h — Class interval of median class

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Ex.5 Find the median of following freq. dist.

class	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50
f	8	30	40	12	10

Sol.

class	f_i	c.f.
0 – 10	8	8
10 – 20	30	38
20 – 30	40	78
30 – 40	12	90
40 – 50	10	100

Here $\frac{N}{2} = \frac{100}{2} = 50$ which lies in the value 78 of c.f. hence corresponding class of this c.f. is 20-30 is the median class, so $\ell = 20$, f = 40, F = 38, h = 10

$$\therefore \text{ Median} = \ell + \frac{\left(\frac{N}{2} - F\right)}{f} \times h = 20 + \frac{(50-38)}{40} \times 10 = 23$$

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3. MODE :

In a frequency distribution the mode is the value of that variate which have the maximum frequency

Method for determining mode :

(i) For ungrouped dist. : The value of that variate which is repeated maximum number of times

(ii) For ungrouped freq. dist. : The value of that variate which have maximum frequency.

(iii) For grouped freq. dist. : First we find the class which have maximum frequency, this is model calss

$$\therefore \text{ Mode} = \ell + \frac{f_0 - f_1}{2f_0 - f_1 - f_2} \times h$$

where ℓ — lower limit of model class

f_0 — freq. of the model class

f_1 — freq. of the class preceeding model class

f_2 — freq. of the class succeeding model class

h — class interval of model class

Ex. 6 Find the mode of the following frequecy dist

class	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
f_i	2	18	30	45	35	20	6	3

Sol. Here the class 30–40 has maximum freq. so this is the model class

$\ell = 30, f_0 = 45, f_1 = 30, f_2 = 35, h = 10$

$$\therefore \text{ Mode} = \ell + \frac{f_0 - f_1}{2f_0 - f_1 - f_2} \times h = 30 + \frac{45 - 30}{2 \times 45 - 30 - 35} \times 10 = 36$$

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4. RELATION BETWEEN MEAN, MEDIAN AND MODE :

In a moderately asymmetric distribution following relation between mean, median and mode of a distribution. It is known as imprical formula.

Mode = 3 Median – 2 Mean

Note (i) Median always lies between mean and mode

(ii) For a symmetric distribution the mean, median and mode are coincide.

5. MEASURES OF DISPERSION :

The dispersion of a statistical distribution is the measure of deviation of its values about the their average (central) value.

It gives an idea of scatteredness of different values from the average value.

Generally the following measures of dispersion are commonly used.

(i) Range (ii) Mean deviation (iii) Variance and standard deviation

(i) Range : The difference between the greatest and least values of variate of a distribution, are called the range of that distribution.

If the distribution is grouped distribution, then its range is the difference between upper limit of the maximum class and lower limit of the minimum class.

$$\text{Also, coefficient of range} = \frac{\text{difference of extreme values}}{\text{sum of extreme values}}$$

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Ex.7 Find the range of following numbers 10, 8, 12, 11, 14, 9, 6

Sol. Here greatest value and least value of the distribution are 14 and 6 resp. therefore

Range = 14 – 6 = 8

(ii) Mean deviation (M.D.) : The mean deviation of a distribution is, the mean of absolute value of deviations of variate from their statistical average (Mean, Median, Mode).

If A is any statistical average of a distribution then mean deviation about A is defined as

$$\text{Mean deviation} = \frac{\sum_{i=1}^{n} |x_i - A|}{n} \qquad \text{(for ungrouped dist.)}$$

$$\text{Mean deviation} = \frac{\sum_{i=1}^{n} f_i |x_i - A|}{N} \qquad \text{(for freq. dist.)}$$

Note :- is minimum when it taken about the median

$$\text{Coefficient of Mean deviation} = \frac{\text{Mean deviation}}{A}$$

(where A is the central tendency about which Mean deviation is taken)

Ex.8 Find the mean deviation of number 3, 4, 5, 6, 7

Sol. Here n = 5, $\bar{x} = 5$

$$\therefore \quad \text{Mean deviation} = \frac{\Sigma |x_i - \bar{x}|}{n}$$

$$= \frac{1}{5}[|3-5| + |4-5| + |5-5| + |6-5| + |7-5|]$$

$$= \frac{1}{5}[2 + 1 + 0 + 1 + 2] = \frac{6}{5} = 1.2$$

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Ex.9 Find the mean deviation about mean from the following data

x_i	3	9	17	23	27
f_i	8	10	12	9	5

Sol.

x_i	f_i	$f_i x_i$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
3	8	24	12	96
9	10	90	6	60
17	12	204	2	24
23	9	207	8	72
27	5	135	12	60
	N = 44	$\Sigma f_i x_i = 660$		$\Sigma f_i \lvert x_i - \bar{x} \rvert = 312$

$$\text{Mean } (\bar{x}) = \frac{\Sigma f_i x_i}{N} = \frac{660}{44} = 15$$

$$\text{Mean deviation} = \frac{\Sigma f_i |x_i - \bar{x}|}{N} = \frac{312}{44} = 7.09$$

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(iii) Variance and standard deviation : The variance of a distribution is, the mean of squares of deviation of variate from their mean. It is denoted by σ^2 or var(x).

The positive square root of the variance are called the standard deviation. It is denoted by σ or S.D.

Hence standard deviation $= +\sqrt{\text{variance}}$

Formulae for variance :

(i) for ungrouped dist. :

$$\sigma_x^2 = \frac{\Sigma(x_i - \bar{x})^2}{n}$$

$$\sigma_x^2 = \frac{\Sigma x_i^2}{n} - \bar{x}^2 = \frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2$$

$$\sigma_d^2 = \frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2, \quad \text{where } d_i = x_i - a$$

(ii) For freq. dist. :

$$\sigma_x^2 = \frac{\Sigma f_i (x_i - \bar{x})^2}{N}$$

$$\sigma_x^2 = \frac{\Sigma f_i x_i^2}{N} - (\bar{x})^2 = \frac{\Sigma f_i x_i^2}{N} - \left(\frac{\Sigma f_i x_i}{N}\right)^2$$

$$\sigma_d^2 = \frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2$$

$$\sigma_u^2 = h^2\left[\frac{\Sigma f_i u_i^2}{N} - \left(\frac{\Sigma f_i u_i}{N}\right)^2\right] \quad \text{where } u_i = \frac{d_i}{h}$$

(iii) Coefficient of S.D. $= \dfrac{\sigma}{\bar{x}}$

Coefficient of variation $= \dfrac{\sigma}{\bar{x}} \times 100$ (in percentage)

Note :- $\sigma^2 = \sigma_x^2 = \sigma_d^2 = h^2\sigma_u^2$

Ex.10 Find the variance of first n natural numbers

Sol. $\sigma^2 = \dfrac{\Sigma x_i^2}{n} - \left(\dfrac{\Sigma x_i}{n}\right)^2 = \dfrac{\Sigma n^2}{n} - \left(\dfrac{\Sigma n}{n}\right)^2 = \dfrac{n(n+1)(2n+1)}{6n} - \left\{\dfrac{n(n+1)}{2n}\right\}^2 = \dfrac{n^2-1}{12}$

Ex.11 If $\sum_{i=1}^{18}(x_i - 8) = 9$ and $\sum_{i=1}^{18}(x_i - 8)^2 = 45$, then find the standard deviation of $x_1, x_2, \ldots x_{18}$

Sol. Let $(x_i - 8) = d_i$

$$\therefore \quad \sigma_x = \sigma_d = \sqrt{\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2} = \sqrt{\frac{45}{18} - \left(\frac{9}{18}\right)^2} = \sqrt{\frac{5}{2} - \frac{1}{4}} = \frac{3}{2}$$

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Ex.12 Find the coefficient of variation of first n natural numbers

Sol. For first n natural numbers.

$$\text{Mean } (\bar{x}) = \frac{n+1}{2}, \text{ S.D.}(\sigma) = \sqrt{\frac{n^2-1}{12}}$$

$$\therefore \text{ coefficient of variance} = \frac{\sigma}{\bar{x}} \times 100 = \sqrt{\frac{n^2-1}{12}} \times \frac{1}{\left(\frac{n+1}{2}\right)} \times 100 = \sqrt{\frac{(n-1)}{3(n+1)}} \times 100$$

6. MEAN SQUARE DEVIATION :

The mean square deviation of a distrubution is the mean of the square of deviations of variate from assumed mean. It is denoted by S^2

Hence $$S^2 = \frac{\Sigma(x_i - a)^2}{n} = \frac{\Sigma d_i^2}{n} \quad \text{(for ungrouped dist.)}$$

$$S^2 = \frac{\Sigma f_i(x_i - a)^2}{N} = \frac{\Sigma f_i d_i^2}{N} \quad \text{(for freq. dist.), where } d_i = (x_i - a)$$

7. RELATION BETWEEN VARIANCE AND MEAN SQUARE DEVIATION :

$$\because \sigma^2 = \frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2$$

$$\Rightarrow \sigma^2 = s^2 - d^2, \quad \text{where } d = \bar{x} - a = \frac{\Sigma f_i d_i}{N}$$

$$\Rightarrow s^2 = \sigma^2 + d^2 \Rightarrow s^2 \geq \sigma^2$$

Hence the variance is the minimum value of mean square deviation of a distribution

Ex.13 Determine the variance of the following frequency dist.

class	0 – 2	2 – 4	4 – 6	6 – 8	8 – 10	10 – 12
f_i	2	7	12	19	9	1

Sol. Let a = 7, h = 2

class	x_i	f_i	$u_i = \frac{x_i - a}{h}$	$f_i u_i$	$f_i u_i^2$
0 – 2	1	2	–3	–6	18
2 – 4	3	7	–2	–14	28
4 – 6	5	12	–1	–12	12
6 – 8	7	19	0	0	0
8 – 10	9	9	1	9	9
10 – 12	11	1	2	2	4
		N = 50		$\Sigma f_i u_i = -21$	$\Sigma f_i u_i^2 = 71$

$$\therefore \sigma^2 = h^2\left[\frac{\Sigma f_i u_i^2}{N} - \left(\frac{\Sigma f_i u_i}{N}\right)^2\right] = 4\left[\frac{71}{50} - \left(\frac{-21}{50}\right)^2\right] = 4[1.42 - 0.1764] = 4.97$$

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8. MATHEMATICAL PROPERTIES OF VARIANCE :

- $Var.(x_i + \lambda) = Var.(x_i)$

  $Var.(\lambda x_i) = \lambda^2.Var(x_i)$

  $Var(ax_i + b) = a^2.Var(x_i)$

  where λ, a, b, are constant

- If means of two series containing n_1, n_2 terms are $\bar{x}_1, \bar{x}_2$ and their variance's are σ_1^2, σ_2^2 respectively and their combined mean is $\bar{x}$ then the variance σ^2 of their combined series is given by following formula

$$\sigma^2 = \frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{(n_1 + n_2)} \quad \text{where } d_i = \bar{x}_1 - \bar{x},\ d_2 = \bar{x}_2 - \bar{x}$$

i.e.

$$\sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{n_1 n_2}{(n_1 + n_2)^2}(\bar{x}_1 - \bar{x}_2)^2$$

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SOLVED EXAMPLES

Ex.1 If in an examination different weights are assigned to different subjects Physics (2), Chemistry (1), English (1), Mathematics (2) A student scores 60 in Physics, 70 in Chemistry, 70 in English and 80 in Mathematics, then weighted mean is-

(1) 60 (2) 70 (3) 80 (4) 85

Sol.(2) Weighted mean = $\dfrac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i} = \dfrac{2\times 60+1\times 70+1\times 70+2\times 80}{6} = 70$

Ex.2 The mean of two groups of sizes 200 and 300 are 25 and 10 respectively.Their standard deviation are 3 and 4 respectively. The variance of combined sample of size 500 is-

(1) 64 (2) 65.2 (3) 67.2 (4) 64.2

Sol.(3) Combined mean $\bar{x} = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1+n_2} = \dfrac{200\times 25+300\times 10}{500} = 16$

Here $d_1 = \bar{x}_1 - \bar{x} = 25-16 = 9$ and $d_2 = \bar{x}_2 - \bar{x} = 10-16 = -6$

We know that $\sigma^2 = \dfrac{n_1(\sigma_1^2+d_1^2)+n_2(\sigma_2^2+d_2^2)}{n_1+n_2} = \dfrac{200(9+81)+300(16+36)}{500} = \dfrac{33600}{500} = 67.2$

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Ex.3 If the mean of the series $x_1, x_2, \ldots\ldots, x_n$ is $\bar{x}$, then the mean of the series $x_i + 2i$, i = 1, 2, ..., n will be-

(1) $\bar{x}+n$ (2) $\bar{x}+n+1$ (3) $\bar{x}+2$ (4) $\bar{x}+2n$

Sol.(2) As given $\bar{x} = \dfrac{x_1+x_2+\ldots\ldots+x_n}{n}$ ....(1)

If the mean of the series $x_i + 2i$, i = 1, 2, ..., n be $\bar{X}$, then

$$\bar{X} = \frac{(x_1+2)+(x_2+2.2)+(x_3+2.3)+\ldots\ldots+(x_n+2.n)}{n}$$

$$= \frac{x_1+x_2+\ldots\ldots+x_n}{n} + \frac{2(1+2+3+\ldots\ldots+n)}{n}$$

$$= \bar{x} + \frac{2n(n+1)}{2n} \quad \text{from (1)}$$

$$= \bar{x} + n + 1$$

Ex.4 The variance of first 20-natural numbers is-

(1) $\dfrac{133}{4}$ (2) $\dfrac{379}{12}$ (3) $\dfrac{133}{2}$ (4) $\dfrac{399}{4}$

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Sol.(1) $\because \quad \sigma^2 = \frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2$

$$= \frac{1}{20}[1^2 + 2^2 + ... + 20^2] - \left[\frac{1}{20}(1 + 2 + ... + 20)\right]^2$$

$$= \frac{1}{20}\frac{20 \times 21(2 \times 20 + 1)}{6} - \left[\frac{1}{20}\frac{20 \times 21}{2}\right]^2 = \frac{7 \times 41}{2} - \frac{441}{4} = \frac{133}{4}.$$

In fact, the variance of first n-natural numbers is $\frac{n^2 - 1}{12}$

Ex.5 The mean of the following freq. table is 50 and $\Sigma f = 120$

class	$0-20$	$20-40$	$40-60$	$60-80$	$80-100$
f	17	f_1	32	f_2	19

the missing frequencies are-

(1) 28, 24 (2) 24, 36 (3) 36, 28 (4) None of these

Sol.(1) $\Sigma f = 120 = 17 + f_1 + 32 + f_2 + 19$

$\Rightarrow f_1 + f_2 = 52$ ..... (1)

and $\Sigma fx = (10 \times 17) + (30 \times f_1) + (50 \times 32) + (70 \times f_2) + (90 \times 19) = 30f_1 + 70f_2 + 3480$

$\therefore \quad \bar{x} = \frac{\Sigma fx}{\Sigma f} \Rightarrow 50 = \frac{30f_1 + 70f_2 + 3480}{120}$

$\Rightarrow 30f_1 + 70f_2 = 2520 \Rightarrow 3f_1 + 7f_2 = 252$ ..... (2)

by (1) and (2) we get $f_1 = 28$, $f_2 = 24$

Ex.6 A student obtained 75%, 80%, 85% marks in three subjects. If the marks of another subject are added then his average marks can not be less than-

(1) 60% (2) 65% (3) 80% (4) 90%

Sol.(1) Total marks obtained from three subjects out of 300 = 75 + 80 + 85 = 240

if the marks of another subject is added then total marks obtained out of 400 is greater than 240

if marks obtained in fourth subject is 0 then

$$\text{minimum average marks} = \frac{240}{400} \times 100 = 60\%$$

Ex.7 The mean and variance of a series containing 5 terms are 8 and 24 respectively. The mean and variance of another series containing 3 terms are also 8 and 24 respectively. The variance of their combined series will be-

(1) 20 (2) 24 (3) 25 (4) 42

Sol.(2) Using $\sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{n_1 n_2}{(n_1 + n_2)^2}(\bar{x}_1 - \bar{x}_2)^2 \Rightarrow \sigma^2 = \frac{5(24) + 3(24)}{5 + 3} + \frac{5(3)}{(5+3)^2}(8 - 8)^2 = 24$

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Ex.8 The mean deviation about median from the following data 340, 150, 210, 240, 300, 310, 320, is-

(1) 52.4 (2) 52.5 (3) 52.8 (4) none of these

Sol.(3) Arranging the observations in ascending order of magnitude, we have 150, 210, 240, 300, 310, 320, 340. Clearly, the middle observation is 300. So, median = 300

Calculation of Mean deviation

x_i	$\lvert x_i - 300 \rvert$
340	40
150	150
210	90
240	60
300	0
310	10
320	20
Total	$\sum \lvert x_i - 300 \rvert = 370$

Mean deviation from median = $\frac{1}{7}\sum \lvert x_i - 300 \rvert = \frac{370}{7} = 52.8$

Ex.9 Variance of the data given below is

Size of item	3.5	4.5	5.5	6.5	7.5	8.5	9.5
Frequency	3	7	22	60	85	32	8

(1) 1.29 (2) 2.19 (3) 1.32 (4) none of these

Sol.(3) Let the assumed mean be a = 6.5

Calculation of variance

x_i	f_i	$d_i = x_i - 6.5$	f_id_i	$f_id_i^2$
3.5	3	–3	–9	27
4.5	7	–2	–14	28
5.5	22	–1	–22	22
6.5	60	0	0	0
7.5	85	1	85	85
8.5	32	2	64	128
9.5	8	3	24	72
	$N = \sum f_i = 217$		**$\sum f_id_i = 128$**	**$\sum f_id_i^2 = 362$**

Here N = 217, $\sum f_id_i = 128$ and $\sum f_id_i^2 = 362$

$$\therefore \text{ Var (X)} = \left(\frac{1}{N}\sum f_id_i^2\right) - \left(\frac{1}{N}\sum f_id_i\right)^2 = \frac{362}{217} - \left(\frac{128}{217}\right)^2 = 1.668 - 0.347 = 1.321$$

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Ex.10 If a variable takes the value 0, 1, 2.....n with frequencies proportional to the bionomial coefficients ${}^nC_0, {}^nC_1, \ldots, {}^nC_n$ then the mean of the distribution is-

(1) $\frac{n(n+1)}{4}$ (2) $\frac{n}{2}$ (3) $\frac{n(n-1)}{2}$ (4) $\frac{n(n+1)}{2}$

Sol.(2) $N = \sum f_i = k\,[{}^nC_0 + {}^nC_1 + \ldots + {}^nC_n] = k2^n$

$$\sum f_i x_i = k\,[1.{}^nC_1 + 2.\,{}^nC_2 + \ldots + n\,{}^nC_n] = k\sum_{r=1}^{n} r.{}^nC_r = kn\sum_{r=1}^{n} {}^{n-1}C_{r-1} = kn2^{n-1}$$

Thus $\bar{x} = \frac{1}{2^n}(n\ \ 2^{n-1}) = \frac{n}{2}$.

Ex.11 The mean and variance of 5 observations of an experiment are 4 and 5.2 respectively. If from these observations three are 1, 2 and 6, then the remaining will be-

(1) 2, 9 (2) 5, 6 (3) 4, 7 (4) 3, 8

Sol.(3) As given $\bar{x} = 4$, n = 5 and $\sigma^2 = 5.2$. If the remaining observations are x_1, x_2 then

$$\sigma^2 = \frac{\sum(x_i - \bar{x})^2}{n} = 5.2$$

$$\Rightarrow \frac{(x_1-4)^2 + (x_2-4)^2 + (1-4)^2 + (2-4)^2 + (6-4)^2}{5} = 5.2$$

$$\Rightarrow (x_1-4)^2 + (x_2-4)^2 = 9 \qquad \ldots(1)$$

Also $\bar{x} = 4 \Rightarrow \frac{x_1 + x_2 + 1 + 2 + 6}{5} = 4 \Rightarrow x_1 + x_2 = 11$ ....(2)

from eq.(1), (2) $x_1, x_2 = 4, 7$

Ex.12 The mean deviation of the series a, a + d, a + 2d, ..., a + 2nd from its mean is-

(1) $\frac{n+1}{2n+1}|d|$ (2) $\frac{n(n+1)}{2n+1}|d|$ (3) $\frac{n(n-1)}{2n+1}|d|$ (4) none of these

Sol.(2) Number of terms in the series = 2n + 1

$$\therefore \text{ mean } \bar{x} = \frac{a + (a+d) + (a+2d) + \ldots + (a+2nd)}{2n+1} = \frac{1}{2n+1}\left[\frac{2n+1}{2}(a + a + 2nd)\right] = a + nd$$

Also $\sum|x_i - \bar{x}| = |-nd| + |(1-n)d| + \ldots + |-d| + 0 + |d| + \ldots + |nd|$

$$= 2|d|[n + (n-1) + \ldots + 1] = 2|d|\frac{n(n+1)}{2} = n(n+1)|d|$$

$$\therefore \text{ mean deviation from mean} = \frac{\sum|x_i - \bar{x}|}{N} = \frac{n(n+1)}{2n+1}|d|$$

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Ex.13 Let $x_1, x_2 ....., x_n$ be values taken by a variable X and $y_1, y_2, ..., y_n$ be the values taken by a variable Y such that $y_i = ax_i + b$, $i = 1, 2, ..., n$. Then-

(1) Var (Y) = a^2Var(X) (2) Var(Y) = a^2Var(X) + b (3) Var(Y) = Var(X) + b (4) None of these

Sol.(1) We have,

$$\text{Var}(Y) = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2$$

$$[\because y_i = ax_i + b; i = 1, 2, ..., n \Rightarrow \bar{Y} = a\bar{X} + b]$$

$$\Rightarrow \text{Var}(Y) = \frac{1}{n}\sum_{i=1}^{n}a^2(x_i - \bar{X})^2$$

$$\Rightarrow \text{Var}(Y) = a^2\left\{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{X})^2\right\} = a^2\text{Var}(X)$$

Ex.14 The mean square deviation of a set of n observations $x_1, x_2, ..., x_n$ about a point c is defined as $\frac{1}{n}\sum_{i=1}^{n}(x_i - c)^2$ The mean square deviation about –2 and 2 are 18 and 10 respectively, then standard deviation of this set of observations is-

(1) 3 (2) 2 (3) 1 (4) None of these

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Sol.(1) $\because \frac{1}{n}\Sigma(x_i + 2)^2 = 18$ and $\frac{1}{n}\Sigma(x_i - 2)^2 = 10$

$\Rightarrow \Sigma(x_i + 2)^2 = 18n$ and $\Sigma(x_i - 2)^2 = 10n$

$\Rightarrow \Sigma(x_i + 2)^2 + \Sigma(x_i - 2)^2 = 28\,n$ and $\Sigma\,(x_i + 2)^2 - \Sigma(x_i - 2)^2 = 8\,n$

$\Rightarrow 2\Sigma\,x_i^2 + 8n = 28\,n$ and $8\Sigma x_i = 8n$

$\Rightarrow \Sigma\,x_i^2 = 10\,n$ and $\Sigma x_i = n$

$\Rightarrow \frac{\Sigma x_i^2}{n} = 10$ and $\frac{\Sigma x_i}{n} = 1$

$$\therefore \sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2} = \sqrt{10 - (1)^2} = 3$$

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CHECK YOUR GRASP STATISTICS EXERCISE-I

Arithmetic mean, weighted mean, Combined mean

1. Mean of the first n terms of the A.P. a, (a + d), (a + 2d), ... is-

(1) $a+\frac{nd}{2}$ (2) $a+\frac{(n-1)d}{2}$

(3) a + (n – 1) d (4) a + nd

SI0001

2. The A.M. of first n even natural number is -

(1) n(n + 1) (2) $\frac{n+1}{2}$ (3) $\frac{n}{2}$ (4) n + 1

SI0002

3. The A.M. of ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ...${}^nC_n$ is -

(1) $\frac{2^n}{n}$ (2) $\frac{2^{n+1}}{n}$ (3) $\frac{2^n}{n+1}$ (4) $\frac{2^{n+1}}{n+1}$

SI0003

4. If the mean of numbers 27, 31, 89, 107, 156 is 82, then the mean of numbers 130, 126, 68, 50, 1 will be-

(1) 80 (2) 82 (3) 75 (4) 157

SI0004

5. If the mean of n observations x_1, x_2, ...x_n is $\bar{x}$, then the sum of deviations of observations from mean is :-

(1) 0 (2) $n\bar{x}$

(3) $\frac{\bar{x}}{n}$ (4) None of these

SI0005

6. The mean of 9 terms is 15. if one new term is added and mean become 16, then the value of new term is :-

(1) 23 (2) 25 (3) 27 (4) 30

SI0006

7. If the mean of first n natural numbers is equal to $\frac{n+7}{3}$, then n is equal to-

(1) 10 (2) 11

(3) 12 (4) none of these

SI0007

8. The mean of first three terms is 14 and mean of next two terms is 18. The mean of all the five terms is-

(1) 15.5 (2) 15.0 (3) 15.2 (4) 15.6

SI0008

9. If the mean of five observations x, x + 2 , x + 4, x+ 6 and x + 8 is 11, then the mean of last three obsevations is-

(1) 11 (2) 13 (3) 15 (4) 17

SI0009

10. The mean of a set of numbers is $\bar{x}$. If each number is decreased by λ, the mean of the new set is-

(1) $\bar{x}$ (2) $\bar{x}+\lambda$ (3) $\lambda-\bar{x}$ (4) $\bar{x}-\lambda$

SI0010

11. The mean of 50 observations is 36. If its two observations 30 and 42 are deleted, then the mean of the remaining observations is-

(1) 48 (2) 36

(3) 38 (4) none of these

SI0011

12. In a frequency dist. , if d_i is deviation of variates from a number ℓ and mean = $\ell+\frac{\Sigma f_i d_i}{\Sigma f_i}$, then ℓ is :-

(1) Lower limit

(2) Assumed mean

(3) Number of observation

(4) Class interval

SI0012

13. The A.M. of n observation is $\bar{x}$. If the sum of n – 4 observations is K, then the mean of remaining observations is-

(1) $\frac{\bar{x}-K}{4}$ (2) $\frac{n\bar{x}-K}{n-4}$

(3) $\frac{n\bar{x}-K}{4}$ (4) $\frac{n\bar{x}-(n-4)K}{4}$

SI0013

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14. The mean of values $1, \frac{1}{2}, \frac{1}{3}, ... \frac{1}{n}$ which have frequencies 1, 2, 3, ... n resp., is :-

(1) $\frac{2n+1}{3}$ (2) $\frac{2}{n}$ (3) $\frac{n+1}{2}$ (4) $\frac{2}{n+1}$

SI0014

15. The sum of squares of deviation of variates from their A.M. is always :-

(1) Zero
(2) Minimum
(3) Maximum
(4) Nothing can be said

SI0015

16. If the mean of following feq. dist. is 2.6, then the value of f is :-

x_i	1	2	3	4	5
f_i	5	4	f	2	3

(1) 1 (2) 3
(3) 8 (4) None of these

SI0016

17. The weighted mean (W.M.) is computed by the formula ?

(1) W.M. = $\frac{\Sigma x_i}{\Sigma w_i}$ (2) W.M. = $\frac{\Sigma w_i}{\Sigma x_i}$

(3) W.M. = $\frac{\Sigma w_i x_i}{\Sigma x_i}$ (4) W.M. = $\frac{\Sigma w_i x_i}{\Sigma w_i}$

SI0017

18. The weighted mean of first n natural numbers when their weights are equal to corresponding natural number, is :-

(1) $\frac{n+1}{2}$ (2) $\frac{2n+1}{3}$

(3) $\frac{(n+1)(2n+1)}{6}$ (4) None of these

SI0018

19. The average income of a group of persons is $\overline{x}$ and that of another group is $\overline{y}$. If the number of persons of both group are in the ratio 4 : 3, then average income of combined group is :-

(1) $\frac{\overline{x}+\overline{y}}{7}$ (2) $\frac{3\overline{x}+4\overline{y}}{7}$

(3) $\frac{4\overline{x}+3\overline{y}}{7}$ (4) None of these

SI0019

20. In a group of students, the mean weight of boys is 65 kg. and mean weight of girls is 55 kg. If the mean weight of all students of group is 61 kg, then the ratio of the number of boys and girls in the group is :-

(1) 2 : 3 (2) 3 : 1 (3) 3 : 2 (4) 4 : 3

SI0020

Median, Mode

21. The median of an arranged series of n even observations, will be :-

(1) $\left(\frac{n+1}{2}\right)$ th term

(2) $\left(\frac{n}{2}\right)$ th term

(3) $\left(\frac{n}{2}+1\right)$ th term

(4) Mean of $\left(\frac{n}{2}\right)$ th and $\left(\frac{n}{2}+1\right)$ th terms

SI0021

22. The median of the numbers 6, 14, 12, 8, 10, 9, 11, is :-

(1) 8 (2) 10 (3) 10.5 (4) 11

SI0022

23. Median of the following freq. dist.

x_i	3	6	10	12	7	15
f_i	3	4	2	8	13	10

(1) 7 (2) 10
(3) 8.5 (4) None of these

SI0023

24. Median is independent of change of :-

(1) only Origin
(2) only Scale
(3) Origin and scale both
(4) Neither origin nor scale

SI0024

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25. A series which have numbers three 4's, four 5's, five 6's, eight 7's, seven 8's and six 9's then the mode of numbers is :-

(1) 9 (2) 8 (3) 7 (4) 6

SI0025

26. Mode of the following freqency distribution

x :	4	5	6	7	8
f :	6	7	10	8	3

(1) 5 (2) 6 (3) 8 (4) 10

SI0026

27. The mode of the following freq. dist is :-

Class	1–10	11–20	21–30	31–40	41–50
f_i	5	7	8	6	4

(1) 24 (2) 23.83

(3) 27.16 (4) None of these

SI0027

Symmetric and asymmetric distribution, Range

28. For a normal dist :-

(1) mean = median

(2) median = mode

(3) mean = mode

(4) mean = median = mode

SI0028

29. The relationship between mean, median and mode for a moderately skewed distribution is-

(1) mode = median – 2 mean

(2) mode = 2 median – mean

(3) mode = 2 median – 3 mean

(4) mode = 3 median – 2 mean

SI0029

30. The range of observations 2, 3, 5, 9, 8, 7, 6, 5, 7, 4, 3 is :-

(1) 6 (2) 7 (3) 5.5 (4) 11

SI0030

Mean Deviation

31. The mean deviation of a frequency dist. is equal to :-

(1) $\frac{\Sigma d_i}{\Sigma f_i}$ (2) $\frac{\Sigma |d_i|}{\Sigma f_i}$ (3) $\frac{\Sigma f_i d_i}{\Sigma f_i}$ (4) $\frac{\Sigma f_i |d_i|}{\Sigma f_i}$

SI0031

32. Mean deviation from the mean for the observation –1, 0, 4 is-

(1) $\sqrt{\frac{14}{3}}$ (2) $\frac{2}{3}$

(3) 2 (4) none of these

SI0032

33. Mean deviation of the observations 70, 42, 63, 34, 44, 54, 55, 46, 38, 48 from median is :-

(1) 7.8 (2) 8.6

(3) 7.6 (4) 8.8

SI0033

34. Mean deviation of 5 observations from their mean 3 is 1.2, then coefficient of mean deviation is :-

(1) 0.24 (2) 0.4

(3) 2.5 (4) None of these

SI0034

35. The mean deviation from median is

(1) greater than the mean deviation from any other central value

(2) less than the mean deviation from any other central value

(3) equal to the mean deviation from any other central value

(4) maximum if all values are positive

SI0035

Variance and Standard Deviation

36. The variate x and u are related by $u = \frac{x-a}{h}$ then correct relation between σ_x and σ_u is :-

(1) $\sigma_x = h\sigma_u$ (2) $\sigma_x = h + \sigma_u$

(3) $\sigma_u = h\sigma_x$ (4) $\sigma_u = h + \sigma_x$

SI0036

37. The S.D. of the first n natural numbers is-

(1) $\sqrt{\frac{n^2-1}{2}}$ (2) $\sqrt{\frac{n^2-1}{3}}$

(3) $\sqrt{\frac{n^2-1}{4}}$ (4) $\sqrt{\frac{n^2-1}{12}}$

SI0037

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38. The variance of observations 112, 116, 120, 125, 132 is :-
(1) 58.8 (2) 48.8
(3) 61.8 (4) None of these
SI0038

39. If $\sum_{i=1}^{10}(x_i - 15) = 12$ and $\sum_{i=1}^{10}(x_i - 15)^2 = 18$ then the S.D. of observations $x_1, x_2 \ldots\ldots\ldots x_{10}$ is :-
(1) $\frac{2}{5}$ (2) $\frac{3}{5}$
(3) $\frac{4}{5}$ (4) None of these
SI0039

40. The S.D. of 7 scored 1, 2, 3, 4, 5, 6, 7 is-
(1) 4 (2) 2
(3) $\sqrt{7}$ (4) none of these
SI0040

41. The variance of series a, a + d, a + 2d, ..., a + 2nd is :-
(1) $\frac{n(n+1)}{2}d^2$ (2) $\frac{n(n+1)}{3}d^2$
(3) $\frac{n(n+1)}{6}d^2$ (4) $\frac{n(n+1)}{12}d^2$
SI0041

42. Variance is independent of change of-
(1) only origin
(2) only scale
(3) origin and scale both
(4) none of these
SI0042

43. If the coefficient of variation and standard deviation of a distribution are 50% and 20 respectively, then its mean is-
(1) 40 (2) 30
(3) 20 (4) None of these
SI0043

44. If each observation of a dist. whose S.D. is σ, is increased by λ, then the variance of the new observations is -
(1) σ (2) $\sigma + \lambda$ (3) σ^2 (4) $\sigma^2 + \lambda$
SI0044

45. The variance of 2, 4, 6, 8, 10 is-
(1) 8 (2) $\sqrt{8}$
(3) 6 (4) none of these
SI0045

46. If each observation of a dist., whose variance is σ^2, is multiplied by λ, then the S.D. of the new new observations is-
(1) σ (2) $\lambda\sigma$
(3) $|\lambda|\sigma$ (4) $\lambda^2\sigma$
SI0046

47. The standard deviation of variate x_i is σ. Then standard deviation of the variate $\frac{ax_i + b}{c}$, where a , b, c are constants is-
(1) $\left(\frac{a}{c}\right)\sigma$ (2) $\left|\frac{a}{c}\right|\sigma$
(3) $\left(\frac{a^2}{c^2}\right)\sigma$ (4) None of these
SI0047

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CHECK YOUR GRASP — ANSWER-KEY — EXERCISE-I

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	2	4	3	3	1	2	2	4	2	4	2	2	3	4	2	1	4	2	3	3
Que.	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
Ans.	4	2	3	4	3	2	2	4	4	2	4	3	2	2	2	1	4	2	2	2
Que.	41	42	43	44	45	46	47													
Ans.	2	1	1	3	1	3	2													

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BRAIN TEASERS STATISTICS EXERCISE-II

1. The A.M. of the series 1, 2, 4, 8, 16, ..., 2^n is-

(1) $\frac{2^n-1}{n}$ (2) $\frac{2^{n+1}-1}{n+1}$

(3) $\frac{2^n-1}{n+1}$ (4) $\frac{2^{n+1}-1}{n}$

SI0048

2. If the mean of n observations $1^2, 2^2, 3^2$, ... n^2 is $\frac{46n}{11}$, then n is equal to-

(1) 11 (2) 12

(3) 23 (4) 22

SI0049

3. The weighted mean of first n natural numbers whose weights are equal, is :-

(1) $\frac{n+1}{2}$ (2) $\frac{2n+1}{2}$

(3) $\frac{2n+1}{3}$ (4) $\frac{(2n+1)(n+1)}{6}$

SI0050

4. The average age of a group of men and women is 30years. If average age of men is 32 and that of women is 27, then the percentage of women in the group is-

(1) 60 (2) 50

(3) 40 (4) 30

SI0051

5. Mean and median of four numbers a, b, c and d $(b < a < d < c)$ is 35 and 25 respectively then the value of b + c – a – d will be :-

(1) 90 (2) 115 (3) 40 (4) 10

SI0052

6. Variance of the group α, $\alpha + 2$, $\alpha + 4$, $\alpha + 6$, ... upto n terms $(\alpha \neq 0)$ is :-

(1) $\frac{n^2-1}{12}+2n+\alpha$ (2) $\frac{n^2-1}{3}+\alpha$

(3) $\frac{n^2-1}{3}$ (4) None

SI0053

7. Product of n positive numbers is unit. The sum of these numbers can not be less than-

(1) 1 (2) n

(3) n^2 (4) none of these

SI0054

8. The A.M. of first n terms of the series 1.3.5, 3.5.7, 5.7.9,....., is-

(1) $3n^3 + 6n^2 + 7n - 1$ (2) $n^3 + 8n^2 + 7n - 1$

(3) $2n^3 + 8n^2 - 7n - 2$ (4) $2n^3 + 8n^2 + 7n - 2$

SI0055

9. The observations 29, 32, 48, 50, x, x + 2, 72, 78, 84, 95 are arranged in ascending order and their median is 63 then the value of x is :-

(1) 61 (2) 62 (3) 62.5 (4) 63

SI0056

10. If the mode of a distribution is 18 and the mean is 24, then median is-

(1) 18 (2) 24 (3) 22 (4) 21

SI0057

11. If the mean and S.D. of n observations x_1, x_2,....x_n are $\bar{x}$ and σ resp, then the sum of squares of observations is :-

(1) $n(\sigma^2 + \bar{x}^2)$ (2) $n(\sigma^2 - \bar{x}^2)$

(3) $n(\bar{x}^2 - \sigma^2)$ (4) None of these

SI0058

12. The variance of observations 8, 12, 13, 15, 22, is :-

(1) 21 (2) 21.2

(3) 21.4 (4) None of these

SI0059

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13. If the mean of a set of observations $x_1, x_2, \ldots\ldots, x_{10}$ is 20, then the mean of $x_1 + 4$, $x_2 + 8$, $x_3 + 12, \ldots\ldots, x_{10} + 40$ is-

(1) 34 (2) 42 (3) 38 (4) 40

SI0060

14. The mean of values 0, 1, 2, ..., n when their weights are $1, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$, resp., is

(1) $\frac{2^n}{n+1}$ (2) $\frac{n+1}{2}$

(3) $\frac{2^{n+1}}{n(n+1)}$ (4) $\frac{n}{2}$

SI0061

15. For 15 observations of x, mean and median were found to be 12 and 20 respectively. Later an observation which was 25 found to be wrong then replaced by its correct value 55, then new mean and median will be :-

(1) 14 and 50 respectively

(2) 12 and 20 respectively

(3) 14 and 20 respectively

(4) Mean is 14 but median can't be determined.

SI0062

16. If a variable takes the discrete values $\alpha + 4$, $\alpha - \frac{7}{2}$, $\alpha - \frac{5}{2}$, $\alpha - 3$, $\alpha - 2$, $\alpha + \frac{1}{2}$, $\alpha - \frac{1}{2}$, $\alpha + 5 (\alpha > 0)$, then the median of these values-

(1) $\alpha - \frac{5}{4}$ (2) $\alpha - \frac{1}{2}$

(3) $\alpha - 2$ (4) $\alpha + \frac{5}{4}$

SI0063

17. The S.D. of first n odd natural numbers is :-

(1) $\sqrt{\frac{n^2-1}{2}}$ (2) $\sqrt{\frac{n^2-1}{3}}$

(3) $\sqrt{\frac{n^2-1}{6}}$ (4) $\sqrt{\frac{n^2-1}{12}}$

SI0064

18. If the sum and sum of squares of 10 observations are 12 and 18 resp., then, The S.D. of observations is :-

(1) $\frac{1}{5}$ (2) $\frac{2}{5}$ (3) $\frac{3}{5}$ (4) $\frac{4}{5}$

SI0065

19. The mean of n values of a distribution is $\bar{x}$. If its first value is increased by 1 , second by 2, ... then the mean of new values will be-

(1) $\bar{x} + n$ (2) $\bar{x} + n/2$

(3) $\bar{x} + \left(\frac{n+1}{2}\right)$ (4) None of these

SI0066

20. The mean of the series $x_1, x_2, \ldots\ldots, x_n$ is $\bar{X}$. If x_2 is replaced by λ, then the new mean is-

(1) $\frac{\bar{X} - x_2 + \lambda}{n}$ (2) $\frac{n\bar{X} + x_2 - \lambda}{n}$

(3) $\frac{(n-1)\bar{X} + \lambda}{n}$ (4) $\frac{n\bar{X} - x_2 + \lambda}{n}$

SI0067

21. The mean square deviation about –1 and +1 of a set of observations are 7 and 3 respectively then standard deviation of the set is :-

(1) $\sqrt{2}$ (2) $\sqrt{3}$ (3) 2 (4) None

SI0068

22. The mean deviation of the numbers 1, 2, 3, 4, 5 is-

(1) 0 (2) 1.2

(3) 2 (4) 1.4

SI0069

23. If mean = (3 median – mode) x, then the value of x is-

(1) 1 (2) 2 (3) 1/2 (4) 3/2

SI0070

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24. A man spends equal ammount on purchasing three kinds of pens at the rate 5 Rs/pen, 10 Rs/pen, 20 Rs/pen, then average cost of one pen is :-

(1) 10 Rs (2) $\frac{35}{3}$ Rs

(3) $\frac{60}{7}$ Rs (4) None of these

SI0071

25. The median of 21 observation is 40. if each observations greater than the median are increased by 6, then the median of the observations will be-

(1) 40 (2) 46

(3) 46 + 40/21 (4) 46 – 40/21

SI0072

26. The coefficient of range of the following distribution 10, 14, 11, 9, 8, 12, 6

(1) 0.4 (2) 2.5

(3) 8 (4) 0.9

SI0073

27. The S.D. of the following freq. dist. :-

Class	0 – 10	10 – 20	20 – 30	30 – 40
f_i	1	3	4	2

(1) 7.8 (2) 9

(3) 8.1 (4) 0.9

SI0074

28. The mean of a dist. is 4. if its coefficient of variation is 58%. Then the S.D. of the dist. is:-

(1) 2.23 (2) 3.23

(3) 2.32 (4) None of these

SI0075

29. The mean of a set of observations is $\bar{x}$. If each observation is divided by α, $(\alpha \neq 0)$ and then is increased by 10, then the mean of the new set is

(1) $\frac{\bar{x}}{\alpha}$ (2) $\frac{\bar{x}+10}{\alpha}$

(3) $\frac{\bar{x}+10\alpha}{\alpha}$ (4) $\frac{\alpha\bar{x}+10}{\alpha}$

SI0076

30. The average age of a teacher and three students is 20 years. If all students are of equal age and the difference between the age of the teacher and that of a student is 20 years, then the age of the teacher is-

(1) 25 years (2) 30 years

(3) 35 years (4) 45 years

SI0077

31. Median of 5 observations i.e.

$3^{\log_9 4}, 5^{\log_{1/2} 8}, e^{2\ell n 3}, \ell n\left(\frac{1}{e}2\right)+3, e^{2\ell n 3+\frac{1}{\log_4 e}}$:-

(1) 1 (2) 2 (3) 9 (4) 36

SI0078

32. Median of ${}^{2n}C_0, {}^{2n}C_1, {}^{2n}C_2, ..., {}^{2n}C_n$ (when n is even) is-

(1) ${}^{2n}C_{\frac{n-1}{2}}$ (2) ${}^{2n}C_{\frac{n}{2}}$

(3) ${}^{2n}C_{\frac{n+1}{2}}$ (4) None of these

SI0079

33. The mean deviation from mean of observations 5, 10, 15, 20, ...85 is :-

(1) 43.71 (2) 21.17

(3) 38.7 (4) None of these

SI0080

34. If standard deviation of variate x_i is 10, then variance of the variate $(50 + 5x_i)$ will be-

(1) 50 (2) 250

(3) 500 (4) 2500

SI0081

35. The S.D. of the numbers 31, 32, 33, ... 47 is-

(1) $2\sqrt{6}$ (2) $4\sqrt{3}$

(3) $\sqrt{\frac{47^2-1}{12}}$ (4) None of these

SI0082

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36. The sum of the squares of deviation of 10 observations from their mean 50 is 250, then coefficient of variation is-
(1) 10% (2) 40%
(3) 50% (4) None of these
SI0083

37. The median and standard deviation (S.D.) of a distribution will be, If each term is increased by 2 -
(1) median and S.D. will increased by 2
(2) median will increased by 2 but S.D. will remain same
(3) median will remain same but S.D. will increased by 2
(4) median and S.D. will remain same
SI0084

38. If $\bar{X}_1$ and $\bar{X}_2$ are the means of two series such that $\bar{X}_1 < \bar{X}_2$ and $\bar{X}$ is the mean of the combined series, then-
(1) $\bar{X} < \bar{X}_1$ (2) $\bar{X} > \bar{X}_2$
(3) $\bar{X}_1 < \bar{X} < \bar{X}_2$ (4) $\bar{X} = \frac{\bar{X}_1 + \bar{X}_2}{2}$
SI0085

39. The median of 19 observations of a group is 30. If two observations with values 8 and 32 are further included, then the median of the new group of 21 observation will be
(1) 28 (2) 30 (3) 32 (4) 34
SI0086

40. The coefficient of mean deviation from median of observations 40, 62, 54, 90, 68, 76 is :-
(1) 2.16 (2) 0.2
(3) 5 (4) None of these
SI0087

41. A group of 10 observations has mean 5 and S.D. $2\sqrt{6}$. another group of 20 observations has mean 5 and S.D. $3\sqrt{2}$, then the S.D. of combined group of 30 observations is :-
(1) $\sqrt{5}$ (2) $2\sqrt{5}$
(3) $3\sqrt{5}$ (4) None of these
SI0088

42. For the values $x_1, x_2 ..... x_{101}$ of a distribution $x_1 < x_2 < x_3 < ..... < x_{100} < x_{101}$. The mean deviation of this distribution with respect to a number k will be minimum when k is equal to-
(1) x_1
(2) x_{51}
(3) x_{50}
(4) $\frac{x_1 + x_2 + ... + x_{101}}{101}$
SI0089

43. In any discrete series (when all the value are not same) the relationship between M.D. about mean and S.D. is-
(1) M.D. = S.D. (2) M.D. > S.D.
(3) M.D. < S.D. (4) M.D. ≤ S.D.
SI0090

44. Median of observations x_i such that $(x_i^2 - 7x_i + 12)(x_i^3 - x_i^2 - 4x_i + 4) = 0$ will be :-
(1) 1 (2) 2 (3) 3 (4) None
SI0091

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BRAIN TEASERS — ANSWER-KEY — EXERCISE-II

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	2	1	1	3	3	3	2	4	2	3	1	2	2	4	3	1	2	3	3	4
Que.	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
Ans.	2	2	3	3	1	1	2	3	3	3	3	2	2	4	1	1	2	3	2	2
Que.	41	42	43	44																
Ans.	2	2	3	2																

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PREVIOUS YEAR QUESTIONS STATISTICS EXERCISE-III

1. All the students of a class performed poorly in Mathematics. The teacher decided to give grace marks of 10 to each of the students. Which of the following statistical measures will not change even after the grace marks were given? **[JEE(Main)-2013]**

(1) mean (2) median
(3) mode (4) variance

SI0092

2. The variance of first 50 even natural numbers is :- **[JEE(Main)-2014]**

(1) $\frac{833}{4}$ (2) 833
(3) 437 (4) $\frac{437}{4}$

SI0093

3. The mean of the data set comprising of 16 observations is 16. If one of the observation valued 16 is deleted and three new observations valued 3, 4 and 5 are added to the data, then the mean of the resultant data, is : **[JEE(Main)-2015]**

(1) 15.8 (2) 14.0
(3) 16.8 (4) 16.0

SI0094

4. If the standard deviation of the numbers 2, 3, a and 11 is 3.5, then which of the following is true ? **[JEE(Main)-2016]**

(1) $3a^2 - 23a + 44 = 0$
(2) $3a^2 - 26a + 55 = 0$
(3) $3a^2 - 32a + 84 = 0$
(4) $3a^2 - 34a + 91 = 0$

SI0095

5. If $\sum_{i=1}^{9}(x_i - 5) = 9$ and $\sum_{i=1}^{9}(x_i - 5)^2 = 45$, then the standard deviation of the 9 items $x_1, x_2, ..., x_9$ is - **[JEE(Main)-2018]**

(1) 4 (2) 2 (3) 3 (4) 9

SI0096

6. 5 students of a class have an average height 150 cm and variance 18 cm^2. A new student, whose height is 156 cm, joined them. The variance (in cm^2) of the height of these six students is: **[JEE(Main)-19]**

(1) 22 (2) 20 (3) 16 (4) 18

SI0097

7. A data consists of n observations : $x_1, x_2, ..., x_n$. If $\sum_{i=1}^{n}(x_i + 1)^2 = 9n$ and $\sum_{i=1}^{n}(x_i - 1)^2 = 5n$, then the standard deviation of this data is : **[JEE(Main)-19]**

(1) 5 (2) $\sqrt{5}$
(3) $\sqrt{7}$ (4) 2

SI0098

8. The mean of five observations is 5 and their variance is 9.20. If three of the given five observations are 1, 3 and 8, then a ratio of other two observations is : **[JEE(Main)-19]**

(1) 4 : 9 (2) 6 : 7
(3) 5 : 8 (4) 10 : 3

SI0099

9. The outcome of each of 30 items was observed; 10 items gave an outcome $\frac{1}{2} - d$ each, 10 items gave outcome $\frac{1}{2}$ each and the remaining 10 items gave outcome $\frac{1}{2} + d$ each. If the variance of this outcome data is $\frac{4}{3}$ then $|d|$ equals :- **[JEE(Main)-19]**

(1) 2 (2) $\frac{\sqrt{5}}{2}$
(3) $\frac{2}{3}$ (4) $\sqrt{2}$

SI0100

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10. The mean and the variance of five observation are 4 and 5.20, respectively. If three of the observations are 3, 4 and 4; then then absolute value of the difference of the other two observations, is : **[JEE(Main)-19]**

(1) 1 (2) 3
(3) 7 (4) 5

SI0101

11. The mean and variance of seven observations are 8 and 16, respectively. If 5 of the observations are 2, 4, 10, 12, 14, then the product of the remaining two observations is : **[JEE(Main)-19]**

(1) 40 (2) 49
(3) 48 (4) 45

SI0102

12. If the standard deviation of the numbers –1, 0, 1, k is $\sqrt{5}$ where k > 0, then k is equal to

(1) $2\sqrt{\frac{10}{3}}$ (2) $2\sqrt{6}$
(3) $4\sqrt{\frac{5}{3}}$ (4) $\sqrt{6}$

SI0103

13. If for some $x \in R$, the frequency distribution of the marks obtained by 20 students in a test is : **[JEE(Main)-19]**

Marks	2	3	5	7
Frequencey	$(x+1)^2$	$2x-5$	x^2-3x	x

then the mean of the marks is :

(1) 2.8 (2) 3.2
(3) 3.0 (4) 2.5

SI0104

14. If both the mean and the standard deviation of 50 observations $x_1, x_2....,x_{50}$ are equal to 16, then the mean of $(x_1-4)^2, (x_2-4)^2,...(x_{50}-4)^2$ is : **[JEE(Main)-19]**

(1) 525 (2) 380 (3) 480 (4) 400

SI0105

15. If the mean and variance of eight numbers 3, 7, 9, 12, 13, 20, x and y be 10 and 25 respectively, then x·y is equal to_____ **[JEE(Main)-20]**

SI0106

16. If the variance of the first n natural numbers is 10 and the variance of the first m even natural numbers is 16, then m + n is equal to _____. **[JEE(Main)-20]**

SI0107

17. The mean and variance of 20 observations are found to be 10 and 4, respectively. On rechecking, it was found that an observation 9 was incorrect and the correct observation was 11. Then the correct variance is **[JEE(Main)-20]**

(1) 3.99 (2) 3.98
(3) 4.02 (4) 4.01

SI0108

18. The mean and the standard deviation (s.d.) of 10 observations are 20 and 2 resepectively. Each of these 10 observations is multiplied by p and then reduced by q, where $p \neq 0$ and $q \neq 0$. If the new mean and new s.d. become half of their original values, then q is equal to **[JEE(Main)-20]**

(1) –20 (2) 10 (3) –10 (4) –5

SI0109

19. Let the observations $x_i (1 \le i \le 10)$ satisfy the equations, $\sum_{i=1}^{10}(x_i-5)=10$ and $\sum_{i=1}^{10}(x_i-5)^2=40$. If μ and λ are the mean and the variance of the observations, $x_1-3, x_2-3, ..., x_{10}-3$, then the ordered pair (μ, λ) is equal to : **[JEE(Main)-20]**

(1) (6, 6) (2) (3, 6) (3) (6, 3) (4) (3, 3)

SI0110

PREVIOUS YEARS QUESTIONS ANSWER-KEY EXERCISE-III

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Ans.	4	2	2	3	2	2	2	1	4	3	3	2	1	4	54.00	18	1	1	4

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MATHEMATICAL REASONING

1. STATEMENT :

A sentence which is either true or false but cannot be both are called a statement. A sentence which is an exclamatory or a wish or an imperative or an interrogative can not be a statement.

If a statement is true then its truth value is T and if it is false then its truth value is F

For ex.

(i) "New Delhi is the capital of India", a true statement

(ii) "3 + 2 = 6", a false statement

(iii) "Where are you going ?" not a statement beasuse

it connot be defined as true or false

Note : A statement cannot be both true and false at a time

2. SIMPLE STATEMENT :

Any statement whose truth value does not depend on other statement are called simple statement

For ex. (i) "$\sqrt{2}$ is an irrational number" (ii) "The set of real number is an infinite set"

3. COMPOUND STATEMENT :

A statement which is a combination of two or more simple statements are called compound statement

Here the simple statements which form a compound statement are known as its sub statements

For ex.

(i) "If x is divisible by 2 then x is even number"

(ii) "ΔABC is equilatral if and only if its three sides are equal"

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4. LOGICAL CONNECTIVES :

The words or phrases which combined simple statements to form a compound statement are called logical connectives.

In the following table some possible connectives, their symbols and the nature of the compound statement formed by them

S.N.	Connectives	symbol	use	operation
1.	and	$\wedge$	$p \wedge q$	conjunction
2.	or	$\vee$	$p \vee q$	disjunction
3.	not	$\sim$ or '	$\sim p$ or p'	negation
4.	If .... then	$\Rightarrow$ or $\rightarrow$	$p \Rightarrow q$ or $p \rightarrow q$	Implication or conditional
5.	If and only if (iff)	$\Leftrightarrow$ or $\leftrightarrow$	$p \Leftrightarrow q$ or $p \leftrightarrow q$	Equivalence or Bi-conditional

Explanation :

(i) $p \wedge q \equiv$ statement p and q

($p \wedge q$ is true only when p and q both are true otherwise it is false)

(ii) $p \vee q \equiv$ statement p or q

($p \vee q$ is true if at least one from p and q is true i.e. $p \vee q$ is false only when p and q both are false)

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(iii) $\sim p \equiv$ not statement p

($\sim$ p is true when p is false and $\sim$ p is false when p is true)

(iv) $p \Rightarrow q \equiv$ statement p then statement q

($p \Rightarrow q$ is false only when p is true and q is false otherwise it is true for all other cases)

(v) $p \Leftrightarrow q \equiv$ statement p if and only if statement q

($p \Leftrightarrow q$ is true only when p and q both are true or false otherwise it is false)

5. TRUTH TABLE :

A table which shows the relationship between the truth value of compound statement S(p, q, r ....) and the truth values of its sub statements p, q, r, ... is said to be truth table of compound statement S

If p and q are two simple statements then truth table for basic logical connectives are given below

Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Negation

p	$(\sim p)$
T	F
F	T

Conditional

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$ or $p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

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Note : If the compound statement contain n sub statements then its truth table will contain 2^n rows.

6. LOGICAL EQUIVALENCE :

Two compound statements S_1(p, q, r...) and S_2(p, q, r ....) are said to be logically equivalent or simply equivalent if they have same truth values for all logically possibilities

Two statements S_1 and S_2 are equivalent if they have identical truth table i.e. the entries in the last column of their truth table are same. If statements S_1 and S_2 are equivalent then we write $S_1 \equiv S_2$

For ex. The truth table for $(p \to q)$ and $(\sim p \vee q)$ given as below

p	q	$(\sim p)$	$p \to q$	$\sim p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

We observe that last two columns of the above truth table are identical hence compound statements $(p \to q)$ and $(\sim p \vee q)$ are equivalent

i.e. $\boxed{p \to q \equiv \sim p \vee q}$

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7. TAUTOLOGY AND CONTRADICTION :

(i) Tautology : A statement is said to be a tautology if it is true for all logical possibilities i.e. its truth value always T. it is denoted by t.

For ex. the statement $p \vee \sim (p \wedge q)$ is a tautology

p	q	$p \wedge q$	$\sim (p \wedge q)$	$p \vee \sim (p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Clearly, The truth value of $p \vee \sim (p \wedge q)$ is T for all values of p and q. so $p \wedge \sim (p \wedge q)$ is a tautology

(ii) Contradiction : A statement is a contradiction if it is false for all logical possibilities.

i.e. its truth value always F. It is denoted by c.

For ex. The statement $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradiction

p	q	$\sim p$	$\sim q$	$p \vee q$	$(\sim p \wedge \sim q)$	$(p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	F

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Clearly, then truth value of $(p \vee q) \wedge (\sim p \wedge \sim q)$ is F for all value of p and q. So $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradiction.

Note : The negation of a tautology is a contradiction and negation of a contradiction is a tautology

8. DUALITY :

Two compound statements S_1 and S_2 are said to be duals of each other if one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$

If a compound statement contains the special variable t (tautology) and c (contradiction) then obtain its dual we replaced t by c and c by t in addition to replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$.

Note :

(i) the connectives $\wedge$ and $\vee$ are also called dual of each other.

(ii) If $S^*(p, q)$ is the dual of the compound statement $S(p, q)$ then

(a) $S^*(\sim p, \sim q) \equiv \sim S(p, q)$ (ii) $\sim S^*(p, q) \equiv S(\sim p, \sim q)$

For ex. The duals of the following statements

(i) $(p \wedge q) \vee (r \vee s)$ (ii) $(p \vee t) \wedge (p \vee c)$

(iii) $\sim(p \wedge q) \vee [p \wedge \sim(q \vee \sim s)]$

are as given below

(i) $(p \vee q) \wedge (r \wedge s)$

(ii) $(p \wedge c) \vee (p \wedge t)$

(iii) $\sim(p \vee q) \wedge [p \vee \sim(q \wedge \sim s)]$

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9. CONVERSE, INVERSE AND CONTRAPOSITIVE OF THE CONDITIONAL STATEMENT ($p \to q$):

(i) Converse : The converse of the conditional statement $p \to q$ is defined as $q \to p$

(ii) Inverse : The inverse of the conditional statement $p \to q$ is defined as $\sim p \to \sim q$

(iii) Contrapositive : The contrapositive of conditional statement $p \to q$ is defined as $\sim q \to \sim p$

10. NEGATION OF COMPOUND STATEMENTS :

If p and q are two statements then

(i) **Negation of conjunction :** $\sim(p \wedge q) \equiv \sim p \vee \sim q$

p	q	$\sim p$	$\sim q$	$(p \wedge q)$	$\sim(p \wedge q)$	$(\sim p \vee \sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

(ii) Negation of disjunction : $\sim(p \vee q) \equiv \sim p \wedge \sim q$

p	q	$\sim p$	$\sim q$	$(p \vee q)$	$(\sim p \vee q)$	$(\sim p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

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(iii) Negation of conditional : $\sim(p \to q) \equiv p \wedge \sim q$

p	q	$\sim q$	$(p \to q)$	$\sim(p \to q)$	$(p \wedge \sim q)$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

(iv) Negation of biconditional : $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

we know that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

$\therefore \sim(p \leftrightarrow q) \equiv \sim[(p \to q) \wedge (q \to p)]$

$\equiv \sim(p \to q) \vee \sim(q \to p)$

$\equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

Note : The above result also can be proved by preparing truth table for $\sim(p \leftrightarrow q)$ and $(p \wedge \sim q) \vee (q \wedge \sim p)$

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11. ALGEBRA OF STATEMENTS :

If p, q, r are any three statements then the some low of algebra of statements are as follow

(i) Idempotent Laws :

(a) $p \wedge p \equiv p$ (b) $p \vee p \equiv p$

i.e. $p \wedge p \equiv p \equiv p \vee p$

p	$(p \wedge p)$	$(p \vee p)$
T	T	T
F	F	F

(ii) Comutative laws :

(a) $p \wedge q \equiv q \wedge p$ (b) $p \vee q \equiv q \vee p$

p	q	$(p \wedge q)$	$(q \wedge p)$	$(p \vee q)$	$(q \vee p)$
T	T	T	T	T	T
T	F	F	F	T	T
F	T	F	F	T	T
F	F	F	F	F	F

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(iii) Associative laws :

(a) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(b) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

p	q	r	$(p \wedge q)$	$(q \wedge r)$	$(p \wedge q) \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	T	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

Similarly we can proved result (b)

(iv) Distributive laws : (a) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ (c) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge (p \wedge r)$

(b) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (d) $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$

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p	q	r	$(q \vee r)$	$(p \wedge q)$	$(p \wedge r)$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Similarly we can prove result (b), (c), (d)

(v) De Morgan Laws : (a) $\sim (p \wedge q) \equiv \sim p \vee \sim q$

(b) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

p	q	$\sim p$	$\sim q$	$(p \wedge q)$	$\sim (p \wedge q)$	$(\sim p \vee \sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Similarly we can proved resulty (b)

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(vi) Involution laws (or Double negation laws) : $\sim(\sim p) \equiv p$

p	$\sim p$	$\sim (\sim p)$
T	F	T
F	T	F

(vii) Identity Laws : If p is a statement and t and c are tautology and contradiction respectively then

(a) $p \wedge t \equiv p$ (b) $p \vee t \equiv t$ (c) $p \wedge c \equiv c$ (d) $p \vee c \equiv p$

p	t	c	$(p \wedge t)$	$(p \vee t)$	$(p \wedge c)$	$(p \vee c)$
T	T	F	T	T	F	T
F	T	F	F	T	F	F

(viii) Complement Laws :

(a) $p \wedge (\sim p) \equiv c$ (b) $p \vee (\sim p) \equiv t$ (c) $(\sim t) \equiv c$ (d) $(\sim c) \equiv t$

p	$\sim p$	$(p \wedge \sim p)$	$(p \vee \sim p)$
T	F	F	T
F	T	F	T

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(ix) Contrapositive laws : $p \to q \equiv \sim q \to \sim p$

p	q	~ p	~ q	p → q	~ q →~ p
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

12. QUANTIFIED STATEMENTS AND QUANTIFIERS :

The words or phrases "All", "Some", "None", "There exists a" are examples of quantifiers.

A statement containing one or more of these words (or phrases) is a quantified statement.

E.g. (1) All dogs are poodles

(2) Some books have hard covers

(3) There exists an odd number which is prime.

Note : Phrases "There exists a" and "Atleast one" and the word "some" have the same meaning.

NEGATION OF QUANTIFIED STATEMENTS :

(1) '**None**' is the negation of '**at least one**' or '**some**' or '**few**'

Statement : Some dogs are poodles.

Negation : No dogs are poodles.

Similarly negation of '**some**' is '**none**'

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(2) The negation of "**some A are B**" or "**There exist A which is B**" is "**No A are (is) B**" or "**There does not exist any A which is B".**

Statement-1 : Some boys in the class are smart

Statement-2 : There exists a boy in the class who is smart

Statement-3 :Alteast one boy in the class is smart

All the three above statements have same meaning as they all indicate "**existence**" of smart boy in the class.

Negation of these statements is

No boy in the class is smart.

or

There does not exist any boy in the class who is smart.

(3) Negation of "**All A are B**" is "**Some A are not B**".

Statement : All boys in the class are smart.

Negation : Some boys in the class are not smart.

or

There exists a boy in the class who is not smart.

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SOLVED EXAMPLES

Ex.1 Which of the following is correct for the statements p and q ?

(1) $p \wedge q$ is true when at least one from p and q is true

(2) $p \to q$ is true when p is true and q is false

(3) $p \leftrightarrow q$ is true only when both p and q are true

(4) $\sim (p \vee q)$ is true only when both p and q are false

Sol.(4) We know that $p \wedge q$ is true only when both p and q are true so option (1) is not correct

we know that $p \to q$ is false only when p is true and q is false so option (2) is not correct

we know that $p \leftrightarrow q$ is true only when either p and q both are true or both are flase

so option (3) is not correct

we know that $\sim(p \vee q)$ is true only when $(p \vee q)$ is false

i.e. p and q both are false

So option (4) is correct

Ex.2 $\sim(p \vee q) \vee (\sim p \wedge q)$ is equivalent to-

(1) p (2) ~p (3) q (4) ~q

Sol.(2) $\because \sim(p \vee q) \vee (\sim p \wedge q) \equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$ (By Demorgan Law)

$\equiv \sim p \wedge (\sim q \vee q)$ (By distributive laws)

$\equiv \sim p \wedge t$ (By complement laws)

$\equiv \sim p$ (By Identity Laws)

Ex.3 Which of the following is logically equivalent to $(p \wedge q)$?

(1) $p \to \sim q$ (2) $\sim p \vee \sim q$ (3) $\sim(p \to \sim q)$ (4) $\sim(\sim p \wedge \sim q)$

Sol.(3) $\because p \to \sim q \equiv \sim p \vee \sim q \equiv \sim(p \wedge q)$ $(\because p \to q \equiv \sim p \vee q)$

i.e. $\sim(p \to \sim q) \equiv p \wedge q$

$\because \sim p \vee \sim q \equiv \sim(p \wedge q)$

and $\sim(\sim p \wedge \sim q) \equiv p \vee q$

Ex.4 If $p \to (q \vee r)$ is false, then the truth values of p, q, r respectively are-

(1) T, F, F (2) F, F, F (3) F, T, T (4) T, T, F

Sol.(1) We know $p \to (q \vee r)$ is false only when p is true and $(q \vee r)$ is false. but $(q \vee r)$ is false only when q and r both are false

Hence truth values of p, q, r are respectively T, F, F

Ex.5 Statement $(p \wedge \sim q) \wedge (\sim p \vee q)$ is

(1) a tautology (2) a contradiction

(3) neither a tautology not a contradiction (4) None of these

Sol.(2) $\because (p \wedge \sim q) \wedge (\sim p \vee q)$

$\equiv (p \wedge \sim q) \wedge \sim(p \wedge \sim q)$ (By Demargon Laws)

$\equiv c$, where c is contradiction (By complement laws)

Ex.6 Negation of the statement $p \to (q \wedge r)$ is-

(1) $\sim p \to \sim(q \wedge r)$ (2) $\sim p \vee (q \wedge r)$ (3) $(q \wedge r) \to p$ (4) $p \wedge (\sim q \vee \sim r)$

Sol.(4) $\sim(p \to (q \wedge r)) \equiv p \wedge \sim(q \wedge r)$ $(\because \sim(p \to q) \equiv p \wedge \sim q)$

$\equiv p \wedge (\sim q \vee \sim r)$

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Ex.7 If x = 5 and y = –2 then x – 2y = 9. The contrapositive of this statement is-

(1) If $x - 2y \neq 9$ then $x \neq 5$ or $y \neq -2$ (2) If $x - 2y \neq 9$ then $x \neq 5$ and $y \neq -2$

(3) If $x - 2y = 9$ then $x = 5$ and $y = -2$ (4) None of these

Sol.(1) Let p, q, r be the three statements such that

$p : x = 5, \quad q : y = -2$ and $r : x - 2y = 9$

Here given statement is $(p \wedge q) \rightarrow r$ and its contrapositive is $\sim r \rightarrow \sim(p \wedge q)$

i.e. $\sim r \rightarrow (\sim p \vee \sim q)$

i.e. if $x - 2y \neq 9$ then $x \neq 5$ or $y \neq -2$

Ex.8 Which of the following is wrong ?

(1) $p \rightarrow q$ is logically equivalent to $\sim p \vee q$

(2) If the $(p \vee q) \wedge (q \vee r)$ is true then truth values of p, q, r are T, F, T respectively

(3) $\sim(p \wedge (q \vee r)) \equiv (\sim p \vee \sim q) \wedge (\sim p \vee \sim r)$

(4) The truth value of $p \wedge \sim(p \vee q)$ is always T

Sol.(4) We know that $p \rightarrow q \equiv \sim p \vee q$

If $(p \vee q) \wedge (q \vee r)$ is true then

$(p \vee q)$ and $(q \vee r)$ both are true.

i.e. truth values of p, q, r may be T, F, T respectively

$\because \quad \sim(p \wedge (q \vee r)) \equiv \sim((p \wedge q) \vee (p \wedge r) \equiv \sim(p \wedge q) \wedge \sim(p \wedge r) \equiv (\sim p \vee \sim q) \wedge (\sim p \vee \sim r)$

If p is true and q is false then $\sim(p \vee q)$ is false i.e. $p \wedge \sim(p \vee q)$ is false

Ex.9 If S*(p, q, r) is the dual of the compound statement S(p, q, r) and $S(p, q, r) = \sim p \wedge [\sim(q \vee r)]$ then S*(~p, ~q, ~r) is equivalent to-

(1) S(p, q, r) (2) ~S(~p, ~q, ~r) (3) ~S(p, q, r) (4) S*(p, q, r)

Sol.(3) $\because \quad S(p, q, r) = \sim p \wedge [\sim(q \vee r)]$

So $S(\sim p, \sim q, \sim r) \equiv \sim(\sim p) \wedge [\sim(\sim q \vee \sim r)] \equiv p \wedge (q \wedge r)$

$S^*(p, q, r) \equiv \sim p \vee [\sim(q \wedge r)]$

$S^*(\sim p, \sim q, \sim r) \equiv p \vee (q \vee r)$

Clearly $S^*(\sim p, \sim q, \sim r) \equiv \sim S(p, q, r)$

Ex.10 The negation of the statement "If a quadrilateral is a square then it is a rhombus"

(1) If a quadrilateral is not a square then is a rhombus it

(2) If a quadrilateral is a square then it is not a rhombus

(3) a quadrilateral is a square and it is not a rhombus

(4) a quadritateral is not a square and it is a rhombus

Sol.(3) Let p and q be the statements as given below

p : a quadrilateral is a square

q : a quadritateral is a rhombus

the given statement is $p \rightarrow q$

$\because \quad \sim(p \rightarrow q) \equiv p \wedge \sim q$

Therefore the negation of the given statement is a quadrilateral is a square and it is not a rhombus

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CHECK YOUR GRASP MATHEMATICAL REASONING EXERCISE-I

1. The inverse of the statement $(p \wedge \sim q) \rightarrow r$ is-
(1) $\sim(p \vee \sim q) \rightarrow \sim r$ (2) $(\sim p \wedge q) \rightarrow \sim r$
(3) $(\sim p \vee q) \rightarrow \sim r$ (4) None of these
MR0001

2. $(\sim p \vee \sim q)$ is logically equivalent to-
(1) $p \wedge q$ (2) $\sim p \rightarrow q$
(3) $p \rightarrow \sim q$ (4) $\sim p \rightarrow \sim q$
MR0002

3. The equivalent statement of $(p \leftrightarrow q)$ is-
(1) $(p \wedge q) \vee (p \vee q)$
(2) $(p \rightarrow q) \vee (q \rightarrow p)$
(3) $(\sim p \vee q) \vee (p \vee \sim q)$
(4) $(\sim p \vee q) \wedge (p \vee \sim q)$
MR0003

4. If the compound statement $p \rightarrow (\sim p \vee q)$ is false then the truth value of p and q are respectively-
(1) T, T (2) T, F (3) F, T (4) F, F
MR0004

5. The statement $(p \rightarrow \sim p) \wedge (\sim p \rightarrow p)$ is-
(1) a tautology
(2) a contradiction
(3) neither a tautology nor a contradiction
(4) None of these
MR0005

6. Negation of the statement $(p \wedge r) \rightarrow (r \vee q)$ is-
(1) $\sim(p \wedge r) \rightarrow \sim(r \vee q)$
(2) $(\sim p \vee \sim r) \vee (r \vee q)$
(3) $(p \wedge r) \wedge (r \wedge q)$
(4) $(p \wedge r) \wedge (\sim r \wedge \sim q)$
MR0006

7. The dual of the statement $\sim p \wedge [\sim q \wedge (p \vee q) \wedge \sim r]$ is-
(1) $\sim p \vee [\sim q \vee (p \vee q) \vee \sim r]$
(2) $p \vee [q \vee (\sim p \wedge \sim q) \vee r]$
(3) $\sim p \vee [\sim q \vee (p \wedge q) \vee \sim r]$
(4) $\sim p \vee [\sim q \wedge (p \wedge q) \wedge \sim r]$
MR0007

8. Which of the following is correct-
(1) $(\sim p \vee \sim q) \equiv (p \wedge q)$
(2) $(p \rightarrow q) \equiv (\sim q \rightarrow \sim p)$
(3) $\sim(p \rightarrow \sim q) \equiv (p \wedge \sim q)$
(4) $\sim(p \leftrightarrow q) \equiv (p \rightarrow q) \vee (q \rightarrow p)$
MR0008

9. The contrapositive of $p \rightarrow (\sim q \rightarrow \sim r)$ is-
(1) $(\sim q \wedge r) \rightarrow \sim p$ (2) $(q \rightarrow r) \rightarrow \sim p$
(3) $(q \vee \sim r) \rightarrow \sim p$ (4) None of these
MR0009

10. The converse of $p \rightarrow (q \rightarrow r)$ is-
(1) $(q \wedge \sim r) \vee p$ (2) $(\sim q \vee r) \vee p$
(3) $(q \wedge \sim r) \wedge \sim p$ (4) $(q \wedge \sim r) \wedge p$
MR0010

11. If p and q are two statement then $(p \leftrightarrow \sim q)$ is true when-
(1) p and q both are true
(2) p and q both are false
(3) p is false and q is true
(4) None of these
MR0011

12. Statement $(p \wedge q) \rightarrow p$ is-
(1) a tautology (2) a contradiction
(3) neither (1) nor (2) (4) None of these
MR0012

13 If statements p, q, r have truth values T, F, T respectively then which of the following statement is true-
(1) $(p \rightarrow q) \wedge r$ (2) $(p \rightarrow q) \vee \sim r$
(3) $(p \wedge q) \vee (q \wedge r)$ (4) $(p \rightarrow q) \rightarrow r$
MR0013

14. If statement $p \rightarrow (q \vee r)$ is true then the truth values of statements p, q, r respectively-
(1) T, F, T (2) F, T, F
(3) F, F, F (4) All of these
MR0014

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15. Which of the following statement is a contradiction-
(1) $(p \wedge q) \wedge (\sim(p \vee q))$ (2) $p \vee (\sim p \wedge q)$
(3) $(p \rightarrow q) \rightarrow p$ (4) $\sim p \vee \sim q$

MR0015

16. The negative of the statement "If a number is divisible by 15 then it is divisible by 5 or 3"
(1) If a number is divisible by 15 then it is not divisible by 5 and 3
(2) A number is divisible by 15 and it is not divisible by 5 or 3
(3) A number is divisible by 15 or it is not divisible by 5 and 3
(4) A number is divisible by 15 and it is not divisible by 5 and 3

MR0016

17. If $x = 5$ and $y = -2$ then $x - 2y = 9$. The contrapositive of this statement is-
(1) If $x - 2y \neq 9$ then $x \neq 5$ or $y \neq -2$
(2) If $x - 2y \neq 9$ then $x \neq 5$ and $y \neq -2$
(3) If $x - 2y = 9$ then $x = 5$ and $y = -2$
(4) None of these

MR0017

18. The negation of the statement "$2 + 3 = 5$ and $8 < 10$" is-
(1) $2 + 3 \neq 5$ and $8 \not< 10$
(2) $2 + 3 \neq 5$ or $8 > 10$
(3) $2 + 3 \neq 5$ or $8 \geq 10$
(4) None of these

MR0018

19. For any three simple statement p, q, r the statement $(p \wedge q) \vee (q \wedge r)$ is true when-
(1) p and r true and q is false
(2) p and r false and q is true
(3) p, q, r all are false
(4) q and r true and p is false

MR0019

20. Which of the following statement is a tautology-
(1) $(\sim p \vee \sim q) \vee (p \vee \sim q)$
(2) $(\sim p \vee \sim q) \wedge (p \vee \sim q)$
(3) $\sim p \wedge (\sim p \vee \sim q)$
(4) $\sim q \wedge (\sim p \vee \sim q)$

MR0020

21. Which of the following statement is a contradiction-
(1) $(\sim p \vee \sim q) \vee (p \vee \sim q)$
(2) $(p \rightarrow q) \vee (p \wedge \sim q)$
(3) $(\sim p \wedge q) \wedge (\sim q)$
(4) $(\sim p \wedge q) \vee (\sim q)$

MR0021

22. The negation of the statement $q \vee (p \wedge \sim r)$ is equivalent to-
(1) $\sim q \wedge (p \rightarrow r)$ (2) $\sim q \wedge \sim(p \rightarrow r)$
(3) $\sim q \wedge (\sim p \wedge r)$ (4) None of these

MR0022

23. Let Q be a non empty subset of N. and q is a statement as given below :-
q : There exists an even number $a \in Q$.
Negation of the statement q will be :-
(1) There is no even number in the set Q.
(2) Every $a \in Q$ is an odd number.
(3) (1) and (2) both
(4) None of these

MR0023

24. The statement $\sim(p \rightarrow q) \leftrightarrow (\sim p \vee \sim q)$ is-
(1) a tautology
(2) a contradiction
(3) neither a tautology nor a contradiction
(4) None of these

MR0024

25. Which of the following is equivalent to $(p \wedge q)$
(1) $p \rightarrow \sim q$ (2) $\sim(\sim p \wedge \sim q)$
(3) $\sim(p \rightarrow \sim q)$ (4) None of these

MR0025

26. The dual of the following statement "Reena is healthy and Meena is beautiful" is-
(1) Reena is beaufiful and Meena is healthy
(2) Reena is beautiful or Meena is healthy
(3) Reena is healthy or Meena is beutiful
(4) None of these

MR0026

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27. If p is any statement, t and c are a tautology and a contradiction respectively then which of the following is not correct-

(1) $p \wedge t \equiv p$ (2) $p \wedge c \equiv c$

(3) $p \vee t \equiv c$ (4) $p \vee c \equiv p$

MR0027

28. If S*(p, q) is the dual of the compound statement S(p, q) then S*(~p, ~q) is equivalent to-

(1) S(~p, ~q) (2) ~S(p, q)

(3) ~S*(p, q) (4) None of these

MR0028

29. If p is any statement, t is a tautology and c is a contradiction then which fo the following is not correct-

(1) $p \wedge (\sim c) \equiv p$

(2) $p \vee (\sim t) \equiv p$

(3) $t \vee c \equiv p \vee t$

(4) $(p \wedge t) \vee (p \vee c) \equiv (t \wedge c)$

MR0029

30. If p, q, r are simple statement with truth values T, F, T respectively then the truth value of $((\sim p \vee q) \wedge \sim r) \rightarrow p$ is-

(1) True

(2) False

(3) True if r is false

(4) True if q is true

MR0030

31. Which of the following is wrong-

(1) $p \vee \sim p$ is a tautology

(2) $\sim(\sim p) \leftrightarrow p$ is a tautology

(3) $p \wedge \sim p$ is a contradiction

(4) $((p \wedge p) \rightarrow q) \rightarrow p$ is a tautology

MR0031

32. The statement "If $2^2 = 5$ then I get first class" is logically equivalent to-

(1) $2^2 = 5$ and I do not get first class

(2) $2^2 = 5$ or I do not get first class

(3) $2^2 \neq 5$ or I get first class

(4) None of these

MR0032

33. If statement $(p \vee \sim r) \rightarrow (q \wedge r)$ is false and statement q is true then statement p is-

(1) true (2) false

(3) may be true or false (4) None of these

MR0033

34. Which of the following statement are not logically equivalent-

(1) $\sim(p \vee \sim q)$ and $(\sim p \wedge q)$

(2) $\sim(p \rightarrow q)$ and $(p \wedge \sim q)$

(3) $(p \rightarrow q)$ and $(\sim q \rightarrow \sim p)$

(4) $(p \rightarrow q)$ and $(\sim p \wedge q)$

MR0034

35. Consider the following statements

p : Virat kohli plays cricket.

q : Virat kohli is good at maths

r : Virat kohli is successful.

then negation of the statement "If virat kohli plays cricket and is not good at maths then he is successful" will be :-

(1) $\sim p \wedge (q \wedge r)$ (2) $(\sim p \vee q) \wedge r$

(3) $p \wedge (\sim q \wedge \sim r)$ (4) None of these

MR0035

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36. Let p statement "If 2 > 5 then earth will not rotate" and q be the statement "$2 \not> 5$ or earth will not rotate".

Statement–1 : p and q are equivalent.

Statement–2 : $m \to n$ and $\sim m \vee n$ are equivalent.

(1) Statement–1 is true, Statement–2 is true; Statement–2 is not the correct explanation of Statement–1.

(2) Statement–1 is false, Statement–2 is true.

(3) Statement–1 is true, Statement–2 is false.

(4) Statement–1 is true, Statement–2 is true; Statement–2 is the correct explanation of Statement–1.

MR0036

37. Which of the following is a tautology :-

(1) $[(\sim p \wedge p) \to q] \longrightarrow (p \wedge p)$

(2) $[(\sim p \wedge p) \to q] \longrightarrow (\sim p \to p)$

(3) $[(\sim p \wedge p) \to q] \longrightarrow (p \to p)$

(4) None of these

MR0037

38. Negation of the statement "No one in the class is fond of music" is :-

(1) everyone in the class is fond of music.

(2) Some of the students in the class are fond of music.

(3) There exists a student in the class who is fond of music.

(4) (2) and (3) both

MR0038

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CHECK YOUR GRASP — ANSWER-KEY — EXERCISE-I

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	3	3	4	2	2	4	3	2	1	1	3	1	4	4	1	4	1	3	4	1
Que.	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38		
Ans.	3	1	3	3	3	3	3	2	4	1	4	3	3	4	3	4	3	4		

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PREVIOUS YEAR QUESTIONS STATISTICS EXERCISE-II

1. The negation of the statement **[JEE(Main)-2012]**

"If I become a teacher, then I will open a school", is :

(1) I will not become a teacher or I will open a school.

(2) I will become a teacher and I will not open a school.

(3) Either I will not become a teacher or I will not open a school.

(4) Neither I will become a teacher nor I will open a school.

MR0039

2. Consider :

Statement-I : $(p\wedge\sim q)\wedge(\sim p\wedge q)$ is a fallacy.

Statement-II : $(p\to q)\leftrightarrow(\sim q\to\sim p)$ is a tuatology. **[JEE(Main)-2013]**

(1) Statement-I is true, Statement-II is true; statement-II is a **correct** explanation for Statement-I.

(2) Statement-I is true, Statement-II is true; statement-II is **not** a correct explanation for Statement-I.

(3) Statement-I is true, Statement-II is false.

(4) Statement-I is false, Statement-II is true.

MR0040

3. The statement $\sim(p\leftrightarrow\sim q)$ is : **[JEE(Main)-2014]**

(1) equivalent to $p\leftrightarrow q$

(2) equivalent to $\sim p\leftrightarrow q$

(3) a tautology

(4) a fallacy

MR0041

4. The negation of $\sim s\vee(\sim r\wedge s)$ is equivalent to : **[JEE(Main)-2015]**

(1) $s\vee(r\vee\sim s)$ (2) $s\wedge r$

(3) $s\wedge\sim r$ (4) $s\wedge(r\wedge\sim s)$

MR0042

5. The Boolean Expression $(p\wedge\sim q)\vee q\vee(\sim p\wedge q)$ is equivalent to :- **[JEE(Main)-2016]**

(1) $p\vee\sim q$ (2) $\sim p\wedge q$ (3) $p\wedge q$ (4) $p\vee q$

MR0043

6. The following statement

$(p\to q)\to[(\sim p\to q)\to q]$ is :

[JEE(Main)-2017]

(1) a fallacy

(2) a tautology

(3) equivalent to $\sim p\to q$

(4) equivalent to $p\to\sim q$

MR0044

7. The Boolean expression $\sim(p\vee q)\vee(\sim p\wedge q)$ is equivalent to : **[JEE(Main)-2018]**

(1) p (2) q (3) ~q (4) ~p

MR0045

8. If the Boolean expression $(p\oplus q)\wedge(\sim p\odot q)$ is equivalent to $p\wedge q$, where $\oplus,\odot\in\{\wedge,\vee\}$, then the ordered pair $(\oplus,\odot)$ is:

[JEE(Main)-19]

(1) $(\wedge,\vee)$ (2) $(\vee,\vee)$ (3) $(\wedge,\wedge)$ (4) $(\vee,\wedge)$

MR0046

9. The logical statement

$$[\sim(\sim p\vee q)\vee(p\wedge r)]\wedge(\sim q\wedge r)$$

is equivalent to : **[JEE(Main)-19]**

(1) $(p\wedge r)\wedge\sim q$ (2) $(\sim p\wedge\sim q)\wedge r$

(3) $\sim p\vee r$ (4) $(p\wedge\sim q)\vee r$

MR0047

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10. Consider the following three statements :

P : 5 is a prime number.

Q : 7 is a factor of 192.

R : L.C.M. of 5 and 7 is 35.

Then the truth value of which one of the following statements is true ?

[JEE(Main)-19]

(1) $(P \wedge Q) \vee (\sim R)$ (2) $(\sim P) \wedge (\sim Q \wedge R)$

(3) $(\sim P) \vee (Q \wedge R)$ (4) $P \vee (\sim Q \wedge R)$

MR0048

11. If q is false and $p \wedge q \leftrightarrow r$ is true, then which one of the following statements is a tautology?

[JEE(Main)-19]

(1) $(p \vee r) \rightarrow (p \wedge r)$ (2) $p \vee r$

(3) $p \wedge r$ (4) $(p \wedge r) \rightarrow (p \vee r)$

MR0049

12. Contrapositive of the statement "If two numbers are not equal, then their squares are not equal." is :-

[JEE(Main)-19]

(1) If the squares of two numbers are equal, then the numbers are equal.

(2) If the squares of two numbers are equal, then the numbers are not equal.

(3) If the squares of two numbers are not equal, then the numbers are equal.

(4) If the squares of two numbers are not equal, then the numbers are not equal.

MR0050

13. The contrapositive of the statement "If you are born in India, then you are a citizen of India", is : **[JEE(Main)-19]**

(1) If you are born in India, then you are not a citizen of India.

(2) If you are not a citizen of India, then you are not born in India.

(3) If you are a citizen of India, then you are born in India.

(4) If you are not born in India, then you are not a citizen of India.

MR0051

14. For any two statements p and q, the negation of the expression $p \vee (\sim p \wedge q)$ is

[JEE(Main)-19]

(1) $p \wedge q$ (2) $p \leftrightarrow q$

(3) $\sim p \vee \sim q$ (4) $\sim p \wedge \sim q$

MR0052

15. If the truth value of the statement $p \rightarrow (\sim q \vee r)$ is false(F), then the truth values of the statements p, q, r are respectively : **[JEE(Main)-19]**

(1) F, T, T (2) T, F, F

(3) T, T, F (4) T, F, T

MR0053

16. The Boolean expression $\sim(p \Rightarrow (\sim q))$ is equivalent to : **[JEE(Main)-19]**

(1) $(\sim p) \Rightarrow q$ (2) $p \vee q$

(3) $q \Rightarrow \sim p$ (4) $p \wedge q$

MR0054

17. Let A, B, C and D be four non-empty sets. The contrapositive statement of "If $A \subseteq B$ and $B \subseteq D$, then $A \subseteq C$" is : **[JEE(Main)-20]**

(1) If $A \subseteq C$, then $B \subset A$ or $D \subset B$

(2) If $A \not\subseteq C$, then $A \not\subseteq B$ or $B \not\subseteq D$

(3) If $A \not\subseteq C$, then $A \subseteq B$ and $B \subseteq D$

(4) If $A \not\subseteq C$, then $A \not\subseteq B$ and $B \subseteq D$

MR0055

18. The logical statement $(p \Rightarrow q) \wedge (q \Rightarrow \sim p)$ is equivalent to : **[JEE(Main)-20]**

(1) p (2) q (3) $\sim p$ (4) $\sim q$

MR0056

19. Which of the following statements is a tautology? **[JEE(Main)-20]**

(1) $\sim(p \vee \sim q) \rightarrow p \vee q$

(2) $\sim(p \wedge \sim q) \rightarrow p \vee q$

(3) $\sim(p \vee \sim q) \rightarrow p \wedge q$

(4) $p \vee (\sim q) \rightarrow p \wedge q$

MR0057

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20. Which one of the following is a tautology ? **[JEE(Main)-20]**

(1) $P \wedge (P \vee Q)$

(2) $P \vee (P \wedge Q)$

(3) $Q \rightarrow (P \wedge (P \rightarrow Q))$

(4) $(P \wedge (P \rightarrow Q)) \rightarrow Q$

MR0058

21. If $p \rightarrow (p \wedge \sim q)$ is false, then the truth values of p and q are respectively : **[JEE(Main)-20]**

(1) F, T (2) T, T (3) F, F (4) T, F

MR0059

22. Negation of the statement : **[JEE(Main)-20]**

$\sqrt{5}$ is an integer or 5 is irrational is :

(1) $\sqrt{5}$ is irrational or 5 is an integer.

(2) $\sqrt{5}$ is not an integer and 5 is not irrational.

(3) $\sqrt{5}$ is an integer and 5 is irrational.

(4) $\sqrt{5}$ is not an integer or 5 is not irrational.

MR0060

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PREVIOUS YEARS QUESTIONS **ANSWER-KEY** **EXERCISE-II**

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	2	2	1	2	4	2	4	1	1	4	4	1	2	4	3
Que.	16	17	18	19	20	21	22								
Ans.	4	2	3	1	4	2	2								

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RATE MEASURE, TANGENT & NORMAL

1. RATE MEASUREMENT :

Whenever one quantity y varies with another quantity x, satisfying some rule y = f(x), then $\frac{dy}{dx}$ (or $f'(x)$) represents the rate of change of y with respect to x and $\left.\frac{dy}{dx}\right]_{x=a}$ (or $f'(a)$) represents the rate of change of y with respect to x at x = a.

Illustration 1 : The volume of a cube is increasing at a rate of $9cm^3/s$. How fast is the surface area increasing when the length of an edge is 10cm ?

Solution : Let x be the length of side, V be the volume and S be the surface area of the cube. Then $V = x^3$ and $S = 6x^2$, where x is a function of time t.

$$\frac{dV}{dt} = 9cm^3/s = \frac{d}{dt}(x^3) = 3x^2\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{3}{x^2}$$

$$\frac{dS}{dt} = \frac{d}{dt}(6x^2) = 12x\left(\frac{3}{x^2}\right) = \frac{36}{x}$$

$$\left.\frac{dS}{dt}\right]_{x=10cm} = 3.6\ cm^2/s.$$

Illustration 2 : x and y are the sides of two squares such that $y = x - x^2$. Find the rate of change of the area of the second square with respect to the first square.

Solution : Given x and y are sides of two squares. Thus the area of two squares are x^2 and y^2

We have to obtain $\frac{d(y^2)}{d(x^2)} = \frac{2y\frac{dy}{dx}}{2x} = \frac{y}{x}.\frac{dy}{dx}$ ..... (i)

where the given curve is, $y = x - x^2 \Rightarrow \frac{dy}{dx} = 1 - 2x$ ..... (ii)

Thus, $\frac{d(y^2)}{d(x^2)} = \frac{y}{x}(1-2x)$ [from (i) and (ii)]

or $\frac{d(y^2)}{d(x^2)} = \frac{(x-x^2)(1-2x)}{x} \Rightarrow \frac{d(y^2)}{d(x^2)} = (2x^2 - 3x + 1)$

The rate of change of the area of second square with respect to first square is $(2x^2 - 3x + 1)$

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Do yourself - 1 :

(i) What is the rate of change of the area of a circle with respect to its radius r at r = 6cm.

(ii) A stone is dropped into a quiet lake and waves move in circles at the speed of 5cm/s. At the instant when the radius of the circular wave is 8cm, how fast is the enclosed area increasing ?

2. APPROXIMATION USING DIFFERENTIALS :

In order to calculate the approximate value of a function, differentials may be used where the differential of a function is equal to its derivative multiplied by the differential of the independent variable. In general $dy = f'(x)dx$ or $df(x) = f'(x)dx$

Note :

(i) For the independent variable 'x', increment Δx and differential dx are equal but this is not the case with the dependent variable 'y' i.e. $\Delta y \neq dy$.

$\therefore$ Approximate value of y when increment Δx is given to independent variable x in $y = f(x)$ is

$$y + \Delta y = f(x + \Delta x) = f(x) + \frac{dy}{dx}.\Delta x$$

(ii) The relation $dy = f'(x)\,dx$ can be written as $\frac{dy}{dx} = f'(x)$; thus the quotient of the differentials of 'y' and 'x' is equal to the derivative of 'y' w.r.t. 'x'.

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Illustration 3 : Find the approximate value of square root of 25.2.

Solution : Let $f(x) = \sqrt{x}$

Now, $f(x + \Delta x) - f(x) = f'(x) \,.\, \Delta x = \frac{\Delta x}{2\sqrt{x}}$

we may write, 25.2 = 25 + 0.2

Taking x = 25 and $\Delta x = 0.2$, we have

$$f(25.2) - f(25) = \frac{0.2}{2\sqrt{25}}$$

or $\quad f(25.2) - \sqrt{25} = \frac{0.2}{10} = 0.02 \quad \Rightarrow \quad f(25.2) = 5.02$

or $\quad \sqrt{(25.2)} = 5.02$

Do yourself - 2 :

(i) Find the approximate value of $(0.009)^{1/3}$.

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3. TANGENT TO THE CURVE AT A POINT :

The tangent to the curve at 'P' is the line through P whose slope is limit of the secant slopes as $Q \to P$ from either side.

4. MYTHS ABOUT TANGENT :

(a) **Myth :** A line meeting the curve only at one point is a tangent to the curve.

Explanation : A line meeting the curve in one point is not necessarily tangent to it.

Here L is not tangent to C

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(b) **Myth :** A line meeting the curve at more than one point is not a tangent to the curve.

Explanation : A line may meet the curve at several points and may still be tangent to it at some point

Here L is tangent to C at P, and cutting it again at Q.

(c) **Myth :** Tangent at a point to the curve can not cross it at the same point.

Explanation : A line may be tangent to the curve and also cross it.

Here X-axis is tangent to $y = x^3$ at origin.

5. NORMAL TO THE CURVE AT A POINT :

A line which is perpendicular to the tangent at the point of contact is called normal to the curve at that point.

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6. POINTS TO REMEMBER :

(a) The value of the derivative at $P(x_1, y_1)$ gives the slope of the tangent to the curve at P. Symbolically

$$f'(x_1) = \left.\frac{dy}{dx}\right]_{(x_1, y_1)} = \text{Slope of tangent at } P(x_1, y_1) = m(\text{say}).$$

(b) Equation of tangent at (x_1, y_1) is ; $\quad y - y_1 = \left.\frac{dy}{dx}\right]_{(x_1, y_1)} (x - x_1)$

(c) Equation of normal at (x_1, y_1) is; $\quad y - y_1 = -\dfrac{1}{\left.\frac{dy}{dx}\right]_{(x_1, y_1)}} (x - x_1)$.

Note :

(i) The point $P(x_1, y_1)$ will satisfy the equation of the curve & the equation of tangent & normal line.

(ii) If the tangent at any point P on the curve is parallel to the axis of x then dy/dx = 0 at the point P.

(iii) If the tangent at any point on the curve is parallel to the axis of y, then dy/dx not defined or dx/dy = 0.

(iv) If the tangent at any point on the curve is equally inclined to both the axes then dy/dx = ±1.

(v) If a curve passing through the origin be given by a rational integral algebraic equation, then the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation. e.g. If the equation of a curve be $x^2 - y^2 + x^3 + 3x^2y - y^3 = 0$, the tangents at the origin are given by $x^2 - y^2 = 0$ i.e. $x + y = 0$ and $x - y = 0$

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Illustration 4 : Find the equation of the tangent to the curve $y = (x^3 - 1)(x - 2)$ at the points where the curve cuts the x-axis.

Solution : The equation of the curve is $y = (x^3 - 1)(x - 2)$ ..... (i)

It cuts x-axis at y = 0. So, putting y = 0 in (i), we get $(x^3 - 1)(x - 2) = 0$

$\Rightarrow (x-1)(x-2)(x^2 + x + 1) = 0 \Rightarrow x - 1 = 0, x - 2 = 0 \qquad [\because x^2 + x + 1 \neq 0]$

$\Rightarrow x = 1, 2$.

Thus, the points of intersection of curve (i) with x-axis are (1, 0) and (2, 0). Now,

$$y = (x^3 - 1)(x - 2) \Rightarrow \frac{dy}{dx} = 3x^2(x - 2) + (x^3 - 1) \Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)} = -3 \text{ and } \left(\frac{dy}{dx}\right)_{(2,0)} = 7$$

The equations of the tangents at (1, 0) and (2, 0) are respectively

$y - 0 = -3(x - 1)$ and $y - 0 = 7(x - 2) \Rightarrow y + 3x - 3 = 0$ and $7x - y - 14 = 0$ **Ans.**

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Illustration 5 : The equation of the normal to the curve $y = x + \sin x \cos x$ at $x = \frac{\pi}{2}$ is -

(A) $x = 2$ (B) $x = \pi$ (C) $x + \pi = 0$ (D) $2x = \pi$

Solution : $\because$ $x = \frac{\pi}{2} \Rightarrow y = \frac{\pi}{2} + 0 = \frac{\pi}{2}$, so the given point $= \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

Now from the given equation $\frac{dy}{dx} = 1 + \cos^2 x - \sin^2 x \Rightarrow \left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{2}, \frac{\pi}{2}\right)} = 1 + 0 - 1 = 0$

$\Rightarrow$ The curve has vertical normal at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The equation to this normal is $x = \frac{\pi}{2}$

$\Rightarrow x - \frac{\pi}{2} = 0 \Rightarrow 2x = \pi$ **Ans. (D)**

Illustration 6 : The equation of normal to the curve $x + y = x^y$, where it cuts x-axis is -

(A) $y = x + 1$ (B) $y = -x + 1$ (C) $y = x - 1$ (D) $y = -x - 1$

Solution : Given curve is $x + y = x^y$ ..... (i)

at x-axis y=0,

$\therefore$ $x + 0 = x^0 \Rightarrow x = 1$

$\therefore$ Point is A(1, 0)

Now to differentiate $x + y = x^y$ take log on both sides

$\Rightarrow \log(x + y) = y \log x$ $\therefore$ $\frac{1}{x+y}\left\{1 + \frac{dy}{dx}\right\} = y.\frac{1}{x} + (\log x)\frac{dy}{dx}$

Putting $x = 1, y = 0$ $\left\{1 + \frac{dy}{dx}\right\} = 0 \Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)} = -1$

$\therefore$ slope of normal = 1

Equation of normal is, $\frac{y - 0}{x - 1} = 1 \Rightarrow y = x - 1$ **Ans. (C)**

Do yourself - 3 :

(i) Find the distance between the point (1,1) and the tangent to the curve $y = e^{2x} + x^2$ drawn at the point where the curve cuts y-axis.

(ii) Find the equation of a line passing through (–2,3) and parallel to tangent at origin for the circle $x^2 + y^2 + x - y = 0$.

7. ANGLE OF INTERSECTION BETWEEN TWO CURVES :

Angle of intersection between two curves is defined as the angle between the two tangents drawn to the two curves at their point of intersection.

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Orthogonal curves :

If the angle between two curves at each point of intersection is 90° then they are called **orthogonal curves.**

For example, the curves $x^2 + y^2 = r^2$ & $y = mx$ are orthogonal curves.

Illustration 7 : The angle of intersection between the curve $x^2 = 32y$ and $y^2 = 4x$ at point (16, 8) is-

(A) 60° (B) 90° (C) $\tan^{-1}\left(\frac{3}{5}\right)$ (D) $\tan^{-1}\left(\frac{4}{3}\right)$

Solution : $x^2 = 32y \Rightarrow \frac{dy}{dx} = \frac{x}{16} \Rightarrow y^2 = 4x \Rightarrow \frac{dy}{dx} = \frac{2}{y}$

$\therefore$ at $(16, 8)$, $\left(\frac{dy}{dx}\right)_1 = 1, \left(\frac{dy}{dx}\right)_2 = \frac{1}{4}$

So required angle $= \tan^{-1}\left(\frac{1-\frac{1}{4}}{1+1\left(\frac{1}{4}\right)}\right) = \tan^{-1}\left(\frac{3}{5}\right)$ **Ans. (C)**

Illustration 8 : Check the orthogonality of the curves $y^2 = x$ & $x^2 = y$.

Solution : Solving the curves simultaneously we get points of intersection as (1, 1) and (0, 0).

At (1,1) for first curve $2y\left(\frac{dy}{dx}\right)_1 = 1 \Rightarrow m_1 = \frac{1}{2}$

& for second curve $2x = \left(\frac{dy}{dx}\right)_2 \Rightarrow m_2 = 2$

$m_1 m_2 \neq -1$ at (1,1).

But at (0, 0) clearly x-axis & y-axis are their respective tangents hence they are orthogonal at (0,0) but not at (1,1). Hence these curves are not said to be orthogonal.

Illustration 9 : If curve $y = 1 - ax^2$ and $y = x^2$ intersect orthogonally then the value of a is -

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) 2 (D) 3

Solution : $y = 1 - ax^2 \Rightarrow \frac{dy}{dx} = -2ax$ $\quad y = x^2 \Rightarrow \frac{dy}{dx} = 2x$

Two curves intersect orthogonally if $\left(\frac{dy}{dx}\right)_1\left(\frac{dy}{dx}\right)_2 = -1$

$\Rightarrow (-2ax)(2x) = -1 \quad \Rightarrow 4ax^2 = 1$ ..... (i)

Now eliminating y from the given equations we have $1 - ax^2 = x^2$

$\Rightarrow (1+a)x^2 = 1$ ..... (ii)

Eliminating x^2 from (i) and (ii) we get $\frac{4a}{1+a} = 1 \Rightarrow a = \frac{1}{3}$ **Ans. (B)**

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Do yourself -4 :

(i) If two curves $y = a^x$ and $y = b^x$ intersect at an angle α, then find the value of $\tan\alpha$.

(ii) Find the angle of intersection of curves $y = 4 - x^2$ and $y = x^2$.

8. LENGTH OF TANGENT, SUBTANGENT, NORMAL & SUBNORMAL :

(a) Length of the tangent (PT) $= \left|\dfrac{y_1\sqrt{1+[f'(x_1)]^2}}{f'(x_1)}\right|$

(b) Length of Subtangent (MT) $= \left|\dfrac{y_1}{f'(x_1)}\right|$

(c) Length of Normal (PN) $= \left|y_1\sqrt{1+\left[f'(x_1)\right]^2}\right|$

(d) Length of Subnormal (MN) $= |y_1 f'(x_1)|$

(e) **Initial ordinate :** Y intercept of tangent at point P(x, y) = OS

(f) **Radius vector (Polar radius) :** Line segment joining origin to point P(x, y) = OP

(g) **Vectorial angle :** Angle made by radius vector with positive direction of x-axis in anticlock wise direction is called vectorial angle. In given figure α is vectorial angle.

Illustration 10 : The length of the normal to the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at $\theta = \dfrac{\pi}{2}$ is -

(A) 2a (B) $\dfrac{a}{2}$ (C) $\sqrt{2}a$ (D) $\dfrac{a}{\sqrt{2}}$

Solution :

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\sin\theta}{a(1+\cos\theta)} = \tan\frac{\theta}{2} \Rightarrow \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{2}} = \tan\left(\frac{\pi}{4}\right) = 1$$

Also at $\theta = \dfrac{\pi}{2}$, $y = a\left(1 - \cos\dfrac{\pi}{2}\right) = a$

$\therefore$ required length of normal $= y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2} = a\sqrt{1+1} = \sqrt{2}a$ **Ans. (C)**

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Illustration 11 : The length of the tangent to the curve $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$, $y = a\sin t$ is

(A) ax (B) ay (C) a (D) xy

Solution : $\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\Big/\left(\frac{dx}{dt}\right) = \frac{a\cos t}{a\left(-\sin t + \frac{1}{\sin t}\right)} = \tan t$

$\therefore$ length of the tangent $= y\frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\left(\frac{dy}{dx}\right)} = a\sin t\frac{\sqrt{1+\tan^2 t}}{\tan t} = a\sin t\left(\frac{\sec t}{\tan t}\right) = a$ **Ans. (C)**

Do yourself - 5 :

(i) Prove that at any point of a curve, the product of the length of sub tangent and the length of sub normal is equal to square of the ordinates of point of contact.

(ii) Find the length of subtangent to the curve $x^2 + y^2 + xy = 7$ at the point $(1, -3)$.

Miscellaneous Illustrations :

Illustration 12 : Find the slope of normal at the point with abcissa $x = -2$ of the graph of the function $f(x) = |x^2 - |x||$

Solution : At $x = -2$, $f(x)$ becomes

$f(x) = x^2 + x$

$\frac{dy}{dx} = 2x + 1 = -3$

Slope of normal $= \frac{1}{3}$

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Illustration 13 : If $y = 4x - 5$ is a tangent to the curve $y^2 = px^3 + q$ at (2, 3), then

(A) $p = 2, q = -7$ (B) $p = -2, q = 7$

(C) $p = -2, q = -7$ (D) $p = 2, q = 7$

Solution : $\frac{dy}{dx} = 4$ & $9 = 8p + q$

$2y\frac{dy}{dx} = 3px^2$

$6\frac{dy}{dx} = 3p(4) \Rightarrow \frac{dy}{dx} = 2p = 4 \Rightarrow p = 2$ & $q = -7$

ANSWERS FOR DO YOURSELF

1 :	**(i)**	12π cm	**(ii)**	$80\ \pi cm^2/s$
2 :	**(i)**	0.208		
3 :	**(i)**	$\frac{2}{\sqrt{5}}$ units	**(ii)**	$x - y + 5 = 0$
4 :	**(i)**	$\left\lvert\frac{\ell na - \ell nb}{1 + \ell na\,\ell nb}\right\rvert$	**(ii)**	$\tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$
5 :	**(ii)**	15		

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TANGENT & NORMAL

EXERCISE (O-1)

1. If the surface area of a sphere of radius r is increasing uniformly at the rate $8cm^2/s$, then the rate of change of its volume is :

(A) proportional to r^2 (B) constant

(C) proportional to r (D) proportional to $\sqrt{r}$

TN0001

2. A Spherical balloon is being inflated at the rate of 35 cc/min. The rate of increase in the surface area (in cm^2/min.) of the balloon when its diameter is 14 cm, is :

(A) $\sqrt{10}$ (B) $10\sqrt{10}$ (C) 100 (D) 10

TN0002

3. Let S be a square with sides of length x. If we approximate the change in size of the area of S by $h \cdot \left.\frac{dA}{dx}\right|_{x=x_0}$, when the sides are changed from x_0 to $x_0 + h$, then the absolute value of the error in our approximation, is

(A) h^2 (B) $2hx_0$ (C) x_0^2 (D) h

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TN0003

4. The slope of the curve $y = \sin x + \cos^2 x$ is zero at the point, where-

(A) $x = \frac{\pi}{4}$ (B) $x = \frac{\pi}{2}$ (C) $x = \pi$ (D) No where

TN0004

5. The equation of tangent at the point (at^2, at^3) on the curve $ay^2 = x^3$ is-

(A) $3tx - 2y = at^3$ (B) $tx - 3y = at^3$

(C) $3tx + 2y = at^3$ (D) None of these

TN0005

6. The equation of normal to the curve $y = x^3 - 2x^2 + 4$ at the point x = 2 is-

(A) $x + 4y = 0$ (B) $4x - y = 0$ (C) $x + 4y = 18$ (D) $4x - y = 18$

TN0006

7. The slope of the normal to the curve $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ at point $\theta = \pi/2$ is-

(A) 0 (B) 1 (C) –1 (D) $1/\sqrt{2}$

TN0007

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8. The coordinates of the points on the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, where tangent is inclined an angle $\pi/4$ to the x-axis are-

(A) (a, a) (B) $\left(a\left(\frac{\pi}{2}-1\right), a\right)$ (C) $\left(a\left(\frac{\pi}{2}+1\right), a\right)$ (D) $\left(a, a\left(\frac{\pi}{2}+1\right)\right)$

TN0008

9. Consider the curve represented parametrically by the equation
$x = t^3 - 4t^2 - 3t$ and $y = 2t^2 + 3t - 5$ where $t \in R$.
If H denotes the number of point on the curve where the tangent is horizontal and V the number of point where the tangent is vertical then

(A) H = 2 and V = 1 (B) H = 1 and V = 2
(C) H = 2 and V = 2 (D) H = 1 and V = 1

TN0009

10. The line $x/a + y/b = 1$ touches the curve $y = be^{-x/a}$ at the point-

(A) (0, a) (B) (0, 0) (C) (0, b) (D) (b, 0)

TN0010

11. If the tangent to the curve $2y^3 = ax^2 + x^3$ at a point (a, a) cuts off intercepts p and q on the coordinates axes, where $p^2 + q^2 = 61$, then a equals-

(A) 30 (B) –30 (C) 0 (D) ±30

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TN0011

12. The sum of the intercepts made by a tangent to the curve $\sqrt{x} + \sqrt{y} = 4$ at point (4, 4) on coordinate axes is-

(A) $4\sqrt{2}$ (B) $6\sqrt{3}$ (C) $8\sqrt{2}$ (D) $\sqrt{256}$

TN0012

13. The curve $x^2 - y^2 = 5$ and $\frac{x^2}{18} + \frac{y^2}{8} = 1$ cut each other at any common point at an angle-

(A) $\pi/4$ (B) $\pi/3$ (C) $\pi/2$ (D) None of these

TN0013

14. The lines tangent to the curve $y^3 - x^2y + 5y - 2x = 0$ and $x^4 - x^3y^2 + 5x + 2y = 0$ at the origin intersect at an angle θ equal to-

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

TN0014

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15. If the tangent at a point P, with parameter t, on the curve $x = 4t^2 + 3$, $y = 8t^3 - 1$, $t \in R$, meets the curve again at a point Q, then the coordinates of Q are :

(A) $(t^2 + 3, t^3 - 1)$ (B) $(t^2 + 3, -t^3 - 1)$

(C) $(16t^2 + 3, -64t^3 - 1)$ (D) $(4t^2 + 3, -8t^3 - 1)$

TN0015

16. The angle of intersection between the curves $y^2 = 8x$ and $x^2 = 4y - 12$ at (2, 4) is-

(A) 90° (B) 60° (C) 45° (D) 0°

TN0016

17. The length of subtangent to the curve $x^2 + xy + y^2 = 7$ at the point (1, –3) is-

(A) 3 (B) 5 (C) 15 (D) 3/5

TN0017

18. For a curve $\frac{(\text{length of normal})^2}{(\text{length of tangent})^2}$ is equal to -

(A) (subnormal)/(subtangent) (B) (subtangent)/(subnormal)

(C) (subtangent × subnormal) (D) constant

TN0018

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19. At any point of a curve (subtangent) × (subnormal) is equal to the square of the-

(A) slope of the tangent at that point (B) slope of the normal at that point

(C) abscissa of that point (D) ordinate of that point

TN0019

EXERCISE (O-2)

[SINGLE CORRECT CHOICE TYPE]

1. Let S be a sphere with radius r. If we approximate the change of volume of S by $h.A\big|_{r_0} + \frac{h^2}{2}\frac{dA}{dr}\Big|_{r=r_0}$ where A is surface area, when radius is changed from r_0 to $(r_0 + h)$, then the absolute value of error in our approximation is

(A) h^3 (B) $4\pi h r_0^2$ (C) $4\pi r_0 h^2$ (D) $\frac{4\pi}{3}h^3$

TN0020

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2. At the point P(a, a^n) on the graph of $y = x^n$ ($n \in N$) in the first quadrant a normal is drawn. The normal intersects the y-axis at the point (0, b). If $\lim_{a \to 0} b = \frac{1}{2}$, then *n* equals

(A) 1 (B) 3 (C) 2 (D) 4

TN0021

3. A circle with centre at (15, –3) is tangent to $y = \frac{x^2}{3}$ at a point in the first quadrant. The radius of the circle is equal to

(A) $5\sqrt{6}$ (B) $8\sqrt{3}$ (C) $9\sqrt{2}$ (D) $6\sqrt{5}$

TN0022

4. A line L is perpendicular to the curve $y = \frac{x^2}{4} - 2$ at its point P and passes through (10, –1). The coordinates of the point P are

(A) (2, –1) (B) (6, 7) (C) (0, –2) (D) (4, 2)

TN0023

5. The point(s) at each of which the tangents to the curve $y = x^3 - 3x^2 - 7x + 6$ cut off on the positive semi axis OX a line segment half that on the negative semi axis OY then the co-ordinates of the point(s) is/are given by :

(A) (– 1, 9) (B) (3, – 15) (C) (1, – 3) (D) none

TN0024

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6. The x-intercept of the tangent at any arbitrary point of the curve $\frac{a}{x^2} + \frac{b}{y^2} = 1$ is proportional to:

(A) square of the abscissa of the point of tangency

(B) square root of the abscissa of the point of tangency

(C) cube of the abscissa of the point of tangency

(D) cube root of the abscissa of the point of tangency.

TN0025

7. A curve is represented by the equations, $x = \sec^2 t$ and $y = \cot t$ where t is a parameter. If the tangent at the point P on the curve where $t = \pi/4$ meets the curve again at the point Q then $|PQ|$ is equal to:

(A) $\frac{5\sqrt{3}}{2}$ (B) $\frac{5\sqrt{5}}{2}$ (C) $\frac{2\sqrt{5}}{3}$ (D) $\frac{3\sqrt{5}}{2}$

TN0026

8. A curve is represented parametrically by the equations $x = t + e^{at}$ and $y = -t + e^{at}$ when $t \in R$ and $a > 0$. If the curve touches the axis of x at the point A, then the coordinates of the point A are

(A) (1, 0) (B) (1/e, 0) (C) (e, 0) (D) (2e, 0)

TN0027

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9. At any two points of the curve represented parametrically by $x = a(2\cos t - \cos 2t)$; $y = a(2\sin t - \sin 2t)$ the tangents are parallel to the axis of x corresponding to the values of the parameter t differing from each other by :

(A) $2\pi/3$ (B) $3\pi/4$ (C) $\pi/2$ (D) $\pi/3$

TN0028

[MULTIPLE CORRECT CHOICE TYPE]

10. If $\frac{x}{a} + \frac{y}{b} = 1$ is a tangent to the curve $x = Kt$, $y = \frac{K}{t}$, $K > 0$ then :

(A) $a > 0,\ b > 0$ (B) $a > 0,\ b < 0$ (C) $a < 0,\ b > 0$ (D) $a < 0,\ b < 0$

TN0029

11. The co-ordinates of the point(s) on the graph of the function, $f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 7x - 4$ where the tangent drawn cut off intercepts from the co-ordinate axes which are equal in magnitude but opposite in sign, is

(A) (2, 8/3) (B) (3, 7/2) (C) (1, 5/6) (D) none

TN0030

12. Given that g(x) is a non constant linear function defined on R-

(A) $y = g(x)$ and $y = g^{-1}(x)$ are orthogonal (B) $y = g(x)$ and $y = g^{-1}(-x)$ are orthogonal

(C) $y = g(-x)$ and $y = g^{-1}(x)$ are orthogonal (D) $y = g(-x)$ and $y = g^{-1}(-x)$ are orthogonal

TN0031

13. If the curves $y = 2(x - a)^2$ and $y = e^{2x}$ touches each other, then 'a' is less than-

(A) –1 (B) 0 (C) 1 (D) 2

TN0032

14. For the curve C : $y = e^{2x}\cos x$, which of the following statement(s) is/are true ?

(A) equation of the tangent where C crosses y-axis is $y = 3x + 1$

(B) equation of the tangent where C crosses y-axis is $y = 2x + 1$

(C) number of points in $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ where tangent on the curve C is parallel to x-axis is 4.

(D) number of points in $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ where tangent on the curve C is parallel to x-axis is 2.

TN0033

EXERCISE (S-1)

1. Water is being poured on to a cylindrical vessel at the rate of 1 m^3/min. If the vessel has a circular base of radius 3 m, find the rate at which the level of water is rising in the vessel.

TN0034

2. A man 1.5 m tall walks away from a lamp post 4.5 m high at the rate of 4 km/hr.

(i) how fast is the farther end of the shadow moving on the pavement ?

(ii) how fast is his shadow lengthening ?

TN0035

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3. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y coordinate is changing 8 times as fast as the x coordinate.

TN0036

4. An inverted cone has a depth of 10 cm & a base of radius 5 cm. Water is poured into it at the rate of 1.5 cm^3/min. Find the rate at which level of water in the cone is rising, when the depth of water is 4 cm.

TN0037

5. Water is dripping out from a conical funnel of semi vertical angle $\pi/4$, at the uniform rate of 2 cm^3/sec through a tiny hole at the vertex at the bottom. When the slant height of the water is 4 cm, find the rate of decrease of the slant height of the water.

TN0038

6. An air force plane is ascending vertically at the rate of 100 km/h. If the radius of the earth is R Km, how fast the area of the earth, visible from the plane increasing at 3min after it started ascending.

Take visible area $A = \frac{2\pi R^2 h}{R+h}$ Where h is the height of the plane in kms above the earth.

TN0039

7. If in a triangle ABC, the side 'c' and the angle 'C' remain constant, while the remaining elements are changed slightly, show that $\frac{da}{\cos A} + \frac{db}{\cos B} = 0$.

TN0040

8. A water tank has the shape of a right circular cone with its vertex down. Its altitude is 10 cm and the radius of the base is 15 cm. Water leaks out of the bottom at a constant rate of 1cu. cm/sec. Water is poured into the tank at a constant rate of C cu. cm/sec. Compute C so that the water level will be rising at the rate of 4 cm/sec at the instant when the water is 2 cm deep.

TN0041

9. Sand is pouring from a pipe at the rate of 12 cc/sec. The falling sand forms a cone on the ground in such a way that the height of the cone is always 1/6th of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm.

TN0042

10. A circular ink blot grows at the rate of 2 cm^2 per second. Find the rate at which the radius is increasing after $2\frac{6}{11}$ seconds. Use $\pi = \frac{22}{7}$.

TN0043

11. Water is flowing out at the rate of 6 m^3/min from a reservoir shaped like a hemispherical bowl of radius R = 13 m. The volume of water in the hemispherical bowl is given by $V = \frac{\pi}{3} \cdot y^2(3R - y)$ when the water is y meter deep. Find

(a) At what rate is the water level changing when the water is 8 m deep.

(b) At what rate is the radius of the water surface changing when the water is 8 m deep.

TN0044

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12. (i) Use differentials to a approximate the values of ; (a) $\sqrt{36.6}$ and (b) $\sqrt[3]{26}$.

TN0045

(ii) If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.

TN0046

13. Find the equation of the normal to the curve $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$ at $x = 0$.

TN0047

14. Find all the lines that pass through the point (1, 1) and are tangent to the curve represented parametrically as $x = 2t - t^2$ and $y = t + t^2$.

TN0048

15. The tangent to $y = ax^2 + bx + \frac{7}{2}$ at (1, 2) is parallel to the normal at the point (–2, 2) on the curve $y = x^2 + 6x + 10$. Find the value of a and b.

TN0049

16. A line is tangent to the curve $f(x) = \frac{41x^3}{3}$ at the point P in the first quadrant, and has a slope of 2009. This line intersects the y-axis at (0, b). Find the value of 'b'.

TN0050

17. Find all the tangents to the curve $y = \cos(x + y)$, $-2\pi \le x \le 2\pi$, that are parallel to the line $x + 2y = 0$.

TN0051

18. The curve $y = ax^3 + bx^2 + cx + 5$, touches the x - axis at P (– 2 , 0) & cuts the y-axis at a point Q where its gradient is 3. Find a , b , c.

TN0052

19. Find the gradient of the line passing through the point (2,8) and touching the curve $y = x^3$.

TN0053

20. (a) Find the value of n so that the subnormal at any point on the curve $xy^n = a^{n+1}$ may be constant.

TN0054

(b) Show that in the curve $y = a.\ln(x^2 - a^2)$, sum of the length of tangent & subtangent varies as the product of the coordinates of the point of contact.

TN0055

21. (a) Show that the curves $\frac{x^2}{a^2+K_1} + \frac{y^2}{b^2+K_1} = 1$ & $\frac{x^2}{a^2+K_2} + \frac{y^2}{b^2+K_2} = 1$ $(K_1 \ne K_2)$ intersect orthogonally.

TN0056

(b) If the two curves $C_1 : x = y^2$ and $C_2 : xy = k$ cut at right angles find the value of k.

TN0057

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22. Show that the angle between the tangent at any point 'A' of the curve $ln\,(x^2 + y^2) = C\tan^{-1}\frac{y}{x}$ and the line joining A to the origin is independent of the position of A on the curve.

TN0058

EXERCISE (S-2)

RATE MEASURE AND APPROXIMATIONS

1. A variable Δ ABC in the xy plane has its orthocentre at vertex 'B' , a fixed vertex 'A' at the origin and the third vertex 'C' restricted to lie on the parabola $y = 1 + \frac{7x^2}{36}$. The point B starts at the point (0, 1) at time t = 0 and moves upward along the y axis at a constant velocity of 2 cm/sec. How fast is the area of the triangle increasing when $t = \frac{7}{2}$ sec.

TN0059

2. Find the equations of the tangents drawn to the curve $y^2 - 2x^3 - 4y + 8 = 0$ from the point (1, 2).

TN0060

3. Find the point of intersection of the tangents drawn to the curve $x^2y = 1 - y$ at the points where it is intersected by the curve $xy = 1 - y$.

TN0061

4. A function is defined parametrically by the equations

$$f(t) = x = \begin{cases} 2t + t^2\sin\frac{1}{t} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases} \quad \text{and } g(t) = y = \begin{cases} \frac{1}{t}\sin t^2 & \text{if } t \neq 0 \\ o & \text{if } t = 0 \end{cases}$$

Find the equation of the tangent and normal at the point for t = 0 if exist.

TN0062

5. There is a point (p,q) on the graph of $f(x) = x^2$ and a point (r,s) on the graph of $g(x) = -8/x$ where $p > 0$ and $r > 0$. If the line through (p,q) and (r,s) is also tangent to both the curves at these points respectively then find the value of (p + q).

TN0063

6. Tangent at a point P_1 [other than (0 , 0)] on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve at P_3 & so on. Show that the abscissae of $P_1, P_2, P_3, \ldots\ldots P_n$, form a GP. Also find the ratio $\frac{\text{area of } \Delta(P_1P_2P_3)}{\text{area of } \Delta(P_2P_3P_4)}$.

TN0064

7. The chord of the parabola $y = -a^2x^2 + 5ax - 4$ touches the curve $y = \frac{1}{1-x}$ at the point x = 2 and is bisected by that point. Find 'a'.

TN0065

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8. Show that the condition that the curves $x^{2/3} + y^{2/3} = c^{2/3}$ & $(x^2/a^2) + (y^2/b^2) = 1$ may touch if $c = a + b$.

TN0066

9. Prove that the segment of the normal to the curve $x = 2a \sin t + a \sin t \cos^2 t$; $y = -a \cos^3 t$ contained between the co-ordinate axes is equal to 2a.

TN0067

EXERCISE (JM)

1. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the x-axis, is :-

[AIEEE-2010]

(1) y = 0 (2) y = 1 (3) y = 2 (4) y = 3

TN0068

2. The normal to the curve, $x^2 + 2xy - 3y^2 = 0$, at (1, 1) : **[JEE-MAIN 2015]**

(1) meets the curve again in the third quadrant

(2) meets the curve again in the fourth quadrant

(3) does not meet the curve again

(4) meets the curve again in the second quadrant

TN0069

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3. Consider $f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$, $x \in \left(0, \frac{\pi}{2}\right)$. A normal to y = f(x) at $x = \frac{\pi}{6}$ also passes through the point : **[JEE-MAIN 2016]**

(1) $\left(\frac{\pi}{4}, 0\right)$ (2) (0, 0) (3) $\left(0, \frac{2\pi}{3}\right)$ (4) $\left(\frac{\pi}{6}, 0\right)$

TN0070

4. The normal to the curve $y(x-2)(x-3) = x + 6$ at the point where the curve intersects the y-axis passes through the point : **[JEE-MAIN 2017]**

(1) $\left(\frac{1}{2}, \frac{1}{3}\right)$ (2) $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ (3) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (4) $\left(\frac{1}{2}, -\frac{1}{3}\right)$

TN0071

5. If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is :

(1) $\frac{7}{2}$ (2) 4 (3) $\frac{9}{2}$ (4) 6 **[JEE-MAIN 2018]**

TN0072

6. The tangent to the curve, $y = xe^{x^2}$ passing through the point (1,e) also passes through the point :

[JEE-MAIN 2019]

(1) $\left(\frac{4}{3}, 2e\right)$ (2) (2,3e) (3) $\left(\frac{5}{3}, 2e\right)$ (4) (3,6e)

TN0073

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7. The length of the perpendicular from the origin, on the normal to the curve, $x^2 + 2xy - 3y^2 = 0$ at the point (2,2) is **[JEE-MAIN 2020]**

(1) $4\sqrt{2}$ (2) $2\sqrt{2}$ (3) 2 (4) $\sqrt{2}$

TN0074

8. Let the normal at a point P on the curve $y^2 - 3x^2 + y + 10 = 0$ intersect the y-axis at $\left(0, \frac{3}{2}\right)$. If m is the slope of the tangent at P to the curve, then |m| is equal to **[JEE-MAIN 2020]**

TN0075

EXERCISE (JA)

1. Find the equation of the straight line which is tangent at one point and normal at another point of the curve, $x = 3t^2$, $y = 2t^3$. **[REE 2000 (Mains) 5 out of 100]**

TN0076

2. If the normal to the curve, $y = f(x)$ at the point (3, 4) makes an angle $\frac{3\pi}{4}$ with the positive x–axis. Then $f'(3) =$

(A) – 1 (B) $-\frac{3}{4}$ (C) $\frac{4}{3}$ (D) 1

[JEE 2000 (Scr.) 1 out of 35]

TN0077

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3. The point(s) on the curve $y^3 + 3x^2 = 12y$ where the tangent is vertical, is(are)

(A) $\left(\pm\frac{4}{\sqrt{3}}, -2\right)$ (B) $\left(\pm\sqrt{\frac{11}{3}}, 1\right)$ (C) (0, 0) (D) $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

[JEE 2002 (Scr.), 3]

TN0078

4. Tangent to the curve $y = x^2 + 6$ at a point P (1, 7) touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q. Then the coordinates of Q are

(A) (– 6, –11) (B) (–9, –13) (C) (– 10, – 15) (D) (–6, –7)

[JEE 2005 (Scr.), 3]

TN0079

5. The tangent to the curve $y = e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c - 1, e^{c-1})$ and $(c + 1, e^{c+1})$

(A) on the left of x = c (B) on the right of x = c

(C) at no point (D) at all points **[JEE 2007, 3]**

TN0080

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ANSWER KEY

(TANGENT & NORMAL)

EXERCISE (O-1)

1. C	**2.** D	**3.** A	**4.** B	**5.** A	**6.** C	**7.** C
8. C	**9.** B	**10.** C	**11.** D	**12.** D	**13.** C	**14.** D
15. B	**16.** D	**17.** C	**18.** A	**19.** D		

EXERCISE (O-2)

1. D	**2.** C	**3.** D	**4.** D	**5.** B	**6.** C	**7.** D
8. D	**9.** A	**10.** A,D	**11.** A,B	**12.** B,C	**13.** B,C,D	**14.** B,D

EXERCISE (S-1)

1. $1/9\pi$ m/min **2.** **(i)** 6 km/h ; **(ii)** 2 km/hr **3.** $(4, 11)$ & $(-4, -31/3)$

4. $3/8\pi$ cm/min **5.** $\frac{\sqrt{2}}{4\pi}$ cm/s **6.** $200\pi r^3/(r+5)^2$ km²/h

8. $1 + 36\pi$ cu. cm/sec **9.** $1/48\pi$ cm/s **10.** $\frac{1}{4}$ cm/sec.

11. (a) $-\frac{1}{24\pi}$ m/min., (b) $-\frac{5}{288\pi}$ m/min. **12.** **(i)** (a) 6.05, (b) $\frac{80}{27}$; (ii) 9.72π cm³

13. $x + y - 1 = 0$ **14.** $x = 1$ when $t = 1$, $m \to \infty$; $5x - 4y = 1$ if $t \neq 1$, $t = 1/3$

15. $a = 1, b = \frac{-5}{2}$ **16.** $-\frac{82 \cdot 7^3}{3}$ **17.** $x + 2y = \pi/2$ & $x + 2y = -3\pi/2$

18. $a = -1/2$; $b = -3/4$; $c = 3$ **19.** 3, 12 **20.** **(a)** $n = -2$

21. **(b)** $\pm\frac{1}{2\sqrt{2}}$ **22.** $\theta = \tan^{-1}\frac{2}{C}$

EXERCISE (S-2)

1. $\frac{66}{7}$ **2.** $2\sqrt{3}\,x - y = 2(\sqrt{3} - 1)$ or $2\sqrt{3}\,x + y = 2(\sqrt{3} + 1)$ **3.** (0, 1)

4. $T : x - 2y = 0$; $N : 2x + y = 0$ **5.** 20 **6.** 1/16 **7.** $a = 1$

EXERCISE (JM)

1. 4	**2.** 2	**3.** 3	**4.** 3	**5.** 3	**6.** 1	**7.** **2**
8. **4.00**						

EXERCISE (JA)

1. $\sqrt{2}\,x + y - 2\sqrt{2} = 0$ or $\sqrt{2}\,x - y - 2\sqrt{2} = 0$ **2.** D **3.** D **4.** D **5.** A

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MONOTONICITY

1. INCREASING / DECREASING / STRICTLY INCREASING / STRICTLY DECREASING NATURE OF A FUNCTION AT A POINT :

I. Increasing at x = a :

If $f(s) \le f(a) \le f(t)$ when ever $s < a < t$, where $s, t \in (a-h, a+h) \cap D_f$ for some $h > 0$, then f is said to be increasing at $x = a$.

(1) When 'a' be left end of the interval

$f(a) \le f(x)\ \forall\ x \in (a, a+h) \cap D_f$ for some $h > 0$

$\Rightarrow$ f is increasing at $x = a$.

(2) When 'a' be right end of the interval

$f(x) \le f(a)\ \forall\ x \in (a-h, a) \cap D_f$ for some $h > 0$

$\Rightarrow$ f is increasing at $x = a$.

II. Strictly increasing at x = a :

If $f(s) < f(a) < f(t)$ when ever $s < a < t$, where $s, t \in (a-h, a+h) \cap D_f$ for some $h > 0$, then f is said to be strictly increasing at $x = a$.

(1) When 'a' be left end of the interval

$f(a) < f(x)\ \forall\ x \in (a, a+h) \cap D_f$ for some $h > 0$

$\Rightarrow$ f is strictly increasing at $x = a$.

(2) When 'a' be right end of the interval

$f(x) < f(a)\ \forall\ x \in (a-h, a) \cap D_f$ for some $h > 0$

$\Rightarrow$ f is strictly increasing at $x = a$.

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III. Decreasing at x = a :

If $f(s) \ge f(a) \ge f(t)$ when ever $s < a < t$, where $s, t \in (a-h, a+h) \cap D_f$ for some $h > 0$, then f is said to be decreasing at $x = a$.

(1) When 'a' be left end of the interval

$f(a) \ge f(x)\ \forall\ x \in (a, a+h) \cap D_f$ for some $h > 0$

$\Rightarrow$ f is decreasing at $x = a$.

(2) When 'a' be right end of the interval

$f(x) \ge f(a)\ \forall\ x \in (a-h, a) \cap D_f$ for some $h > 0$

$\Rightarrow$ f is decreasing at $x = a$.

IV. Strictly decreasing at x = a :

If $f(s) > f(a) > f(t)$ when ever $s < a < t$, where $s, t \in (a-h, a+h) \cap D_f$ for some $h > 0$, then f is said to be strictly decreasing at $x = a$.

(1) When 'a' be left end of the interval

$f(a) > f(x)\ \forall\ x \in (a, a+h) \cap D_f$ for some $h > 0$

$\Rightarrow$ f is strictly decreasing at $x = a$.

(2) When 'a' be right end of the interval

$f(x) > f(a)\ \forall\ x \in (a-h, a) \cap D_f$ for some $h > 0$

$\Rightarrow$ f is strictly decreasing at $x = a$.

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Strictly increasing at x = a | Strictly decreasing at x = a | Strictly increasing at x = a | Strictly increasing at x = a

2. INCREASING & DECREASING NATURE OF A FUNCTION OVER AN INTERVAL :

Consider an interval $I \subseteq D_f$

I. Increasing Over an Interval I :

$\forall\ x_1, x_2 \in I,\ x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$, then f is increasing over interval I.

II. Decreasing Over an Interval I :

$\forall\ x_1, x_2 \in I,\ x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$, then f is decreasing over interval I.

III. Strictly increasing over an Interval I :

$\forall\ x_1, x_2 \in I,\ x_1 < x_2 \Leftrightarrow f(x_1) < f(x_2)$, then f is strictly increasing over interval I.

IV. Strictly decreasing over an Interval I :

$\forall\ x_1, x_2 \in I,\ x_1 < x_2 \Leftrightarrow f(x_1) > f(x_2)$, then f is strictly decreasing over interval I.

3. MONOTONIC FUNCTION :

If a function is either increasing or decreasing over an interval then it is said to be monotonic function over the interval.

If a function is either strictly increasing or strictly decreasing over an interval then it is said to be strictly monotonic function over the interval.

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4. FOR DIFFERENTIABLE FUNCTIONS :

Consider an interval I ($\subseteq D_f$) that can be [a, b] or (a, b) or [a, b) or (a, b].

(1) $f'(x) > 0\ \forall\ x \in I \Rightarrow f$ is strictly increasing function over the interval I.

(2) $f'(x) \ge 0\ \forall\ x \in I \Rightarrow f$ is increasing function over the interval I.

(3) $f'(x) \ge 0\ \forall\ x \in I$ and $f'(x) = 0$ do not form any interval (that means $f'(x) = 0$ at discrete points) $\Rightarrow f$ is strictly increasing function over the interval I.

(4) $f'(x) < 0\ \forall\ x \in I \Rightarrow f$ is strictly decreasing function over the interval I.

(5) $f'(x) \le 0\ \forall\ x \in I \Rightarrow f$ is decreasing function over the interval I.

(6) $f'(x) \le 0\ \forall\ x \in I$ and $f'(x) = 0$ do not form any interval (that means $f'(x) = 0$ at discrete points) $\Rightarrow f$ is strictly decreasing function over the interval I.

Illustration 1 : Let $f(x) = x^3 - 3x + 2$. Examine the monotonicity of function at points x = 0, 1 & 2.

Solution : at x = 0,

$f(0) = 2,\ f(0 + h) = h(h - 3) + 2 < 2$

$f(0 - h) = h(3 - h) + 2 > 2$

$\Rightarrow f(0 - h) > f(0) > f(0 + h)$

f is decreasing or strictly decreasing at x = 0.

at x = 1, $f(1 - h) > f(1) < f(1 + h)$ neither increasing nor decreasing

Similarly at x = 2, $f(2 - h) < f(2) < f(2 + h)$ increasing at x = 2.

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Illustration 2 : Prove that the function $f(x)=\log\left(x^3+\sqrt{x^6+1}\right)$ is strictly increasing.

Solution : Now, $f(x)=\log\left(x^3+\sqrt{x^6+1}\right)$

$$f'(x)=\frac{1}{x^3+\sqrt{x^6+1}}\left(3x^2+\frac{6x^5}{2\sqrt{x^6+1}}\right)=\frac{3x^2}{\sqrt{x^6+1}}\geq 0$$

$\Rightarrow$ f(x) is strictly increasing.

Illustration 3 : Find the intervals of monotonicity of the function $y=x^2-\log_e|x|$, $(x\neq 0)$.

Solution : Let $y=f(x)=x^2-\log_e|x|$

$f'(x)=2x-\frac{1}{x}$; for all $x(x\neq 0)$

$$f'(x)=\frac{2x^2-1}{x} \Rightarrow f'(x)=\frac{\left(\sqrt{2}x-1\right)\left(\sqrt{2}x+1\right)}{x}$$

So $f'(x)\geq 0$ when $x\in\left[-\frac{1}{\sqrt{2}},0\right)\cup\left[\frac{1}{\sqrt{2}},\infty\right)$ and $f'(x)\leq 0$

when $x\in\left(-\infty,-\frac{1}{\sqrt{2}}\right]\cup\left(0,\frac{1}{\sqrt{2}}\right]$

$\therefore$ f(x) is strictly increasing when $x\in\left[-\frac{1}{\sqrt{2}},0\right);\left[\frac{1}{\sqrt{2}},\infty\right)$

and strictly decreasing when $x\in\left(-\infty,-\frac{1}{\sqrt{2}}\right];\left(0,\frac{1}{\sqrt{2}}\right]$

Do yourself - 1 :

(i) If function $f(x)=x^3+\lambda x^2-\lambda x+1$ is increasing at $x=0$ & decreasing at $x=1$, then find the greatest integral value of λ.

(ii) If $f(x)=\sin x+\ell n|\sec x+\tan x|-2x$ for $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ then check the monotonicity of $f(x)$

(iii) Prove that $y=e^x+\sin x$ is strictly increasing in $x\in R^+$

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5. GREATEST AND LEAST VALUE OF A FUNCTION :

(a) If a continuous function $y = f(x)$ is increasing in the closed interval [a, b], then $f(a)$ is the least value and $f(b)$ is the greatest value of $f(x)$ in [a, b] (figure-1)

figure-1

(b) If a continuous function $y = f(x)$ is decreasing in [a, b], then $f(b)$ is the least and $f(a)$ is the greatest value of $f(x)$ in [a, b]. (figure-2)

figure-2

(c) If a continuous function $y = f(x)$ is increasing/decreasing in the (a,b), then no greatest and least value exist.

Illustration 4 : Show that $f(x) = \sin^{-1}\frac{x}{\sqrt{1+x^2}} - \ell nx$ is strictly decreasing in $x \in \left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$. Also find its range.

Solution : $f(x) = \sin^{-1}\frac{x}{\sqrt{1+x^2}} - \ell nx = \tan^{-1}x - \ell nx \Rightarrow f'(x) = \frac{1}{1+x^2} - \frac{1}{x} = \frac{-(1+x^2-x)}{x(1+x^2)}$

$\therefore \quad f'(x) \le 0 \ \forall \ x \in \left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$

$\Rightarrow$ f(x) is strictly decreasing .

$f(x)\big|_{max} = f\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} + \frac{1}{2}\ell n3 \quad \& \quad f(x)\big|_{min} = f\left(\sqrt{3}\right) = \frac{\pi}{3} - \frac{1}{2}\ell n3$

$\therefore \quad$ Range of $f(x) = \left[\frac{\pi}{3} - \frac{1}{2}\ell n3, \ \frac{\pi}{6} + \frac{1}{2}\ell n3\right]$ **Ans.**

Illustration 5 : Find the greatest and least value of $f(x) = x^3 + 5x + e^x$ in [1, 3]

Solution : $f'(x) = 3x^2 + 5 + e^x \Rightarrow f(x)$ is strictly increasing.

Least value $= f(1) = 6 + e$

greatest value $= f(3) = (42 + e^3)$

---

6. PROVING INEQUALITIES USING MONOTONICITY :

Comparison of two functions f(x) and g(x) can be done by analysing their monotonic behaviour.

Illustration 6 : Let f (x) and g (x) are two function which are defined and differentiable for all $x \ge x_0$. If $f(x_0) = g(x_0)$ and $f'(x) > g'(x)$ for all $x > x_0$ then

(A) $f(x) < g(x)$ for some $x > x_0$ (B) $f(x) = g(x)$ for some $x > x_0$

(C) $f(x) > g(x)$ only for some $x > x_0$ (D) $f(x) > g(x)$ for all $x > x_0$

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Solution : Let $h(x) = f(x) - g(x)$

$h'(x) = f'(x) - g'(x) > 0 \ \forall \ x > x_0$

$h(x_0) = 0$

$\Rightarrow \ h(x) > 0 \ \forall \ x > x_0$

Ans. (D)

Do yourself - 2 :

(i) Let $f(x) = \frac{x^3}{3} - \frac{x^2}{2} + 2$ in [–2, 2]. Find the greatest and least value of $f(x)$ in [–2, 2]

(ii) Prove that $x > \sin x > x - \frac{x^3}{6}$ for all $x > 0$.

7. CONCAVITY OR CONVEXITY OF FUNCTION :

Convex function (concave upward) :

If $f(rx_1 + (1-r)x_2) \le rf(x_1) + (1-r)f(x_2)$

$\forall \ x_1 , x_2 \in I$ and $\forall \ r \in [0, 1]$

then f is said to be convex function over I (or concave upward).

Strictly Convex function (strictly concave upward) :

If $f(rx_1 + (1-r)x_2) < rf(x_1) + (1-r)f(x_2) \ \forall \ x_1 , x_2 \in I$ where $x_1 \ne x_2$ and $\forall \ r \in (0, 1)$

then f is said to be strictly convex function over I (or strictly concave upward).

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Concave function (convex upward) :

If $f(rx_1 + (1-r)x_2) \ge rf(x_1) + (1-r)f(x_2)$

$\forall \ x_1 , x_2 \in I$ and $\forall \ r \in [0, 1]$

then f is said to be concave function over I (or convex upward).

Strictly Concave function (Strictly convex upward) :

If $f(rx_1 + (1-r)x_2) > rf(x_1) + (1-r)f(x_2) \ \forall \ x_1 , x_2 \in I$ where $x_1 \ne x_2$ and $\forall \ r \in (0, 1)$

then f is said to be Strictly concave function over I (or Strictly convex upward).

Note :

1. Straight line or linear function is said to be concave up as well as concave down.

2. The sign of the 2nd order derivative determines the concavity of the curve.

i.e. If $f''(x) \ge 0 \ \forall \ x \in (a, b)$ then graph of f(x) is concave upward in (a, b).

Similarly if $f''(x) \le 0 \ \forall \ x \in (a, b)$ then graph of f(x) is concave downward in (a, b).

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8. ROLLE'S THEOREM :

Let f be a function that satisfies the following three conditions:

(a) f is continuous on the closed interval [a, b].

(b) f is differentiable on the open interval (a, b)

(c) f(a) = f(b)

Then there exist at least one number c in (a, b) such that f '(c) = 0.

Note : If f is differentiable function then between any two consecutive roots of f(x) = 0, there is atleast one root of the equation f '(x) = 0.

(d) Geometrical Interpretation :

Geometrically, the Rolle's theorem says that somewhere between A and B the curve has at least one tangent parallel to x-axis.

Illustration 7 : Verify Rolle's theorem for the function $f(x) = x^3 - 3x^2 + 2x$ in the interval [0, 2].

Solution : Here we observe that

(a) f(x) is polynomial and since polynomial are always continuous, as well as differentiable. Hence f(x) is continuous in the [0,2] and differentiable in the (0, 2).

&

(b) $f(0) = 0$, $f(2) = 2^3 - 3.(2)^2 + 2(2) = 0$

$\therefore$ $f(0) = f(2)$

Thus, all the condition of Rolle's theorem are satisfied.

So, there must exists some $c \in (0, 2)$ such that $f'(c) = 0$

$$\Rightarrow \quad f'(c) = 3c^2 - 6c + 2 = 0 \Rightarrow c = 1 \pm \frac{1}{\sqrt{3}}$$

where both $c = 1 \pm \frac{1}{\sqrt{3}} \in (0, 2)$ thus Rolle's theorem is verified.

Illustration 8 : Let Rolle's theorem holds for $f(x) = x^3 + bx^2 + ax$, when $1 \le x \le 2$ at the point $c = \frac{4}{3}$, then find a + b.

Solution : $f(1) = f(2) \Rightarrow 1 + b + a = 8 + 4b + 2a$

$a + 3b + 7 = 0$ .....(1)

$f'(c) = 3x^2 + 2bx + a = 0$

$\frac{16}{3} + \frac{8b}{3} + a = 0 \Rightarrow 3a + 8b + 16 = 0$ .....(2)

By solving a = 8, b = –5

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Do yourself - 3 :

(i) Verify Rolle's theorem for $y = 1 - (x^4)^{1/3}$ on the interval $[-1,1]$

(ii) (a) Let $f(x) = 1 - (x^2)^{1/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1,1)$ such that $f'(c) = 0$.
Why does this not contradict Rolle's Theorem ?

(b) Let $f(x) = (x-1)^{-2}$. Show that $f(0) = f(2)$ but there is no number c in $(0,2)$ such that $f'(c) = 0$.
Why does this not contradict Rolle's Theorem ?

9. LAGRANGE'S MEAN VALUE THEOREM (LMVT) :

Let f be a function that satisfies the following conditions:

(i) f is continuous in [a, b]

(ii) f is differentiable in (a, b).

Then there is a number c in (a, b) such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

(a) Geometrical Interpretation :

Geometrically, the Mean Value Theorem says that somewhere between A and B the curve has at least one tangent parallel to chord AB.

(b) Physical Interpretations :

If we think of the number $(f(b) - f(a))/(b - a)$ as the average change in f over [a, b] and $f'(c)$ as an instantaneous change, then the Mean Value Theorem says that at some interior point the instantaneous change must equal the average change over the entire interval.

Illustration 9 : Find c of the Lagrange's mean value theorem for the function $f(x) = 3x^2 + 5x + 7$ in the interval [1, 3].

Solution : Given $f(x) = 3x^2 + 5x + 7$ ..... (i)

$\Rightarrow$ $f(1) = 3 + 5 + 7 = 15$ and $f(3) = 27 + 15 + 7 = 49$

Again $f'(x) = 6x + 5$

Here $a = 1, b = 3$

Now from Lagrange's mean value theorem

$$f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow 6c + 5 = \frac{f(3)-f(1)}{3-1} = \frac{49-15}{2} = 17 \text{ or } c = 2.$$

Illustration 10 : If f(x) is continuous and differentiable over $[-2, 5]$ and $-4 \le f'(x) \le 3$ for all x in $(-2, 5)$, then the greatest possible value of $f(5) - f(-2)$ is -

(A) 7 (B) 9 (C) 15 (D) 21

Solution : Apply LMVT

$$f'(x) = \frac{f(5)-f(-2)}{5-(-2)} \text{ for some x in } (-2, 5)$$

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Now, $-4 \le \dfrac{f(5)-f(-2)}{7} \le 3$

$-28 \le f(5) - f(-2) \le 21$

$\therefore$ Greatest possible value of $f(5) - f(-2)$ is 21.

Do yourself - 4 :

(i) If $f(x) = x^2$ in [a, b], then show that there exist atleast one c in (a, b) such that a, c, b are in A.P.

(ii) Find C of LMVT for $f(x) = |x|^3$ in [2, 5].

10. SPECIAL NOTE :

Use of Monotonicity in identifying the number of roots of the equation in a given interval. Suppose a and b are two real numbers such that,

(a) Let f(x) is differentiable & either strictly increasing or strictly decreasing for $a \le x \le b$.

&

(b) f(a) and f(b) have opposite signs.

Then there is one & only one root of the equation f(x) = 0 in (a, b).

Miscellaneous Illustrations :

Illustration 11 : If $g(x) = f(x) + f(1 - x)$ and $f''(x) < 0$; $0 \le x \le 1$, show that g(x) strictly increasing in $x \in (0, 1/2)$ and strictly decreasing in $x \in (1/2, 1)$

Solution : $\therefore$ $f''(x) < 0 \Rightarrow f'(x)$ is strictly decreasing function.

Now, $g(x) = f(x) + f(1 - x)$ $\therefore$ $g'(x) = f'(x) - f'(1 - x)$ ..... (i)

Case I : If $x > (1 - x) \Rightarrow$ $x > 1/2$

$\therefore$ $f'(x) < f'(1 - x) \Rightarrow f'(x) - f'(1 - x) < 0$

$\Rightarrow g'(x) < 0$

$\therefore$ g(x) strictly decreases in $x \in \left(\dfrac{1}{2}, 1\right)$

Case II : If $x < (1 - x) \Rightarrow x < 1/2$

$\therefore$ $f'(x) > f'(1 - x) \Rightarrow f'(x) - f'(1 - x) > 0$

$\Rightarrow g'(x) > 0$

$\therefore$ g(x) strictly increases in $x \in (0, 1/2)$

Illustration 12 : Which of the following functions are strictly decreasing on $\left(0, \dfrac{\pi}{2}\right)$

(A) cosx (B) cos2x (C) cos3x (D) tanx

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Solution :

$f(x) = \cos x$

$f(x) = \cos 2x$

$f(x) = \cos 3x$

Non monotonic

$f(x) = \tan x$ is strictly increasing in $\left(0, \frac{\pi}{2}\right)$

Option A & B are correct.

Illustration 13 : Prove that the equation $e^{(x-1)} + x = 2$ has one solution.

Solution : Let $f(x) = e^{(x-1)} + x$

$f'(x) = e^{(x-1)} + 1$

$f(x)$ is strictly increasing funtion

$\lim_{x\to\infty} f(x) = \infty$ & $\lim_{x\to-\infty} f(x) = 0$

$f(x) = 2$ has exactly one solution.

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1 : **(i)** –4

(ii) Strictly increasing

2 : **(i)** Greatest is $\frac{8}{3}$ and least value is $-\frac{8}{3}$.

3 : **(i)** Rolles theorem is valid

(ii) (a) $f(x)$ is non-differentiable at $x = 0$ in $(-1,1)$ (b) $f(x)$ is discontinuous at $x = 1$ in $(0,2)$

4 : **(ii)** $C = \sqrt{\frac{117}{9}}$

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MONOTONICITY

EXERCISE (O-1)

1. Let f (x) and g (x) be two continuous functions defined from R → R, such that $f(x_1) > f(x_2)$ and $g(x_1) < g(x_2)$, $\forall\ x_1 > x_2$, then solution set of $f\left(g(\alpha^2 - 2\alpha)\right) > f\left(g(3\alpha - 4)\right)$ is

(A) R (B) ϕ (C) (1, 4) (D) R – [1, 4]

MN0001

2. When $0 \le x \le 1$, $f(x) = |x| + |x - 1|$ is-

(A) strictly increasing (B) strictly decreasing (C) constant (D) None of these

MN0002

3. Complete set of values of K in order that f(x) = sinx – cosx – Kx + b decreases for all real values is given by-

(A) K < 1 (B) K ≥ 1 (C) $K \ge \sqrt{2}$ (D) $K < \sqrt{2}$

MN0003

4. Function $f(x) = x^2(x-2)^2$ is-

(A) increasing in [0, 1]; [2, ∞) (B) decreasing in [0, 1]; [2, ∞]

(C) decreasing function (D) increasing function

MN0004

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5. If f and g are two strictly decreasing function such that fog is defined, then fog will be-

(A) strictly increasing function (B) strictly decreasing function

(C) neither increasing nor decreasing (D) None of these

MN0005

6. If the function $f(x) = 2x^2 - kx + 5$ is strictly increasing in [1, 2], then complete set of values of k is

(A) (–∞, 4) (B) (4, ∞) (C) (–∞, 8] (D) (8, ∞)

MN0006

7. The function x^x strictly decreases on the interval-

(A) (0, e) (B) (0, 1) (C) $\left(0, \frac{1}{e}\right)$ (D) None of these

MN0007

8. If function $f(x) = 2x^2 + 3x - m\log x$ is monotonic decreasing in the interval (0, 1), then the least value of the parameter m is-

(A) 7 (B) $\frac{15}{2}$ (C) $\frac{31}{4}$ (D) 8

MN0008

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9. If $f(x) = x^3 - 10x^2 + 200x - 10$, then f(x) is-

(A) strictly decreasing in $(-\infty, 10]$ and strictly increasing in $[10, \infty)$

(B) strictly increasing in $(-\infty, 10]$ and strictly decreasing in $[10, \infty)$

(C) strictly increasing for every value of x

(D) strictly decreasing for every value of x

MN0009

10. Which one of the following statements does not hold good for the function $f(x) = \cos^{-1}(2x^2 - 1)$?

(A) f is not differentiable at x = 0 (B) f is monotonic

(C) f is even (D) f has an extremum

MN0010

11. The function $f(x) = \tan^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is -

(A) strictly increasing in its domain

(B) strictly decreasing in its domain

(C) strictly decreasing in $(-\infty, 0)$ and strictly increasing in $(0, \infty)$

(D) strictly increasing in $(-\infty, 0)$ and strictly decreasing in $(0, \infty)$

MN0011

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12. The function $f : [a, \infty) \to R$ where R denotes the range corresponding to the given domain, with rule $f(x) = 2x^3 - 3x^2 + 6$, will have an inverse provided

(A) $a \geq 1$ (B) $a \geq 0$ (C) $a \leq 0$ (D) $a \leq 1$

MN0012

13. If $2a + 3b + 6c = 0$, then at least one root of the equation $ax^2 + bx + c = 0$ lies in the interval-

(A) (0, 1) (B) (1, 2) (C) (2, 3) (D) none

MN0013

14. If the equation $a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x = 0$ has a positive root $x = \alpha$, then the equation $na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + ... + a_1 = 0$ has a positive root, which is-

(A) Smaller than α (B) Greater than α

(C) Equal to α (D) Greater than or equal to α

MN0014

15. A value of c for which the conclusion of Mean values theorem holds for the function $f(x) = \log_e x$ on the interval [1, 3] is-

(A) $2\log_3 e$ (B) $\frac{1}{2}\log_e 3$ (C) $\log_3 e$ (D) $\log_e 3$

MN0015

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16. The value of c in Lagrange's theorem for the function $f(x) = \begin{cases} x\cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ in the interval $[-1, 1]$ is-

(A) 0 (B) $\frac{1}{2}$

(C) $-\frac{1}{2}$ (D) Non existent in the interval

MN0016

17. If the function $f(x) = x^3 - 6x^2 + ax + b$ defined on [1, 3], satisfies the rolle's theorem for $c = \frac{2\sqrt{3}+1}{\sqrt{3}}$, then-

(A) a = 11, b = 6 (B) a = –11, b = 6 (C) a = 11, b ∈ R (D) None of these

MN0017

18. Given: $f(x) = 4 - \left(\frac{1}{2} - x\right)^{2/3}$ $g(x) = \begin{cases} \frac{\tan[x]}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

$h(x) = \{x\}$ $k(x) = 5^{\log_2(x+3)}$

then in [0, 1], Lagranges Mean Value Theorem is NOT applicable to

(A) f, g, h (B) h, k (C) f, g (D) g, h, k

where [x] and {x} denotes the greatest integer and fractional part function.

MN0018

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19. Given $f'(1) = 1$ and $\frac{d}{dx}(f(2x)) = f'(x)\ \forall\ x > 0$. If $f'(x)$ is differentiable then there exists a number $c \in (2, 4)$ such that $f''(c)$ equals

(A) – 1/4 (B) – 1/8 (C) 1/4 (D) 1/8

MN0019

20. If the function $f(x) = 2x^2 + 3x + 5$ satisfies LMVT at x = 2 on the closed interval [1, a], then the value of 'a' is equal to

(A) 3 (B) 4 (C) 6 (D) 1

MN0020

EXERCISE (O-2)

[SINGLE CORRECT CHOICE TYPE]

1. If $f : R \to R^+$ be a differentiable function and $g(x) = e^{2x}(2f(x) - 3(f(x))^2 + 2(f(x))^3)\ \forall\ x \in R$, then which of the following is/are always correct -

(A) g(x) is strictly increasing wherever $f(x)$ is strictly increasing

(B) g(x) is strictly increasing wherever $f(x)$ is strictly decreasing

(C) g(x) is strictly decreasing wherever $f(x)$ is strictly decreasing

(D) g(x) is strictly decreasing wherever $f(x)$ is strictly increasing

MN0021

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2. The equation sin x + x cos x = 0 has at least one root in

(A) $\left(-\frac{\pi}{2}, 0\right)$ (B) $(0, \pi)$ (C) $\left(\pi, \frac{3\pi}{2}\right)$ (D) $\left(0, \frac{\pi}{2}\right)$

MN0022

3. Let a, b, c, d are non-zero real numbers such that 6a + 4b + 3c + 3d = 0, then the equation $ax^3 + bx^2 + cx + d = 0$ has

(A) atleast one root in [–2, 0]
(B) atleast one root in [0, 2]
(C) atleast two roots in [–2, 2]
(D) no root in [–2, 2]

MN0023

[MULTIPLE CORRECT CHOICE TYPE]

4. Which of the following is/are correct ?

(A) Between any two roots of $e^x \cos x = 1$, there exists atleast one root of tanx = 1.

(B) Between any two roots of $e^x \sin x = 1$, there exists atleast one root of tanx = –1.

(C) Between any two roots of $e^x \cos x = 1$, there exists atleast one root of $e^x \sin x = 1$.

(D) Between any two roots of $e^x \sin x = 1$, there exists atleast one root of $e^x \cos x = -1$.

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MN0024

5. Let $f(x) = \sin^2 x$ & $0 < A < B$, where $A, B \in \left(0, \frac{\pi}{4}\right)$, then

(A) $(B - A)\sin 2A < \sin^2 B - \sin^2 A < (B - A)\sin 2B$

(B) $(B - A)\sin 2A > \sin^2 B - \sin^2 A > (B - A)\sin 2B$

(C) $\cos 2B < \cos 2A$

(D) $B\cos 2A > A\cos 2B$

MN0025

6. For the equation $\frac{e^{-x}}{x+1} = a$; which of the following statement(s) is/are correct ?

(A) If $a \in (0,\infty)$, then equation has 2 real and distinct roots
(B) If $a \in (-\infty,-e^2)$, then equation has 2 real & distinct roots.
(C) If $a \in (0,\infty)$, then equation has 1 real root
(D) If $a \in (-e,0)$, then equation has no real root.

MN0026

7. The number of roots of the equation $x^2 - 2x - \log_2|1 - x| = 3$ is

(A) 4 (B) 2 (C) 1 (D) 0

MN0027

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EXERCISE (S-1)

1. Find the intervals of monotonicity for the following functions & represent your solution set on the number line.

(a) $f(x) = 2.\ e^{x^2-4x}$ **MN0028**

(b) $f(x) = e^x/x$ **MN0029**

(c) $f(x) = x^2 e^{-x}$ **MN0030**

(d) $f(x) = 2x^2 - \ln|x|$ **MN0031**

Also plot the graphs in each case & state their range.

2. Find the intervals of monotonocity of the functions in $[0, 2\pi]$

(a) $f(x) = \sin x - \cos x$ in $x \in [0, 2\pi]$ **MN0032**

(b) $g(x) = 2\sin x + \cos 2x$ in $(0 \le x \le 2\pi)$. **MN0033**

(c) $f(x) = \dfrac{4\sin x - 2x - x\cos x}{2 + \cos x}$ **MN0034**

3. Let $f(x) = 1 - x - x^3$. Find all real values of x satisfying the inequality, $1 - f(x) - f^3(x) > f(1 - 5x)$ **MN0035**

4. Find the greatest & the least values of the following functions in the given interval if they exist.

(a) $f(x) = \sin^{-1}\dfrac{x}{\sqrt{x^2+1}} - \ln x$ in $\left[\dfrac{1}{\sqrt{3}}, \sqrt{3}\right]$ **MN0036**

(b) $f(x) = 12x^{4/3} - 6x^{1/3}$, $x \in [-1, 1]$ **MN0037**

(c) $y = x^5 - 5x^4 + 5x^3 + 1$ in $[-1, 2]$ **MN0038**

5. If $f(x) = 2e^x - ae^{-x} + (2a+1)x - 3$ increases for every $x \in R$ then find the range of values of 'a'. **MN0039**

6. If $b > a$, find the minimum value of $|(x-a)^3| + |(x-b)^3|$, $x \in R$. **MN0040**

7. Find the set of values of x for which the inequality $\ln(1+x) > x/(1+x)$ is valid. **MN0041**

8. (a) Let f, g be differentiable on R and suppose that $f(0) = g(0)$ and $f'(x) \le g'(x)$ for all $x \ge 0$. Show that $f(x) \le g(x)$ for all $x \ge 0$. **MN0042**

(b) Show that exactly two real values of x satisfy the equation $x^2 = x\sin x + \cos x$. **MN0043**

(c) Prove that inequality $e^x > (1 + x)$ for all $x \in R_0$ and use it to determine which of the two numbers e^π and π^e is greater. **MN0044**

9. Verify Rolles theorem for $f(x) = (x-a)^m (x-b)^n$ on $[a, b]$; m, n being positive integer. **MN0045**

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10. Let $f(x) = 4x^3 - 3x^2 - 2x + 1$, use Rolle's theorem to prove that there exist c, $0 < c < 1$ such that $f(c) = 0$.

MN0046

11. Assume that f is continuous on [a, b], $a > 0$ and differentiable on an open interval (a, b). Show that if $\frac{f(a)}{a} = \frac{f(b)}{b}$, then there exist $x_0 \in (a, b)$ such that $x_0 f'(x_0) = f(x_0)$.

MN0047

12. f(x) and g(x) are differentiable functions for $0 \le x \le 2$ such that $f(0) = 5$, $g(0) = 0$, $f(2) = 8$, $g(2) = 1$. Show that there exists a number c satisfying $0 < c < 2$ and $f'(c) = 3g'(c)$.

MN0048

13. For what value of a, m and b does the function $f(x) = \begin{cases} 3 & x = 0 \\ -x^2 + 3x + a & 0 < x < 1 \\ mx + b & 1 \le x \le 2 \end{cases}$

satisfy the hypothesis of the mean value theorem for the interval [0, 2].

MN0049

14. Let f be continuous on [a, b] and assume the second derivative f '' exists on (a, b). Suppose that the graph of f and the line segment joining the point $(a, f(a))$ and $(b, f(b))$ intersect at a point $(x_0, f(x_0))$ where $a < x_0 < b$. Show that there exists a point $c \in (a, b)$ such that $f''(c) = 0$.

MN0050

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EXERCISE (S-2)

1. Find the values of 'a' for which the function $f(x) = \sin x - a\sin 2x - \frac{1}{3}\sin 3x + 2ax$ strictly increases throughout the number line.

MN0051

2. Find the set of values of 'a' for which the function,

$f(x) = \left(1 - \frac{\sqrt{21 - 4a - a^2}}{a + 1}\right) x^3 + 5x + \sqrt{7}$ is strictly increasing at every point of its domain.

MN0052

3. Let $f(x)$ be a strictly increasing function defined on $(0, \infty)$. If $f(2a^2 + a + 1) > f(3a^2 - 4a + 1)$. Find the range of a.

MN0053

4. Find the minimum value of the function $f(x) = x^{3/2} + x^{-3/2} - 4\left(x + \frac{1}{x}\right)$ for all permissible real x.

MN0054

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5. Prove that, $x^2 - 1 > 2x\, ln\, x > 4(x - 1) - 2\, ln\, x$ for $x > 1$.

MN0055

6. If f, ϕ, ψ are continuous in [a, b] and derivable in]a, b[then show that there is a value of c lying between a & b such that,

$$\begin{vmatrix} f(a) & f(b) & f'(c) \\ \phi(a) & \phi(b) & \phi'(c) \\ \Psi(a) & \Psi(b) & \Psi'(c) \end{vmatrix} = 0$$

MN0056

7. Prove that if f is differentiable on [a, b] and if $f(a) = f(b) = 0$ then for any real α there is an $x \in (a, b)$ such that $\alpha f(x) + f'(x) = 0$.

MN0057

8. Let $a > 0$ and f be continuous in [–a, a]. Suppose that $f'(x)$ exists and $f'(x) \le 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, show that $f(0) = 0$.

MN0058

9. Let f be continuous on [a, b] and differentiable on (a, b). If $f(a) = a$ and $f(b) = b$, show that there exist distinct c_1, c_2 in (a, b) such that $f'(c_1) + f'(c_2) = 2$.

MN0059

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10. Using LMVT prove that : **(a)** $\tan x > x$ in $\left(0, \frac{\pi}{2}\right)$, **(b)** $\sin x < x$ for $x > 0$

MN0060

EXERCISE (JM)

1. A function is matched below against an interval where it is supposed to be increasing. which of the following pairs is incorrectly matched ? **[AIEEE-2005]**

interval	function
(1) $(-\infty, \infty)$	$x^3 - 3x^2 + 3x + 3$
(2) $[2, \infty)$	$2x^3 - 3x^2 - 12x + 6$
(3) $\left(-\infty, \frac{1}{3}\right]$	$3x^2 - 2x + 1$
(4) $(-\infty, -4)$	$x^3 + 6x^2 + 6$

MN0061

2. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in- **[AIEEE-2007]**

(1) $(\pi/4, \pi/2)$ (2) $(-\pi/2, \pi/4)$ (3) $(0, \pi/2)$ (4) $(-\pi/2, \pi/2)$

MN0062

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3. **Statement 1 :** The function $x^2(e^x + e^{-x})$ is increasing for all $x > 0$. **[JEE-MAIN Online 2013]**

Statement 2 : The functions x^2e^x and x^2e^{-x} are increasing for all $x > 0$ and the sum of two increasing functions in any interval (a, b) is an increasing function in (a, b).

(1) Statement 1 is false; Statement 2 is true.

(2) Statement 1 is true; Statement 2 is false.

(3) Statement 1 is true; Statement 2 is true; Statement 2 is a correct explanation for Statement 1.

(4) Statement 1 is true ; Statement 2 is true; Statement 2 is not a correct explanation for Statement 1.

MN0063

4. If f and g are differentiable functions in [0, 1] satisfying f(0) = 2 = g(1), g(0) = 0 and f(1) = 6, then for some $c \in]0, 1[$: **[JEE-MAIN 2014]**

(1) $2f'(c) = g'(c)$ (2) $2f'(c) = 3g'(c)$

(3) $f'(c) = g'(c)$ (4) $f'(c) = 2g'(c)$

MN0064

5. If the function *f* given by $f(x) = x^3 - 3(a-2)x^2 + 3ax + 7$, for some $a \in R$ is increasing in (0, 1] and decreasing in [1, 5), then a root of the equation, $\frac{f(x)-14}{(x-1)^2} = 0 (x \neq 1)$ is : **[JEE-MAIN 2019]**

(1) 6 (2) 5 (3) 7 (4) –7

MN0065

6. Let $f(x) = e^x - x$ and $g(x) = x^2 - x$, $\forall x \in R$. Then the set of all $x \in R$, where the function $h(x) = (fog)(x)$ is increasing, is : **[JEE-MAIN 2019]**

(1) $\left[-1, \frac{-1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$ (2) $\left[0, \frac{1}{2}\right] \cup [1, \infty)$ (3) $\left[\frac{-1}{2}, 0\right] \cup [1, \infty)$ (4) $[0, \infty)$

MN0066

7. If m is the minimum value of k for which the function $f(x) = x\sqrt{kx - x^2}$ is increasing in the interval [0, 3] and M is the maximum value of f in [0, 3] when k = m, then the ordered pair (m, M) is equal to : **[JEE-MAIN 2019]**

(1) $(4, 3\sqrt{2})$ (2) $(4, 3\sqrt{3})$ (3) $(3, 3\sqrt{3})$ (4) $(5, 3\sqrt{6})$

MN0067

8. The value of c in the Lagrange's mean value theorem for the function $f(x) = x^3 - 4x^2 + 8x + 11$, when $x \in [0, 1]$ is : **[JEE-MAIN 2020]**

(1) $\frac{2}{3}$ (2) $\frac{\sqrt{7}-2}{3}$ (3) $\frac{4-\sqrt{5}}{3}$ (4) $\frac{4-\sqrt{7}}{3}$

MN0068

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9. Let the function, $f : [-7, 0] \to R$ be continuous on $[-7, 0]$ and differentiable on $(-7, 0)$. If $f(-7) = -3$ and $f'(x) \le 2$, for all $x \in (-7, 0)$, then for all such functions f, $f(-1) + f(0)$ lies in the interval : **[JEE-MAIN 2020]**

(1) $[-6, 20]$ (2) $(-\infty, 20]$ (3) $(-\infty, 11]$ (4) $[-3, 11]$

MN0069

10. Let $f(x) = x\cos^{-1}(-\sin|x|)$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then which of the following is true ? **[JEE-MAIN 2020]**

(1) f' is decreasing in $\left(-\frac{\pi}{2}, 0\right)$ and increasing in $\left(0, \frac{\pi}{2}\right)$

(2) f is not differentiable at $x = 0$

(3) $f'(0) = -\frac{\pi}{2}$

(4) f' is increasing in $\left(-\frac{\pi}{2}, 0\right)$ and decreasing in $\left(0, \frac{\pi}{2}\right)$

MN0070

11. Let f be any function continuous on $[a, b]$ and twice differentiable on (a, b). If for all $x \in (a, b)$, $f'(x) > 0$ and $f''(x) < 0$, then for any $c \in (a, b)$, $\frac{f(c) - f(a)}{f(b) - f(c)}$ is greater than: **[JEE-MAIN 2020]**

(1) $\frac{b+a}{b-a}$ (2) $\frac{b-c}{c-a}$ (3) $\frac{c-a}{b-c}$ (4) 1

MN0071

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EXERCISE (JA)

1. Let $f : (0,1) \to R$ be defined by $f(x) = \frac{b-x}{1-bx}$, where b is a constant such that $0 < b < 1$. Then

(A) f is not invertible on $(0,1)$

(B) $f \ne f^{-1}$ on $(0,1)$ and $f'(b) = \frac{1}{f'(0)}$

(C) $f = f^{-1}$ on $(0,1)$ and $f'(b) = \frac{1}{f'(0)}$

(D) f^{-1} is differentiable on $(0,1)$ **[JEE 2011, 4M]**

MN0072

2. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is **[JEE 2011, 4M]**

MN0073

3. The number of points in $(-\infty, \infty)$, for which $x^2 - x\sin x - \cos x = 0$, is **[JEE 2013, 2M]**

(A) 6 (B) 4 (C) 2 (D) 0

MN0074

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4. Let $f(x) = x \sin \pi x$, $x > 0$. Then for all natural numbers n, $f'(x)$ vanishes at - **[JEE 2013, 4M, –1M]**

(A) a unique point in the interval $\left(n, n+\frac{1}{2}\right)$ (B) a unique point in the interval $\left(n+\frac{1}{2}, n+1\right)$

(C) a unique point in the interval (n, n + 1) (D) two points in the interval (n, n + 1)

MN0075

5. Let $a \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^5 - 5x + a$. Then **[JEE(Advanced)-2014, 3]**

(A) $f(x)$ has three real roots if $a > 4$ (B) $f(x)$ has only one real roots if $a > 4$

(C) $f(x)$ has three real roots if $a < -4$ (D) $f(x)$ has three real roots if $-4 < a < 4$

MN0076

6. Let f, $g : [-1, 2] \to \mathbb{R}$ be continuous function which are twice differentiable on the interval (–1, 2). Let the values of f and g at the points –1, 0 and 2 be as given in the following table :

	$x = -1$	$x = 0$	$x = 2$
$f(x)$	3	6	0
$g(x)$	0	1	–1

[JEE 2015, 4M, –2M]

In each of the intervals (–1, 0) and (0, 2) the function $(f - 3g)''$ never vanishes. Then the correct statement(s) is(are)

(A) $f'(x) - 3g'(x) = 0$ has exactly three solutions in $(-1, 0) \cup (0, 2)$

(B) $f'(x) - 3g'(x) = 0$ has exactly one solution in (–1, 0)

(C) $f'(x) - 3g'(x) = 0$ has exactly one solutions in (0, 2)

(D) $f'(x) - 3g'(x) = 0$ has exactly two solutions in (–1, 0) and exactly two solutions in (0, 2)

MN0077

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Answer Q.7, Q.8 and Q.9 by appropriately matching the information given in the three columns of the following table. **[JEE(Advanced)-2017]**

Let $f(x) = x + \log_e x - x \log_e x$, $x \in (0,\infty)$.

* Column 1 contains information about zeros of $f(x)$, $f'(x)$ and $f''(x)$.
* Column 2 contains information about the limiting behavior of $f(x)$, $f'(x)$ and $f''(x)$ at infinity.
* Column 3 contains information about increasing/decreasing nature of $f(x)$ and $f'(x)$.

Column 1	Column 2	Column 3
(I) $f(x) = 0$ for some $x \in (1,e^2)$	(i) $\lim_{x\to\infty} f(x) = 0$	(P) f is increasing in (0,1)
(II) $f'(x) = 0$ for some $x \in (1,e)$	(ii) $\lim_{x\to\infty} f(x) = -\infty$	(Q) f is decreasing in (e,e^2)
(III) $f'(x) = 0$ for some $x \in (0,1)$	(iii) $\lim_{x\to\infty} f'(x) = -\infty$	(R) f' is increasing in (0,1)
(IV) $f''(x) = 0$ for some $x \in (1,e)$	(iv) $\lim_{x\to\infty} f''(x) = 0$	(S) f' is decreasing in (e,e^2)

7. Which of the following options is the only **CORRECT** combination ?

(A) (IV) (i) (S) (B) (I) (ii) (R) (C) (III) (iv) (P) (D) (II) (iii) (S)

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8. Which of the following options is the only **CORRECT** combination ?

(A) (III) (iii) (R) (B) (I) (i) (P) (C) (IV) (iv) (S) (D) (II) (ii) (Q)

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9. Which of the following options is the only **INCORRECT** combination ?

(A) (II) (iii) (P) (B) (II) (iv) (Q) (C) (I) (iii) (P) (D) (III) (i) (R)

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10. If $f:\mathbb{R}\to\mathbb{R}$ is a twice differentiable function such that $f''(x)>0$ for all $x\in\mathbb{R}$, and $f\left(\frac{1}{2}\right)=\frac{1}{2}$, $f(1)=1$, then **[JEE(Advanced)-2017]**

(A) $0<f'(1)\le\frac{1}{2}$ (B) $f'(1)\le 0$ (C) $f'(1)>1$ (D) $\frac{1}{2}<f'(1)\le 1$

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ANSWER KEY

(MONOTONICITY)

EXERCISE (O-1)

1. C	**2.** C	**3.** C	**4.** A	**5.** A	**6.** A	**7.** C	**8.** A
9. C	**10.** B	**11.** D	**12.** A	**13.** A	**14.** A	**15.** A	**16.** D
17. C	**18.** A	**19.** B	**20.** A				

EXERCISE (O-2)

1. A	**2.** B	**3.** B	**4.** A,B,C,D	**5.** A,C,D	**6.** B,C,D	**7.** A

EXERCISE (S-1)

1. **(a)** Strictly increasing in $[2, \infty)$ & Strictly decreasing in $(-\infty, 2]$

(b) Strictly increasing in $[1, \infty)$ & Strictly decreasing in $(-\infty, 0)$; $(0, 1]$

(c) Strictly increasing in $[0, 2]$ & Strictly decreasing in $(-\infty, 0]$; $[2, \infty)$

(d) Strictly increasing for $x \geq \frac{1}{2}$ or $-\frac{1}{2} \leq x < 0$ & Strictly decreasing for $x \leq -\frac{1}{2}$ or $0 < x \leq \frac{1}{2}$

2. **(a)** Strictly increasing in $[0, 3\pi/4]$; $[7\pi/4, 2\pi]$ & Strictly decreasing in $[3\pi/4, 7\pi/4]$

(b) Strictly increasing in $[0, \pi/6]$; $[\pi/2, 5\pi/6]$; $[3\pi/2, 2\pi]$ & Strictly decreasing in $[\pi/6, \pi/2]$; $[5\pi/6, 3\pi/2]$

(c) Strictly increasing in $[0, \pi/2] \cup [3\pi/2, 2\pi]$ and Strictly decreasing in $[\pi/2, 3\pi/2]$

3. $(-2, 0) \cup (2, \infty)$

4. **(a)** $(\pi/6)+(1/2)\ln 3$, $(\pi/3) - (1/2)\ln 3$

(b) Maximum at $x = 1$ and $f(-1) = 18$; Minimum at $x = 1/8$ and $f(1/8) = -9/4$

(c) 2 & -10

5. $a \geq 0$ **6.** $(b-a)^3/4$ **7.** $(-1, 0)$; $(0, \infty)$ **9.** $c = \frac{mb + na}{m + n}$ which lies between a & b

13. $a = 3$, $b = 4$ and $m = 1$

EXERCISE (S-2)

1. $[1, \infty)$ **2.** $[-7, -1) \cup [2, 3]$ **3.** $(0, 1/3) \cup (1, 5)$ **4.** -10

EXERCISE (JM)

1. 3	**2.** 2	**3.** 2	**4.** 4	**5.** 3	**6.** 2	**7.** 2	**8.** 4
9. 2	**10.** 1	**11.** 3					

EXERCISE (JA)

1. A	**2.** 2	**3.** C	**4.** B,C	**5.** B,D	**6.** B,C	**7.** D
8. D	**9.** D	**10.** C				

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MAXIMA-MINIMA

1. INTRODUCTION :

Some of the most important applications of differential calculus are optimization problems, in which we are required to find the optimal (best) way of doing something. Here are examples of such problems that we will solve in this chapter

- What is the shape of a vessel that can with-stand maximum pressure ?
- What is the maximum acceleration of a space shuttle ? (This is an important question to the astronauts who have to withstand the effects of acceleration)
- What is the radius of a contracted windpipe that expels air most rapidly during a cough ?

These problems can be reduced to finding the maximum or minimum values of a function. Let's first explain exactly what we mean by maxima and minima.

MAXIMA & MINIMA :

(a) Local Maxima/Relative maxima :

A function $f(x)$ is said to have a local maxima at $x = a$ if $f(a) \geq f(x)\ \forall\ x \in (a-h, a+h) \cap D_{f(x)}$

Where h is some positive real number.

(b) Local Minima/Relative minima :

A function $f(x)$ is said to have a local minima at $x = a$ if $f(a) \leq f(x)\ \forall\ x \in (a-h, a+h) \cap D_{f(x)}$

Where h is some positive real number.

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(c) Absolute maxima (Global maxima) :

A function f has an absolute maxima (or global maxima) at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f. The number f(c) is called the maximum value of f on D.

(d) Absolute minima (Global minima) :

A function f has an absolute minima at c if $f(c) \leq f(x)$ for all x in D and the number f(c) is called the minimum value of f on D.

Note :

(i) The term 'extrema' is used for both maxima or minima.

(ii) A local maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.

(iii) A function can have several extreme values such that local minimum value may be greater than a local maximum value.

(iv) It is not necessary that $f(x)$ always has local maxima/minima at end points of the given interval when they are included.

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2. DERIVATIVE TEST FOR ASCERTAINING MAXIMA/MINIMA :

(a) First derivative test :

If $f'(x) = 0$ at a point (say $x = a$) and

(i) If $f'(x)$ changes sign from positive to negative in the neighbourhood of $x = a$ then $x = a$ is said to be a point **local maxima.**

(ii) If $f'(x)$ changes sign from negative to positive in the neighbourhood of $x = a$ then $x = a$ is said to be a point **local minima.**

Note : If $f'(x)$ does not change sign i.e. has the same sign in a certain complete neighbourhood of a, then $f(x)$ is either increasing or decreasing throughout this neighbourhood implying that $x=a$ is not a point of extremum of f.

Illustration 1 : Let $f(x) = x + \frac{1}{x}$; $x \neq 0$. Discuss the local maximum and local minimum values of $f(x)$.

Solution : Here, $f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2} = \frac{(x-1)(x+1)}{x^2}$ $\quad + \quad -1 \quad - \quad 1 \quad +$

Using number line rule, $f(x)$ will have local maximum at $x = -1$ and local minimum at $x = 1$

$\therefore$ local maximum value of $f(x) = -2$ at $x = -1$

and local minimum value of $f(x) = 2$ at $x = 1$ **Ans.**

Illustration 2 : If $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \le x \le 2 \\ 37 - x, & 2 < x \le 3, \end{cases}$, then

(A) $f(x)$ is increasing on $[-1, 2)$
(B) $f(x)$ is continuous on $[-1, 3]$
(C) $f'(x)$ does not exist at $x = 2$
(D) $f(x)$ has the maximum value at $x = 2$

Solution : Given, $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \le x \le 2 \\ 37 - x, & 2 < x \le 3 \end{cases}$

$\Rightarrow \quad f'(x) = \begin{cases} 6x + 12, & -1 \le x \le 2 \\ -1, & 2 < x \le 3 \end{cases}$

(A) which shows $f'(x) > 0$ for $x \in [-1, 2)$

So, $f(x)$ is increasing on $[-1, 2)$

Hence, (A) is correct.

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(B) for continuity of f(x). (check at x = 2)

RHL = 35, LHL = 35 and f(2) = 35

So, (B) is correct

(C) $Rf'(2) = -1$ and $Lf'(2) = 24$

so, not differentiable at x = 2.

Hence, (C) is correct.

(D) we know f(x) is increasing on [–1, 2) and decreasing on (2, 3],

Thus maximum at x = 2,

Hence, (D) is correct.

$\therefore$ (A), (B), (C), (D) all are correct. **Ans.**

Do yourself - 1 :

(i) Find local maxima and local minima for the function $f(x) = x^3 - 3x$.

(ii) If function $f(x) = x^3 - 62x^2 + ax + 9$ has local maxima at x = 1, then find the value of a.

(b) Second derivative test :

If f(x) is continuous and differentiable at x = a where $f'(a) = 0$ and $f''(a)$ also exists then for ascertaining maxima/minima at x = a, 2nd derivative test can be used -

(i) If $f''(a) > 0 \Rightarrow x = a$ is a point of local minima

(ii) If $f''(a) < 0 \Rightarrow x = a$ is a point of local maxima

(iii) If $f''(a) = 0 \Rightarrow$ second derivative test fails. To identify maxima/minima at this point either first derivative test or higher derivative test can be used.

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Illustration 3 : If $f(x) = 2x^3 - 3x^2 - 36x + 6$ has local maximum and minimum at x = a and x = b respectively, then ordered pair (a, b) is -

(A) (3, –2) (B) (2, –3) (C) (–2, 3) (D) (–3, 2)

Solution : $f(x) = 2x^3 - 3x^2 - 36x + 6$

$f'(x) = 6x^2 - 6x - 36$ & $f''(x) = 12x - 6$

Now $f'(x) = 0 \Rightarrow 6(x^2 - x - 6) = 0 \Rightarrow (x - 3)(x + 2) = 0 \Rightarrow x = -2, 3$

$f''(-2) = -30$

$\therefore$ x = –2 is a point of local maximum

$f''(3) = 30$

$\therefore$ x = 3 is a point of local minimum

Hence, (–2, 3) is the required ordered pair. **Ans. (C)**

Illustration 4 : Find the point of local maxima of f(x) = sinx (1+cosx) in $x \in (0, \pi/2)$.

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Solution : Let $f(x) = \sin x\,(1+\cos x) = \sin x + \frac{1}{2}\sin 2x$

$\Rightarrow \quad f'(x) = \cos x + \cos 2x$

$f''(x) = -\sin x - 2\sin 2x$

Now $f'(x) = 0 \Rightarrow \cos x + \cos 2x = 0$

$\Rightarrow \quad \cos 2x = \cos(\pi - x) \quad \Rightarrow \quad x = \pi/3$

Also $f''(\pi/3) = -\sqrt{3}/2 - \sqrt{3} < 0 \qquad \therefore$ f(x) has a maxima at $x = \pi/3$ **Ans.**

Illustration 5 : Find the global maximum and global minimum of $f(x) = \frac{e^x + e^{-x}}{2}$ in $[-\log_e 2, \log_e 7]$.

Solution : $f(x) = \frac{e^x + e^{-x}}{2}$ is differentiable at all x in its domain.

Then $\quad f'(x) = \frac{e^x - e^{-x}}{2}$, $f''(x) = \frac{e^x + e^{-x}}{2}$

$f'(x) = 0 \Rightarrow \frac{e^x - e^{-x}}{2} = 0 \Rightarrow \quad e^{2x} = 1 \Rightarrow x = 0$

$f''(0) = 1 \quad \therefore \quad x = 0$ is a point of local minimum

Points $x = -\log_e 2$ and $x = \log_e 7$ are extreme points.

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Now, check the value of f(x) at all these three points $x = -\log_e 2, 0, \log_e 7$

$\Rightarrow f(-\log_e 2) = \frac{e^{-\log_e 2} + e^{+\log_e 2}}{2} = \frac{5}{4}$

$f(0) = \frac{e^0 + e^{-0}}{2} = 1$

$f(\log_e 7) = \frac{e^{\log_e 7} + e^{-\log_e 7}}{2} = \frac{25}{7}$

$\therefore \quad x = 0$ is absolute minima & $x = \log_e 7$ is absolute maxima

Hence, absolute/global minimum value of f(x) is 1 at x = 0

and absolute/global maximum value of f(x) is $\frac{25}{7}$ at $x = \log_e 7$ **Ans.**

Do yourself - 2 :

(i) Find local maximum value of function $f(x) = \frac{\ell nx}{x}$

(ii) If $f(x) = x^2 e^{-2x}$ $(x > 0)$, then find the local maximum value of f(x).

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Illustration 6 : Identify a point of maxima/minima in $f(x) = (x + 1)^4$.

Solution : $f(x) = (x + 1)^4$

$f'(x) = 4(x + 1)^3$

$-ve$ | $+ve$ $\quad \frac{dy}{dx}$

-1

at $x = -1$ $f(x)$ is having local minima

$\therefore$ at $x = -1$ $f(x)$ has point of minima.

Illustration 7 : Find point of local maxima and minima of $f(x) = x^5 - 5x^4 + 5x^3 - 1$

Solution : $f(x) = x^5 - 5x^4 + 5x^3 - 1$

$f'(x) = 5x^4 - 20x^3 + 15x^2 = 5x^2 (x^2 - 4x + 3)$

$= 5x^2 (x - 1)(x - 3)$

$f'(x) = 0 \quad \Rightarrow \quad x = 0, 1, 3$

$f''(x) = 10x(2x^2 - 6x + 3)$

But at $x = 0$, deriavative sign is positive in its neighbourhood.

Now $f''(1) < 0 \quad \Rightarrow$ Maxima at $x = 1$

$f''(3) > 0 \quad \Rightarrow$ Minima at $x = 3$

$\Rightarrow$ Neither maxima nor minima at $x = 0$.

Do yourself - 3 :

(i) Identify the point of local maxima/minima in $f(x) = (x - 3)^{10}$.

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3. USEFUL FORMULAE OF MENSURATION TO REMEMBER :

(a) Volume of a cuboid = ℓbh.

(b) Surface area of a cuboid = $2(\ell b + bh + h\ell)$.

(c) Volume of a prism = area of the base $\times$ height.

(d) Lateral surface area of prism = perimeter of the base $\times$ height.

(e) Total surface area of a prism = lateral surface area + 2 area of the base
(Note that lateral surfaces of a prism are all rectangles).

(f) Volume of a pyramid = $\frac{1}{3}$ area of the base x height.

(g) Curved surface area of a pyramid = $\frac{1}{2}$ (perimeter of the base) x slant height.
(Note that slant surfaces of a pyramid are triangles).

(h) Volume of a cone = $\frac{1}{3}\pi r^2 h$.

(i) Curved surface area of a cylinder = $2\pi rh$.

(j) Total surface area of a cylinder = $2\pi rh + 2\pi r^2$.

(k) Volume of a sphere = $\frac{4}{3}\pi r^3$.

(l) Surface area of a sphere = $4\pi r^2$.

(m) Area of a circular sector = $\frac{1}{2} r^2 \theta$, when θ is in radians.

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4. SUMMARY OF WORKING RULES FOR SOLVING REAL LIFE OPTIMIZATION PROBLEM :

First : When possible, draw a figure to illustrate the problem & label those parts that are important in the problem. Constants & variables should be clearly distinguished.

Second : Write an equation for the quantity that is to be maximized or minimized. If this quantity is denoted by 'y', it must be expressed in terms of a single independent variable x. This may require some algebraic manipulations.

Third : If $y = f(x)$ is a quantity to be maximum or minimum, find those values of x for which $dy/dx = f'(x) = 0$.

Fourth : Using derivative test, test each value of x for which $f'(x) = 0$ to determine whether it provides a maximum or minimum or neither.

Fifth : If the derivative fails to exist at some point, examine this point as possible maximum or minimum.

Sixth : If the function $y = f(x)$ is defined only for $x \in [a,b]$ then examine $x = a$ & $x = b$ for possible extreme values.

Illustration 8 : Determine the largest area of the rectangle whose base is on the x-axis and two of its vertices lie on the curve $y = e^{-x^2}$.

Solution : Area of the rectangle will be $A = 2a.\ e^{-a^2}$

For max. area, $\frac{dA}{da} = \frac{d}{da}(2ae^{-a^2}) = e^{-a^2}[2 - 4a^2]$

$\frac{dA}{da} = 0 \Rightarrow a = \pm\frac{1}{\sqrt{2}}$

& sign of $\frac{dA}{da}$ changes from positive to negative at $a = +\frac{1}{\sqrt{2}}$

$\Rightarrow \quad x = \frac{1}{\sqrt{2}}$ are points of maxima $\Rightarrow A_{max} = \frac{2}{\sqrt{2}} \cdot e^{-\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{\sqrt{2}}{e^{1/2}}$ sq units. **Ans.**

Illustration 9 : A box of maximum volume with top open is to be made by cutting out four equal squares from four corners of a square tin sheet of side length a ft, and then folding up the flaps. Find the side of the square base cut off.

Solution : Volume of the box is, $V = x(a - 2x)^2$ i.e., squares of side x are cut out then we will get a box with a square base of side (a – 2x) and height x.

$\therefore \quad \frac{dV}{dx} = (a - 2x)^2 + x \cdot 2(a - 2x)(-2)$

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$$\frac{dV}{dx} = (a - 2x)(a - 6x)$$

For V to be extremum $\frac{dV}{dx} = 0 \quad \Rightarrow \quad x = a/2, a/6$

But when x = a/2; V = 0 (minimum) and we know minimum and maximum occurs alternately in a continuous function.

Hence, V is maximum when x = a/6. **Ans.**

Illustration 10 : If a right circular cylinder is inscribed in a given cone. Find the dimension of the cylinder such that its volume is maximum.

Solution : Let x be the radius of cylinder and y be its height

$V = \pi x^2 y$

x, y can be related by using similar triangles

$$\frac{y}{r-x} = \frac{h}{r} \quad \Rightarrow \quad y = \frac{h}{r}(r - x)$$

$$\Rightarrow \quad V(x) = \pi x^2 \frac{h}{r}(r - x) \quad \Rightarrow \quad V(x) = \frac{\pi h}{r}(rx^2 - x^3)$$

$$V'(x) = \frac{\pi h}{r}(2rx - 3x^2)$$

$$V'(x) = 0 \quad \Rightarrow \quad x = 0, \frac{2r}{3}$$

$$V''(x) = \frac{\pi h}{r}(2r - 6x)$$

$V''(0) = 2\pi h \quad \Rightarrow$ x = 0 is point of minima

$V''\left(\frac{2r}{3}\right) = -2\pi h \quad \Rightarrow \quad x = \frac{2r}{3}$ is point of maxima

Thus volume is maximum at $x = \left(\frac{2r}{3}\right)$ and $y = \frac{h}{3}$.

Do yourself - 4 :

(i) Find the two positive numbers x & y such that their sum is 60 and xy^3 is maximum.

(ii) If from a wire of length 36 metre, a rectangle of greatest area is made, then find its two adjacent sides in metre.

(iii) If $ab = 2a + 3b$ where $a > 0, b > 0$, then find the minimum value of ab.

(iv) Of all closed right circular cylinders of a given volume of 100 cubic centimetres, find the dimensions of cylinder which has minimum surface area.

Important note :

(i) If the sum of two real numbers x and y is constant then their product is maximum if they are equal.

i.e. $xy = \frac{1}{4}[(x + y)^2 - (x - y)^2]$

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(ii) If the product of two positive numbers is constant then their sum is least if they are equal.

i.e. $(x + y)^2 = (x - y)^2 + 4xy$

5. LEAST/GREATEST DISTANCE BETWEEN TWO CURVES :

Least/Greatest distance between two non-intersecting curves usually lies along the common normal. (Wherever defined)

Note : Given a fixed point A(a, b) and a moving point P(x, f (x)) on the curve y = f (x). Then

AP will be maximum or minimum if it is normal to the curve at P.

Proof : $F(x) = (x - a)^2 + (f(x) - b))^2 \Rightarrow F'(x) = 2(x - a) + 2(f(x) - b) . f'(x)$

$$\therefore \quad f'(x) = -\left(\frac{x-a}{f(x)-b}\right). \text{ Also } m_{AP} = \frac{f(x)-b}{x-a}. \text{ Hence } f'(x) . m_{AP} = 1.$$

Illustration 11 : Find the co-ordinates of the point on the curve $x^2 = 4y$, which is at least distance from the line y = x – 4.

Solution : Let $P(x_1 y_1)$ be a point on the curve $x^2 = 4y$

at which normal is also a perpendicular to the line y = x – 4.

Slope of the tangent at (x_1, y_1) is $2x = 4\frac{dy}{dx} \Rightarrow \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = \frac{x_1}{2}$

$\therefore \quad \frac{x_1}{2} = 1 \Rightarrow x_1 = 2$

$\because \quad x_1^2 = 4y_1 \Rightarrow y_1 = 1$

Hence required point is (2, 1)

Do yourself - 5 :

(i) Find the coordinates of point on the curve $y^2 = 8x$, which is at minimum distance from the line x + y = –2.

6. SOME SPECIAL POINTS ON A CURVE :

(a) **Stationary points:** The stationary points are the points of domain where f'(x) = 0 .

(b) **Critical points :** There are three kinds of critical points as follows :

(i) The point at which $f'(x) = 0$

(ii) The point at which $f'(x)$ does not exists

(iii) The end points of interval (if included)

These points belongs to domain of the function.

Note : Local maxima and local minima occurs at critical points only but not all critical points will correspond to local maxima/local minima.

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(c) **Point of inflection :** A point where the graph of a function has a tangent line and where the strict concavity changes is called a point of inflection. For finding point of inflection of any function, compute the points (x-coordinate) where $\frac{d^2y}{dx^2} = 0$ or $\frac{d^2y}{dx^2}$ does not exist. Let the solution is x = a, if $\frac{d^2y}{dx^2} = 0$ at x = a and sign of $\frac{d^2y}{dx^2}$ changes about this point then it is called point of inflection.

if $\frac{d^2y}{dx^2}$ does not exist at x = a and sign of $\frac{d^2y}{dx^2}$ changes about this point and tangent exist at this point then it is called point of inflection.

Illustration 12 : Find the critical point(s) & stationary point(s) of the function $f(x) = (x-2)^{2/3}(2x+1)$

Solution : $f(x) = (x-2)^{2/3}(2x+1)$

$$f'(x) = (x-2)^{2/3}.2 + (2x+1)\frac{2}{3}(x-2)^{-1/3} = 2(x-2)^{2/3} + \frac{2}{3}(2x+1)\frac{1}{(x-2)^{1/3}}$$

$$= \left[2(x-2) + \frac{2}{3}(2x+1)\right]\frac{1}{(x-2)^{1/3}} = \frac{2(5x-5)}{3(x-2)^{1/3}}$$

f '(x) does not exist at x = 2 and f '(x) = 0 at x = 1

$\therefore$ x = 1, 2 are critical points and x = 1 is stationary point.

Illustration 13 : The point of inflection for the curve $y = x^{\frac{5}{3}}$ is -

(A) (1, 1) (B) (0, 0) (C) (1, 0) (D) (0, 1)

Solution : Here $\frac{d^2y}{dx^2} = \frac{10}{9x^{1/3}}$

From the given points we find that (0, 0) is the point of the curve where $\frac{d^2y}{dx^2}$ does not exist but sign of $\frac{d^2y}{dx^2}$ changes about this point.

$\therefore$ (0, 0) is the required point **Ans. (B)**

Illustration 14 : Find the inflection point of $f(x) = 3x^4 - 4x^3$. Also draw the graph of f(x) giving due importance to maxima, minima and concavity.

Solution : $f(x) = 3x^4 - 4x^3$

$f'(x) = 12x^3 - 12x^2$

$f'(x) = 12x^2(x-1)$

$f'(x) = 0 \Rightarrow x = 0, 1$

examining sign change of f'(x)

thus x = 1 is a point of local minima

$f''(x) = 12(3x^2 - 2x)$

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$f''(x) = 12x(3x - 2)$

$f''(x) = 0 \quad \Rightarrow \quad x = 0, \frac{2}{3}$

Again examining sign of $f''(x)$

thus $x = 0, \frac{2}{3}$ are the inflection points

Hence the graph of f(x) is

Do yourself - 6 :

(i) Find the critical points and stationary point of the function $f(x) = \frac{e^x}{x}$

(ii) Find the point of inflection for the curve $y = x^3 - 6x^2 + 12x + 5$

(iii) For the function $f(x) = \frac{x^4}{12} - \frac{5x^3}{6} + 3x^2 + 7$

(a) Find the interval in which $f''(x) > 0$

(b) Find the interval in which $f''(x) < 0$

(c) Find the points of inflection of f(x).

Miscellaneous Illustrations :

Illustration 15 : Let a cuboid having square base has area 6. Then find its maximum volume.

Solution : Total area $= 2a^2 + 4ah = 6$

$h = \frac{(6-2a^2)}{4a} \quad \Rightarrow \quad V = a^2h = \frac{a^2(6-2a^2)}{4a}$

$\frac{dv}{da} = 6 - 6a^2 = 0$

$a = 1 \quad \& \quad \frac{d^2v}{da^2} = -ve$

Minimum V = 1

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ANSWERS FOR DO YOURSELF

1 : **(i)** local max. at x = –1, local min. at x = 1 **(ii)** 121

2 : **(i)** 1/e **(ii)** $1/e^2$

3 : **(i)** local minima at x = 3

4 : **(i)** x = 15 & y = 45 **(ii)** 9 & 9 **(iii)** 24 **(iv)** $r = \left(\frac{50}{\pi}\right)^{1/3}$ cm. & $h = 2\left(\frac{50}{\pi}\right)^{1/3}$ cm.

5 : **(i)** (2,–4)

6 : **(i)** x = 1 is a critical point as well as stationary point (Note x = 0 is not in the domain of $f(x)$)

(ii) x = 2 **(iii)** **(a)** $(-\infty, 2) \cup (3, \infty)$ **(b)** (2, 3) **(c)** x = 2 & x = 3

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MAXIMA - MINIMA

EXERCISE (O-1)

1. The least area of a circle circumscribing any right triangle of area S is :

(A) πS (B) $2\pi S$ (C) $\sqrt{2}\pi S$ (D) $4\pi S$

MM0001

2. A minimum value of sinx cos2x is-

(A) 1 (B) –1 (C) $-2/3\sqrt{6}$ (D) None of these

MM0002

3. $f(x) = 1 + [\cos x]x$, in $0 \le x \le \frac{\pi}{2}$ (where [.] denotes greatest integer function)

(A) has a minimum value 0 (B) has a maximum value 2

(C) is continuous in $\left[0, \frac{\pi}{2}\right]$ (D) is not differentiable at $x = \frac{\pi}{2}$

MM0003

4. Minimum value of the function $f(x) = \sum_{k=1}^{5}(x-k)^2$ is at-

(A) x = 2 (B) x = 5/2 (C) x = 3 (D) x = 5

MM0004

5. If $f(x) = x^3 + ax^2 + bx + c$ is minimum at x = 3 and maximum at x = –1, then-

(A) a = –3, b = –9, c = 0 (B) a = 3, b = 9, c = 0

(C) a = –3, b = –9, c ∈ R (D) none of these

MM0005

6. The greatest value of $x^3 - 18x^2 + 96x$ in the interval (0, 9) is-

(A) 128 (B) 60 (C) 160 (D) 120

MM0006

7. Difference between the greatest and the least values of the function $f(x) = x(\ln x - 2)$ on $[1, e^2]$ is

(A) 2 (B) e (C) e^2 (D) 1

MM0007

8. The maximum area of a right angled triangle with hypotenuse h is :-

(A) $\frac{h^2}{\sqrt{2}}$ (B) $\frac{h^2}{2}$ (C) $\frac{h^2}{4}$ (D) $\frac{h^2}{2\sqrt{2}}$

MM0008

9. If $f(x) = |x| + |x-1| + |x-2|$, then-

(A) f(x) has minima at x = 1 (B) f(x) has maxima at x = 0

(C) f(x) has neither maxima nor minima at x = 3 (D) None of these

MM0009

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10. Consider the function $f(x) = x \cos x - \sin x$, then identify the statement which is correct .

(A) f is neither odd nor even
(B) f is monotonic decreasing at $x = 0$
(C) f has a maxima at $x = \pi$
(D) f has a minima at $x = -\pi$

MM0010

11. Range of the function $f(x) = \frac{\ell nx}{\sqrt{x}}$ is

(A) $(-\infty, e)$
(B) $(-\infty, e^2)$
(C) $\left(-\infty, \frac{2}{e}\right]$
(D) $\left(-\infty, \frac{1}{e}\right)$

MM0011

12. If $(x-a)^{2m}(x-b)^{2n+1}$, where m and n are positive integers and $a > b$, is the derivative of a function f, then-

(A) $x = a$ gives neither a maximum, nor a minimum
(B) $x = a$ gives a maximum
(C) $x = b$ gives neither a maximum nor a minimum
(D) None of these

MM0012

13. The minimum value of $a\sec x + b\operatorname{cosec} x$, $0 < a < b$, $0 < x < \pi/2$ is-

(A) $a + b$
(B) $a^{2/3} + b^{2/3}$
(C) $(a^{2/3} + b^{2/3})^{3/2}$
(D) None of these

MM0013

14. The rate of change of the function $f(x) = 3x^5 - 5x^3 + 5x - 7$ is minimum when

(A) $x = 0$
(B) $x = 1/\sqrt{2}$
(C) $x = -1/\sqrt{2}$
(D) $x = \pm 1/\sqrt{2}$

MM0014

15. The cost of running a bus from A to B, is Rs. $\left(av + \frac{b}{v}\right)$, where v km/h is the average speed of the bus. When the bus travels at 30 km/h, the cost comes out to be Rs. 75 while at 40 km/h, it is Rs. 65. Then the most economical speed (in km/h) of the bus is :

(A) 40
(B) 60
(C) 45
(D) 50

MM0015

16. The sum of lengths of the hypotenuse and another side of a right angled triangle is given. The area of the triangle will be maximum if the angle between them is :

(A) $\pi/6$
(B) $\pi/4$
(C) $\pi/3$
(D) $5\pi/12$

MM0016

17. If $ax + \frac{b}{x} \geq c$ for all positive x, where a, b, c > 0, then-

(A) $ab < \frac{c^2}{4}$
(B) $ab \geq \frac{c^2}{4}$
(C) $ab \geq \frac{c}{4}$
(D) None of these

MM0017

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18. Two sides of a triangle are to have lengths 'a' cm & 'b' cm. If the triangle is to have the maximum area, then the length of the median from the vertex containing the sides 'a' and 'b' is

(A) $\frac{1}{2}\sqrt{a^2+b^2}$ (B) $\frac{2a+b}{3}$ (C) $\sqrt{\frac{a^2+b^2}{2}}$ (D) $\frac{a+2b}{3}$

MM0018

19. A rectangle has one side on the positive y-axis and one side on the positive x - axis. The upper right hand vertex of the rectangle lies on the curve $y = \frac{\ell nx}{x^2}$. The maximum area of the rectangle is

(A) e^{-1} (B) $e^{-½}$ (C) 1 (D) $e^{½}$

MM0019

20. P is a point on positive x-axis, Q is a point on the positive y-axis and 'O' is the origin. If the line passing through P and Q is tangent to the curve $y = 3 - x^2$ then the minimum area of the triangle OPQ, is

(A) 2 (B) 4 (C) 8 (D) 9

MM0020

21. The set of value(s) of 'a' for which the function $f(x) = \frac{ax^3}{3} + (a+2)x^2 + (a-1)x + 2$ posses a negative point of inflection.

(A) $(-\infty, -2) \cup (0, \infty)$ (B) $\{-4/5\}$ (C) $(-2, 0)$ (D) empty set

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MM0021

EXERCISE (O-2)

[SINGLE CORRECT CHOICE TYPE]

1 If $f(x) = \begin{cases} e^{x+1} - e^x & x \le 0 \\ e^{1-x} - 1 & 0 < x < 1 \\ x + \ell nx & x \ge 1 \end{cases}$, then -

(A) x = 0 is point of local maxima, x = 1 is neither local maxima nor local minima.
(B) x = 1 is point of local minima, x = 0 is point of local maxima
(C) x = 0 and x = 1 both are points of local maxima
(D) x = 0 and x = 1 both are points of local minima

MM0022

2. Let $f(x) = ax^2 - b|x|$, where a and b are constants. Then at x = 0, f(x) has

(A) a maxima whenever a > 0, b > 0
(B) a maxima whenever a > 0, b < 0
(C) minima whenever a > 0, b > 0
(D) neither a maxima nor minima whenever a > 0, b < 0

MM0023

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3. The bottom of the legs of a three legged table are the vertices of an isosceles triangle with sides 5, 5 and 6. The legs are to be braced at the bottom by three wires in the shape of a Y. The minimum length of the wire needed for this purpose, is

(A) $4+3\sqrt{3}$ (B) 10 (C) $3+4\sqrt{3}$ (D) $1+6\sqrt{2}$

MM0024

4. Let $f(x)=\dfrac{\tan^n x}{1+\sum_{r=1}^{2n}\tan^r x}$, $n\in N$, where $x\in\left[0,\dfrac{\pi}{2}\right)$

(A) f(x) is bounded and it takes both of it's bounds and the range of f(x) contains exactly one integral point.

(B) f(x) is bounded and it takes both of it's bounds and the range of f(x) contains more than one integral point.

(C) f(x) is bounded but minimum and maximum does not exists.

(D) f(x) is not bounded as the upper bound does not exist.

MM0025

5. If (a, b) be the point on the curve $y=|x^2-4x+3|$ which is nearest to the circle $x^2+y^2-4x-4y+7=0$, then (a + b) is equal to -

(A) $\dfrac{7}{4}$ (B) 0 (C) 3 (D) 2

MM0026

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6. Let $f(x)=\begin{cases} x^{3/5} & \text{if } x\le 1 \\ -(x-2)^3 & \text{if } x>1\end{cases}$, then the number of critical points on the graph of the function is

(A) 1 (B) 2 (C) 3 (D) 4

MM0027

7. The set of all values of 'a' for which the function,

$f(x)=(a^2-3a+2)\left(\cos^2\dfrac{x}{4}-\sin^2\dfrac{x}{4}\right)+(a-1)x+\sin 1$ does not possess critical points is:

(A) $[1,\infty)$ (B) $(0,1)\cup(1,4)$ (C) $(-2,4)$ (D) $(1,3)\cup(3,5)$

MM0028

8. There are 50 apple trees in an orchard. Each tree produces 800 apples. For each additional tree planted in the orchard, per tree the output drop by 10 times the number of additional trees . Number of trees that should be added to the existing orchard for maximising the output of the trees, is

(A) 5 (B) 10 (C) 15 (D) 20

MM0029

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[MULTIPLE CORRECT CHOICE TYPE]

9. Give the correct order of initials **T** or **F** for following statements. Use **T** if statement is true and **F** if it is false.

Statement-1: If $f : R \to R$ and $c \in R$ is such that f is increasing in $(c - \delta, c)$ and f is decreasing in $(c, c + \delta)$ then f has a local maximum at c. Where δ is a sufficiently small positive quantity.

Statement-2 : Let $f : (a, b) \to R$, $c \in (a, b)$. Then f can not have both a local maximum and a point of inflection at $x = c$.

Statement-3 : The function $f(x) = x^2 |x|$ is twice differentiable at $x = 0$.

Statement-4 : Let $f : [c - 1, c + 1] \to [a, b]$ be bijective map such that f is differentiable at c then f^{-1} is also differentiable at $f(c)$.

(A) FFTF (B) TTFT (C) FTTF (D) TTTF

MM0030

10. Let $f(x) = ax^3 + bx^2 + cx + 1$ have extrema at $x = \alpha, \beta$ such that $\alpha\beta < 0$ and $f(\alpha) \cdot f(\beta) < 0$, then which of the following can be true for the equation $f(x) = 0$?

(A) three equal roots
(B) three distinct real roots
(C) one positive root if $f(\alpha) < 0$ and $f(\beta) > 0$
(D) one negative root if $f(\alpha) > 0$ and $f(\beta) < 0$.

MM0031

11. Graph of y = f(x) is given, then

(A) $y = H(x) = \max\{f(t) : t \le x\}\ \forall\ x \in R$ has a local maximum as well as a local minimum
(B) $y = |f(x)|$ has 5 point of extrema
(C) $y = f(|x|)$ has 5 points of extrema
(D) $y = f(|x|)$ has 3 points of extrema

MM0032

12. If $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} [f(x)]$ ('a' is a finite quantity), where [.] denotes greatest integer function and f(x) is a non constant continuous function, then

(A) $\lim\limits_{x \to a} f(x)$ is an integer.
(B) $\lim\limits_{x \to a} f(x)$ need not be an integer.
(C) f(x) has a local minimum at x = a
(D) f(x) has a local maximum at x = a.

MM0033

13. Let $f(x)$ be a cubic polynomial such that it has point of inflection at x = 2 and local minima at x = 4, then-

(A) $f(x)$ has local minima at x = 0
(B) $f(x)$ has local maxima at x = 0
(C) $\lim\limits_{x \to \infty} f(x) \to \infty$
(D) $\lim\limits_{x \to \infty} f(x) \to -\infty$

MM0034

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14. Let $f(x,y)=\sqrt{x^2+y^2}+\sqrt{x^2+y^2-2x+1}+\sqrt{x^2+y^2-2y+1}+\sqrt{x^2+y^2-6x-8y+25}$ $\forall x,y\in R$, then-

(A) Minimum value of $f(x,y)=5+\sqrt{2}$ (B) Minimum value of $f(x,y)=5-\sqrt{2}$

(C) Minimum value occurs of $f(x,y)$ for $x=\frac{3}{7}$ (D) Minimum value occurs of $f(x,y)$ for $y=\frac{4}{7}$

MM0035

15. Let A(p,q) and B(h,k) are points on the curve $4x^2+9y^2=1$, which are nearest and farthest from the line $72+8x=9y$ respectively, then -

(A) $p+q=-\frac{1}{5}$ (B) $p+q=\frac{1}{5}$ (C) $h+k=-\frac{1}{5}$ (D) $h+k=\frac{1}{5}$

MM0036

EXERCISE (S-1)

1. The graph of the derivative $f'(x)$ of a continuous function $f(x)$ in (0,9). If

(i) f is strictly increasing in the interval (a,b]; [c,d];[e,f) and strictly decreasing in [p, q];[r, s].

(ii) f has a local minima at $x = x_1$ and $x = x_2$.

(iii) $f''(x) > 0$ in (l, m) ; (n, t)

(iv) f has inflection point at $x = k$

(v) number of critical points of $y = f(x)$ is 'w'.

Find the value of $(a + b + c + d + e) + (p + q + r + s) + (l + m + n) + (x_1 + x_2) + (k + w)$.

MM0037

2. Let f(x) be a cubic polynomial which has local maximum at $x = -1$ and f'(x) has a local minimum at $x = 1$. If $f(-1) = 10$ and $f(3) = -22$, then find the distance between its two horizontal tangents.

MM0038

3. Find the greatest & least value for the function ;

(a) $y = x + \sin 2x$, $0 \le x \le 2\pi$ **MM0039**

(b) $y = 2\cos 2x - \cos 4x$, $0 \le x \le \pi$ **MM0040**

4. Let $f(x) = x^3 - x^2 + x + 1$ and $g(x) = \begin{cases} \max\{f(t) : 0 \le t \le x\} & , 0 \le x \le 1 \\ 3 - x & , 1 < x \le 2 \end{cases}$

Discuss the continuity & differentiability of g(x) in the interval (0,2).

MM0041

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5. Let P(x) be a polynomial of degree 5 having extremum at x = –1, 1 and $\lim_{x\to 0}\left(\frac{P(x)}{x^3}-2\right)=4$. If M and m are the maximum and minimum value of the function y = P'(x) on the set $A = \{x | x^2 + 6 \le 5x\}$ then find $\frac{m}{M}$.

MM0042

6. If $y = \frac{ax+b}{(x-1)(x-4)}$ has a turning value at (2, –1) find a & b and show that the turning value is a maximum.

MM0043

7. Find the area of the largest rectangle with lower base on the x-axis & upper vertices on the curve $y = 12 - x^2$.

MM0044

8. Of all the lines tangent to the graph of the curve $y = \frac{6}{x^2+3}$, find the equations of the tangent lines of minimum and maximum slope.

MM0045

9. The plan view of a swimming pool consists of a semicircle of radius r attached to a rectangle of length '2r' and width 's'. If the surface area A of the pool is fixed, for what value of 'r' and 's' the perimeter 'P' of the pool is minimum.

MM0046

10. The length of three sides of a trapezium are equal, each being 10 cms. Find the maximum area of such a trapezium.

MM0047

11. By the post office regulations, the combined length & girth of a parcel must not exceed 3 metre. Find the volume of the biggest cylindrical (right circular) packet that can be sent by the parcel post.

MM0048

12. A running track of 440 ft. is to be laid out enclosing a football field, the shape of which is a rectangle with semi circle at each end. If the area of the rectangular portion is to be maximum, find the length of its sides.
Use : $\pi \approx 22/7$.

MM0049

13. A closed rectangular box with a square base is to be made to contain 1000 cubic feet. The cost of the material per square foot for the bottom is 15 paise, for the top 25 paise and for the sides 20 paise. The labour charges for making the box are Rs. 3/-. Find the dimensions of the box when the cost is minimum.

MM0050

14. What are the dimensions of the rectangular plot of the greatest area which can be laid out within a triangle of base 36 ft. & altitude 12 ft ? Assume that one side of the rectangle lies on the base of the triangle.

MM0051

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EXERCISE (S-2)

1. The graph of the derivative f' of a continuous function f is shown with f (0) = 0

(i) On what intervals f is strictly increasing or strictly decreasing?

(ii) At what values of x does f have a local maximum or minimum?

(iii) On what intervals is $f'' > 0$ or $f'' < 0$

(iv) State the x-coordinate(s) of the point(s) of inflection.

(v) Assuming that f (0) = 0, sketch a graph of f.

MM0052

2. The value of 'a' for which $f(x) = x^3 + 3(a-7)x^2 + 3(a^2-9)x - 1$ have a positive point of maximum lies in the interval $(a_1, a_2) \cup (a_3, a_4)$. Find the value of $a_2 + 11a_3 + 70a_4$.

MM0053

3. The function f (x) defined for all real numbers x has the following properties
$f(0) = 0$, $f(2) = 2$ and $f'(x) = k(2x - x^2)e^{-x}$ for some constant $k > 0$. Find

(a) the intervals on which f is strictly increasing and strictly decreasing and any local maximum or minimum values.

(b) the intervals on which the graph f is concave down and concave up.

(c) the function f (x) and plot its graph.

MM0054

4. Find all the values of the parameter 'a' for which the function ;
$f(x) = 8ax - a\sin 6x - 7x - \sin 5x$ increasing & has no critical points for all $x \in R$.

MM0055

5. Use calculus to prove the inequality, $\sin x \geq 2x/\pi$ in $0 \leq x \leq \pi/2$.

Use this inequality to prove that, $\cos x \leq 1 - x^2/\pi$ in $0 \leq x \leq \pi/2$.

MM0056

6. Consider the function $f(x) = \begin{cases} \sqrt{x}\,\ell nx & \text{when } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$

(a) Find whether f is continuous at x = 0 or not.

(b) Find the absolute minima and absolute maxima (if they exist).

(c) Does $f'(0)$? Find $\lim\limits_{x \to 0} f'(x)$.

(d) Find the inflection points of the graph of y = f (x)..

MM0057

7. Given two points A (– 2 , 0) & B (0 , 4) and a line y = x. Find the co-ordinates of a point M on this line so that the perimeter of the Δ AMB is least.

MM0058

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8. Find the maximum perimeter of a triangle on a given base 'a' and having the given vertical angle α.

MM0059

9. A statue 4 metres high sits on a column 5.6 metres high. How far from the column must a man, whose eye level is 1.6 metres from the ground, stand in order to have the most favourable view of statue.

MM0060

10. A perpendicular is drawn from the centre to a tangent to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Find the greatest value of the intercept between the point of contact and the foot of the perpendicular.

MM0061

11. A beam of rectangular cross section must be sawn from a round log of diameter d. What should the width x and height y of the cross section be for the beam to offer the greatest resistance (a) to compression; (b) to bending. Assume that the compressive strength of a beam is proportional to the area of the cross section and the bending strength is proportional to the product of the width of section by the square of its height.

MM0062

12. A given quantity of metal is to be casted into a half cylinder i.e. with a rectangular base and semicircular ends. Show that in order that total surface area may be minimum , the ratio of the height of the cylinder to the diameter of the semi circular ends is $\pi/(\pi+2)$.

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MM0063

13. Find the set of value of m for the cubic $x^3-\frac{3}{2}x^2+\frac{5}{2}=\log_{1/4}(m)$ has 3 distinct solutions.

MM0064

14. A cylinder is obtained by revolving a rectangle about the $x-$axis , the base of the rectangle lying on the x-axis and the entire rectangle lying in the region between the curve $y=\frac{x}{x^2+1}$ & the x-axis. Find the maximum possible volume of the cylinder.

MM0065

15. The circle $x^2+y^2=1$ cuts the x-axis at P & Q. Another circle with centre at Q and variable radius intersects the first circle at R above the x-axis & the line segment PQ at S. Find the maximum area of the triangle QSR.

MM0066

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EXERCISE (JM)

1. Let f : R → R be defined by $f(x)=\begin{cases} k-2x, & \text{if} \quad x \le -1 \\ 2x+3, & \text{if} \quad x > -1 \end{cases}$ **[AIEEE-2010]**

If f has a local minimum at x = – 1, then a possible value of k is :

(1) 1 (2) 0 (3) $-\frac{1}{2}$ (4) –1

MM0067

2. The shortest distance between line y – x = 1 and curve $x = y^2$ is :- **[AIEEE-2011]**

(1) $\frac{8}{3\sqrt{2}}$ (2) $\frac{4}{\sqrt{3}}$ (3) $\frac{\sqrt{3}}{4}$ (4) $\frac{3\sqrt{2}}{8}$

MM0068

3. Let f be a function defined by $f(x)=\begin{cases} \frac{\tan x}{x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$

Statement - 1: x = 0 is point of minima of f.

Statement - 2: f'(0) = 0. **[AIEEE-2011]**

(1) Statement-1 is false, statement-2 is true.

(2) Statement-1 is true, statement-2 is true; Statement-2 is correct explanation for statement-1.

(3) Statement-1 is true, statement-2 is true; Statement-2 is not a correct explanation for statement-1.

(4) Statement-1 is true, statement-2 is false.

MM0069

4. A spherical balloon is filled with 4500π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is : **[AIEEE-2012]**

(1) 9/2 (2) 9/7 (3) 7/9 (4) 2/9

MM0070

5. Let a, b ∈ R be such that the function f given by $f(x) = \ln|x| + bx^2 + ax$, $x \ne 0$ has extreme values at x = – 1 and x = 2.

Statement–1 : f has local maximum at x = –1 and at x = 2.

Statement–2 : $a = \frac{1}{2}$ and $b = \frac{-1}{4}$. **[AIEEE-2012]**

(1) Statement–1 is true, Statement–2 is false.

(2) Statement–1 is false, Statement–2 is true.

(3) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.

(4) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for Statement–1.

MM0071

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6. The real number k for which the equation $2x^3 + 3x + k = 0$ has two distinct real roots in [0, 1] **[JEE-MAIN 2013]**

(1) lies between 1 and 2. (2) lies between 2 and 3.
(3) lies between –1 and 0 (4) does not exist

MM0072

7. If x = – 1 and x = 2 are extreme points of $f(x) = \alpha \log|x| + \beta x^2 + x$ then : **[JEE-MAIN 2014]**

(1) $\alpha = -6$, $\beta = \frac{1}{2}$ (2) $\alpha = -6$, $\beta = -\frac{1}{2}$
(3) $\alpha = 2$, $\beta = -\frac{1}{2}$ (4) $\alpha = 2$, $\beta = \frac{1}{2}$

MM0073

8. Let $f(x)$ be a polynomial of degree four having extreme values at x = 1 and x = 2. If $\lim_{x\to 0}\left[1+\frac{f(x)}{x^2}\right] = 3$, then $f(2)$ is equal to : **[JEE-MAIN 2015]**

(1) 0 (2) 4 (3) –8 (4) –4

MM0074

9. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side = x units and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then : **[JEE-MAIN 2016]**

(1) 2x = r (2) $2x = (\pi + 4)r$ (3) $(4 - \pi)x = \pi r$ (4) x = 2r

MM0075

10. Twenty meters of wire is available for fencing off a flower-bed in the form of a circular sector. Then the maximum area (in sq. m) of the flower -bed, is :- **[JEE-MAIN 2017]**

(1) 30 (2) 12.5 (3) 10 (4) 25

MM0076

11. Let $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}$, $x \in R - \{-1, 0, 1\}$. If $h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of h(x) is : **[JEE-MAIN 2018]**

(1) –3 (2) $-2\sqrt{2}$ (3) $2\sqrt{2}$ (4) 3

MM0077

12. A helicopter is flying along the curve given by $y - x^{3/2} = 7$, $(x \geq 0)$. A soldier positioned at the point $\left(\frac{1}{2}, 7\right)$ wants to shoot down the helicopter when it is nearest to him. Then this nearest distance is : **[JEE-MAIN 2019]**

(1) $\frac{1}{2}$ (2) $\frac{1}{3}\sqrt{\frac{7}{3}}$ (3) $\frac{1}{6}\sqrt{\frac{7}{3}}$ (4) $\frac{\sqrt{5}}{6}$

MM0078

13. Let $a_1, a_2, a_3, \ldots\ldots$ be an A. P. with $a_6 = 2$. Then the common difference of this A. P., which maximises the produce $a_1a_4a_5$, is : **[JEE-MAIN 2019]**

(1) $\frac{6}{5}$ (2) $\frac{8}{5}$ (3) $\frac{2}{3}$ (4) $\frac{3}{2}$

MM0079

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14. Let f(x) be a polynomial of degree 5 such that x = ±1 are its critical points. If $\lim_{x\to 0}\left(2+\frac{f(x)}{x^3}\right)=4$, then which one of the following is not true? **[JEE-MAIN 2020]**

(1) f is an odd function

(2) x = 1 is a point of minima and x = –1 is a point of maxima of f.

(3) x = 1 is a point of maxima and x = –1 is a point of minimum of f.

(4) f(1) – 4f(–1) = 4

MM0080

15. Let $f(x)$ be a polynomial of degree 3 such that $f(-1) = 10$, $f(1) = -6$, $f(x)$ has a critical point at $x = -1$ and $f'(x)$ has a critical point at $x = 1$. Then $f(x)$ has a local minima at x = _____. **[JEE-MAIN 2020]**

MM0081

16. A spherical iron ball of 10 cm radius is coated with a layer of ice of uniform thickness the melts at a rate of 50 cm^3/min. When the thickness of ice is 5 cm, then the rate (in cm/min.) at which of the thickness of ice decreases, is : **[JEE-MAIN 2020]**

(1) $\frac{1}{36\pi}$ (2) $\frac{5}{6\pi}$ (3) $\frac{1}{18\pi}$ (4) $\frac{1}{54\pi}$

MM0082

EXERCISE (JA)

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1. **(a)** Let f, g and h be real-valued functions defined on the interval [0, 1] by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2e^{x^2} + e^{-x^2}$. If a, b and c denote respectively, the absolute maximum of f, g and h on [0, 1], then

(A) a = b and c ≠ b (B) a = c and a ≠ b (C) a ≠ b and c ≠ b (D) a = b = c

MM0083

(b) Let f be a function defined *on* ***R*** (the set of all real numbers) such that

$f'(x) = 2010\,(x-2009)\,(x-2010)^2\,(x-2011)^3\,(x-2012)^4$, for all $x \in R$.

If g is a function defined on R with values in the interval $(0,\infty)$ such that

$f(x) = \ell n\,(g(x))$, for all $x \in R$,

then the number of points in R at which g has a local maximum is **[JEE 2010, 3 + 3]**

MM0084

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is **[JEE 2012, 4M]**

MM0085

3. Let p(x) be a real polynomial of least degree which has a local maximum at x = 1 and a local minimum at x = 3. If p(1) = 6 and p(3) = 2, then p'(0) is **[JEE 2012, 4M]**

MM0086

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4. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio of 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. Then the lengths of the sides of the rectangular sheet are **[JEE 2013, 4M,–1M]**

(A) 24 (B) 32 (C) 45 (C) 60

MM0087

5. The function $f(x) = 2|x| + |x + 2| - ||x + 2| - 2|x||$ has a local minimum or a local maximum at x = **[JEE 2013, 3M,–1M]**

(A) –2 (B) $\frac{-2}{3}$ (C) 2 (D) $\frac{2}{3}$

MM0088

6. A cylindrical container is to be made from certain solid material with the following constraints. It has a fixed inner volume of V mm^3, has a 2 mm thick solid wall and is open at the top. The bottom of the container is a solid circular disc of thickness 2 mm and is of radius equal to the outer radius of the container.

If the volume of the material used to make the container is minimum when the inner radius of the container is 10mm, then the value of $\frac{V}{250\pi}$ is **[JEE 2015, 4M, 0M]**

MM0089

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7. The least value of $\alpha \in \mathbb{R}$ for which $4\alpha x^2 + \frac{1}{x} \geq 1$, for all x > 0, is -

[JEE(Advanced)-2016, 3(–1)]

(A) $\frac{1}{64}$ (B) $\frac{1}{32}$ (C) $\frac{1}{27}$ (D) $\frac{1}{25}$

MM0090

8. Let $f : \mathbb{R} \to (0, \infty)$ and $g : \mathbb{R} \to \mathbb{R}$ be twice differentiable function such that f'' and g'' ar continuous functions on $\mathbb{R}$. Suppose $f'(2) = g(2) = 0$, $f''(2) \neq 0$ and $g'(2) \neq 0$. If $\lim_{x \to 2} \frac{f(x)g(x)}{f'(x)g'(x)} = 1$, then

[JEE(Advanced)-2016, 4(–2)]

(A) f has a local minimum at x = 2

(B) f has a local maximum at x = 2

(C) $f''(2) > f(2)$

(D) $f(x) - f''(x) = 0$ for at least one $x \in \mathbb{R}$

MM0091

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9. If $f(x) = \begin{vmatrix} \cos(2x) & \cos(2x) & \sin(2x) \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$, then **[JEE(Advanced)-2017]**

(A) $f'(x) = 0$ at exactly three points in $(-\pi, \pi)$

(B) $f(x)$ attains its maximum at $x = 0$

(C) $f(x)$ attains its minimum at $x = 0$

(D) $f'(x) = 0$ at more than three points in $(-\pi, \pi)$

MM0092

10. For every twice differentiable function $f : \mathbb{R} \to [-2, 2]$ with $(f(0))^2 + (f'(0))^2 = 85$, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018]**

(A) There exist $r, s \in \mathbb{R}$, where $r < s$, such that f is one-one on the open interval (r, s).

(B) There exists $x_0 \in (-4, 0)$ such that $|f'(x_0)| \le 1$

(C) $\lim_{x\to\infty} f(x) = 1$

(D) There exists $\alpha \in (-4, 4)$ such that $f(\alpha) + f''(\alpha) = 0$ and $f'(\alpha) \ne 0$

MM0093

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11. Let $f(x) = \dfrac{\sin \pi x}{x^2}, x > 0$

Let $x_1 < x_2 < x_3 < \ldots < x_n < \ldots$ be all the points of local maximum of f and $y_1 < y_2 < y_3 < \ldots < y_n < \ldots$ be all the points of local minimum of f.
Then which of the following options is/are correct ? **[JEE(Advanced)-2019, 4(–1)]**

(1) $|x_n - y_n| > 1$ for every n

(2) $x_1 < y_1$

(3) $x_n \in \left(2n, 2n + \frac{1}{2}\right)$ for every n

(4) $x_{n+1} - x_n > 2$ for every n

MM0094

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ANSWER KEY

(MAXIMA-MINIMA)

EXERCISE (O-1)

1. A **2.** B **3.** C **4.** C **5.** C **6.** C **7.** B **8.** C **9.** A,C
10. B **11.** C **12.** A **13.** C **14.** D **15.** B **16.** C **17.** B **18.** A
19. A **20.** B **21.** A

EXERCISE (O-2)

1. A **2.** A **3.** A **4.** A **5.** C **6.** C **7.** B **8.** C **9.** C
10. B,C,D **11.** A,B,D **12.** A,C **13.** B,C **14.** A,C,D **15.** A,D

EXERCISE (S-1)

1. 74 **2.** 32

3. **(a)** Max at $x = 2\pi$, Max value $= 2\pi$, Min. at $x = 0$, Min value $= 0$
(b) Max at $x = \pi/6$ & also at $x = 5\pi/6$ and Max value $= 3/2$, Min at $x = \pi/2$, Min value $= -3$

4. Continuous but not diff. at $x = 1$ **5.** 6 **6.** $a = 1, b = 0$ **7.** 32 sq. units

8. $3x + 4y - 9 = 0$; $3x - 4y + 9 = 0$. **9.** $r = \sqrt{\frac{2A}{\pi+4}}$, $s = \sqrt{\frac{2A}{\pi+4}}$ **10.** $75\sqrt{3}$ sq. units

11. $1/\pi$ cu m **12.** 110', 70' **13.** side 10', height 10' **14.** $6' \times 18'$

EXERCISE (S-2)

1. **(i)** I in [1, 6] ; [8, 9] and D in [0, 1] ; [6, 8]; **(ii)** L.Min. at $x = 1$ and $x = 8$; L.Max. at $x = 0, 6, 9$
(iii) $(0, 2) \cup (3, 5) \cup (7, 9)$ and $(2, 3) \cup (5, 7)$; **(iv)** $x = 2, 3, 5, 7$

(v) Graph is

2. 320

3. **(a)** strictly increasing in [0,2] and strictly decreasing in $(-\infty, 0]$; $[2, \infty)$, local min. value = 0 and local max. value = 2

(b) concave up for $(-\infty, 2-\sqrt{2}]$; $[2+\sqrt{2}, \infty)$ and concave down in $[2-\sqrt{2}, 2+\sqrt{2}]$

(c) $f(x) = \frac{1}{2}e^{2-x} \cdot x^2$

4. $(6, \infty)$

6. **(a)** f is continuous at $x = 0$; **(b)** absolute minima $= -2/e$; absolute maxima does not exist
(c) does not exist, does not exist ; **(d)** pt. of inflection $x = 1$

7. (0 , 0) **8.** $P_{max} = a\left(1 + \operatorname{cosec}\frac{\alpha}{2}\right)$ **9.** $4\sqrt{2}$ m **10.** $|a - b|$

11. **(a)** $x = y = \frac{d}{\sqrt{2}}$, **(b)** $x = \frac{d}{\sqrt{3}}$, $y = \sqrt{\frac{2}{3}}d$ **13.** $m \in \left(\frac{1}{32}, \frac{1}{16}\right)$ **14.** $\pi/4$ **15.** $\frac{4}{3\sqrt{3}}$

EXERCISE (JM)

1. 4 **2.** 4 **3.** 3 **4.** 4 **5.** 3 **6.** 4 **7.** 3 **8.** 1 **9.** 4
10. 4 **11.** 3 **12.** 3 **13.** 2 **14.** 2 **15.** 3 **16.** 3

EXERCISE (JA)

1. (a) D, (b) 1 **2.** 5 **3.** 9 **4.** A,C **5.** A,B **6.** 4
7. C **8.** A,D **9.** B,D **10.** A,B,D **11.** 1,3,4

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JEE (Main/Advanced) Syllabus

JEE (Main) Syllabus :

Ordinary differential equations, their order and degree. Formation of differential equations. Solution of differential equations by the method of separation of variables, solution of homogeneous and linear differential equations of the type : dy + p(x)y = q(x)dx. Determining areas of the regions bounded by simple curves in standard form.

JEE (Advanced) Syllabus :

Formation of ordinary differential equations, solution of homogeneous differential equations, separation of variables method, linear first order differential equations.

Application of definite integrals to the determination of areas involving simple curves.

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DIFFERENTIAL EQUATION

1. DIFFERENTIAL EQUATION :

An equation that involves independent and dependent variables and the derivatives of the dependent variables is called a **differential equation**.

A differential equation is said to be **ordinary**, if the differential coefficients have reference to a single independent variable only e.g. $\frac{d^2y}{dx^2}-\frac{2dy}{dx}+\cos x=0$ and it is said to be **partial** if there are two or more independent variables. e.g. $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$ is a partial differential equation. We are concerned with ordinary differential equations only.

2. ORDER OF DIFFERENTIAL EQUATION :

The order of a differential equation is the order of the highest differential coefficient occurring in it.

3. DEGREE OF DIFFERENTIAL EQUATION :

The exponent of the highest order differential coefficient, when the differential equation is expressed as a polynomial in all the differential coefficient.

Thus the differential equation :

$$f(x,y)\left[\frac{d^m y}{dx^m}\right]^p+\phi(x,y)\left[\frac{d^{m-1}(y)}{dx^{m-1}}\right]^q+\ldots\ldots\ldots=0 \text{ is of order } m \text{ \& degree } p.$$

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Note :

(i) The exponents of all the differential coefficient should be free from radicals and fraction.

(ii) The degree is always positive natural number.

(iii) The degree of differential equation may or may not exist.

Illustration 1 : Find the order and degree of the following differential equation :

(i) $\sqrt{\frac{d^2y}{dx^2}}=\sqrt[3]{\frac{dy}{dx}+3}$ (ii) $\frac{d^2y}{dx^2}=\sin\left(\frac{dy}{dx}\right)$ (iii) $\frac{dy}{dx}=\sqrt{3x+5}$

Solution : (i) The given differential equation can be re-written as $\left(\frac{d^2y}{dx^2}\right)^3=\left(\frac{dy}{dx}+3\right)^2$

Hence order is 2 and degree is 3.

(ii) The given differential equation has the order 2. Since the given differential equation cannot be written as a polynomial in the differential coefficients, the degree of the equation is not defined.

(iii) Its order is 1 and degree 1. **Ans.**

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Illustration 2 : The order and degree of the differential equation $\left(\frac{d^2s}{dt^2}\right)^2 + 3\left(\frac{ds}{dt}\right)^3 + 4 = 0$ are -

(A) 2 , 2 (B) 2 , 3 (C) 3 , 2 (D) none of these

Solution : Clearly order is 2 and degree is 2 (from the definition of order and degree of differential equations). **Ans. (A)**

Do yourself - 1 :

Find the order and degree of following differential equations

(i) $[1 + (y')^2]^{1/2} = x^2 + y$ **(ii)** $(1 + y')^{1/2} = y''$ **(iii)** $y' = \sin y$

4. FORMATION OF A DIFFERENTIAL EQUATION :

In order to obtain a differential equation whose solution is

$$f(x_1, y_1, c_1, c_2, c_3.....,c_n) = 0$$

where $c_1, c_2,.....c_n$ are 'n' arbitrary constants, we have to eliminate the 'n' constants for which we require (n+1) equations.

A differential equation is obtained as follows :

(a) Differentiate the given equation w.r.t the independent variable (say x) as many times as the number of independent arbitrary constants in it.

(b) Eliminate the arbitrary constants.

(c) The eliminant is the required differential equation.

Note :

(i) A differential equation represents a family of curves all satisfying some common properties. This can be considered as the geometrical interpretation of the differential equation.

(ii) For there being n differentiation, the resulting equation must contain a derivative of nth order i.e. equal to number of independent arbitrary constant.

Illustration 3 : Find the differential equation of all parabolas whose axes is parallel to the x-axis and having latus rectum a.

Solution : Equation of parabola whose axes is parallel to x-axis and having latus rectum 'a' is $(y - \beta)^2 = a(x - \alpha)$

Differentiating both sides, we get $2(y - \beta)\frac{dy}{dx} = a$

Again differentiating, we get

$$\Rightarrow 2(y - \beta)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0 \quad \Rightarrow \quad a\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^3 = 0.$$ **Ans.**

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Illustration 4 : Find the differential equation whose solution represents the family : $c\,(y + c)^2 = x^3$

Solution : $c\,(y + c)^2 = x^3$ ...(i)

Differentiating, we get, $c.\left[2(y+c)\right]\dfrac{dy}{dx} = 3x^2$

Writing the value of c from (i), we have

$$\frac{2x^3}{(y+c)^2}(y + c)\frac{dy}{dx} = 3x^2 \qquad \Rightarrow \frac{2x^3}{y+c}\frac{2x^3}{y+c} = 3x^2$$

i.e. $\dfrac{2x}{y+c}\dfrac{dy}{dx} = 3 \Rightarrow \dfrac{2x}{3}\left[\dfrac{dy}{dx}\right] = y + c$

Hence $c = \dfrac{2x}{3}\left[\dfrac{dy}{dx}\right] - y$

Substituting value of c in equation (i), we get $\left[\dfrac{2x}{3}\left(\dfrac{dy}{dx}\right) - y\right]\left[\dfrac{2x}{3}\dfrac{dy}{dx}\right]^2 = x^3$,

which is the required differential equation. **Ans.**

Illustration 5 : Find the differential equation whose solution represents the family : $y = a\cos\theta x + b\sin\theta x$, where θ = fixed constant

Solution : $y = a\cos\theta x + b\sin\theta x$, θ = fixed constant ....(i)

Differentiating, we get $\dfrac{dy}{dx} = -\theta a\sin\theta x + \theta b\cos\theta x$

Again differentiating, we get $\dfrac{d^2y}{dx^2} = -\theta^2 a\cos\theta x - \theta^2 b\sin\theta x$

using equation (i), we get $\dfrac{d^2y}{dx^2} = -\theta^2 y$ **Ans.**

Do yourself - 2

Eliminate the arbitrary constants and obtain the differential equation satisfied by it.

(i) $y = 2x + ce^x$ **(ii)** $y = \left(\dfrac{a}{x^2}\right) + bx$ **(iii)** $y = ae^{2x} + be^{-2x} + c$

5. SOLUTION OF DIFFERENTIAL EQUATION :

The solution of the differential equation is a relation between the variables of the equation not containing the derivatives, but satisfying the given differential equation (i.e., from which the given differential equation can be derived).

Thus, the solution of $\dfrac{dy}{dx} = e^x$ could be obtained by simply integrating both sides, i.e., $y = e^x + c$ and that of, $\dfrac{dy}{dx} = px + q$ is $y = \dfrac{px^2}{2} + qx + c$, where c is arbitrary constant.

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(i) **A general solution** or an integral of a differential equation is a relation between the variables (not involving the derivatives) which contains the same number of the arbitrary constants as the order of the differential equation.

For example, a general solution of the differential equation $\frac{d^2x}{dt^2} = -4x$ is $x = A\cos 2t + B\sin 2t$ where A and B are the arbitrary constants.

(ii) **Particular solution or particular integral** is that solution of the differential equation which is obtained from the general solution by assigning particular values to the arbitrary constant in the general solution.

For example, $x = 10\cos 2t + 5\sin 2t$ is a particular solution of differential equation $\frac{d^2x}{dt^2} = -4x$.

Note :

(i) The general solution of a differential equation can be expressed in different (but equivalent) forms. For example

$$\log x - \log(y+2) = k \qquad ....(i)$$

where k is an arbitrary constant is the general solution of the differential equation $xy' = y + 2$. The solution given by equation (i) can also be re-written as

$$\log\left(\frac{x}{y+2}\right) = k \text{ or } \frac{x}{y+2} = e^k = c_1 \qquad ...(ii)$$

or $$x = c_1(y+2) \qquad ...(iii)$$

where $c_1 = e^k$ is another arbitrary constant. The solution (iii) can also be written as

$$y + 2 = c_2 x$$

where $c_2 = 1/c_1$ is another arbitrary constant.

(ii) All differential equations that we come across have unique solutions or a family of solutions.

For example, the differential equation $\left|\frac{dy}{dx}\right| + |y| = 0$ has only the trivial solution, i.e. $y = 0$.

The differential equation $\left|\frac{dy}{dx}\right| + |y| + c = 0,\ c > 0$ has no solution.

6. ELEMENTARY TYPES OF FIRST ORDER & FIRST DEGREE DIFFERENTIAL EQUATIONS :

(a) **Separation of Variables :**

Some differential equations can be solved by the method of separation of variables (or "variable separable"). This method is only possible, if we can express the differential equation in the form $A(x)dx + B(y)\,dy = 0$

where A(x) is a function of 'x' only and B(y) is a function of 'y' only.

A general solution of this is given by,

$$\int A(x)\,dx + \int B(y)dy = c$$

where 'c' is the arbitrary constant.

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Illustration 6 : Solve the differential equation $xy\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}(1 + x + x^2)$.

Solution : Differential equation can be rewritten as

$$xy\frac{dy}{dx} = (1 + y^2)\left(1+\frac{x}{1+x^2}\right) \Rightarrow \frac{y}{1+y^2}dy = \left(\frac{1}{x}+\frac{1}{1+x^2}\right)dx$$

Integrating, we get

$$\frac{1}{2}\ell n(1 + y^2) = \ell n\, x + \tan^{-1} x + \ell n\, c \Rightarrow \sqrt{1+y^2} = cxe^{\tan^{-1}x}.$$ **Ans.**

Illustration 7 : Solve the differential equation $(x^3 - y^2x^3)\dfrac{dy}{dx} + y^3 + x^2y^3 = 0$.

Solution : The given equation $(x^3 - y^2x^3)\dfrac{dy}{dx} + y^3 + x^2y^3 = 0$

Case-I : y = 0 is one solution of differential equation

Case-II : If $y \neq 0$, then

$$\Rightarrow \frac{1-y^2}{y^3}dy + \frac{1+x^2}{x^3}dx = 0 \quad \Rightarrow \int\left(\frac{1}{y^3}-\frac{1}{y}\right)dy + \int\left(\frac{1}{x^3}+\frac{1}{x}\right)dx = 0$$

$$\Rightarrow \log\left(\frac{x}{y}\right) = \frac{1}{2}\left(\frac{1}{y^2}+\frac{1}{x^2}\right) + c$$

$$y = 0 \text{ or } \log\left(\frac{x}{y}\right) = \frac{1}{2}\left(\frac{1}{y^2}+\frac{1}{x^2}\right) + c$$ **Ans.**

Illustration 8 : Solve : $\dfrac{dy}{dx} = (x - 3)(y + 1)^{2/3}$

Solution : Case-I : y = –1 is one solution of differential equation

Case-II : If $y \neq -1$, then

$$\frac{dy}{dx} = (x-3)(y+1)^{2/3}$$

$$\int\frac{dy}{(y+1)^{2/3}} = \int(x-3)dx$$

Integrate and solve for y : $3(y + 1)^{1/3} = \dfrac{1}{2}(x - 3)^2 + C$

$$(y + 1)^{1/3} = \frac{1}{6}(x-3)^2 + C_0 \Rightarrow y+1 = \left(\frac{1}{6}(x-3)^2 + C_0\right)^3$$

$$y = -1 \text{ or } y = \left(\frac{1}{6}(x-3)^2 + C_0\right)^3 - 1$$ **Ans.**

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Do yourself - 3 :
Solve the following differential equations :

(i) $\frac{2dy}{dx} = \frac{y(x+1)}{x}$ **(ii)** $\sqrt{1+4x^2}\,dy = y^3 x dx$ **(iii)** $(\tan y)\frac{dy}{dx} = \sin(x+y) + \sin(x-y)$

(i) Equation of the form :

$\Rightarrow$ **$y' = f(ax + by + c), b \neq 0$**

To solve this, substitute $t = ax + by + c$. Then the equation reduces to separable type in the variable t and x which can be solved.

Illustration 9 : Solve $\frac{dy}{dx} = \cos(x+y) - \sin(x+y)$.

Solution : $\frac{dy}{dx} = \cos(x+y) - \sin(x+y)$

Substituting, $x + y = t$, we get $\frac{dy}{dx} = \frac{dt}{dx} - 1$

Therefore $\frac{dt}{dx} - 1 = \cos t - \sin t$

$$\Rightarrow \int \frac{dt}{1+\cos t - \sin t} = \int dx \Rightarrow \int \frac{\sec^2 \frac{t}{2} dt}{2\left(1-\tan\frac{t}{2}\right)} = \int dx \Rightarrow -\ell n\left|1-\tan\frac{x+y}{2}\right| = x + c.$$

Ans.

Illustration 10: Solve : $y' = (x + y + 1)^2$

Solution : $y' = (x + y + 1)^2$ ....(i)

Let $t = x + y + 1$

$$\frac{dt}{dx} = 1 + \frac{dy}{dx}$$

Substituting in equation (i) we get

$$\frac{dt}{dx} = t^2 + 1 \Rightarrow \int \frac{dt}{1+t^2} = \int dx \Rightarrow \tan^{-1} t = x + C \Rightarrow t = \tan(x + C)$$

$x + y + 1 = \tan(x + C) \Rightarrow y = \tan(x + C) - x - 1$ **Ans.**

Do yourself - 4 :
Solve the following differential equations :

(i) $\frac{dy}{dx} = (y - 4x)^2$ **(ii)** $\tan^2(x + y)dx - dy = 0$

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(ii) Equation of the form : $\Rightarrow \dfrac{dy}{dx} = \dfrac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$

Case I : If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$ **then**

Let $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \lambda$ then $a_1 = \lambda a_2$ .....(i) ; $b_1 = \lambda b_2$ .....(ii)

from (i) and (ii), differential equation becomes

$$\frac{dy}{dx} = \frac{\lambda a_2x + \lambda b_2y + c_1}{a_2x + b_2y + c_2} \Rightarrow \frac{dy}{dx} = \frac{\lambda(a_2x + b_2y) + c_1}{a_2x + b_2y + c_2}$$

or we can say, $\dfrac{dy}{dx} = f(a_2x + b_2y)$

which can be solved by substituting $t = a_2x + b_2y$

Illustration 11 : Solve : $(x + y)dx + (3x + 3y - 4)\, dy = 0$

Solution : Let $t = x + y$

$\Rightarrow$ $dy = dt - dx$

So we get, $tdx + (3t - 4)(dt - dx) = 0$

$$2dx + \left(\frac{3t-4}{2-t}\right)dt = 0 \Rightarrow 2dx - 3dt + \frac{2}{2-t}dt = 0$$

Integrating and replacing t by x + y, we get

$2x - 3t - 2[\ell n|(2 - t)|] = c_1$

$\Rightarrow$ $2x - 3(x + y) - 2[\ell n|(2 - x - y)|] = c_1$

$\Rightarrow$ $x + 3y + 2\ell n|(2 - x - y)| = c$ **Ans.**

Case II : **If $a_2 + b_1 = 0$, then** a simple cross multiplication and substituting d(xy) for xdy + ydx and integrating term by term, yield the results easily.

Illustration 12 : Solve $\dfrac{dy}{dx} = \dfrac{x - 2y + 1}{2x + 2y + 3}$

Solution : $\dfrac{dy}{dx} = \dfrac{x - 2y + 1}{2x + 2y + 3}$

$\Rightarrow$ $2xdy + 2y\, dy + 3dy = xdx - 2y\, dx + dx$

$\Rightarrow$ $(2y + 3)\, dy = (x + 1)\, dx - 2(xdy + ydx)$

On integrating, we get

$$\Rightarrow \int (2y+3)dy = \int (x+1)dx - \int 2d(xy)$$

Solving : $y^2 + 3y = \dfrac{x^2}{2} + x - 2xy + c$ **Ans.**

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Do yourself - 5 :

Solve the following differential equations :

(i) $\frac{dy}{dx} = \frac{2x - y + 2}{2y - 4x + 1}$ **(ii)** $\frac{dy}{dx} = \frac{3x - 5y}{5x + y + 3}$

(iii) Equation of the form :

$\Rightarrow$ **$yf(xy)dx + xg(xy)dy = 0$** ..... (i)

The substitution xy = z, reduces differential equation of this form to the form in which the variables are separable.

Let $xy = z$ ..... (ii)

$dy = \left[\frac{xdz - zdx}{x^2}\right]$ ..... (iii)

using equation (ii) & (iii), equation (i) becomes $\frac{z}{x}f(z)dx + xg(z)\left[\frac{xdz - zdx}{x^2}\right] = 0$

$\Rightarrow \frac{z}{x}f(z)dx + g(z)dz - \frac{z}{x}g(z)dx = 0 \quad \Rightarrow \quad \frac{z}{x}\{f(z) - g(z)\}dx + g(z)dz = 0$

$\Rightarrow \frac{1}{x}dx + \frac{g(z)dz}{z\{f(z) - g(z)\}} = 0$

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Illustration 13 : Solve $y(xy + 1)dx + x(1 + xy + x^2y^2)dy = 0$

Solution : Let xy = v

$y = \frac{v}{x} \Rightarrow dy = \frac{xdv - vdx}{x^2}$

Now, differential equation becomes $\Rightarrow \frac{v}{x}(v+1)dx + x(1 + v + v^2)\left(\frac{xdv - vdx}{x^2}\right) = 0$

On solving, we get

$v^3dx - x(1 + v + v^2)\,dv = 0$

separating the variables & integrating we get

$\Rightarrow \int\frac{dx}{x} - \int\left(\frac{1}{v^3} + \frac{1}{v^2} + \frac{1}{v}\right)dv = 0 \quad \Rightarrow \quad \ell nx + \frac{1}{2v^2} + \frac{1}{v} - \ell nv = c$

$\Rightarrow 2v^2\ell n\left(\frac{v}{x}\right) - 2v - 1 = -2cv^2 \quad \Rightarrow \quad 2x^2y^2\ell ny - 2xy - 1 = Kx^2y^2$ where $K = -2c$

Do yourself - 6 :

Solve the following differential equations :

(i) $(y - xy^2)dx - (x + x^2y)dy = 0$ **(ii)** $y(1 + 2xy)dx + x(1 - xy)dy = 0$

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(iv) Transformation to polar-co-ordinates :

Sometimes conversion of cartesian co-ordinates into polar coordinates helps us in separating the variables.

(1) $x = r\cos\theta,\ y = r\sin\theta$

then $x^2 + y^2 = r^2$

$xdx + ydy = rdr$

$xdy - ydx = r^2 d\theta$

(2) $x = r\sec\theta,\ y = r\tan\theta$

then $x^2 - y^2 = r^2$

$xdx - ydy = rdr$

$xdy - ydx = r^2 \sec\theta\, d\theta$

Illustration 14 : Solve : $\dfrac{x + y\dfrac{dy}{dx}}{y - x\dfrac{dy}{dx}} = x^2 + 2y^2 + \dfrac{y^4}{x^2}$

Solution : The given equation can be reduced to

$$\frac{xdx + ydy}{ydx - xdy} = \frac{(x^2 + y^2)^2}{x^2}$$

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Substituting $x = r\cos\theta$

$y = r\sin\theta$

we get, $\dfrac{rdr}{r^2 d\theta} = \dfrac{-(r^2)^2}{r^2\cos^2\theta} \Rightarrow \displaystyle\int \frac{dr}{r^3} = -\int \sec^2\theta d\theta \Rightarrow -\frac{1}{2r^2} = -\tan\theta + c$

Substituting, $\dfrac{1}{2(x^2 + y^2)} = \dfrac{y}{x} + K$ **Ans.**

Do yourself - 7 :

Solve the following differential equations :

(i) $xdx + ydy = xdy - ydx$ **(ii)** $ydx - xdy = xy\,dy - x^2dx$

(b) Homogeneous equations :

A function $f(x,y)$ is said to be a homogeneous function of degree n, if the substitution $x = \lambda x,\ y = \lambda y,\ \lambda > 0$ produces the equality

$f(\lambda x, \lambda y) = \lambda^n f(x,y)$

The degree of homogeneity 'n' can be any real number.

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Illustration 15 : Find the degree of homogeneity of function

(i) $f(x,y) = x^2 + y^2$ (ii) $f(x,y) = (x^{3/2} + y^{3/2})/(x + y)$ (iii) $f(x,y) = \sin\left(\frac{x}{y}\right)$

Solution : (i) $f(\lambda x, \lambda y) = \lambda^2x^2 + \lambda^2y^2 = \lambda^2(x^2 + y^2) = \lambda^2 f(x,y)$

degree of homogeneity $\rightarrow 2$

(ii) $f(\lambda x, \lambda y) = \dfrac{\lambda^{3/2}x^{3/2} + \lambda^{3/2}y^{3/2}}{\lambda x + \lambda y}$

$f(\lambda x, \lambda y) = \lambda^{1/2} f(x,y)$

degree of homogeneity $\rightarrow 1/2$

(iii) $f(\lambda x, \lambda y) = \sin\left(\dfrac{\lambda x}{\lambda y}\right) = \lambda^\circ \sin\left(\dfrac{x}{y}\right) = \lambda^\circ f(x,y)$

degree of homogeneity $\rightarrow 0$

Illustration 16 : Determine whether or not each of the following functions is homogeneous.

(i) $f(x,y) = x^2 - xy$ (ii) $f(x,y) = \dfrac{xy}{x + y^2}$ (iii) $f(x,y) = \sin xy$

Solution : (i) $f(\lambda x, \lambda y) = \lambda^2x^2 - \lambda^2xy = \lambda^2(x^2 - xy) = \lambda^2 f(x,y)$ homogeneous.

(ii) $f(\lambda x, \lambda y) = \dfrac{\lambda^2 xy}{\lambda x + \lambda^2 y^2} \neq \lambda^n f(x,y)$ not homogeneous.

(iii) $f(\lambda x, \lambda y) = \sin(\lambda^2 xy) \neq \lambda^n f(x,y)$ not homogeneous.

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Do yourself - 8 :

(i) Find the degree of homogeneity of function $f(x,y) = x^3 \ell n\left[\sqrt{x+y}/\sqrt{x-y}\right]$

(ii) Find the degree of homogeneity of function $f(x,y) = ax^{2/3} + hx^{1/3}y^{1/3} + by^{2/3}$

(iii) Determine whether or not each of the following functions is homogeneous.

(a) $f(x,y) = \sqrt{x^2 + 2xy + 3y^2}$ (b) $f(x,y) = x + y\cos\dfrac{y}{x}$ (c) $f(x,y) = x\sin y + y\sin x$.

(i) Homogeneous first order differential equation

A differential equation of the form $\dfrac{dy}{dx} = \dfrac{f(x,y)}{g(x,y)}$

where $f(x,y)$ and $g(x,y)$ are homogeneous functions of x,y and of the same degree, is said to be homogeneous. Such equations can be solved by substituting

$$y = vx,$$

so that the dependent variable y is changed to another variable v.

Since $f(x,y)$ and $g(x,y)$ are homogeneous functions of the same degree say, n, they can be written as

$$f(x,y) = x^n f_1\left(\frac{y}{x}\right) \quad \text{and} \quad g(x,y) = x^n g_1\left(\frac{y}{x}\right).$$

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As y = vx, we have $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

The given differential equation, therefore, becomes

$$v + x\frac{dv}{dx} = \frac{f_1(v)}{g_1(v)} \Rightarrow \frac{g_1(v)dv}{f_1(v) - vg_1(v)} = \frac{dx}{x},$$

so that the variables v and x are now separable.

Note : Sometimes homogeneous equation can be solved by substituting x = vy or by using polar coordinate substitution.

Illustration 17 : The solution of the differential equation $\frac{dy}{dx} = \frac{\sin y + x}{\sin 2y - x\cos y}$ is -

(A) $\sin^2 y = x \sin y + \frac{x^2}{2} + c$ (B) $\sin^2 y = x \sin y - \frac{x^2}{2} + c$

(C) $\sin^2 y = x + \sin y + \frac{x^2}{2} + c$ (D) $\sin^2 y = x - \sin y + \frac{x^2}{2} + c$

Solution : Here, $\frac{dy}{dx} = \frac{\sin y + x}{\sin 2y - x\cos y}$

$\Rightarrow \cos y\frac{dy}{dx} = \frac{\sin y + x}{2\sin y - x}$, (put sin y = t)

$\Rightarrow \frac{dt}{dx} = \frac{t + x}{2t - x}$ (put t = vx)

$$\frac{xdv}{dx} + v = \frac{vx + x}{2vx - x} = \frac{v+1}{2v-1}$$

$$\therefore \quad x\frac{dv}{dx} = \frac{v+1}{2v-1} - v = \frac{v + 1 - 2v^2 + v}{2v-1}$$

or $\frac{2v-1}{-2v^2 + 2v + 1}dv = \frac{dx}{x}$ on solving, we get

$\sin^2 y = x \sin y + \frac{x^2}{2} + c.$ **Ans. (A)**

Illustration 18 : Solve the differential equation $(1 + 2e^{x/y})\,dx + 2e^{x/y}(1 - x/y)\,dy = 0$.

Solution : The equation is homogeneous of degree 0.

Put x = vy, dx = v dy + y dv,

Then, differential equation becomes

$(1 + 2e^v)(v\,dy + y\,dv) + 2e^v(1 - v)\,dy = 0 \Rightarrow (v + 2e^v)\,dy + y(1 + 2e^v)\,dv = 0$

$$\frac{dy}{y} + \frac{1 + 2e^v}{v + 2e^v}dv = 0$$

Integrating and replacing v by x/y, we get $\ell n\, y + \ell n(v + 2e^v) = \ell nc$ and $x + 2ye^{x/y} = c$

Ans.

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Do yourself - 9 :

Solve the following differential equations :

(i) $y' = \dfrac{3x - y}{x + y}$

(ii) $(x - y\ell ny + y\ell nx)\,dx + x(\ell ny - \ell nx)\,dy = 0$

(iii) $(3xy + y^2)dx + (x^2 + xy)dy = 0,\ y(1) = 1$

(ii) Equations reducible to homgeneous form

The equation of the form $\dfrac{dy}{dx} = \dfrac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$ where $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$

can be reduced to homgeneous form by changing the variable x, y to u,v as

$x = u + h\ ,\ y = v + k$

where h,k are the constants to be chosen so as to make the given equation homgeneous.

We have

$$\frac{dy}{dx} = \frac{dv}{du}$$

$\therefore$ The equation becomes, $\dfrac{dv}{du} = \dfrac{a_1u + b_1v + (a_1h + b_1k + c_1)}{a_2u + b_2v + (a_2h + b_2k + c_2)}$

Let h and k be chosen so as to satisfy the equation

$a_1h + b_1k + c_1 = 0$ ...(i)

$a_2h + b_2k + c_2 = 0$ ...(ii)

Solve for h and k from (i) and (ii)

Now $\dfrac{du}{dv} = \dfrac{a_1u + b_1v}{a_2u + b_2v}$

is a homgeneous equation and can be solved by substituting v = ut.

Illustration 19 : Solve $\dfrac{dy}{dx} = \dfrac{x + 2y + 3}{2x + 3y + 4}$

Solution : Put x = X + h, y = Y + k

We have $\dfrac{dY}{dX} = \dfrac{X + 2Y + (h + 2k + 3)}{2X + 3Y + (2h + 3k + 4)}$

To determine h and k, we write

$h + 2k + 3 = 0,\ 2h + 3k + 4 = 0 \Rightarrow h = 1,\ k = -2$

So that $\dfrac{dY}{dX} = \dfrac{X + 2Y}{2X + 3Y}$

Putting Y = VX, we get

$$V + X\frac{dV}{dX} = \frac{1 + 2V}{2 + 3V} \Rightarrow \frac{2 + 3V}{3V^2 - 1}dV = -\frac{dX}{X}$$

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$$\Rightarrow \quad \left[\frac{2+\sqrt{3}}{2(\sqrt{3}V-1)}-\frac{2-\sqrt{3}}{2(\sqrt{3}V+1)}\right]dV = -\frac{dX}{X}$$

$$\Rightarrow \quad \frac{2+\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}V-1)-\frac{2-\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}V+1) = (-\log X + c)$$

$$\frac{2+\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}Y-X)-\frac{2-\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}Y+X) = A \text{ where A is another constant}$$

and $X = x - 1$, $Y = y + 2$. **Ans.**

Do yourself - 10 :

(i) Solve the differential equation : $\frac{dy}{dx}=\frac{x+2y-5}{2x+y-4}$

(c) Linear differential equations :

A differential equation is said to be linear if the dependent variable & its differential coefficients occur in the first degree only and are not multiplied together.

The nth order linear differential equation is of the form ;

$a_0(x)\frac{d^ny}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+.....+a_n(x)\,y = \phi(x)$, where $a_0(x)$, $a_1(x)$ .... $a_n(x)$ are called the coefficients of the differential equation.

Note that a linear differential equation is always of the first degree but every differential equation of the first degree need not be linear . e.g. the differential equation $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3+y^2=0$ is not linear, though its degree is 1.

Illustration 20: Which of the following equation is linear ?

(A) $\frac{dy}{dx}+xy^2=1$ (B) $x^2\frac{dy}{dx}+y=e^x$ (C) $\frac{dy}{dx}+3y=xy^2$ (D) $x\frac{dy}{dx}+y^2=\sin x$

Solution : Clearly answer is (B)

Illustration 21 : Which of the following equation is non-linear ?

(A) $\frac{dy}{dx}=\cos x$ (B) $\frac{d^2y}{dx^2}+y=0$ (C) $dx+dy=0$ (D) $x\left(\frac{dy}{dx}\right)+\frac{3}{\left(\frac{dy}{dx}\right)}=y^2$

Solution : Clearly answer is (D)

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(i) Linear differential equations of first order :

The most general form of a linear differential equation of first order is $\frac{dy}{dx}+Py=Q$, where P & Q are functions of x.

To solve such an equation multiply both sides by $e^{\int Pdx}$.

So that we get $$e^{\int Pdx}\left[\frac{dy}{dx}+Py\right]=Qe^{\int Pdx} \quad ....(i)$$

$$\frac{d}{dx}\left(e^{\int Pdx}.y\right)=Qe^{\int Pdx} \quad ...(ii)$$

On integrating equation (ii), we get

$$ye^{\int Pdx}=\int Qe^{\int Pdx}dx+c$$

This is the required general solution.

Note :

(i) The factor $e^{\int Pdx}$ on multiplying by which the left hand side of the differential equation becomes the differential coefficient of some function of x & y, is called integrating factor of the differential equation popularly abbreviated as I.F.

(ii) Sometimes a given differential equation becomes linear if we take y as the independent variable and x as the dependent variable. e.g. the equation ; $(x+y+1)\frac{dy}{dx}=y^2+3$ can be written as $(y^2+3)\frac{dx}{dy}=x+y+1$ which is a linear differential equation.

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Illustration 22 : Solve $(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$.

Solution : Differential equation can be rewritten as $(1+y^2)\frac{dx}{dy}+x=e^{\tan^{-1}y}$

or $$\frac{dx}{dy}+\frac{1}{1+y^2}.x=\frac{e^{\tan^{-1}y}}{1+y^2} \quad ...(i)$$

I. F = $e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$

so solution is $xe^{\tan^{-1}y}=\int\frac{e^{\tan^{-1}y}e^{\tan^{-1}y}}{1+y^2}dy$

Let $e^{\tan^{-1}y}=t\Rightarrow\frac{e^{\tan^{-1}y}}{1+y^2}dy=dt$

$xe^{\tan^{-1}y}=\int t\,dt$ [Putting $e^{\tan^{-1}y}=t$]

or $xe^{\tan^{-1}y}=\frac{t^2}{2}+\frac{c}{2}\Rightarrow 2xe^{\tan^{-1}y}=e^{2\tan^{-1}y}+c.$ **Ans.**

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Illustration 23 : The solution of differential equation $(x^2-1)\dfrac{dy}{dx}+2xy=\dfrac{1}{x^2-1}$ is -

(A) $y(x^2-1)=\dfrac{1}{2}\log\left|\dfrac{x-1}{x+1}\right|+C$ (B) $y(x^2+1)=\dfrac{1}{2}\log\left|\dfrac{x-1}{x+1}\right|-C$

(C) $y(x^2-1)=\dfrac{5}{2}\log\left|\dfrac{x-1}{x+1}\right|+C$ (D) none of these

Solution : The given differential equation is

$$(x^2-1)\frac{dy}{dx}+2\ xy=\frac{1}{x^2-1} \quad\Rightarrow\quad \frac{dy}{dx}+\frac{2x}{x^2-1}y=\frac{1}{(x^2-1)^2} \qquad ...(i)$$

This is linear differential equation of the form

$$\frac{dy}{dx}+Py=Q,\ \text{where } P=\frac{2x}{x^2-1} \text{ and } Q=\frac{1}{(x^2-1)^2}$$

$$\therefore\ \text{I.F.}=e^{\int P\,dx}=e^{\int 2x/(x^2-1)dx}=e^{\log(x^2-1)}=(x^2-1)$$

multiplying both sides of (i) by I.F. = (x^2-1), we get

$$(x^2-1)\frac{dy}{dx}+2\ xy=\frac{1}{x^2-1}$$

integrating both sides we get

$$y(x^2-1)=\int\frac{1}{x^2-1}dx+C \qquad [\text{Using : } y\,(\text{I.F.})=\int Q.(\text{I.F.})\ dx+C]$$

$$\Rightarrow\ y\,(x^2-1)=\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+C.$$

This is the required solution. **Ans.(A)**

Do yourself - 11 :
Solve the following differential equations :

(i) $\dfrac{xdy}{dx}=2y+x^4+6x^2+2x,\ x\neq 0$ **(ii)** $(x-a)\dfrac{dy}{dx}+3y=12(x-a)^3,\ x>a>0$

(iii) $y\,\ell n\,y\,dx+(x-\ell n\,y)\,dy=0$

(ii) Equation reducible to linear form :

The equation of the form $\dfrac{dy}{dx}+Py=Qy^n$, where P and Q are functions of x, is called **Bernoulli's equation.**

On dividing by y^n, we get $y^{-n}\dfrac{dy}{dx}+Py^{-n+1}=Q$

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Let $y^{-n+1} = t$, so that $(-n+1)y^{-n}\dfrac{dy}{dx} = \dfrac{dt}{dx}$

then equation becomes $\dfrac{dt}{dx} + P(1-n)t = Q(1-n)$

which is linear with t as a dependent variable.

Illustration 24: Solve the differential equation $x\dfrac{dy}{dx} + y = x^3y^6$.

Solution : The given differential equation can be written as

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{xy^5} = x^2$$

Putting $y^{-5} = v$ so that

$$-5\,y^{-6}\frac{dy}{dx} = \frac{dv}{dx} \text{ or } y^{-6}\frac{dy}{dx} = -\frac{1}{5}\frac{dv}{dx} \text{ we get}$$

$$-\frac{1}{5}\frac{dv}{dx} + \frac{1}{x}v = x^2 \Rightarrow \frac{dv}{dx} - \frac{5}{x}v = -5x^2 \qquad .....(i)$$

This is the standard form of the linear deferential equation having integrating factor

$$\text{I.F} = e^{\int -\frac{5}{x}dx} = e^{-5\log x} = \frac{1}{x^5}$$

Multiplying both sides of (i) by I.F. and integrating w.r.t. x

We get $v\,.\,\dfrac{1}{x^5} = \displaystyle\int -5x^2\,.\,\frac{1}{x^5}dx$

$$\Rightarrow \frac{v}{x^5} = \frac{5}{2}x^{-2} + c$$

$$\Rightarrow y^{-5}x^{-5} = \frac{5}{2}x^{-2} + c$$ which is the required solution. **Ans.**

Illustration 25 : Find the solution of differential equation $\dfrac{dy}{dx} - y\tan x = -y^2\sec x$.

Solution : $\dfrac{1}{y^2}\dfrac{dy}{dx} - \dfrac{1}{y}\tan x = -\sec x$

$\dfrac{1}{y} = v$; $\dfrac{-1}{y^2}\dfrac{dy}{dx} = \dfrac{dv}{dx}$ $\qquad\therefore \dfrac{-dv}{dx} - v\tan x = -\sec x$

$\dfrac{dv}{dx} + v\tan x = \sec x$, $\qquad$ Here P = tan x, Q = sec x

I.F. = $e^{\int \tan x dx} = |\sec x|$ $\qquad$ $v\,|\sec x| = \displaystyle\int \sec^2 x\,dx + c$

Hence the solution is $y^{-1}\,|\sec x| = \tan x + c$ **Ans.**

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Do yourself - 12 :
Solve the following differential equations :

(i) $y' + 3y = e^{3x} y^2$ **(ii)** $xdy - \{y + xy^3 (1 + \ell nx)\}dx = 0$ **(iii)** $\frac{dy}{dx} + y = y^2(\cos x - \sin x)$

7. TRAJECTORIES :

A curve which cuts every member of a given family of curves according to a given law is called a Trajectory of the given family.

The trajectory will be called **Orthogonal** if each trajectory cuts every member of given family at right angle.

Working rule for finding orthogonal trajectory

1. Form the differential equation of family of curves

2. Write $-\frac{1}{dy/dx}$ for $\frac{dy}{dx}$ or $-\frac{r^2 d\theta}{dr}$ for $\frac{dr}{d\theta}$ if differential equation is in the polar form.

3. Solve the new differential equation to get the equation of orthogonal trajectories.

Note: A family of curves is self-orthogonal if it is its own orthogonal family.

Illustration 26: Find the value of k such that the family of parabolas $y = cx^2 + k$ is the orthogonal trajectory of the family of ellipses $x^2 + 2y^2 - y = c$.

Solution : Differentiate both sides of $x^2 + 2y^2 - y = c$ w.r.t. x, We get

$$2x + 4y\frac{dy}{dx} - \frac{dy}{dx} = 0$$

or $2x + (4y - 1)\frac{dy}{dx} = 0$, is the differential equation of the given family of curves.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ to obtain the differential equation of the orthogonal trajectories, we get

$$2x + \frac{(1-4y)}{\frac{dy}{dx}} = 0 \Rightarrow \frac{dy}{dx} = \frac{4y-1}{2x}$$

$\Rightarrow \int \frac{dy}{4y-1} = \int \frac{dx}{2x} \Rightarrow \frac{1}{4}\ell n(4y - 1) = \frac{1}{2}\ell n\, x + \frac{1}{2}\ell n\, a$, where a is any constant.

$\Rightarrow \ell n(4y - 1) = 2\,\ell n\, x + 2\,\ell n\, a$ or , $4y - 1 = a^2x^2$

or, $y = \frac{1}{4}a^2x^2 + \frac{1}{4}$, is the required orthogonal trajectory, which is of the form

$y = cx^2 + k$ where $c = \frac{a^2}{4}$, $k = \frac{1}{4}$. **Ans.**

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Illustration 27 : Prove that $\frac{x^2}{1+\lambda}+\frac{y^2}{4+\lambda}=1$ are self orthogonal family of curves.

Solution :

$$\frac{x^2}{1+\lambda}+\frac{y^2}{4+\lambda}=1 \qquad ...(i)$$

Differentiating (i) with respect to x, we have

$$\frac{x}{1+\lambda}+\frac{y}{4+\lambda}\frac{dy}{dx}=0 \qquad ...(ii)$$

From (i) and (ii), we have to eliminate λ.

Now, (ii) gives

$$\lambda=\frac{-\left[4x+y\frac{dy}{dx}\right]}{x+y\frac{dy}{dx}}$$

$$\Rightarrow 1+\lambda=\frac{(1-4)x}{x+y(dy/dx)},\quad 4+\lambda=\frac{-(1-4)y(dy/dx)}{x+y(dy/dx)}$$

Substituting these values in (i), we get

$$\left(x+y\frac{dy}{dx}\right)\left(x-y\frac{dx}{dy}\right)=-3 \qquad ....(iii)$$

as the differential equation of the given family.

Changing dy/dx to –dx/dy in (iii), we obtain

$$\left(x-y\frac{dx}{dy}\right)\left(x+y\frac{dy}{dx}\right)=-3 \qquad ...(iv)$$

which is the same as (iii). Thus we see that the family (i) as self-orthogonal, i.e., every member of the family (i) cuts every other member of the same family orthogonally.

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Do yourself - 13 :

(i) Find the orthogonal trajectories of the following families of curves :

(a) $x + 2y = C$ **(b)** $y = Ce^{-2x}$

Note :

Following exact differentials must be remembered :

(i) $xdy + y\,dx = d(xy)$ (ii) $\frac{xdy-ydx}{x^2}=d\left(\frac{y}{x}\right)$ (iii) $\frac{ydx-xdy}{y^2}=d\left(\frac{x}{y}\right)$

(iv) $\frac{xdy+ydx}{xy}=d(\ell nxy)$ (v) $\frac{dx+dy}{x+y}=d(\ell n(x+y))$ (vi) $\frac{xdy-ydx}{xy}=d\left(\ell n\frac{y}{x}\right)$

(vii) $\frac{ydx-xdy}{xy}=d\left(\ell n\frac{x}{y}\right)$ (viii) $\frac{xdy-ydx}{x^2+y^2}=d\left(\tan^{-1}\frac{y}{x}\right)$ (ix) $\frac{ydx-xdy}{x^2+y^2}=d\left(\tan^{-1}\frac{x}{y}\right)$

(x) $\frac{xdx+ydy}{x^2+y^2}=d\left[\ell n\sqrt{x^2+y^2}\right]$ (xi) $d\left(-\frac{1}{xy}\right)=\frac{xdy+ydx}{x^2y^2}$ (xii) $d\left(\frac{e^x}{y}\right)=\frac{ye^xdx-e^xdy}{y^2}$

(xiii) $d\left(\frac{e^y}{x}\right)=\frac{xe^ydy-e^ydx}{x^2}$

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Illustration 28 : Solve $\dfrac{y+\sin x\cos^2(xy)}{\cos^2(xy)}dx + \left(\dfrac{x}{\cos^2(xy)}+\sin y\right)dy = 0.$

Solution : The given differential equation can be written as;

$$\frac{y\,dx + x\,dy}{\cos^2(xy)} + \sin x\,dx + \sin y\,dy = 0.$$

$\Rightarrow$ $\sec^2(xy)\,d(xy) + \sin x\,dx + \sin y\,dy = 0$

$\Rightarrow$ $d(\tan(xy)) + d(-\cos x) + d(-\cos y) = 0$

$\Rightarrow$ $\tan(xy) - \cos x - \cos y = c.$ **Ans.**

Do yourself - 14 :

Solve the following differential equations :

(i) $x\,dx + y\,dy + 4y^3(x^2+y^2)dy = 0.$ **(ii)** $x\,dy - y\,dx - (1-x^2)dx = 0.$

8. APPLICATION OF DIFFERENTIAL EQUATIONS :

(a) Mixing Problems

A chemical in a liquid solution with given concentration c_{in} gm/lit. (or dispersed in a gas) runs into a container with a rate of a_{in} lit/min. holding the liquid (or the gas) with, possibly, a specified amount of the chemical dissolved as well. The mixture is kept uniform by stirring and flows out of the container at a known rate (a_{out} litre/min.). In this process it is often important to know the concentration of the chemical in the container at any given time. The differential equation describing the process is based on the formula.

$$\begin{matrix}\text{Rate of change}\\ \text{of amount}\\ \text{in container}\end{matrix} = \begin{pmatrix}\text{rate at which}\\ \text{chemical}\\ \text{arrives}\end{pmatrix} - \begin{pmatrix}\text{rate at which}\\ \text{chemical}\\ \text{departs}\end{pmatrix} \qquad .....(i)$$

Arrival rate = (conc. in) × (inflow rate) = $c_{in} \times a_{in}$

If y(t) denotes the amount of substance in the tank at time t & V(t) denotes the amount of mixture in tank at that time

$$\text{Departure rate} = \begin{pmatrix}\text{concentration in}\\ \text{container at time t}\end{pmatrix}.(\text{outflow rate})$$

$$= \frac{y(t)}{V(t)}\cdot(a_{out})$$

where volume of mixture at time t, V(t) = initial volume + (inflow rate – outflow rate) × t

$$= V_0 + (a_{in} - a_{out})t$$

Accordingly, Equation (i) becomes

$$\frac{dy(t)}{dt} = (\text{chemical's given arrival rate}) - \frac{y(t)}{V(t)}\cdot(\text{out flow rate}) \qquad .....(ii)$$

$$\frac{d(y(t))}{dt} = c_{in}a_{in} - \frac{y(t)}{V_0 + (a_{in} - a_{out})t}.a_{out}$$

This leads to a first order linear D.E. which can be solved to obtain y(t) i.e. amount of chemical at time 't'.

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Illustration 29 : A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour ?

Solution : Let y(t) be the amount of salt after t min.

Given y(0) = 20 kg

$$\text{rate in} = \left(\frac{0.03\text{kg}}{\text{L}}\right)\left(\frac{25\text{L}}{\text{min.}}\right) = \frac{0.75\text{kg}}{\text{min.}}$$

As $a_{in} = a_{out}$, so the tank always contains 5000 L of liquid so the conc. at time 't' is $\left(\frac{y(t)}{5000}\right)\frac{\text{kg}}{\text{L}}$

$$\text{so} \quad \text{rate out} = \left(\frac{y(t)}{5000}\frac{\text{kg}}{\text{L}}\right)\left(\frac{25\text{L}}{\text{min}}\right) = \frac{y(t)}{200}\frac{\text{kg}}{\text{min}}$$

$$\frac{dy(t)}{dt} = 0.75 - \frac{y(t)}{200}$$

by solving as linear D.E. or variable separable and using initial condition, we get

$$y(t) = 150 - 130\, e^{-t/200}$$

The amount of salt after 30 min is

$$y(30) = 150 - 130\, e^{-30/100} = 38.1 \text{ kg}$$

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Do yourself - 15 :

(i) A tank initially holds 10 lit. of fresh water. At t = 0, a brine solution containing $\frac{1}{2}$ kg of salt per lit. is poured into the tank at a rate of 2 lit/min. while the well-stirred mixture leaves the tank at the same rate. Find

(a) the amount and

(b) the concentration of salt in the tank at any time t.

(b) Exponential Growth and Decay :

In general, if y(t) is the value of quantity y at time t and if the rate of change of y with respect to t is proportional to its value y(t) at that time, then

$$\frac{dy(t)}{dt} = ky(t), \text{ where k is a constant} \qquad ....(i)$$

$$\int \frac{dy(t)}{y(t)} = \int k\,dt$$

Solving, we get $y(t) = Ae^{kt}$

equation (i) is sometimes called the law of natural growth (if k > 0) or law of natural decay (if k < 0).

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In the context of population growth, we can write

$$\frac{dP}{dt} = kP \quad \text{or} \quad \frac{1}{P}\frac{dP}{dt} = k$$

where k is growth rate divided by the population size; it is called **the relative growth rate.**

Illustration 30 : A certain radioactive material is known to decay at a rate proportional to the amount present. If initially there is 50 kg of the material present and after two hours it is observed that the material has lost 10 percent of its original mass, find **(a)** an expression for the mass of the material remaining at any time t, **(b)** the mass of the material after four hours, and **(c)** the time at which the material has decayed to one half of its initial mass.

Solution : **(a)** Let N denote the amount of material present at time t.

So, $\frac{dN}{dt} - kN = 0$

This differential equation is separable and linear, its solution is

$N = ce^{kt}$ ....(i)

At t=0, we are given that N = 50. Therefore, from (i), $50 = ce^{k(0)}$ or c = 50. Thus,

$N = 50e^{kt}$ ....(ii)

At t = 2, 10 percent of the original mass of 50kg or 5kg has decayed. Hence, at t = 2, N = 50 – 5 = 45. Substituting these values into (ii) and solving for k, we have

$45 = 50e^{2k}$ or $k = \frac{1}{2}\ell n\frac{45}{50}$

Substituting this value into (ii), we obtain the amount of mass present at any time t as

$N = 50e^{\frac{1}{2}(\ell n 0.9)t}$ .....(iii)

where t is measured in hours.

(b) We require N at t = 4. Substituting t = 4 into (iii) and then solving for N, we find $N = 50e^{-2\,\ell n\,(0.9)}$ kg

(c) We require when N = 50/2 = 25. Substituting N = 25 into (iii) and solving for t, we find

$$25 = 50e^{\frac{1}{2}(\ell n 0.9)t} \Rightarrow t = \ell n\left(\frac{1}{2}\right)\Big/\left[\frac{1}{2}\ell n\,(0.9)\right] \text{hours}$$

(c) Geometrical applications :

Let $P(x_1, y_1)$ be any point on the curve $y = f(x)$, then slope of the tangent at point P is $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

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(i) The equation of the tangent at P is $y - y_1 = \frac{dy}{dx}(x - x_1)$

x-intercept of the tangent $= x_1 - y_1\left(\frac{dx}{dy}\right)$

y-intercept of the tangent $= y_1 - x_1\left(\frac{dy}{dx}\right)$

(ii) The equation of normal at P is $y - y_1 = -\frac{1}{(dy/dx)}(x - x_1)$

x and y-intercepts of normal are ; $x_1 + y_1\frac{dy}{dx}$ and $y_1 + x_1\frac{dx}{dy}$

(iii) Length of tangent $= PT = |y_1|\sqrt{1 + (dx/dy)^2_{(x_1, y_1)}}$

(iv) Length of normal $= PN = |y_1|\sqrt{1 + (dy/dx)^2_{(x_1, y_1)}}$

(v) Length of sub-tangent $= ST = \left|y_1\left(\frac{dx}{dy}\right)_{(x_1, y_1)}\right|$

(vi) Length of sub-normal $= SN = \left|y_1\left(\frac{dy}{dx}\right)_{(x_1, y_1)}\right|$

(vii) Length of radius vector $= \sqrt{x_1^2 + y_1^2}$

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Do yourself - 16 :

(i) At each point (x,y) of a curve the intercept of the tangent on the y-axis is equal to $2xy^2$. Find the curve.

(ii) Find the equation of the curve for which the normal at any point (x,y) passes through the origin.

Miscellaneous Illustrations :

Illustration 31 : Solve $(y \log x - 1)\, ydx = xdy$.

Solution : The given differential equation can be written as

$$x\frac{dy}{dx} + y = y^2 \log x \qquad .....(i)$$

Divide by xy^2 . Hence $\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \frac{1}{x}\log x$

Let $\frac{1}{y} = v \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$ so that $\frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x}\log x \qquad .....(ii)$

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(ii) is the standard linear differential equation with $P = -\frac{1}{x}$, $Q = -\frac{1}{x}\log x$

I.F. = $e^{\int p dx} = e^{\int -1/x\, dx} = 1/x$

The solution is given by

$$v \cdot \frac{1}{x} = \int \frac{1}{x}\left(-\frac{1}{x}\log x\right)dx = -\int \frac{\log x}{x^2}dx = \frac{\log x}{x} - \int \frac{1}{x}\cdot\frac{1}{x}dx = \frac{\log x}{x} + \frac{1}{x} + C$$

$\Rightarrow$ $v = 1 + \log x + cx = \log ex + cx$

or $\frac{1}{y} = \log ex + cx$ or $y(\log ex + cx) = 1$. **Ans.**

Illustration 32: For a certain curve $y = f(x)$ satisfying $\frac{d^2y}{dx^2} = 6x - 4$, $f(x)$ has a local minimum value 5 when $x = 1$. Find the equation of the curve and also the global maximum and global minimum values of $f(x)$ given that $0 \le x \le 2$.

Solution : Integrating $\frac{d^2y}{dx^2} = 6x - 4$, we get $\frac{dy}{dx} = 3x^2 - 4x + A$

When $x = 1$, $\frac{dy}{dx} = 0$, so that $A = 1$. Hence

$$\frac{dy}{dx} = 3x^2 - 4x + 1 \qquad ...(i)$$

Integrating, we get $y = x^3 - 2x^2 + x + B$

When $x = 1$, $y = 5$, so that $B = 5$.

Thus we have $y = x^3 - 2x^2 + x + 5$.

From (i), we get the critical points $x = 1/3$, $x = 1$

At the critical point $x = \frac{1}{3}$, $\frac{d^2y}{dx^2}$ is negative.

Therefore at $x = 1/3$, y has a local maximum.

At $x = 1$, $\frac{d^2y}{dx^2}$ is positive.

Therefore at $x = 1$, y has a local minimum.

Also $f(1) = 5$, $f\left(\frac{1}{3}\right) = \frac{139}{27}$. $f(0) = 5$, $f(2) = 7$

Hence the global maximum value = 7, and the global minimum value = 5.

Ans.

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Illustration 33 : Solve $\frac{dy}{dx}$ = tany cotx – secy cosx.

Solution : $\frac{dy}{dx}$ = tany cotx – secy cosx.

Rearrange it :

(sin x – siny)cos x dx + sin x cosy dy = 0.

Put u = sin y, So, du = cos y dy :

Substituting, we get

(sin x – u)cos x dx + sin x du = 0, $\frac{du}{dx} - u\frac{\cos x}{\sin x} = -\cos x$

The equation is first-order linear in u.

The integrating factor is

$$I = \exp\int -\frac{\cos x}{\sin x}dx = \exp\{-\ln(\sin x)\} = \frac{1}{\sin x}.$$

Hence, $u\frac{1}{\sin x} = -\int \frac{\cos x}{\sin x}dx = -\ln|\sin x| + C$.

Solve for u : u = –sin x ln |sin x|+ C sin x.

Put y back : siny = – sin x ln |sin x|+ C sin x. **Ans.**

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Illustration 34 : Solve the equation $x\int_0^x y(t)dt = (x+1)\int_0^x t\,y(t)dt, x > 0$

Solution : Differentiating the equation with respect to x, we get

$$xy(x) + 1.\int_0^x y(t)dt = (x+1)xy(x) + 1.\int_0^x ty(t)dt$$

i.e., $\int_0^x y(t)dt = x^2y(x) + \int_0^x ty(t)dt$

Differentiating again with respect to x, we get $y(x) = x^2 y'(x) + 2xy(x) + xy(x)$

i.e., $(1-3x)y(x) = \frac{x^2dy(x)}{dx}$

i.e., $\frac{(1-3x)dx}{x^2} = \frac{dy(x)}{y(x)}$, integrating we get

i.e., $y = \frac{c}{x^3}e^{-1/x}$ **Ans.**

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ANSWERS FOR DO YOURSELF

1 : (i) one, two **(ii)** two, two **(iii)** one, one

2 : (i) $y' - y = 2(1 - x)$ **(ii)** $x^2y'' + 2xy' - 2y = 0$ **(iii)** $y''' = 4y'$

3 : (i) $\ell ny^2 = x + \ell n|x| + k$ **(ii)** $-\frac{1}{2y^2} = \frac{1}{4}\sqrt{1+4x^2} + k$ **(iii)** $\sec y = -2\cos x + C$

4 : (i) $\frac{y-4x+2}{y-4x-2} = ce^{-4x}$ **(ii)** $2(x - y) = c + \sin 2(x + y)$

5 : (i) $x + 2y + \ell n|2x - y| + c = 0$ **(ii)** $\frac{y^2}{2} + 3y - \frac{3x^2}{2} + 5xy = 0$

6 : (i) $x = cye^{xy}$ **(ii)** $y = cx^2e^{-1/xy}$

7 : (i) $\ell n(x^2 + y^2) = 2\tan^{-1}\left(\frac{y}{x}\right) + c$ **(ii)** $\sqrt{x^2 - y^2} = \sin^{-1}\left(\frac{y}{x}\right) + c$

8 : (i) 3 **(ii)** 2/3 **(iii) (a)** homogeneous **(b)** homogeneous **(c)** not homogeneous

9 : (i) $(3x + y)(x - y) = c_0$ **(ii)** $y\ell n\left(\frac{y}{x}\right) - y + x\ell nx + cx = 0$ **(iii)** $x^2y(2x + y) = 3$

10 : (i) $x + y - 3 = C(x - y + 1)^3$

11 : (i) $y = \frac{x^4}{2} + 6x^2\ell n|x| - 2x + cx^2$ **(ii)** $y = 2(x-a)^3 + \frac{c}{(x-a)^3}$ **(iii)** $2x\,\ell n\,y = \ell n^2\,y + C$.

12 : (i) $y = \frac{1}{(c-x)e^{3x}}$ **(ii)** $\frac{x^2}{y^2} = -\frac{2}{3}x^3\left(\frac{2}{3} + \ell nx\right) + c$ **(iii)** $\frac{1}{y} = -\sin x + ce^x$.

13 : (i) (a) $y - 2x = K$ **(b)** $y^2 = x + K$

14 : (i) ½ $\ell n(x^2 + y^2) + y^4 = C$ **(ii)** $y + x^2 + 1 = Cx$

15 : (i) (a) $-5e^{-0.2t} + 5$ kg **(b)** $\frac{1}{2}(-e^{0.2t} + 1)$ kg/ℓ

16 : (i) $\frac{x}{y} = x^2 + C$ **(ii)** $x^2 + y^2 = C$

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EXERCISE (O-1)

1. The differential equation corresponding to the family of curves $y = e^x (ax + b)$ is

(A) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - y = 0$ (B) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$ (C) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ (D) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - y = 0$

DE0001

2. Number of values of $m \in N$ for which $y = e^{mx}$ is a solution of the differential equation $D^3y - 3D^2y - 4Dy + 12y = 0$, is

(A) 0 (B) 1 (C) 2 (D) more than 2

DE0002

3. Number of straight lines which satisfy the differential equation $\frac{dy}{dx} + x\left(\frac{dy}{dx}\right)^2 - y = 0$ is:

(A) 1 (B) 2 (C) 3 (D) 4

DE0003

4. The value of the constant 'm' and 'c' for which $y = mx + c$ is a solution of the differential equation $D^2y - 3Dy - 4y = -4x$.

(A) is $m = -1$; $c = 3/4$ (B) is $m = 1$; $c = -3/4$ (C) no such real m, c (D) is $m = 1$; $c = 3/4$

DE0004

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5. Consider the two statements

Statement-1: $y = \sin kt$ satisfies the differential equation $y'' + 9y = 0$.

Statement-2: $y = e^{kt}$ satisfy the differential equation $y'' + y' - 6y = 0$

The value of k for which both the statements are correct is

(A) -3 (B) 0 (C) 2 (D) 3

DE0005

6. The general solution of the differential equation $\frac{dy}{dx} = \frac{1-x}{y}$ is a family of curves which looks most like which of the following?

(A)

(B)

(C)

(D)

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7. If $y = \dfrac{x}{\ln|cx|}$ (where c is an arbitrary constant) is the general solution of the differential equation $\dfrac{dy}{dx} = \dfrac{y}{x} + \phi\left(\dfrac{x}{y}\right)$ then the function $\phi\left(\dfrac{x}{y}\right)$ is :

(A) $\dfrac{x^2}{y^2}$ (B) $-\dfrac{x^2}{y^2}$ (C) $\dfrac{y^2}{x^2}$ (D) $-\dfrac{y^2}{x^2}$

DE0007

8. Water is drained from a vertical cylindrical tank by opening a valve at the base of the tank. It is known that the rate at which the water level drops is proportional to the square root of water depth y, where the constant of proportionality k > 0 depends on the acceleration due to gravity and the geometry of the hole. If t is measured in minutes and $k = \dfrac{1}{15}$ then the time to drain the tank if the water is 4 meter deep to start with is

(A) 30 min (B) 45 min (C) 60 min (D) 80 min

DE0008

9. Spherical rain drop evaporates at a rate proportional to its surface area. The differential equation corresponding to the rate of change of the radius of the rain drop if the constant of proportionality is K > 0, is

(A) $\dfrac{dr}{dt} + K = 0$ (B) $\dfrac{dr}{dt} - K = 0$ (C) $\dfrac{dr}{dt} = Kr$ (D) none

DE0009

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10. The x-intercept of the tangent to a curve is equal to the ordinate of the point of contact. The equation of the curve through the point (1, 1) is

(A) $ye^{\frac{x}{y}} = e$ (B) $xe^{\frac{x}{y}} = e$ (C) $xe^{\frac{y}{x}} = e$ (D) $ye^{\frac{y}{x}} = e$

DE0010

11. A function $f(x)$ satisfying $\int_0^1 f(tx)\,dt = n f(x)$, where x > 0, is

(A) $f(x) = c \cdot x^{\frac{1-n}{n}}$ (B) $f(x) = c \cdot x^{\frac{n}{n-1}}$ (C) $f(x) = c \cdot x^{\frac{1}{n}}$ (D) $f(x) = c \cdot x^{(1-n)}$

DE0011

12. Which one of the following curves represents the solution of the initial value problem
$Dy = 100 - y$, where $y(0) = 50$

(A)

(B)

(C)

(D)

DE0012

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13. The real value of m for which the substitution, $y = u^m$ will transform the differential equation, $2x^4y\frac{dy}{dx} + y^4 = 4x^6$ into a homogeneous equation is :

(A) m = 0 (B) m = 1 (C) m = 3/2 (D) no value of m

DE0013

14. A curve C passes through origin and has the property that at each point (x, y) on it the normal line at that point passes through (1, 0). The equation of a common tangent to the curve C and the parabola $y^2 = 4x$ is

(A) x = 0 (B) y = 0 (C) y = x + 1 (D) x + y + 1 = 0

DE0014

15. A function $y = f(x)$ satisfies $(x + 1) . f'(x) - 2(x^2 + x) f(x) = \frac{e^{x^2}}{(x+1)}, \forall\ x > -1$

If $f(0) = 5$, then $f(x)$ is

(A) $\left(\frac{3x+5}{x+1}\right).e^{x^2}$ (B) $\left(\frac{6x+5}{x+1}\right).e^{x^2}$ (C) $\left(\frac{6x+5}{(x+1)^2}\right).e^{x^2}$ (D) $\left(\frac{5-6x}{x+1}\right).e^{x^2}$

DE0015

16. The equation to the orthogonal trajectories of the system of parabolas $y = ax^2$ is

(A) $\frac{x^2}{2} + y^2 = c$ (B) $x^2 + \frac{y^2}{2} = c$ (C) $\frac{x^2}{2} - y^2 = c$ (D) $x^2 - \frac{y^2}{2} = c$

DE0016

17. If $\int_a^x t\,y(t)dt = x^2 + y(x)$ then y as a function of x is

(A) $y = 2 - (2 + a^2)e^{\frac{x^2-a^2}{2}}$ (B) $y = 1 - (2 + a^2)e^{\frac{x^2-a^2}{2}}$

(C) $y = 2 - (1 + a^2)e^{\frac{x^2-a^2}{2}}$ (D) none

DE0017

18. If $f(x)$ and $g(x)$ are two differentiable functions on R^+ such that $xf'(x) + g(x) = 0$ and $xg'(x) + f(x) = 0$ for all $x \in R^+$ and $f(1) + g(1) = 4$, then the value of $f''(2).g''(2)$ is -

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$ (C) $\frac{1}{8}$ (D) $\frac{1}{16}$

DE0018

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EXERCISE (O-2)

One or more than one correct

1. The differential equation, $x\dfrac{dy}{dx} + \dfrac{3}{\frac{dy}{dx}} = y^2$:

(A) is of order 1 (B) is of degree 2 (C) is linear (D) is non linear

DE0019

2. Which of the following pair of curves is/are orthogonal?

(A) $16x^2 + y^2 = c$ and $y^{16} = kx$ (B) $y = x + ce^{-x}$ and $x + 2 = y + ke^{-y}$

(C) $y = cx^2$ and $x^2 + 2y^2 = k$ (D) $x^2 - y^2 = c$ and $xy = k$

where c and k arbitrary constant.

DE0020

3. The function f(x) satisfying the equation, $f^2(x) + 4f'(x) . f(x) + [f'(x)]^2 = 0$.

(A) $f(x) = c . e^{(2-\sqrt{3})x}$ (B) $f(x) = c . e^{(2+\sqrt{3})x}$

(C) $f(x) = c . e^{(\sqrt{3}-2)x}$ (D) $f(x) = c . e^{-(2+\sqrt{3})x}$

where c is an arbitrary constant .

DE0021

4. A curve $y = f(x)$ has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of the point P from the x-axis. Then the differential equation of the curve

(A) is homogeneous.

(B) can be converted into linear differential equation with some suitable substitution.

(C) is the family of circles touching the x-axis at the origin.

(D) the family of circles touching the y-axis at the origin.

DE0022

5. Consider $g(x) = \begin{cases} \sin x & 0 \le x < \dfrac{\pi}{2} \\ \cos x & x \ge \dfrac{\pi}{2} \end{cases}$ and a continuous function $y = f(x)$ satisfies $\dfrac{5dy}{dx} + 5y = g(x)$,

$f(0) = 0$, then-

(A) $f\left(\dfrac{\pi}{4}\right) = \dfrac{e^{-\pi/4}}{10}$ (B) $f\left(\dfrac{\pi}{4}\right) = \dfrac{e^{-\pi/4} - 1}{10}$

(C) $f\left(\dfrac{\pi}{2}\right) = \dfrac{e^{-\pi/2} + 1}{10}$ (D) $f\left(\dfrac{\pi}{2}\right) = e^{-\pi/2}$

DE0023

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6. A function $y = f(x)$ satisfying the differential equation $\frac{dy}{dx} . \sin x - y \cos x + \frac{\sin^2 x}{x^2} = 0$ is such that, $y \to 0$ as $x \to \infty$ then the statement which is correct is

(A) $\underset{x \to 0}{\text{Limit}}\ f(x) = 1$

(B) $\int_0^{\pi/2} f(x)\,dx$ is less than $\frac{\pi}{2}$

(C) $\int_0^{\pi/2} f(x)\,dx$ is greater than unity

(D) $f(x)$ is an odd function

DE0024

7. If a function $y = f(x)$ satisfies the differential equation $f(x) \cdot \sin 2x - \cos x + (1 + \sin^2 x) f'(x) = 0$ with initial condition $y(0) = 0$. Then-

(A) Range of $f(x)$ is $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(B) $f(1) < f(2)$

(C) $f(1) > f(2)$

(D) $f(x)$ is odd function

DE0025

8. A function $y = f(x)$ satisfies the condition $f'(x) \sin x + f(x) \cos x = 1$, $f(x)$ being bounded when $x \to 0$. If $I = \int_0^{\pi/2} f(x)\,dx$ then

(A) $\frac{\pi}{2} < I < \frac{\pi^2}{4}$

(B) $\frac{\pi}{4} < I < \frac{\pi^2}{2}$

(C) $f\left(\frac{\pi}{4}\right) < f\left(\frac{\pi}{3}\right)$

(D) $f\left(\frac{\pi}{4}\right) > f\left(\frac{\pi}{3}\right)$

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DE0026

9. If $y = f(x)$ is solution of the differential equation, $x^2 \frac{dy}{dx} . \cos\frac{1}{x} - y \sin\frac{1}{x} = -1$, where $y \to -1$ as $x \to \infty$, then

(A) $f(x) = \sin\frac{1}{x} - \cos\frac{1}{x}$

(B) $\lim_{x \to \infty} \left(-f(x)\right)^x = \frac{1}{e}$

(C) $\lim_{x \to \infty} \left(-f(x)\right)^x = e$

(D) $f'(x) + 2xf'(x) + x^2 f''(x) = -\frac{2}{x^2} \sin\frac{1}{x}$

DE0027

10. Consider the differential equation $\frac{dy}{dx} + y \tan x = x \tan x + 1$. Then

(A) The integral curves satisfying the differential equation and given by $y = x + c \cos x$.

(B) The angle at which the integral curves cut the y-axis is $\frac{\pi}{4}$.

(C) Tangents to all the integral curves at their point of intersection with y-axis are parallel.

(D) none of these

DE0028

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EXERCISE (S-1)

(E-1)

[FORMATION & VARIABLES SEPARABLE]

1. State the order and degree of the following differential equations:

(i) $\left[\frac{d^2x}{dt^2}\right]^3 + \left[\frac{dx}{dt}\right]^4 - xt = 0$ (ii) $\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$

DE0029

2. (a) Form a differential equation for the family of curves represented by $ax^2 + by^2 = 1$, where a & b are arbitrary constants.

DE0030

(b) Obtain the differential equation of the family of circles $x^2 + y^2 + 2gx + 2fy + c = 0$; where g , f & c are arbitrary constants.

DE0031

(c) Obtain the differential equation associated with the primitive,

$y = c_1 e^{3x} + c_2 e^{2x} + c_3 e^x$, where c_1, c_2, c_3 are arbitrary constants.

DE0032

Solve the following differential equation for Q.3 to Q.9.

3. $\frac{\ell n(\sec x + \tan x)}{\cos x}dx = \frac{\ell n(\sec y + \tan y)}{\cos y}dy$

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DE0033

4. $(1 - x^2)(1 - y)\,dx = xy(1 + y)\,dy$

DE0034

5. $\frac{dy}{dx} + \frac{\sqrt{(x^2-1)(y^2-1)}}{xy} = 0$

DE0035

6. $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

DE0036

7. $\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$

DE0037

8. $\frac{dy}{dx} = \frac{x(2\ell n x + 1)}{\sin y + y\cos y}$

DE0038

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9. (a) $\frac{dy}{dx}+\sin\frac{x+y}{2}=\sin\frac{x-y}{2}$

DE0039

(b) $\sin x.\frac{dy}{dx}=y$. *l*ny if y = e, when $x=\frac{\pi}{2}$

DE0040

10. The population P of a town decreases at a rate proportional to the number by which the population exceeds 1000, proportionality constant being k > 0. Find

(a) Population at any time t, given initial population of the town being 2500.

(b) If 10 years later the population has fallen to 1900, find the time when the population will be 1500.

(c) Predict about the population of the town in the long run.

DE0041

11. It is known that the decay rate of radium is directly proportional to its quantity at each given instant. Find the law of variation of a mass of radium as a function of time if at $t = 0$, the mass of the radius was m_0 and during time t_0 α % of the original mass of radium decay.

DE0042

12. A normal is drawn at a point P(x,y) of a curve. It meets the x-axis at Q. If PQ is of constant length k, then show that the differential equation describing such curves is, $y\frac{dy}{dx}=\pm\sqrt{k^2-y^2}$. Find the equation of such a curve passing through (0,k).

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DE0043

13. A curve is such that the length of the polar radius of any point on the curve is equal to the length of the tangent drawn at this point. Form the differential equation and solve it to find the equation of the curve.

DE0044

14. Given y(0) = 2000 and $\frac{dy}{dx}=32000-20y^2$, then find the value of $\lim_{x\to\infty} y(x)$.

DE0045

15. Let f(x) is a continuous function which takes positive values for $x \geq 0$ and satisfy $\int_0^x f(t)dt = x\sqrt{f(x)}$ with $f(1)=\frac{1}{2}$. Find the value of $f\left(\sqrt{2}+1\right)$.

DE0046

16. $\frac{dy}{dx}=y+\int_0^1 y\,dx$ given $y=1$, where $x=0$

DE0047

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(E-2)
[HOMOGENEOUS]

1. $\frac{dy}{dx} = \frac{x^2 + xy}{x^2 + y^2}$

DE0048

2. $(x - y)\, dy = (x + y + 1)\, dx$

DE0049

3. $\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$

DE0050

4. $\frac{dy}{dx} = \frac{y - x + 1}{y + x + 5}$

DE0051

5. $\frac{dy}{dx} = \frac{x + y + 1}{2x + 2y + 3}$

DE0052

6. $\frac{dy}{dx} = \frac{2\,(y + 2)^2}{(x + y - 1)^2}$

DE0053

7. Find the equation of a curve such that the projection of its ordinate upon the normal is equal to its abscissa.

DE0054

8. The perpendicular from the origin to the tangent at any point on a curve is equal to the abscissa of the point of contact. Find the equation of the curve satisfying the above condition and which passes through (1 , 1).

DE0055

9. Find the curve for which any tangent intersects the y-axis at the point equidistant from the point of tangency and the origin.

DE0056

(E-3)
[LINEAR]

1. If y = y(x) is solution of differential equation $\frac{dy}{dx} - y = 1 - e^{-x}$ and $y(0) = y_0$ has a finite value, when $x \to \infty$, then find y_0.

DE0057

2. $\frac{dy}{dx} + \frac{x}{1 + x^2} y = \frac{1}{2x(1 + x^2)}$

DE0058

3. $(1 - x^2)\frac{dy}{dx} + 2xy = x\,(1 - x^2)^{1/2}$

DE0059

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4. $x(x-1)\dfrac{dy}{dx} - (x-2)y = x^3(2x-1)$

DE0060

5. $\sin x \dfrac{dy}{dx} + 3y = \cos x$

DE0061

6. $x(x^2+1)\dfrac{dy}{dx} = y(1-x^2) + x^3 . \ell nx$

DE0062

7. $x\dfrac{dy}{dx} - y = 2x^2 \operatorname{cosec} 2x$

DE0063

8. $(1+y^2)\,dx = (\tan^{-1} y - x)dy$

DE0064

9. (a) Find the curve such that the area of the trapezium formed by the co-ordinate axes, ordinate of an arbitrary point & the tangent at this point equals half the square of its abscissa.

(b) A curve in the first quadrant is such that the area of the triangle formed in the first quadrant by the x-axis, a tangent to the curve at any of its point P and radius vector of the point P is 2sq. units. If the curve passes through (2,1), find the equation of the curve.

DE0065

10. Let the function $ln\, f(x)$ is defined where $f(x)$ exists for $x \geq 2$ & k is fixed positive real number, prove that if $\dfrac{d}{dx}(x . f(x)) \leq -k f(x)$ then $f(x) \leq A x^{-1-k}$ where A is independent of x.

DE0066

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11. Find the differentiable function which satisfies the equation $f(x) = -\int_0^x f(t)\tan t\,dt + \int_0^x \tan(t-x)dt$ where $x \in (-\pi/2, \pi/2)$

DE0067

12. $y - x\,Dy = b(1 + x^2 Dy)$

DE0068

13. Find all functions f(x) defined on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ with real values and has a primitive F(x) such that

$f(x) + \cos x . F(x) = \dfrac{\sin 2x}{(1+\sin x)^2}$.

DE0069

14. $2\dfrac{dy}{dx} - y \sec x = y^3 \tan x$

DE0070

15. $x^2 y - x^3 \dfrac{dy}{dx} = y^4 \cos x$

DE0071

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16. $y(2xy + e^x)\,dx - e^x\,dy = 0$

DE0072

17. A tank contains 100 litres of fresh water. A solution containing 1 gm/litre of soluble lawn fertilizer runs into the tank at the rate of 1 lit/min and the mixture in pumped out of the tank at the rate of 3 litres/min. Find the time when the amount of fertilizer in the tank is maximum.

DE0073

(E-4)

(GENERAL – CHANGE OF VARIABLE BY A SUITABLE SUBSTITUTION)

1. $(x - y^2)\,dx + 2xy\,dy = 0$

DE0074

2. $(x^3 + y^2 + 2)\,dx + 2y\,dy = 0$

DE0075

3. $x\dfrac{dy}{dx} + y\ln y = xye^x$

DE0076

4. $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1 + x)\,e^x \sec y$

DE0077

5. $\dfrac{dy}{dx} = \dfrac{e^y}{x^2} - \dfrac{1}{x}$

DE0078

6. $\left(\dfrac{dy}{dx}\right)^2 - (x+y)\dfrac{dy}{dx} + xy = 0$

DE0079

7. $\dfrac{dy}{dx} = \dfrac{y^2 - x}{2y\,(x+1)}$

DE0080

8. $(1 - xy + x^2y^2)\,dx = x^2\,dy$

DE0081

9. $\dfrac{dy}{dx} = e^{x-y}(e^x - e^y)$

DE0082

10. $(x^2 + y)\dfrac{dy}{dx} = 6x$

DE0083

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EXERCISE (S-2)

(MISCELLANEOUS)

1. $\frac{dy}{dx} - y\,ln2 = 2^{sinx}.\,(cosx - 1)ln2$, y being bounded when $x \to +\infty$.

DE0084

2. The light rays emanating from a point source situated at origin when reflected from the mirror of a search light are reflected as beam parallel to the x-axis. Show that the surface is parabolic, by first forming the differential equation and then solving it.

DE0085

3. $\frac{ydx - xdy}{(x-y)^2} = \frac{dx}{2\sqrt{1-x^2}}$, given that y = 2 when x = 1.

DE0086

4. Consider the differential equation, $\frac{dy}{dx} + P(x)y = Q(x)$

(i) If two particular solutions of given equation u(x) and v(x) are known, find the general solution of the same equation in terms of u(x) and v(x).

(ii) If α and β are constants such that the linear combinations $\alpha \cdot u(x) + \beta \cdot v(x)$ is a solution of the given equation, find the relation between α and β.

(iii) If w(x) is the third particular solution different from u(x) and v(x) then find the ratio $\frac{v(x) - u(x)}{w(x) - u(x)}$.

DE0087

5. Find the integral curve of the differential equation, $x(1 - x\,\ell n\,y).\frac{dy}{dx} + y = 0$ which passes through $\left(1, \frac{1}{e}\right)$.

DE0088

6. A tank consists of 50 litres of fresh water. Two litres of brine each litre containing 5 gms of dissolved salt are run into tank per minute; the mixture is kept uniform by stirring, and runs out at the rate of one litre per minute. If 'm' grams of salt are present in the tank after t minute, express 'm' in terms of t and find the amount of salt present after 10 minutes.

DE0089

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7. Find the orthogonal trajectories for the given family of curves when 'a' is the parameter.

(i) $y = ax^2$

DE0090

(ii) $\cos y = a\, e^{-x}$

DE0091

(iii) $x^k + y^k = a^k$

DE0092

(iv) Find the isogonal trajectories for the family of rectangular hyperbolas $x^2 - y^2 = a^2$ which makes with it an angle of 45°.

DE0093

8. Let $f(x, y, c_1) = 0$ and $f(x, y, c_2) = 0$ define two integral curves of a homogeneous first order differential equation. If P_1 and P_2 are respectively the points of intersection of these curves with an arbitrary line, $y = mx$ then prove that the slopes of these two curves at P_1 and P_2 are equal.

DE0094

9. Let f(x) be a differentiable function and satisfy f(0) = 2, f'(0) = 3 and f''(x) = f(x). Find

(a) the range of the function f(x)

(b) the value of the function when $x = \ln 2$

(c) the area enclosed by y = f(x) in the 2nd quadrant

DE0095

10. Show that the curve such that the distance between the origin and the tangent at an arbitrary point is equal to the distance between the origin and the normal at the same point, $\sqrt{x^2 + y^2} = c\, e^{\pm \tan^{-1}\frac{y}{x}}$.

DE0096

11. Find the curve possessing the property that the intercept, the tangent at any point of a curve cuts off on the y-axis is equal to the square of the abscissa of the point of tangency.

DE0097

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EXERCISE (JM)

1. If $\frac{dy}{dx} = y + 3 > 0$ and $y(0) = 2$, then $y(\ln 2)$ is equal to :- **[AIEEE-2011]**

(1) 13 (2) –2 (3) 7 (4) 5

DE0098

2. Let I be the purchase value of an equipment and V(t) be the value after it has been used for t years. The value V(t) depreciates at a rate given by differential equation $\frac{dV(t)}{dt} = -k(T - t)$, where $k > 0$ is a constant and T is the total life in years of the equipment. Then the scrap value V(T) of the equipment is :- **[AIEEE-2011]**

(1) $I - \frac{k(T-t)^2}{2}$ (2) e^{-kT} (3) $T^2 - \frac{I}{k}$ (4) $I - \frac{kT^2}{2}$

DE0099

3. The curve that passes through the point (2, 3), and has the property that the segment of any tangent to it lying between the coordinate axes is bisected by the point of contact, is given by : **[AIEEE-2011]**

(1) $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 2$ (2) $2y - 3x = 0$ (3) $y = \frac{6}{x}$ (4) $x^2 + y^2 = 13$

DE0100

4. Consider the differential equation $y^2dx + \left(x - \frac{1}{y}\right)dy = 0$. If $y(1) = 1$, then x is given by : **[AIEEE-2011]**

(1) $1 - \frac{1}{y} + \frac{e^{\frac{1}{y}}}{e}$ (2) $4 - \frac{2}{y} - \frac{e^{\frac{1}{y}}}{e}$ (3) $3 - \frac{1}{y} + \frac{e^{\frac{1}{y}}}{e}$ (4) $1 + \frac{1}{y} - \frac{e^{\frac{1}{y}}}{e}$

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DE0101

5. The population p(t) at time t of a certain mouse species satisfies the differential equation $\frac{dp(t)}{dt} = 0.5\,p(t) - 450$. If $p(0) = 850$, then the time at which the population becomes zero is : **[AIEEE-2012]**

(1) ln18 (2) 2 ln18 (3) ln9 (4) $\frac{1}{2}$ ln18

DE0102

6. At present a firm is manufacturing 2000 items. It is estimated that the rate of change of production P w.r.t. additional number of workers x is given by $\frac{dP}{dx} = 100 - 12\sqrt{x}$. If the firm employs 25 more workers, then the new level of production of items is : **[JEE (Main)-2013]**

(1) 2500 (2) 3000 (3) 3500 (4) 4500

DE0103

7. If the surface area of a sphere of radius r is increasing uniformly at the rate $8 cm^2/s$, then the rate of change of its volume is : **[JEE-Main (On line)-2013]**

(1) proportional to r^2 (2) constant (3) proportional to r (4) proportional to $\sqrt{r}$

DE0104

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8. Consider the differential equation $\frac{dy}{dx}=\frac{y^3}{2(xy^2-x^2)}$:

Statement 1 : The substitution $z = y^2$ transforms the above equation into a first order homogenous differential equation.

Statement 2 : The solution of this differential equation is $y^2e^{-\frac{y^2}{x}}=C$. **[JEE-Main (On line)-2013]**

(1) Statement 1 is false and statement 2 is true. (2) Both statements are true.

(3) Statement 1 is true and statement 2 is false. (4) Both statements are false.

DE0105

9. If a curve passes through the point $\left(2, \frac{7}{2}\right)$ and has slope $\left(1-\frac{1}{x^2}\right)$ at any point (x, y) on it, then the ordinate of the point on the curve whose abscissa is – 2 is : **[JEE-Main (On line)-2013]**

(1) $-\frac{5}{2}$ (2) $\frac{5}{2}$ (3) $-\frac{3}{2}$ (4) $\frac{3}{2}$

DE0106

10. The equation of the curve passing through the origin and satisfying the differential equation $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$ is : **[JEE-Main (On line)-2013]**

(1) $(1 + x^2) y = x^3$ (2) $3(1 + x^2) y = 4x^3$ (3) $3(1+ x^2) y = 2x^3$ (4) $(1 + x^2) y = 3x^3$

DE0107

11. Let the population of rabbits surviving at a time t be governed by the differential equation $\frac{dp(t)}{dt}=\frac{1}{2}p(t)-200$. If p(0) = 100, then p(t) equals : **[JEE(Main)-2014]**

(1) $400 - 300 e^{t/2}$ (2) $300 - 200 e^{-t/2}$ (3) $600 - 500 e^{t/2}$ (4) $400 - 300 e^{-t/2}$

DE0108

12. Let y(x) be the solution of the differential equation $(x \log x)\frac{dy}{dx} + y = 2x \log x$, $(x \geq 1)$. Then y(e) is equal to : **[JEE(Main)-2015]**

(1) 2 (2) 2e (3) e (4) 0

DE0109

13. If a curve y = f(x) passes through the point (1, –1) and satisfies the differential equation, $y(1 + xy)\,dx = x\,dy$, then $f\left(-\frac{1}{2}\right)$ is equal to : **[JEE(Main)-2016]**

(1) $\frac{4}{5}$ (2) $-\frac{2}{5}$ (3) $-\frac{4}{5}$ (4) $\frac{2}{5}$

DE0110

14. If $(2+\sin x)\frac{dy}{dx}+(y+1)\cos x=0$ and y(0) = 1, then $y\left(\frac{\pi}{2}\right)$ is equal to :- **[JEE(Main)-2017]**

(1) $\frac{4}{3}$ (2) $\frac{1}{3}$ (3) $-\frac{2}{3}$ (4) $-\frac{1}{3}$

DE0111

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15. Let y = y(x) be the solution of the differential equation $\sin x \frac{dy}{dx} + y \cos x = 4x$, $x \in (0, \pi)$. If $y\left(\frac{\pi}{2}\right) = 0$, then $y\left(\frac{\pi}{6}\right)$ is equal to : **[JEE(Main)-2018]**

(1) $\frac{-8}{9\sqrt{3}}\pi^2$ (2) $-\frac{8}{9}\pi^2$ (3) $-\frac{4}{9}\pi^2$ (4) $\frac{4}{9\sqrt{3}}\pi^2$

DE0112

16. Let $f:[0,1]\to R$ be such that $f(xy) = f(x).f(y)$ for all $x,y,\varepsilon[0,1]$, and $f(0)\neq 0$. If y = y(x) satisfies the differential equation, $\frac{dy}{dx} = f(x)$ with y(0) = 1, then $y\left(\frac{1}{4}\right)+y\left(\frac{3}{4}\right)$ is equal to **[JEE(Main)-2019]**

(1) 4 (2) 3 (3) 5 (4) 2

DE0113

17. If $\frac{dy}{dx}+\frac{3}{\cos^2 x}y=\frac{1}{\cos^2 x}, x\in\left(\frac{-\pi}{3},\frac{\pi}{3}\right)$ and $y\left(\frac{\pi}{4}\right)=\frac{4}{3}$, then $y\left(-\frac{\pi}{4}\right)$ equals : **[JEE(Main)-2019]**

(1) $\frac{1}{3}+e^6$ (2) $\frac{1}{3}$ (3) $-\frac{4}{3}$ (4) $\frac{1}{3}+e^3$

DE0114

18. The curve amongst the family of curves, represented by the differential equation, $(x^2 - y^2)dx + 2xy\, dy = 0$ which passes through (1,1) is : **[JEE(Main)-2019]**

(1) A circle with centre on the y-axis

(2) A circle with centre on the x-axis

(3) An ellipse with major axis along the y-axis

(4) A hyperbola with transverse axis along the x-axis

DE0115

19. The solution of the differential equation $\frac{dy}{dx}=(x-y)^2$, when y(1) = 1, is :- **[JEE(Main)-2019]**

(1) $\log_e\left|\frac{2-y}{2-x}\right| = 2(y-1)$ (2) $\log_e\left|\frac{2-x}{2-y}\right| = x-y$

(3) $-\log_e\left|\frac{1+x-y}{1-x+y}\right| = x+y-2$ (4) $-\log_e\left|\frac{1-x+y}{1+x-y}\right| = 2(x-1)$

DE0116

20. If a cuve passes through the point (1, –2) and has slope of the tangent at any point (x, y) on it as $\frac{x^2-2y}{x}$, then the curve also passes through the point : **[JEE(Main)-2019]**

(1) $\left(-\sqrt{2},1\right)$ (2) $\left(\sqrt{3},0\right)$ (3) (–1, 2) (4) (3, 0)

DE0117

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21. Let y = y(x) be the solution curve of the differential equation, $(y^2 - x)\frac{dy}{dx} = 1$, satisfying y(0) = 1. This curve intersects the x-axis at a point whose abscissa is : **[JEE(Main)-2020]**

(1) 2 + e (2) 2 (3) 2 – e (4) –e

DE0118

22. If for x ≥ 0, y = y(x) is the solution of the differential equation **[JEE(Main)-2020]**
$(x + 1)dy = ((x + 1)^2 + y - 3)dx$, y(2) = 0, then y(3) is equal to ———.

DE0119

EXERCISE (JA)

1. **(a)** Let $f : [1,\infty) \to [2,\infty)$ be a differentiable function such that f(1) = 2. If $6\int_1^x f(t)dt = 3x f(x) - x^3$ for all $x \geq 1$, then the value of $f(2)$ is **[JEE 2011, 4]**

DE0120

(b) Let y'(x) + y(x)g'(x) = g(x)g'(x), y(0) = 0, x ∈ R, where $f'(x)$ denotes $\frac{df(x)}{dx}$ and g(x) is a given non-constant differentiable function on R with g(0) = g(2) = 0. Then the value of y(2) is **[JEE 2011, 4]**

DE0121

2. If y(x) satisfies the differential equation y' – ytanx = 2x sec x and y(0) = 0, then **[JEE 2012, 4M]**

(A) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$ (B) $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$ (C) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{9}$ (D) $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$

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DE0122

3. Let $f : \left[\frac{1}{2}, 1\right] \to R$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right) = 1$. Then the value of $\int_{1/2}^{1} f(x)dx$ lies in the interval

[JEE(Advanced) 2013, 2M]

(A) (2e – 1, 2e) (B) (e – 1, 2e – 1) (C) $\left(\frac{e-1}{2}, e-1\right)$ (D) $\left(0, \frac{e-1}{2}\right)$

DE0123

4. A curve passes through the point $\left(1, \frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x} + \sec\left(\frac{y}{x}\right), x > 0$. Then the equation of the curve is [JEE(Advanced) 2013, 2M]

(A) $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$ (B) $\operatorname{cosec}\left(\frac{y}{x}\right) = \log x + 2$

(C) $\sec\left(\frac{2y}{x}\right) = \log x + 2$ (D) $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$

DE0124

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Paragraph for Question 5 and 6

Let $f : [0,1] \to \mathbb{R}$ (the set of all real numbers) be a function. Suppose the function f is twice differentiable, $f(0) = f(1) = 0$ and satisfies $f''(x) - 2f'(x) + f(x) \geq e^x$, $x \in [0,1]$.

5. If the function $e^{-x}f(x)$ assumes its minimum in the interval [0,1] at $x = \frac{1}{4}$, which of the following is true ? [JEE(Advanced) 2013, 3, (–1)]

(A) $f'(x) < f(x),\ \frac{1}{4} < x < \frac{3}{4}$
(B) $f'(x) > f(x),\ 0 < x < \frac{1}{4}$
(C) $f'(x) < f(x),\ 0 < x < \frac{1}{4}$
(D) $f'(x) < f(x),\ \frac{3}{4} < x < 1$

DE0125

6. Which of the following is true for $0 < x < 1$? [JEE(Advanced) 2013, 3, (–1)]

(A) $0 < f(x) < \infty$
(B) $-\frac{1}{2} < f(x) < \frac{1}{2}$
(C) $-\frac{1}{4} < f(x) < 1$
(D) $-\infty < f(x) < 0$

DE0125

7. The function $y = f(x)$ is the solution of the differential equation $\frac{dy}{dx} + \frac{xy}{x^2-1} = \frac{x^4+2x}{\sqrt{1-x^2}}$ in (–1, 1) satisfying $f(0) = 0$. Then $\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)dx$ is [JEE(Advanced)-2014, 3(–1)]

(A) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$
(B) $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$
(C) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$
(D) $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$

DE0126

8. Let y(x) be a solution of the differential equation $(1 + e^x)y' + ye^x = 1$. If $y(0) = 2$, then which of the following statements is(are) true ? **[JEE 2015, 4M, –2M]**

(A) $y(-4) = 0$
(B) $y(-2) = 0$
(C) y(x) has a critical point in the interval (–1,0)
(D) y(x) has no critical point in the interval (–1,0)

DE0127

9. Consider the family of all circles whose centers lie on the straight line y = x. If this family of circles is represented by the differential equation $Py'' + Qy' + 1 = 0$, where P,Q are functions of x,y and y' (here $y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}$), then which of the following statements is (are) true?**[JEE 2015, 4M, –2M]**

(A) $P = y + x$
(B) $P = y - x$
(C) $P + Q = 1 - x + y + y' + (y')^2$
(D) $P - Q = x + y - y' - (y')^2$

DE0128

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10. A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0, x > 0$, passes through the point (1,3). The the solution curve-

(A) intersects y = x + 2 exactly at one point

(B) intersects y = x + 2 exactly at two points

(C) intersects $y = (x + 2)^2$

(D) does NOT intersect $y = (x + 3)^2$ **[JEE(Advanced)-2016, 4(–2)]**

DE0129

11. Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function such that $f'(x) = 2 - \frac{f(x)}{x}$ for all $x \in (0,\infty)$ and $f(1) \neq 1$. Then **[JEE(Advanced)-2016, 4(–2)]**

(A) $\lim_{x\to 0^+} f'\left(\frac{1}{x}\right) = 1$

(B) $\lim_{x\to 0^+} x f\left(\frac{1}{x}\right) = 2$

(C) $\lim_{x\to 0^+} x^2 f'(x) = 0$

(D) $|f(x)| \leq 2$ for all $x \in (0,2)$

DE0130

12. If y = y(x) satisfies the differential equation $8\sqrt{x}\left(\sqrt{9+\sqrt{x}}\right)dy = \left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1} dx,\ x > 0$ and $y(0) = \sqrt{7}$, then y(256) = **[JEE(Advanced)-2017, 3(–1)]**

(A) 80 (B) 3 (C) 16 (D) 9

DE0131

13. If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $f'(x) > 2f(x)$ for all $x \in \mathbb{R}$, and $f(0) = 1$, then **[JEE(Advanced)-2017, 3(–2)]**

(A) $f(x) > e^{2x}$ in $(0,\infty)$

(B) $f(x)$ is decreasing in $(0,\infty)$

(C) $f(x)$ is increasing in $(0,\infty)$

(D) $f'(x) < e^{2x}$ in $(0,\infty)$

DE0132

14. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two non-constant differentiable functions. If $f'(x) = (e^{(f(x)-g(x))})g'(x)$ for all $x \in \mathbb{R}$, and $f(1) = g(2) = 1$, then which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018, 4(–2)]**

(A) $f(2) < 1 - \log_e 2$

(B) $f(2) > 1 - \log_e 2$

(C) $g(1) > 1 - \log_e 2$

(D) $g(1) < 1 - \log_e 2$

DE0133

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15. Let $f : (0, \pi) \to \mathbb{R}$ be a twice differentiable function such that

$$\lim_{t \to x} \frac{f(x)\sin t - f(t)\sin x}{t - x} = \sin^2 x \text{ for all } x \in (0, \pi).$$

If $f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}$, then which of the following statement(s) is (are) TRUE ?

[JEE(Advanced)-2018, 4(–2)]

(A) $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4\sqrt{2}}$

(B) $f(x) < \frac{x^4}{6} - x^2$ for all $x \in (0, \pi)$

(C) There exists $\alpha \in (0, \pi)$ such that $f'(\alpha) = 0$

(D) $f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 0$

DE0134

16. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f(0) = 0$. If $y = f(x)$ satisfies the differential equation $\frac{dy}{dx} = (2 + 5y)(5y - 2)$,

then the value of $\lim_{x \to -\infty} f(x)$ is _____ . **[JEE(Advanced)-2018, 3(0)]**

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DE0135

17. Let Γ denote a curve $y = y(x)$ which is in the first quadrant and let the point (1, 0) lie on it. Let the tangent to Γ at a point P intersect the y-axis at Y_P. If PY_P has length 1 for each point P on Γ, then which of the following options is/are correct ? **[JEE(Advanced)-2019, 4(–1)]**

(1) $y = \log_e\left(\frac{1 + \sqrt{1 - x^2}}{x}\right) - \sqrt{1 - x^2}$

(2) $xy' - \sqrt{1 - x^2} = 0$

(3) $y = -\log_e\left(\frac{1 + \sqrt{1 - x^2}}{x}\right) + \sqrt{1 - x^2}$

(4) $xy' + \sqrt{1 - x^2} = 0$

DE0136

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ANSWER KEY

DIFFERENTIAL EQUATION

EXERCISE (O-1)

1.	B	**2.**	C	**3.**	B	**4.**	B	**5.**	A	**6.**	B	**7.**	D	**8.**	C	**9.**	A
10.	A	**11.**	A	**12.**	B	**13.**	C	**14.**	A	**15.**	B	**16.**	A	**17.**	A	**18.**	B

EXERCISE (O-2)

1.	A,B,D	**2.**	A,B,C,D	**3.**	C,D	**4.**	A,B,D	**5.**	A,C
6.	A,B,C	**7.**	A,B,D	**8.**	A,B,C	**9.**	A,B,D	**10.**	A,B,C

EXERCISE (S-1)

(E-1)

[FORMATION & VARIABLES SEPARABLE]

1. **(i)** order 2 & degree 3 **(ii)** order 2 & degree 2

2. **(a)** $xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}$; **(b)** $[1+(y')^2].\ y'''-3y'(y'')^2=0$ **(c)** $\frac{d^3y}{dx^3}-6\frac{d^2y}{dx^2}+11\frac{dy}{dx}-6y=0$

3. $ln^2(\sec x+\tan x)-ln^2(\sec y+\tan y)=c$

4. $lnx\,(1-y)^2=c-\frac{1}{2}y^2-2y+\frac{1}{2}x^2$

5. $\sqrt{x^2-1}-\sec^{-1}x+\sqrt{y^2-1}=c$

6. $y=c\,(1-ay)\,(x+a)$

7. $ln\left[1+\tan\frac{x+y}{2}\right]=x+c$

8. $y\sin y=x^2lnx+c$

9. **(a)** $ln\left|\tan\frac{y}{4}\right|=c-2\sin\frac{x}{2}$, **(b)** $y=e^{\tan(x/2)}$

10. (a) $P=1000+1500e^{-kt}$ where $k=\frac{1}{10}ln\left(\frac{5}{3}\right)$; (b) $T=10\log_{5/3}(3)$; (c) $P=1000$ as $t\to\infty$

11. $m=m_0\,e^{-kt}$ where $k=-\frac{1}{t_0}ln\left(1-\frac{\alpha}{100}\right)$

12. $x^2+y^2=k^2$

13. $y=kx$ or $xy=c$

14. 40

15. 1/4

16. $y=\frac{1}{3-e}\,(2e^x-e+1)$

(E-2)

[HOMOGENEOUS]

1. $c(x-y)^{2/3}(x^2+xy+y^2)^{1/6}=\exp\left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{x+2y}{x\sqrt{3}}\right]$, where $\exp x\equiv e^x$

2. $\text{arc}\tan\frac{2y+1}{2x+1}=ln\,c\sqrt{x^2+y^2+x+y+\frac{1}{2}}$

3. $(x+y-2)=c\,(y-x)^3$

4. $\tan^{-1}\frac{y+3}{x+2}+lnc\sqrt{(y+3)^2+(x+2)^2}=0$

5. $x+y+\frac{4}{3}=ce^{3(x-2y)}$

6. $e^{-2\tan^{-1}\frac{y+2}{x-3}}=c.(y+2)$

7. $\frac{y^2\pm y\sqrt{y^2-x^2}}{x^2}=\ell n\left|\left(y\pm\sqrt{y^2-x^2}\right).\frac{c^2}{x^3}\right|$, where same sign has to be taken.

8. $x^2+y^2-2x=0$

9. $x^2+y^2=cx$

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(E-3)

[LINEAR]

1. $-\frac{1}{2}$ **2.** $y\sqrt{1+x^2} = c + \frac{1}{2} ln\left[\tan\frac{1}{2}\arctan x\right]$. Another form is $y\sqrt{1+x^2} = c + \frac{1}{2} ln \frac{\sqrt{1+x^2}-1}{x}$

3. $y = c(1-x^2) + \sqrt{1-x^2}$ **4.** $y(x-1) = x^2(x^2 - x + c)$ **5.** $\left(\frac{1}{3} + y\right)\tan^3\frac{x}{2} = c + 2\tan\frac{x}{2} - x$

6. $4(x^2+1)y + x^3(1 - 2\,ln x) = cx$ **7.** $y = cx + x\,ln\tan x$ **8.** $x = ce^{-\text{arc}\tan y} + \text{arc}\tan y - 1$

9. (a) $y = cx^2 \pm x$; (b) $xy = 2$ **11.** $\cos x - 1$ **12.** $y(1 + bx) = b + cx$

13. $f(x) = -\frac{2\cos x}{(1+\sin x)^2} - Ce^{-\sin x}\cdot\cos x$ **14.** $\frac{1}{y^2} = -1 + (c+x)\cot\left(\frac{x}{2}+\frac{\pi}{4}\right)$

15. $x^3 y^{-3} = 3\sin x + c$ **16.** $y^{-1}e^x = c - x^2$ **17.** $27\frac{7}{9}$ minutes

(E-4)

1. $y^2 + x\,ln\,ax = 0$ **2.** $y^2 = 3x^2 - 6x - x^3 + ce^{-x} + 4$ **3.** $x\,ln\,y = e^x(x-1) + c$

4. $\sin y = (e^x + c)(1 + x)$ **5.** $cx^2 + 2xe^{-y} = 1$ **6.** $y = ce^x$; $y = c + \frac{x^2}{2}$

7. $y^2 = -1 + (x+1)ln\frac{c}{x+1}$ or $x + (x+1)\,ln\frac{c}{x+1}$ **8.** $y = \frac{1}{x}\tan\left(\ell n|cx|\right)$

9. $e^y = c.\exp(-e^x) + e^x - 1$ **10.** $y = 3\,ln(x^2 + y + 3) + C$

EXERCISE (S-2)

(MISCELLANEOUS)

1. $y = 2^{\sin x}$ **3.** $\frac{\sin^{-1}x}{2} + \frac{y}{x-y} = \frac{\pi}{4} - 2$

4. (i) $y = u(x) + K(u(x) - v(x))$, where K is any constant ; (ii) $\alpha + \beta = 1$; (iii) constant

5. $x(ey + ln\,y + 1) = 1$ **6.** $y = 5t\left(1 + \frac{50}{50+t}\right)$ gms; $91\frac{2}{3}$ gms

7. (i) $x^2 + 2y^2 = c$, (ii) $\sin y = ce^{-x}$, (iii) $y = cx$ if $k = 2$ and $\frac{1}{x^{k-2}} - \frac{1}{y^{k-2}} = \frac{1}{c^{k-2}}$ if $k \neq 2$

(iv) $x^2 - y^2 + 2xy = c$; $x^2 - y^2 - 2xy = c$

9. (a) $(-\infty,\infty)$ (b) $\frac{19}{4}$ (c) $3 - \sqrt{5}$ **11.** $y = cx - x^2$

EXERCISE (JM)

1.	3	**2.**	4	**3.**	3	**4.**	4	**5.**	2	**6.**	3	**7.**	3	**8.**	2	**9.**	3
10.	2	**11.**	1	**12.**	1	**13.**	1	**14.**	2	**15.**	2	**16.**	2	**17.**	1	**18.**	2
19.	4	**20.**	2	**21.**	3	**22.**	3.00										

EXERCISE (JA)

1.	**(a)** Bonus; **(b)** 0			**2.**	A,D	**3.**	D	**4.**	A	**5.**	C	**6.**	D	**7.**	B
8.	A,C	**9.**	B,C	**10.**	A,D	**11.**	A	**12.**	B	**13.**	A,C	**14.**	B,C		
15.	B,C,D	**16.**	0.4	**17.**	1,4										

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AREA UNDER THE CURVE

1. AREA UNDER THE CURVES :

(a) Area bounded by the curve y = f(x), the x-axis and the ordinates at x = a and x = b is given by $A = \int_a^b y\,dx$, where y = f (x) lies above the x-axis and b > a. Here vertical strip of thickness dx is considered at distance x.

(b) If y = f (x) lies completely below the x-axis then A is negative and we consider the magnitude only, i.e. $A = \left|\int_a^b y\,dx\right|$

(c) If curve crosses the x-axis at x = c, then $A = \left|\int_a^c y\,dx\right| + \int_c^b y\,dx$

(d) Sometimes integration w.r.t. y is very useful (horizontal strip) :

Area bounded by the curve, y-axis and the two abscissae at y = a & y = b is written as $A = \int_a^b x\,dy$.

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Note : If the curve is symmetric and suppose it has 'n' symmetric portions, then total area = n (Area of one symmetric portion).

Illustration 1 : Find the area bounded by y = $\sec^2$ x, x = $\frac{\pi}{6}$, x = $\frac{\pi}{3}$ & x-axis

Solution : Area bounded $= \int_{\pi/6}^{\pi/3} y\,dx = \int_{\pi/6}^{\pi/3} \sec^2 x\,dx = [\tan x]_{\pi/6}^{\pi/3} = \tan\frac{\pi}{3} - \tan\frac{\pi}{6} = \sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$ sq.units.

Illustration 2 : Find the area in the first quadrant bounded by y = $4x^2$, x = 0, y = 1 and y = 4.

Solution : Required area $= \int_1^4 x\,dy = \int_1^4 \frac{\sqrt{y}}{2}\,dy = \frac{1}{2}\left[\frac{2}{3}y^{3/2}\right]_1^4$

$= \frac{1}{3}[4^{3/2} - 1] = \frac{1}{3}[8 - 1]$

$= \frac{7}{3} = 2\frac{1}{3}$ sq.units.

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Illustration 3 : Find the area bounded by the curve y = sin2x, x-axis and the lines $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$

Solution : Required area $= \int_{\pi/4}^{\pi/2} \sin 2x dx + \left|\int_{\pi/2}^{3\pi/4} \sin 2x dx\right| = \left(-\frac{\cos 2x}{2}\right)\Big|_{\pi/4}^{\pi/2} + \left|\left(-\frac{\cos 2x}{2}\right)\Big|_{\pi/2}^{3\pi/4}\right|$

$= -\frac{1}{2}[-1-0] + \left|\frac{1}{2}(0+(-1))\right| = 1$ sq. unit

Do yourself - 1 :

(i) Find the area bounded by $y = x^2 + 2$ above x-axis between x = 2 & x = 3.

(ii) Using integration, find the area of the curve $y = \sqrt{1-x^2}$ with co-ordinate axes bounded in first quadrant.

(iii) Find the area bounded by the curve y = 2cosx and the x-axis from x = 0 to x = 2π.

(iv) Find the area bounded by the curve y = x|x|, x-axis and the ordinates $x = -\frac{1}{2}$ and x=1.

2. AREA ENCLOSED BETWEEN TWO CURVES :

(a) Area bounded by two curves y = f (x) & y = g(x) such that f (x) > g(x) is

$$A = \int_{x_1}^{x_2} (y_1 - y_2)dy$$

$$A = \int_{x_1}^{x_2} [f(x) - g(x)]dx$$

(b) In case horizontal strip is taken we have

$$A = \int_{y_1}^{y_2} (x_1 - x_2)dy$$

$$A = \int_{y_1}^{y_2} [f(y) - g(y)]dy$$

(c) If the curves $y_1 = f$ (x) and y_2 = g(x) intersect at x = c, then required area

$$A = \int_a^c (g(x) - f(x))dx + \int_c^b (f(x) - g(x))dx = \int_a^b |f(x) - g(x)|dx$$

Note : Required area must have all the boundaries indicated in the problem.

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Illustration 4 : Find the area bounded by the curve $y = (x-1)(x-2)(x-3)$ lying between the ordinates $x = 0$ and $x = 3$ and x-axis

Solution : To determine the sign, we follow the usual rule of change of sign.

$y = +ve$ for $x > 3$

$y = -ve$ for $2 < x < 3$

$y = +ve$ for $1 < x < 2$

$y = -ve$ for $x < 1$.

$$\int_0^3 |y|\,dx = \int_0^1 |y|\,dx + \int_1^2 |y|\,dx + \int_2^3 |y|\,dx$$

$$= \int_0^1 -y\,dx + \int_1^2 y\,dx + \int_2^3 -y\,dx$$

Now let $F(x) = \int (x-1)(x-2)(x-3)\,dx = \int (x^3 - 6x^2 + 11x - 6)\,dx = \frac{1}{4}x^4 - 2x^3 + \frac{11}{2}x^2 - 6x.$

$\therefore \quad F(0) = 0,\ F(1) = -\frac{9}{4},\ F(2) = -2,\ F(3) = -\frac{9}{4}.$

Hence required Area $= -[F(1) - F(0)] + [F(2) - F(1)] - [F(3) - F(2)] = 2\frac{3}{4}$ sq.units.

Illustration 5 : Compute the area of the figure bounded by the straight lines $x = 0$, $x = 2$ and the curves $y = 2^x$, $y = 2x - x^2$.

Solution : The required area $= \int_0^2 (y_1 - y_2)\,dx$

where $y_1 = 2^x$ and $y_2 = 2x - x^2 = \int_0^2 (2^x - 2x + x^2)dx$

$$= \left[\frac{2^x}{\ln 2} - x^2 + \frac{1}{3}x^3\right]_0^2 = \left(\frac{4}{\ln 2} - 4 + \frac{8}{3}\right) = \frac{1}{\ln 2} = \frac{3}{\ln 2} - \frac{4}{3} \text{ sq.units.}$$

Illustration 6 : Compute the area of the figure bounded by the parabolas $x = -2y^2$, $x = 1 - 3y^2$.

Solution : Solving the equations $x = -2y^2$, $x = 1 - 3y^2$, we find that ordinates of the points of intersection of the two curves as $y_1 = -1$, $y_2 = 1$.

The points are $(-2, -1)$ and $(-2, 1)$.

The required area

$$2\int_0^1 (x_1 - x_2)\,dy = 2\int_0^1 [(1-3y^2) - (-2y^2)]dy$$

$$= 2\int_0^1 (1-y^2)dy = 2\left[y - \frac{y^3}{3}\right]_0^1 = \frac{4}{3} \text{ sq.units.}$$

Do yourself - 2 :

(i) Find the area bounded by $y = \sqrt{x}$ and $y = x$.

(ii) Find the area bounded by the curves $x = y^2$ and $x = 3 - 2y^2$.

(iii) Find the area of the region bounded by the curves $x = \frac{1}{2}$, $x = 2$, $y = \log x$ and $y = 2^x$.

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3. CURVE TRACING :

The following procedure is to be applied in sketching the graph of a function y = f(x) which in turn will be extremely useful to quickly and correctly evaluate the area under the curves.

(a) Symmetry : The symmetry of the curve is judged as follows :

(i) If all the powers of y in the equation are even then the curve is symmetrical about the axis of x.

(ii) If all the powers of x are even, the curve is symmetrical about the axis of y.

(iii) If powers of x & y both are even, the curve is symmetrical about the axis of x as well as y.

(iv) If the equation of the curve remains unchanged on interchanging x and y, then the curve is symmetrical about y = x.

(v) If on interchanging the signs of x & y both, the equation of the curve is unaltered then there is symmetry in opposite quadrants.

(b) Find dy/dx & equate it to zero to find the points on the curve where you have horizontal tangents.

(c) Find the points where the curve crosses the x–axis & also the y–axis.

(d) Examine if possible the intervals when f(x) is increasing or decreasing. Examine what happens to 'y' when $x \to \infty$ or $-\infty$.

Illustration 7 : Find the area of a loop as well as the whole area of the curve $a^2y^2 = x^2(a^2 - x^2)$.

Solution : The curve is symmetrical about both the axes. It cuts x-axis at (0, 0), (–a, 0), (a, 0)

$$\text{Area of a loop} = 2\int_0^a y\,dx = 2\int_0^a \frac{x}{a}\sqrt{a^2 - x^2}\,dx$$

$$= -\frac{1}{a}\int_0^a \sqrt{a^2 - x^2}(-2x)dx = \frac{1}{a}\left[\frac{2}{3}(a^2 - x^2)^{3/2}\right]_0^a = \frac{2}{3}a^2$$

$$\text{Total area} = 2 \times \frac{2}{3}a^2 = \frac{4}{3}a^2 \text{ sq.units.}$$

Illustration 8 : Find the whole area included between the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptotes.

Solution : (i) The curve is symmetric about both the axes (even powers of x & y)

(ii) Asymptotes are x = ± a

$$A = 4\int_0^a y\,dx$$

$$= 4\int_0^a \frac{ax}{\sqrt{a^2 - x^2}}dx$$

$$= 4a\left|-\sqrt{a^2 - x^2}\right|_0^a$$

$$= 4a^2$$

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Illustration 9 : Find the area bounded by the curve $xy^2 = 4a^2(2a–x)$ and its asymptote.

Solution : (i) The curve is symmetrical about the x-axis as it contains even powers of y.

(ii) It passes through (2a,0).

(iii) Its asymptote is x = 0, i.e., y-axis.

$$A = 2\int_0^{2a} y\,dx = 2\int_0^{2a} 2a\sqrt{\frac{2a-x}{x}}\,dx$$

Put $x = 2a\sin^2\theta$

$$A = 16a^2\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$= 4\pi a^2$$

---

4. IMPORTANT POINTS :

(a) Since area remains invariant even if the co-ordinate axes are shifted, hence shifting of origin in many cases proves to be very convenient in computing the area.

Illustration 10 : Find the area enclosed by $|x – 1| + |y + 1| = 1$.

Solution : Shift the origin to (1, –1).

$X = x – 1$ $\qquad$ $Y = y + 1$

$|X| + |Y| = 1$

Area $= \sqrt{2}\times\sqrt{2} = 2$ sq. units

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Illustration 11 : Find the area of the region common to the circle $x^2 + y^2 + 4x + 6y – 3 = 0$ and the parabola $x^2 + 4x = 6y + 14$.

Solution : Circle is $x^2 + y^2 + 4x + 6y – 3 = 0$

$\Rightarrow (x + 2)^2 + (y + 3)^2 = 16$

Shifting origin to (–2,–3).

$X^2 + Y^2 = 16$

equation of parabola $\rightarrow (x + 2)^2 = 6(y + 3)$

$\Rightarrow X^2 = 6Y$

Solving circle & parabola, we get $X = \pm 2\sqrt{3}$

Hence they intersect at $\left(-2\sqrt{3},2\right)$ & $\left(2\sqrt{3},2\right)$

$$A = 2\left[\int_0^2\sqrt{6Y}\,dY + \int_2^4\sqrt{16-Y^2}\,dY\right]$$

$$= 2\left[\frac{2}{3}\sqrt{6}\left[Y^{3/2}\right]_0^2 + \left[\frac{1}{2}Y\sqrt{16-Y^2} + \frac{16}{2}\sin^{-1}\frac{Y}{4}\right]_2^4\right] = \left(\frac{4\sqrt{3}}{3} + \frac{16\pi}{3}\right)\text{ sq. units}$$

Do yourself : 3

(i) Find the area inside the circle $x^2–2x + y^2 – 4y + 1 = 0$ and outside the ellipse $x^2–2x+4y^2–16y+13= 0$

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(b) If y = f (x) is a monotonic function in (a, b), then the area bounded by the ordinates at x = a, x = b, y = f (x) and y = f (c) [where c $\in$ (a, b)] is minimum when $c = \frac{a+b}{2}$.

Proof : Let the function y = f (x) be monotonically increasing.

Required area $A = \int_a^c [f(c) - f(x)]dx + \int_c^b [f(x) - f(c)]dx$

For minimum area, $\frac{dA}{dc} = 0$

$\Rightarrow \quad [f'(c).c + f(c) - f'(c)a - f(c)] + [-f(c) - f'(c).b + f'(c).c + f(c)] = 0$

$\Rightarrow \quad f'(c)\left\{c - \frac{a+b}{2}\right\} = 0$

$\Rightarrow \quad c = \frac{a+b}{2} \quad (\because f'(c) \neq 0)$

Illustration 12 : Find the value of 'a' for which area bounded by x = 1, x=2, y=6x^2 and y=f(a) is minimum.

Solution : Let b = f (a).

$A = \int_1^a (b - 6x^2)dx + \int_a^2 (6x^2 - b)dx = \left|bx - 2x^3\right|_1^a + \left|2x^3 - bx\right|_a^2$

= 8a^3 – 18a^2 + 18

For minimum area $\frac{dA}{da} = 0$

$\Rightarrow \quad 24a^2 - 36a = 0 \quad \Rightarrow \quad a = 1.5$

Alternatively, y = 6x^2 $\Rightarrow \quad \frac{dy}{dx} = 12x$

Hence y = f (x) is monotonically increasing. Hence bounded area is minimum when

$a = \left(\frac{1+2}{2}\right) = 1.5$

Do yourself - 4 :

(i) Find the value of 'a' $(0 < a < \frac{\pi}{2})$ for which the area bounded by the curve f (x) = sin^{3}x + sinx, y = f(a) between x = 0 & x = π/2 is minimum.

(c) The area bounded by a curve & an axis is equal to the area bounded by the inverse of that curve & the other axis, i.e., the area bounded by y = f (x) and x-axis (say) is equal to the area bounded by y = f^{-1}(x) and y-axis.

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Illustration 13 : If y = g(x) is the inverse of a bijective mapping $f : R \to R$, $f(x) = 6x^5 + 4x^3 + 2x$, find the area bounded by g(x), the x-axis and the ordinate at x = 12.

Solution : $f(x) = 12$

$\Rightarrow \quad 6x^5 + 4x^3 + 2x = 12 \quad \Rightarrow \quad x = 1$

$$\int_0^{12} g(x)dx = \text{area of rectangle OEDF} - \int_0^1 f(x)dx$$

$$= 1 \times 12 - \int_0^1 (6x^5 + 4x^3 + 2x)dx = 12 - 3 = 9 \text{ sq. units.}$$

Do yourself - 5 :

(i) Find the area bounded by the inverse of bijective function $f(x) = 4x^3 + 6x$, the x-axis and the ordinates x = 0 & x = 44.

5. USEFUL RESULTS :

(a) Whole area of the ellipse, $x^2/a^2 + y^2/b^2 = 1$ is πab sq.units.

(b) Area enclosed between the parabolas $y^2 = 4ax$ & $x^2 = 4by$ is 16ab/3 sq.units.

(c) Area included between the parabola $y^2 = 4ax$ & the line y = mx is $8a^2/3m^3$ sq.units.

(d) The area of the region bounded by one arch of sin ax (or cos ax) and x-axis is 2/a sq.units.

(e) Average value of a function y = f(x) over an interval $a \le x \le b$ is defined as : $y(av) = \frac{1}{b-a}\int_a^b f(x)dx$.

Miscellaneous Illustration :

Illustration 14 : Find the smaller of the areas bounded by the parabola $4y^2 - 3x - 8y + 7 = 0$ and the ellipse $x^2 + 4y^2 - 2x - 8y + 1 = 0$.

Solution : C_1 is $4(y^2 - 2y) = 3x - 7$

or $\quad 4(y-1)^2 = 3x - 3 = 3(x-1) \quad$ .....(i)

Above is parabola with vertex at (1, 1)

C_2 is $(x^2 - 2x) + 4(y^2 - 2y) = -1$

or $\quad (x-1)^2 + 4(y-1)^2 = -1 + 1 + 4$

or $\quad \frac{(x-1)^2}{2^2} + \frac{(y-1)^2}{1^2} = 1 \quad$ .....(ii)

Above represents an ellipse with centre at (1, 1). Shift the origin to (1, 1) and this will not affect the magnitude of required area but will make the calculation simpler.

Thus the two curves are

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$4Y^2 = 3X$ and $\frac{X^2}{2^2}+\frac{Y^2}{1}=1$ They meet at $\left(1,\pm\frac{\sqrt{3}}{2}\right)$

Required area = 2(A + B) = $2\left[\int Y_1 dX+\int Y_2 dX\right]$

$$=2\left[\frac{\sqrt{3}}{2}\int_0^1\sqrt{X}dX+\int_1^2\frac{\sqrt{4-X^2}}{2}dX\right]=\left[\frac{\sqrt{3}}{6}+\frac{2\pi}{3}\right] \text{ sq.units.}$$

Illustration 15 : Find the area bounded by the regions $y \ge \sqrt{x}$, $x > -\sqrt{y}$ & curve $x^2 + y^2 = 2$.

Solution : Common region is given by the diagram

If area of region OAB = λ

then area of OCD = λ

Because $y=\sqrt{x}$ & $x=-\sqrt{y}$

will bound same area with x & y axes respectively.

$y=\sqrt{x} \Rightarrow y^2 = x$

$x=-\sqrt{y} \Rightarrow x^2 = y$ and hence both the curves are symmetric with respect to the line y = x

Area of first quadrant OBC = $\frac{\pi r^2}{4}=\frac{\pi}{2}$ $\quad(\because\ r=\sqrt{2})$

Area of region OCA = $\frac{\pi}{2}-\lambda$

Area of shaded region = $(\frac{\pi}{2}-\lambda)+\lambda=\frac{\pi}{2}$ sq.units.

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Illustration 16 : Find the equation of line passing through the origin & dividing the curvilinear triangle with vertex at the origin, bounded by the curves $y = 2x - x^2$, y = 0 & x = 1 in two parts of equal areas.

Solution : Area of region OBA = $\int_0^1(2x-x^2)dx$

$$=\left[x^2-\frac{x^3}{3}\right]_0^1=\frac{2}{3}$$

$\frac{2}{3}=A_1+A_1 \Rightarrow A_1=\frac{1}{3}$

Let pt. C has coordinates (1, y)

Area of $\Delta OCB = \frac{1}{2}\times 1\times y=\frac{1}{3}$

$y=\frac{2}{3}$

C has coordinates $\left(1,\frac{2}{3}\right)$

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Line OC has slope $m = \dfrac{\frac{2}{3}-0}{1-0} = \dfrac{2}{3}$

Equation of line OC is $y = mx \Rightarrow y = \dfrac{2}{3}x$.

Illustration 17 : The area bounded by $y = x^2 + 1$ and the tangents to it drawn from the origin is :-

(A) 8/3 sq. units (B) 1/3 sq. units (C) 2/3 sq. units (D) none of these

Solution : The parabola is even function & let the equation of tangent is y= mx

Now we calculate the point of intersection of parabola & tangent

$mx = x^2 + 1$

$x^2 - mx + 1 = 0 \Rightarrow D = 0$

$\Rightarrow m^2 - 4 = 0 \Rightarrow m = \pm 2$

Two tangents are possible $y = 2x$ & $y = -2x$

Intersection of $y = x^2 + 1$ & $y = 2x$ is $x = 1$ & $y = 2$

Area of shaded region OAB $= \int_0^1 (y_2 - y_1)\,dx = \int_0^1 \left((x^2+1) - 2x\right)dx = \dfrac{1}{3}$ sq. units

Area of total shaded region $= 2\left(\dfrac{1}{3}\right) = \dfrac{2}{3}$ sq. units

Illustration 18 : Find the area bounded by x-axis and the curve given by $x = a\sin t$, $y = a\cos t$ for $0 \le t \le \pi$.

Solution : Area $= \int_0^\pi y\dfrac{dx}{dt}.dt = \int_0^\pi a\cos t(a\cos t)dt = \dfrac{a^2}{2}\int_0^\pi (1+\cos 2t)dt = \dfrac{a^2}{2}\left|t + \dfrac{\sin 2t}{2}\right|_0^\pi = \dfrac{a^2}{2}|\pi| = \dfrac{\pi a^2}{2}$

Alternatively,

Area $= \int_0^\pi x\dfrac{dy}{dt}.dt = \left|\int_0^\pi a\sin t(-a\sin t)dt\right| = \dfrac{a^2}{2}\left|\int_0^\pi (\cos 2t - 1)dt\right| = \dfrac{a^2}{2}\left|-\dfrac{\sin 2t}{2} - t\right|_0^\pi = \dfrac{\pi a^2}{2}$

ANSWERS FOR DO YOURSELF

1 : **(i)** $\dfrac{25}{3}$ sq. units **(ii)** $\dfrac{\pi}{4}$ sq. units. **(iii)** 8 sq. units. **(iv)** $\dfrac{3}{8}$ sq.units

2 : **(i)** $\dfrac{1}{6}$ sq. units **(ii)** 4 sq. units **(iii)** $\dfrac{4-\sqrt{2}}{\log 2} - \dfrac{5}{2}\log 2 + \dfrac{3}{2}$ sq. units

3 : **(i)** 2π sq. units

4 : **(i)** $\dfrac{\pi}{4}$

5 : **(i)** 60 sq. units.

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EXERCISE (O-1)

1. The area bounded in the first quadrant by the normal at $(1, 2)$ on the curve $y^2 = 4x$, x-axis & the curve is given by :

(A) $\frac{10}{3}$ (B) $\frac{7}{3}$ (C) $\frac{4}{3}$ (D) $\frac{9}{2}$

AU0001

2. Suppose y = f (x) and y = g(x) are two functions whose graphs intersect at the three points (0, 4), (2, 2) and (4, 0) with f (x) > g (x) for 0 < x < 2 and f (x) < g (x) for 2 < x < 4. If $\int_0^4 [f(x) - g(x)]dx = 10$ and $\int_2^4 [g(x) - f(x)]dx = 5$, the area between two curves for 0 < x < 2, is

(A) 5 (B) 10 (C) 15 (D) 20

AU0002

3. Let 'a' be a positive constant number. Consider two curves C_1: $y = e^x$, C_2 : $y = e^{a-x}$. Let S be the area of the part surrounding by C_1, C_2 and the y-axis, then $\lim_{a \to 0} \frac{S}{a^2}$ equals

(A) 4 (B) 1/2 (C) 0 (D) 1/4

AU0003

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4. The area of the region(s) enclosed by the curves $y = x^2$ and $y = \sqrt{|x|}$ is

(A) 1/3 (B) 2/3 (C) 1/6 (D) 1

AU0004

5. Area enclosed by the graph of the function $y = \ln^2 x - 1$ lying in the 4th quadrant is

(A) $\frac{2}{e}$ (B) $\frac{4}{e}$ (C) $2\left(e + \frac{1}{e}\right)$ (D) $4\left(e - \frac{1}{e}\right)$

AU0005

6. The area bounded by the curve $y = f(x)$ (where $f(x) \geq 0$), the co-ordinate axes & the line $x = x_1$ is given by $x_1 \cdot e^{x_1}$. Therefore f(x) equals :

(A) e^x (B) xe^x (C) $xe^x - e^x$ (D) $xe^x + e^x$

AU0006

7. The slope of the tangent to a curve $y = f(x)$ at $(x, f(x))$ is $2x + 1$. If the curve passes through the point $(1, 2)$ then the area of the region bounded by the curve, the x-axis and the line $x = 1$ is

(A) $\frac{5}{6}$ (B) $\frac{6}{5}$ (C) $\frac{1}{6}$ (D) 1

AU0007

8. The area bounded by the curves $y = x(x-3)^2$ and $y = x$ is (in sq. units) :

(A) 28 (B) 32 (C) 4 (D) 8

AU0008

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9. Area of the region enclosed between the curves $x = y^2 - 1$ and $x = |y|\sqrt{1-y^2}$ is

(A) 1 (B) 4/3 (C) 2/3 (D) 2

AU0009

10. The curve $y = ax^2 + bx + c$ passes through the point (1, 2) and its tangent at origin is the line $y = x$. The area bounded by the curve, the ordinate of the curve at minima and the tangent line is

(A) $\frac{1}{24}$ (B) $\frac{1}{12}$ (C) $\frac{1}{8}$ (D) $\frac{1}{6}$

AU0010

EXERCISE (O-2)

[SINGLE CHOICE OBJECTIVE TYPE]

1. The area bounded by the curve $y = x\,e^{-x}$; $xy = 0$ and $x = c$ where c is the x-coordinate of the curve's inflection point, is

(A) $1 - 3e^{-2}$ (B) $1 - 2e^{-2}$ (C) $1 - e^{-2}$ (D) 1

AU0011

2. A function $y = f(x)$ satisfies the differential equation $\frac{dy}{dx} - y = \cos x - \sin x$, with initial condition that y is bounded when $x \to \infty$. The area enclosed by $y = f(x)$, $y = \cos x$ and the y-axis in the 1st quadrant

(A) $\sqrt{2} - 1$ (B) $\sqrt{2}$ (C) 1 (D) $\frac{1}{\sqrt{2}}$

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AU0012

3. If the area bounded between x-axis and the graph of $y = 6x - 3x^2$ between the ordinates $x = 1$ and $x = a$ is 19 square units then 'a' can take the value

(A) 4 or – 2 (B) two values are in (2, 3) and one in (–1, 0)

(C) two values one in (3,4) and one in (–2,–1) (D) none of these

AU0013

[MULTIPLE OBJECTIVE TYPE]

4. Let T be the triangle with vertices (0, 0), $(0, c^2)$ and (c, c^2) and let R be the region between $y = cx$ and $y = x^2$ where $c > 0$ then

(A) Area (R) = $\frac{c^3}{6}$ (B) Area of R = $\frac{c^3}{3}$

(C) $\lim_{c\to 0^+} \frac{\text{Area (T)}}{\text{Area (R)}} = 3$ (D) $\lim_{c\to 0^+} \frac{\text{Area (T)}}{\text{Area (R)}} = \frac{3}{2}$

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5. Suppose f is defined from $R \to [-1, 1]$ as $f(x) = \dfrac{x^2-1}{x^2+1}$ where R is the set of real number. Then the statement which does not hold is

(A) f is many one onto

(B) f increases for $x > 0$ and decrease for $x < 0$

(C) minimum value is not attained even though f is bounded

(D) the area included by the curve $y = f(x)$ and the line $y = 1$ is π sq. units.

AU0015

6. Consider $f(x) = \begin{cases} \cos x & 0 \le x < \dfrac{\pi}{2} \\ \left(\dfrac{\pi}{2} - x\right)^2 & \dfrac{\pi}{2} \le x < \pi \end{cases}$ such that f is periodic with period π, then

(A) The range of f is $\left[0, \dfrac{\pi^2}{4}\right)$

(B) f is continuous for all real x, but not differentiable for some real x

(C) f is continuous for all real x

(D) The area bounded by $y = f(x)$ and the X-axis from $x = -n\pi$ to $x = n\pi$ is $2n\left(1 + \dfrac{\pi^3}{24}\right)$ for a given $n \in N$

AU0016

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7. Which of the following statement(s) is/are True for the function $f(x) = (x-1)^2(x-2) + 1$ defined on $[0, 2]$?

(A) Range of f is $\left[\dfrac{23}{27}, 1\right]$.

(B) The coordinates of the turning point of the graph of $y = f(x)$ occur at $(1, 1)$ and $\left(\dfrac{5}{3}, \dfrac{23}{27}\right)$.

(C) The value of p for which the equation $f(x) = p$ has 3 distinct solutions lies in interval $\left(\dfrac{23}{27}, 1\right)$.

(D) The area enclosed by $y = f(x)$, the lines $x = 0$ and $y = 1$ as x varies from 0 to 1 is $\dfrac{7}{12}$.

AU0017

8. If any curve passing through $(-2, 0)$ whose length of sub normal at any point is 4 unit, then -

(A) there are two possible such curves those touches each other

(B) area bounded by any possible curve and y-axis is $\dfrac{32}{3}$

(C) there will be a unique such curve

(D) line $x + 2 = 0$ is tangent to the curve

AU0018

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9. Let $f(x) = 2 - |x - 1|$ & $g(x) = (x - 1)^2$, then -

(A) area bounded by $f(x)$ & $g(x)$ is $\frac{7}{6}$ (B) area bounded by $f(x)$ & $g(x)$ is $\frac{7}{3}$

(C) area bounded by $f(x)$, $g(x)$ & x-axis is $\frac{5}{3}$ (D) area bounded by $f(x)$, $g(x)$ & x-axis is $\frac{5}{6}$

AU0019

10. The figure shows a horizontal line $y = c$ passing through (b, c) intersecting the curve $y = 8x - 27x^3$. If the shaded areas are equal, then

(A) $b = \frac{1}{9}$ (B) $b = \frac{4}{9}$

(C) $c = \frac{32}{27}$ (D) $c = \frac{23}{27}$

AU0020

11. If A_1 denotes area of the region bounded by the curves $C_1 : y = (x - 1)e^x$, tangent to C_1 at (1,0) & y-axis and A_2 denotes the area of the region bounded by C_1 and co-ordinate axes in fourth quadrant, then -

(A) $A_1 > A_2$ (B) $A_1 < A_2$ (C) $2A_1 + A_2 = 2$ (D) $A_1 + 2A_2 = 4$

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AU0021

12. Area bounded by $y = \sec^{-1} x, y = \cot^{-1} x$ and line $x = 1$ is given by-

(A) $\int_1^{\sqrt{\frac{1+\sqrt{5}}{2}}} \left(\cot^{-1} x - \sec^{-1} x\right) dx$

(B) $\int_0^{\alpha} \sec x\, dx + \int_{\alpha}^{\pi/4} \cot x\, dx - \frac{\pi}{4}$, where $\sin\alpha = \cos^2\alpha$

(C) $\int_0^{\alpha} \sec x\, dx + \int_{\alpha}^{\pi/4} \cot x\, dx - \frac{\pi}{4} + 1$, where $\sin\alpha = \cos^2\alpha$

(D) $\int_1^{\frac{1+\sqrt{5}}{2}} \left(\cot^{-1} x - \sec^{-1} x\right) dx$

AU0022

13. Area bounded by the curve $y = \cot x$, $x = \frac{\pi}{4}$ and $y = 0$ is-

(A) $\int_0^{\pi/4} \tan\left(\frac{\pi}{4} - x\right) dx$ (B) $\frac{\pi}{4} - \int_0^1 \tan^{-1} x dx$ (C) $1 - \int_0^1 \tan^{-1} x dx$ (D) $\int_0^{\pi.4} \tan^{-1} x dx$

AU0023

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14. If y = $f(x)$ is the solution of equation $ydx + dy = -e^x y^2 dy$, $f(0) = 1$ and area bounded by the curve $y = f(x)$, $y = e^x$ and x = 1 is A, then -

(A) curve y = $f(x)$ is passing through (–2,e)

(B) curve y = $f(x)$ is passing through $\left(1, \frac{1}{e}\right)$

(C) $A = e - \frac{2}{\sqrt{e}} + 3$

(D) $A = e + \frac{2}{\sqrt{e}} - 3$

AU0024

15. The function $f(x) = x^{1/3}(x - 1)$

(A) has 2 inflection points.

(B) is strictly increasing for x > 1/4 and strictly decreasing for x < 1/4.

(C) is concave down in (– 1/2, 0).

(D) Area enclosed by the curve lying in the fourth quadrant is $9/28$.

AU0025

EXERCISE (S-1)

1. Find the area bounded on the right by the line x + y = 2, on the left by the parabola $y = x^2$ and below by the x-axis.

AU0026

2. Find the area of the region $\{(x, y) : 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$.

AU0027

3. Find the area of the region bounded by curves $f(x) = (x - 4)^2$, $g(x) = 16 - x^2$ and the x-axis.

AU0028

4. A figure is bounded by the curves $y = \left|\sqrt{2}\sin\frac{\pi x}{4}\right|$, y = 0, x = 2 & x = 4. At what angles to the positive x-axis straight lines must be drawn through (4, 0) so that these lines partition the figure into three parts of the same area.

AU0029

5. Find the area bounded by the curves $y = \sqrt{1 - x^2}$ and $y = x^3 - x$. Also find the ratio in which the y-axis divided this area.

AU0030

6. If the area enclosed by the parabolas $y = a - x^2$ and $y = x^2$ is $18\sqrt{2}$ sq. units. Find the value of 'a'.

AU0031

7. The line 3x + 2y = 13 divides the area enclosed by the curve, $9x^2 + 4y^2 - 18x - 16y - 11 = 0$ into two parts. Find the ratio of the larger area to the smaller area.

AU0032

8. Consider two curves $C_1 : y = \frac{1}{x}$ and $C_2 : y = \ln x$ on the xy plane. Let D_1 denotes the region surrounded by C_1, C_2 and the line x = 1 and D_2 denotes the region surrounded by C_1, C_2 and the line x = a. If $D_1 = D_2$. Find the value of 'a'.

AU0033

9. Find the area enclosed between the curves : $y = \log_e(x + e)$, $x = \log_e(1/y)$ & the x-axis.

AU0034

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10. For what value of 'a' is the area bounded by the curve $y = a^2x^2 + ax + 1$ and the straight line $y = 0$, $x = 0$ & $x = 1$ the least ?

AU0035

11. Find the positive value of 'a' for which the parabola $y = x^2 + 1$ bisects the area of the rectangle with vertices $(0, 0)$, $(a, 0)$, $(0, a^2 + 1)$ and $(a, a^2 + 1)$.

AU0036

12. Compute the area of the curvilinear triangle bounded by the y-axis & the curve, $y = \tan x$ & $y=(2/3)\cos x$.

AU0037

13. Find the value of 'c' for which the area of the figure bounded by the curve, $y = 8x^2 - x^5$, the straight lines $x = 1$ & $x = c$ & the abscissa axis is equal to 16/3.

AU0038

14. Find the area bounded by the curve $y = x\,e^{-x^2}$, the x-axis, and the line $x = c$ where $y(c)$ is maximum.

AU0039

15. A polynomial function $f(x)$ satisfies the condition $f(x + 1) = f(x) + 2x + 1$. Find $f(x)$ if $f(0) = 1$. Find also the equations of the pair of tangents from the origin on the curve $y = f(x)$ and compute the area enclosed by the curve and the pair of tangents.

AU0040

16. The figure shows two regions in the first quadrant.

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$A(t)$ is the area under the curve $y = \sin x^2$ from 0 to t and $B(t)$ is the area of the triangle with vertices O, P and M(t, 0). Find $\lim\limits_{t\to 0} \dfrac{A(t)}{B(t)}$.

AU0041

EXERCISE (S-2)

1. Compute the area of the region bounded by the curves $y = e.\,x.\,\ln x$ & $y = \ln x/(e.\,x)$ where $\ln e=1$.

AU0042

2. Find the values of m $(m > 0)$ for which the area bounded by the line $y = mx + 2$ and $x = 2y - y^2$ is, (i) 9/2 square units & (ii) minimum. Also find the minimum area.

AU0043

3. Let A_n be the area bounded by the curve $y = (\tan x)^n$ & the lines $x = 0$, $y = 0$ & $x = \pi/4$. Prove that for $n > 2$, $A_n + A_{n-2} = 1/(n-1)$ & deduce that $1/(2n+2) < A_n < 1/(2n-2)$.

AU0044

4. Find the area bounded by the curve $y = x\,e^{-x}$; $xy = 0$ and $x = c$ where c is the x-coordinate of the curve's inflection point.

AU0045

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5. Consider the curve $y = x^n$ where $n > 1$ in the 1st quadrant. If the area bounded by the curve, the x-axis and the tangent line to the graph of $y = x^n$ at the point (1, 1) is maximum then find the value of n.

AU0046

6. Consider the two curves $y = 1/x^2$ & $y = 1/[4(x-1)]$.

(i) At what value of 'a' ($a > 2$) is the reciprocal of the area of the fig. bounded by the curves, the lines $x = 2$ & $x = a$ equal to 'a' itself ?

(ii) At what value of 'b' ($1 < b < 2$) the area of the figure bounded by these curves, the lines $x = b$ & $x = 2$ equal to $1 - 1/b$.

AU0047

7. Show that the area bounded by the curve $y = \frac{\ln x - c}{x}$, the x-axis and the vertical line through the maximum point of the curve is independent of the constant c.

AU0048

8. For what value of 'a' is the area of the figure bounded by the lines, $y = \frac{1}{x}$, $y = \frac{1}{2x-1}$, $x = 2$ & $x = a$ equal to $\ln\frac{4}{\sqrt{5}}$?

AU0049

9. For the curve $f(x) = \frac{1}{1+x^2}$, let two points on it are $A(\alpha, f(\alpha))$, $B\left(-\frac{1}{\alpha}, f\left(-\frac{1}{\alpha}\right)\right)$ ($\alpha > 0$). Find the minimum area bounded by the line segments OA, OB and $f(x)$, where 'O' is the origin.

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AU0050

10. Let 'c' be the constant number such that $c > 1$. If the least area of the figure given by the line passing through the point (1, c) with gradient 'm' and the parabola $y = x^2$ is 36 sq. units find the value of $(c^2 + m^2)$.

AU0051

11. If f (x) is monotonic in (a, b) then prove that the area bounded by the ordinates at $x = a$; $x = b$; $y = f(x)$ and $y = f(c)$, $c \in (a, b)$ is minimum when $c = \frac{a+b}{2}$.

Hence if the area bounded by the graph of $f(x) = \frac{x^3}{3} - x^2 + a$, the straight lines $x = 0$, $x = 2$ and the x-axis is minimum then find the value of 'a'.

AU0052

12. Consider the two curves $C_1 : y = 1 + \cos x$ & $C_2 : y = 1 + \cos(x - \alpha)$ for $\alpha \in (0, \pi/2)$; $x \in [0, \pi]$. Find the value of α, for which the area of the figure bounded by the curves C_1, C_2 & $x = 0$ is same as that of the figure bounded by C_2, $y = 1$ & $x = \pi$. For this value of α, find the ratio in which the line $y = 1$ divides the area of the figure by the curves C_1, C_2 & $x = \pi$.

AU0053

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13. For what values of $a \in [0, 1]$ does the area of the figure bounded by the graph of the function $y = f(x)$ and the straight lines $x = 0$, $x = 1$ & $y = f(a)$ is at a minimum & for what values it is at a maximum if $f(x) = \sqrt{1-x^2}$. Find also the maximum & the minimum areas.

AU0054

14. Let C_1 & C_2 be two curves passing through the origin as shown in the figure. A curve C is said to "bisect the area" the region between C_1 & C_2, if for each point P of C, the two shaded regions A & B shown in the figure have equal areas. Determine the upper curve C_2, given that the bisecting curve C has the equation $y = x^2$ & that the lower curve C_1 has the equation $y = x^2/2$.

AU0055

EXERCISE (JM)

1. The area of the region enclosed by the curves $y = x$, $x = e$, $y = \frac{1}{x}$ and the positive x-axis is:- **[AIEEE-2011]**

(1) $\frac{3}{2}$ square units (2) $\frac{5}{2}$ square units (3) $\frac{1}{2}$ square units (4) 1 square units

AU0056

2. The area bounded by the curves $y^2 = 4x$ and $x^2=4y$ is :- **[AIEEE-2011]**

(1) 0 (2) $\frac{32}{3}$ (3) $\frac{16}{3}$ (4) $\frac{8}{3}$

AU0057

3. The area bounded between the parabolas $x^2 = \frac{y}{4}$ and $x^2 = 9y$, and the straight line $y = 2$ is : **[AIEEE-2012]**

(1) $10\sqrt{2}$ (2) $20\sqrt{2}$ (3) $\frac{10\sqrt{2}}{3}$ (4) $\frac{20\sqrt{2}}{3}$

AU0058

4. The area (in square units) bounded by the curves $y = \sqrt{x}$, $2y - x + 3 = 0$, x-axis and lying in the first quadrant is : **[JEE (Main)-2013]**

(1) 9 (2) 36 (3) 18 (4) $\frac{27}{4}$

AU0059

5. The area bounded by the curve $y = ln(x)$ and the lines $y = 0$, $y = ln(3)$ and $x = 0$ is equal to : **[JEE-Main (On line)-2013]**

(1) $3\,ln(3) - 2$ (2) 3 (3) 2 (4) $3\,ln(3) + 2$

AU0060

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6. The area of the region (in sq. units), in the first quadrant, bounded by the parabola $y = 9x^2$ and the lines x = 0, y = 1 and y = 4, is :- **[JEE-Main (On line)-2013]**

(1) 7/9 (2) 14/3 (3) 14/9 (4) 7/3

AU0061

7. The area under the curve $y = |\cos x - \sin x|$, $0 \le x \le \frac{\pi}{2}$, and above x-axis is :

[JEE-Main (On line)-2013]

(1) $2\sqrt{2}$ (2) $2\sqrt{2} + 2$ (3) 0 (4) $2\sqrt{2} - 2$

AU0062

8. Let $f : [-2, 3] \to [0, \infty)$ be a continuous function such that $f(1-x) = f(x)$ for all $x \in [-2, 3]$. If R_1 is the numerical value of the area of the region bounded by y = f(x), x = –2, x = 3 and the axis of x and $R_2 = \int_{-2}^{3} x\, f(x)\, dx$, then : **[JEE-Main (On line)-2013]**

(1) $2R_1 = 3R_2$ (2) $R_1 = R_2$ (3) $3R_1 = 2R_2$ (4) $R_1 = 2R_2$

AU0063

9. The area of the region described by $A = \{(x, y) : x^2 + y^2 \le 1 \text{ and } y^2 \le 1 - x\}$ is :

[JEE(Main)-2014]

(1) $\frac{\pi}{2} + \frac{4}{3}$ (2) $\frac{\pi}{2} - \frac{4}{3}$ (3) $\frac{\pi}{2} - \frac{2}{3}$ (4) $\frac{\pi}{2} + \frac{2}{3}$

AU0064

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10. The area (in sq.units) of the region $\{(x, y) : y^2 \ge 2x \text{ and } x^2 + y^2 \le 4x, x \ge 0, y \ge 0\}$ is :-

[JEE(Main)-2016]

(1) $\frac{\pi}{2} - \frac{2\sqrt{2}}{3}$ (2) $\pi - \frac{4}{3}$ (3) $\pi - \frac{8}{3}$ (4) $\pi - \frac{4\sqrt{2}}{3}$

AU0065

11. The area (in sq. units) of the region $\{(x, y\} : x \ge 0, x + y \le 3, x^2 \le 4y \text{ and } y \le 1 + \sqrt{x}\}$ is :

[JEE(Main)-2017]

(1) $\frac{5}{2}$ (2) $\frac{59}{12}$ (3) $\frac{3}{2}$ (4) $\frac{7}{3}$

AU0066

12. Let $g(x) = \cos x^2$, $f(x) = \sqrt{x}$ and α, β $(\alpha < \beta)$ be the roots of the quadratic equation $18x^2 - 9\pi x + \pi^2 = 0$. Then the area (in sq. units) bounded by the curve y = (gof) (x) and the lines $x = \alpha$, $x = \beta$ and y = 0 is- **[JEE(Main)-2018]**

(1) $\frac{1}{2}(\sqrt{3}+1)$ (2) $\frac{1}{2}(\sqrt{3}-\sqrt{2})$ (3) $\frac{1}{2}(\sqrt{2}-1)$ (4) $\frac{1}{2}(\sqrt{3}-1)$

AU0067

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13. The area (in sq. units) bounded by the parabola $y = x^2 - 1$, the tangent at the point (2, 3) to it and the y-axis is : **[JEE(Main)-2019]**

(1) $\frac{14}{3}$ (2) $\frac{56}{3}$ (3) $\frac{8}{3}$ (4) $\frac{32}{3}$

AU0068

14. The area of the region $A = \left[(x,y) : 0 \le y \le x|x| + 1 \text{ and } -1 \le x \le 1\right]$ in sq. units, is : **[JEE(Main)-2019]**

(1) $\frac{2}{3}$ (2) $\frac{1}{3}$ (3) 2 (4) $\frac{4}{3}$

AU0069

15. The area (in sq. units) of the region $A = \{(x, y) \in R \times R | 0 \le x \le 3, 0 \le y \le 4, y \le x^2 + 3x\}$ is :

[JEE(Main)-2019]

(1) $\frac{53}{6}$ (2) $\frac{59}{6}$ (3) 8 (4) $\frac{26}{3}$

AU0070

16. The area (in sq. units) of the region bounded by the curves $y = 2^x$ and $y = |x + 1|$, in the first quadrant is : **[JEE(Main)-2019]**

(1) $\frac{3}{2} - \frac{1}{\log_e 2}$ (2) $\frac{1}{2}$ (3) $\log_e 2 + \frac{3}{2}$ (4) $\frac{3}{2}$

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AU0071

17. If the area (in sq. units) of the region $\{(x, y) : y^2 \le 4x, x + y \le l, x \ge 0, y \ge O\}$ is $a\sqrt{2} + b$, then a – b is equal to : **[JEE(Main)-2019]**

(1) $\frac{8}{3}$ (2) $\frac{10}{3}$ (3) 6 (4) $-\frac{2}{3}$

AU0072

18. The area (in sq. units) of the region $\{(x, y) \in R^2 | 4x^2 \le y \le 8x + 12)$ is :

[JEE(Main)-2020]

(1) $\frac{127}{3}$ (2) $\frac{125}{3}$ (3) $\frac{124}{3}$ (4) $\frac{128}{3}$

AU0073

19. The area of the region, enclosed by the circle $x^2 + y^2 = 2$ which is not common to the region bounded by the parabola $y^2 = x$ and the straight line $y = x$, is : **[JEE(Main)-2020]**

(1) $\frac{1}{3}(12\pi - 1)$ (2) $\frac{1}{6}(12\pi - 1)$ (3) $\frac{1}{6}(24\pi - 1)$ (4) $\frac{1}{3}(6\pi - 1)$

AU0074

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20. The area (in sq. units) of the region $\{(x,y) \in R^2 : x^2 \le y \le 3 - 2x\}$, is **[JEE(Main)-2020]**

(1) $\frac{29}{3}$ (2) $\frac{31}{3}$ (3) $\frac{34}{3}$ (4) $\frac{32}{3}$

AU0075

21. Given : $f(x) = \begin{cases} x & , \ 0 \le x < \frac{1}{2} \\ \frac{1}{2} & , \ x = \frac{1}{2} \\ 1 - x & , \ \frac{1}{2} < x \le 1 \end{cases}$ and $g(x) = \left(x - \frac{1}{2}\right)^2$, $x \in R$. Then the area (in sq. units) of the region bounded by the curves, $y = f(x)$ and $y = g(x)$ between the lines, $2x = 1$ and $2x = \sqrt{3}$, is :

[JEE(Main)-2020]

(1) $\frac{1}{3} + \frac{\sqrt{3}}{4}$ (2) $\frac{\sqrt{3}}{4} - \frac{1}{3}$ (3) $\frac{1}{2} + \frac{\sqrt{3}}{4}$ (4) $\frac{1}{2} - \frac{\sqrt{3}}{4}$

AU0076

EXERCISE (JA)

1. **(a)** Let the straight line $x = b$ divide the area enclosed by $y = (1 - x)^2$, $y = 0$ and $x = 0$ into two parts $R_1 (0 \le x \le b)$ d and $R_2 (b \le x \le 1)$ such that $R_1 - R_2 = \frac{1}{4}$. Then b equals

(A) $\frac{3}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$

AU0077

(b) Let $f:[-1,2] \to [0,\infty)$ be a continuous function such that $f(x) = f(1-x)$ for all $x \in [-1,2]$. Let $R_1 = \int_{-1}^{2} x f(x) dx$, and R_2 be the area of the region bounded by $y = f(x)$, $x=-1$, $x=2$, and the x-axis. Then -

(A) $R_1 = 2R_2$ (B) $R_1 = 3R_2$ (C) $2R_1 = R_2$ (D) $3R_1 = R_2$

[JEE 2011, 3+3]

AU0078

2. The area enclosed by the curve $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $\left[0, \frac{\pi}{2}\right]$ is

[JEE(Advanced) 2013, 2M]

(A) $4(\sqrt{2} - 1)$ (B) $2\sqrt{2}(\sqrt{2} - 1)$ (C) $2(\sqrt{2} + 1)$ (D) $2\sqrt{2}(\sqrt{2} + 1)$

AU0079

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3. Let $F(x) = \int_{x}^{x^2+\frac{\pi}{6}} 2\cos^2 t\,dt$ for all $x \in \mathbb{R}$ and $f : \left[0, \frac{1}{2}\right] \to [0, \infty)$ be a continuous function. For $a \in \left[0, \frac{1}{2}\right]$, if F'(a) + 2 is the area of the region bounded by x = 0, y = 0, y = f(x) and x = a, then f(0) is **[JEE 2015, 4M, –0M]**

AU0080

4. If the line x = α divides the area of region R = {(x, y) ∈ $\mathbb{R}^2$: $x^3 \le y \le x$, $0 \le x \le 1$} into two equal parts, then **[JEE(Advanced)-2017, 3(–2)]**

(A) $\frac{1}{2} < \alpha < 1$

(B) $\alpha^4 + 4\alpha^2 - 1 = 0$

(C) $0 < \alpha \le \frac{1}{2}$

(D) $2\alpha^4 - 4\alpha^2 + 1 = 0$

AU0081

5. Let f : [0, ∞) → ℝ be a continuous function such that $f(x) = 1 - 2x + \int_0^x e^{x-t} f(t)\,dt$ for all x ∈ [0, ∞). Then, which of the following statement(s) is (are) TRUE ?

(A) The curve y = f(x) passes through the point (1, 2) **[JEE(Advanced)-2018, 4(–2)]**

(B) The curve y = f(x) passes through the point (2, –1)

(C) The area of the region {(x, y) ∈ [0, 1] × ℝ : $f(x) \le y \le \sqrt{1-x^2}$ } is $\frac{\pi - 2}{4}$

(D) The area of the region {(x, y) ∈ [0, 1] × ℝ : $f(x) \le y \le \sqrt{1-x^2}$ } is $\frac{\pi - 1}{4}$

AU0082

6. A farmer F_1 has a land in the shape of a triangle with vertices at P(0, 0), Q(1, 1) and R(2, 0). From this land, a neighbouring farmer F_2 takes away the region which lies between the side PQ and a curve of the form y = x^n(n > 1). If the area of the region taken away by the farmer F_2 is exactly 30% of the area of ΔPQR, then the value of n is _____ **[JEE(Advanced)-2018, 3(0)]**

AU0083

7. The area of the region {(x, y) : xy ≤ 8, 1 ≤ y ≤ x^2} is **[JEE(Advanced)-2019, 3(–1)]**

(1) $8\log_e 2 - \frac{14}{3}$ (2) $16\log_e 2 - \frac{14}{3}$ (3) $16\log_e 2 - 6$ (4) $8\log_e 2 - \frac{7}{3}$

AU0084

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ANSWER KEY

AREA UNDER THE CURVE

EXERCISE (O-1)

1.	A	**2.**	C	**3.**	D	**4.**	B	**5.**	B	**6.**	D	**7.**	A	**8.**	D	**9.**	D
10.	**A**																

EXERCISE (O-2)

1.	A	**2.**	A	**3.**	C	**4.**	A,C	**5.**	A,C,D	**6.**	A,D
7.	B,C,D	**8.**	A,B,D	**9.**	B,C	**10.**	B,C	**11.**	B,C	**12.**	A,B
13.	A,B	**14.**	A,D	**15.**	A,B,C,D						

EXERCISE (S-1)

1. 5/6 sq. units **2.** 23/6 sq. units **3.** 64 **4.** $\pi - \tan^{-1}\frac{2\sqrt{2}}{3\pi}$; $\pi - \tan^{-1}\frac{4\sqrt{2}}{3\pi}$

5. $\frac{\pi}{2}$; $\frac{\pi-1}{\pi+1}$ **6.** a = 9 **7.** $\frac{3\pi+2}{\pi-2}$ **8.** e **9.** 2 sq. units

10. a = – 3/4 **11.** $\sqrt{3}$ **12.** $\frac{1}{3} + \ell n\left(\frac{\sqrt{3}}{2}\right)$ sq. units

13. C = – 1 or $\left(8-\sqrt{17}\right)^{1/3}$ **14.** $\frac{1}{2}(1 - e^{-1/2})$

15. $f(x) = x^2 + 1$; $y = \pm 2x$; $A = \frac{2}{3}$ sq. units **16.** 2/3

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EXERCISE (S-2)

1. $(e^2 - 5)/4$ e sq. units **2.** (i) m = 1, (ii) m = ∞ ; $A_{min} = 4/3$ **4.** $1 - 3e^{-2}$

5. $\sqrt{2}+1$ **6.** $a = 1 + e^2$, $b = 1 + e^{-2}$ **7.** 1/2

8. a = 8 or $\frac{2}{5}\left(6-\sqrt{21}\right)$ **9.** $\frac{(\pi-1)}{2}$ **10.** 104 **11.** $a = \frac{2}{3}$

12. $\alpha = \pi/3$, ratio = $2 : \sqrt{3}$

13. a = 1/2 gives minima, $A\left(\frac{1}{2}\right) = \frac{3\sqrt{3}-\pi}{12}$; a = 0 gives local maxima $A(0) = 1 - \frac{\pi}{4}$;

a = 1 gives maximum value , $A(1) = \pi/4$

14. $(16/9)\,x^2$

EXERCISE (JM)

1.	1	**2.**	3	**3.**	4	**4.**	1	**5.**	3	**6.**	3	**7.**	4	**8.**	4	**9.**	1
10.	3	**11.**	1	**12.**	4	**13.**	3	**14.**	3	**15.**	2	**16.**	1	**17.**	3	**18.**	4
19.	2	**20.**	4	**21.**	2												

EXERCISE (JA)

1. (a) B; (b) C **2.** B **3.** 3 **4.** A,D **5.** B,C **6.** 4 **7.** 2

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VECTOR

1. INTRODUCTION :

Vectors constitute one of the several Mathematical systems which can be usefully employed to provide mathematical handling for certain types of problems in Geometry, Mechanics and other branches of Applied Mathematics.

Vectors facilitate mathematical study of such physical quantities as possess Direction in addition to Magnitude. Velocity of a particle, for example, is one such quantity.

2. Physical quantities are broadly divided in two categories viz **(a) Vector Quantities** & **(b) Scalar quantities.**

(a) Vector quantities :

Any quantity, such as velocity, momentum, or force, that has both magnitude and direction and for which vector addition is defined and meaningful; is treated as vector quantities.

Note :

Quantities having magnitude and direction but not obeying the vector law of addition will not be treated as vectors.

For example, the rotations of a rigid body through finite angles have both magnitude & direction but do not satisfy the law of vector addition therefore not a vector.

(b) Scalar quantities :

A quantity, such as mass, length, time, density or energy, that has size or magnitude but does not involve the concept of direction is called scalar quantity.

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3. MATHEMATICAL DESCRIPTION OF VECTOR & SCALAR :

To understand vectors mathematically we will first understand directed line segment.

Directed line segment :

Any given portion of a given straight line where the two end points are distinguished as **Initial** and **Terminal** is called a **Directed Line Segment**.

The directed line segment with initial point A and terminal point B is denoted by the symbol $\overrightarrow{AB}$.

The two end points of a directed line segment are not interchangeable and the directed line segments. $\overrightarrow{AB}$ and $\overrightarrow{BA}$ must be thought of as different.

(a) Vector :

A directed line segment is called vector. Every directed line segment have three essential characteristics.

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(i) **Length :** The length of $\overrightarrow{AB}$ will be denoted by the symbol $|\overrightarrow{AB}|$

Clearly, we have $|\overrightarrow{AB}| = |\overrightarrow{BA}|$

(ii) **Support :** The line of unlimited length of which a directed line segment is a part is called its line of support or simply the Support.

(iii) **Sense :** The sense of $\overrightarrow{AB}$ is from A to B and that of $\overrightarrow{BA}$ from B to A so that the sense of a directed line segment is from its initial to the terminal point.

(b) Scalar :

Any real number is a scalar.

4. EQUALITY OF TWO VECTORS :

Two vectors are said to be equal if they have

(a) the same length,

(b) the same or parallel supports and

(c) the same sense.

Note : Components of two equal vectors taken in any arbitrary direction are equal. i.e. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, where $\hat{i}$, $\hat{j}$ & $\hat{k}$ are the unit vectors taken along co-ordinate axes, then $\vec{a} = \vec{b} \Leftrightarrow a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3$.

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Illustration 1 : Let $\vec{r} = 3\hat{i} + 2\hat{j} - 5\hat{k}$, $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{c} = -2\hat{i} + \hat{j} - 3\hat{k}$, $\vec{r} = \lambda\vec{a} + \mu\vec{b} + \nu\vec{c}$, then find $\lambda + \mu + \nu$.

Solution : $3\hat{i} + 2\hat{j} - 5\hat{k} = \lambda(2\hat{i} - \hat{j} + \hat{k}) + \mu(\hat{i} + 3\hat{j} - 2\hat{k}) + \nu(-2\hat{i} + \hat{j} - 3\hat{k})$

$$= (2\lambda + \mu - 2\nu)\hat{i} + (-\lambda + 3\mu + \nu)\hat{j} + (\lambda - 2\mu - 3\nu)\hat{k}$$

Equating components of equal vectors

$2\lambda + \mu - 2\nu = 3$ .....(i)

$-\lambda + 3\mu + \nu = 2$ .....(ii)

$\lambda - 2\mu - 3\nu = -5$ .....(iii)

on solving (i), (ii) & (iii)

we get $\lambda = 3,\ \mu = 1,\ \nu = 2$

So $\lambda + \mu + \nu = 6$. **Ans.**

Do yourself - 1 :

(i) If $\vec{a} = 2\hat{i} + \mu\hat{j} - 7\hat{k}$ and $\vec{b} = \lambda\hat{i} + \sqrt{3}\,\hat{j} - 7\hat{k}$ are two equal vectors, then find $\lambda^2 + \mu^2$.

(ii) If $\vec{a}$, $\vec{b}$ are two vectors then which of the following statements is/are correct -

(A) $\vec{a} = -\vec{b} \Rightarrow |\vec{a}| = |\vec{b}|$ (B) $|\vec{a}| = |\vec{b}| \Rightarrow \vec{a} = \pm\vec{b}$

(C) $|\vec{a}| = |\vec{b}| \Rightarrow \vec{a} = \vec{b}$ (D) $|\vec{a}| = |\vec{b}| \Rightarrow \vec{a} = \pm 2\vec{b}$

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5. LEFT AND RIGHT - HANDED ORIENTATION (CONFIGURATIONS) :

For each hand take the directions Ox, Oy and Oz as shown in the figure. Thus we get two rectangular coordinate systems. Can they be made congruent ? They cannot be, because the two hands have different orientations. Therefore these two systems are different.

A rectangular coordinate system which can be made congruent with the system formed with the help of right hand (or left hand) is called a right handed (or left handed) rectangular coordinates system.

Thus we have the following condition to identify these two systems using sense of rotation :

(a) If the rotation from Ox to Oy is in the anticlockwise direction and Oz is directed upwards (see right hand), then the system is right handed.

(b) If the rotation from Ox to Oy is clockwise and Oz is directed upward (see left hand) then the system is left handed.

Here after we shall use the right-handed rectangular Cartesian coordinate system (or Ortho-normal system).

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6. ALGEBRA OF VECTORS :

It is possible to develop an Algebra of Vectors which proves useful in the study of Geometry, Mechanics and other branches of Applied Mathematics.

(a) Addition of two vectors :

The vectors have magnitude as well as direction, therefore their addition is different than addition of real numbers.

Let $\vec{a}$ and $\vec{b}$ be two vectors in a plane, which are represented by $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Their addition can be performed in the following two ways :

(i) Triangle law of addition of vectors : If two vectors can be represented in magnitude and direction by the two sides of a triangle, taken in order, then their sum will be represented by the third side in reverse order.

Let O be the fixed point in the plane of vectors. Draw a line segment $\overrightarrow{OE}$ from O, equal and parallel to $\overrightarrow{AB}$, which represents the vector $\vec{a}$. Now from E, draw a line segment $\overrightarrow{EF}$, equal and parallel to $\overrightarrow{CD}$, which represents the vector $\vec{b}$. Line segment $\overrightarrow{OF}$ obtained by joining O and F represents the sum of vectors $\vec{a}$ and $\vec{b}$.

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i.e. $\overrightarrow{OE}+\overrightarrow{EF}=\overrightarrow{OF}$

or $\vec{a}+\vec{b}=\overrightarrow{OF}$

This method of addition of two vectors is called **Triangle law of addition of vectors.**

(ii) Parallelogram law of addition of vectors : If two vectors be represented in magnitude and direction by the two adjacent sides of a parallelogram then their sum will be represented by the diagonal through the co-initial point.

Let $\vec{a}$ and $\vec{b}$ be vectors drawn from point O denoted by line segments $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Now complete the parallelogram OPRQ. Then the vector represented by the diagonal OR will represent the sum of the vectors $\vec{a}$ and $\vec{b}$.

i.e. $\overrightarrow{OP}+\overrightarrow{OQ}=\overrightarrow{OR}$

or $\vec{a}+\vec{b}=\overrightarrow{OR}$

This method of addition of two vectors is called **Parallelogram law of addition of vectors.**

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(iii) Properties of vector addition :

(1) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ (commutative) (2) $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ (associativity)

(3) $\vec{a}+\vec{0}=\vec{a}=\vec{0}+\vec{a}$ (additive identity) (4) $\vec{a}+(-\vec{a})=\vec{0}=(-\vec{a})+\vec{a}$ (additive inverse)

(b) Polygon law of vector Addition (Addition of more than two vectors):

Addition of more than two vectors is found to be by repetition of triangle law. Suppose we have to find the sum of five vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ and $\vec{e}$. If these vectors be represented by line segment $\overrightarrow{OA}, \overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}$ and $\overrightarrow{DE}$ respectively, then their sum will be denoted by $\overrightarrow{OE}$. This is the vector represented by rest (last) side of the polygon OABCDE in reverse order. We can also make it clear this way :

By triangle's law

$\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}$ or $\vec{a}+\vec{b}=\overrightarrow{OB}$

$\overrightarrow{OB}+\vec{c}=\overrightarrow{OC}$ or $(\vec{a}+\vec{b})+\vec{c}=\overrightarrow{OC}$

$\overrightarrow{OC}+\vec{d}=\overrightarrow{OD}$ or $(\vec{a}+\vec{b}+\vec{c})+\vec{d}=\overrightarrow{OD}$

$\overrightarrow{OD}+\vec{e}=\overrightarrow{OE}$ or $(\vec{a}+\vec{b}+\vec{c}+\vec{d})+\vec{e}=\overrightarrow{OE}$

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Here, we see that $\overrightarrow{OE}$ is represented by the line segment joining the initial point O of the first vector $\vec{a}$ and the final point of the last vector $\vec{e}$.

In order to find the sum of more that two vectors by this method, a polygon is formed. Therefore this method is known as the **polygon law of addition.**

Note : If the initial point of the first vector and the final point of the last vector are the same, then the sum of the vectors will be a null vector.

(c) Subtraction of Vectors :

Vector $-\vec{b}$ has length equals to vector $\vec{b}$ but its direction is opposite. Subtraction of vector $\vec{a}$ and $\vec{b}$ is defined as addition of $\vec{a}$ and $(-\vec{b})$. It is written as follows :

$\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$

Geometrical representation :

In the given diagram, $\vec{a}$ and $\vec{b}$ are represented by $\overrightarrow{OA}$ and $\overrightarrow{AB}$. We extend the line AB in opposite direction upto C, where AB = AC. The line segment $\overrightarrow{AC}$ will represent the vector $-\vec{b}$. By joining the points O and C, the vector represented by $\overrightarrow{OC}$ is $\vec{a}+(-\vec{b})$. i.e. denotes the vector $\vec{a}-\vec{b}$.

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Note :

(i) $\vec{a}-\vec{a}=\vec{a}+(-\vec{a})=\vec{0}$

(ii) $\vec{a}-\vec{b}\neq\vec{b}-\vec{a}$

Hence subtraction of vectors does not obey the commutative law.

(iii) $\vec{a}-(\vec{b}-\vec{c})\neq(\vec{a}-\vec{b})-\vec{c}$

i.e. subtraction of vectors does not obey the associative law.

(d) Multiplication of vector by scalars :

If $\vec{a}$ is a vector & m is a scalar, then m($\vec{a}$) is a vector parallel to $\vec{a}$ whose modulus is |m| times that of $\vec{a}$. This multiplication is called SCALAR MULTIPLICATION. If $\vec{a}$ & $\vec{b}$ are vectors & m, n are scalars, then :

(i) $m(\vec{a})=(\vec{a})m=m\vec{a}$

(ii) $m(n\vec{a})=n(m\vec{a})=(mn)\vec{a}$

(iii) $(m+n)\vec{a}=m\vec{a}+n\vec{a}$

(iv) $m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}$

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Illustration 2 : ABCD is a parallelogram whose diagonals meet at P. If O is a fixed point, then $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}$ equals :-

(A) $\overrightarrow{OP}$ (B) $2\overrightarrow{OP}$ (C) $3\overrightarrow{OP}$ (D) $4\overrightarrow{OP}$

Solution : Since, P bisects both the diagonal AC and BD, so

$\therefore\ \overrightarrow{OA}+\overrightarrow{OC}=2\overrightarrow{OP}$ and $\overrightarrow{OB}+\overrightarrow{OD}=2\overrightarrow{OP}\ \Rightarrow\ \overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}=4\overrightarrow{OP}$ **Ans. [D]**

Illustration 3 : A, B, P, Q, R are five points in any plane. If forces $\overrightarrow{AP},\overrightarrow{AQ},\overrightarrow{AR}$ acts on point A and force $\overrightarrow{PB},\overrightarrow{QB},\overrightarrow{RB}$ acts on point B then resultant is :-

(A) $3\overrightarrow{AB}$ (B) $3\overrightarrow{BA}$ (C) $3\overrightarrow{PQ}$ (D) $3\overrightarrow{PR}$

Solution : From figure

$\overrightarrow{AP}+\overrightarrow{PB}=\overrightarrow{AB}$

$\overrightarrow{AQ}+\overrightarrow{QB}=\overrightarrow{AB}$

$\overrightarrow{AR}+\overrightarrow{RB}=\overrightarrow{AB}$

So $(\overrightarrow{AP}+\overrightarrow{AQ}+\overrightarrow{AR})+(\overrightarrow{PB}+\overrightarrow{QB}+\overrightarrow{RB})=3\overrightarrow{AB}$

so required resultant $=3\overrightarrow{AB}$. **Ans. [A]**

Illustration 4 : Prove that the line joining the middle points of two sides of a triangle is parallel to the third side and is of half its length.

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Solution : Let the middle points of side AB and AC of a ΔABC be D and E respectively.

$\overrightarrow{BA}=2\overrightarrow{DA}$ and $\overrightarrow{AC}=2\overrightarrow{AE}$

Now in ΔABC, by triangle law of addition

$\overrightarrow{BA}+\overrightarrow{AC}=\overrightarrow{BC}$

$2\overrightarrow{DA}+2\overrightarrow{AE}=\overrightarrow{BC}\ \Rightarrow\ \overrightarrow{DA}+\overrightarrow{AE}=\frac{1}{2}\overrightarrow{BC}$

$\overrightarrow{DE}=\frac{1}{2}\overrightarrow{BC}$

Hence, line DE is parallel to third side BC of triangle and half of it.

Do yourself - 2 :

(i) If $\vec{a}$, $\vec{b}$, $\vec{c}$ be the vectors represented by the sides of a triangle taken in order, then prove that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

(ii) If $\overrightarrow{PO}+\overrightarrow{OQ}=\overrightarrow{QO}+\overrightarrow{OR}$, then prove that the points P, Q and R are collinear.

(iii) For any two vectors $\vec{a}$ and $\vec{b}$ prove that

(a) $|\vec{a}+\vec{b}|\le|\vec{a}|+|\vec{b}|$ (b) $|\vec{a}-\vec{b}|\le|\vec{a}|+|\vec{b}|$ (c) $|\vec{a}+\vec{b}|\ge|\vec{a}|-|\vec{b}|$

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Note : In general for any non-zero vectors $\vec{a}$, $\vec{b}$ & $\vec{c}$ one may note that although $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ but it will not always represent the three sides of a triangle.

7. COLLINEAR VECTORS :

Two vectors are said to be collinear if their supports are parallel disregards to their direction. Collinear vectors are also called **Parallel vectors**. If they have the same direction they are named as **like vectors** otherwise **unlike vectors.**

Note :

(i) Symbolically two non zero vectors $\vec{a}$ & $\vec{b}$ are collinear if and only if, $\vec{a}=K\vec{b}$, where $K \in R$

(ii) If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ are two collinear vectors then $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$.

(iii) If $\vec{a}$ & $\vec{b}$ are two non-zero, non-collinear vectors such that $x\vec{a}+y\vec{b}=\vec{0} \Rightarrow x=y=0$

8. CO-INITIAL VECTORS :

Vectors having same initial point are called Co-initial Vectors.

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Illustration 5 : If $\vec{a}$ and $\vec{b}$ are non-collinear vectors, then find the value of x for which vectors :

$\vec{\alpha}=(x-2)\vec{a}+\vec{b}$ and $\vec{\beta}=(3+2x)\vec{a}-2\vec{b}$ are collinear.

Solution : Since the vectors $\vec{\alpha}$ and $\vec{\beta}$ are collinear.

$\therefore$ there exist scalar λ such that $\vec{\alpha}=\lambda\vec{\beta}$

$\Rightarrow$ $(x-2)\vec{a}+\vec{b}=\lambda\{(3+2x)\vec{a}-2\vec{b}\} \Rightarrow (x-2-\lambda(3+2x))\vec{a}+(1+2\lambda)\vec{b}=\vec{0}$

$\Rightarrow$ $x-2-\lambda(3+2x)=0$ and $1+2\lambda=0$

$x-2-\lambda(3+2x)=0$ and $\lambda=-\frac{1}{2}$

$\Rightarrow$ $x-2+\frac{1}{2}(3+2x)=0 \Rightarrow 4x-1=0 \Rightarrow x=\frac{1}{4}$. **Ans.**

Illustration 6 : If $A \equiv (2\hat{i}+3\hat{j})$, $B \equiv (p\hat{i}+9\hat{j})$ and $C \equiv (\hat{i}-\hat{j})$ are collinear, then the value of p is :-

(A) 1/2 (B) 3/2 (C) 7/2 (D) 5/2

Solution : $\overrightarrow{AB}=(p-2)\hat{i}+6\hat{j}$, $\overrightarrow{AC}=-\hat{i}-4\hat{j}$

Now A, B, C are collinear $\Leftrightarrow \overrightarrow{AB} \parallel \overrightarrow{AC} \Leftrightarrow \frac{p-2}{-1}=\frac{6}{-4} \Leftrightarrow p=7/2$ **Ans. [C]**

Illustration 7 : The value of λ when $\vec{a}=2\hat{i}-3\hat{j}+\hat{k}$ and $\vec{b}=8\hat{i}+\lambda\hat{j}+4\hat{k}$ are parallel is :-

(A) 4 (B) – 6 (C) – 12 (D) 1

Solution : Since $\vec{a}$ & $\vec{b}$ are parallel $\Rightarrow \frac{2}{8}=-\frac{3}{\lambda}=\frac{1}{4} \Rightarrow \lambda=-12$ **Ans. [C]**

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Do yourself - 3 :

(i) In the given figure of regular hexagon, which vectors are (provided vertices A, B, C, D, E, F are all fixed) :

(a) Parallel (b) Equal

(c) Coinitial (d) Parallel but not equal.

(ii) If $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = 8\hat{i} - 12\hat{j} + 16\hat{k}$ such that $\vec{a} = \lambda\vec{b}$, then λ equals to

(iii) If $3\vec{a} + 2\vec{b} = 5\vec{c}$ and $8\vec{a} - 7\vec{b} = 4\vec{c}$, then which statement is/are true :

(A) $|\vec{a}| > |\vec{b}|$ (B) $|\vec{c}| > |\vec{b}|$

(C) $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear vectors. (D) $|\vec{a}| = |\vec{b}|$

9. COPLANAR VECTORS :

A given number of vectors are called coplanar if their supports are all parallel to the same plane. Note that "TWO VECTORS ARE ALWAYS COPLANAR ".

Note : Coplanar vectors may have any directions or magnitude.

10. REPRESENTATION OF A VECTOR IN SPACE IN TERMS OF 3 ORTHONORMAL TRIAD OF UNIT VECTORS :

Let P(x, y, z) be a point in space with reference to OX, OY and OZ as the coordinate axes, then OA = x, OB = y and OC = z

Let $\hat{i}$, $\hat{j}$, $\hat{k}$ be unit vectors along OX, OY and OZ respectively, then

$\overrightarrow{OA} = x\hat{i}$, $\overrightarrow{OB} = y\hat{j}$, $\overrightarrow{OC} = z\hat{k}$

$\overrightarrow{OP} = \overrightarrow{OC'} + \overrightarrow{C'P} = \overrightarrow{OB} + \overrightarrow{OA} + \overrightarrow{OC}$ $[\because \overrightarrow{C'P} = \overrightarrow{OC}]$

$= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = x\hat{i} + y\hat{j} + z\hat{k}$

If $\overrightarrow{OP} = \vec{r}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\vec{r}| = \overrightarrow{OP} = \sqrt{x^2 + y^2 + z^2}$

11. POSITION VECTOR :

Let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ & $\vec{b}$ are position vectors of two point A and B, then $\overrightarrow{AB} = \vec{b} - \vec{a}$ = pv of B – pv of A.

12. ZERO VECTOR OR NULL VECTOR :

A vector of zero magnitude i.e. which has the same initial & terminal point is called a zero vector. It is denoted by $\vec{0}$. It can have any arbitrary direction and any line as its line of support.

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13. UNIT VECTOR :

A vector of unit magnitude in direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$ symbolically $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$ (provided $|\vec{a}| \neq 0$)

14. SECTION FORMULA :

If $\vec{a}$ & $\vec{b}$ are the position vectors of two points A & B then the p.v. of a point $C(\vec{r})$ which divides AB in the ratio m : n is given by :

(a) Internal Division :

$$\overrightarrow{OC} = \vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

Note : Position vector of mid point of **AB** $= \frac{\vec{a}+\vec{b}}{2}$

(b) External division :

$$\overrightarrow{OC} = \vec{r} = \frac{m\vec{b} - n\vec{a}}{m-n}$$

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Illustration 8 : Prove that the medians of a triangle are concurrent.

Solution : Let ABC be a triangle and position vectors of three vertices A, B and C with respect to the origin O be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.

$\therefore \quad \overrightarrow{OA} = \vec{a}, \ \overrightarrow{OB} = \vec{b}, \ \overrightarrow{OC} = \vec{c}$

Again, let D be the middle point of the side BC,

so the position vector of point D is $\overrightarrow{OD} = \frac{\vec{b}+\vec{c}}{2}$

Now take a point G, which divides the median AD in the ratio 2 : 1.

Position vector of point G is $\overrightarrow{OG} = \frac{1.\overrightarrow{OA} + 2.\overrightarrow{OD}}{1+2} = \frac{1.\vec{a} + 2.\frac{1}{2}(\vec{b}+\vec{c})}{1+2} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$

Similarly, the position vector of the middle points of the other two medians, which divide the medians in the ratio 2 : 1 will comes out to the same $\frac{\vec{a}+\vec{b}+\vec{c}}{3}$, which is the position vector of G.

Hence, the medians of the triangles meet in G i.e. are concurrent.

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Illustration 9 : If the middle points of sides BC, CA & AB of triangle ABC are respectively D, E, F then position vector of centroid of triangle DEF, when position vector of A,B,C are respectively $\hat{i}+\hat{j},\ \hat{j}+\hat{k},\ \hat{k}+\hat{i}$ is -

(A) $\frac{1}{3}(\hat{i}+\hat{j}+\hat{k})$ (B) $(\hat{i}+\hat{j}+\hat{k})$ (C) $2(\hat{i}+\hat{j}+\hat{k})$ (D) $\frac{2}{3}(\hat{i}+\hat{j}+\hat{k})$

Solution : The position vector of points D, E, F are respectively $\frac{\hat{i}+\hat{j}}{2}+\hat{k},\ \hat{i}+\frac{\hat{k}+\hat{j}}{2}$ and $\frac{\hat{i}+\hat{k}}{2}+\hat{j}$

So, position vector of centroid of $\Delta DEF = \frac{1}{3}\left[\frac{\hat{i}+\hat{j}}{2}+\hat{k}+\hat{i}+\frac{\hat{k}+\hat{j}}{2}+\frac{\hat{i}+\hat{k}}{2}+\hat{j}\right] = \frac{2}{3}[\hat{i}+\hat{j}+\hat{k}]$.

Ans. [D]

Do yourself - 4 :

(i) Find the position vectors of the points which divide the join of the points $2\vec{a}-3\vec{b}$ and $3\vec{a}-2\vec{b}$ internally and externally in the ratio 2 : 3,

(ii) ABCD is a parallelogram and P is the point of intersection of its diagonals. If O is the origin of reference, show that $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}=4\overrightarrow{OP}$

(iii) Find the unit vector in the direction of $3\hat{i}-6\hat{j}+2\hat{k}$.

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15. VECTOR EQUATION OF A LINE :

Parametric vector equation of a line passing through two points $A(\vec{a})$ & $B(\vec{b})$ is given by, $\mathbf{\vec{r}=\vec{a}+t(\vec{b}-\vec{a})}$ where t is a parameter. If the line passes through the point $A(\vec{a})$ & is parallel to the vector $\vec{b}$ then its equation is $\mathbf{\vec{r}=\vec{a}+t\vec{b}}$.

Note :

(i) Equations of the bisectors of the angles between the lines $\vec{r}=\vec{a}+\lambda\vec{b}$ & $\vec{r}=\vec{a}+\mu\vec{c}$ is, $\mathbf{\vec{r}=\vec{a}+t(\hat{b}+\hat{c})}$ & $\mathbf{\vec{r}=\vec{a}+p(\hat{c}-\hat{b})}$.

(ii) In a plane, two lines are either intersecting or parallel.

(iii) Two non parallel nor intersecting lines are called **skew lines**.

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Illustration 10 : In a triangle ABC, D and E are points on BC and AC respectively, such that BD = 2DC and AE = 3EC. Let P be the point of intersection of AD and BE. Find BP/PE using vector methods. **(JEE-1993)**

Solution : Let the position vectors of A and B be a and b respectively. Equations of AD and BE are

$\vec{r} = \vec{a} + t(\vec{b}/3 - \vec{a})$ ..... (i)

$\vec{r} = \vec{b} + s(\vec{a}/4 - \vec{b})$ ..... (ii)

If they intersect at P we must have identical values of r.

Comparing the coefficients of a and b in (i) and (ii), we get

$1 - t = \frac{s}{4}, \frac{t}{3} = 1 - s$

solving we get $t = \frac{9}{11}, s = \frac{8}{11}$.

Putting for t or s in (i) or (ii), we get the point P as $\frac{2\vec{a} + 3\vec{b}}{11}$.

Let P divide BE in the ratio k : 1, then P is $\frac{k.\frac{\vec{a}}{4} + \vec{b}}{k+1} = \frac{2\vec{a} + 3\vec{b}}{11}$.

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Comparing $\vec{a}$ and $\vec{b}$, we get $11k = 8(k + 1)$ and $11 = 3(k + 1)$ $\therefore$ $k = \frac{8}{3}$

and this satisfies the 2nd relation also. Hence the required ratio is 8 : 3. **Ans.**

Illustration 11 : Find whether the given lines are coplanar or not

$\vec{r} = \hat{i} - \hat{j} - 10\hat{k} + \lambda(2\hat{i} - 3\hat{j} + 8\hat{k})$; $\vec{r} = 4\hat{i} - 3\hat{j} - \hat{k} + \mu(\hat{i} - 4\hat{j} + 7\hat{k})$

Solution : $L_1 : \vec{r} = (2\lambda + 1)\hat{i} - (1 + 3\lambda)\hat{j} + (8\lambda - 10)\hat{k}$

$L_2 : \vec{r} = (4 + \mu)\hat{i} - (4\mu + 3)\hat{j} + (7\mu - 1)\hat{k}$

The given lines are not parallel. For coplanarity, the lines must intersect.

$\therefore$ $(2\lambda + 1)\hat{i} - (1 + 3\lambda)\hat{j} + (8\lambda - 10)\hat{k} = (4 + \mu)\hat{i} - (4\mu + 3)\hat{j} + (7\mu - 1)\hat{k}$

$2\lambda + 1 = 4 + \mu$ .....(i)

$1 + 3\lambda = 4\mu + 3$ .....(ii)

$8\lambda - 10 = 7\mu - 1$ .....(iii)

Solving (i) & (ii), $\lambda = 2, \mu = 1$ and $\lambda = 2, \mu = 1$ satisfies equation (iii)

Given lines are intersecting & hence coplanar. **Ans.**

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16. TEST OF COLLINEARITY OF THREE POINTS :

(a) 3 points A B C will be collinear if $\overrightarrow{AB} = \lambda\overrightarrow{BC}$, where $\lambda \in R$

(b) Three points A, B, C with position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively are collinear, if & only if there exist scalars x, y, z not all zero simultaneously such that ; $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, where $x + y + z = 0$

(c) Collinearly can also be checked by first finding the equation of line through two points and satisfying the third point.

Illustration 12 : Prove that the points with position vectors $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} + 3\hat{j} - 4\hat{k}$ & $-7\hat{j} + 10\hat{k}$ are collinear.

Solution : If we find, three scalars ℓ, m & n such that $\ell\vec{a} + m\vec{b} + n\vec{c} = \vec{0}$, where $\ell + m + n = 0$ then points are collinear.

$\ell(\hat{i} - 2\hat{j} + 3\hat{k}) + m(2\hat{i} + 3\hat{j} - 4\hat{k}) + n(-7\hat{j} + 10\hat{k}) = \vec{0}$

$\Rightarrow \quad (\ell + 2m)\hat{i} + (-2\ell + 3m - 7n)\hat{j} + (3\ell - 4m + 10n)\hat{k} = \vec{0}$

$\Rightarrow \quad \ell + 2m = 0, -2\ell + 3m - 7n = 0, 3\ell - 4m + 10n = 0$

Solving, we get $\ell = 2$, $m = -1$, $n = -1$

since $\ell + m + n = 0$

Hence, the points are collinear.

Aliter :

$\overrightarrow{AB} = \vec{b} - \vec{a} = \left(2\hat{i} + 3\hat{j} - 4\hat{k}\right) - \left(\hat{i} - 2\hat{j} + 3\hat{k}\right) = \hat{i} + 5\hat{j} - 7\hat{k}$

$\overrightarrow{BC} = \vec{c} - \vec{b} = \left(-7\hat{j} + 10\hat{k}\right) - \left(2\hat{i} + 3\hat{j} - 4\hat{k}\right) = -2\hat{i} - 10\hat{j} + 14\hat{k} = -2\left(\hat{i} + 5\hat{j} - 7\hat{k}\right)$

$\therefore \quad \overrightarrow{AB} = -2\overrightarrow{BC}$

Hence $\vec{a}, \vec{b}$ & $\vec{c}$ are collinear.

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Do yourself - 5 :

(i) The position vectors of the points P, Q, R are $\hat{i} + 2\hat{j} + 3\hat{k}$, $-2\hat{i} + 3\hat{j} + 5\hat{k}$ and $7\hat{i} - \hat{k}$ respectively. Prove that P, Q and R are collinear.

17. SCALAR PRODUCT OF TWO VECTORS (DOT PRODUCT) :

Definition : Let $\vec{a}$ and $\vec{b}$ be two non zero vectors inclined at an angle θ. Then the scalar product of $\vec{a}$ with $\vec{b}$ is denoted by $\vec{a} \cdot \vec{b}$ and is defined as $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$; $0 \le \theta \le \pi$.

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Geometrical Interpretation of Scalar product :

$|\overrightarrow{OA}| = |\vec{a}|, |\overrightarrow{OB}| = |\vec{b}|$

Now $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$= |\vec{a}|(OB\cos\theta)$

$=$ (magnitude of $\vec{a}$) (Projection of $\vec{b}$ on $\vec{a}$)

Again, $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$= |\vec{b}|(|\vec{a}|\cos\theta)$

$=$ (Magnitude of $\vec{b}$) (Projection of $\vec{a}$ on $\vec{b}$)

(a) $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta (0 \le \theta \le \pi)$

Note that if θ is acute then $\vec{a}.\vec{b} > 0$ & if θ is obtuse then $\vec{a}.\vec{b} < 0$

(b) (i) $\vec{a}.\vec{a} = |\vec{a}|^2 = \vec{a}^2$ (ii) $\vec{a}.\vec{b} = \vec{b}.\vec{a}$ (commutative)

(c) $\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$ (distributive)

(d) $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$; $(\vec{a}, \vec{b} \ne \vec{0})$

(e) $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1; \hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

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(f) Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$. (Provided $|\vec{b}| \ne 0$)

Projection of $\vec{a}$ on $\vec{b}$

Note :

(i) The vector component of $\vec{a}$ along $\vec{b} = \left(\dfrac{\vec{a}.\vec{b}}{\vec{b}^2}\right)\vec{b}$ and perpendicular to $\vec{b} = \vec{a} - \left(\dfrac{\vec{a}.\vec{b}}{\vec{b}^2}\right)\vec{b}$ [by triangle law of vector Addition]

(ii) The angle ϕ between $\vec{a}$ & $\vec{b}$ is given by $\cos\phi = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$ $0 \le \phi \le \pi$

(iii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}, |\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

(iv) Maximum value of $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|$

(v) Minimum values of $\vec{a}.\vec{b} = -|\vec{a}||\vec{b}|$

(vi) Any vector $\vec{a}$ can be written as, $\vec{a} = (\vec{a}.\hat{i})\hat{i} + (\vec{a}.\hat{j})\hat{j} + (\vec{a}.\hat{k})\hat{k}$

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(g) Vector equation of angle bisector :

A vector in the direction of the bisector of the angle between the two vectors $\vec{a}$ & $\vec{b}$ is $\frac{\vec{a}}{|\vec{a}|}+\frac{\vec{b}}{|\vec{b}|}$. Hence bisector of the angle between the two vectors $\vec{a}$ & $\vec{b}$ is $\lambda(\hat{a}+\hat{b})$, where $\lambda \in R^+$.

Illustration 13 : The vector $\vec{c}$, directed along the bisector of the angle between the vector $7\hat{i}-4\hat{j}-4\hat{k}$ and $-2\hat{i}-\hat{j}+2\hat{k}$ with $|\vec{c}|=5\sqrt{6}$ is -

(A) $\frac{5}{3}(\hat{i}-7\hat{j}+2\hat{k})$ (B) $\frac{5}{3}(5\hat{i}+5\hat{j}+2\hat{k})$ (C) $\frac{5}{3}(\hat{i}+7\hat{j}+2\hat{k})$ (D) none of these

Solution : Let $\vec{a}=7\hat{i}-4\hat{j}-4\hat{k}$

and $\vec{b}=-2\hat{i}-\hat{j}+2\hat{k}$

Angle bisector of A divides the BC in the ratio of $|\overrightarrow{AB}|:|\overrightarrow{AC}|$,

$|\overrightarrow{AB}|=9$, $|\overrightarrow{AC}|=3$

$$\overrightarrow{AD}=\left(\frac{9(-2\hat{i}-\hat{j}+2\hat{k})+3(7\hat{i}-4\hat{j}-4\hat{k})}{9+3}\right)=\frac{\hat{i}-7\hat{j}+2\hat{k}}{4}$$

$$\vec{c}=\left(\frac{\overrightarrow{AD}}{|\overrightarrow{AD}|}\right)5\sqrt{6}=\frac{5}{3}(\hat{i}-7\hat{j}+2\hat{k})$$ **Ans.[A]**

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Illustration 14 : If p^{th}, q^{th}, r^{th} terms of a G.P. are the positive numbers a, b, c then angle between the vectors $\log a^2\hat{i}+\log b^2\hat{j}+\log c^2\hat{k}$ and $(q-r)\hat{i}+(r-p)\hat{j}+(p-q)\hat{k}$ is :-

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$

(C) $\sin^{-1}\left(\frac{1}{\sqrt{a^2+b^2+c^2}}\right)$ (D) none of these

Solution : Let x_0 be first term and x the common ratio of the G.P.

$\therefore$ $a=x_0\,x^{p-1}$, $b=x_0\,x^{q-1}$, $c=x_0\,x^{r-1}$ $\Rightarrow$ $\log a=\log x_0+(p-1)\log x$;

$\log b=\log x_0+(q-1)\log x$; $\log c=\log x_0+(r-1)\log x$

If $\vec{a}=\log a^2\hat{i}+\log b^2\hat{j}+\log c^2\hat{k}$ and $\vec{b}=(q-r)\hat{i}+(r-p)\hat{j}+(p-q)\hat{k}$

$\therefore$ $\vec{a}.\vec{b}=\Sigma 2(\log a)(q-r)=2\sum(\log x_0+(p-1)\log x)(q-r)=0 \Rightarrow \vec{a}\wedge\vec{b}=\frac{\pi}{2}$. **Ans.**

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Illustration 15: Find the distance of the point $B(\hat{i}+2\hat{j}+3\hat{k})$ from the line which is passing through $A(4\hat{i}+2\hat{j}+2\hat{k})$ and which is parallel to the vector $\vec{C}=2\hat{i}+3\hat{j}+6\hat{k}$. **(Roorkee 1993)**

Solution : $AB=\sqrt{3^2+1^2}=\sqrt{10}$

$AM=\overrightarrow{AB}.\hat{c}=(-3\hat{i}+\hat{k}).\dfrac{(2\hat{i}+3\hat{j}+6\hat{k})}{7}$

$=-6+6=0$

$BM^2=AB^2-AM^2$

So, $BM=AB=\sqrt{10}$ **Ans.**

Illustration 16 : Prove that the medians to the base of an isosceles triangle is perpendicular to the base.

Solution : The triangle being isosceles, we have

$AB=AC$ ..... (i)

Now $\overrightarrow{AP}=\dfrac{\vec{b}+\vec{c}}{2}$ where P is mid-point of BC.

Also $\overrightarrow{BC}=\vec{c}-\vec{b}$

$\therefore$ $\overrightarrow{AP}.\overrightarrow{BC}=\dfrac{\vec{b}+\vec{c}}{2}.(\vec{c}-\vec{b})=\dfrac{1}{2}(c^2-b^2)$

$=\dfrac{1}{2}(AC^2-AB^2)=0$ {by (i)}

$\therefore$ Median AP is perpendicular to base BC.

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Do yourself - 6 :

(i) Find the angle between two vectors $\vec{a}$ & $\vec{b}$ with magnitude 2 and 1 respectively and such that $\vec{a}.\vec{b}=\sqrt{3}$.

(ii) Find the value of $(\vec{a}+3\vec{b}).(2\vec{a}-\vec{b})$ if $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$, $\vec{b}=3\hat{i}+2\hat{j}-\hat{k}$.

(iii) The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of the vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to 1, find λ.

(iv) Find the projection of the vector $\vec{a}=4\hat{i}-2\hat{j}+\hat{k}$ on the vector $\vec{b}=3\hat{i}+6\hat{j}+2\hat{k}$. Also find vector component of $\vec{a}$ along $\vec{b}$ and perpendicular to $\vec{b}$.

(v) Find the unit vectors along the angle bisectors between the vectors $\vec{a}=\hat{i}+2\hat{j}-2\hat{k}$ and $\vec{b}=-3\hat{i}+6\hat{j}+2\hat{k}$.

18. LINEAR COMBINATIONS :

Given a finite set of vectors $\vec{a}, \vec{b}, \vec{c}$,..... then the vector $\vec{r}=x\vec{a}+y\vec{b}+z\vec{c}+$..... is called a **linear combination** of $\vec{a}, \vec{b}, \vec{c}$,..... for any x, y, z..... $\in$ R.

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FUNDAMENTAL THEOREM IN PLANE :

Let $\vec{a}, \vec{b}$ be non zero, non collinear vectors. then any vector $\vec{r}$ coplanar with $\vec{a}, \vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}, \vec{b}$ i.e. there exist some unique $x,\ y \in R$ such that $\mathbf{x\vec{a} + y\vec{b} = \vec{r}}$.

Illustration 17 : Find a vector $\vec{c}$ in the plane of $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} + \hat{k}$ such that $\vec{c}$ is perpendicular to $\vec{b}$ and $\vec{c}.(-2\hat{i} + 3\hat{j} - \hat{k}) = -1$

Solution : Any vector in the plane of $\vec{a}$ & $\vec{b}$ can be written as $x\vec{a} + y\vec{b}$

let $\vec{c} = x\vec{a} + y\vec{b}$ [by fundamental theorem in plane]

Now, given that

$\vec{c}.\vec{b} = 0 \Rightarrow (x\vec{a} + y\vec{b}).\vec{b} = 0$

$x\vec{a}.\vec{b} + y\vec{b}^2 = 0$

$\Rightarrow$ $x(-2 + 1 - 1) + y(3) = 0$

$-2x + 3y = 0$ .....(i)

Also $(x\vec{a} + y\vec{b}).(-2\hat{i} + 3\hat{j} - \hat{k}) = -1$

$\Rightarrow$ $x\vec{a}.(-2\hat{i} + 3\hat{j} - \hat{k}) + y\vec{b}.(-2\hat{i} + 3\hat{j} - \hat{k}) = -1$

$\Rightarrow$ $x(-4 + 3 + 1) + y(2 + 3 - 1) = -1$

$y = -\frac{1}{4}$

$x = \frac{3y}{2} = -\frac{3}{8}$

Hence the required vector $\vec{c} = -\frac{3}{8}(2\hat{i} + \hat{j} - \hat{k}) - \frac{1}{4}(-\hat{i} + \hat{j} + \hat{k})$

$= \frac{1}{8}[-6\hat{i} - 3\hat{j} + 3\hat{k} + 2\hat{i} - 2\hat{j} - 2\hat{k}] = \frac{1}{8}[-4\hat{i} - 5\hat{j} + \hat{k}]$ **Ans.**

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Do yourself - 7 :

(i) Find a vector $\vec{r}$ in the plane of $\vec{p} = -\hat{i} + \hat{j}$ and $\vec{q} = -\hat{j} + \hat{k}$ such that $\vec{r}$ is perpendicular to $\vec{p}$ and $\vec{r}.\vec{q} = -2$.

19. VECTOR PRODUCT OF TWO VECTORS (CROSS PRODUCT) :

(a) If $\vec{a}$ & $\vec{b}$ are two vectors & θ is the angle between them, then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \hat{n}$, where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ & $\vec{b}$ such that $\vec{a}, \vec{b}$ & $\vec{n}$ forms a right handed screw system.

Sign convention :

Right handed screw system : $\vec{a}, \vec{b}$ and $\hat{n}$ form a right handed system it means that if we rotate vector $\vec{a}$ towards the direction of $\vec{b}$ through the angle θ, then $\hat{n}$ advances in the same direction as a right handed screw would, if turned in the same way.

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(b) **Lagranges Identity :** For any two vectors $\vec{a}$ & $\vec{b}$; $(\vec{a}\times\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$

(c) Formulation of vector product in terms of scalar product : The vector product $\vec{a}\times\vec{b}$ is the vector $\vec{c}$, such that

(i) $|\vec{c}| = \sqrt{\vec{a}^2\vec{b}^2 - (\vec{a}.\vec{b})^2}$ (ii) $\vec{c}.\vec{a} = 0; \vec{c}.\vec{b} = 0$ and (iii) $\vec{a}, \vec{b}, \vec{c}$ form a right handed system

(d) (i) $\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow \vec{a}$ & $\vec{b}$ are parallel (collinear) $(\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0})$ i.e. $\vec{a} = K\vec{b}$, where K is a scalar

(ii) $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ **(not commutative)**

(iii) $(m\vec{a})\times\vec{b} = \vec{a}\times(m\vec{b}) = m(\vec{a}\times\vec{b})$ where m is a scalar.

(iv) $\vec{a}\times(\vec{b}+\vec{c}) = (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c})$ **(distributive over addition)**

(v) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$

(vi) $\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j}$

(e) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$.

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(f) Geometrically $|\vec{a} \times \vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ & $\vec{b}$.

(g) (i) Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\hat{n} = \pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

(ii) A vector of magnitude 'r' & perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\pm\dfrac{r(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$

(iii) If θ is the angle between $\vec{a}$ & $\vec{b}$, then $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

(h) Vector area :

(i) If $\vec{a}, \vec{b}$ and $\vec{c}$ are the pv's of 3 points A, B & C then the vector area of triangle ABC $= \dfrac{1}{2}\left[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}\right]$.

(ii) The points A, B & C are collinear if $\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = 0$

(iii) Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ & $\vec{d}_2$ is given by $\dfrac{1}{2}|\vec{d}_1\times\vec{d}_2|$.

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Illustration 18 : Find the vectors of magnitude 5 which are perpendicular to the vectors $\vec{a} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.

Solution : Unit vectors perpendicular to $\vec{a}$ & $\vec{b} = \pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

$$\because \quad \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -3 \\ 1 & -2 & 1 \end{vmatrix} = -5\hat{i} - 5\hat{j} - 5\hat{k}$$

$$\therefore \quad \text{Unit vectors} = \pm\frac{(-5\hat{i} - 5\hat{j} - 5\hat{k})}{5\sqrt{3}}$$

Hence the required vectors are $\pm\dfrac{5\sqrt{3}}{3}(\hat{i} + \hat{j} + \hat{k})$ **Ans.**

Illustration 19 : If $\vec{a}, \vec{b}, \vec{c}$ are three non zero vectors such that $\vec{a}\times\vec{b} = \vec{c}$ and $\vec{b}\times\vec{c} = \vec{a}$, prove that $\vec{a}, \vec{b}, \vec{c}$ are mutually at right angles and $|\vec{b}| = 1$ and $|\vec{c}| = |\vec{a}|$.

Solution : $\vec{a}\times\vec{b} = \vec{c}$ and $\vec{a} = \vec{b}\times\vec{c}$

$\Rightarrow \quad \vec{c} \perp \vec{a},\ \vec{c} \perp \vec{b}$ and $\vec{a} \perp \vec{b},\ \vec{a} \perp \vec{c}$

$\Rightarrow \quad \vec{a} \perp \vec{b},\ \vec{b} \perp \vec{c}$ and $\vec{c} \perp \vec{a}$

$\Rightarrow \quad \vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors.

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Again, $\vec{a}\times\vec{b} = \vec{c}$ and $\vec{b}\times\vec{c} = \vec{a}$

$\Rightarrow \quad |\vec{a}\times\vec{b}| = |\vec{c}|$ and $|\vec{b}\times\vec{c}| = |\vec{a}|$

$\Rightarrow \quad |\vec{a}||\vec{b}|\sin\dfrac{\pi}{2} = |\vec{c}|$ and $|\vec{b}||\vec{c}|\sin\dfrac{\pi}{2} = |\vec{a}|$ $\quad (\because \vec{a} \perp \vec{b}$ and $\vec{b} \perp \vec{c})$

$\Rightarrow \quad |\vec{a}||\vec{b}| = |\vec{c}|$ and $|\vec{b}||\vec{c}| = |\vec{a}|$

$\Rightarrow \quad |\vec{b}|^2 |\vec{c}| = |\vec{c}|$

$\Rightarrow \quad |\vec{b}|^2 = 1$

$\Rightarrow \quad |\vec{b}| = 1$

putting in $|\vec{a}||\vec{b}| = |\vec{c}|$

$\Rightarrow \quad |\vec{a}| = |\vec{c}|$

Illustration 20 : Show that the area of the triangle formed by joining the extremities of an oblique side of a trapezium to the midpoint of opposite side is half that of the trapezium.

Solution : Let ABCD be the trapezium and E be the midpoint of BC. Let A be the initial point and let $\vec{b}$ be the position vector of B and $\vec{d}$ that of D. Since DC is parallel to AB, $t\vec{b}$ is a vector along DC, so that the position vector of c is $\vec{d} + t\vec{b}$.

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$\Rightarrow$ the position vector of E is $\dfrac{\vec{b}+\vec{d}+t\vec{b}}{2}=\dfrac{\vec{d}+(1+\vec{t})\vec{b}}{2}$

Area of $\Delta AED=\dfrac{1}{2}\left|\dfrac{\vec{d}+(1+t)\vec{b}}{2}\times\vec{d}\right|=\dfrac{1}{4}(1+t)|\vec{b}\times\vec{d}|$

Area of the trapezium = Area (ΔACD) + Area (ΔABC).

$$=\frac{1}{2}|\vec{b}\times(\vec{d}+t\vec{b})|+\frac{1}{2}|(\vec{d}+t\vec{b})\times\vec{d}|$$

$$=\frac{1}{2}|\vec{b}\times\vec{d}|+\frac{t}{2}|\vec{b}\times\vec{d}|=\frac{1}{2}(1+t)|\vec{b}\times\vec{d}|=2\Delta AED$$

Illustration 21 : Let $\vec{a}$ & $\vec{b}$ be two non-collinear unit vectors. If $\vec{u}=\vec{a}-(\vec{a}.\vec{b})\vec{b}$ & $\vec{v}=(\vec{a}\times\vec{b})$, then $|\vec{v}|$ is - **[JEE 99]**

(A) $|\vec{u}|$ (B) $|\vec{u}|+|\vec{u}.\vec{a}|$ (C) $|\vec{u}|+|\vec{u}.\vec{b}|$ (D) $\vec{u}+\vec{u}.(\vec{a}+\vec{b})$

Solution : $\vec{u}.\vec{a}=\vec{a}.\vec{a}-(\vec{a}.\vec{b})(\vec{a}.\vec{b})$

$=1-|\vec{a}|^2|\vec{b}|^2\cos^2\theta$ (where θ is the angle between $\vec{a}$ and $\vec{b}$)

$=1-\cos^2\theta=\sin^2\theta$

$|\vec{v}|=|\vec{a}\times\vec{b}|=\sin\theta$

$|\vec{u}|=\sqrt{\vec{u}.\vec{u}}$

$=\sqrt{\vec{a}.\vec{a}-2(\vec{a}.\vec{b})^2+(\vec{a}.\vec{b})^2|\vec{b}|^2}=\sqrt{1-(\vec{a}.\vec{b})^2}=\sin\theta$

$\therefore$ $|\vec{v}|=|\vec{u}|$ also $\vec{u}.\vec{b}=0$

Hence, $|\vec{v}|=|\vec{u}|=|\vec{u}|+|\vec{u}.\vec{b}|$ **Ans. (A, C)**

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Do yourself - 8 :

(i) If $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$, then show that $(\vec{a}-\vec{d})$ is parallel to $(\vec{b}-\vec{c})$ when $\vec{a}\neq\vec{d}$ and $\vec{b}\neq\vec{c}$.

(ii) Find $\vec{a}\times\vec{b}$, if $\vec{a}=2\hat{i}+\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}+\hat{k}$.

(iii) For any two vectors $\vec{u}$ & $\vec{v}$, prove that **[JEE 98]**

(a) $(\vec{u}.\vec{v})^2+|\vec{u}\times\vec{v}|^2=|\vec{u}|^2|\vec{v}|^2$ (b) $(1+|\vec{u}|^2)(1+|\vec{v}|^2)=(1-\vec{u}.\vec{v})^2+|\vec{u}+\vec{v}+(\vec{u}\times\vec{v})|^2$

20. SHORTEST DISTANCE BETWEEN TWO LINES :

If two lines in space intersect at a point, then obviously the shortest distance between them is zero. Lines which do not intersect & are also not parallel are called **skew lines**. **In other words the lines which are not coplanar are skew lines**. For Skew lines the direction of the shortest distance vector would be perpendicular to both the lines. The magnitude of the shortest distance vector would be equal to that of the projection of $\vec{AB}$ along the direction of the line of shortest distance, $\vec{LM}$ is parallel to $\vec{p}\times\vec{q}$

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i.e. $\vec{LM} = |\text{Projection of } \vec{AB} \text{ on } \vec{LM}| = |\text{Projection of } \vec{AB} \text{ on } \vec{p}\times\vec{q}|$

$$= \left|\frac{\overrightarrow{AB}.(\vec{p}\times\vec{q})}{\vec{p}\times\vec{q}}\right| = \left|\frac{(\vec{b}-\vec{a}).(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$$

(a) **The two lines directed along $\vec{p}$ & $\vec{q}$ will intersect only if shortest distance = 0 i.e.** $(\vec{b}-\vec{a}).(\vec{p}\times\vec{q}) = 0$ i.e. $(\vec{b}-\vec{a})$ lies in the plane containing $\vec{p}$ & $\vec{q} \Rightarrow \left[(\vec{b}-\vec{a})\ \ \vec{p}\ \ \vec{q}\right] = 0$

(b) If two lines are given by $\vec{r_1} = \vec{a_1} + K_1\vec{b}$ & $\vec{r_2} = \vec{a_2} + K_2\vec{b}$ i.e. they are parallel then, $d = \left|\frac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|$

Illustration 22 : Find the shortest distance between the lines

$\vec{r} = (4\hat{i}-\hat{j}) + \lambda(\hat{i}+2\hat{j}-3\hat{k})$ and $\vec{r} = (\hat{i}-\hat{j}+2k) + \mu(2\hat{i}+4\hat{j}-5\hat{k})$

Solution : We known, the shortest distance between the lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ & $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ is given by

$$d = \left|\frac{(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$$

Comparing the given equation with the equations $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $r = \vec{a}_2 + \lambda\vec{b}_2$ respectively,

we have $\vec{a}_1 = 4\hat{i}-\hat{j}, \vec{a}_2 = \hat{i}-\hat{j}+2\hat{k}$, $\vec{b}_1 = \hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b}_2 = 2\hat{i}+4\hat{j}-5\hat{k}$

Now, $\vec{a}_2-\vec{a}_1 = -3\hat{i}+0\hat{j}+2\hat{k}$ and $\vec{b}_1\times\vec{b}_2 = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -3\\ 2 & 4 & -5\end{vmatrix} = 2\hat{i}-\hat{j}+0\hat{k}$

$\therefore\ (\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2) = (-3\hat{i}+0\hat{j}+2\hat{k}).(2\hat{i}-\hat{j}+0\hat{k}) = -6$ and $|\vec{b}_1\times\vec{b}_2| = \sqrt{4+1+0} = \sqrt{5}$

$\therefore$ Shortest distance $d = \left|\frac{(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right| = \left|\frac{-6}{\sqrt{5}}\right| = \frac{6}{\sqrt{5}}$. **Ans.**

Do yourself - 9 :

(i) Find the shortest distance between the lines :

$\vec{r}_1 = (\hat{i}+2\hat{j}+3\hat{k}) + \lambda(2\hat{i}+3\hat{j}+4\hat{k})$ & $\vec{r}_2 = (2\hat{i}+4\hat{j}+5\hat{k}) + \mu(3\hat{i}+4\hat{j}+5\hat{k})$.

21. SCALAR TRIPLE PRODUCT / BOX PRODUCT / MIXED PRODUCT :

(i) The scalar triple product of three vectors $\vec{a}$, $\vec{b}$ & $\vec{c}$ is defined as : $(\vec{a}\times\vec{b}).\vec{c} = |\vec{a}||\vec{b}||\vec{c}|\sin\theta\cos\phi$ where θ is the angle between $\vec{a}$ & $\vec{b}$ & ϕ is the angle between $\vec{a}\times\vec{b}$ & $\vec{c}$. It is also defined as $[\vec{a}\ \ \vec{b}\ \ \vec{c}]$, spelled as box product.

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(ii) Scalar triple product geometrically represents the volume of the parallelopiped whose three coterminous edges are represented by $\vec{a}$, $\vec{b}$ & $\vec{c}$ i.e. $V = [\vec{a}\,\vec{b}\,\vec{c}]$

(iii) In a scalar triple product the position of dot & cross can be interchanged i.e. $\vec{a}.(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b}).\vec{c}$ OR $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

(iv) $\vec{a}.(\vec{b}\times\vec{c}) = -\vec{a}.(\vec{c}\times\vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\,\vec{c}\,\vec{b}]$

If $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$; $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ & $\vec{c} = c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$, then $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

In general , if $\vec{a} = a_1\vec{l}+a_2\vec{m}+a_3\vec{n}$; $\vec{b} = b_1\vec{l}+b_2\vec{m}+b_3\vec{n}$ & $\vec{c} = c_1\vec{l}+c_2\vec{m}+c_3\vec{n}$

then $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}[\vec{l}\,\vec{m}\,\vec{n}]$; where $\vec{l}, \vec{m}$ & $\vec{n}$ are non coplanar vectors.

(v) **If $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0 \Rightarrow$ $\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly dependent.**

(vi) Scalar product of three vectors, two of which are equal or parallel is 0 i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

Note : If $\vec{a}$, $\vec{b}$, $\vec{c}$ are non- coplanar then $[\vec{a}\ \vec{b}\ \vec{c}] > 0$ for right handed system & $[\vec{a}\ \vec{b}\ \vec{c}] < 0$ for left handed system.

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(vii) $[\hat{i}\,\hat{j}\,\hat{k}] = 1$ **(viii)** $[K\vec{a}\ \vec{b}\ \vec{c}] = K[\vec{a}\ \vec{b}\ \vec{c}]$ **(ix)** $[(\vec{a}+\vec{b})\ \vec{c}\ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}]+[\vec{b}\ \vec{c}\ \vec{d}]$

(viii) The Volume of the tetrahedron OABC with O as origin & the pv's of A, B and C being $\vec{a}$, $\vec{b}$ & $\vec{c}$ are given by $V = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$

The position vector of the centroid of a tetrahedron if the pv's of its angular vertices are $\vec{a}$, $\vec{b}$, $\vec{c}$ & $\vec{d}$ are given by $\frac{1}{4}(\vec{a}+\vec{b}+\vec{c}+\vec{d})$

Note that this is also the point of concurrency of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron.

(ix) $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}] = 0$ & $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$

Illustration 23 : For any three vectors $\vec{a},\vec{b},\vec{c}$ prove that $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\,\vec{b}\,\vec{c}]$

Solution : We have $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]$

$= \{(\vec{a}+\vec{b})\times(\vec{b}+\vec{c})\}.(\vec{c}+\vec{a}) = \{\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{b}+\vec{b}\times\vec{c}\}.(\vec{c}+\vec{a})$ $\{\because \vec{b}\times\vec{b} = 0\}$

$= \{\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}\}.(\vec{c}+\vec{a}) = (\vec{a}\times\vec{b}).\vec{c}+(\vec{a}\times\vec{c}).\vec{c}+(\vec{b}\times\vec{c}).\vec{c}+(\vec{a}\times\vec{b}).\vec{a}+(\vec{a}\times\vec{c}).\vec{a}+(\vec{b}\times\vec{c}).\vec{a}$

$= [\vec{a}\,\vec{b}\,\vec{c}]+0+0+0+0+[\vec{b}\,\vec{c}\,\vec{a}]$ $\{\because [\vec{a}\,\vec{c}\,\vec{c}] = 0, [\vec{b}\,\vec{c}\,\vec{c}] = 0, [\vec{a}\,\vec{b}\,\vec{a}] = 0, [\vec{a}\,\vec{c}\,\vec{a}] = 0\}$

$= [\vec{a}\,\vec{b}\,\vec{c}]+[\vec{a}\,\vec{b}\,\vec{c}] = 2[\vec{a}\,\vec{b}\,\vec{c}]$. **Ans.**

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Illustration 24 : If $\vec{a}, \vec{b}$ are non-zero and non-collinear vectors then show $\vec{a}\times\vec{b} = [\vec{a}\,\vec{b}\,\hat{i}]\hat{i} + [\vec{a}\,\vec{b}\,\hat{j}]\hat{j} + [\vec{a}\,\vec{b}\,\hat{k}]\hat{k}$

Solution : Let $\vec{a}\times\vec{b} = x\hat{i} + y\hat{j} + z\hat{k}$

$(\vec{a}\times\vec{b}).\hat{i} = (x\hat{i} + y\hat{j} + z\hat{k}).\hat{i}$

$(\vec{a}\times\vec{b}).\hat{i} = x$

also $(\vec{a}\times\vec{b}).\hat{j} = y$ & $(\vec{a}\times\vec{b}).\hat{k} = z$

$\therefore$ $\vec{a}\times\vec{b} = [\vec{a}\,\vec{b}\,\hat{i}]\hat{i} + [\vec{a}\,\vec{b}\,\hat{j}]\hat{j} + [\vec{a}\,\vec{b}\,\hat{k}]\hat{k}$ **Ans.**

Do yourself - 10 :

(i) If $\vec{a}, \vec{b}, \vec{c}$ are three non coplanar mutually perpendicular unit vectors then find $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$.

(ii) If $\vec{r}$ be a vector perpendicular to $\vec{a}+\vec{b}+\vec{c}$, where $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = z$ and $\vec{r} = \ell(\vec{b}\times\vec{c}) + m(\vec{c}\times\vec{a}) + n(\vec{a}\times\vec{b})$, then find $l + m + n$.

(iii) Find the volume of the parallelepiped whose coterminous edges are represented by $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$

(iv) Examine whether the vectors $\vec{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$, $\vec{b} = \hat{i} - \hat{j} + 2\hat{k}$ and $\vec{c} = 3\hat{i} + 2\hat{j} - 4\hat{k}$ form a left handed or right handed system.

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22. VECTOR TRIPLE PRODUCT :

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three vectors, then the expression $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector & is called a **vector triple product**.

Geometrical interpretation of $\vec{a}\times(\vec{b}\times\vec{c})$

Consider the expression $\vec{a}\times(\vec{b}\times\vec{c})$ which itself is a vector, since it is a cross product of two vectors $\vec{a}$ & $(\vec{b}\times\vec{c})$. Now $\vec{a}\times(\vec{b}\times\vec{c})$ is vector perpendicular to the plane containing $\vec{a}$ & $(\vec{b}\times\vec{c})$ but $(\vec{b}\times\vec{c})$ is a vector perpendicular to the plane $\vec{b}$ & $\vec{c}$, therefore $\vec{a}\times(\vec{b}\times\vec{c})$ is vector lies in the plane of $\vec{b}$ & $\vec{c}$ and perpendicular to $\vec{a}$. Hence we can express $\vec{a}\times(\vec{b}\times\vec{c})$ in terms of $\vec{b}$ & $\vec{c}$ i.e. $\vec{a}\times(\vec{b}\times\vec{c}) = x\vec{b} + y\vec{c}$ where x & y are scalars.

(a) $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$ (b) $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

(c) $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$

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Illustration 25 : Prove that $\left[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\right]=\left[\vec{a}\vec{b}\vec{c}\right]^2$

Solution : We have, $\left[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\right]=\left\{\left(\vec{a}\times\vec{b}\right)\times\left(\vec{b}\times\vec{c}\right)\right\}.\left(\vec{c}\times\vec{a}\right)$

$=\left\{\vec{d}\times\left(\vec{b}\times\vec{c}\right)\right\}.\left(\vec{c}\times\vec{a}\right)$ (where, $\vec{d}=(\vec{a}\times\vec{b})$)

$=\left\{\left(\vec{d}.\vec{c}\right)\vec{b}-\left(\vec{d}.\vec{b}\right)\vec{c}\right\}.\left(\vec{c}\times\vec{a}\right)=\left\{\left(\left(\vec{a}\times\vec{b}\right).\vec{c}\right)\vec{b}-\left(\left(\vec{a}\times\vec{b}\right).\vec{b}\right)\vec{c}\right\}.\left(\vec{c}\times\vec{a}\right)$

$=\left\{\left[\vec{a}\vec{b}\vec{c}\right]\vec{b}-0\right\}.\left(\vec{c}\times\vec{a}\right)$ $(\because\ [\vec{a}\vec{b}\vec{b}]=0)$

$=\left[\vec{a}\vec{b}\vec{c}\right]\left\{\vec{b}.\left(\vec{c}\times\vec{a}\right)\right\}=\left[\vec{a}\vec{b}\vec{c}\right]\left[\vec{b}\vec{c}\vec{a}\right]=\left[\vec{a}\vec{b}\vec{c}\right]\left[\vec{a}\vec{b}\vec{c}\right]=\left[\vec{a}\vec{b}\vec{c}\right]^2$

Illustration 26 : Show that $(\vec{b}\times\vec{c}).(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a}).(\vec{b}\times\vec{d})+(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})=0$

Solution : Let $\vec{b}\times\vec{c}=\vec{u},\vec{c}\times\vec{a}=\vec{v},\vec{c}\times\vec{d}=\vec{w}$

$\therefore$ L.H.S $=\vec{u}.(\vec{a}\times\vec{d})+\vec{v}.(\vec{b}\times\vec{d})+(\vec{a}\times\vec{b}).\vec{w}=(\vec{u}\times\vec{a}).\vec{d}+(\vec{v}\times\vec{b}).\vec{d}+\vec{a}.(\vec{b}\times\vec{w})$

$=[(\vec{b}\times\vec{c})\times\vec{a}].\vec{d}+[(\vec{c}\times\vec{a})\times\vec{b}].\vec{d}+\vec{a}.[\vec{b}\times(\vec{c}\times\vec{d})]$

$=[(\vec{b}.\vec{a})\vec{c}-(\vec{c}.\vec{a})\vec{b}].\vec{d}+[(\vec{c}.\vec{b})\vec{a}-(\vec{a}.\vec{b})\vec{c}].\vec{d}+\vec{a}.[(\vec{b}.\vec{d})\vec{c}-(\vec{b}.\vec{c})\vec{d}]$

$=\{(\vec{a}.\vec{b})(\vec{c}.\vec{d})\}-\{(\vec{c}.\vec{a})(\vec{b}.\vec{d})\}+\{(\vec{c}.\vec{b})(\vec{a}.\vec{d})\}-\{(\vec{a}.\vec{b})(\vec{c}.\vec{d})\}+\{(\vec{a}.\vec{c})(\vec{b}.\vec{d})\}-\{(\vec{a}.\vec{d})(\vec{b}.\vec{c})\}=0$

= R.H.S.

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Do yourself - 11 :

(i) If $\vec{a}=2\hat{i}-4\hat{j}+7\hat{k}$, $\vec{b}=3\hat{i}+5\hat{j}-9\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}+\hat{k}$, then find $[\vec{a}\ \ \vec{b}\ \ \vec{c}]$ and also $\vec{a}\times(\vec{b}\times\vec{c})$.

23. LINEAR INDEPENDENCE AND DEPENDENCE OF VECTORS :

(a) If $\vec{x}_1, \vec{x}_2, ...\vec{x}_n$ are n non zero vectors, & $k_1, k_2, ...k_n$ are n scalars & if the linear combination $\mathbf{k_1\vec{x}_1+k_2\vec{x}_2+....k_n\vec{x}_n=\vec{0}\Rightarrow k_1=0,\ k_2=0....k_n=0}$, then we say that vectors $\vec{x}_1, \vec{x}_2, ...\vec{x}_n$ are **linearly independent vectors.**

(b) If $\vec{x}_1, \vec{x}_2, ...\vec{x}_n$ are not **linearly independent** then they are said to be **linearly dependent vectors**. i.e. if $\mathbf{k_1\vec{x}_1+k_2\vec{x}_2+.....k_n\vec{x}_n=\vec{0}}$ & if there exists at least one $\mathbf{k_r\neq 0}$ then $\mathbf{\vec{x}_1, \vec{x}_2, ...\vec{x}_n}$ are said to be **linearly dependent.**

FUNDAMENTAL THEOREM IN SPACE :

Let $\vec{a}, \vec{b}, \vec{c}$ be non-zero, non-coplanar vectors in space. Then any vector $\vec{r}$, can be uniquely expressed as a linear combination of $\vec{a}, \vec{b}, \vec{c}$ i.e. There exist some unique x, y, z $\in$ R such that $\mathbf{\vec{r}=x\vec{a}+y\vec{b}+z\vec{c}}$.

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Note :

(i) If $\vec{a} = 3\hat{i}+2\hat{j}+5\hat{k}$ then $\vec{a}$ is expressed as a linear combination of vectors $\hat{i}, \hat{j}, \hat{k}$. Also, $\vec{a}, \hat{i}, \hat{j}, \hat{k}$ form a linearly dependent set of vectors. In general, every set of four vectors is a linearly dependent system.

(ii) If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero, non-coplanar vectors then $x\vec{a}+y\vec{b}+z\vec{c}=\vec{0} \Rightarrow x=y=z=0$

(iii) $\hat{i}, \hat{j}, \hat{k}$ are linearly independent set of vectors. For $K_1\hat{i}+K_2\hat{j}+K_3\hat{k}=\vec{0} \Rightarrow K_1=0=K_2=K_3$

(iv) Two vectors $\vec{a}$ & $\vec{b}$ are **linearly dependent** $\Rightarrow \vec{a}$ is a parallel to $\vec{b}$ i.e. $\vec{a}\times\vec{b}=\vec{0} \Rightarrow$ linear dependence of $\vec{a}$ & $\vec{b}$. Conversely if $\vec{a}\times\vec{b}\neq\vec{0}$, then $\vec{a}$ & $\vec{b}$ are **linearly independent**.

(v) If three vectors $\vec{a}, \vec{b}, \vec{c}$ are **linearly dependent**, then they are **coplanar** i.e. $[\vec{a}\ \vec{b}\ \vec{c}]=0$ conversely, if $[\vec{a}\ \vec{b}\ \vec{c}]\neq 0$, then the vectors are linearly independent.

Illustration 27 : Show that points with position vectors $\vec{a}-2\vec{b}+3\vec{c}, -2\vec{a}+3\vec{b}-\vec{c}, 4\vec{a}-7\vec{b}+7\vec{c}$ are collinear. It is given that vectors $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar.

Solution : The three points are collinear, if we can find λ_1, λ_2 and λ_3, such that

$$\lambda_1(\vec{a}-2\vec{b}+3\vec{c})+\lambda_2(-2\vec{a}+3\vec{b}-\vec{c})+\lambda_3(4\vec{a}-7\vec{b}+7\vec{c})=\vec{0}$$

with $\lambda_1+\lambda_2+\lambda_3=0$

equating the coefficients $\vec{a}, \vec{b}$ and $\vec{c}$ separately to zero, we get

$\lambda_1-2\lambda_2+4\lambda_3=0, -2\lambda_1+3\lambda_2-7\lambda_3=0$ and $3\lambda_1-\lambda_2+7\lambda_3=0$

on solving we get, $\lambda_1=-2, \lambda_2=1, \lambda_3=1$

So that $\lambda_1+\lambda_2+\lambda_3=0$

Hence the given vectors are collinear.

---

24. COPLANARITY OF FOUR POINTS :

Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if there exist scalars x, y, z, w not all zero simultaneously such that $x\vec{a}+y\vec{b}+z\vec{c}+w\vec{d}=\vec{0}$, where $x+y+z+w=0$

25. RECIPROCAL SYSTEM OF VECTORS :

If $\vec{a}, \vec{b}, \vec{c}$ & $\vec{a}', \vec{b}', \vec{c}'$ are two sets of non coplanar vectors such that $\vec{a}\cdot\vec{a}'=\vec{b}\cdot\vec{b}'=\vec{c}\cdot\vec{c}'=1$, then the two systems are called Reciprocal System of vectors.

Note : $a'=\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$; $b'=\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$; $c'=\dfrac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

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26. PROPERTIES OF RECIPROCAL SYSTEM OF VECTORS :

(a) $\vec{a}.\vec{b}' = \vec{a}.\vec{c}' = \vec{b}.\vec{c}' = \vec{c}.\vec{a}' = \vec{c}.\vec{b}' = 0$

(b) The scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}]$ formed by three non-coplanar vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ is the reciprocal of the scalar triple product formed from reciprocal system.

Illustration 28 : Find a set of vectors reciprocal to the vectors $\vec{a}, \vec{b}$ and $\vec{a}\times\vec{b}$.

Solution : Let the given vectors be denoted by $\vec{a}, \vec{b}$ and $\vec{c}$ where $\vec{c} = \vec{a}\times\vec{b}$

$\therefore \quad [\vec{a}\,\vec{b}\,\vec{c}] = (\vec{a}\times\vec{b}).\vec{c} = (\vec{a}\times\vec{b}).(\vec{a}\times\vec{b}) = (\vec{a}\times\vec{b})^2$

and let the reciprocal system of vectors be $\vec{a}'\vec{b}'$ and $\vec{c}'$

$$\therefore \quad \vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]} = \frac{\vec{b}\times(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|^2}; \vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]} = \frac{(\vec{a}\times\vec{b})\times\vec{a}}{|\vec{a}\times\vec{b}|^2}; \vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|^2}$$

$\therefore \quad \vec{a}', \vec{b}', \vec{c}'$ are required reciprocal system of vectors for $\vec{a}, \vec{b}$ and $\vec{a}\times\vec{b}$. **Ans.**

Illustration 29 : If $\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \vec{b}' = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \vec{c}' = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, then shown that ; $\vec{a}\times\vec{a}' + \vec{b}\times\vec{b}' + \vec{c}\times\vec{c}' = \vec{0}$

where $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors.

Solution : Here $\vec{a}\times\vec{a}' = \dfrac{\vec{a}\times(\vec{b}\times\vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$$\vec{a}\times\vec{a}' = \frac{(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Similarly $\vec{b}\times\vec{b}' = \dfrac{(\vec{b}.\vec{a})\vec{c} - (\vec{b}.\vec{c})\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ & $\vec{c}\times\vec{c}' = \dfrac{(\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$$\vec{a}\times\vec{a}' + \vec{b}\times\vec{b}' + \vec{c}\times\vec{c}' = \frac{(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} + (\vec{b}.\vec{a})\vec{c} - (\vec{b}.\vec{c})\vec{a} + (\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

$[\because \vec{a}.\vec{b} = \vec{b}.\vec{a}$ etc.$]$

$= \vec{0}$. **Ans.**

Do yourself - 12 :

(i) If $\vec{p} = \dfrac{\vec{b}\times\vec{c}}{[\vec{b}\ \vec{c}\ \vec{a}]}, \vec{q} = \dfrac{\vec{c}\times\vec{a}}{[\vec{c}\ \vec{a}\ \vec{b}]}, \vec{r} = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$,

then find the value of $(\vec{a}+\vec{b}).\vec{p} + (\vec{b}+\vec{c}).\vec{q} + (\vec{c}+\vec{a}).\vec{r}$.

(ii) If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non zero, non coplanar vectors determine whether the vectors $\vec{r}_1 = 2\vec{a} - 3\vec{b} + \vec{c}$, $\vec{r}_2 = 3\vec{a} - 5\vec{b} + 2\vec{c}$ and $\vec{r}_3 = 4\vec{a} - 5\vec{b} + \vec{c}$ are linearly independent or dependent.

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Miscellaneous Illustrations :

Illustration 30 : Let $\vec{u}$ and $\vec{v}$ be unit vectors. If $\vec{w}$ is a vector such that $\vec{w}+(\vec{w}\times\vec{u})=\vec{v}$, then prove that $|(\vec{u}\times\vec{v})\,.\,\vec{w}|\le\frac{1}{2}$ and that the equality holds if and only if $\vec{u}$ is perpendicular to $\vec{v}$.

[JEE 99]

Solution : $\vec{w}+(\vec{w}\times\vec{u})=\vec{v}$ .....(i)

$\Rightarrow$ $\vec{w}\times\vec{u}=\vec{v}-\vec{w}$ $\Rightarrow$ $(\vec{w}\times\vec{u})^2=v^2+w^2-2\vec{v}.\vec{w}$

$\Rightarrow$ $2\vec{v}.\vec{w}=1+w^2-(\vec{u}\times\vec{w})^2$ .....(ii)

also taking dot product of (i) with $\vec{v}$, we get

$\vec{w}.\vec{v}+(\vec{w}\times\vec{u}).\vec{v}=\vec{v}.\vec{v}$

$\Rightarrow$ $\vec{v}.(\vec{w}\times\vec{u})=1-\vec{w}.\vec{v}$ .....(iii) $\{\because \vec{v}.\vec{v}=|\vec{v}|^2=1\}$

Now ; $\vec{v}.(\vec{w}\times\vec{u})=1-\frac{1}{2}(1+w^2-(\vec{u}\times\vec{w})^2)$ (using (ii) and (iii))

$=\frac{1}{2}-\frac{w^2}{2}+\frac{(\vec{u}\times\vec{w})^2}{2}$ $(\therefore\ 0\le\cos^2\theta\le 1)$

$=\frac{1}{2}(1-w^2+w^2\sin^2\theta)$ .....(iv)

as we know ; $0\le w^2\cos^2\theta\le w^2$

$\therefore$ $\frac{1}{2}\ge\frac{1-w^2\cos^2\theta}{2}\ge\frac{1-w^2}{2}$

$\Rightarrow$ $\frac{1-w^2\cos^2\theta}{2}\le\frac{1}{2}$ .....(v)

from (iv) and (v)

$|\vec{v}.(\vec{w}\times\vec{u})|\le\frac{1}{2}$

Equality holds only when $\cos^2\theta=0$ $\Rightarrow$ $\theta=\frac{\pi}{2}$

i.e., $\vec{u}\perp\vec{w}\Rightarrow\vec{u}.\vec{w}=0$ $\Rightarrow$ $\vec{w}+(\vec{w}\times\vec{u})=\vec{v}$

$\Rightarrow$ $\vec{u}\cdot\vec{w}+\vec{u}.(\vec{w}\times\vec{u})=\vec{u}.\vec{v}$ (taking dot with $\vec{u}$)

$\Rightarrow$ $0+0=\vec{u}\cdot\vec{v}$ $\Rightarrow$ $\vec{u}\cdot\vec{v}=0$ $\Rightarrow$ $\vec{u}\perp\vec{v}$

Illustration 31 : A point $A(x_1, y_1)$ with abscissa $x_1 = 1$ and a point $B(x_2, y_2)$ with ordinate $y_2 = 11$ are given in a rectangular cartesian system of co-ordinates OXY on the part of the curve $y = x^2 - 2x + 3$ which lies in the first quadrant. Find the scalar product of $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

Solution : Since (x_1, y_1) and (x_2, y_2) lies on $y = x^2 - 2x + 3$.

$\therefore$ $y_1 = x_1^2 - 2x_1 + 3$

$y_1 = 1^2 - 2(1) + 3$ (as $x_1 = 1$)

$y_1 = 2$

so the co-ordinates of A(1, 2)

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Also, $y_2 = x_2^2 - 2x_2 + 3$

$11 = x_2^2 - 2x_2 + 3 \Rightarrow x_2 = 4,\ x_2 \neq -2$ (as B lie in 1st quadrant)

$\therefore$ co-ordinates of B (4, 11).

Hence, $\overrightarrow{OA} = \hat{i} + 2\hat{j}$ and $\overrightarrow{OB} = 4\hat{i} + 11\hat{j}$

$\Rightarrow \quad \overrightarrow{OA}.\overrightarrow{OB} = 4 + 22 = 26.$

Illustration 32 : If 'a' is real constant and A, B, C are variable angles and $\sqrt{a^2-4}\tan A + a\tan B + \sqrt{a^2+4}\tan C = 6a$, then find the least value of $\tan^2 A + \tan^2 B + \tan^2 C$

Solution : The given relation can be re-written as :

$(\sqrt{a^2-4}\hat{i} + a\hat{j} + \sqrt{a^2+4}\hat{k}).(\tan A\hat{i} + \tan B\hat{j} + \tan C\hat{k}) = 6a$

$\Rightarrow \quad \sqrt{(a^2-4)+a^2+(a^2+4)}.\sqrt{\tan^2 A + \tan^2 B + \tan^2 C}\ .\cos\theta = 6a$

(as $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$)

$\Rightarrow \quad \sqrt{3}\,a\,.\sqrt{\tan^2 A + \tan^2 B + \tan^2 C}\cos\theta = 6a$

$\Rightarrow \quad \tan^2 A + \tan^2 B + \tan^2 C = 12\sec^2\theta$ .....(i)

also, $12\sec^2\theta \geq 12$ (as, $\sec^2\theta \geq 1$) .....(ii)

from (i) and (ii),

$\tan^2 A + \tan^2 B + \tan^2 C \geq 12$

$\therefore$ least value of $\tan^2 A + \tan^2 B + \tan^2 C = 12$.

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Illustration 33 : $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar unit vectors such that angle between any two is α. If $\vec{a}\times\vec{b} + \vec{b}\times\vec{c} = \ell\vec{a} + m\vec{b} + n\vec{c}$, then determine ℓ, m, n in terms of α. **(JEE-1997)**

Solution : $a^2 = b^2 = c^2 = 1$, $[abc] \neq 0$

$\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = \cos\alpha$ ..... (i)

Multiply both sides of given relation scalarly by $\vec{a}$, $\vec{b}$ and $\vec{c}$, we get

$0 + [\vec{a}\ \vec{b}\ \vec{c}] = \ell.1 + (m+n)\cos\alpha$ ..... (ii)

$0 = m + (n+\ell)\cos\alpha$ ..... (iii)

$[\vec{a}\ \vec{b}\ \vec{c}] + 0 = (\ell + m)\cos\alpha + n$ ..... (iv)

Adding, we get

$2[\vec{a}\ \vec{b}\ \vec{c}] = (\ell + m + n) + 2(\ell + m + n)\cos\alpha$

or $\quad 2[\vec{a}\ \vec{b}\ \vec{c}] = (\ell + m + n)(1 + 2\cos\alpha)$ ..... (v)

From (ii), $(m+n) = \dfrac{[\vec{a}\,\vec{b}\,\vec{c}] - \ell}{\cos\alpha}$

Putting in (v), we get $2[\vec{a}\,\vec{b}\,\vec{c}] = \left\{\ell + \dfrac{[\vec{a}\,\vec{b}\,\vec{c}] - \ell}{\cos\alpha}\right\}(1 + 2\cos\alpha)$

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or $\quad [\vec{a}\,\vec{b}\,\vec{c}]\left\{2-\dfrac{1+2\cos\alpha}{\cos\alpha}\right\}=\ell\left(1-\dfrac{1}{\cos\alpha}\right)(1+2\cos\alpha)$

$\therefore \quad \ell=\dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{(1+2\cos\alpha)(1-\cos\alpha)}=n \qquad$ {as above}

and $\quad m=-(n+\ell)\cos\alpha=\dfrac{-2[\vec{a}\,\vec{b}\,\vec{c}]\cos\alpha}{(1+2\cos\alpha)(1-\cos\alpha)}$

Thus the values of ℓ, m, n depend on $[\vec{a}\ \vec{b}\ \vec{c}]$

Hence we now find the value of scalar $[\vec{a}\ \vec{b}\ \vec{c}]$ in terms of α.

Now $[\vec{a}\ \vec{b}\ \vec{c}]^2=\begin{vmatrix}\vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c}\\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c}\\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c}\end{vmatrix}=\begin{vmatrix}1 & \cos\alpha & \cos\alpha\\ \cos\alpha & 1 & \cos\alpha\\ \cos\alpha & \cos\alpha & 1\end{vmatrix}$ (Apply $C_1+C_2+C_3$)

$=(1+2\cos\alpha)\begin{vmatrix}1 & \cos\alpha & \cos\alpha\\ 1 & 1 & \cos\alpha\\ 1 & \cos\alpha & 1\end{vmatrix}$ (Apply R_2-R_1 and R_3-R_1)

$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}]^2=(1+2\cos\alpha)(1-\cos\alpha)^2$

$\therefore \quad \dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{1-\cos\alpha}=\sqrt{1+2\cos\alpha}$

Putting in the value of ℓ, m, n we have $\ell=\dfrac{1}{\sqrt{(1+2\cos\alpha)}}=n, m=\dfrac{-2\cos\alpha}{\sqrt{(1+2\cos\alpha)}}$ **Ans.**

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ANSWERS FOR DO YOURSELF

1 : **(i)** 7 **(ii)** A

3 : **(i)** **(a)** $\vec{a},\vec{d};\vec{b},\ \vec{x},\ \vec{z};\vec{c},\ \vec{y}$ **(b)** $\vec{b},\ \vec{x};\vec{a},\ \vec{d};\vec{c},\ \vec{y}$ **(c)** $\vec{a},\ \vec{y},\ \vec{z}$ **(d)** $\vec{b},\ \vec{z};\ \vec{x},\ \vec{z}$

(ii) $\dfrac{1}{4}$ **(iii)** A,B,C

4 : **(i)** $\dfrac{12\vec{a}-13\vec{b}}{5},-5\vec{b}$ **(iii)** $\dfrac{3}{7}\hat{i}-\dfrac{6}{7}\hat{j}+\dfrac{2}{7}\hat{k}$

6 : **(i)** $\dfrac{\pi}{6}$ **(ii)** – 15 **(iii)** 1 **(iv)** $\dfrac{2}{7},\ \dfrac{2}{49}(3\hat{i}+6\hat{j}+2\hat{k})$ and $\dfrac{190\hat{i}-110\hat{j}+45\hat{k}}{49}$

(v) $-\dfrac{2}{21}\hat{i}+\dfrac{32}{21}\hat{j}-\dfrac{8}{21}\hat{k}$

7 : **(i)** $\vec{r}=\dfrac{2}{3}\left(\hat{i}+\hat{j}-2\hat{k}\right)$ **8 :** **(ii)** $-\hat{i}-\hat{j}+2\hat{k}$ **9 :** **(i)** $\dfrac{1}{\sqrt{6}}$

10 : **(i)** ±1 **(ii)** 0 **(iii)** 7 **(iv)** Right handed system

11 : **(i)** 62, $92\hat{i}+102\hat{j}+32\hat{k}$ **12 :** **(i)** 3 **(ii)** linearly dependent.

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EXERCISE (O-1)

[STRAIGHT OBJECTIVE TYPE]

1. A (1, –1, –3), B (2, 1, –2) & C (–5, 2, –6) are the position vectors of the vertices of a triangle ABC. The length of the bisector of its internal angle at A is :

(A) $\sqrt{10}/4$ (B) $3\sqrt{10}/4$ (C) $\sqrt{10}$ (D) none

VT0001

2. Let $\vec{p}$ is the p.v. of the orthocentre & $\vec{g}$ is the p.v. of the centroid of the triangle ABC where circumcentre is the origin. If $\vec{p} = K\vec{g}$, then K =

(A) 3 (B) 2 (C) 1/3 (D) 2/3

VT0002

3. A vector $\vec{a}$ has components 2p & 1 with respect to a rectangular cartesian system. The system is rotated through a certain angle about the origin in the counterclockwise sense. If with respect to the new system, $\vec{a}$ has components p + 1 & 1 then ,

(A) p = 0 (B) p = 1 or p = – 1/3 (C) p = – 1 or p = 1/3 (D) p = 1 or p = – 1

VT0003

4. The number of vectors of unit length perpendicular to vectors $\vec{a} = (1, 1, 0)$ & $\vec{b}$ (0, 1, 1) is:

(A) 1 (B) 2 (C) 3 (D) ∞

VT0004

5. Four points A(+1, –1, 1) ; B(1, 3, 1) ; C(4, 3, 1) and D(4, – 1, 1) taken in order are the vertices of

(A) a parallelogram which is neither a rectangle nor a rhombus

(B) rhombus

(C) an isosceles trapezium

(D) a cyclic quadrilateral.

VT0005

6. Let α, β & γ be distinct real numbers. The points whose position vector's are $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$; $\beta\hat{i} + \gamma\hat{j} + \alpha\hat{k}$ and $\gamma\hat{i} + \alpha\hat{j} + \beta\hat{k}$

(A) are collinear (B) form an equilateral triangle

(C) form a scalene triangle (D) form a right angled triangle

VT0006

7. If the vectors $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$, $\vec{b} = -\hat{i} + 3\hat{j} + 4\hat{k}$ & $\vec{c} = 4\hat{i} - 2\hat{j} - 6\hat{k}$ constitute the sides of a Δ ABC, then the length of the median bisecting the vector $\vec{c}$ is

(A) $\sqrt{2}$ (B) $\sqrt{14}$ (C) $\sqrt{74}$ (D) $\sqrt{6}$

VT0007

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8. Let A(0, –1, 1), B(0, 0, 1), C(1, 0, 1) are the vertices of a ΔABC. If R and r denotes the circumradius and inradius of ΔABC, then $\frac{r}{R}$ has value equal to

(A) $\tan\frac{3\pi}{8}$ (B) $\cot\frac{3\pi}{8}$ (C) $\tan\frac{\pi}{12}$ (D) $\cot\frac{\pi}{12}$

VT0008

9. $\vec{a}, \vec{b}, \vec{c}$ are three non-zero vectors, no two of which are collinear and the vector $\vec{a}+\vec{b}$ is collinear with $\vec{c}$, $\vec{b}+\vec{c}$ is collinear with $\vec{a}$, then $\vec{a}+\vec{b}+\vec{c}$ is equal to -

(A) $\vec{a}$ (B) $\vec{b}$ (C) $\vec{c}$ (D) none of these

VT0009

10. If the three points with position vectors (1, a, b) ; (a, 2, b) and (a, b, 3) are collinear in space, then the value of a + b is

(A) 3 (B) 4 (C) 5 (D) none

VT0010

11. Consider the following 3 lines in space

$L_1 : \vec{r} = 3\hat{i} - \hat{j} + 2\hat{k} + \lambda(2\hat{i} + 4\hat{j} - \hat{k})$

$L_2 : \vec{r} = \hat{i} + \hat{j} - 3\hat{k} + \mu(4\hat{i} + 2\hat{j} + 4\hat{k})$

$L_3 : \vec{r} = 3\hat{i} + 2\hat{j} - 2\hat{k} + t(2\hat{i} + \hat{j} + 2\hat{k})$

Then which one of the following pair(s) are in the same plane.

(A) only L_1L_2 (B) only L_2L_3 (C) only L_3L_1 (D) L_1L_2 and L_2L_3

VT0011

12. The acute angle between the medians drawn from the acute angles of an isosceles right angled triangle is:

(A) $\cos^{-1}(2/3)$ (B) $\cos^{-1}(3/4)$ (C) $\cos^{-1}(4/5)$ (D) none

VT0012

13. The vectors $3\hat{i} - 2\hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} + 5\hat{k}$ & $2\hat{i} + \hat{j} - 4\hat{k}$ form the sides of a triangle. Then triangle is

(A) an acute angled triangle (B) an obtuse angled triangle

(C) an equilateral triangle (D) a right angled triangle

VT0013

14. If the vectors $3\bar{p} + \bar{q}$; $5\bar{p} - 3\bar{q}$ and $2\bar{p} + \bar{q}$; $4\bar{p} - 2\bar{q}$ are pairs of mutually perpendicular vectors then $\sin(\bar{p} \wedge \bar{q})$ is

(A) $\sqrt{55}/4$ (B) $\sqrt{55}/8$ (C) 3/16 (D) $\sqrt{247}/16$

VT0014

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15. Consider the points A, B and C with position vectors $\left(-2\hat{i}+3\hat{j}+5\hat{k}\right)$, $\left(\hat{i}+2\hat{j}+3\hat{k}\right)$ and $7\hat{i}-\hat{k}$ respectively.

Statement-1: The vector sum, $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{0}$

because

Statement-2: A, B and C form the vertices of a triangle.

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

VT0015

16. The set of values of c for which the angle between the vectors $cx\hat{i}-6\hat{j}+3\hat{k}$ & $x\hat{i}-2\hat{j}+2cx\hat{k}$ is acute for every $x \in R$ is

(A) (0, 4/3) (B) [0, 4/3] (C) (11/9, 4/3) (D) [0, 4/3)

VT0016

17. Let $\vec{u}=\hat{i}+\hat{j}$, $\vec{v}=\hat{i}-\hat{j}$ and $\vec{w}=\hat{i}+2\hat{j}+3\hat{k}$. If $\hat{n}$ is a unit vector such that $\vec{u}\cdot\hat{n}=0$ and $\vec{v}\cdot\hat{n}=0$, then $|\vec{w}\cdot\hat{n}|$ is equal to

(A) 1 (B) 2 (C) 3 (D) 0

VT0017

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18. If the vector $6\hat{i}-3\hat{j}-6\hat{k}$ is decomposed into vectors parallel and perpendicular to the vector $\hat{i}+\hat{j}+\hat{k}$ then the vectors are :

(A) $-\left(\hat{i}+\hat{j}+\hat{k}\right)$ & $7\hat{i}-2\hat{j}-5\hat{k}$ (B) $-2\left(\hat{i}+\hat{j}+\hat{k}\right)$ & $8\hat{i}-\hat{j}-4\hat{k}$

(C) $+2\left(\hat{i}+\hat{j}+\hat{k}\right)$ & $4\hat{i}-5\hat{j}-8\hat{k}$ (D) none

VT0018

19. Let $\vec{r}=\vec{a}+\lambda\vec{l}$ and $\vec{r}=\vec{b}+\mu\vec{m}$ be two lines in space where $\vec{a}=5\hat{i}+\hat{j}+2\hat{k}$, $\vec{b}=-\hat{i}+7\hat{j}+8\hat{k}$, $\vec{l}=-4\hat{i}+\hat{j}-\hat{k}$ and $\vec{m}=2\hat{i}-5\hat{j}-7\hat{k}$ then the p.v. of a point which lies on both of these lines, is

(A) $\hat{i}+2\hat{j}+\hat{k}$ (B) $2\hat{i}+\hat{j}+\hat{k}$

(C) $\hat{i}+\hat{j}+2\hat{k}$ (D) non existent as the lines are skew

VT0019

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20. Let A(1, 2, 3), B(0, 0, 1), C(–1, 1, 1) are the vertices of a ΔABC.

(i) The equation of internal angle bisector through A to side BC is

(A) $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \mu(3\hat{i} + 2\hat{j} + 3\hat{k})$ (B) $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \mu(3\hat{i} + 4\hat{j} + 3\hat{k})$

(C) $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \mu(3\hat{i} + 3\hat{j} + 2\hat{k})$ (D) $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \mu(3\hat{i} + 3\hat{j} + 4\hat{k})$

(ii) The equation of median through C to side AB is

(A) $\vec{r} = -\hat{i} + \hat{j} + \hat{k} + p(3\hat{i} - 2\hat{k})$ (B) $\vec{r} = -\hat{i} + \hat{j} + \hat{k} + p(3\hat{i} + 2\hat{k})$

(C) $\vec{r} = -\hat{i} + \hat{j} + \hat{k} + p(-3\hat{i} + 2\hat{k})$ (D) $\vec{r} = -\hat{i} + \hat{j} + \hat{k} + p(3\hat{i} + 2\hat{j})$

(iii) The area (ΔABC) is equal to

(A) $\frac{9}{2}$ (B) $\frac{\sqrt{17}}{2}$ (C) $\frac{17}{2}$ (D) $\frac{7}{2}$

VT0020

21. If $\vec{a} + \vec{b} + \vec{c} = 0$, $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 7$, then the angle between $\vec{a}$ & $\vec{b}$ is :

(A) $\pi/6$ (B) $2\pi/3$ (C) $5\pi/3$ (D) $\pi/3$

VT0021

22. A line passes through the point $A\left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$ and is parallel to the vector $\vec{V}\left(\hat{i} + \hat{j} + \hat{k}\right)$. The shortest distance from the origin, of the line is -

(A) $\sqrt{2}$ (B) $\sqrt{4}$ (C) $\sqrt{5}$ (D) $\sqrt{6}$

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VT0022

23. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be vectors of length 3, 4, 5 respectively. Let $\vec{a}$ be perpendicular to $\vec{b} + \vec{c}$, $\vec{b}$ to $\vec{c} + \vec{a}$ & $\vec{c}$ to $\vec{a} + \vec{b}$. Then $|\vec{a} + \vec{b} + \vec{c}|$ is :

(A) $2\sqrt{5}$ (B) $2\sqrt{2}$ (C) $10\sqrt{5}$ (D) $5\sqrt{2}$

VT0023

24. The set of values of x for which the angle between the vectors $\vec{a} = x\hat{i} - 3\hat{j} - \hat{k}$ and $\vec{b} = 2x\hat{i} + x\hat{j} - \hat{k}$ acute and the angle between the vector $\vec{b}$ and the axis of ordinates is obtuse, is

(A) $1 < x < 2$ (B) $x > 2$ (C) $x < 1$ (D) $x < 0$

VT0024

25. If a vector $\vec{a}$ of magnitude 50 is collinear with vector $\vec{b} = 6\hat{i} - 8\hat{j} - \frac{15}{2}\hat{k}$ and makes an acute angle with positive z-axis then :

(A) $\vec{a} = 4\vec{b}$ (B) $\vec{a} = -4\vec{b}$ (C) $\vec{b} = 4\vec{a}$ (D) none

VT0025

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26. A, B, C & D are four points in a plane with pv's $\vec{a}, \vec{b}, \vec{c}$ & $\vec{d}$ respectively such that $(\vec{a}-\vec{d})\cdot(\vec{b}-\vec{c})=(\vec{b}-\vec{d})\cdot(\vec{c}-\vec{a}) = 0$. Then for the triangle ABC, D is its

(A) incentre (B) circumcentre (C) orthocentre (D) centroid

VT0026

27. $\vec{a}$ and $\vec{b}$ are unit vectors inclined to each other at an angle α, $\alpha \in (0, \pi)$ and $|\vec{a}+\vec{b}| < 1$. Then $\alpha \in$

(A) $\left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$ (B) $\left(\frac{2\pi}{3}, \pi\right)$ (C) $\left(0, \frac{\pi}{3}\right)$ (D) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

VT0027

28. Image of the point P with position vector $7\hat{i} - \hat{j} + 2\hat{k}$ in the line whose vector equation is, $\vec{r} = 9\hat{i}+5\hat{j}+5\hat{k}+\lambda(\hat{i}+3\hat{j}+5\hat{k})$ has the position vector

(A) $(-9, 5, 2)$ (B) $(9, 5, -2)$ (C) $(9, -5, -2)$ (D) none

VT0028

29. Let $\hat{a}, \hat{b}, \hat{c}$ are three unit vectors such that $\hat{a} + \hat{b} + \hat{c}$ is also a unit vector. If pairwise angles between $\hat{a}, \hat{b}, \hat{c}$ are θ_1, θ_2 and θ_3 respectively then $\cos\theta_1 + \cos\theta_2 + \cos\theta_3$ equals

(A) 3 (B) -3 (C) 1 (D) -1

VT0029

30. A tangent is drawn to the curve $y = \frac{8}{x^2}$ at a point $A(x_1, y_1)$, where $x_1 = 2$. The tangent cuts the x-axis at point B. Then the scalar product of the vectors $\vec{AB}$ & $\vec{OB}$ is

(A) 3 (B) -3 (C) 6 (D) -6

VT0030

31. Cosine of an angle between the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ if $|\vec{a}| = 2$, $|\vec{b}| = 1$ and $\vec{a} \wedge \vec{b} = 60°$ is

(A) $\sqrt{3/7}$ (B) $9/\sqrt{21}$ (C) $3/\sqrt{7}$ (D) none

VT0031

32. An arc AC of a circle subtends a right angle at the centre O. The point B divides the arc in the ratio 1 : 2. If $\vec{OA} = \vec{a}$ & $\vec{OB} = \vec{b}$, then the vector $\vec{OC}$ in terms of $\vec{a}$ & $\vec{b}$, is

(A) $\sqrt{3}\vec{a} - 2\vec{b}$ (B) $-\sqrt{3}\vec{a} + 2\vec{b}$ (C) $2\vec{a} - \sqrt{3}\vec{b}$ (D) $-2\vec{a} + \sqrt{3}\vec{b}$

VT0032

33. Given three vectors $\vec{a}, \vec{b}$ & $\vec{c}$ each two of which are non collinear. Further if $(\vec{a}+\vec{b})$ is collinear with $\vec{c}$, $(\vec{b}+\vec{c})$ is collinear with $\vec{a}$ & $|\vec{a}| = |\vec{b}| = |\vec{c}| = \sqrt{2}$. Then the value of $\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}$:

(A) is 3 (B) is -3 (C) is 0 (D) cannot be evaluated

VT0033

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34. The vector equations of two lines L_1 and L_2 are respectively

$\vec{r}=17\hat{i}-9\hat{j}+9\hat{k}+\lambda(3\hat{i}+\hat{j}+5\hat{k})$ and $\vec{r}=15\hat{i}-8\hat{j}-\hat{k}+\mu(4\hat{i}+3\hat{j})$

I L_1 and L_2 are skew lines

II $(11, -11, -1)$ is the point of intersection of L_1 and L_2

III $(-11, 11, 1)$ is the point of intersection of L_1 and L_2

IV $\cos^{-1}\left(3/\sqrt{35}\right)$ is the acute angle between L_1 and L_2

then , which of the following is true?

(A) II and IV (B) I and IV (C) IV only (D) III and IV

VT0034

35. For two particular vectors $\vec{A}$ and $\vec{B}$ it is known that $\vec{A}\times\vec{B} = \vec{B}\times\vec{A}$. What must be true about the two vectors?

(A) At least one of the two vectors must be the zero vector.

(B) $\vec{A}\times\vec{B} = \vec{B}\times\vec{A}$ is true for any two vectors.

(C) One of the two vectors is a scalar multiple of the other vector.

(D) The two vectors must be perpendicular to each other.

VT0035

36. For some non zero vector $\vec{V}$, if the sum of $\vec{V}$ and the vector obtained from $\vec{V}$ by rotating it by an angle 2α equals to the vector obtained from $\vec{V}$ by rotating it by α then the value of α, is

(A) $2n\pi \pm \frac{\pi}{3}$ (B) $n\pi \pm \frac{\pi}{3}$ (C) $2n\pi \pm \frac{2\pi}{3}$ (D) $n\pi \pm \frac{2\pi}{3}$

where n is an integer.

VT0036

37. Let $\vec{u}, \vec{v}, \vec{w}$ be such that $|\vec{u}|=1$, $|\vec{v}|=2$, $|\vec{w}|=3$. If the projection of $\vec{v}$ along $\vec{u}$ is equal to that of $\vec{w}$ along $\vec{u}$ and vectors $\vec{v}$, $\vec{w}$ are perpendicular to each other then $|\vec{u}-\vec{v}+\vec{w}|$ equals

(A) 2 (B) $\sqrt{7}$ (C) $\sqrt{14}$ (D) 14

VT0037

38. If $\vec{a}$ and $\vec{b}$ are non zero, non collinear, and the linear combination

$(2x-y)\vec{a}+4\vec{b}=5\vec{a}+(x-2y)\vec{b}$ holds for real x and y then x + y has the value equal to

(A) – 3 (B) 1 (C) 17 (D) 3

VT0038

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39. Given an equilateral triangle ABC with side length equal to 'a'. Let M and N be two points respectively on the side AB and AC such that $\overrightarrow{AN} = K\overrightarrow{AC}$ and $\overrightarrow{AM} = \frac{\overrightarrow{AB}}{3}$. If $\overrightarrow{BN}$ and $\overrightarrow{CM}$ are orthogonal then the value of K is equal to

(A) $\frac{1}{5}$ (B) $\frac{1}{4}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$

VT0039

40. If $\vec{p}$ & $\vec{s}$ are not perpendicular to each other and $\vec{r} \times \vec{p} = \vec{q} \times \vec{p}$ & $\vec{r} \cdot \vec{s} = 0$, then $\vec{r} =$

(A) $\vec{p} \cdot \vec{s}$ (B) $\vec{q} + \left(\frac{\vec{q} \cdot \vec{p}}{\vec{p} \cdot \vec{s}}\right)\vec{p}$ (C) $\vec{q} - \left(\frac{\vec{q} \cdot \vec{s}}{\vec{p} \cdot \vec{s}}\right)\vec{p}$ (D) $\vec{q} + \mu\vec{p}$ for all scalars μ

VT0040

41. If $\vec{u}$ and $\vec{v}$ are two vectors such that $|\vec{u}| = 3$; $|\vec{v}| = 2$ and $|\vec{u} \times \vec{v}| = 6$ then the correct statement is

(A) $\vec{u} \wedge \vec{v} \in (0, 90°)$ (B) $\vec{u} \wedge \vec{v} \in (90°, 180°)$ (C) $\vec{u} \wedge \vec{v} = 90°$ (D) $(\vec{u} \times \vec{v}) \times \vec{u} = 6\vec{v}$

VT0041

42. Given a parallelogram OACB. The lengths of the vectors $\vec{OA}$, $\vec{OB}$ & $\vec{AB}$ are a, b & c respectively. The scalar product of the vectors $\vec{OC}$ & $\vec{OB}$ is :

(A) $\frac{a^2 - 3b^2 + c^2}{2}$ (B) $\frac{3a^2 + b^2 - c^2}{2}$ (C) $\frac{3a^2 - b^2 + c^2}{2}$ (D) $\frac{a^2 + 3b^2 - c^2}{2}$

VT0042

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43. Vectors $\vec{a}$ & $\vec{b}$ make an angle $\theta = \frac{2\pi}{3}$. If $|\vec{a}| = 1$, $|\vec{b}| = 2$ then $\left\{\left(\vec{a} + 3\vec{b}\right) \times \left(3\vec{a} - \vec{b}\right)\right\}^2 =$

(A) 225 (B) 250 (C) 275 (D) 300

VT0043

44. If the vector product of a constant vector $\vec{OA}$ with a variable vector $\vec{OB}$ in a fixed plane OAB be a constant vector, then locus of B is :

(A) a straight line perpendicular to $\vec{OA}$ (B) a circle with centre O radius equal to $|\vec{OA}|$

(C) a straight line parallel to $\vec{OA}$ (D) none of these

VT0044

45. For non-zero vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $|\vec{a} \times \vec{b}.\vec{c}| = |\vec{a}||\vec{b}||\vec{c}|$ holds if and only if ;

(A) $\vec{a}.\vec{b} = 0$, $\vec{b}.\vec{c} = 0$ (B) $\vec{c}.\vec{a} = 0$, $\vec{a}.\vec{b} = 0$

(C) $\vec{a}.\vec{c} = 0$, $\vec{b}.\vec{c} = 0$ (D) $\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$

VT0045

46. The vectors $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$; $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ & $\vec{c} = 3\hat{i} + \hat{j} + 4\hat{k}$ are so placed that the end point of one vector is the starting point of the next vector. Then the vectors are -

(A) not coplanar (B) coplanar but cannot form a triangle

(C) coplanar but can form a triangle (D) coplanar & can form a right angled triangle

VT0046

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47. Given the vectors

$$\vec{u} = 2\hat{i} - \hat{j} - \hat{k}$$
$$\vec{v} = \hat{i} - \hat{j} + 2\hat{k}$$
$$\vec{w} = \hat{i} - \hat{k}$$

If the volume of the parallelopiped having $-c\vec{u}$, $\vec{v}$ and $c\vec{w}$ as concurrent edges, is 8 then 'c' can be equal to

(A) ± 2 (B) 4 (C) 8 (D) can not be determined

VT0047

48. Given $\bar{a} = x\hat{i} + y\hat{j} + 2\hat{k}$, $\bar{b} = \hat{i} - \hat{j} + \hat{k}$, $\bar{c} = \hat{i} + 2\hat{j}$; $(\bar{a} \wedge \bar{b}) = \pi/2, \bar{a} \cdot \bar{c} = 4$ then

(A) $[\bar{a}\ \bar{b}\ \bar{c}]^2 = |\bar{a}|$ (B) $[\bar{a}\ \bar{b}\ \bar{c}] = |\bar{a}|$ (C) $[\bar{a}\ \bar{b}\ \bar{c}] = 0$ (D) $[\bar{a}\ \bar{b}\ \bar{c}] = |\bar{a}|^2$

VT0048

49. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$; $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ & $\vec{b}$. If the angle between $\vec{a}$ & $\vec{b}$ is $\frac{\pi}{6}$, then $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 =$

(A) 0

(B) 1

(C) $\frac{1}{4}(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$

(D) $\frac{3}{4}(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)(c_1^2 + c_2^2 + c_3^2)$

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VT0049

50. For three vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ which of the following expressions is not equal to any of the remaining three?

(A) $\vec{u} \cdot (\vec{v} \times \vec{w})$ (B) $(\vec{v} \times \vec{w}) \cdot \vec{u}$ (C) $\vec{v} \cdot (\vec{u} \times \vec{w})$ (D) $(\vec{u} \times \vec{v}) \cdot \vec{w}$

VT0050

51. Let $\vec{a} = \hat{i} + \hat{j}$, $\vec{b} = \hat{j} + \hat{k}$ & $\vec{c} = \alpha\vec{a} + \beta\vec{b}$. If the vectors, $\hat{i} - 2\hat{j} + \hat{k}$, $3\hat{i} + 2\hat{j} - \hat{k}$ & $\vec{c}$ are coplanar then $\frac{\alpha}{\beta}$ is

(A) 1 (B) 2 (C) 3 (D) -3

VT0051

52. A rigid body rotates with constant angular velocity ω about the line whose vector equation is, $\vec{r} = \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$. The speed of the particle at the instant it passes through the point with p.v. $2\hat{i} + 3\hat{j} + 5\hat{k}$ is :

(A) $\omega\sqrt{2}$ (B) 2ω (C) $\omega/\sqrt{2}$ (D) none

VT0052

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53. Given 3 vectors $\vec{V}_1 = a\hat{i} + b\hat{j} + c\hat{k}$; $\vec{V}_2 = b\hat{i} + c\hat{j} + a\hat{k}$; $\vec{V}_3 = c\hat{i} + a\hat{j} + b\hat{k}$

In which one of the following conditions $\vec{V}_1$, $\vec{V}_2$ and $\vec{V}_3$ are linearly independent?

(A) $a + b + c = 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$
(B) $a + b + c = 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$
(C) $a + b + c \neq 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$
(D) $a + b + c \neq 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$

VT0053

54. Given unit vectors $\vec{m}$, $\vec{n}$ & $\vec{p}$ such that angle between $\vec{m}$ & $\vec{n}$ = angle between $\vec{p}$ and $(\vec{m} \times \vec{n}) = \pi/6$, then $[\vec{n}\ \vec{p}\ \vec{m}]$ =

(A) $\sqrt{3}/4$ (B) 3/4 (C) 1/4 (D) none

VT0054

55. Let $\overrightarrow{AB} = 3\hat{i} - \hat{j}$, $\overrightarrow{AC} = 2\hat{i} + 3\hat{j}$ and $\overrightarrow{DE} = 4\hat{i} - 2\hat{j}$. The area of the shaded region in the adjacent figure, is-

(A) 5 (B) 6
(C) 7 (D) 8

VT0055

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56. The altitude of a parallelopiped whose three coterminous edges are the vectors, $\vec{A} = \hat{i} + \hat{j} + \hat{k}$; $\vec{B} = 2\hat{i} + 4\hat{j} - \hat{k}$ & $\vec{C} = \hat{i} + \hat{j} + 3\hat{k}$ with $\vec{A}$ and $\vec{B}$ as the sides of the base of the parallelopiped, is

(A) $2/\sqrt{19}$ (B) $4/\sqrt{19}$ (C) $2\sqrt{38}/19$ (D) none

VT0056

57. Consider Δ ABC with $A \equiv (\bar{a})$; $B \equiv (\bar{b})$ & $C \equiv (\bar{c})$. If $\bar{b}.(\bar{a} + \bar{c}) = \bar{b}.\bar{b} + \bar{a}.\bar{c}$; $|\bar{b} - \bar{a}| = 3$; $|\bar{c} - \bar{b}| = 4$ then the angle between the medians $\overrightarrow{AM}$ & $\overrightarrow{BD}$ is

(A) $\pi - \cos^{-1}\left(\frac{1}{5\sqrt{13}}\right)$ (B) $\pi - \cos^{-1}\left(\frac{1}{13\sqrt{5}}\right)$
(C) $\cos^{-1}\left(\frac{1}{5\sqrt{13}}\right)$ (D) $\cos^{-1}\left(\frac{1}{13\sqrt{5}}\right)$

VT0057

58. If A (– 4, 0, 3) ; B (14, 2, –5) then which one of the following points lie on the bisector of the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ ('O' is the origin of reference)

(A) (2, 1, –1) (B) (2, 11, 5) (C) (10, 2, –2) (D) (1, 1, 2)

VT0058

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59. Position vectors of the four angular points of a tetrahedron ABCD are A(3, – 2, 1); B(3, 1, 5); C(4, 0, 3) and D(1, 0, 0). Acute angle between the plane faces ADC and ABC is

(A) $\tan^{-1}(5/2)$ (B) $\cos^{-1}(2/5)$ (C) $\text{cosec}^{-1}(5/2)$ (D) $\cot^{-1}(3/2)$

VT0059

60. The volume of the tetrahedron formed by the coterminus edges $\vec{a}, \vec{b}, \vec{c}$ is 3. Then the volume of the parallelepiped formed by the coterminus edges $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ is

(A) 6 (B) 18 (C) 36 (D) 9

VT0060

61. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ & $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$, then the vector $\vec{c}$ such that $\vec{a}.\vec{c}=2$ & $\vec{a}\times\vec{c}=\vec{b}$ is -

(A) $\frac{1}{3}\left(3\hat{i}-2\hat{j}+5\hat{k}\right)$ (B) $\frac{1}{3}\left(-\hat{i}+2\hat{j}+5\hat{k}\right)$

(C) $\frac{1}{3}\left(\hat{i}+2\hat{j}-5\hat{k}\right)$ (D) $\frac{1}{3}\left(3\hat{i}+2\hat{j}+\hat{k}\right)$

VT0061

62. $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors having magnitudes 1,1 and 2 respectively. If $\vec{a}\times(\vec{a}\times\vec{c})+\vec{b}=0$, then the acute angle between $\vec{a}$ & $\vec{c}$ is :

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $5\pi/12$

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VT0062

63. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}=4\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{c}=\hat{i}+\alpha\hat{j}+\beta\hat{k}$ are linearly dependent vectors & $|\vec{c}|=\sqrt{3}$, then

(A) $\alpha = 1, \beta = -1$ (B) $\alpha = 1, \beta = \pm 1$ (C) $\alpha = -1, \beta = \pm 1$ (D) $\alpha = \pm 1, \beta = 1$

VT0063

64. A vector of magnitude $5\sqrt{5}$ coplanar with vectors $\hat{i}+2\hat{j}$ & $\hat{j}+2\hat{k}$ and the perpendicular vector $2\hat{i}+\hat{j}+2\hat{k}$ is

(A) $\pm 5\left(5\hat{i}+6\hat{j}-8\hat{k}\right)$ (B) $\pm\sqrt{5}\left(5\hat{i}+6\hat{j}-8\hat{k}\right)$

(C) $\pm 5\sqrt{5}\left(5\hat{i}+6\hat{j}-8\hat{k}\right)$ (D) $\pm\left(5\hat{i}+6\hat{j}-8\hat{k}\right)$

VT0064

65. Let $\vec{\alpha}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{\beta}=\hat{i}+\hat{j}$. If $\vec{\gamma}$ is a unit vector, then the maximum value of $\left[\vec{\alpha}\times\vec{\beta}\ \ \vec{\beta}\times\vec{\gamma}\ \ \vec{\gamma}\times\vec{\alpha}\right]$ is equal to

(A) 2 (B) 3 (C) 4 (D) 9

VT0065

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[MATRIX MATCH TYPE]

66. If A(0, 1, 0), B(0, 0, 0), C(1, 0, 1) are the vertices of a ΔABC. Match the entries of column-I with column-II.

	Column-I		Column-II
(A)	Orthocentre of ΔABC.	(P)	$\frac{\sqrt{2}}{2}$
(B)	Circumcentre of ΔABC.	(Q)	$\frac{\sqrt{3}}{2}$
(C)	Area (ΔABC).	(R)	$\frac{\sqrt{3}}{3}$
(D)	Distance between orthocentre and centroid.	(S)	$\frac{\sqrt{3}}{6}$
(E)	Distance between orthocentre and circumcentre.	(T)	(0, 0, 0)
(F)	Distance between circumcentre and centroid.	(U)	$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$
(G)	Incentre of ΔABC.	(V)	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$
(H)	Centroid of ΔABC	(W)	$\left(\frac{1}{\sqrt{1}+\sqrt{2}+\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{1}+\sqrt{2}+\sqrt{3}}, \frac{1}{\sqrt{1}+\sqrt{2}+\sqrt{3}}\right)$

VT0066

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EXERCISE (O-2)

[STRAIGHT OBJECTIVE TYPE]

1. Given a parallelogram ABCD. If $|\vec{AB}| = a$, $|\vec{AD}| = b$ & $|\vec{AC}| = c$, then $\vec{DB} \cdot \vec{AB}$ has the value

(A) $\frac{3a^2 + b^2 - c^2}{2}$ (B) $\frac{a^2 + 3b^2 - c^2}{2}$ (C) $\frac{a^2 - b^2 + 3c^2}{2}$ (D) none

VT0067

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2. L_1 and L_2 are two lines whose vector equations are

$$L_1 : \vec{r} = \lambda\left[\left(\cos\theta + \sqrt{3}\right)\hat{i} + \left(\sqrt{2}\sin\theta\right)\hat{j} + \left(\cos\theta - \sqrt{3}\right)\hat{k}\right]$$

$$L_2 : \vec{r} = \mu\left(a\hat{i} + b\hat{j} + c\hat{k}\right),$$

where λ and μ are scalars and α is the acute angle between L_1 and L_2.
If the angle 'α' is independent of θ then the value of 'α' is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

VT0068

3. In the isosceles triangle ABC $\left|\vec{AB}\right| = \left|\vec{BC}\right| = 8$, a point E divides AB internally in the ratio 1 : 3, then the cosine of the angle between $\vec{CE}$ & $\vec{CA}$ is (where $\left|\vec{CA}\right| = 12$)

(A) $-\frac{3\sqrt{7}}{8}$ (B) $\frac{3\sqrt{8}}{17}$ (C) $\frac{3\sqrt{7}}{8}$ (D) $\frac{-3\sqrt{8}}{17}$

VT0069

4. If $\vec{p} = 3\vec{a} - 5\vec{b}$; $\vec{q} = 2\vec{a} + \vec{b}$; $\vec{r} = \vec{a} + 4\vec{b}$; $\vec{s} = -\vec{a} + \vec{b}$ are four vectors such that $\sin\left(\vec{p} \wedge \vec{q}\right) = 1$ and $\sin\left(\vec{r} \wedge \vec{s}\right) = 1$ then $\cos\left(\vec{a} \wedge \vec{b}\right)$ is :

(A) $-\frac{19}{5\sqrt{43}}$ (B) 0 (C) 1 (D) $\frac{19}{5\sqrt{43}}$

VT0070

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5. In a quadrilateral ABCD, $\vec{AC}$ is the bisector of the $\left(\vec{AB} \wedge \vec{AD}\right)$ which is $\frac{2\pi}{3}$,

$15\left|\vec{AC}\right| = 3\left|\vec{AB}\right| = 5\left|\vec{AD}\right|$ then $\cos\left(\vec{BA} \wedge \vec{CD}\right)$ is

(A) $-\frac{\sqrt{14}}{7\sqrt{2}}$ (B) $-\frac{\sqrt{21}}{7\sqrt{3}}$ (C) $\frac{2}{\sqrt{7}}$ (D) $\frac{2\sqrt{7}}{14}$

VT0071

6. If the two adjacent sides of two rectangles are represented by the vectors $\vec{p} = 5\vec{a} - 3\vec{b}$; $\vec{q} = -\vec{a} - 2\vec{b}$ and $\vec{r} = -4\vec{a} - \vec{b}$; $\vec{s} = -\vec{a} + \vec{b}$ respectively, then the angle between the vectors $\vec{x} = \frac{1}{3}\left(\vec{p} + \vec{r} + \vec{s}\right)$ and $\vec{y} = \frac{1}{5}\left(\vec{r} + \vec{s}\right)$

(A) is $-\cos^{-1}\left(\frac{19}{5\sqrt{43}}\right)$ (B) is $\cos^{-1}\left(\frac{19}{5\sqrt{43}}\right)$

(C) is $\pi - \cos^{-1}\left(\frac{19}{5\sqrt{43}}\right)$ (D) cannot be evaluated

VT0072

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7. A rigid body rotates about an axis through the origin with an angular velocity $10\sqrt{3}$ radians/sec. If $\vec{\omega}$ points in the direction of $\hat{i}+\hat{j}+\hat{k}$ then the equation to the locus of the points having tangential speed 20 m/sec. is

(A) $x^2+y^2+z^2-xy-yz-zx-1=0$
(B) $x^2+y^2+z^2-2xy-2yz-2zx-1=0$
(C) $x^2+y^2+z^2-xy-yz-zx-2=0$
(D) $x^2+y^2+z^2-2xy-2yz-2zx-2=0$

VT0073

[MULTIPLE OBJECTIVE TYPE]

8. If $\vec{a}, \vec{b}, \vec{c}$ be three non zero vectors satisfying the condition $\vec{a}\times\vec{b}=\vec{c}$ & $\vec{b}\times\vec{c}=\vec{a}$ then which of the following always hold(s) good?

(A) $\vec{a}, \vec{b}, \vec{c}$ are orthogonal in pairs
(B) $\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\left|\vec{b}\right|$
(C) $\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\left|\vec{c}\right|^2$
(D) $\left|\vec{b}\right|=\left|\vec{c}\right|$

VT0074

9. Given the following information about the non zero vectors $\vec{A}, \vec{B}$ and $\vec{C}$

(i) $(\vec{A}\times\vec{B})\times\vec{A}=\vec{0}$ (ii) $\vec{B}\cdot\vec{B}=4$ (iii) $\vec{A}\cdot\vec{B}=-6$ (iv) $\vec{B}\cdot\vec{C}=6$

Which one of the following holds good?

(A) $\vec{A}\times\vec{B}=\vec{0}$
(B) $\vec{A}\cdot(\vec{B}\times\vec{C})=0$
(C) $\vec{A}\cdot\vec{A}=8$
(D) $\vec{A}\cdot\vec{C}=-9$

VT0075

10. If $\vec{A}, \vec{B}, \vec{C}$ and $\vec{D}$ are four non zero vectors in the same plane no two of which are collinear then which of the following hold(s) good?

(A) $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D})=0$
(B) $(\vec{A}\times\vec{C})\cdot(\vec{B}\times\vec{D})\neq 0$
(C) $(\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D})=\vec{0}$
(D) $(\vec{A}\times\vec{C})\times(\vec{B}\times\vec{D})\neq\vec{0}$

VT0076

11. If $\vec{a}, \vec{b}, \vec{c}$ & $\vec{d}$ are the pv's of the points A, B, C & D respectively in three dimensional space & satisfy the relation $3\vec{a}-2\vec{b}+\vec{c}-2\vec{d}=0$, then :

(A) A, B, C & D are coplanar
(B) the line joining the points B & D divides the line joining the point A & C in the ratio 2 : 1.
(C) the line joining the points A & C divides the line joining the points B & D in the ratio 1 : 1
(D) the four vectors $\vec{a}, \vec{b}, \vec{c}$ & $\vec{d}$ are linearly dependent.

VT0077

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12. The vectors $\vec{u} = \begin{bmatrix} 6 \\ -3 \\ 2 \end{bmatrix}$; $\vec{v} = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}$; $\vec{w} = \begin{bmatrix} 3 \\ 2 \\ -6 \end{bmatrix}$

(A) form a left handed system

(B) form a right handed system

(C) are linearly independent

(D) are such that each is perpendicular to the plane containing the other two.

VT0078

13. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-zero, non-collinear vectors such that a vector

$\vec{p} = a\,b\cos\left(2\pi - \left(\vec{a} \wedge \vec{b}\right)\right)\vec{c}$ and a vector $\vec{q} = a\,c\cos\left(\pi - \left(\vec{a} \wedge \vec{c}\right)\right)\vec{b}$ then $\vec{p} + \vec{q}$ is

(A) parallel to $\vec{a}$

(B) perpendicular to $\vec{a}$

(C) coplanar with $\vec{b}$ & $\vec{c}$

(D) coplanar with $\vec{a}$ and $\vec{c}$

VT0079

14. Which of the following statement(s) hold good ?

(A) if $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c} \Rightarrow \vec{b} = \vec{c}$ $(\vec{a} \neq 0)$

(B) if $\vec{a}\times\vec{b} = \vec{a}\times\vec{c} \Rightarrow \vec{b} = \vec{c}$ $(\vec{a} \neq 0)$

(C) if $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$ and $\vec{a}\times\vec{b} = \vec{a}\times\vec{c} \Rightarrow \vec{b} = \vec{c}$ $(\vec{a} \neq 0)$

(D) if $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are non coplanar vectors and $\vec{k}_1 = \dfrac{\vec{v}_2 \times \vec{v}_3}{\vec{v}_1\cdot(\vec{v}_2 \times \vec{v}_3)}$; $\vec{k}_2 = \dfrac{\vec{v}_3 \times \vec{v}_1}{\vec{v}_1\cdot(\vec{v}_2 \times \vec{v}_3)}$

and $\vec{k}_3 = \dfrac{\vec{v}_1 \times \vec{v}_2}{\vec{v}_1\cdot(\vec{v}_2 \times \vec{v}_3)}$ then $\vec{k}_1\cdot(\vec{k}_2 \times \vec{k}_3) = \dfrac{1}{\vec{v}_1\cdot(\vec{v}_2 \times \vec{v}_3)}$

VT0080

15. If the line $\vec{r} = 2\hat{i} - \hat{j} + 3\hat{k} + \lambda(\hat{i} + \hat{j} + \sqrt{2}\,\hat{k})$ makes angles α, β, γ with xy, yz and zx planes respectively then which of the following are not possible?

(A) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$ & $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

(B) $\tan^2\alpha + \tan^2\beta + \tan^2\gamma = 7$ & $\cot^2\alpha + \cot^2\beta + \cot^2\gamma = 5/3$

(C) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 1$ & $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 2$

(D) $\sec^2\alpha + \sec^2\beta + \sec^2\gamma = 10$ & $\text{cosec}^2\alpha + \text{cosec}^2\beta + \text{cosec}^2\gamma = 14/3$

VT0081

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16. If a, b, c are different real numbers and $a\hat{i}+b\hat{j}+c\hat{k}$; $b\hat{i}+c\hat{j}+a\hat{k}$ & $c\hat{i}+a\hat{j}+b\hat{k}$ are position vectors of three non-collinear points A, B & C then :

(A) centroid of triangle ABC is $\frac{a+b+c}{3}\left(\hat{i}+\hat{j}+\hat{k}\right)$

(B) $\hat{i}+\hat{j}+\hat{k}$ is equally inclined to the three vectors

(C) perpendicular from the origin to the plane of triangle ABC meet at centroid

(D) triangle ABC is an equilateral triangle.

VT0082

17. A vector of magnitude 10 along the normal to the curve $3x^2+8xy+2y^2-3=0$ at its point P(1, 0) can be

(A) $6\hat{i}+8\hat{j}$ (B) $-6\hat{i}+8\hat{j}$ (C) $6\hat{i}-8\hat{j}$ (D) $-6\hat{i}-8\hat{j}$

VT0083

18. Let OAB be a regular triangle with side length unity (O being the origin). Also M,N are the points of trisection of AB,M being closer to A and N closer to B. Position vectors of A,B,M and N are $\vec{a},\vec{b},\vec{m}$ and $\vec{n}$ respectively. Which of the following hold(s) good ?

(A) $\vec{m}=x\vec{a}+y\vec{b}\Rightarrow\frac{2}{3}$ and $y=\frac{1}{3}$ (B) $\vec{m}=x\vec{a}+y\vec{b}\Rightarrow x=\frac{5}{6}$ and $y=\frac{1}{6}$

(C) $\vec{m}.\vec{n}$ equals $\frac{13}{18}$ (D) $\vec{m}.\vec{n}$ equals $\frac{15}{18}$

VT0084

19. If $A(\bar{a});B(\bar{b});C(\bar{c})$ and $D(\bar{d})$ are four points such that

$\bar{a}=-2\hat{i}+4\hat{j}+3\hat{k}$; $\bar{b}=2\hat{i}-8\hat{j}$; $\bar{c}=\hat{i}-3\hat{j}+5\hat{k}$; $\bar{d}=4\hat{i}+\hat{j}-7\hat{k}$

d is the shortest distance between the lines AB and CD, then which of the following is True?

(A) d = 0, hence AB and CD intersect (B) $d=\frac{[\overrightarrow{AB}\ \overrightarrow{CD}\ \overrightarrow{BD}]}{|\overrightarrow{AB}\times\overrightarrow{CD}|}$

(C) AB and CD are skew lines and $d=\frac{23}{13}$ (D) $d=\frac{[\overrightarrow{AB}\ \overrightarrow{CD}\ \overrightarrow{AC}]}{|\overrightarrow{AB}\times\overrightarrow{CD}|}$

VT0085

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20. Which of the following statement(s) is(are) incorrect ?

(A) The relation $|(\vec{u}\times\vec{v})|=|\vec{u}.\vec{v}|$ is only possible if atleast one of the vectors $\vec{u}$ and $\vec{v}$ is null vector.

(B) Every vector contained in the line $\vec{r}(t)=\langle 1+2t, 1+3t, 1+4t\rangle$ is parallel to the vector $\langle 1,1,1\rangle$.

(C) If scalar triple product of three vectors, $\vec{u},\vec{v},\vec{w}$ is larger than $|\vec{u}\times\vec{v}|$ then $|\vec{w}|>1$.

(D) The distance between the x-axis and the line x = y = 1 is $\sqrt{2}$.

VT0086

21. Given three vectors $\vec{U}=2\hat{i}+3\hat{j}-6\hat{k}$; $\vec{V}=6\hat{i}+2\hat{j}+3\hat{k}$; $\vec{W}=3\hat{i}-6\hat{j}-2\hat{k}$

Which of the following hold good for the vectors $\vec{U}$, $\vec{V}$ and $\vec{W}$?

(A) $\vec{U}$, $\vec{V}$ and $\vec{W}$ are linearly dependent

(B) $(\vec{U}\times\vec{V})\times\vec{W}=\vec{0}$

(C) $\vec{U}$, $\vec{V}$ and $\vec{W}$ form a triplet of mutually perpendicular vectors

(D) $\vec{U}\times(\vec{V}\times\vec{W})=\vec{0}$

VT0087

22. Which of the following statement(s) is/are true in respect of the lines

$\vec{r}=\vec{a}+\lambda\vec{b}$; $\vec{r}=\vec{c}+\mu d$ where $\vec{b}\times\vec{d}\neq 0$

(A) acute angle between the lines is $\cos^{-1}\left(\dfrac{|\vec{b}.\vec{d}|}{|\vec{b}||\vec{d}|}\right)$

(B) The lines would intersect if $[\vec{c}\ \vec{b}\ \vec{d}]=[\vec{a}\ \vec{b}\ \vec{d}]$

(C) The lines will be skew if $[\vec{c}-\vec{a}\ \vec{b}\ \vec{d}]\neq 0$

(D) If the lines intersect at $\vec{r}=\vec{r}_0$, then the equation of the plane containing the lines is $[\vec{r}-\vec{r}_0\ \vec{b}\ \vec{d}]=0$

VT0088

23. Let $\vec{a}$ and $\vec{b}$ be two non-zero and non-collinear vectors then which of the following is/are always correct?

(A) $\vec{a}\times\vec{b}=[\vec{a}\ \vec{b}\ \hat{i}]\hat{i}+[\vec{a}\ \vec{b}\ \hat{j}]\hat{j}+[\vec{a}\ \vec{b}\ \vec{k}]\hat{k}$

(B) $\vec{a}.\vec{b}=(\vec{a}.\hat{i})(\vec{b}.\hat{i})+(\vec{a}.\hat{j}).(\vec{b}.\hat{j})+(\vec{a}.\hat{k})(\vec{b}.\hat{k})$

(C) if $\vec{u}=\hat{a}-(\hat{a}.\hat{b})\hat{b}$ and $\vec{v}=\hat{a}\times\hat{b}$ then $|\vec{u}|=|\vec{v}|$

(D) if $\vec{c}=\vec{a}\times(\vec{a}\times\vec{b})$ and $\vec{d}=\vec{b}\times(\vec{a}\times\vec{b})$ then $\vec{c}+\vec{d}=\vec{0}$

VT0089

[COMPREHENSION TYPE]

Paragraph for questions nos. 24 to 26

Consider three vectors $\vec{p}=\hat{i}+\hat{j}+\hat{k}$, $\vec{q}=2\hat{i}+4\hat{j}-\hat{k}$ and $\vec{r}=\hat{i}+\hat{j}+3\hat{k}$ and let $\vec{s}$ be a unit vector, then

24. $\vec{p}$, $\vec{q}$ and $\vec{r}$ are

(A) linearly dependent

(B) can form the sides of a possible triangle

(C) such that the vectors $(\vec{q}-\vec{r})$ is orthogonal to $\vec{p}$

(D) such that each one of these can be expressed as a linear combination of the other two

VT0090

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25. If $(\vec{p}\times\vec{q}) \times \vec{r} = u\vec{p}+v\vec{q}+w\vec{r}$, then (u + v + w) equals to

(A) 8 (B) 2 (C) – 2 (D) 4

VT0090

26. the magnitude of the vector $(\vec{p}\cdot\vec{s})(\vec{q}\times\vec{r}) + (\vec{q}\cdot\vec{s})(\vec{r}\times\vec{p}) + (\vec{r}\cdot\vec{s})(\vec{p}\times\vec{q})$ is

(A) 4 (B) 8 (C) 18 (D) 2

VT0090

[MATRIX MATCH TYPE]

27.

	Column-I		Column-II
(A)	P is point in the plane of the triangle ABC. pv's of A,B and C are $\vec{a},\vec{b}$ and $\vec{c}$ respectively with respect to P as the origin. If $(\vec{b}+\vec{c}).(\vec{b}-\vec{c})=0$ and $(\vec{c}+\vec{a}).(\vec{c}-\vec{a})=0$, then w.r.t. the triangle ABC,P is its	(P)	centroid
		(Q)	orthocentre
(B)	If $\vec{a},\vec{b},\vec{c}$ are the position vectors of the three non collinear points A,B and C respectively such that the vector $\vec{V}=\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}$ is a null vector then w.r.t. the ΔABC, P is its	(R)	Incentre
(C)	If P is a point inside the ΔABC such that the vector $\vec{R}=(BC)\overrightarrow{PA}+(CA)\left(\overrightarrow{PB}\right)+(AB)\left(\overrightarrow{PC}\right)$ is a null vector then w.r.t. the ΔABC, P is its	(S)	circumcentre
(D)	If P is a point in the plane of the triangle ABC such that the scalar product $\overrightarrow{PA}.\overrightarrow{CB}$ and $\overrightarrow{PB}.\overrightarrow{AC}$ vanishes, then w.r.t. the ΔABC, P is its		

VT0091

VT0092

VT0093

VT0094

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EXERCISE (S-1)

1. Given the vector $\overrightarrow{PQ} = -6\mathbf{i} - 4\mathbf{j}$ and **Q** is the point (3, 3), find the point **P**.

VT0095

2. Find the unit vector (in xy plane) obtained by rotating **j** counterclockwise 3π/4 radian about the origin.

VT0096

3. Show that the vector **v** = a**i** + b**j** is perpendicular to the line ax + by = c.

VT0097

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4. In ΔABC, a point P is chosen on side $\overrightarrow{AB}$ so that AP : PB = 1 : 4 and a point Q is chosen on the side $\overrightarrow{BC}$ so that CQ : QB = 1:3. Segment $\overrightarrow{CP}$ and $\overrightarrow{AQ}$ intersect at M. If the ratio $\frac{MC}{PC}$ is expressed as a rational numbers in the lowest term as $\frac{a}{b}$, find (a + b).

VT0098

5. Let O be an interior point of ΔABC such that $2\overrightarrow{OA}+5\overrightarrow{OB}+10\overrightarrow{OC}=\vec{0}$. If the ratio of the area of ΔABC to the area of ΔAOC is t, where 'O' is the origin. Find [t].

(where [ ] denotes greatest integer function)

VT0099

6. If the distance from the point P(1, 1, 1) to the line passing through the points Q(0, 6, 8) and R(–1, 4, 7) is expressed in the form $\sqrt{p/q}$ where p and q are coprime, then the value of $\frac{(p+q)(p+q-1)}{2}$.

VT0100

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7. Let S(t) be the area of the ΔOAB with O(0,0,0), A(2,2,1) and B(t,1,t+1).

The value of the definite integral $\int_1^e (S(t))^2 \ln t\, dt$, is equal to $\left(\frac{e^3+a}{b}\right)$ where a,b $\in$N, find (a + b).

VT0101

8. Given $f^2(x) + g^2(x) + h^2(x) \le 9$ and U(x) = 3f(x) + 4g(x) + 10h(x), where f(x),g(x) and h(x) are continuous $\forall$ x $\in$ R. If maximum value of U(x) is $\sqrt{N}$, then find N.

VT0102

9. If $\vec{a}$ & $\vec{b}$ are non collinear vectors such that $\vec{p}=(x+4y)\vec{a}+(2x+y+1)\vec{b}$ & $\vec{q}=(y-2x+2)\vec{a}+(2x-3y-1)\vec{b}$, find x & y such that $3\vec{p}=2\vec{q}$.

VT0103

10. (a) Show that the points $\vec{a}-2\vec{b}+3\vec{c};2\vec{a}+3\vec{b}-4\vec{c}$ & $-7\vec{b}+10\vec{c}$ are collinear.

(b) Prove that the points A (1, 2, 3), B(3, 4, 7), C(–3, –2, –5) are collinear & find the ratio in which B divides AC.

VT0104

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11. Find out whether the following pairs of lines are parallel, non-parallel & intersecting, or non-parallel non-intersecting.

(a) $\vec{r}_1 = \hat{i} + \hat{j} + 2\hat{k} + \lambda(3\hat{i} - 2\hat{j} + 4\hat{k})$
$\vec{r}_2 = 2\hat{i} + \hat{j} + 3\hat{k} + \mu(-6\hat{i} + 4\hat{j} - 8\hat{k})$

(b) $\vec{r}_1 = \hat{i} - \hat{j} + 3\hat{k} + \lambda(\hat{i} - \hat{j} + \hat{k})$
$\vec{r}_2 = 2\hat{i} + 4\hat{j} + 6\hat{k} + \mu(2\hat{i} + \hat{j} + 3\hat{k})$

(c) $\vec{r}_1 = \hat{i} + \hat{k} + \lambda(\hat{i} + 3\hat{j} + 4\hat{k})$
$\vec{r}_2 = 2\hat{i} + 3\hat{j} + \mu(4\hat{i} - \hat{j} + \hat{k})$

VT0105

12. If $\vec{r}$ and $\vec{s}$ are non zero constant vectors and the scalar b is chosen such that $|\vec{r} + b\vec{s}|$ is minimum, then show that the value of $|b\vec{s}|^2 + |\vec{r} + b\vec{s}|^2$ is equal to $|\vec{r}|^2$.

VT0106

13. In a unit cube. Find

(a) The angle between the diagonal of the cube and a diagonal of a face skew to it.

(b) The angle between the diagonals of two faces of the cube through the same vertex.

(c) The angle between a diagonal of a cube and a diagonal of a face intersecting it.

VT0107

Instruction for question nos. 14 to 16 :

Suppose the three vectors $\vec{a}, \vec{b}, \vec{c}$ on a plane satisfy the condition that

$|\vec{a}| = |\vec{b}| = |\vec{c}| = |\vec{a} + \vec{b}| = 1$; $\vec{c}$ is perpendicular to $\vec{a}$ and $\vec{b}.\vec{c} > 0$, then

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14. Find the angle formed by $2\vec{a} + \vec{b}$ and $\vec{b}$.

VT0108

15. If the vector $\vec{c}$ is expressed as a linear combination $\lambda\vec{a} + \mu\vec{b}$ then find the ordered pair (λ, μ).

VT0109

16. For real numbers x,y the vector $\vec{p} = x\vec{a} + y\vec{c}$ satisfies the condition $0 \le \vec{p}.\vec{a} \le 1$ and $0 \le \vec{p}.\vec{b} \le 1$. Find the maximum value of $\vec{p}.\vec{c}$.

VT0110

17. **(a)** Find the minimum area of the triangle whose vertices are A(–1,1,2); B(1,2,3) and C(t,1,1) where t is a real number.

VT0111

(b) Let $\overrightarrow{OA} = \vec{a}; \overrightarrow{OB} = 100\vec{a} + 2\vec{b}$ and $\overrightarrow{OC} = \vec{b}$ where O,A and C are non collinear points. Let P denotes the area of the parallelogram with $\overrightarrow{OA}$ and $\overrightarrow{OC}$ as adjacent sides and Q denotes the area of the quadrilateral OABC. If Q = λP. Find the value of λ.

VT0112

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18. Given that $\vec{a}$ and $\vec{b}$ are two unit vectors such that angle between $\vec{a}$ and $\vec{b}$ is $\cos^{-1}\left(\frac{1}{4}\right)$. If $\vec{c}$ be a vector in the plane of $\vec{a}$ and $\vec{b}$, such that $|\vec{c}| = 4, \vec{c}\times\vec{b} = 2\vec{a}\times\vec{b}$ and $\vec{c} = \lambda\vec{a} + \mu\vec{b}$ then, find

(a) the value of λ, **(b)** the sum of values of μ and **(c)** the product of all possible values of μ.

VT0113

19. Let $\vec{A} = \hat{i} - 2\hat{j} + 3\hat{k}, \vec{B} = 2\hat{i} + \hat{j} - \hat{k}, \vec{C} = \hat{j} + \hat{k}$.

If the vector $\vec{B}\times\vec{C}$ can be expressed as a linear combination $\vec{B}\times\vec{C} = x\vec{A} + y\vec{B} + z\vec{C}$ where x,y,z are scalars, then find the value of (100x + 10y + 8z).

VT0114

20. The base vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ are given in terms of base vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3$ as $\vec{a}_1 = 2\vec{b}_1 + 3\vec{b}_2 - \vec{b}_3$; $\vec{a}_2 = \vec{b}_1 - 2\vec{b}_2 + 2\vec{b}_3$ & $\vec{a}_3 = -2\vec{b}_1 + \vec{b}_2 - 2\vec{b}_3$. If $\vec{F} = 3\vec{b}_1 - \vec{b}_2 + 2\vec{b}_3$, then express $\vec{F}$ in terms of $\vec{a}_1, \vec{a}_2$ & $\vec{a}_3$.

VT0115

21. The vector $\overrightarrow{OP} = \hat{i} + 2\hat{j} + 2\hat{k}$ turns through a right angle, passing through the positive x-axis on the way. Find the vector in its new position.

VT0116

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22. The pv's of the four angular points of a tetrahedron are $A(\hat{j} + 2\hat{k}); B(3\hat{i} + \hat{k}); C(4\hat{i} + 3\hat{j} + 6\hat{k})$ & $D(2\hat{i} + 3\hat{j} + 2\hat{k})$. Find :

(i) the perpendicular distance from A to the line BC.

VT0117

(ii) the volume of the tetrahedron ABCD.

VT0118

(iii) the perpendicular distance from D to the plane ABC.

VT0119

(iv) the shortest distance between the lines AB & CD.

VT0120

23. Let a 3 dimensional vector $\vec{V}$ satisfies the condition $2\vec{V} + \vec{V}\times(\hat{i} + 2\hat{j}) = 2\hat{i} + \hat{k}$.

If $3|\vec{V}| = \sqrt{m}$, where $m \in N$, then find m.

VT0121

24. If $\vec{x}, \vec{y}$ are two non-zero and non-collinear vectors satisfying $[(a-2)\alpha^2 + (b-3)\alpha + c]\,\vec{x} + [(a-2)\beta^2 + (b-3)\beta + c]\,\vec{y} + [(a-2)\gamma^2 + (b-3)\gamma + c]\,(\vec{x}\times\vec{y}) = 0$ where α, β, γ are three distinct real numbers, then find the value of $(a^2 + b^2 + c^2)$.

VT0122

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25. Solve the simultaneous vector equations for the vectors $\vec{x}$ and $\vec{y}$.

$\vec{x}+\vec{c}\times\vec{y}=\vec{a}$ and $\vec{y}+\vec{c}\times\vec{x}=\vec{b}$ where $\vec{c}$ is a non zero vector.

VT0123

26. Vector $\vec{V}$ is perpendicular to the plane of vectors $\vec{a}=2\hat{i}-3\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+3\hat{k}$ and satisfies the condition $\vec{V}.(\hat{i}+2\hat{j}-7\hat{k})=10$. Find $|\vec{V}|^2$.

VT0124

27. Let two non-collinear vectors $\vec{a}$ and $\vec{b}$ inclined at an angle $\frac{2\pi}{3}$ be such that $|\vec{a}|=3$ and $|\vec{b}|=4$. A point P moves so that at any time t the position vector $\overrightarrow{OP}$ (where O is the origin) is given as $\overrightarrow{OP}=(e^t+e^{-t})\vec{a}+(e^t-e^{-t})\vec{b}$. If the least distance of P from origin is $\sqrt{2}\sqrt{\sqrt{p}-q}$ where p,q $\in$ N then find the value of (p + q).

VT0125

EXERCISE (S-2)

1. Given a tetrahedron D-ABC with AB = 12 , CD = 6. If the shortest distance between the skew lines AB and CD is 8 and the angle between them is $\frac{\pi}{6}$, then find the volume of tetrahedron.

VT0126

2. A vector $\vec{V}=v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$ satisfies the following conditions :

(i) magnitude of $\vec{V}$ is $7\sqrt{2}$

(ii) $\vec{V}$ is parallel to the plane x – 2y + z = 6

(iii) $\vec{V}$ is orthogonal to the vector $2\hat{i}-3\hat{j}+6\hat{k}$ and (iv) $\vec{V}.\hat{i}>0$

Find the value of $(v_1+v_2+v_3)$.

VT0127

3. Let $(\vec{p}\times\vec{q})\times\vec{r}+(\vec{q}.\vec{r})\vec{q}=(x^2+y^2)\vec{q}+(14-4x-6y)\vec{p}$ and $(\vec{r}.\vec{r})\vec{p}=\vec{r}$ where $\vec{p}$ and $\vec{q}$ are two non-zero non-collinear vectors and x and y are scalars. Find the value of (x + y).

VT0128

4. In a ΔABC, points E and F divide sides AC and AB respectively so that $\frac{AE}{EC}=4$ and $\frac{AF}{FB}=1$. Suppose D is a point on side BC. Let G be the intersection of EF and AD and suppose D is situated so that $\frac{AG}{GD}=\frac{3}{2}$. If the ratio $\frac{BD}{DC}=\frac{a}{b}$, where a and b are in their lowest form, find the value of (a + b).

VT0129

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5. Let $\vec{u}$ be a vector on rectangular coordinate system with sloping angle 60°. Suppose that $|\vec{u}-\hat{i}|$ is geometric mean of $|\vec{u}|$ and $|\vec{u}-2\hat{i}|$ where $\hat{i}$ is the unit vector along x-axis then find the value of $|\vec{u}|$.

VT0130

6. $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ are the position vectors of the points A ≡ (x, y, z); B ≡ (y, –2z, 3x) ; C ≡ (2z, 3x, –y) and D ≡ (1, –1, 2) respectively. If $|\vec{a}|=2\sqrt{3};(\vec{a}\wedge\vec{b})=(\vec{a}\wedge\vec{c});(\vec{a}\wedge\vec{d})=\frac{\pi}{2}$ and $(\vec{a}\wedge\hat{j})$ is obtuse, then find x, y, z.

VT0131

7. The length of the edge of the regular tetrahedron D–ABC is 'a'. Point E and F are taken on the edges AD and BD respectively such that E divides $\overrightarrow{DA}$ and F divides $\overrightarrow{BD}$ in the ratio 2 : 1 each. Then find the area of triangle CEF.

VT0132

8. The position vectors of the points A, B, C are respectively (1, 1, 1) ; (1, –1, 2) ; (0, 2, –1). Find a unit vector parallel to the plane determined by ABC & perpendicular to the vector (1, 0, 1).

VT0133

9. The position vectors of the vertices A,B and C of a tetrahedron are (1,1,1), (1,0,0) and (3,0,0) respectively. The altitude from the vertex D to the opposite face ABC meets the median line through A of the triangle ABC at a point E. If the length of side AD is 4 and volume of the tetrahedron is $2\sqrt{2}/3$ then find the all possible position vectors of the point E.

VT0134

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10. Given non zero number x_1,x_2,x_3 ;y_1,y_2,y_3 and z_1,z_2 and z_3

(i) Can the given numbers satisfy

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}=0 \text{ and } \begin{cases} x_1x_2+y_1y_2+z_1z_2=0 \\ x_2x_3+y_2y_3+z_2z_3=0 \\ x_3x_1+y_3y_1+z_3z_1=0 \end{cases}$$

(ii) If $x_i>0$ and $y_i<0$ for all i = 1,2,3 and P (x_1,x_2,x_3); Q(y_1,y_2,y_3) and O(0,0,0) can the triangle POQ be a right angled triangle ?

VT0135

11. Given that $\vec{a},\vec{b},\vec{p},\vec{q}$ are four vectors such that $\vec{a}+\vec{b}=\mu\vec{p},\vec{b}.\vec{q}=0$ & $(\vec{b})^2=1$, where μ is a scalar then prove that $|(\vec{a}.\vec{q})\vec{p}-(\vec{p}.\vec{q})\vec{a}|=|\vec{p}.\vec{q}|$.

VT0136

12. Let $g(\theta)=\int_{-(\hat{a}.\hat{b})^2}^{|\hat{a}\times\hat{b}|^2}(2t+1)dt$, where θ is the angle between $\hat{a}$ and $\hat{b}$. If volume of the parallelopiped whose coterminous edges are represented by vectors $\hat{a},\hat{a}\times\hat{b}$ and $\hat{a}\times(\hat{a}\times\hat{b})$ (where angle between $\hat{a}$ and $\hat{b}$ is taken from the equation 2g(θ) – 1 = 0), is $\frac{p}{q}$ then find the least value of (p + q).

VT0137

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13. (a) Find a unit vector $\hat{a}$ which makes an angle $(\pi/4)$ with axis of z & is such that $\hat{a}+\hat{i}+\hat{j}$ is a unit vector.

VT0138

(c) If $\vec{a}$ and $\vec{b}$ are any two unit vectors, then find the range of $\frac{3|\vec{a}+\vec{b}|}{2}+2|\vec{a}-\vec{b}|$.

VT0139

14. Given four non zero vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$. The vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar but not collinear pair by pair and vector $\vec{d}$ is not coplanar with vectors $\vec{a}, \vec{b}$ and $\vec{c}$ and

$(\vec{a}\,\hat{\wedge}\,\vec{b})=(\vec{b}\,\hat{\wedge}\,\vec{c})=\frac{\pi}{3},(\vec{d}\,\hat{\wedge}\,\vec{a})=\alpha,(\vec{d}\,\hat{\wedge}\,\vec{b})=\beta$, then prove that $(\vec{d}\,\hat{\wedge}\,\vec{c})=\cos^{-1}(\cos\beta-\cos\alpha)$

VT0140

15. Given three points on the xy plane on O(0, 0), A(1, 0) and B(–1, 0). Point P is moving on the plane satisfying the condition $(\overrightarrow{PA}.\overrightarrow{PB})+3(\overrightarrow{OA}.\overrightarrow{OB})=0$. If the maximum and minimum values of $|\overrightarrow{PA}||\overrightarrow{PB}|$ are M and m respectively then find the values of M^2+m^2.

VT0141

16. Let $\vec{a},\vec{b},\vec{c}$ are unit vectors where $|\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|^2+|\vec{c}+\vec{a}|^2=3$, then $|\vec{a}+2\vec{b}+3\vec{c}|^2$ is equal to

VT0142

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EXERCISE (JM)

1. If $\vec{u}, \vec{v}, \vec{w}$ are non-coplanar vectors and p, q are real numbers, then the equality **[AIEEE-2009]**
$[3\vec{u}\ p\vec{v}\ p\vec{w}] - [p\vec{v}\ \vec{w}\ q\vec{u}] - [2\vec{w}\ q\vec{v}\ q\vec{u}] = 0$ holds for :-

(1) More than two but not all values of (p,q) (2) All values of (p, q)
(3) Exactly one value of (p, q) (4) Exactly two values of (p, q)

VT0143

2. Let $\vec{a}=\hat{j}-\hat{k}$ and $\vec{c}=\hat{i}-\hat{j}-\hat{k}$. Then the vector $\vec{b}$ satisfying $\vec{a}\times\vec{b}+\vec{c}=\vec{0}$ and $\vec{a}\,.\,\vec{b}=3$ is : **[AIEEE-2010]**

(1) $-\hat{i}+\hat{j}-2\hat{k}$ (2) $2\hat{i}-\hat{j}+2\hat{k}$ (3) $\hat{i}-\hat{j}-2\hat{k}$ (4) $\hat{i}+\hat{j}-2\hat{k}$

VT0144

3. The vectors $\vec{a}$ and $\vec{b}$ are not perpendicular and $\vec{c}$ and $\vec{d}$ are two vectors satisfying : $\vec{b}\times\vec{c}=\vec{b}\times\vec{d}$ and $\vec{a}.\vec{d}=0$. Then the vector $\vec{d}$ is equal to :- **[AIEEE-2011]**

(1) $\vec{b}+\left(\frac{\vec{b}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{c}$ (2) $\vec{c}-\left(\frac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{b}$ (3) $\vec{b}-\left(\frac{\vec{b}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{c}$ (4) $\vec{c}+\left(\frac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{b}$

VT0145

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4. If $\vec{a}=\frac{1}{\sqrt{10}}(3\hat{i}+\hat{k})$ and $\vec{b}=\frac{1}{7}(2\hat{i}+3\hat{j}-6\hat{k})$, then the value of $(2\vec{a}-\vec{b}).[(\vec{a}\times\vec{b})\times(\vec{a}+2\vec{b})]$ is :- **[AIEEE-2011]**

(1) 5 (2) 3 (3) – 5 (4) – 3

VT0146

5. Let $\vec{a},\vec{b},\vec{c}$ be three non-zero vectors which are pairwise non-collinear. If $\vec{a}+3\vec{b}$ is collinear with $\vec{c}$ and $\vec{b}+2\vec{c}$ is colliner with $\vec{a}$, then $\vec{a}+3\vec{b}+6\vec{c}$ is : **[AIEEE-2011]**

(1) $\vec{a}+\vec{c}$ (2) $\vec{a}$ (3) $\vec{c}$ (4) $\vec{0}$

VT0147

6. Let $\hat{a}$ and $\hat{b}$ be two unit vectors. If the vectors $\vec{c}=\hat{a}+2\hat{b}$ and $\vec{d}=5\hat{a}-4\hat{b}$ are perpendicular to each other, then the angle between $\hat{a}$ and $\hat{b}$ is : **[AIEEE-2012]**

(1) $\frac{\pi}{4}$ (2) $\frac{\pi}{6}$ (3) $\frac{\pi}{2}$ (4) $\frac{\pi}{3}$

VT0148

7. Let ABCD be a parallelogram such that $\overrightarrow{AB}=\vec{q}$, $\overrightarrow{AD}=\vec{p}$ and $\angle BAD$ be an acute angle. If $\vec{r}$ is the vector that coincides with the altitude directed from the vertex B to the side AD, then $\vec{r}$ is given by : **[AIEEE-2012]**

(1) $\vec{r}=-3\vec{q}+\frac{3(\vec{p}\cdot\vec{q})}{(\vec{p}\cdot\vec{p})}\vec{p}$ (2) $\vec{r}=3\vec{q}-\frac{3(\vec{p}\cdot\vec{q})}{(\vec{p}\cdot\vec{p})}\vec{p}$

(3) $\vec{r}=-\vec{q}+\left(\frac{\vec{p}\cdot\vec{q}}{\vec{p}\cdot\vec{p}}\right)\vec{p}$ (4) $\vec{r}=\vec{q}-\left(\frac{\vec{p}\cdot\vec{q}}{\vec{p}\cdot\vec{p}}\right)\vec{p}$

VT0149

8. If the vectors $\overrightarrow{AB}=3\hat{i}+4\hat{k}$ and $\overrightarrow{AC}=5\hat{i}-2\hat{j}+4\hat{k}$ are the sides of a triangle ABC, then the length of the median through A is : **[JEE-MAINS 2013]**

(1) $\sqrt{18}$ (2) $\sqrt{72}$ (3) $\sqrt{33}$ (4) $\sqrt{45}$

VT0150

9. Let $\vec{a}=2\hat{i}-\hat{j}+\hat{k}, \vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}-2\hat{k}$ be three vectors. A vectors of the type $\vec{b}+\lambda\vec{c}$ for some scalar λ, whose projection on $\vec{a}$ is of magnitude $\sqrt{\frac{2}{3}}$, is : **[JEE-MAINS Online 2013]**

(1) $2\hat{i}+3\hat{j}-3\hat{k}$ (2) $2\hat{i}+\hat{j}+5\hat{k}$ (3) $2\hat{i}-\hat{j}+5\hat{k}$ (4) $2\hat{i}+3\hat{j}+3\hat{k}$

VT0151

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10. Let $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$, $\vec{b} = \hat{i} + \hat{j}$. If $\vec{c}$ is a vector such that $\vec{a} \bullet \vec{c} = |\vec{c}|$, $|\vec{c} - \vec{a}| = 2\sqrt{2}$ and the angle between $\vec{a} \times \vec{b}$ and $\vec{c}$ is 30º, then $\left|(\vec{a} \times \vec{b}) \times \vec{c}\right|$ equals : **[JEE-MAINS Online 2013]**

(1) $\frac{3}{2}$ (2) 3 (3) $\frac{1}{2}$ (4) $\frac{3\sqrt{3}}{2}$

VT0152

11. If $\left[\vec{a} \times \vec{b}\ \vec{b} \times \vec{c}\ \vec{c} \times \vec{a}\right] = \lambda\left[\vec{a}\,\vec{b}\,\vec{c}\right]^2$ then λ is equal to : **[JEE(Main)-2014]**

(1) 2 (2) 3 (3) 0 (4) 1

VT0153

12. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non-zero vectors such that no two of them are collinear and $(\vec{a} \times \vec{b}) \times \vec{c} = \frac{1}{3}|\vec{b}||\vec{c}|\,\vec{a}$. If θ is the angle between vectors $\vec{b}$ and $\vec{c}$, then a value of sin θ is :

[JEE(Main)-2015]

(1) $\frac{2}{3}$ (2) $\frac{-2\sqrt{3}}{3}$ (3) $\frac{2\sqrt{2}}{3}$ (4) $\frac{-\sqrt{2}}{3}$

VT0154

13. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three unit vectors such that $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \frac{\sqrt{3}}{2}\left(\vec{b} + \vec{c}\right)$. If $\vec{b}$ is not parallel to $\vec{c}$, then the angle between $\vec{a}$ and $\vec{b}$ is :- **[JEE(Main)-2016]**

(1) $\frac{5\pi}{6}$ (2) $\frac{3\pi}{4}$ (3) $\frac{\pi}{2}$ (4) $\frac{2\pi}{3}$

VT0155

14. Let $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$. Let $\vec{c}$ be a vector such that $|\vec{c} - \vec{a}| = 3$, $|(\vec{a} \times \vec{b}) \times \vec{c}| = 3$ and the angle between $\vec{c}$ and $\vec{a} \times \vec{b}$ be 30º. Then $\vec{a} \cdot \vec{c}$ is equal to : **[JEE(Main)-2017]**

(1) $\frac{1}{8}$ (2) $\frac{25}{8}$ (3) 2 (4) 5

VT0156

15. Let $\vec{u}$ be a vector coplanar with the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$. If $\vec{u}$ is perpendicular to $\vec{a}$ and $\vec{u}.\vec{b} = 24$, then $|\vec{u}|^2$ is equal to - **[JEE(Main)-2018]**

(1) 315 (2) 256 (3) 84 (4) 336

VT0157

16. Let $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{c}$ be a vector such that $\vec{a} \times \vec{c} + \vec{b} = \vec{0}$ and $\vec{a}.\vec{c} = 4$, then $|\vec{c}|^2$ is equal to :-

[JEE(Main) 19]

(1) $\frac{19}{2}$ (2) 8 (3) $\frac{17}{2}$ (4) 9

VT0158

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17. Let $\sqrt{3}\hat{i}+\hat{j}$, $\hat{i}+\sqrt{3}\hat{j}$ and $\beta\hat{i}+(1-\beta)\hat{j}$ respectively be the position vectors of the points A, B and C with respect to the origin O. If the distance of C from the bisector of the acute angle between OA and OB is $\frac{3}{\sqrt{2}}$, then the sum of all possible values of β is :- **[JEE(Main)-19]**

(1) 2 (2) 1 (3) 3 (4) 4

VT0159

18. Let $\vec{\alpha}=3\hat{i}+\hat{j}$ and $\vec{\beta}=2\hat{i}-\hat{j}+3\hat{k}$. If $\vec{\beta}=\vec{\beta}_1-\vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$, then $\vec{\beta}_1\times\vec{\beta}_2$ is equal to **[JEE(Main)-19]**

(1) $-3\hat{i}+9\hat{j}+5\hat{k}$ (2) $3\hat{i}-9\hat{j}-5\hat{k}$ (3) $\frac{1}{2}\left(-3\hat{i}+9\hat{j}+5\hat{k}\right)$ (4) $\frac{1}{2}\left(3\hat{i}-9\hat{j}+5\hat{k}\right)$

VT0160

19. The distance of the point having position vector $-\hat{i}+2\hat{j}+6\hat{k}$ from the straight line passing through the point (2, 3, –4) and parallel to the vector, $6\hat{i}+3\hat{j}-4\hat{k}$ is : **[JEE(Main)-19]**

(1) 7 (2) $4\sqrt{3}$ (3) $2\sqrt{13}$ (4) 6

VT0161

20. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three units vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. If $\lambda = \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$ and $\vec{d}=\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$, then the ordered pair, $(\lambda,\vec{d})$ is equal to : **[JEE(Main)-20]**

(1) $\left(-\frac{3}{2},3\vec{a}\times\vec{b}\right)$ (2) $\left(-\frac{3}{2},3\vec{c}\times\vec{b}\right)$ (3) $\left(\frac{3}{2},3\vec{b}\times\vec{c}\right)$ (4) $\left(\frac{3}{2},3\vec{a}\times\vec{c}\right)$

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VT0162

21. Let $\vec{a}=\hat{i}-2\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-\hat{j}+\hat{k}$ be two vectors. If $\vec{c}$ is a vector such that $\vec{b}\times\vec{c}=\vec{b}\times\vec{a}$ and $\vec{c}\cdot\vec{a}=0$, then $\vec{c}\cdot\vec{b}$ is equal to **[JEE(Main)-20]**

(1) $\frac{1}{2}$ (2) –1 (3) $-\frac{1}{2}$ (4) $-\frac{3}{2}$

VT0163

22. Let the volume of a parallelopiped whose coterminous edges are given by $\vec{u}=\hat{i}+\hat{j}+\lambda\hat{k}, \vec{v}=\hat{i}+\hat{j}+3\hat{k}$ and $\vec{w}=2\hat{i}+\hat{j}+\hat{k}$ be 1 cu. unit. If θ be the angle between the edges $\vec{u}$ and $\vec{w}$, then $\cos\theta$ can be **[JEE(Main)-20]**

(1) $\frac{7}{6\sqrt{3}}$ (2) $\frac{5}{7}$ (3) $\frac{7}{6\sqrt{6}}$ (4) $\frac{5}{3\sqrt{3}}$

VT0164

23. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors such that $|\vec{a}|=\sqrt{3}$, $|\vec{b}|=5$, $\vec{b}\cdot\vec{c}=10$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{3}$. If $\vec{a}$ is perpendicular to the vector $\vec{b}\times\vec{c}$, then $\left|\vec{a}\times(\vec{b}\times\vec{c})\right|$ is equal to _____. **[JEE(Main)-20]**

VT0165

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24. If the vectors, $\vec{p}=(a+1)\hat{i}+a\hat{j}+a\hat{k}$, **[JEE(Main)-20]**

$\vec{q}=a\hat{i}+(a+1)\hat{j}+a\hat{k}$ and

$\vec{r}=a\hat{i}+a\hat{j}+(a+1)\hat{k}$ $(a\in R)$ are coplanar and $3(\vec{p}.\vec{q})^2-\lambda|\vec{r}\times\vec{q}|^2=0$, then the value of λ is ———.

VT0166

25. The projection of the line segment joining the points (1, –1, 3) and (2, –4, 11) on the line joining the points (–1, 2, 3) and (3, –2, 10) is ———. **[JEE(Main)-20]**

VT0167

EXERCISE (JA)

1. **(a)** Two adjacent sides of a parallelogram ABCD are given by $\overrightarrow{AB}=2\hat{i}+10\hat{j}+11\hat{k}$ and $\overrightarrow{AD}=-\hat{i}+2\hat{j}+2\hat{k}$. The side AD is rotated by an acute angle α in the plane of the parallelogram so that AD becomes AD'. If AD' makes a right angle with the side AB, then the cosine of the angle α is given by -

(A) $\frac{8}{9}$ (B) $\frac{\sqrt{17}}{9}$ (C) $\frac{1}{9}$ (D) $\frac{4\sqrt{5}}{9}$

VT0168

(b) If $\vec{a}$ and $\vec{b}$ are vectors in space given by $\vec{a}=\frac{\hat{i}-2\hat{j}}{\sqrt{5}}$ and $\vec{b}=\frac{2\hat{i}+\hat{j}+3\hat{k}}{\sqrt{14}}$, then the value of $(2\vec{a}+\vec{b}).\left[(\vec{a}\times\vec{b})\times(\vec{a}-2\vec{b})\right]$ is **[JEE 2010, 5+3]**

VT0169

2. **(a)** Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}-\hat{j}-\hat{k}$ be three vectors. A vector $\vec{v}$ in the plane of $\vec{a}$ and $\vec{b}$, whose projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$, is given by

(A) $\hat{i}-3\hat{j}+3\hat{k}$ (B) $-3\hat{i}-3\hat{j}-\hat{k}$ (C) $3\hat{i}-\hat{j}+3\hat{k}$ (D) $\hat{i}+3\hat{j}-3\hat{k}$

VT0170

(b) The vector(s) which is/are coplanar with vectors $\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$, and perpendicular to the vector $\hat{i}+\hat{j}+\hat{k}$ is/are

(A) $\hat{j}-\hat{k}$ (B) $-\hat{i}+\hat{j}$ (C) $\hat{i}-\hat{j}$ (D) $-\hat{j}+\hat{k}$

VT0171

(c) Let $\vec{a}=-\hat{i}-\hat{k}$, $\vec{b}=-\hat{i}+\hat{j}$ and $\vec{c}=\hat{i}+2\hat{j}+3\hat{k}$ be three given vectors. If $\vec{r}$ is a vector such that $\vec{r}\times\vec{b}=\vec{c}\times\vec{b}$ and $\vec{r}.\vec{a}=0$, then the value of $\vec{r}.\vec{b}$ is **[JEE 2011, 3+4+4]**

VT0172

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3. **(a)** If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors satisfying $|\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|^2+|\vec{c}-\vec{a}|^2=9$, then $|2\vec{a}+5\vec{b}+5\vec{c}|$ is **VT0173**

(b) If $\vec{a}$ and $\vec{b}$ are vectors such that $|\vec{a}+\vec{b}|=\sqrt{29}$ and $\vec{a}\times(2\hat{i}+3\hat{j}+4\hat{k})=(2\hat{i}+3\hat{j}+4\hat{k})\times\vec{b}$, then a possible value of $(\vec{a}+\vec{b}).(-7\hat{i}+2\hat{j}+3\hat{k})$ is **[JEE 2012, 4+3]**

(A) 0 (B) 3 (C) 4 (D) 8

VT0174

4. Let $\overrightarrow{PR}=3\hat{i}+\hat{j}-2\hat{k}$ and $\overrightarrow{SQ}=\hat{i}-3\hat{j}-4\hat{k}$ determine diagonals of a parallelogram PQRS and $\overrightarrow{PT}=\hat{i}+2\hat{j}+3\hat{k}$ be another vector. Then the volume of the parallelepiped determined by the vectors $\overrightarrow{PT}, \overrightarrow{PQ}$ and $\overrightarrow{PS}$ is **[JEE-Advanced 2013, 2M]**

(A) 5 (B) 20 (C) 10 (D) 30

VT0175

5. Consider the set of eight vectors $V=\{a\hat{i}+b\hat{j}+c\hat{k} : a,b,c\in\{-1,1\}\}$. Three non-coplanar vectors can be chosen from V in 2^p ways. Then p is **[JEE-Advanced 2013, 4, (–1)]**

VT0176

6. Match List-I with List-II and select the correct answer using the code given below the lists.

	List-I		**List-II**
P.	Volume of parallelepiped determined by vectors $\vec{a},\vec{b}$ and $\vec{c}$ is 2. Then the volume of the parallelepiped determined by vectors $2(\vec{a}\times\vec{b}),3(\vec{b}\times\vec{c})$ and $(\vec{c}\times\vec{a})$ is	1.	100
Q.	Volume of parallelepiped determined by vectors $\vec{a},\vec{b}$ and $\vec{c}$ is 5. Then the volume of the parallelepiped determined by vectors $3(\vec{a}+\vec{b}),(\vec{b}+\vec{c})$ and $2(\vec{c}+\vec{a})$ is	2.	30
R.	Area of a triangle with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 20. Then the area of the triangle with adjacent sides determined by vectors $(2\vec{a}+3\vec{b})$ and $(\vec{a}-\vec{b})$ is	3.	24
S.	Area of a parallelogram with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 30. Then the area of the parallelogram with adjacent sides determined by vectors $(\vec{a}+\vec{b})$ and $\vec{a}$ is	4.	60

Codes :

	P	Q	R	S
(A)	4	2	3	1
(B)	2	3	1	4
(C)	3	4	1	2
(D)	1	4	3	2

[JEE-Advanced 2013, 3, (–1)]

VT0177

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7. Let $\vec{x}, \vec{y}$ and $\vec{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\frac{\pi}{3}$. If $\vec{a}$ is a nonzero vector perpendicular to $\vec{x}$ and $\vec{y} \times \vec{z}$ and $\vec{b}$ is nonzero vector perpendicular to $\vec{y}$ and $\vec{z} \times \vec{x}$, then **[JEE(Advanced)-2014, 3]**

(A) $\vec{b} = (\vec{b}.\vec{z})(\vec{z} - \vec{x})$ (B) $\vec{a} = (\vec{a}.\vec{y})(\vec{y} - \vec{z})$ (C) $\vec{a}.\vec{b} = -(\vec{a}.\vec{y})(\vec{b}.\vec{z})$ (D) $\vec{a} = (\vec{a}.\vec{y})(\vec{z} - \vec{y})$

VT0178

8. Let $\vec{a}, \vec{b}$, and $\vec{c}$ be three non-coplanar unit vectors such that the angle between every pair of them is $\frac{\pi}{3}$. If $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\vec{a} + q\vec{b} + r\vec{c}$, where p,q and r are scalars, then the value of $\frac{p^2 + 2q^2 + r^2}{q^2}$ is **[JEE(Advanced)-2014, 3]**

VT0179

9. Let ΔPQR be a triangle. Let $\vec{a} = \overrightarrow{QR}, \vec{b} = \overrightarrow{RP}$ and $\vec{c} = \overrightarrow{PQ}$. If $|\vec{a}| = 12, |\vec{b}| = 4\sqrt{3}$ and $\vec{b}.\vec{c} = 24$, then which of the following is (are) true ? **[JEE 2015, 4M, –2M]**

(A) $\frac{|\vec{c}|^2}{2} - |\vec{a}| = 12$ (B) $\frac{|\vec{c}|^2}{2} + |\vec{a}| = 30$ (C) $|\vec{a} \times \vec{b} + \vec{c} \times \vec{a}| = 48\sqrt{3}$ (D) $\vec{a}.\vec{b} = -72$

VT0180

10. Suppose that $\vec{p}, \vec{q}$ and $\vec{r}$ are three non-coplanar vectors in $\mathbb{R}^3$. Let the components of a vector $\vec{s}$ along $\vec{p}, \vec{q}$ and $\vec{r}$ be 4,3 and 5, respectively. If the components of this vector $\vec{s}$ along $(-\vec{p} + \vec{q} + \vec{r})$, $(\vec{p} - \vec{q} + \vec{r})$ and $(-\vec{p} - \vec{q} + \vec{r})$ are x,y and z, respectively, then the value of 2x + y + z is **[JEE 2015, 4M, –0M]**

VT0181

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11. Let $\hat{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ be a unit vector in $\mathbb{R}^3$ and $\hat{w} = \frac{1}{\sqrt{6}}(\hat{i} + \hat{j} + 2\hat{k})$. Given that there exists a vector $\vec{v}$ in $\mathbb{R}^3$ such that $|\hat{u} \times \vec{v}| = 1$ and $\hat{w}.(\hat{u} \times \vec{v}) = 1$. Which of the following statement(s) is(are) correct?

(A) There is exactly one choice for such $\vec{v}$

(B) There are infinitely many choice for such $\vec{v}$

(C) If $\hat{u}$ lies in the xy-plane then $|u_1| = |u_2|$

(D) If $\hat{u}$ lies in the xz-plane then $2|u_1| = |u_3|$ **[JEE(Advanced)-2016, 4(–2)]**

VT0182

12. Let O be the origin and let PQR be an arbitrary triangle. The point S is such that $\overrightarrow{OP}.\overrightarrow{OQ} + \overrightarrow{OR}.\overrightarrow{OS} = \overrightarrow{OR}.\overrightarrow{OP} + \overrightarrow{OQ}.\overrightarrow{OS} = \overrightarrow{OQ}.\overrightarrow{OR} + \overrightarrow{OP}.\overrightarrow{OS}$. Then the triangle PQR has S as its **[JEE(Advanced)-2017]**

(A) incentre (B) orthocenter (C) circumcentre (D) centroid

VT0183

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PARAGRAPH :

Let O be the origin, and $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ be three unit vectors in the directions of the sides $\overrightarrow{QR}, \overrightarrow{RP}, \overrightarrow{PQ}$, respectively, of a triangle PQR. **[JEE(Advanced)-2017]**

13. $\left|\overrightarrow{OX} \times \overrightarrow{OY}\right| =$

(A) sin(Q + R) (B) sin(P + R) (C) sin 2R (D) sin(P + Q)

VT0184

14. If the triangle PQR varies, then the minimum value of cos(P + Q) + cos(Q + R) + cos(R + P) is

(A) $\frac{3}{2}$ (B) $-\frac{3}{2}$ (C) $\frac{5}{3}$ (D) $-\frac{5}{3}$

VT0184

15. Let $\vec{a}$ and $\vec{b}$ be two unit vectors such that $\vec{a}.\vec{b} = 0$. For some x, y $\in \mathbb{R}$, let $\vec{c} = x\vec{a} + y\vec{b} + (\vec{a} \times \vec{b})$. If $|\vec{c}| = 2$ and the vector $\vec{c}$ is inclined at the same angle α to both $\vec{a}$ and $\vec{b}$, then the value of $8\cos^2\alpha$ is _____ **[JEE(Advanced)-2018, 3(0)]**

VT0185

16. Three lines $L_1 : \vec{r} = \lambda\hat{i},\ \lambda \in \mathbb{R}$,

$L_2 : \vec{r} = \vec{k} + \mu\hat{j},\ \mu \in \mathbb{R}$ and

$L_3 : \vec{r} = \hat{i} + \hat{j} + v\hat{k},\ v \in \mathbb{R}$

are given. For which point(s) Q on L_2 can we find a point P on L_1 and a point R on L_3 so that P, Q and R are collinear ? **[JEE(Advanced)-2019, 4(–1)]**

(1) $\hat{k} + \hat{j}$ (2) $\hat{k}$ (3) $\hat{k} + \frac{1}{2}\hat{j}$ (4) $\hat{k} - \frac{1}{2}\hat{j}$

VT0186

17. Let $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$ be two vectors. Consider a vector $\vec{c} = \alpha\vec{a} + \beta\vec{b}$, $\alpha, \beta \in \mathbb{R}$. If the projection of $\vec{c}$ on the vector $(\vec{a} + \vec{b})$ is $3\sqrt{2}$, then the minimum value of $\left(\vec{c} - (\vec{a} \times \vec{b})\right).\vec{c}$ equals

[JEE(Advanced)-2019, 3(0)]

VT0187

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ANSWER KEY

EXERCISE (O-1)

1. B	2. A	3. B	4. B	5. D	6. B	7. D	8. B
9. D	10. B	11. D	12. C	13. D	14. B	15. C	16. D
17. C	18. A	19. A	20. (i) D, (ii) B, (iii) B		21. D	22. A	23. D
24. D	25. B	26. C	27. B	28. B	29. D	30. A	31. A
32. B	33. B	34. A	35. C	36. A	37. C	38. B	39. A
40. C	41. C	42. D	43. D	44. C	45. D	46. B	47. A
48. D	49. C	50. C	51. D	52. A	53. D	54. A	55. C
56. C	57. A	58. D	59. A	60. C	61. B	62. A	63. D

64. D **65.** B **66.** (A) T; (B) U ; (C) P; (D) R ; (E) Q; (F) S; (G) W; (H) V

EXERCISE (O-2)

1. A	2. A	3. C	4. D	5. C	6. B	7. C	8. A,C
9. A,B,D	10. B,C	11. A,C,D	12. A,C,D	13. B,C	14. C,D	15. A,B,D	
16. A,B,C,D	17. A,D	18. A,C	19. B,C,D	20. A,B,D	21. B,C,D	22. A,B,C,D	
23. A,B,C	24. C	25. B	26. A	27. (A) - S ; (B) - P ; (C) - R ; (D) - Q			

EXERCISE (S-1)

1. (9, 7) **2.** $-\frac{1}{\sqrt{2}}(\hat{i}+\hat{j})$ **4.** 13 **5.** 3 **6.** 4950 **7.** 7

8. 1125 **9.** $x = 2, y = -1$ **10.** (b) externally in the ratio 1 : 3

11. (i) parallel (ii) the lines intersect at the point p.v. $-2\hat{i}+2\hat{j}$ **(iii)** lines are skew

13. (a) $\cot^{-1}(0)$; (b) $\cot^{-1}\frac{1}{\sqrt{3}}$; (c) $\cot^{-1}\sqrt{2}$ **14.** $\frac{\pi}{2}$ **15.** $\left(\frac{1}{\sqrt{3}},\frac{2}{\sqrt{3}}\right)$ **16.** $\sqrt{3}$

17. (a) $\frac{\sqrt{3}}{2}$, (b) 51 **18.** (a) 2, (b) – 1, (c) –12 **19.** 101 **20.** $F = 2\vec{a}_1 + 5\vec{a}_2 + 3\vec{a}_3$

21. $\frac{4}{\sqrt{2}}\hat{i}-\frac{1}{\sqrt{2}}\hat{j}-\frac{1}{\sqrt{2}}\hat{k}$ **22.** (i) $\frac{6}{7}\sqrt{14}$ (ii) 6 (iii) $\frac{3}{5}\sqrt{10}$ (iv) $\sqrt{6}$ **23.** 6 **24.** 13

25. $\vec{x}=\frac{\vec{a}+(\vec{c}.\vec{a})\vec{c}+\vec{b}\times\vec{c}}{1+\vec{c}^2}, y=\frac{\vec{b}+(\vec{c}.\vec{b})\vec{c}+\vec{a}\times\vec{c}}{1+\vec{c}^2}$ **26.** 75 **27.** 488

EXERCISE (S-2)

1. 48 **2.** 12 **3.** 5 **4.** 9 **5.** $\sqrt{2}-1$ **6.** $x = 2, y = -2, z = -2$

7. $\frac{5a^2}{12\sqrt{3}}$ sq. units **8.** $\pm\frac{1}{3\sqrt{3}}\left(\hat{i}+5\hat{j}-\hat{k}\right)$ **9.** (–1,3,3) & (3,–1,–1) **10.** NO, NO

12. 5 **13.** (a) $\frac{-1}{2}\hat{i}-\frac{1}{2}\hat{j}+\frac{1}{\sqrt{2}}\hat{k}$, (c) Range : [3,5] **15.** 34 **16.** 19

EXERCISE (JM)

1. 3	2. 1	3. 2	4. 3	5. 4	6. 4	7. 3	8. 3
9. 1	10. 1	11. 4	12. 3	13. 1	14. 3	15. 4	16. 1
17. 2	18. 3	19. 1	20. 1	21. 3	22. 1	23. 30	24. 1.00
25. 8.00							

EXERCISE (JA)

1. (a) B; (b) 5	2. (a) C; (b) A,D; (c) 9	3. (a) 3; (b) C	4. C	5. 5			
6. C	7. A,B,C	8. 4	9. A,C,D	10. Bonus	11. B,C	12. B	
13. D	14. B	15. 3	16. 3,4	17. 18.00			

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3D-COORDINATE GEOMETRY

POINT

1. INTRODUCTION :

In earlier classes we have learnt about points, lines, circles and conic section in two dimensional geometry. In two dimensions a point represented by an ordered pair (x, y) (where x & y are both real numbers)

In space, each body has length, breadth and height i.e. each body exist in three dimensional space. Therefore three independent quantities are essential to represent any point in space. Three axes are required to represent these three quantities.

2. RECTANGULAR CO-ORDINATE SYSTEM :

In cartesian system of three lines which are mutually perpendicular, such a system is called rectangular cartesian co-ordinate system.

Co-ordinate axes and co-ordinate planes :

When three mutually perpendicular planes intersect at a point, then mutually perpendicular lines are obtained and these lines also pass through that point. If we assume the point of intersection as origin, then the three planes are known as co-ordinate planes and the three lines are known as co-ordinate axes.

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Octants :

Every plane bisects the space. Hence three co-ordinate plane divide the space in eight parts. These parts are known as octants.

3. COORDINATES OF A POINT IN SPACE :

Let O be a fixed point, known as origin and let OX, OY and OZ be three mutually perpendicular lines, taken as x-axis, y-axis and z-axis respectively, in such a way that they form a right handed system.

The planes XOY, YOZ and ZOX are known as xy-plane, yz-plane and zx-plane respectively.

Let P be a point in space and distances of P from yz, zx and xy planes be x, y, z respectively (with proper signs) then we say that coordinates of P are (x, y, z). Also OA = |x|, OB = |y|, OC = |z|

4. DISTANCE FORMULA :

The distance between two points A (x_1, y_1, z_1) and B (x_2, y_2, z_2) is given by

$$AB = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$$

(a) Distance from Origin :

Let O be the origin and P (x, y, z) be any point, then $OP = \sqrt{(x^2 + y^2 + z^2)}$

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(b) Distance of a point from coordinate axes :

Let P(x, y, z) be any point in the space. Let PA, PB and PC be the perpendiculars drawn from P to the axes OX, OY and OZ respectively. Then

$$PA = \sqrt{(y^2+z^2)}\,;\ PB = \sqrt{(z^2+x^2)}\,;\ PC = \sqrt{(x^2+y^2)}$$

Illustration 1 : Prove by using distance formula that the points P (1, 2, 3), Q (–1, –1, –1) and R(3, 5, 7) are collinear.

Solution : We have PQ = $\sqrt{(-1-1)^2+(-1-2)^2+(-1-3)^2} = \sqrt{4+9+16} = \sqrt{29}$

QR = $\sqrt{(3+1)^2+(5+1)^2+(7+1)^2} = \sqrt{16+36+64} = \sqrt{116} = 2\sqrt{29}$

and PR = $\sqrt{(3-1)^2+(5-2)^2+(7-3)^2} = \sqrt{4+9+16} = \sqrt{29}$

Since QR = PQ + PR. Therefore the given points are collinear. **Ans.**

Illustration 2 : Find the locus of a point the sum of whose distances from (1, 0, 0) and (–1, 0, 0) is equal to 10.

Solution : Let the points A(1,0,0), B (–1,0,0) and P(x,y,z)

Given : PA + PB =10

$$\sqrt{(x-1)^2+(y-0)^2+(z-0)^2}+\sqrt{(x+1)^2+(y-0)^2+(z-0)^2} = 10$$

$$\Rightarrow \quad \sqrt{(x-1)^2+y^2+z^2} = 10-\sqrt{(x+1)^2+y^2+z^2}$$

Squaring both sides, we get ;

$$\Rightarrow \quad (x-1)^2+y^2+z^2 = 100+(x+1)^2+y^2+z^2-20\sqrt{(x+1)^2+y^2+z^2}$$

$$\Rightarrow \quad -4x-100 = -20\sqrt{(x+1)^2+y^2+z^2} \quad \Rightarrow \quad x+25 = 5\sqrt{(x+1)^2+y^2+z^2}$$

Again squaring both sides we get $x^2 + 50x + 625 = 25\ \{(x^2+2x+1)+y^2+z^2\}$

$\Rightarrow \quad 24x^2 + 25y^2 + 25z^2 - 600 = 0$

i.e. required equation of locus **Ans.**

5. SECTION FORMULAE :

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points and let R (x, y, z) divide PQ in the ratio $m_1 : m_2$. Then co-ordinates of R(x, y, z) = $\left(\dfrac{m_1x_2+m_2x_1}{m_1+m_2}, \dfrac{m_1y_2+m_2y_1}{m_1+m_2}, \dfrac{m_1z_2+m_2z_1}{m_1+m_2}\right)$

If (m_1/m_2) is positive, R divides PQ internally and if (m_1/m_2) is negative, then externally.

Mid-Point : Mid point of PQ is given by $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}, \dfrac{z_1+z_2}{2}\right)$

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Illustration 3 : Find the ratio in which the plane x – 2y + 3z = 17 divides the line joining the points (–2, 4, 7) and (3, –5, 8).

Solution : Let the required ratio be k : 1

The co-ordinates of the point which divides the join of (–2, 4, 7) and (3, –5, 8) in the ratio k : 1 are $\left(\frac{3k-2}{k+1}, \frac{-5k+4}{k+1}, \frac{8k+7}{k+1}\right)$

Since this point lies on the plane x – 2y + 3z – 17 = 0

$\therefore \quad \left(\frac{3k-2}{k+1}\right) - 2\left(\frac{-5k+4}{k+1}\right) + 3\left(\frac{8k+7}{k+1}\right) - 17 = 0$

$\Rightarrow \quad (3k - 2) - 2(-5k + 4) + 3(8k + 7) = 17k + 17$

$\Rightarrow \quad 3k + 10k + 24k - 17k = 17 + 2 + 8 - 21$

$\Rightarrow \quad 37k - 17k = 6 \quad \Rightarrow \quad 20k = 6 ; \quad k = \frac{6}{20} = \frac{3}{10}$

Hence the required ratio = k : 1 $= \frac{3}{10} : 1 = 3 : 10$ **Ans.**

6. CENTROID OF A TRIANGLE :

Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ be the vertices of a triangle ABC. Then its centroid G is given by $G = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$

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Illustration 4 : If the centre of a tetrahedron OABC where A, B, C, are given by (a, 2, 3), (1, b, 2) and (2, 1, c) respectively is (1, 2, – 1), then distance of P (a, b, c) from origin is -

(A) $\sqrt{107}$ (B) $\sqrt{14}$ (C) $\sqrt{107}/14$ (D) none of these

Solution : Centre is $\left(\frac{1}{4}\Sigma x, \frac{1}{4}\Sigma y, \frac{1}{4}\Sigma z\right) = (1, 2, -1)$

$\Rightarrow \quad \frac{a+1+2+0}{4} = 1, \frac{2+b+1+0}{4} = 2, \frac{3+2+c+0}{4} = -1 \quad \Rightarrow \quad a = 1, b = 5, c = -9$

$\therefore \quad OP = \sqrt{a^2+b^2+c^2} = \sqrt{107}$ **Ans. (A)**

Do yourself 1:

(i) Find the distance between the points P(3, 4, 5) and Q(–1, 2, –3).

(ii) Show that the points A(0, 7, 10), B(–1, 6, 6) and C(–4, 9, 6) are vertices of an isosceles right angled triangle.

(iii) Find the locus of a point such that the difference of the square of its distance from the points A(3, 4, 5) and B(–1, 3, –7) is equal to $2k^2$.

(iv) Find the co-ordinates of points which trisects the line joining the points A(–3, 2, 4) and B(0, 4, 7)

(v) Find the ratio in which the planes (a) xy (b) yz divide the line joining the points P(–2, 4, 7) and Q(3, –5, 8).

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7. DIRECTION COSINES OF LINE :

If α, β, γ be the angles made by a line with x-axis, y-axis & z-axis respectively then $\cos\alpha$, $\cos\beta$ & $\cos\gamma$ are called direction cosines of a line, denoted by ℓ, m & n respectively.

Note :

(i) If line makes angles α, β, γ with x, y & z axis respectively then $\pi - \alpha$, $\pi - \beta$ & $\pi - \gamma$ is another set of angle that line makes with principle axes. Hence if ℓ, m & n are direction cosines of line then $-\ell$, $-$m & $-$n are also direction cosines of the same line.

(ii) Since parallel lines have same direction. So, in case of lines, which do not pass through the origin. We can draw a parallel line passing through the origin and direction cosines of that line can be found.

Important points :

(i) Direction cosines of a line :

Take a vector $\vec{A} = a\hat{i} + b\hat{j} + c\hat{k}$ parallel to a line whose D.C's are to be found out.

$\vec{A}.\hat{i} = a$

$|\vec{A}|\cos\alpha = a$

$\cos\alpha = \dfrac{a}{|\vec{A}|}$ similarly, $\cos\beta = \dfrac{b}{|\vec{A}|}$; $\cos\gamma = \dfrac{c}{|\vec{A}|}$

$\Rightarrow \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \quad \Rightarrow \quad \ell^2 + m^2 + n^2 = 1$

(ii) Direction cosine of axes :

Since the positive x-axes makes angle 0°, 90°, 90° with axes of x, y and z respectively,

$\therefore$ D.C.'s of x axes are 1, 0, 0.

D.C.'s of y-axis are 0, 1, 0

D.C.'s of z-axis are 0, 0, 1

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8. DIRECTION RATIOS :

Any three numbers a, b, c proportional to direction cosines ℓ, m, n are called direction ratios of the line. i.e. $\dfrac{\ell}{a} = \dfrac{m}{b} = \dfrac{n}{c}$

There can be infinitely many sets of direction ratios for a given line.

Direction ratios and Direction cosines of the line joining two points :

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two points, then d.r.'s of AB are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ and the d.c.'s of AB are $\dfrac{1}{r}(x_2 - x_1)$, $\dfrac{1}{r}(y_2 - y_1)$, $\dfrac{1}{r}(z_2 - z_1)$ where $r = \sqrt{[\Sigma(x_2 - x_1)^2]}$

9. RELATION BETWEEN D.C'S & D.R'S :

$$\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$$

$$\therefore \quad \frac{\ell^2}{a^2} = \frac{m^2}{b^2} = \frac{n^2}{c^2} = \frac{\ell^2 + m^2 + n^2}{a^2 + b^2 + c^2}$$

$$\therefore \quad \ell = \frac{\pm a}{\sqrt{a^2 + b^2 + c^2}} \; ; m = \frac{\pm b}{\sqrt{a^2 + b^2 + c^2}} \; ; n = \frac{\pm c}{\sqrt{a^2 + b^2 + c^2}}$$

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Important point :

Direction cosines of a line have two sets but direction ratios of a line have infinite possible sets.

7. PROJECTIONS :

(a) Projection of line segment OP on co-ordinate axes :

Let line segment make angle α with x-axis

Thus, the projections of line segment OP on axes are the absolute values of the co-ordinates of P. i.e.

Projection of OP on x-axis $= |x|$

Projection of OP on y-axis $= |y|$

Projection of OP on z-axis $= |z|$

Now, in ΔOAP, angle A is a right angle and $OA = x$

$$OP = \sqrt{x^2 + y^2 + z^2}$$

$$\therefore \quad \cos\alpha = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{|OP|}$$

if $|OP| = r$, then $x = |OP|\cos\alpha = \ell r$

Similarly $y = |OP|\cos\beta = mr$, $z = nr$, where ℓ, m, n are DC's of line

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(b) Projection of a line segment AB on coordinate axes :

Projection of the point $A(x_1, y_1, z_1)$ on x-axis is $E(x_1, 0, 0)$. Projection of point $B(x_2, y_2, z_2)$ on x-axis is $F(x_2, 0, 0)$.

Hence projection of AB on x-axis is $EF = |x_2 - x_1|$.

Similarly, projection of AB on y and z-axis are $|y_2 - y_1|$, $|z_2 - z_1|$ respectively.

(c) Projection of line segment AB on a line having direction cosines ℓ, m, n :

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$.

Now projection of AB on EF = CD = AB $\cos\theta$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \times \frac{|(x_2 - x_1)\ell + (y_2 - y_1)m + (z_2 - z_1)n|}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

$$= |(x_2 - x_1)\ell + (y_2 - y_1)m + (z_2 - z_1)n|$$

Illustration 5 : A line OP makes with the x-axis an angle of measure 120° and with y-axis an angle of measure 60°. Find the angle made by the line with the z-axis.

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Solution : $\alpha = 120°$ and $\beta = 60°$

$\therefore \cos\alpha = \cos 120° = -\frac{1}{2}$ and $\cos\beta = \cos 60° = \frac{1}{2}$ but $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\therefore \left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2\gamma = 1$

$\Rightarrow \cos^2\gamma = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2} \Rightarrow \cos\gamma = \pm\frac{1}{\sqrt{2}}$

$\therefore \gamma = 45°$ or $135°$ **Ans.**

Illustration 6 : Find the length of projection of the line segment joining the points (–1, 0, 3) and (2, 5, 1) on the line whose direction ratios are 6, 2, 3.

Solution : The direction cosines ℓ, m, n of the line are given by $\frac{\ell}{6} = \frac{m}{2} = \frac{n}{3} = \frac{\sqrt{\ell^2+m^2+n^2}}{\sqrt{6^2+2^2+3^2}} = \frac{1}{\sqrt{49}} = \frac{1}{7}$

$\therefore \ell = \frac{6}{7}, m = \frac{2}{7}, n = \frac{3}{7}$

The required length of projection is given by

$= |\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)| = \left|\frac{6}{7}[2-(-1)] + \frac{2}{7}(5-0) + \frac{3}{7}(1-3)\right|$

$= \left|\frac{6}{7}\times 3 + \frac{2}{7}\times 5 + \frac{3}{7}\times -2\right| = \left|\frac{18}{7} + \frac{10}{7} - \frac{6}{7}\right| = \left|\frac{18+10-6}{7}\right| = \frac{22}{7}$ **Ans.**

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Do yourself - 2 :

(i) Find the length of projections of the line segment joining the origin O to the point P(3, 2, –5) on the axes.

(ii) Find the length of projections of the line joining the points P(3, 2, 5) and Q(0, –2, 8) on the axes.

(iii) Find the direction ratios & direction cosines of the line joining the points O(0, 0, 0) and P(2, 3, 4).

11. ANGLE BETWEEN TWO LINES :

Let θ be the angle between the lines with d.c.'s ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 then $\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$.

If a_1, b_1, c_1 and a_2, b_2, c_2 be D.R.'s of two lines then angle θ between them is given by

$$\cos\theta = \frac{(a_1a_2 + b_1b_2 + c_1c_2)}{\sqrt{(a_1^2+b_1^2+c_1^2)}\sqrt{(a_2^2+b_2^2+c_2^2)}}$$

Illustration 7 : If a line makes angles $\alpha, \beta, \gamma, \delta$ with four diagonals of a cube, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$ equals -

(A) 3 (B) 4 (C) 4/3 (D) 3/4

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Solution : Let OA, OB, OC be coterminous edges of a cube and OA = OB = OC = a, then co-ordinates of its vertices are O(0, 0, 0), A(a, 0, 0), B(0, a, 0), C(0, 0, a), L(0, a, a), M(a, 0, a), N(a, a, 0) and P (a, a, a)

Direction ratio of diagonal AL, BM, CN and OP are

$$\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right),\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$

Let ℓ, m, n be the direction cosines of the given line, then

$$\cos\alpha = \ell\left(-\frac{1}{\sqrt{3}}\right)+m\left(\frac{1}{\sqrt{3}}\right)+n\left(\frac{1}{\sqrt{3}}\right)=\frac{-\ell+m+n}{\sqrt{3}}$$

Similarly $\cos\beta = \dfrac{\ell-m+n}{\sqrt{3}}$, $\cos\gamma = \dfrac{\ell+m-n}{\sqrt{3}}$ and $\cos\delta = \dfrac{\ell+m+n}{\sqrt{3}}$

$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \dfrac{4}{3}$ **Ans. (C)**

Illustration 8 : (a) Find the acute angle between two lines whose direction ratios are 2, 3, 6 and 1, 2, 2 respectively.

(b) Find the measure of the angle between the lines whose direction ratios are 1, –2, 7 and 3, –2, –1.

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Solution : (a) $a_1 = 2, b_1 = 3, c_1 = 6; a_2 = 1, b_2 = 2, c_2 = 2.$

If θ be the angle between two lines whose d.r's are given, then

$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} = \frac{2\times1+3\times2+6\times2}{\sqrt{2^2+3^2+6^2}\sqrt{1^2+2^2+2^2}} = \frac{2+6+12}{7\times3} = \frac{20}{21}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{20}{21}\right)$$

(b) $\sqrt{1^2+(-2)^2+7^2} = \sqrt{54}$

$\sqrt{3^2+(-2)^2+(-1)^2} = \sqrt{14}$

$\therefore$ The actual direction cosines of the lines are

$$\frac{1}{\sqrt{54}},\frac{-2}{\sqrt{54}},\frac{7}{\sqrt{54}} \quad \text{and} \quad \frac{3}{\sqrt{14}},\frac{-2}{\sqrt{14}},\frac{-1}{\sqrt{14}}$$

If θ is the angle between the lines, then

$$\cos\theta = \left(\frac{1}{\sqrt{54}}\right)\left(\frac{3}{\sqrt{14}}\right)+\left(\frac{-2}{\sqrt{54}}\right)\left(\frac{-2}{\sqrt{14}}\right)+\left(\frac{7}{\sqrt{54}}\right)\left(\frac{-1}{\sqrt{14}}\right)$$

$$= \frac{3+4-7}{\sqrt{54}.\sqrt{14}} = 0 \Rightarrow \theta = 90^\circ$$ **Ans.**

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12. PERPENDICULAR AND PARALLEL LINES :

Let the two lines have their d.c.'s given by ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 respectively then they are perpendicular if $\theta = 90°$ i.e. $\cos\theta = 0$, i.e. $\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$.

Also the two lines are parallel if $\theta = 0$ i.e. $\sin\theta = 0$, i.e. $\frac{\ell_1}{\ell_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

Note: If instead of d.c.'s, d.r.'s a_1, b_1, c_1 and a_2, b_2, c_2 are given, then the lines are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ and parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Illustration 9 : If the lines whose direction cosines satisfies the equations $a\ell + bm + cn = 0$ and $fmn + gn\ell + h\ell m = 0$ are perpendicular, then $\frac{f}{a} + \frac{g}{b} + \frac{h}{c}$ equals -

(A) 0 (B) –1 (C) 1 (D) none of these

Solution : Eliminating n between the given relations, we find that $(fm + g\ell)\left(\frac{-a\ell - bm}{c}\right) + h\ell m = 0$

or $ag\left(\frac{\ell}{m}\right)^2 + (af + bg - ch)\left(\frac{\ell}{m}\right) + bf = 0$ .....(i)

Let $\frac{\ell_1}{m_1}$ and $\frac{\ell_2}{m_2}$, are roots of (i), then $\frac{\ell_1}{m_1} \cdot \frac{\ell_2}{m_2} = \frac{bf}{ag}$

$\Rightarrow \frac{\ell_1\ell_2}{f/a} = \frac{m_1m_2}{g/b}$ .....(ii)

Similarly $\frac{m_1m_2}{g/b} = \frac{n_1n_2}{h/c}$ .....(iii)

From (ii) and (iii), we get $\frac{\ell_1\ell_2}{f/a} = \frac{m_1m_2}{g/b} = \frac{n_1n_2}{h/c} = \lambda$

$\Rightarrow \ell_1\ell_2 = \lambda.f/a$; $m_1m_2 = \lambda.g/b$; $n_1n_2 = \lambda.h/c$

$\Rightarrow \ell_1\ell_2 + m_1m_2 + n_1n_2 = \lambda\left(\frac{f}{a} + \frac{g}{b} + \frac{h}{c}\right)$

$\Rightarrow \frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$ $\{\because \ell_1\ell_2 + m_1m_2 + n_1n_2 = 0\}$ **Ans. (A)**

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Do yourself - 3 :

(i) Find the angle between the lines whose direction ratios are 1, –2, 1 and 4, 3, 2.

(ii) If a line makes α, β and γ angle with axes, then prove that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.

(iii) Find the direction cosines of the line which is perpendicular to the lines with direction cosines proportional to (1, –2, –2) & (0, 2, 1).

PLANE

13. DEFINITION :

A plane is a surface such that a line segment joining any two points on the surface lies wholly on it.

14. EQUATIONS OF A PLANE :

The equation of every plane is of the first degree i.e. of the form ax + by + cz + d = 0, in which a, b, c are constants, not all zero simultaneously.

(a) Equation of plane passing through a fixed point :

Vector form : If $\vec{a}$ is the position vector of a point on the plane and $\vec{n}$ is a vector normal to the plane then its vectorial equation is given by $(\vec{r}-\vec{a}).\vec{n}=0 \Rightarrow \mathbf{\vec{r}.\vec{n}=d}$, where $d=\vec{a}.\vec{n}=$constant.

Cartesian form : If $\vec{a}(x_1, y_1, z_1)$ and $\vec{n}=a\hat{i}+b\hat{j}+c\hat{k}$, then cartesian equation of plane will be

$\mathbf{a(x-x_1)+b(y-y_1)+c(z-z_1)=0}$

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(b) Plane Parallel to the Coordinate Planes :

(i) Equation of yz plane is x = 0.

(ii) Equation of zx plane is y = 0.

(iii) Equation of xy plane is z = 0.

(iv) Equation of the plane parallel to xy plane at a distance c is z = c or z = –c.

(v) Equation of the plane parallel to yz plane at a distance c is x = c or x = –c

(vi) Equation of the plane parallel to zx plane at a distance c is y = c or y = –c.

(c) Equations of Planes Parallel to the Axes :

If a = 0, the plane is parallel to x-axis i.e. equation of the plane parallel to x-axis is **by + cz + d = 0**.

Similarly, equations of planes parallel to y-axis and parallel to z-axis are **ax + cz + d = 0** and **ax + by + d = 0**, respectively.

(d) Equation of a Plane in Intercept Form :

Equation of the plane which cuts off intercepts a, b, c from the axes x, y, z respectively is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

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(e) Equation of a Plane in Normal Form :

Vector form : If $\hat{n}$ is a unit vector normal to the plane from the origin and d be the perpendicular distance of plane from origin then its vector equation is $\vec{r}.\hat{n} = d$.

Cartesian form : If the length of the perpendicular distance of the plane from the origin is p and direction cosines of this perpendicular are (ℓ, m, n), then the equation of the plane is $\ell x + my + nz = p$.

(f) Equation of a Plane through three points :

Vector form : If A, B, C are three points having P.V.'s $\vec{a}$, $\vec{b}$, $\vec{c}$ respectively, then vector equation of the plane is $[\vec{r}\ \vec{a}\ \vec{b}]+[\vec{r}\ \vec{b}\ \vec{c}]+[\vec{r}\ \vec{c}\ \vec{a}]=[\vec{a}\ \vec{b}\ \vec{c}]$.

Cartesian form : The equation of the plane through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$

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Illustration 10 : Find the equation of the plane through the points A(2, 2, –1), B(3, 4, 2) and C(7, 0, 6).

Solution : The general equation of a plane passing through (2, 2, –1) is

$$a(x-2) + b(y-2) + c(z+1) = 0 \quad .....(i)$$

It will pass through B (3, 4, 2) and C (7, 0, 6) if

$$a(3-2) + b(4-2) + c(2+1) = 0 \quad \text{or} \quad a + 2b + 3c = 0 \quad .....(ii)$$

and $a(7-2) + b(0-2) + c(6+1) = 0$ or $5a - 2b + 7c = 0$ .....(iii)

Solving (ii) and (iii) by cross-multiplication, we have

$$\frac{a}{14+6} = \frac{b}{15-7} = \frac{c}{-2-10} \text{ or } \frac{a}{5} = \frac{b}{2} = \frac{c}{-3} = \lambda \quad \text{(say)}$$

$\Rightarrow$ $a = 5\lambda$, $b = 2\lambda$ and $c = -3\lambda$

Substituting the values of a, b and c in (i), we get

$$5\lambda(x-2) + 2\lambda(y-2) - 3\lambda(z+1) = 0$$

or $5(x-2) + 2(y-2) - 3(z+1) = 0$

$\Rightarrow$ $5x + 2y - 3z = 17$, which is the required equation of the plane **Ans.**

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Illustration 11 : A plane meets the co-ordinates axes in A,B,C such that the centroid of the Δ ABC is the point (p,q,r) show that the equation of the plane is $\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3$

Solution : Let the required equation of plane be :

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad .....(i)$$

Then, the co-ordinates of A, B and C are A(a, 0, 0), B(0, b, 0), C(0, 0, c) respectively

So the centroid of the triangle ABC is $\left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)$

But the co-ordinate of the centroid are (p,q,r)

$$\frac{a}{3}=p,\ \frac{b}{3}=q,\ \frac{c}{3}=r$$

Putting the values of a, b and c in (i), we get the required plane as $\frac{x}{3p}+\frac{y}{3q}+\frac{z}{3r}=1$

$\Rightarrow \quad \frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3$ **Ans.**

Do yourself - 4 :

(i) Equation of a plane is 3x + 4y + 5z = 7.

(a) Find the direction cosines of its normal

(b) Find the points where it intersects the axes.

(c) Find its intercept form.

(d) Find its equation in normal form (in cartesian as well as in vector form)

(ii) Find the equation of the plane passing through the points (2, 3, 1), (3, 0, 2) and (–1, 2, 3).

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15. ANGLE BETWEEN TWO PLANES :

Vector form : If $\vec{r}.\vec{n}_1=d_1$ and $\vec{r}.\vec{n}_2=d_2$ be two planes, then angle between these planes is the angle between their normals

$$\cos\theta=\frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

$\therefore$ Planes are perpendicular if $\vec{n}_1.\vec{n}_2=0$ and they are parallel if $\vec{n}_1=\lambda\vec{n}_2$.

Cartesian form : Consider two planes ax + by + cz + d = 0 and a' x + b' y + c' z + d' = 0. Angle between these planes is the angle between their normals.

$$\cos\theta=\frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}$$

$\therefore$ Planes are perpendicular if aa' + bb' + cc' = 0 and they are parallel if $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$.

Planes parallel to a given Plane :

Equation of a plane parallel to the plane ax + by + cz + d = 0 is ax + by + cz + d' = 0. d' is to be found by other given condition.

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Illustration 12 : Find the angle between the planes $x + y + 2z = 9$ and $2x - y + z = 15$

Solution : We know that the angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by $\cos\theta = \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

Therefore, angle between $x + y + 2z = 9$ and $2x - y + z = 15$ is given by

$$\cos\theta = \frac{(1)(2) + (1)(-1) + (2)(1)}{\sqrt{1^2 + 1^2 + 2^2}\sqrt{2^2 + (-1)^2 + 1^2}} = \frac{1}{2} \quad \Rightarrow \theta = \frac{\pi}{3}$$ **Ans.**

Illustration 13 : Find the equation of the plane through the point (1, 4, –2) and parallel to the plane $-2x + y - 3z = 7$.

Solution : Let the equation of a plane parallel to the plane $-2x + y - 3z = 7$ be $-2x + y - 3z + k = 0$

This passes through (1, 4, –2), therefore $(-2)(1) + 4 - 3(-2) + k = 0$

$\Rightarrow \quad -2 + 4 + 6 + k = 0 \quad \Rightarrow \quad k = -8$

Putting $k = -8$ in (i), we obtain $\quad -2x + y - 3z - 8 = 0 \quad$ or $\quad -2x + y - 3z = 8$

Ans.

This is the equation of the required plane.

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Do yourself - 5 :

(i) Prove that the planes $3x - 2y + z + 17 = 0$ and $4x + 3y - 6z - 25 = 0$ are perpendicular.

(ii) Find the angle between the planes $3x + 4y + z + 7 = 0$ and $-x + y - 2z = 5$

16. A PLANE THROUGH THE LINE OF INTERSECTION OF TWO GIVEN PLANES :

Consider two planes $u \equiv ax + by + cz + d = 0$ and $v \equiv a'x + b'y + c'z + d' = 0$.

The equation $u + \lambda v = 0$, λ a real parameter, represents the plane passing through the line of intersection of given planes and if planes are parallel, this represents a plane parallel to them.

Illustration 14 : Find the equation of plane containing the line of intersection of the plane $x + y + z - 6 = 0$ and $2x + 3y + 4z + 5 = 0$ and passing through (1,1,1).

Solution : The equation of the plane through the line of intersection of the given planes is,

$$(x + y + z - 6) + \lambda(2x + 3y + 4z + 5) = 0 \qquad .....(i)$$

If it is passes through (1,1,1)

$\Rightarrow \quad (1 + 1 + 1 - 6) + \lambda(2 + 3 + 4 + 5) = 0 \quad \Rightarrow \quad \lambda = \dfrac{3}{14}$

Putting $\lambda = 3/14$ in (i); we get $(x + y + z - 6) + \dfrac{3}{14}(2x + 3y + 4z + 5) = 0$

$\Rightarrow \quad 20x + 23y + 26z - 69 = 0$ **Ans.**

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17. PERPENDICULAR DISTANCE OF A POINT FROM THE PLANE :

Vector form : If $\vec{r}.\vec{n} = d$ be the plane, then perpendicular distance p, of the point A($\vec{a}$)

$$p = \frac{|\vec{a}.\vec{n} - d|}{|\vec{n}|}$$

Distance between two parallel planes $\vec{r}.\vec{n} = d_1$ & $\vec{r}.\vec{n} = d_2$ is $\left|\frac{d_1 - d_2}{|\vec{n}|}\right|$.

Cartesian form : Perpendicular distance p, of the point $A(x_1, y_1, z_1)$ from the plane $ax + by + cz + d = 0$ is given by $p = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{(a^2 + b^2 + c^2)}}$

Distance between two parallel planes $ax + by + cz + d_1 = 0$ & $ax + by + cz + d_2 = 0$ is $\left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right|$

Illustration 15 : Find the perpendicular distance of the point (2, 1, 0) from the plane $2x + y + 2z + 5 = 0$

Solution : We know that the perpendicular distance of the point (x_1, y_1, z_1) from the plane

$ax + by + cz + d = 0$ is $\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$

so required distance = $\frac{|2 \times 2 + 1 \times 1 + 2 \times 0 + 5|}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{10}{3}$ **Ans.**

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Illustration 16: Find the distance between the parallel planes $2x - y + 2z + 3 = 0$ and $4x - 2y + 4z + 5 = 0$.

Solution : Let $P(x_1, y_1, z_1)$ be any point on $2x - y + 2z + 3 = 0$, then $2x_1 - y_1 + 2z_1 + 3 = 0$

The length of the perpendicular from $P(x_1, y_1, z_1)$ to $4x - 2y + 4z + 5 = 0$ is

$$\left|\frac{4x_1 - 2y_1 + 4z_1 + 5}{\sqrt{4^2 + (-2)^2 + 4^2}}\right| = \left|\frac{2(2x_1 - y_1 + 2z_1) + 5}{\sqrt{36}}\right| = \frac{|2(-3) + 5|}{6} = \frac{1}{6} \text{ [using (i)]}$$

Therefore, the distance between the two given parallel planes is $\frac{1}{6}$ **Ans.**

Do yourself - 6 :

(i) Find the perpendicular distance of the point P(1, 2, 3) from the plane $2x + y + z + 1 = 0$.

(ii) Find the equation of the plane passing through the line of intersection of the planes $x + y + z = 5$ and $2x + 3y + z + 5 = 0$ and passing through the point (0, 0, 0).

18. BISECTORS OF ANGLES BETWEEN TWO PLANES :

Let the equations of the two planes be $ax + by + cz + d = 0$ and $a_1x + b_1y + c_1z + d_1 = 0$.

Then equations of bisectors of angles between them are given by

$$\frac{ax + by + cz + d}{\sqrt{(a^2 + b^2 + c^2)}} = \pm \frac{a_1x + b_1y + c_1z + d_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}}$$

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(a) **Equation of bisector of the angle containing origin :** First make both constant terms positive. Then positive sign give the bisector of the angle which contains the origin.

(b) **Bisector of acute/obtuse angle :** First making both constant terms positive,

$aa_1 + bb_1 + cc_1 > 0 \quad \Rightarrow$ origin lies in obtuse angle

$aa_1 + bb_1 + cc_1 < 0 \quad \Rightarrow$ origin lies in acute angle

---

Illustration 17: Find the equation of the bisector planes of the angles between the planes $2x - y + 2z + 3 = 0$ and $3x - 2y + 6z + 8 = 0$ and specify the plane which bisects the acute angle and the plane which bisects the obtuse angle.

Solution : The two given planes are $2x - y + 2z + 3 = 0$ and $3x - 2y + 6z + 8 = 0$

where $d_1, d_2 > 0$

and $a_1a_2 + b_1b_2 + c_1c_2 = 6 + 2 + 12 > 0$

$$\therefore \quad \frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = -\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \text{ (acute angle bisector)}$$

$$\text{and} \quad \frac{a_1x + b_1y + c_1z + d}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \text{ (obtuse angle bisector)}$$

$$\text{i.e.,} \quad \frac{2x - y + 2z + 3}{\sqrt{4 + 1 + 4}} = \pm\frac{3x - 2y + 6z + 8}{\sqrt{9 + 4 + 36}}$$

$\Rightarrow \quad (14x - 7y + 14z + 21) = \pm (9x - 6y + 18z + 24)$

Taking positive sign on the right hand side,

we get $\quad 5x - y - 4z - 3 = 0 \quad$ (obtuse angle bisector)

and taking negative sign on the right hand side,

we get $\quad 23x - 13y + 32z + 45 = 0 \quad$ (acute angle bisector) **Ans.**

---

19. POSITION OF TWO POINTS W.R.T. A PLANE :

Two points $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$ are on the same or opposite sides of a plane $ax + by + cz + d = 0$ according to $ax_1 + by_1 + cz_1 + d$ & $ax_2 + by_2 + cz_2 + d$ are of same or opposite signs. The plane divides the line joining the points P & Q externally or internally according to P and Q lying on same or opposite sides of the plane.

Do yourself - 7 :

(i) Find the position of the point P(2, –2, 1), Q(3, 0, 1) and R(–12, 1, 8) w.r.t. the plane $2x - 3y + 4z - 7 = 0$.

(ii) Two given planes are $-2x + y - 2z + 5 = 0$ and $6x - 2y + 3z - 7 = 0$. Find

(a) equation of plane bisecting the angle between the planes.

(b) equation of a plane parallel to the plane bisecting the angle between both the two planes and passing through the point (3, 2, 0).

(c) specify which plane is acute angle bisector and which one is obtuse angle bisector.

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STRAIGHT LINE

20. DEFINITION :

A straight line in space is characterised by the intersection of two planes which are not parallel and, therefore, the equation of a straight line is present as a solution of the system constituted by the equations of the two planes : $a_1 x + b_1 y + c_1 z + d_1 = 0$; $a_2 x + b_2 y + c_2 z + d_2 = 0$

This form is also known as **unsymmetrical form**.

Some particular straight lines :

	Straight lines	Equation
(i)	Through the origin	$y = mx,\ z = nx$
(ii)	x-axis	$y = 0,\ z = 0$ or $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$
(iii)	y-axis	$x = 0,\ z = 0$ or $\frac{x}{0} = \frac{y}{1} = \frac{z}{0}$
(iv)	z-axis	$x = 0,\ y = 0$ or $\frac{x}{0} = \frac{y}{0} = \frac{z}{1}$
(v)	parallel to x-axis	$y = p,\ z = q$
(vi)	parallel to y-axis	$x = h,\ z = q$
(vii)	parallel to z-axis	$x = h,\ y = p$

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21. EQUATION OF A STRAIGHT LINE IN SYMMETRICAL FORM :

(a) **One point form :** Let $A(x_1, y_1, z_1)$ be a given point on the straight line and ℓ, m, n be the d.c's of the line, then its equation is

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \quad \text{(say)}$$

It should be noted that $P(x_1 + \ell r, y_1 + mr, z_1 + nr)$ is a general point on this line at a distance r from the point $A(x_1, y_1, z_1)$ i.e. AP = r. One should note that for AP = r; ℓ, m, n must be d.c.'s not d.r.'s. If a, b, c are direction ratios of the line, then equation of the line is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = r \quad \text{but here } AP \neq r$$

(b) Equation of the line through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Illustration 18 : Find the co-ordinates of those points on the line $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{6}$ which is at a distance of 3 units from point (1,–2, 3).

Solution : Here, $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{6}$ .....(i)

is the given straight line

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Let, P = (1,–2,3) on the straight line
Here direction ratios of line (i) are (2,3,6)

$\therefore$ Direction cosines of line (i) are : $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$

$\Rightarrow$ Equations of line (i) can be written as

$$\frac{x-1}{2/7} = \frac{y+2}{3/7} = \frac{z-3}{6/7} \qquad .....(ii)$$

Co-ordinates of any point on the line (ii) can be taken as $\left(\frac{2}{7}r+1, \frac{3}{7}r-2, \frac{6}{7}r+3\right)$

Let, $Q\left(\frac{2}{7}r+1, \frac{3}{7}r-2, \frac{6}{7}r+3\right)$

Given $|\vec{r}| = 3, \therefore r = \pm 3$

Putting the value of r, we have $Q\left(\frac{13}{7}, -\frac{5}{7}, \frac{39}{7}\right)$ or $Q = \left(\frac{1}{7}, -\frac{23}{7}, \frac{3}{7}\right)$ **Ans.**

---

22. ANGLE BETWEEN A LINE AND A PLANE :

Let equations of the line and plane be $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and $ax + by + cz + d = 0$ respectively and θ be the angle which line makes with the plane. Then $(\pi/2 - \theta)$ is the angle between the line and the normal to the plane.

So, $\sin\theta = \frac{a\ell + bm + cn}{\sqrt{(a^2+b^2+c^2)}\sqrt{(\ell^2+m^2+n^2)}}$

Line is parallel to plane if $\theta = 0$ i.e. if $a\ell + bm + cn = 0$.

Line is perpendicular to the plane if line is parallel to the normal of the plane i.e. if $\frac{a}{\ell} = \frac{b}{m} = \frac{c}{n}$.

Illustration 19 : Find the angle between the line $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-3}{-2}$ and the plane $3x + 4y + z + 5 = 0$.

Solution : The given line is $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-3}{-2}$ ..... (i)

and the given plane is $3x + 4y + z + 5 = 0$ ..... (ii)

If the line (i) makes angle θ with the plane (ii), then the line (i) will make angle $(90° - \theta)$ with the normal to the plane (i). Now direction-ratios of line (i) are 3, –1, –2 and direction-ratios of normal to plane (ii) are 3, 4, 1

$$\therefore \quad \cos(90° - \theta) = \frac{(3)(3)+(-1)(4)+(-2)(1)}{\sqrt{9+1+4}\sqrt{9+16+1}} \Rightarrow \sin\theta = \frac{9-4-2}{\sqrt{14}\sqrt{26}} = \frac{3}{\sqrt{14}\sqrt{26}}$$

Hence $\theta = \sin^{-1}\left(\frac{3}{\sqrt{14}\sqrt{26}}\right)$ **Ans.**

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Do yourself - 8 :

(i) Find the equation of the line passing through the point (4, 2, 3) and having direction ratios 1, –1, 2

(ii) Find the symmetrical form of the line $x - y + 2z = 5$, $3x + y + z = 6$.

(iii) Find the angle between the plane $3x + 4y + 5 = 0$ and the line $\frac{x-1}{2} = \frac{y-2}{0} = \frac{z-1}{1}$.

(iv) Prove that the line $\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-5}{4}$ is parallel to the plane $4x + 4y - 5z + 2 = 0$.

23. CONDITION THAT A LINE LIES ON THE GIVEN PLANE :

The line $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ will lie on the plane $Ax + By + Cz + D = 0$ if

(a) $A\ell + Bm + Cn = 0$ and **(b)** $Ax_1 + By_1 + Cz_1 + D = 0$

24. IMAGE OF A POINT IN THE PLANE :

In order to find the image of a point $P(x_1, y_1, z_1)$ in a plane $ax + by + cz + d = 0$, assume it as a mirror. Let $Q(x_2, y_2, z_2)$ be the image of the point $P(x_1, y_1, z_1)$ in the plane, then

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(a) Line PQ is perpendicular to the plane. Hence equation of PQ is $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = r$

(b) Hence, Q satisfies the equation of line then $\frac{x_2-x_1}{a} = \frac{y_2-y_1}{b} = \frac{z_2-z_1}{c} = r$. The plane passes through the middle point of line PQ, therefore the middle point satisfies the equation of the plane i.e. $a\left(\frac{x_2+x_1}{2}\right) + b\left(\frac{y_2+y_1}{2}\right) + c\left(\frac{z_2+z_1}{2}\right) + d = 0$. The co-ordinates of Q can be obtained by solving these equations.

25. FOOT, LENGTH AND EQUATION OF PERPENDICULAR FROM A POINT TO A LINE:

Let equation of the line be $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$ .....(i)

and A (α, β, γ) be the point. Any point on the line (i) is $P(\ell r + x_1, mr + y_1, nr + z_1)$.....(ii)

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If it is the foot of the perpendicular, from A on the line, then AP is ⊥ to the line, so

$$\ell(\ell r + x_1 - \alpha) + m(mr + y_1 - \beta) + n(nr + z_1 - \gamma) = 0$$

i.e. $r = (\alpha - x_1)\ell + (\beta - y_1)m + (\gamma - z_1)n$

since $\ell^2 + m^2 + n^2 = 1$

Putting this value of r in (ii), we get the foot of perpendicular from point A to the line.

Length : Since foot of perpendicular P is known, length of perpendicular,

$$AP = \sqrt{[(\ell r + x_1 - \alpha)^2 + (mr + y_1 - \beta)^2 + (nr + z_1 - \gamma)^2]}$$

Equation of perpendicular is given by

$$\frac{x-\alpha}{\ell r + x_1 - \alpha} = \frac{y-\beta}{mr + y_1 - \beta} = \frac{z-\gamma}{nr + z_1 - \gamma}$$

Illustration 20 : Find the co-ordinates of the foot of the perpendicular from (1, 1, 1) on the line joining (5, 4, 4) and (1, 4, 6).

Solution : Let A (1, 1, 1), B (5, 4, 4) and C (1, 4, 6) be the given points. Let M be the foot of the perpendicular from A on BC.

If M divides BC in the ratio λ : 1, then

co-ordinates of M are $\left(\frac{\lambda+5}{\lambda+1}, \frac{4\lambda+4}{\lambda+1}, \frac{6\lambda+4}{\lambda+1}\right)$

Direction ratios of BC are 1 – 5, 4 – 4, 6 – 4

i.e. –4, 0, 2

D.R.'s of AM are $\frac{\lambda+5}{\lambda+1} - 1, \frac{4\lambda+4}{\lambda+1} - 1, \frac{6\lambda+4}{\lambda+1} - 1$

$\Rightarrow \quad \frac{4}{\lambda+1}, \frac{3\lambda+3}{\lambda+1}, \frac{5\lambda+3}{\lambda+1} \quad \Rightarrow \quad 4, 3\lambda + 3, 5\lambda + 3$

Since AM ⊥ BC

$\therefore \quad 2(4) + 0(3\lambda + 3) - 1(5\lambda + 3) = 0 \quad \Rightarrow 8 - 5\lambda - 3 = 0 \quad \Rightarrow \quad \lambda = 1$

Hence the co-ordinates of M are (3, 4, 5) **Ans.**

Illustration 21 : Find the length of perpendicular from P(2, – 3,1) to the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1}$

Solution : Given line is $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1}$ .....(i)

and P(2, – 3,1)

Co-ordinates of any point on (i) may be taken as

(2r–1,3r+3,–r–2)

Let Q = (2r –1, 3r + 3,–r – 2)

Direction ratio's of PQ are : (2r –3, 3r +6, –r –3)

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Direction ratio's of AB are : (2, 3, –1)

Since, $PQ \perp AB$

$$2(2r-3)+3(3r+6)-1(-r-3)=0$$

$$\Rightarrow \quad r=-\frac{15}{14}$$

$$\therefore \quad Q=\left(-\frac{22}{7},-\frac{3}{14},-\frac{13}{14}\right)$$

$$PQ^2=\left(2+\frac{22}{7}\right)^2+\left(-3+\frac{3}{14}\right)^2+\left(1+\frac{13}{14}\right)^2=\frac{531}{14}$$

$$PQ=\sqrt{\frac{531}{14}} \text{ units}$$ **Ans.**

Do yourself - 9 :

(i) Find the image of point P(1, 3, 2) in the plane $2x-y+z+3=0$ as well as the foot of the perpendicular drawn from the point (1, 3, 2).

(ii) Find the distance of the point (1, –2, 3) from the plane $x-y+z=5$ measured parallel to the line $\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$

(iii) Prove that $\frac{x+1}{-2}=\frac{y+2}{3}=\frac{z+5}{4}$ lies in the plane $x+2y-z=0$.

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26. EQUATION OF PLANE CONTAINING TWO INTERSECTING LINES :

Let the two lines be

$$\frac{x-\alpha_1}{\ell_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1} \quad \text{..... (i)}$$

and

$$\frac{x-\alpha_2}{\ell_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2} \quad \text{..... (ii)}$$

These lines will coplanar if $\begin{vmatrix} \alpha_2-\alpha_1 & \beta_2-\beta_1 & \gamma_2-\gamma_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix}=0$ (It is condition for intersection of two lines)

the plane containing the two lines is $\begin{vmatrix} x-\alpha_1 & y-\beta_1 & z-\gamma_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix}=0$

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Illustration 22 : Find the equation of the plane containing the line $\frac{x-1}{3}=\frac{y+6}{4}=\frac{z+1}{2}$ and parallel to the line $\frac{x-4}{2}=\frac{y-1}{-3}=\frac{z+3}{5}$.

Solution : Any plane containing the line $\frac{x-1}{3}=\frac{y+6}{4}=\frac{z+1}{2}$ is

$a(x-1)+b(y+6)+c(z+1)=0$ ..... (i)

where, $3a+4b+2c=0$ ..... (ii)

Also, it is parallel to the second line and hence, its normal is perpendicular to this line

$\therefore$ $2a-3b+5c=0$ ..... (iii)

Solving (ii) & (iii) by cross multiplication, we get $\frac{a}{26}=\frac{b}{-11}=\frac{c}{-17}=k$

$\Rightarrow$ $a=26k,\ b=-11k\ \&\ c=-17k$

Putting these values in (i), we get $26k(x-1)-11k(y+6)-17k(z+1)=0$

$\Rightarrow$ $26x-11y-17z=109$, which is the required equation of the plane.

27. LINE OF GREATEST SLOPE :

Consider two planes G-plane and H-plane. H-plane is treated as a horizontal plane or reference plane. G-plane is a given plane. Let AB be the line of intersection of G-plane & H-plane. Line of greatest slope is a line which is contained by G-plane & perpendicular to line of intersection of G-plane & H-plane. Obviously, infinitely many such lines of greatest slopes are contained by G-plane. Generally an additional information is given in problem so that a unique line of greatest slope can be found out.

Illustration 23 : Assuming the plane $4x-3y+7z=0$ to be horizontal, find the equation of the line of greatest slope on the plane $2x+y-5z=0$, passing through the point (2,1,1).

Solution : The required line passing through the point P (2,1,1) in the plane $2x+y-5z=0$ and is having greatest slope, so it must be perpendicular to the line of intersection of the planes

$2x+y-5z=0$ .....(i)

and $4x-3y+7z=0$ .....(ii)

Let the d.r'.s of the line of intersection of (i) and (ii) be a, b, c

$\Rightarrow$ $2a+b-5c=0$ and $4a-3b+7c=0$

{as dr'.s of straight line (a, b,c) is perpendicular to d.r'.s of normal to both the planes}

$\Rightarrow$ $\frac{a}{4}=\frac{b}{17}=\frac{c}{5}$

Now let the direction ratio of required line be proportional to ℓ, m, n then its equation be $\frac{x-2}{\ell}=\frac{y-1}{m}=\frac{z-1}{n}$

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where $2\ell + m - 5n = 0$ and $4\ell + 17m + 5n = 0$

so, $\dfrac{\ell}{3} = \dfrac{m}{-1} = \dfrac{n}{1}$

Thus the required line is $\dfrac{x-2}{3} = \dfrac{y-1}{-1} = \dfrac{z-1}{1}$ **Ans.**

28. AREA OF TRIANGLE :

To find the area of a triangle in terms of its projections on the co-ordinates planes.

Let Δ_x, Δ_y, Δ_z be the projections of the plane area of the triangle on the planes yOz, zOx, xOy respectively.Let ℓ, m, n be the direction cosines of the normal to the plane of the triangle.

Then the angle between the plane of the triangle and yOz plane is the angle between the normal to the plane of the triangle and the x-axis.

$\therefore \quad \Delta_x = \Delta\ell$

Similarly $\quad \Delta_y = \Delta m$; $\Delta_z = \Delta n \quad \Rightarrow \Delta = \sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2}$

If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ be the three vertices of the triangle then

$$\Delta_x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}, \Delta_y = \frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}, \Delta_z = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

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Do yourself - 10 :

(i) Prove that the lines $\dfrac{x+1}{3} = \dfrac{y+3}{5} = \dfrac{z+5}{7}$ and $\dfrac{x-2}{1} = \dfrac{y-4}{3} = \dfrac{z-6}{5}$ are coplanar. Find their point of intersection.

(ii) Find the area of the triangle whose vertices are the points (1, 2, 3), (–2, 1, –4), (3, 4, –2).

Miscellaneous Illustrations :

Illustration 24 : If a variable plane cuts the coordinate axes in A, B and C and is at a constant distance p from the origin, find the locus of the centre of the tetrahedron OABC.

Solution : Let $A \equiv (a, 0, 0)$, $B \equiv (0, b, 0)$ and $C \equiv (0, 0, c)$

$\therefore$ Equation of plane ABC is $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$

Now p = length of perpendicular from O to this plane

$$= \frac{1}{\sqrt{\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}}} \quad \text{or} \quad p^2 = \frac{1}{\left(\dfrac{1}{a}\right)^2 + \left(\dfrac{1}{b}\right)^2 + \left(\dfrac{1}{c}\right)^2} \quad ...(i)$$

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Let G(α, β, γ) be the centre of the tetrahedron OABC, then

$$\alpha = \frac{a}{4},\ \beta = \frac{b}{4},\ \gamma = \frac{c}{4} \qquad \left[\because\ \alpha = \frac{a+0+0+0}{4} = \frac{a}{4}\right]$$

or, $a = 4\alpha,\ b = 4\beta,\ c = 4\gamma$

Putting these values of a, b, c in equation (i), we get

$$p^2 = \frac{16}{\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right)} \quad \text{or} \quad \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{16}{p^2}$$

$\therefore$ locus of (α, β, γ) is $x^{-2} + y^{-2} + z^{-2} = 16\,p^{-2}$ **Ans.**

Illustration 25 : Through a point P(h, k, ℓ) a plane is drawn at right angles to OP to meet the coordinate axes in A, B and C. If OP = p, show that the area of ΔABC is $\frac{p^5}{2|hk\ell|}$.

Solution : $OP = \sqrt{h^2+k^2+\ell^2} = p$

Direction cosines of OP are $\frac{h}{\sqrt{h^2+k^2+\ell^2}}, \frac{k}{\sqrt{h^2+k^2+\ell^2}}, \frac{\ell}{\sqrt{h^2+k^2+\ell^2}}$

Since OP is normal to the plane, therefore, equation of the plane will be,

$$\frac{h}{\sqrt{h^2+k^2+\ell^2}}x + \frac{k}{\sqrt{h^2+k^2+\ell^2}}y + \frac{\ell}{\sqrt{h^2+k^2+\ell^2}}z = \sqrt{h^2+k^2+\ell^2}$$

or, $hx + ky + \ell z = h^2 + k^2 + \ell^2 = p^2$

$$\therefore\quad A \equiv \left(\frac{p^2}{h},\ 0,\ 0\right),\ B \equiv \left(0,\ \frac{p^2}{k},\ 0\right),\ C \equiv \left(0,\ 0,\ \frac{p^2}{\ell}\right)$$

Now area of ΔABC, $\Delta^2 = A_{xy}^2 + A_{yz}^2 + A_{zx}^2$

Now A_{xy} = area of projection of ΔABC on xy-plane = area of ΔAOB

$$= \text{Mod of } \frac{1}{2}\begin{vmatrix} \frac{p^2}{h} & 0 & 1 \\ 0 & \frac{p^2}{k} & 1 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{2}\frac{p^4}{|hk|}$$

Similarly, $A_{yz} = \frac{1}{2}\frac{p^4}{|k\ell|}$ and $A_{zx} = \frac{1}{2}\frac{p^4}{|\ell h|}$

$$\therefore\quad \Delta^2 = \frac{1}{4}\frac{p^8}{h^2k^2} + \frac{1}{4}\frac{p^8}{k^2\ell^2} + \frac{1}{4}\frac{p^8}{h^2\ell^2} = \frac{p^{10}}{4h^2k^2\ell^2}$$

or $\Delta = \frac{p^5}{2|hk\ell|}$ **Ans.**

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Illustration 26 : Find the locus of a point, the sum of squares of whose distances from the planes : $x - z = 0$, $x - 2y + z = 0$ and $x + y + z = 0$ is 36

Solution : Given planes are $x - z = 0$, $x - 2y + z = 0$ and, $x + y + z = 0$

Let the point whose locus is required be $P(\alpha,\beta,\gamma)$. According to question

$$\frac{|\alpha-\gamma|^2}{2}+\frac{|\alpha-2\beta+\gamma|^2}{6}+\frac{|\alpha+\beta+\gamma|^2}{3}=36$$

or $3(\alpha^2+\gamma^2-2\alpha\gamma)+\alpha^2+4\beta^2+\gamma^2-4\alpha\beta-4\beta\gamma+2\alpha\gamma+2(\alpha^2+\beta^2+\gamma^2+2\alpha\beta+2\beta\gamma+2\alpha\gamma)=36\times 6$

or $6\alpha^2+6\beta^2+6\gamma^2=36\times 6$

or $\alpha^2+\beta^2+\gamma^2=36$

Hence, the required equation of locus is $x^2+y^2+z^2=36$ **Ans.**

Illustration 27 : Direction ratios of normal to the plane which passes through the point (1, 0, 0) and (0, 1, 0) which makes angle $\pi/4$ with $x + y = 3$ are -

(A) 1, 1, 2 (B) $\sqrt{2}$, 1, 1 (C) 1, $\sqrt{2}$, 1 (D) 1, 1, $\sqrt{2}$

Solution : The plane by intercept form is $\frac{x}{1}+\frac{y}{1}+\frac{z}{c}=1$

direction ratios of normal of this plane are $1, 1, \frac{1}{c}$ and that of given plane are 1, 1, 0.

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$$\therefore \quad \cos\frac{\pi}{4}=\frac{1.1+1.1+0\cdot\frac{1}{c}}{\sqrt{1+1+\frac{1}{c^2}}\sqrt{1+1+0}}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2+\frac{1}{c^2}}\sqrt{2}} \Rightarrow 2+\frac{1}{c^2}=4 \Rightarrow \quad c=\pm\frac{1}{\sqrt{2}}$$

$\therefore$ d.r.'s are 1, 1, $\sqrt{2}$ **Ans. (D)**

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ANSWERS FOR DO YOURSELF

1: **(i)** $2\sqrt{21}$ **(iii)** $8x + 2y + 24z \pm 2k^2 + 9 = 0$ **(iv)** $\left(-2, \frac{8}{3}, 5\right)$ & $\left(-1, \frac{10}{3}, 6\right)$

(v) **(a)** 7 : 8, externally **(b)** 2 : 3 internally

2 : **(i)** 3, 2, 5 **(ii)** 3, 4, 3 **(iii)** 2, 3, 4 & $\frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}$

3 : **(i)** $\theta = \frac{\pi}{2}$ **(iii)** $\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}$

4 : **(i)** **(a)** $\frac{3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}}$ **(b)** $\left(\frac{7}{3}, 0, 0\right), \left(0, \frac{7}{4}, 0\right)$ & $\left(0, 0, \frac{7}{5}\right)$ **(c)** $\frac{x}{7/3} + \frac{y}{7/4} + \frac{z}{7/5} = 1$

(d) $\frac{3x}{5\sqrt{2}} + \frac{4y}{5\sqrt{2}} + \frac{z}{\sqrt{2}} = \frac{7}{5\sqrt{2}}$ & $\vec{r}.\left(\frac{3}{5\sqrt{2}}\hat{i} + \frac{4}{5\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}\right) = \frac{7}{5\sqrt{2}}$

(ii) $x + y + 2z = 7$

5: **(ii)** $\theta = \cos^{-1}\left(\frac{1}{\sqrt{156}}\right)$

6 : **(i)** $\frac{8}{\sqrt{6}}$ **(ii)** $3x + 4y + 2z = 0$

7 : **(i)** P, Q same side & R opposite side

(ii) **(a)** $4x + y - 5z + 14 = 0$ & $32x - 13y + 23z - 56 = 0$

(b) $4x + y - 5z - 14 = 0$ & $32x - 13y + 23z - 70 = 0$

(c) $4x + y - 5z + 14 = 0$ (obtuse angle bisector) & $32x - 13y + 23z - 56 = 0$ (acute angle bisector)

8 : **(i)** $\frac{x-4}{1} = \frac{y-2}{-1} = \frac{z-3}{2}$ **(ii)** $\frac{x - 11/4}{-3} = \frac{y + 9/4}{5} = \frac{z-0}{4}$ **(iii)** $\theta = \sin^{-1}\left(\frac{6}{5\sqrt{5}}\right)$

9 : **(i)** $\left(\frac{-5}{3}, \frac{13}{3}, \frac{2}{3}\right)$ & $\left(\frac{-1}{3}, \frac{11}{3}, \frac{4}{3}\right)$ **(ii)** 1

10: **(i)** $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$ **(ii)** $\frac{\sqrt{1218}}{2}$

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EXERCISE (O-1)

[STRAIGHT OBJECTIVE TYPE]

1. Consider three vectors $\vec{p} = \hat{i} + \hat{j} + \hat{k}$, $\vec{q} = 2\hat{i} + 4\hat{j} - \hat{k}$ and $\vec{r} = \hat{i} + \hat{j} + 3\hat{k}$. If $\vec{p}$, $\vec{q}$ and $\vec{r}$ denotes the position vector of three non-collinear points then the equation of the plane containing these points is

(A) $2x - 3y + 1 = 0$ (B) $x - 3y + 2z = 0$ (C) $3x - y + z - 3 = 0$ (D) $3x - y - 2 = 0$

TD0001

2. The intercept made by the plane $\vec{r}.\vec{n} = q$ on the x-axis is

(A) $\dfrac{q}{\hat{i}.\vec{n}}$ (B) $\dfrac{\hat{i}.\vec{n}}{q}$ (C) $(\hat{i}.\vec{n})q$ (D) $\dfrac{q}{|\vec{n}|}$

TD0002

3. If the distance between the planes

$8x + 12y - 14z = 2$

and $4x + 6y - 7z = 2$

can be expressed in the form $\dfrac{1}{\sqrt{N}}$ where N is natural then the value of $\dfrac{N(N+1)}{2}$ is

(A) 4950 (B) 5050 (C) 5150 (D) 5151

TD0003

4. A plane passes through the point P(4, 0, 0) and Q(0, 0, 4) and is parallel to the y-axis. The distance of the plane from the origin is

(A) 2 (B) 4 (C) $\sqrt{2}$ (D) $2\sqrt{2}$

TD0004

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5. If from the point P (f, g, h) perpendiculars PL, PM be drawn to yz and zx planes then the equation to the plane OLM is

(A) $\dfrac{x}{f} + \dfrac{y}{g} - \dfrac{z}{h} = 0$ (B) $\dfrac{x}{f} + \dfrac{y}{g} + \dfrac{z}{h} = 0$ (C) $\dfrac{x}{f} - \dfrac{y}{g} + \dfrac{z}{h} = 0$ (D) $-\dfrac{x}{f} + \dfrac{y}{g} + \dfrac{z}{h} = 0$

TD0005

6. If the plane $2x - 3y + 6z - 11 = 0$ makes an angle $\sin^{-1}(k)$ with x-axis, then k is equal to

(A) $\sqrt{3}/2$ (B) 2/7 (C) $\sqrt{2}/3$ (D) 1

TD0006

7. The plane XOZ divides the join of (1, –1, 5) and (2, 3, 4) in the ratio $\lambda : 1$, then λ is

(A) – 3 (B) – 1/3 (C) 3 (D) 1/3

TD0007

8. A variable plane forms a tetrahedron of constant volume 64 K^3 with the coordinate planes and the origin, then locus of the centroid of the tetrahedron is

(A) $x^3 + y^3 + z^3 = 6K^3$ (B) $xyz = 6k^3$

(C) $x^2 + y^2 + z^2 = 4K^2$ (D) $x^{-2} + y^{-2} + z^{-2} = 4K^{-2}$

TD0008

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9. Let ABCD be a tetrahedron such that the edges AB, AC and AD are mutually perpendicular. Let the area of triangles ABC, ACD and ADB be 3, 4 and 5 sq. units respectively. Then the area of the triangle BCD, is

(A) $5\sqrt{2}$ (B) 5 (C) $5/\sqrt{2}$ (D) 5/2

TD0009

10. Equation of the line which passes through the point with p. v. (2, 1, 0) and perpendicular to the plane containing the vectors $\hat{i}+\hat{j}$ and $\hat{j}+\hat{k}$ is

(A) $\vec{r} = (2, 1, 0) + t(1, -1, 1)$ (B) $\vec{r} = (2, 1, 0) + t(-1, 1, 1)$

(C) $\vec{r} = (2, 1, 0) + t(1, 1, -1)$ (D) $\vec{r} = (2, 1, 0) + t(1, 1, 1)$

where t is a parameter

TD0010

11. Which of the following planes are parallel but not identical?

$P_1 : 4x - 2y + 6z = 3$

$P_2 : 4x - 2y - 2z = 6$

$P_3 : -6x + 3y - 9z = 5$

$P_4 : 2x - y - z = 3$

(A) P_2 & P_3 (B) P_2 & P_4 (C) P_1 & P_3 (D) P_1 & P_4

TD0011

12. A parallelopiped is formed by planes drawn through the points (1, 2, 3) and (9, 8, 5) parallel to the coordinate planes then which of the following is not the length of an edge of this rectangular parallelopiped

(A) 2 (B) 4 (C) 6 (D) 8

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TD0012

13. Vector equation of the plane $\vec{r} = \hat{i} - \hat{j} + \lambda(\hat{i}+\hat{j}+\hat{k}) + \mu(\hat{i} - 2\hat{j} + 3\hat{k})$ in the scalar dot product form is

(A) $\vec{r}.(5\hat{i} - 2\hat{j} + 3\hat{k}) = 7$ (B) $\vec{r}.(5\hat{i} + 2\hat{j} - 3\hat{k}) = 7$

(C) $\vec{r}.(5\hat{i} - 2\hat{j} - 3\hat{k}) = 7$ (D) $\vec{r}.(5\hat{i} + 2\hat{j} + 3\hat{k}) = 7$

TD0013

14. The vector equations of the two lines L_1 and L_2 are given by

$L_1 : \vec{r} = 2\hat{i} + 9\hat{j} + 13\hat{k} + \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$; $L_2 : \vec{r} = -3\hat{i} + 7\hat{j} + p\hat{k} + \mu(-\hat{i} + 2\hat{j} - 3\hat{k})$

then the lines L_1 and L_2 are

(A) skew lines for all $p \in R$

(B) intersecting for all $p \in R$ and the point of intersection is (–1, 3, 4)

(C) intersecting lines for $p = -2$

(D) intersecting for all real $p \in R$

TD0014

15. Consider the plane $(x, y, z) = (0, 1, 1) + \lambda(1, -1, 1) + \mu(2, -1, 0)$. The distance of this plane from the origin is

(A) 1/3 (B) $\sqrt{3}/2$ (C) $\sqrt{3/2}$ (D) $2/\sqrt{3}$

TD0015

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16. The value of 'a' for which the lines $\frac{x-2}{1}=\frac{y-9}{2}=\frac{z-13}{3}$ and $\frac{x-a}{-1}=\frac{y-7}{2}=\frac{z+2}{-3}$ intersect, is

(A) – 5 (B) – 2 (C) 5 (D) – 3

TD0016

17. Given A (1, –1, 0) ; B(3, 1, 2) ; C(2, –2, 4) and D(–1, 1, –1) which of the following points neither lie on AB nor on CD?

(A) (2, 2, 4) (B) (2, –2, 4) (C) (2, 0,1) (D) (0, –2, –1)

TD0017

18. For the line $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$, which one of the following is incorrect?

(A) it lies in the plane x – 2y + z = 0
(B) it is same as line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$
(C) it passes through (2, 3, 5)
(D) it is parallel to the plane x – 2y + z – 6 = 0

TD0018

19. Given planes

$P_1 : cy + bz = x$
$P_2 : az + cx = y$
$P_3 : bx + ay = z$

P_1, P_2 and P_3 pass through one line, if

(A) $a^2 + b^2 + c^2 = ab + bc + ca$
(B) $a^2 + b^2 + c^2 + 2abc = 1$
(C) $a^2 + b^2 + c^2 = 1$
(D) $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca + 2abc = 1$

TD0019

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20. The line $\frac{x-x_1}{0}=\frac{y-y_1}{1}=\frac{z-z_1}{2}$ is

(A) parallel to x-axis
(B) perpendicular to x-axis
(C) perpendicular to YOZ plane
(D) parallel to y-axis

TD0020

21. The lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar if

(A) k = 0 or – 1 (B) k = 1 or – 1 (C) k = 0 or – 3 (D) k = 3 or – 3

TD0021

22. The line which contains all points (x, y, z) which are of the form (x, y, z) = (2, –2, 5) + λ(1, –3, 2) intersects the plane 2x – 3y + 4z = 163 at P and intersects the YZ plane at Q. If the distance PQ is $a\sqrt{b}$ where a, b ∈ N and a > 3 then (a + b) equals

(A) 23 (B) 95 (C) 27 (D) none

TD0022

23. Let L_1 be the line $\vec{r}_1 = 2\hat{i}+\hat{j}-\hat{k}+\lambda(\hat{i}+2\hat{k})$ and let L_2 be the line $\vec{r}_2 = 3\hat{i}+\hat{j}+\mu(\hat{i}+\hat{j}-\hat{k})$. Let Π be the plane which contains the line L_1 and is parallel to L_2. The distance of the plane Π from the origin is

(A) 1/7 (B) $\sqrt{2/7}$ (C) $\sqrt{6}$ (D) none

TD0023

24. The value of m for which straight line 3x – 2y + z + 3 = 0 = 4x – 3y + 4z + 1 is parallel to the plane 2x – y + mz – 2 = 0 is

(A) –2 (B) 8 (C) – 18 (D) 11

TD0024

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25. The distance of the point (–1, –5, – 10) from the point of intersection of the line $\frac{x-2}{2}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane x – y + z = 5 is

(A) $2\sqrt{11}$ (B) $\sqrt{126}$ (C) 13 (D) 14

TD0025

26. $P(\vec{p})$ and $Q(\vec{q})$ are the position vectors of two fixed points and $R(\vec{r})$ is the position vector of a variable point. If R moves such that $(\vec{r}-\vec{p})\times(\vec{r}-\vec{q})=0$ then the locus of R is

(A) a plane containing the origin 'O' and parallel to two non collinear vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$

(B) the surface of a sphere described on PQ as its diameter.

(C) a line passing through the points P and Q

(D) a set of lines parallel to the line PQ.

TD0026

[MATRIX MATCH TYPE]

27. Consider the following four pairs of lines in **column-I** and match them with one or more entries in **column-II**.

	Column-I		**Column-II**
(A)	$L_1 : x = 1 + t,\ y = t,\ z = 2 - 5t$ $L_2 : \vec{r} = (2,1,-3) + \lambda(2, 2, -10)$	(P)	non coplanar lines
(B)	$L_1 : \frac{x-1}{2} = \frac{y-3}{2} = \frac{z-2}{-1}$ $L_2 : \frac{x-2}{1} = \frac{y-6}{-1} = \frac{z+2}{3}$	(Q)	lines lie in a unique plane
(C)	$L_1 : x = -6t,\ y = 1 + 9t,\ z = -3t$ $L_2 : x = 1 + 2s,\ y = 4 - 3s,\ z = s$	(R)	infinite planes containing both the lines
(D)	$L_1 : \frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ $L_2 : \frac{x-3}{-4} = \frac{y-2}{-3} = \frac{z-1}{2}$	(S)	lines are not intersecting at a unique point

TD0027

TD0028

TD0029

TD0030

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EXERCISE (O-2)

[MULTIPLE OBJECTIVE TYPE]

1. The volume of a right triangular prism $ABCA_1B_1C_1$ is equal to 3. If the position vectors of the vertices of the base ABC are A(1, 0, 1) ; B(2,0, 0) and C(0, 1, 0) then the position vectors of the vertex A_1 can be:

(A) (2, 2, 2) (B) (0, 2, 0)

(C) (0, – 2, 2) (D) (0, – 2, 0)

TD0031

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2. Consider Lines $L_1 : \frac{x-\alpha}{1} = \frac{y}{-2} = \frac{z+\beta}{2}$, $L_2 : x = \alpha, \frac{y}{-\alpha} = \frac{z+\alpha}{2-\beta}$, plane $P : 2x + 2y + z + 7 = 0$. Let line L_2 lies in plane P, then

(A) $\alpha = -7$

(B) $\alpha = 7$

(C) minimum distance between line L_1 and plane P is 11.

(D) minimum distance between line L_1 and plane P is $\frac{23}{3}$

TD0032

3. The point $\vec{A}$ (3,4,7), $\vec{B}$ (4,5,9) and $\vec{C}$ (1,2,–1) are three vertices of a parallelogram ABCD, then -

(A) vector equation of line AB is $\vec{r} = 4\hat{i} + 5\hat{j} + 9\hat{k} + \lambda\left(\hat{i} + \hat{j} + 2\hat{k}\right)$

(B) cartesian equation of line BC is $\frac{x-4}{3} = \frac{x-5}{3} = \frac{z-9}{10}$

(C) coordinates of D are (0,1,–3)

(D) ABCD is rectangle

TD0033

4. Let two planes $P_1 : 2x - y + z - 2 = 0$ and $P_2 : x + 2y - z - 3 = 0$ are given then-

(A) The equation of the plane through line of intersection of $P_1 = 0$ and $P_2 = 0$ and the point (3,2,1) is $x - 3y + 2z + 1 = 0$

(B) The equation of the plane through line of intersection of $P_1 = 0$ and $P_2 = 0$ and the point (3,2,1) is $3x - y + 2z - 9 = 0$

(C) The equation of acute angle bisector plane of $P_1 = 0$ and $P_2 = 0$ is $x - 3y + 2z + 1 = 0$

(D) The equation of acute angle bisector plane of $P_1 = 0$ and $P_2 = 0$ is $x + 3y + 2z + 2 = 0$

TD0034

5. A variable point P(4 cost, 4sint, 4sint) moves in space, now which of the following holds good ?

(A) Point P moves on plane $ax + by + cz + d = 0$

(B) Point 'P' traces a circle.

(C) Area enclosed by P is $16\sqrt{2}\pi$

(D) Point P cannot lie on a fixed plane.

TD0035

6. The projection of line $\frac{x}{2} = \frac{y-1}{2} = \frac{z-1}{1}$ on a plane 'P' is $\frac{x}{1} = \frac{y-1}{1} = \frac{z-1}{-1}$. If the plane P passes through (k, –2, 0), then k is greater than -

(A) 2 (B) 3 (C) 5 (D) 4

TD0036

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7. A line segment has length 6 and direction ratios are –3,4,6, then the component of the line vector are-

(A) $\frac{-18}{\sqrt{61}}, \frac{24}{\sqrt{61}}, \frac{36}{\sqrt{61}}$ (B) 27,–18,–54 (C) 27,–18,54 (D) $\frac{18}{\sqrt{61}}, \frac{-24}{\sqrt{61}}, \frac{-36}{\sqrt{61}}$

TD0037

8. Which of the following is (are) correct -

(A) If two lines in space are not intersecting, then they must be skew lines.

(B) If two lines are parallel to a plane 'P', then their direction ratios will be proportional

(C) If two lines are perpendicular to a plane 'P', then their direction ratios will be proportional

(D) Equation $\frac{x+1}{a} = \frac{y-1}{b} = \frac{z}{c}$, where a, b, c are real parameters, represents a family of concurrent lines in space

TD0038

9. Given the equations of the line $3x - y + z + 1 = 0,\ 5x + y + 3z = 0$.

Then which of the following is correct ?

(A) Symmetrical form of the equations of line is $\frac{x}{2} = \frac{y - \frac{1}{8}}{-1} = \frac{z + \frac{5}{8}}{1}$

(B) symmetrical form of the equations of line is $\frac{x + \frac{1}{8}}{1} = \frac{y - \frac{5}{8}}{1} = \frac{z}{-2}$

(C) equation of the plane through (2, 1, 4) and prependicular to the given lines is $2x - y + z - 7 = 0$

(D) equation of the plane through (2, 1, 4) and prependicular to the given lines is $x + y - 2z + 5 = 0$

TD0039

10. Consider the family of planes $x + y + z = c$ where is a parameter intersecting the coordinate axes at P, Q,R and α,β,γ are the angles made by each member of this family with positive x,y and z axis. Which of the following interpretations hold good for this family -

(A) each member of this family is equally inclined with the coordinate axes.

(B) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 1$

(C) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 2$

(D) for c = 3 area of the triangle PQR is $3\sqrt{3}$ sq. units.

TD0040

11. Consider a plane P passing through $A(\lambda,3,\mu)$, B(–1,3,2) and C(7,5,10) and a straight line L with positive direction cosines passing through A, bisecting BC and makes equal angles with the coordinate axes. Let L_1 be a line parallel to L and passing through origin. Which of the following is(are) correct?

(A) The value of $(\lambda + \mu)$ is equal to 5.

(B) Equation of straight line L_1 is $\frac{x-1}{1} = \frac{y-1}{1} = \frac{z-1}{1}$.

(C) Equation of the plane perpendicular to the plane P and containing line L_1 is $x - 2y + z = 0$

(D) Area of triangle ABC is equal to $3\sqrt{2}$.

TD0041

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12. A line L passing through the point P(1,4,3), is perpendicular to both the lines

$$\frac{x-1}{2}=\frac{y+3}{1}=\frac{z-2}{4} \text{ and } \frac{x+2}{3}=\frac{y-4}{2}=\frac{z+1}{-2}.$$

If the position vector of point Q on L is(a_1,a_2,a_3) such that $(PQ)^2 = 357$, then $(a_1 + a_2 + a_3)$ can be-

(A) 16 (B) 15 (C) 2 (D) 1

TD0042

[MATRIX MATCH TYPE]

13. P(0, 3, – 2); Q(3, 7, – 1) and R(1, – 3, – 1) are 3 given points. Let L_1 be the line passing through P and Q and L_2 be the line through R and parallel to the vector $\vec{V}=\hat{i}+\hat{k}$.

	Column-I		Column-II
(A)	perpendicular distance of P from L_2	(P)	$7\sqrt{3}$
(B)	shortest distance between L_1 and L_2	(Q)	2
(C)	area of the triangle PQR	(R)	6
(D)	distance from (0, 0, 0) to the plane PQR	(S)	$\frac{19}{\sqrt{147}}$

TD0043

EXERCISE (S-1)

1. Find the angle between the two straight lines whose direction cosines ℓ, m, n are given by $2\ell + 2m - n = 0$ and $mn + n\ell + \ell m = 0$.

TD0044

2. The plane denoted by $\Pi_1 : 4x + 7y + 4z + 81 = 0$ is rotated through a right angle about its line of intersection with the plane $\Pi_2 : 5x + 3y + 10z = 25$. If the plane in its new position be denoted by Π, and the distance of this plane from the origin is $\sqrt{k}$ where $k \in N$, then find k.

TD0045

3. Find the equations of the straight line passing through the point (1,2,3) to intersect the straight line $x + 1 = 2(y - 2) = z + 4$ and parallel to the plane $x + 5y + 4z = 0$.

TD0046

4. A variable plane is at a constant distance p from the origin and meets the coordinate axes in points A,B and C respectively. Through these points, planes are drawn parallel to the coordinates planes. Find the locus of their point of intersection.

TD0047

5. Find the value of p so that the lines $\frac{x+1}{-3}=\frac{y-p}{2}=\frac{z+2}{1}$ and $\frac{x}{1}=\frac{y-7}{-3}=\frac{z+7}{2}$ are in the same plane. for this value of p, find the coordinates of their point of intersection and the equation of the plane containing them.

TD0048

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6. Find the equations to the line of greatest slope through the point (7, 2, –1) in the plane $x-2y+3z=0$ assuming that the axes are so placed that the plane $2x+3y-4z=0$ is horizontal.

TD0049

7. Let L be the line given by $\vec{r}=\begin{bmatrix}2\\-2\\-1\end{bmatrix}+\lambda\begin{bmatrix}-1\\0\\1\end{bmatrix}$ and let P be the point (2,–1,1). Also suppose that E be the plane containing three non collinear points A(0,1,1) ; B(1,2,2) and C(1,0,1).

Find

(a) Distance between the point P and the line L.

(b) Equation of the plane E.

(c) Equation the plane F containing the line L and the point P.

(d) Acute angle between the plane E and F.

(e) Volume of the parallelopiped by A,B,C and the point D(–3,0,1).

TD0050

8. The position vectors of the four angular points of a tetrahedron OABC are (0, 0, 0); (0, 0, 2); (0, 4, 0) and (6, 0, 0) respectively. A point P inside the tetrahedron is at the same distance 'r' from the four plane faces of the tetrahedron. Find the value of 'r'.

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TD0051

9. Let the equation of the plane containing the line $x-y-z-4=0=x+y+2z-4$ and is parallel to the line of intersection of the planes $2x+3y+z=1$ and $x+3y+2z=2$ be $x+Ay+Bz+C=0$ Compute the value of $|A+B+C|$.

TD0052

10. Find the equation of the line which is reflection of the line $\frac{x-1}{9}=\frac{y-2}{-1}=\frac{z+3}{-3}$ in the plane $3x-3y+10z=26$.

TD0053

11. Find the equation of the plane containing the line $\frac{x-1}{2}=\frac{y}{3}=\frac{z}{2}$ and parallel to the line $\frac{x-3}{2}=\frac{y}{5}=\frac{z-2}{4}$.

Find also the S.D. between the two lines.

TD0054

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12. Consider the plane

$$E : \vec{r} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Let F be the plane containing the point A(–4,2,2) and parallel to E.

Suppose the point B is on the plane E such that B has a minimum distance from the point A. If C(–3,0,4) lies in the plane F. Find the area of the triangle ABC.

TD0055

13. The equation of the plane which has the property that the point Q(5,4,5) is the reflection of point P(1,2,3) through that plane, is $ax + by + cz = d$ where $a,b,c,d \in N$. Find the least value of $(a + b + c + d)$.

TD0056

14. Find the equation of the line passing through the point (4,–14,4) and intersecting the line of intersection of the planes : $3x + 2y - z = 5$ and $x - 2y - 2z = -1$ at right angles.

TD0057

15. Find the point where the line of intersection of the planes $x - 2y + z = 1$ and $x + 2y - 2z = 5$, intersects the plane $2x + 2y + z + 6 = 0$.

TD0058

16. Feet of the perpendicular drawn from the point P(2,3,–5) on the axes of coordinates are A,B and C. Find the equation of the plane passing through their feet and the area of ΔABC.

TD0059

17. Find the equation of the plane containing the straight line $\frac{x-1}{2} = \frac{y+2}{-3} = \frac{z}{5}$ and perpendicular to the plane $x - y + z + 2 = 0$.

TD0060

EXERCISE (S-2)

1. Find the equations of the two lines through the origin which intersect the line $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$ at an angle of $\frac{\pi}{3}$.

TD0061

2. Let $\Pi : x + y - z - 4 = 0$ be the equation of a plane and A be the point with position vector $\hat{i} + 2\hat{j} - 3\hat{k}$. L is a line which passes through the point (1,2,3) with direction ratios 3,–1 and 4. If the distance of the point A from the line L measured parallel to the plane Π is d_1 and the distance of the point A from the plane Π measured parallel to the line L is d_2, then find the value of $\sqrt{d_1^2 - d_2^2}$.

TD0062

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3. The three planes $kx + y + z = 2$, $x + y - z = 3$, $x + 2z = 2$ form a triangular prism and area of the normal section (where normal section of the triangular prism is the plane parallel to the base of the triangular prism) be k_1. Then value of $2\sqrt{14}\,(k.k_1)$ is

TD0063

4. The line $\frac{x+6}{5} = \frac{y+10}{3} = \frac{z+14}{8}$ is the hypotenuse of an isosceles right angled triangle whose opposite vertex is (7, 2, 4). Find the equation of the remaining sides.

TD0064

5. **(a)** Consider a plane passing through three points A(a,0,0), B(0,b,0), C(0,0,c) with $a > 0$, $b > 0$, $c > 0$. Let d be the distance between the origin O and the plane and m be the distance between the origin O and the point M(a,b,c). If a,b,c vary in the range of any positive numbers, then find the minimum value of $\left(\frac{m}{d}\right)^2$.

TD0065

(b) Let A_1, A_2, A_3, A_4 be the areas of the triangular faces of a tetrahedron and h_1, h_2, h_3, h_4 be corresponding altitudes of the tetrahedron. If volume of tetrahedron is 5 cubic units then find the minimum value of $(A_1 + A_2 + A_3 + A_4)(h_1 + h_2 + h_3 + h_4)$ (in cubic units).

TD0066

6. If the angle between the planes given by $6x^2 + 4y^2 - 10z^2 + 3yz + 4zx - 11xy = 0$ is $\cos^{-1}(k)$, then the value of 'k' is equal to

TD0067

7. Planes P_1', P_2', P_3' are drawn parallel to the planes $P_1 : x + y + z = 3$, $P_2 : x - y + z = 1$ & $P_3 : x + y - z = 2$ respectively from the point (2,2,3). If d_1, d_2, d_3 are distances of P_1', P_2', P_3' from (1,1,2) respectively then $\left(d_1^2 + \frac{1}{d_2^2} + \frac{1}{d_3^2}\right)$ is equal to

TD0068

8. Faces ABC and BCD of a tetrahedron ABCD meet at an angle of 30°. The area of face ABC is 120 and the area of face BCD is 80 and BC = 10, then the volume of tetrahedron is

TD0069

9. Through a point $P(\alpha, \beta, \gamma)$ a plane is drawn at right angle to OP to meet the axes in A, B, C. If the area of ΔABC can be written as $\frac{(OP)^m}{n.\alpha.\beta.\gamma}$ (where O is origin, $m, n \in N$), then the value of $(m^2 + n^2)$ is

TD0070

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10. **(i)** Points P_1, P_2, P_3 ..... P_{10} are either lying along vertices or midpoints of the edges of a tetrahedron as shown in the diagram, then the number of groups of four distinct points (where each group of four points contains point P_1) which lies on the same plane is equal to

TD0071

(ii) Let A, B, C, D be four non-coplanar points. Then the number of planes which are equidistant from all the four points is equal to

TD0072

EXERCISE (JM)

1. An equation of a plane parallel to the plane $x - 2y + 2z - 5 = 0$ and at a unit distance from the origin is : **[AIEEE-2012]**

(1) $x - 2y + 2z + 5 = 0$ (2) $x - 2y + 2z - 3 = 0$
(3) $x - 2y + 2z + 1 = 0$ (4) $x - 2y + 2z - 1 = 0$

TD0073

2. If the lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then k is equal to : **[AIEEE-2012]**

(1) 0 (2) – 1 (3) $\frac{2}{9}$ (4) $\frac{9}{2}$

TD0074

3. Distance between two parallel planes $2x + y + 2z = 8$ and $4x + 2y + 4z + 5 = 0$ is :- **[JEE-MAIN 2013]**

(1) $\frac{3}{2}$ (2) $\frac{5}{2}$ (3) $\frac{7}{2}$ (4) $\frac{9}{2}$

TD0075

4. If the lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar, then k can have : **[JEE-MAIN 2013]**

(1) any value (2) exactly one value
(3) exactly two values (4) exactly three values.

TD0076

5. A vector $\vec{n}$ is inclined to x-axis at 45°, to y –axis at 60° and at an acute angle to z-axis. If $\vec{n}$ is a normal to a plane passing through the point $(\sqrt{2}, -1, 1)$, then the equation of the plane is : **[JEE-MAIN Online 2013]**

(1) $\sqrt{2}x - y - z = 2$ (2) $\sqrt{2}x + y + z = 2$
(3) $3\sqrt{2}x - 4y - 3z = 7$ (4) $4\sqrt{2}x + 7y + z = 2$

TD0077

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6. The acute angle between two lines such that the direction cosines ℓ, m, n of each of them satisfy the equations $\ell + m + n = 0$ and $\ell^2 + m^2 - n^2 = 0$ is :- **[JEE-MAIN Online 2013]**

(1) 30° (2) 45° (3) 60° (4) 15°

TD0078

7. Let Q be the foot of perpendicular from the origin to the plane $4x - 3y + z + 13 = 0$ and R be a point (–1, 1, –6) on the plane. Then length QR is :- **[JEE-MAIN Online 2013]**

(1) $3\sqrt{\frac{7}{2}}$ (2) $\sqrt{14}$ (3) $\sqrt{\frac{19}{2}}$ (4) $\frac{3}{\sqrt{2}}$

TD0079

8. If the projections of a line segment on thex, y and z-axes in 3-dimensional space are 2, 3 and 6 respectively, then the length ofthe line segment is : **[JEE-MAIN Online 2013]**

(1) 7 (2) 9 (3) 12 (4) 6

TD0080

9. If two lines L_1 and L_2 in space, are definedby **[JEE-MAIN Online 2013]**

$L_1 = \left\{x = \sqrt{\lambda}\, y + \left(\sqrt{\lambda} - 1\right)\right.$

$\left. z = \left(\sqrt{\lambda} - 1\right) y + \sqrt{\lambda}\right\}$ and

$L_2 = \left\{x = \sqrt{\mu}\, y + \left(1 - \sqrt{\mu}\right)\right.$

$\left. z = \left(1 - \sqrt{\mu}\right) y + \sqrt{\mu}\right\}$, then L_1 is perpendicular to L_2, for all non-negative reals λ and μ, such that :

(1) $\lambda = \mu$ (2) $\lambda \neq \mu$ (3) $\sqrt{\lambda} + \sqrt{\mu} = 1$ (4) $\lambda + \mu = 0$

TD0081

10. The equation of a plane through the line of intersection of the planes $x + 2y = 3$, $y - 2z + 1 = 0$, and perpendicular to the first plane is : **[JEE-MAIN Online 2013]**

(1) $2x - y + 7z = 11$ (2) $2x - y + 10z = 11$ (3) $2x - y - 9z = 10$ (4) $2x - y - 10z = 9$

TD0082

11. Let ABC be a triangle with vertices at points A (2, 3, 5), B (-1, 3, 2) and C $(\lambda, 5, \mu)$ in three dimensional space. If the median through A is equally inclined with the axes, then $(\lambda, \mu.)$ is equal to : **[JEE-MAIN Online 2013]**

(1) (10, 7) (2) (7.5) (3) (7, 10) (4) (5,7)

TD0083

12. The angle between the lines whose direction cosines satisfy the equations $\ell + m + n = 0$ and $\ell^2 = m^2 + n^2$ is : **[JEE-MAIN 2014]**

(1) $\frac{\pi}{3}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{6}$ (4) $\frac{\pi}{2}$

TD0084

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13. The image of the line $\frac{x-1}{3}=\frac{y-3}{1}=\frac{z-4}{-5}$ in the plane $2x-y+z+3=0$ is the line : **[JEE-MAIN 2014]**

(1) $\frac{x+3}{3}=\frac{y-5}{1}=\frac{z-2}{-5}$ (2) $\frac{x+3}{-3}=\frac{y-5}{-1}=\frac{z+2}{5}$

(3) $\frac{x-3}{3}=\frac{y+5}{1}=\frac{z-2}{-5}$ (4) $\frac{x-3}{-3}=\frac{y+5}{-1}=\frac{z-2}{5}$

TD0085

14. The equation of the plane containing the line $2x - 5y + z = 3$; $x + y + 4z = 5$, and parallel to the plane, $x + 3y + 6z = 1$, is : **[JEE(Main)-2015]**

(1) $x + 3y + 6z = 7$ (2) $2x + 6y + 12z = -13$

(3) $2x + 6y + 12z = 13$ (4) $x + 3y + 6z = -7$

TD0086

15. The distance of the point (1, 0, 2) from the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane $x - y + z = 16$, is : **[JEE(Main)-2015]**

(1) $3\sqrt{21}$ (2) 13 (3) $2\sqrt{14}$ (4) 8

TD0087

16. The distance of the point (1, –5, 9) from the plane $x - y + z = 5$ measured along the line $x = y = z$ is : **[JEE(Main)-2016]**

(1) $\frac{20}{3}$ (2) $3\sqrt{10}$ (3) $10\sqrt{3}$ (4) $\frac{10}{\sqrt{3}}$

TD0088

17. If the image of the point P(1, –2, 3) in the plane, $2x + 3y - 4z + 22 = 0$ measured parallel to line, $\frac{x}{1}=\frac{y}{4}=\frac{z}{5}$ is Q, then PQ is equal to :- **[JEE(Main)-2017]**

(1) $6\sqrt{5}$ (2) $3\sqrt{5}$ (3) $2\sqrt{42}$ (4) $\sqrt{42}$

TD0089

18. The distantce of the point (1, 3, –7) from the plane passing through the point (1, –1, –1), having normal perpendicular to both the lines $\frac{x-1}{1}=\frac{y+2}{-2}=\frac{z-4}{3}$ and $\frac{x-2}{2}=\frac{y+1}{-1}=\frac{z+7}{-1}$, is :- **[JEE(Main)-2017]**

(1) $\frac{10}{\sqrt{74}}$ (2) $\frac{20}{\sqrt{74}}$ (3) $\frac{10}{\sqrt{83}}$ (4) $\frac{5}{\sqrt{83}}$

TD0090

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19. The length of the projection of the line segment joining the points (5, –1, 4) and (4, –1, 3) on the plane, x + y + z = 7 is : **[JEE(Main)-2018]**

(1) $\frac{2}{3}$ (2) $\frac{1}{3}$ (3) $\sqrt{\frac{2}{3}}$ (4) $\frac{2}{\sqrt{3}}$

TD0091

20. If L_1 is the line of intersection of the planes 2x – 2y + 3z – 2 = 0, x – y + z + 1 = 0 and L_2 is the line of intersection of the planes x + 2y – z – 3 =0, 3x – y + 2z – 1 = 0, then the distance of the origin from the plane, containing the lines L_1 and L_2 is : **[JEE(Main)-2018]**

(1) $\frac{1}{3\sqrt{2}}$ (2) $\frac{1}{2\sqrt{2}}$ (3) $\frac{1}{\sqrt{2}}$ (4) $\frac{1}{4\sqrt{2}}$

TD0092

21. The equation of the line passing through (–4, 3, 1), parallel to the plane x + 2y – z – 5 = 0 and intersecting the line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z-2}{-1}$ is : **[JEE(Main)-19]**

(1) $\frac{x+4}{-1}=\frac{y-3}{1}=\frac{z-1}{1}$ (2) $\frac{x+4}{3}=\frac{y-3}{-1}=\frac{z-1}{1}$ (3) $\frac{x+4}{1}=\frac{y-3}{1}=\frac{z-1}{3}$ (4) $\frac{x-4}{2}=\frac{y+3}{1}=\frac{z+1}{4}$

TD0093

22. If the lines x = ay + b, z = cy + d and x=a'z + b', y = c'z + d' are perpendicular, then : **[JEE(Main)-19]**

(1) cc' + a + a' = 0 (2) aa' + c + c' = 0

(3) ab' + bc' + 1 = 0 (4) bb' + cc' + 1 = 0

TD0094

23. A tetrahedron has vertices P(1, 2, 1), Q(2, 1, 3), R(–1,1,2) and O(0, 0, 0). The angle between the faces OPQ and PQR is : **[JEE(Main)- 19]**

(1) $\cos^{-1}\left(\frac{9}{35}\right)$ (2) $\cos^{-1}\left(\frac{19}{35}\right)$ (3) $\cos^{-1}\left(\frac{17}{31}\right)$ (4) $\cos^{-1}\left(\frac{7}{31}\right)$

TD0095

24. A plane which bisects the angle between the two given planes 2x – y + 2z – 4 = 0 and x + 2y + 2z – 2 = 0, passes through the point : **[JEE(Main)- 19]**

(1) (2,4,1) (2) (2, – 4, 1) (3) (1, 4, – 1) (4) (1, – 4, 1)

TD0096

25. If the foot of the perpendicular drawn from the point (1, 0, 3) on a line passing through (α, 7, 1) is $\left(\frac{5}{3},\frac{7}{3},\frac{17}{3}\right)$, then α is equal to_____ **[JEE(Main)- 20]**

TD0097

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26. Let P be a plane passing through the points (2, 1, 0), (4, 1, 1) and (5, 0, 1) and R be any point (2, 1, 6). Then the image of R in the plane P is : **[JEE(Main)- 20]**

(1) (6, 5, –2) (2) (4, 3, 2) (3) (3, 4, –2) (4) (6, 5, 2)

TD0098

27. The mirror image of the point (1,2,3) in a plane is $\left(-\frac{7}{3},-\frac{4}{3},-\frac{1}{3}\right)$. Which of the following points lies on this plane ? **[JEE(Main)- 20]**

(1) (–1, –1, –1) (2) (–1, –1, 1) (3) (1, 1, 1) (4) (1, –1, 1)

TD0099

28. The shortest distance between the lines **[JEE(Main)- 20]**

$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$ is

(1) $\frac{7}{2}\sqrt{30}$ (2) $3\sqrt{30}$ (3) 3 (4) $2\sqrt{30}$

TD0100

29. If the distance between the plane, 23x – 10y – 2z + 48 = 0 and the plane containing the lines

$\frac{x+1}{2}=\frac{y-3}{4}=\frac{z+1}{3}$ and $\frac{x+3}{2}=\frac{y+2}{6}=\frac{z-1}{\lambda}(\lambda\in R)$

is equal to $\frac{k}{\sqrt{633}}$, then k is equal to _____. **[JEE(Main)- 20]**

TD0101

30. If for some α and β in R, the intersection of the following three places

x + 4y – 2z = 1

x + 7y – 5z = β

x + 5y + αz = 5

is a line in R^3, then α + β is equal to : **[JEE(Main)- 20]**

(1) 10 (2) –10 (3) 2 (4) 0

TD0102

EXERCISE (JA)

1. **(a)** Equation of the plane containing the straight line $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3}=\frac{y}{4}=\frac{z}{2}$ and $\frac{x}{4}=\frac{y}{2}=\frac{z}{3}$ is

(A) x + 2y – 2z = 0 (B) 3x + 2y – 2z = 0

(C) x – 2y + z = 0 (D) 5x + 2y – 4z = 0

TD0103

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(b) If the distance of the point P(1,–2,1) from the plane $x + 2y - 2z = \alpha$, where $\alpha > 0$, is 5, then the foot of the perpendicular from P to the plane is-

(A) $\left(\frac{8}{3},\frac{4}{3},-\frac{7}{3}\right)$ (B) $\left(\frac{4}{3},-\frac{4}{3},\frac{1}{3}\right)$ (C) $\left(\frac{1}{3},\frac{2}{3},\frac{10}{3}\right)$ (D) $\left(\frac{2}{3},-\frac{1}{3},\frac{5}{2}\right)$

TD0104

(c) If the distance between the plane $Ax - 2y + z = d$ and the plane containing the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ is $\sqrt{6}$, then $|d|$ is

TD0105

(d) Match the statements in **Column-I** with the values in **Column-II**.

Column-I	**Column-II**
(A) A line from the origin meets the lines $\frac{x-2}{1}=\frac{y-1}{-2}=\frac{z+1}{1}$ and $\frac{x-\frac{8}{3}}{2}=\frac{y+3}{-1}=\frac{z-1}{1}$ at P and Q respectively. If length PQ = d, then d^2 is	(p) –4
	TD0106
(B) The values of x satisfying $\tan^{-1}(x+3)-\tan^{-1}(x-3)=\sin^{-1}\left(\frac{3}{5}\right)$ are	(q) 0
	TD0107
(C) Non-zero vectors $\vec{a},\vec{b}$ and $\vec{c}$ satisfy $\vec{a}.\vec{b}=0$, $(\vec{b}-\vec{a}).(\vec{b}+\vec{c})=0$ and $2\left\|(\vec{b}+\vec{c})\right\|=\left\|(\vec{b}-\vec{a})\right\|$. If $\vec{a}=\mu\vec{b}+4\vec{c}$, then the possible values of μ are	(r) 4
	TD0108
(D) Let f be the function on $[-\pi,\pi]$ given by $f(0) = 9$ and $f(x) = \sin\left(\frac{9x}{2}\right)\Big/\sin\left(\frac{x}{2}\right)$ for $x \neq 0$. The value of $\frac{2}{\pi}\int_{-\pi}^{\pi} f(x)dx$ is	(s) 5 (t) 6
	[JEE 2010, 3+5+3+(2+2+2+2)]
	TD0109

2. **(a)** The point P is the intersection of the straight line joining the points Q(2,3,5) and R(1,–1,4) with the plane $5x - 4y - z = 1$. If S is the foot of the perpendicular drawn from the point T(2,1,4) to QR, then the length of the line segment PS is -

(A) $\frac{1}{\sqrt{2}}$ (B) $\sqrt{2}$ (C) 2 (D) $2\sqrt{2}$

TD0110

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(b) The equation of a plane passing through the line of intersection of the planes $x + 2y + 3z = 2$ and $x - y + z = 3$ and at a distance $\frac{2}{\sqrt{3}}$ from the point (3, 1, –1) is

(A) $5x - 11y + z = 17$ (B) $\sqrt{2}x + y = 3\sqrt{2} - 1$

(C) $x + y + z = \sqrt{3}$ (D) $x - \sqrt{2}y = 1 - \sqrt{2}$

TD0111

(c) If the straight lines $\frac{x-1}{2} = \frac{y+1}{k} = \frac{z}{2}$ and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{k}$ are coplanar, then the plane(s) containing these two lines is(are)

(A) $y + 2z = -1$ (B) $y + z = -1$ (C) $y - z = -1$ (D) $y - 2z = -1$

[JEE 2012, 3+3+4]

TD0112

3. Perpendiculars are drawn from points on the line $\frac{x+2}{2} = \frac{y+1}{-1} = \frac{z}{3}$ to the plane $x + y + z = 3$. The feet of perpendiculars lie on the line **[JEE-Advanced 2013, 2]**

(A) $\frac{x}{5} = \frac{y-1}{8} = \frac{z-2}{-13}$ (B) $\frac{x}{2} = \frac{y-1}{3} = \frac{z-2}{-5}$

(C) $\frac{x}{4} = \frac{y-1}{3} = \frac{z-2}{-7}$ (D) $\frac{x}{2} = \frac{y-1}{-7} = \frac{z-2}{5}$

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TD0113

4. A line ℓ passing through the origin is perpendicular to the lines

$\ell_1 : (3+t)\hat{i} + (-1+2t)\hat{j} + (4+2t)\hat{k}, -\infty < t < \infty$

$\ell_2 : (3+2s)\hat{i} + (3+2s)\hat{j} + (2+s)\hat{k}, -\infty < s < \infty$

Then , the coordinate(s) of the point(s) on ℓ_2 at a distance of $\sqrt{17}$ from the point of intersection of ℓ and ℓ_1 is(are) - **[JEE-Advanced 2013, 4, (–1)]**

(A) $\left(\frac{7}{3}, \frac{7}{3}, \frac{5}{3}\right)$ (B) (–1,–1,0) (C) (1,1,1) (D) $\left(\frac{7}{9}, \frac{7}{9}, \frac{8}{9}\right)$

TD0114

5. Two lines $L_1 : x = 5, \frac{y}{3-\alpha} = \frac{z}{-2}$ and $L_2 : x = \alpha, \frac{y}{-1} = \frac{z}{2-\alpha}$ are coplanar. Then α can take value(s)

[JEE-Advanced 2013, 3, (–1)]

(A) 1 (B) 2 (C) 3 (D) 4

TD0115

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6. Consider the lines $L_1 : \frac{x-1}{2} = \frac{y}{-1} = \frac{z+3}{1}, L_2 : \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2}$ and the planes $P_1 : 7x+y+2z=3$, $P_2 : 3x+5y-6z=4$. Let $ax+by+cz=d$ be the equation of the plane passing through the point of intersection of lines L_1 and L_2 and perpendicular to planes P_1 and P_2.

Match List-I with List-II and select the correct answer using the code given below the lists.

	List-I		List-II
P.	a =	1.	13
Q.	b =	2.	–3
R.	c =	3.	1
S.	d =	4.	–2

Codes :

	P	Q	R	S
(A)	3	2	4	1
(B)	1	3	4	2
(C)	3	2	1	4
(D)	2	4	1	3

[JEE-Advanced 2013, 3, (–1)]

TD0116

7. From a point $P(\lambda,\lambda,\lambda)$, perpendiculars PQ and PR are drawn respectively on the lines $y = x, z = 1$ and $y = -x, z = -1$. If P is such that $\angle QPR$ is a right angle, then the possible value(s) of λ is(are)

(A) $\sqrt{2}$ (B) 1 (C) –1 (D) $-\sqrt{2}$

[JEE(Advanced)-2014, 3]

TD0117

8. In $\mathbb{R}^3$, consider the planes $P_1 : y = 0$ and $P_2 : x + z = 1$. Let P_3 be a plane, different from P_1 and P_2, which passes through the intersection of P_1 and P_2. If the distance of the point (0,1,0) from P_3 is 1 and the distance of a point (α,β,γ) from P_3 is 2, then which of the following relations is (are) true ?

[JEE 2015, 4M, –2M]

(A) $2\alpha + \beta + 2\gamma + 2 = 0$ (B) $2\alpha - \beta + 2\gamma + 4 = 0$

(C) $2\alpha + \beta - 2\gamma - 10 = 0$ (D) $2\alpha - \beta + 2\gamma - 8 = 0$

TD0118

9. In $\mathbb{R}^3$, let L be a straight line passing through the origin. Suppose that all the points on L are at a constant distance from the two planes $P_1 : x + 2y - z + 1 = 0$ and $P_2 : 2x - y + z - 1 = 0$. Let M be the locus of the feet of the perpendiculars drawn from the points on L to the plane P_1. Which of the following points lie(s) on M ? **[JEE 2015, 4M, –2M]**

(A) $\left(0,-\frac{5}{6},-\frac{2}{3}\right)$ (B) $\left(-\frac{1}{6},-\frac{1}{3},\frac{1}{6}\right)$ (C) $\left(-\frac{5}{6},0,\frac{1}{6}\right)$ (D) $\left(-\frac{1}{3},0,\frac{2}{3}\right)$

TD0119

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10. Consider a pyramid OPQRS located in the first octant ($x \geq 0, y \geq 0, z \geq 0$) with O as origin, and OP and OR along the x-axis and the y-axis, respectively. The base OPQR of the pyramid is a square with OP= 3. The point S is directly above the mid-point T of diagonal OQ such that TS = 3. Then-

(A) the acute angle between OQ and OS is $\frac{\pi}{3}$. **[JEE(Advanced)-2016, 4(–2)]**

(B) the equaiton of the plane containing the triangle OQS is $x - y = 0$

(C) the length of the perpendicular from P to the plane containing the triangle OQS is $\frac{3}{\sqrt{2}}$

(D) the perpendicular distance from O to the straight line containing RS is $\sqrt{\frac{15}{2}}$

TD0120

11. Let P be the image of the point (3, 1, 7) with respect to the plane $x - y + z = 3$. Then the equation of the plane passing through P and containing the straight line $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$ is **[JEE(Adv.)-2016, 3(–1)]**

(A) $x + y - 3z = 0$ (B) $3x + z = 0$ (C) $x - 4y + 7z = 0$ (D) $2x - y = 0$

TD0121

12. The equation of the plane passing through the point (1,1,1) and perpendicular to the planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$, is- **[JEE(Advanced)-2017]**

(A) $14x + 2y + 15z = 31$ (B) $14x + 2y - 15z = 1$

(C) $-14x + 2y + 15z = 3$ (D) $14x - 2y + 15z = 27$

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TD0122

13. Let $P_1 : 2x + y - z = 3$ and $P_2 : x + 2y + z = 2$ be two planes. Then, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018, 4(–2)]**

(A) The line of intersection of P_1 and P_2 has direction ratios 1, 2, –1

(B) The line $\frac{3x-4}{9} = \frac{1-3y}{9} = \frac{z}{3}$ is perpendicular to the line of intersection of P_1 and P_2

(C) The acute angle between P_1 and P_2 is 60°

(D) If P_3 is the plane passing through the point (4, 2, –2) and perpendicular to the line of intersection of P_1 and P_2, then the distance of the point (2, 1, 1) from the plane P_3 is $\frac{2}{\sqrt{3}}$

TD0123

14. Let P be a point in the first octant, whose image Q in the plane $x + y = 3$ (that is, the line segment PQ is perpendicular to the plane $x + y = 3$ and the mid-point of PQ lies in the plane $x + y = 3$) lies on the z-axis. Let the distance of P from the x-axis be 5. If R is the image of P in the xy-plane, then the length of PR is _____ . **[JEE(Advanced)-2018, 3(0)]**

TD0124

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15. Consider the cube in the first octant with sides OP, OQ and OR of length 1, along the x-axis, y-axis and z-axis, respectively, where O(0, 0, 0) is the origin. Let $S\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ be the centre of the cube and T be the vertex of the cube opposite to the origin O such that S lies on the diagonal OT. If $\vec{p}=\overrightarrow{SP}$, $\vec{q}=\overrightarrow{SQ}$, $\vec{r}=\overrightarrow{SR}$ and $\vec{t}=\overrightarrow{ST}$, then the value of $\left|(\vec{p}\times\vec{q})\times(\vec{r}\times\vec{t})\right|$ is _____ .

[JEE(Advanced)-2018, 3(0)]

TD0125

16. Let L_1 and L_2 denotes the lines

$$\vec{r}=\hat{i}+\lambda(-\hat{i}+2\hat{j}+2\hat{k}),\lambda\in\mathbb{R}$$

$$\text{and}\quad \vec{r}=\mu(2\hat{i}-\hat{j}+2\hat{k}),\mu\in\mathbb{R}$$

respectively. If L_3 is a line which is perpendicular to both L_1 and L_2 and cuts both of them, then which of the following options describe(s) L_3 ? **[JEE(Advanced)-2019, 4(–1)]**

(1) $\vec{r}=\frac{1}{3}(2\hat{i}+\hat{k})+t(2\hat{i}+2\hat{j}-\hat{k}),t\in\mathbb{R}$

(2) $\vec{r}=\frac{2}{9}(2\hat{i}-\hat{j}+2\hat{k})+t(2\hat{i}+2\hat{j}-\hat{k}),t\in\mathbb{R}$

(3) $\vec{r}=t(2\hat{i}+2\hat{j}-\hat{k}),t\in\mathbb{R}$

(4) $\vec{r}=\frac{2}{9}(4\hat{i}+\hat{j}+\hat{k})+t(2\hat{i}+2\hat{j}-\hat{k}),t\in\mathbb{R}$

TD0126

17. Three lines are given by

$$\vec{r}=\lambda\hat{i},\ \lambda\in\mathbb{R}$$

$$\vec{r}=\mu(\hat{i}+\hat{j}),\mu\in\mathbb{R}\ \text{and}$$

$$\vec{r}=\nu(\hat{i}+\hat{j}+\hat{k}),\nu\in\mathbb{R}.$$

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Let the lines cut the plane $x+y+z=1$ at the points A, B and C respectively. If the area of the triangle ABC is Δ then the value of $(6\Delta)^2$ equals ___ **[JEE(Advanced)-2019, 3(0)]**

TD0127

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ANSWER KEY

EXERCISE (O-1)

1. D **2.** A **3.** D **4.** D **5.** A **6.** B **7.** D **8.** B

9. A **10.** A **11.** C **12.** B **13.** C **14.** C **15.** C **16.** D

17. A **18.** C **19.** B **20.** B **21.** C **22.** A **23.** B **24.** A

25. C **26.** C **27.** (A) R,S; (B) Q; (C) Q,S; (D) P,S

EXERCISE (O-2)

1. A,D **2.** A,D **3.** A,B,C **4.** A,C **5.** A,C **6.** A,B,D **7.** A,D **8.** C,D

9. B,D **10.** A,B,C **11.** B,C,D **12.** B,D **13.** (A) R; (B) Q; (C) P ; (D) S

EXERCISE (S-1)

1. $\theta = 90°$ **2.** 212 **3.** $\frac{x-1}{2} = \frac{y-2}{2} = \frac{z-3}{-3}$ **4.** $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$

5. $p = 3$, $(2,1,-3)$; $x + y + z = 0$ **6.** $\frac{x-7}{22} = \frac{y-2}{5} = \frac{z+1}{-4}$

7. **(a)** $\sqrt{3}$; **(b)** $x + y - 2z + 1 = 0$; **(c)** $x - 2y + z = 5$; **(d)** $\pi/3$; **(e)** 4 **8.** $\frac{2}{3}$ **9.** 11

10. $\frac{x-4}{9} = \frac{y+1}{-1} = \frac{z-7}{-3}$ **11.** $x - 2y + 2z - 1 = 0$; 2 units **12.** 9/2 **13.** 17

14. $\frac{x-4}{3} = \frac{y+14}{10} = \frac{z-4}{4}$ **15.** $(1, -2, -4)$ **16.** $\frac{x}{2} + \frac{y}{3} + \frac{z}{-5} = 1$, Area $= \frac{19}{2}$ sq. units

17. $2x + 3y + z + 4 = 0$

EXERCISE (S-2)

1. $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$ or $\frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}$ **2.** 10 **3.** 18 **4.** $\frac{x-7}{3} = \frac{y-2}{6} = \frac{z-4}{2}$; $\frac{x-7}{2} = \frac{y-2}{-3} = \frac{z-4}{6}$

5. **(a)** 9; **(b)** 240 **6.** 0 **7.** 9 **8.** 320 **9.** 29 **10.** **(i)** 33 **(ii)** 7

EXERCISE (JM)

1. 2 **2.** 4 **3.** 3 **4.** 3 **5.** 2 **6.** 3 **7.** 1 **8.** 1

9. 1,4 **10.** 2 **11.** 3 **12.** 1 **13.** 1 **14.** 1 **15.** 2 **16.** 3

17. 3 **18.** 3 **19.** 3 **20.** 1 **21.** 2 **22.** 2 **23.** 2 **24.** 2

25. 4.00 **26.** 1 **27.** 4 **28.** 2 **29.** 3 **30.** 1

EXERCISE (JA)

1. **(a)** C; **(b)** A; **(c)** 6; **(d)** (A) t (B) p,r (C) q (D) r **2.** **(a)** A; **(b)** A; **(c)** B,C **3.** D

4. B,D **5.** A,D **6.** A **7.** C **8.** B,D **9.** A,B **10.** B,C,D **11.** C

12. A **13.** C,D **14.** 8 **15.** 0.5 **16.** 1,2,4 **17.** 0.75

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PARABOLA

1. CONIC SECTIONS :

A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.

(a) The fixed point is called the **focus**.

(b) The fixed straight line is called the **directrix**.

(c) The constant ratio is called the **eccentricity** denoted by e.

(d) The line passing through the focus & perpendicular to the directrix is called the **axis**.

(e) A point of intersection of a conic with its axis is called a **vertex**.

2. GENERAL EQUATION OF A CONIC : FOCAL DIRECTRIX PROPERTY :

The general equation of a conic with focus **(p, q)** & directrix $\mathbf{lx + my + n = 0}$ is :

$$(l^2 + m^2)\,[(x - p)^2 + (y - q)^2] = e^2\,(lx + my + n)^2 \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

3. DISTINGUISHING BETWEEN THE CONIC :

The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise.

Case (i) When the focus lies on the directrix :

In this case $D \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines and if :

$e > 1$ the lines will be real & distinct intersecting at S.

$e = 1$ the lines will coincident.

$e < 1$ the lines will be imaginary.

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Case (ii) When the focus does not lie on the directrix :

The conic represents:

a parabola	an ellipse	a hyperbola	a rectangular hyperbola
$e = 1$; $D \neq 0$ $h^2 = ab$	$0 < e < 1$; $D \neq 0$ $h^2 < ab$	$D \neq 0$; $e > 1$; $h^2 > ab$	$e > 1$; $D \neq 0$ $h^2 > ab$; $a + b = 0$

4. PARABOLA :

A parabola is the locus of a point which moves in a plane, such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix).

Standard equation of a parabola is $\mathbf{y^2 = 4ax}$. For this parabola :

(i) Vertex is **(0, 0)** (ii) Focus is **(a, 0)** (iii) Axis is $\mathbf{y = 0}$ (iv) Directrix is $\mathbf{x + a = 0}$

(a) Focal distance :

The distance of a point on the parabola from the focus is called the **focal distance of the point.**

(b) Focal chord :

A chord of the parabola, which passes through the focus is called a **focal chord**.

(c) Double ordinate :

A chord of the parabola perpendicular to the axis of the symmetry is called a **double ordinate**.

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(d) Latus rectum :

A double ordinate passing through the focus or a focal chord perpendicular to the axis of parabola is called the **latus rectum**. For $y^2 = 4ax$.

- Length of the latus rectum = **4a.**
- Length of the semi latus rectum = **2a**.
- Ends of the latus rectum are **L(a, 2a)** & **L'(a, – 2a)**

Note that :

(i) Perpendicular distance from focus on directrix = half the latus rectum.

(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.

(iii) Two parabolas are said to be equal if they have the same latus rectum.

5. PARAMETRIC REPRESENTATION :

The simplest & the best form of representing the co-ordinates of a point on the parabola is **$(at^2, 2at)$** . The equation **$x = at^2$** & **$y = 2at$** together represents the parabola **$y^2 = 4ax$** , t being the parameter.

6. TYPE OF PARABOLA :

Four standard forms of the parabola are **$y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$**

$y^2 = 4ax$ **$y^2 = -4ax$** **$x^2 = 4ay$** **$x^2 = -4ay$**

Parabola	Vertex	Focus	Axis	Directrix	Length of Latus rectum	Ends of Latus rectum	Parametric equation	Focal length
$y^2 = 4ax$	(0,0)	(a,0)	y=0	x=–a	4a	(a, ±2a)	$(at^2,2at)$	x + a
$y^2 = -4ax$	(0,0)	(–a,0)	y=0	x=a	4a	(–a, ±2a)	$(-at^2,2at)$	x–a
$x^2 = +4ay$	(0,0)	(0,a)	x=0	y=–a	4a	(±2a, a)	$(2at,at^2)$	y+a
$x^2 = -4ay$	(0,0)	(0,–a)	x=0	y=a	4a	(±2a, –a)	$(2at, -at^2)$	y–a
$(y-k)^2 = 4a(x-h)$	(h,k)	(h+a,k)	y=k	x+a– h =0	4a	(h+a, k±2a)	$(h+at^2,k+2at)$	x–h+a
$(x-p)^2 = 4b(y-q)$	(p,q)	(p, b+q)	x=p	y+b–q=0	4b	(p±2a,q+a)	$(p+2at,q+at^2)$	y–q+b

Illustration 1 : Find the vertex, axis, directrix, focus, latus rectum and the tangent at vertex for the parabola $9y^2 - 16x - 12y - 57 = 0$.

Solution : The given equation can be rewritten as $\left(y-\frac{2}{3}\right)^2 = \frac{16}{9}\left(x+\frac{61}{16}\right)$ which is of the form $Y^2 = 4AX$. Hence the vertex is $\left(-\frac{61}{16}, \frac{2}{3}\right)$

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The axis is $y - \frac{2}{3} = 0 \Rightarrow y = \frac{2}{3}$

The directrix is $X + A = 0 \Rightarrow x + \frac{61}{16} + \frac{4}{9} = 0 \Rightarrow x = -\frac{613}{144}$

The focus is $X = A$ and $Y = 0 \Rightarrow x + \frac{61}{16} = \frac{4}{9}$ and $y - \frac{2}{3} = 0$

$\Rightarrow$ focus $= \left(-\frac{485}{144}, \frac{2}{3}\right)$

Length of the latus rectum $= 4A = \frac{16}{9}$

The tangent at the vertex is $X = 0 \Rightarrow x = -\frac{61}{16}$. **Ans.**

Illustration 2 : The length of latus rectum of a parabola, whose focus is (2, 3) and directrix is the line $x - 4y + 3 = 0$ is -

(A) $\frac{7}{\sqrt{17}}$ (B) $\frac{14}{\sqrt{21}}$ (C) $\frac{7}{\sqrt{21}}$ (D) $\frac{14}{\sqrt{17}}$

Solution : The length of latus rectum $= 2 \times$ perp. from focus to the directrix

$$= 2 \times \left|\frac{2 - 4(3) + 3}{\sqrt{(1)^2 + (4)^2}}\right| = \frac{14}{\sqrt{17}}$$ **Ans. (D)**

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Illustration 3 : Find the equation of the parabola whose focus is (–6, –6) and vertex (–2, 2).

Solution : Let S(–6, –6) be the focus and A(–2, 2) is vertex of the parabola. On SA take a point $K(x_1, y_1)$ such that SA = AK. Draw KM perpendicular on SK. Then KM is the directrix of the parabola. Since A bisects SK, $\left(\frac{-6 + x_1}{2}, \frac{-6 + y_1}{2}\right) = (-2, 2)$

$\Rightarrow$ $-6 + x_1 = -4$ and $-6 + y_1 = 4$ or $(x_1, y_1) = (2, 10)$

Hence the equation of the directrix KM is

$y - 10 = m(x - 2)$ ..... (i)

Also gradient of SK $= \frac{10 - (-6)}{2 - (-6)} = \frac{16}{8} = 2; \Rightarrow m = \frac{-1}{2}$

$y - 10 = \frac{-1}{2}(x - 2)$ (from (i))

$\Rightarrow$ $x + 2y - 22 = 0$ is the directrix

Next, let PM be a perpendicular on the directrix KM from any point P(x, y) on the parabola. From SP = PM, the equation of the parabola is $\sqrt{\left\{(x+6)^2 + (y+6)^2\right\}} = \frac{|x + 2y - 22|}{\sqrt{(1^2 + 2^2)}}$

or $5(x^2 + y^2 + 12x + 12y + 72) = (x + 2y - 22)^2$

or $4x^2 + y^2 - 4xy + 104x + 148y - 124 = 0$ or $(2x - y)^2 + 104x + 148y - 124 = 0$.

Ans.

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Illustration 4 : The extreme points of the latus rectum of a parabola are (7, 5) and (7, 3). Find the equation of the parabola.

Solution : Focus of the parabola is the mid-point of the latus rectum.

$\Rightarrow$ S is (7, 4). Also axis of the parabola is perpendicular to the latus rectum and passes through the focus. Its equation is

$$y - 4 = \frac{0}{5-3}(x-7) \Rightarrow y = 4$$

Length of the latus rectum = (5 – 3) = 2

Hence the vertex of the parabola is at a distance 2/4 = 0.5 from the focus. We have two parabolas, one concave rightwards and the other concave leftwards.

The vertex of the first parabola is (6.5, 4) and its equation is $(y-4)^2 = 2(x-6.5)$ and it meets the x-axis at (14.5, 0). The equation of the second parabola is $(y-4)^2 = -2(x-7.5)$. It meets the x-axis at (–0.5, 0). **Ans.**

Do yourself - 1 :

(i) Name the conic represented by the equation $\sqrt{ax} + \sqrt{by} = 1$, where a, b $\in$ R, a, b, > 0.

(ii) Find the vertex, axis, focus, directrix, latus rectum of the parabola $4y^2 + 12x - 20y + 67 = 0$.

(iii) Find the equation of the parabola whose focus is (1, –1) and whose vertex is (2, 1). Also find its axis and latus rectum.

(iv) Find the equation of the parabola whose latus rectum is 4 units, axis is the line 3x + 4y = 4 and the tangent at the vertex is the line 4x – 3y + 7 = 0.

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7. POSITION OF A POINT RELATIVE TO A PARABOLA :

The point (x_1, y_1) lies **outside**, **on** or **inside** the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is **positive**, **zero** or **negative**.

Illustration 5 : Find the value of α for which the point $(\alpha - 1, \alpha)$ lies inside the parabola $y^2 = 4x$.

Solution : $\because$ Point $(\alpha - 1, \alpha)$ lies inside the parabola $y^2 = 4x$

$\therefore$ $y_1^2 - 4ax_1 < 0$

$\Rightarrow$ $\alpha^2 - 4(\alpha - 1) < 0$

$\Rightarrow$ $\alpha^2 - 4\alpha + 4 < 0$

$(\alpha - 2)^2 < 0 \Rightarrow \alpha \in \phi$ **Ans.**

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8. CHORD JOINING TWO POINTS :

The equation of a chord of the parabola $y^2 = 4ax$ joining its two points $P(t_1)$ and $Q(t_2)$ is

$\mathbf{y(t_1 + t_2) = 2x + 2at_1t_2}$

Note :

(i) If PQ is focal chord then $\mathbf{t_1t_2 = -1}$.

(ii) Extremities of focal chord can be taken as $\mathbf{(at^2, 2at)}$ & $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$

Illustration 6 : Through the vertex O of a parabola $y^2 = 4x$ chords OP and OQ are drawn at right angles to one another. Show that for all position of P, PQ cuts the axis of the parabola at a fixed point.

Solution : The given parabola is $y^2 = 4x$ ..... (i)

Let $P \equiv (t_1^2, 2t_1)$, $Q \equiv (t_2^2, 2t_2)$

Slope of OP $= \frac{2t_1}{t_1^2} = \frac{2}{t_1}$ and slope of OQ $= \frac{2}{t_2}$

Since $OP \perp OQ$, $\frac{4}{t_1t_2} = -1$ or $t_1t_2 = -4$ ..... (ii)

The equation of PQ is $y(t_1 + t_2) = 2(x + t_1t_2)$

$\Rightarrow y\left(t_1 - \frac{4}{t_1}\right) = 2(x - 4)$ [from (ii)]

$\Rightarrow 2(x - 4) - y\left(t_1 - \frac{4}{t_1}\right) = 0 \Rightarrow L_1 + \lambda L_2 = 0$

$\therefore$ variable line PQ passes through a fixed point which is point of intersection of $L_1 = 0$ & $L_2 = 0$

i.e. (4, 0) **Ans.**

9. LINE & A PARABOLA :

(a) The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a >=< cm \Rightarrow$ condition of tangency is, $\mathbf{c = \frac{a}{m}}$.

Note : Line $y = mx + c$ will be tangent to parabola $x^2 = 4ay$ if $\mathbf{c = -am^2}$.

(b) Length of the chord intercepted by the parabola $y^2 = 4ax$ on the line $y = mx + c$ is : $\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}$.

Note : Length of the focal chord making an angle α with the x - axis is $4a \operatorname{cosec}^2 \alpha$.

Illustration 7 : If the line $y = 3x + \lambda$ intersect the parabola $y^2 = 4x$ at two distinct points then set of values of λ is -

(A) $(3, \infty)$ (B) $(-\infty, 1/3)$ (C) $(1/3, 3)$ (D) none of these

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Solution : Putting value of y from the line in the parabola -

$(3x + \lambda)^2 = 4x$

$\Rightarrow \quad 9x^2 + (6\lambda - 4)x + \lambda^2 = 0$

$\because$ line cuts the parabola at two distinct points

$\therefore \quad D > 0$

$\Rightarrow \quad 4(3\lambda - 2)^2 - 4.9\lambda^2 > 0$

$\Rightarrow \quad 9\lambda^2 - 12\lambda + 4 - 9\lambda^2 > 0$

$\Rightarrow \quad \lambda < 1/3$

Hence, $\lambda \in (-\infty, 1/3)$ **Ans.(B)**

Do yourself - 2 :

(i) Find the value of 'a' for which the point $(a^2 - 1, a)$ lies inside the parabola $y^2 = 8x$.

(ii) The focal distance of a point on the parabola $(x - 1)^2 = 16(y - 4)$ is 8. Find the co-ordinates.

(iii) Show that the focal chord of parabola $y^2 = 4ax$ makes an angle α with x-axis is of length $4a \operatorname{cosec}^2\alpha$.

(iv) Find the condition that the straight line $ax + by + c = 0$ touches the parabola $y^2 = 4kx$.

(v) Find the length of the chord of the parabola $y^2 = 8x$, whose equation is $x + y = 1$.

10. LENGTH OF SUBTANGENT & SUBNORMAL :

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PT and PG are the tangent and normal respectively at the point P to the parabola $y^2 = 4ax$. Then

TN = length of subtangent = twice the abscissa of the point P

(Subtangent is always bisected by the vertex)

NG = length of subnormal which is constant for all points on the parabola & equal to its semilatus rectum (2a).

11. TANGENT TO THE PARABOLA $y^2 = 4ax$:

(a) Point form :

Equation of tangent to the given parabola at its point (x_1, y_1) is

$\mathbf{yy_1 = 2a\,(x + x_1)}$

(b) Slope form :

Equation of tangent to the given parabola whose slope is 'm', is

$$y = mx + \frac{a}{m}, \ (m \neq 0)$$

Point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

(c) Parametric form :

Equation of tangent to the given parabola at its point P(t), is $\mathbf{ty = x + at^2}$

Note : Point of intersection of the tangents at the point $\mathbf{t_1}$ **&** $\mathbf{t_2}$ **is** $\mathbf{[\,at_1t_2, a(t_1 + t_2)]}$.

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Illustration 8 : A tangent to the parabola $y^2 = 8x$ makes an angle of 45° with the straight line $y = 3x + 5$. Find its equation and its point of contact.

Solution : Let the slope of the tangent be m

$$\therefore \quad \tan 45° = \left|\frac{3-m}{1+3m}\right| \Rightarrow \quad 1+3m = \pm(3-m)$$

$$\therefore \quad m = -2 \quad \text{or} \quad \frac{1}{2}$$

As we know that equation of tangent of slope m to the parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$

and point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

for $m = -2$, equation of tangent is $y = -2x - 1$ and point of contact is $\left(\frac{1}{2}, -2\right)$

for $m = \frac{1}{2}$, equation of tangent is $y = \frac{1}{2}x + 4$ and point of contact is (8, 8)

Ans.

Illustration 9 : Find the equation of the tangents to the parabola $y^2 = 9x$ which go through the point (4, 10).

Solution : Equation of tangent to parabola $y^2 = 9x$ is

$$y = mx + \frac{9}{4m}$$

Since it passes through (4, 10)

$$\therefore \quad 10 = 4m + \frac{9}{4m} \quad \Rightarrow \quad 16m^2 - 40m + 9 = 0$$

$$m = \frac{1}{4}, \frac{9}{4}$$

$\therefore$ equation of tangent's are $y = \frac{x}{4} + 9$ & $y = \frac{9}{4}x + 1$ **Ans.**

Illustration 10: Find the locus of the point P from which tangents are drawn to the parabola $y^2 = 4ax$ having slopes m_1 and m_2 such that -

(i) $m_1^2 + m_2^2 = \lambda$ (constant) (ii) $\theta_1 - \theta_2 = \theta_0$ (constant)

where θ_1 and θ_2 are the inclinations of the tangents from positive x-axis.

Solution : Equation of tangent to $y^2 = 4ax$ is $y = mx + a/m$

Let it passes through P(h, k)

$\therefore \quad m^2h - mk + a = 0$

(i) $m_1^2 + m_2^2 = \lambda$

$$(m_1 + m_2)^2 - 2m_1m_2 = \lambda$$

$$\frac{k^2}{h^2} - 2.\frac{a}{h} = \lambda$$

$\therefore$ locus of P(h, k) is $y^2 - 2ax = \lambda x^2$

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(ii) $\theta_1 - \theta_2 = \theta_0$

$\tan(\theta_1 - \theta_2) = \tan\theta_0$

$$\frac{m_1 - m_2}{1 + m_1 m_2} = \tan\theta_0$$

$$(m_1 + m_2)^2 - 4m_1m_2 = \tan^2\theta_0(1 + m_1m_2)^2$$

$$\frac{k^2}{h^2} - \frac{4a}{h} = \tan^2\theta_0\left(1 + \frac{a}{h}\right)^2$$

$k^2 - 4ah = (h + a)^2 \tan^2\theta_0$

$\therefore$ locus of P(h, k) is $y^2 - 4ax = (x + a)^2\tan^2\theta_0$ **Ans.**

Do yourself - 3 :

(i) Find the equation of the tangent to the parabola $y^2 = 12x$, which passes through the point (2, 5). Find also the co-ordinates of their points of contact.

(ii) Find the equation of the tangents to the parabola $y^2 = 16x$, which are parallel and perpendicular respectively to the line $2x - y + 5 = 0$. Find also the co-ordinates of their points of contact.

(iii) Prove that the locus of the point of intersection of tangents to the parabola $y^2 = 4ax$ which meet at an angle θ is $(x + a)^2 \tan^2\theta = y^2 - 4ax$.

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12. NORMAL TO THE PARABOLA $y^2 = 4ax$:

(a) Point form :

Equation of normal to the given parabola at its point (x_1, y_1) is

$$\mathbf{y - y_1 = -\frac{y_1}{2a}(x - x_1)}$$

(b) Slope form :

Equation of normal to the given parabola whose slope is 'm', is

$\mathbf{y = mx - 2am - am^3}$

foot of the normal is $\mathbf{(am^2, -2am)}$

(c) Parametric form :

Equation of normal to the given parabola at its point P(t), is

$\mathbf{y + tx = 2at + at^3}$

Note :

(i) Point of intersection of normals at t_1 & t_2 is $\mathbf{(a(t_1^2 + t_2^2 + t_1t_2 + 2), -at_1t_2(t_1 + t_2))}$.

(ii) If the normal to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then $\mathbf{t_2 = -\left(t_1 + \frac{2}{t_1}\right)}$.

(iii) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $\mathbf{t_1t_2 = 2 ; t_3 = -(t_1 + t_2)}$ and the line joining t_1 & t_2 passes through a fixed point (–2a, 0).

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(iv) If normal drawn to a parabola passes through a point P(h,k) then $k = mh - 2am - am^3$, i.e. $\mathbf{am^3 + m(2a - h) + k = 0}$.

This gives $\mathbf{m_1 + m_2 + m_3 = 0}$; $\mathbf{m_1m_2+m_2m_3+m_3m_1 = \frac{2a-h}{a}}$; $\mathbf{m_1m_2m_3 = \frac{-k}{a}}$

where m_1, m_2, & m_3 are the slopes of the three concurrent normals :

- Algebraic sum of slopes of the three concurrent normals is zero.
- Algebraic sum of ordinates of the three co-normal points on the parabola is zero.
- Centroid of the Δ formed by three co-normal points lies on the axis of parabola (x-axis).

Illustration 11 : Prove that the normal chord to a parabola $y^2 = 4ax$ at the point whose ordinate is equal to abscissa subtends a right angle at the focus.

Solution : Let the normal at $P\left(at_1^2, 2at_1\right)$ meet the curve at $Q\left(at_2^2, 2at_2\right)$

$\therefore$ PQ is a normal chord.

and $t_2 = -t_1 - \frac{2}{t_1}$ .....(i)

By given condition $2at_1 = at_1^2$

$\therefore$ $t_1 = 2$ from equation (i), $t_2 = -3$

then P(4a, 4a) and Q(9a, –6a)

but focus S(a, 0)

$\therefore$ Slope of SP $= \frac{4a-0}{4a-a} = \frac{4a}{3a} = \frac{4}{3}$

and Slope of SQ $= \frac{-6a-0}{9a-a} = \frac{-6a}{8a} = -\frac{3}{4}$

$\therefore$ Slope of SP $\times$ Slope of SQ $= \frac{4}{3} \times -\frac{3}{4} = -1$

$\therefore$ $\angle PSQ = \pi/2$

i.e. PQ subtends a right angle at the focus S.

Illustration 12 : If two normals drawn from any point to the parabola $y^2 = 4ax$ make angle α and β with the axis such that $\tan\alpha \cdot \tan\beta = 2$, then find the locus of this point.

Solution : Let the point is (h, k). The equation of any normal to the parabola $y^2 = 4ax$ is

$y = mx - 2am - am^3$

passes through (h, k)

$k = mh - 2am - am^3$

$am^3 + m(2a - h) + k = 0$ ....(i)

m_1, m_2, m_3 are roots of the equation, then $m_1 . m_2 . m_3 = -\frac{k}{a}$

but $m_1m_2 = 2$, $m_3 = -\frac{k}{2a}$

m_3 is root of (i) $\therefore$ $a\left(-\frac{k}{2a}\right)^3 - \frac{k}{2a}(2a-h) + k = 0 \Rightarrow k^2 = 4ah$

Thus locus is $y^2 = 4ax$. **Ans.**

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Illustration 13 : Three normals are drawn from the point (14, 7) to the curve $y^2 - 16x - 8y = 0$. Find the coordinates of the feet of the normals.

Solution : The given parabola is $y^2 - 16x - 8y = 0$ ..... (i)

Let the co-ordinates of the feet of the normal from (14, 7) be $P(\alpha, \beta)$. Now the equation of the tangent at $P(\alpha, \beta)$ to parabola (i) is

$y\beta - 8(x + \alpha) - 4(y + \beta) = 0$

or $(\beta - 4)y = 8x + 8a + 4\beta$ ..... (ii)

Its slope $= \dfrac{8}{\beta - 4}$

Equation of the normal to parabola (i) at (α, β) is $y - \beta = \dfrac{4-\beta}{8}(x - \alpha)$

It passes through (14, 7)

$\Rightarrow \quad 7 - \beta = \dfrac{4-\beta}{8}(14 - \alpha) \Rightarrow \alpha = \dfrac{6\beta}{\beta - 4}$ ..... (iii)

Also (α, β) lies on parabola (i) i.e. $\beta^2 - 16\alpha - 8\beta = 0$ ..... (iv)

Putting the value of α from (iii) in (iv), we get $\beta^2 - \dfrac{96\beta}{\beta - 4} - 8\beta = 0$

$\Rightarrow \quad \beta^2(\beta - 4) - 96\beta - 8\beta(\beta - 4) = 0 \Rightarrow \beta(\beta^2 - 4\beta - 96 - 8\beta + 32) = 0$

$\Rightarrow \quad \beta(\beta^2 - 12\beta - 64) = 0 \Rightarrow \beta(\beta - 16)(\beta + 4) = 0$

$\Rightarrow \quad \beta = 0, 16, -4$

from (iii), $\alpha = 0$ when $\beta = 0$; $\alpha = 8$, when $\beta = 16$; $\alpha = 3$ when $\beta = -4$

Hence the feet of the normals are (0, 0), (8, 16) and (3, –4) **Ans.**

Do yourself - 4 :

(i) If three distinct and real normals can be drawn to $y^2 = 8x$ from the point (a, 0), then -

(A) $a > 2$ (B) $a \in (2, 4)$ (C) $a > 4$ (D) none of these

(ii) Find the number of distinct normal that can be drawn from (–2, 1) to the parabola $y^2 - 4x - 2y - 3 = 0$.

(iii) If $2x + y + k = 0$ is a normal to the parabola $y^2 = -16x$, then find the value of k.

(iv) Three normals are drawn from the point (7, 14) to the parabola $x^2 - 8x - 16y = 0$. Find the co-ordinates of the feet of the normals.

13. AN IMPORTANT CONCEPT :

If a family of straight lines can be represented by an equation $\boldsymbol{\lambda^2 P + \lambda Q + R = 0}$ where λ is a parameter and P, Q, R are linear functions of x and y then the family of lines will be tangent to the curve $\mathbf{Q^2 = 4PR}$.

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Illustration 14 : If the equation $m^2(x + 1) + m(y - 2) + 1 = 0$ represents a family of lines, where 'm' is parameter then find the equation of the curve to which these lines will always be tangents.

Solution : $m^2(x + 1) + m(y - 2) + 1 = 0$

The equation of the curve to which above lines will always be tangents can be obtained by equating its discriminant to zero.

$\therefore \quad (y - 2)^2 - 4(x + 1) = 0$

$y^2 - 4y + 4 - 4x - 4 = 0$

$y^2 = 4(x + y)$ **Ans.**

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14. PAIR OF TANGENTS :

The equation of the pair of tangents which can be drawn from any point $P(x_1, y_1)$ outside the parabola to the parabola $y^2 = 4ax$ is given by : $\mathbf{SS_1 = T^2}$ where :

$\mathbf{S \equiv y^2 - 4ax}$; $\mathbf{S_1 \equiv y_1^2 - 4ax_1}$; $\mathbf{T \equiv yy_1 - 2a(x + x_1)}$.

15. DIRECTOR CIRCLE :

Locus of the point of intersection of the perpendicular tangents to the parabola $y^2 = 4ax$ is called the **director circle.** It's equation is $\mathbf{x + a = 0}$ which is parabola's own directrix.

Illustration 15 : The angle between the tangents drawn from a point $(-a, 2a)$ to $y^2 = 4ax$ is -

(A) $\pi/4$ (B) $\pi/2$ (C) $\pi/3$ (D) $\pi/6$

Solution : The given point $(-a, 2a)$ lies on the directrix $x = -a$ of the parabola $y^2 = 4ax$. Thus, the tangents are at right angle. **Ans.(B)**

Illustration 16 : The circle drawn with variable chord $x + ay - 5 = 0$ (a being a parameter) of the parabola $y^2 = 20x$ as diameter will always touch the line -

(A) $x + 5 = 0$ (B) $y + 5 = 0$ (C) $x + y + 5 = 0$ (D) $x - y + 5 = 0$

Solution : Clearly $x + ay - 5 = 0$ will always pass through the focus of $y^2 = 20x$ i.e. (5, 0). Thus the drawn circle will always touch the directrix of the parabola i.e. the line $x + 5 = 0$. **Ans.(A)**

Do yourself - 5 :

(i) If the equation $\lambda^2x + \lambda y - \lambda^2 + 2\lambda + 7 = 0$ represents a family of lines, where 'λ' is parameter, then find the equation of the curve to which these lines will always be tangents.

(ii) Find the angle between the tangents drawn from the origin to the parabola, $y^2 = 4a(x - a)$.

16. CHORD OF CONTACT :

Equation of the chord of contact of tangents drawn from a point $P(x_1, y_1)$ is $\mathbf{yy_1 = 2a(x + x_1)}$

Note : The area of the triangle formed by the tangents from the point (x_1, y_1) & the chord of contact is $\dfrac{\left(y_1^2 - 4ax_1\right)^{3/2}}{2a}$ i.e. $\dfrac{(S_1)^{3/2}}{2a}$, **also note that the chord of contact exists only if the point P is not inside.**

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Illustration 17 : If the line $x - y - 1 = 0$ intersect the parabola $y^2 = 8x$ at P & Q, then find the point of intersection of tangents at P & Q.

Solution : Let (h, k) be point of intersection of tangents then chord of contact is

$$yk = 4(x + h)$$

$$4x - yk + 4h = 0 \qquad \text{..... (i)}$$

But given line is

$$x - y - 1 = 0 \qquad \text{..... (ii)}$$

Comparing (i) and (ii)

$$\therefore \quad \frac{4}{1} = \frac{-k}{-1} = \frac{4h}{-1} \quad \Rightarrow \quad h = -1,\ k = 4$$

$\therefore$ point $\equiv (-1, 4)$ **Ans.**

Illustration 18 : Find the locus of point whose chord of contact w.r.t. to the parabola $y^2 = 4bx$ is the tangent of the parabola $y^2 = 4ax$.

Solution : Equation of tangent to $y^2 = 4ax$ is $y = mx + \frac{a}{m}$ ..... (i)

Let it is chord of contact for parabola $y^2 = 4bx$ w.r.t. the point P(h, k)

$\therefore$ Equation of chord of contact is $yk = 2b(x + h)$

$$y = \frac{2b}{k}x + \frac{2bh}{k} \qquad \text{..... (ii)}$$

From (i) & (ii)

$$m = \frac{2b}{k},\ \frac{a}{m} = \frac{2bh}{k} \Rightarrow a = \frac{4b^2h}{k^2}$$

locus of P is $y^2 = \frac{4b^2}{a}x$. **Ans.**

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17. CHORD WITH A GIVEN MIDDLE POINT :

Equation of the chord of the parabola $y^2 = 4ax$ whose middle point is (x_1, y_1) is $\mathbf{y - y_1 = \frac{2a}{y_1}(x - x_1)}$.

This reduced to $\mathbf{T = S_1}$, where $\mathbf{T \equiv yy_1 - 2a(x + x_1)}$ & $\mathbf{S_1 \equiv y_1^2 - 4ax_1}$.

Illustration 19 : Find the locus of middle point of the chord of the parabola $y^2 = 4ax$ which pass through a given (p, q).

Solution : Let P(h, k) be the mid point of chord of the parabola $y^2 = 4ax$,

so equation of chord is $yk - 2a(x + h) = k^2 - 4ah$.

Since it passes through (p, q)

$\therefore$ $qk - 2a(p + h) = k^2 - 4ah$

$\therefore$ Required locus is $y^2 - 2ax - qy + 2ap = 0$.

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Illustration 20 : Find the locus of the middle point of a chord of a parabola $y^2 = 4ax$ which subtends a right angle at the vertex.

Solution : The equation of the chord of the parabola whose middle point is (α, β) is

$y\beta - 2a(x + \alpha) = \beta^2 - 4a\alpha$

$\Rightarrow \quad y\beta - 2ax = \beta^2 - 2a\alpha$

or $\quad \dfrac{y\beta - 2ax}{\beta^2 - 2a\alpha} = 1 \qquad$ ..... (i)

Now, the equation of the pair of the lines OP and OQ joining the origin O i.e. the vertex to the points of intersection P and Q of the chord with the parabola $y^2 = 4ax$ is obtained by making the equation homogeneous by means of (i). Thus the equation of lines OP and OQ is $y^2 = \dfrac{4ax(y\beta - 2ax)}{\beta^2 - 2a\alpha}$

$\Rightarrow \quad y^2(\beta^2 - 2a\alpha) - 4a\beta xy + 8a^2x^2 = 0$

If the lines OP and OQ are at right angles, then the coefficient of x^2 + the coefficient of y^2 = 0

Therefore, $\beta^2 - 2a\alpha + 8a^2 = 0 \Rightarrow \beta^2 = 2a(\alpha - 4a)$

Hence the locus of (α, β) is $y^2 = 2a(x - 4a)$

Do yourself - 6 :

(i) Find the equation of the chord of contacts of tangents drawn from a point (2, 1) to the parabola $x^2 = 2y$.

(ii) Find the co-ordinates of the middle point of the chord of the parabola $y^2 = 16x$, the equation of which is $2x - 3y + 8 = 0$

(iii) Find the locus of the mid-point of the chords of the parabola $y^2 = 4ax$ such that tangent at the extremities of the chords are perpendicular.

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18. DIAMETER :

The locus of the middle points of a system of parallel chords of a Parabola is called a DIAMETER. Equation to the diameter of a parabola is **y = 2a/m**, where **m = slope of parallel chords**.

19. IMPORTANT HIGHLIGHTS :

(a) If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G then ST = SG = SP where 'S' is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between the focal radius SP & the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of the parabola after reflection.

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(b) The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle at the **focus**.

(c) The tangents at the extremities of a focal chord intersect at right angles on the **directrix,** and a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point P (at^2, 2at) as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1+t^2}$ on a normal at the point P.

(d) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangent at the vertex.

(e) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola is ; **2a =** $\frac{2bc}{b+c}$ **i.e.** $\frac{1}{b}+\frac{1}{c}=\frac{1}{a}$.

(f) If the tangents at P and Q meet in T, then :

(i) TP and TQ subtend equal angles at the focus S.

(ii) $ST^2 = SP \cdot SQ$ &

(iii) The triangles SPT and STQ are similar.

(g) Tangents and Normals at the extremities of the latus rectum of a parabola $y^2 = 4ax$ constitute a square, their points of intersection being (–a, 0) & (3a, 0).

Note :

(i) The two tangents at the extremities of focal chord meet on the foot of the directrix.

(ii) Figure LNL'G is square of side $2\sqrt{2}a$

(h) The circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.

Do yourself - 7 :

(i) The parabola $y^2 = 4x$ and $x^2 = 4y$ divide the square region bounded by the line x = 4, y = 4 and the co-ordinates axes. If S_1, S_2, S_3 are respectively the areas of these parts numbered from top to bottom; then find $S_1 : S_2 : S_3$.

(ii) Let P be the point (1, 0) and Q a point on the parabola $y^2 = 8x$, then find the locus of the mid point of PQ.

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Miscellaneous Illustrations :

Illustration 21 : The common tangent of the parabola $y^2 = 8ax$ and the circle $x^2 + y^2 = 2a^2$ is -

(A) $y = x + a$ (B) $x + y + a = 0$ (C) $x + y + 2a = 0$ (D) $y = x + 2a$

Solution : Any tangent to parabola is $y = mx + \frac{2a}{m}$

Solving with the circle $x^2 + (mx + \frac{2a}{m})^2 = 2a^2 \Rightarrow x^2(1 + m^2) + 4ax + \frac{4a^2}{m^2} - 2a^2 = 0$

$B^2 - 4AC = 0$ gives $m = \pm 1$

Tangent $y = \pm x \pm 2a$ **Ans. (C,D)**

Illustration 22 : If the tangent to the parabola $y^2 = 4ax$ meets the axis in T and tangent at the vertex A in Y and the rectangle TAYG is completed, show that the locus of G is $y^2 + ax = 0$.

Solution : Let $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$.

Then tangent at $P(at^2, 2at)$ is $ty = x + at^2$

Since tangent meet the axis of parabola in T and tangent at the vertex in Y.

$\therefore$ Co-ordinates of T and Y are $(-at^2, 0)$ and $(0, at)$ respectively.

Let co-ordinates of G be (x_1, y_1).

Since TAYG is rectangle.

$\therefore$ Mid-points of diagonals TY and GA is same

$\Rightarrow \frac{x_1 + 0}{2} = \frac{-at^2 + 0}{2} \Rightarrow x_1 = -at^2$ ..... (i)

and $\frac{y_1 + 0}{2} = \frac{0 + at}{2} \Rightarrow y_1 = at$ ..... (ii)

Eliminating t from (i) and (ii) then we get $x_1 = -a\left(\frac{y_1}{a}\right)^2$

or $y_1^2 = -ax_1$ or $y_1^2 + ax_1 = 0$

$\therefore$ The locus of $G(x_1, y_1)$ is $y^2 + ax = 0$

Illustration 23 : If P(–3, 2) is one end of the focal chord PQ of the parabola $y^2 + 4x + 4y = 0$, then the slope of the normal at Q is -

(A) –1/2 (B) 2 (C) 1/2 (D) –2

Solution : The equation of the tangent at (–3, 2) to the parabola $y^2 + 4x + 4y = 0$ is

$2y + 2(x - 3) + 2(y + 2) = 0$

or $2x + 4y - 2 = 0 \Rightarrow x + 2y - 1 = 0$

Since the tangent at one end of the focal chord is parallel to the normal at the other end, the slope of the normal at the other end of the focal chord is $-\frac{1}{2}$. **Ans.(A)**

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Illustration 24 : Prove that the two parabolas $y^2 = 4ax$ and $y^2 = 4c(x - b)$ cannot have common normal, other than the axis unless $b/(a - c) > 2$.

Solution : Given parabolas $y^2 = 4ax$ and $y^2 = 4c(x - b)$ have common normals. Then equation of normals in terms of slopes are $y = mx - 2am - am^3$ and $y = m(x - b) - 2cm - cm^3$ respectively then normals must be identical, compare the co-efficients

$$1 = \frac{2am + am^3}{mb + 2cm + cm^3}$$

$\Rightarrow$ $m[(c - a)m^2 + (b + 2c - 2a)] = 0,\ m \neq 0$ ($\because$ other than axis)

and $m^2 = \dfrac{2a - 2c - b}{c - a}$, $m = \pm\sqrt{\dfrac{2(a-c) - b}{c - a}}$

or $m = \pm\sqrt{\left(-2 - \dfrac{b}{c - a}\right)}$

$\therefore$ $-2 - \dfrac{b}{c - a} > 0$

or $-2 + \dfrac{b}{a - c} > 0 \Rightarrow \dfrac{b}{a - c} > 2$

Illustration 25 : If r_1, r_2 be the length of the perpendicular chords of the parabola $y^2 = 4ax$ drawn through the vertex, then show that $(r_1 r_2)^{4/3} = 16a^2\left(r_1^{2/3} + r_2^{2/3}\right)$.

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Solution : Since chord are perpendicular, therefore if one makes an angle θ then the other will make an angle $(90° - \theta)$ with x-axis

Let $AP = r_1$ and $AQ = r_2$

If $\angle PAX = \theta$

then $\angle QAX = 90° - \theta$

$\therefore$ Co-ordinates of P and Q are $(r_1 \cos\theta, r_1 \sin\theta)$

and $(r_2 \sin\theta, -r_2 \cos\theta)$ respectively.

Since P and Q lies on $y^2 = 4ax$

$\therefore$ $r_1^2 \sin^2\theta = 4ar_1 \cos\theta$ and $r_2^2 \cos^2\theta = 4ar_2 \sin\theta$

$\Rightarrow$ $r_1 = \dfrac{4a\cos\theta}{\sin^2\theta}$ and $r_2 = \dfrac{4a\sin\theta}{\cos^2\theta}$

$\therefore$ $(r_1 r_2)^{4/3} = \left(\dfrac{4a\cos\theta}{\sin^2\theta} \cdot \dfrac{4a\sin\theta}{\cos^2\theta}\right)^{4/3} = \left(\dfrac{16a^2}{\sin\theta\cos\theta}\right)^{4/3}$ ..... (i)

and $16a^2 \cdot \left(r_1^{2/3} + r_2^{2/3}\right) = 16a^2\left\{\left(\dfrac{4a\cos\theta}{\sin^2\theta}\right)^{2/3} + \left(\dfrac{4a\sin\theta}{\cos^2\theta}\right)^{2/3}\right\}$

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$$=16a^2.(4a)^{2/3}\left\{\frac{(\cos\theta)^{2/3}}{(\sin\theta)^{4/3}}+\frac{(\sin\theta)^{2/3}}{(\cos\theta)^{4/3}}\right\} = 16a^2.(4a)^{2/3}\left\{\frac{\cos^2\theta+\sin^2\theta}{(\sin\theta)^{4/3}(\cos\theta)^{4/3}}\right\}$$

$$=\frac{16a^2.(4a)^{2/3}}{(\sin\theta\cos\theta)^{4/3}}=\left(\frac{16a^2}{\cos\theta\cos\theta}\right)^{4/3} = (r_1r_2)^{4/3} \qquad \{\text{from (i)}\}$$

Illustration 26 : The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

Solution : Let the three points on the parabola be

$(at_1^2, 2at_1), (at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$

The area of the triangle formed by these points

$$\Delta_1 = \frac{1}{2}[\,at_1^2(2at_2-2at_3) + at_2^2(2at_3-2at_1) + at_2^2(2a_1-2at_2)]$$

$$=-a^2(t_2-t_3)(t_3-t_1)(t_1-t_2).$$

The points of intersection of the tangents at these points are

$(at_2t_3, a(t_2+t_3)), (at_3t_1, a(t_3+t_1))$ and $(at_1t_2, a(t_1+t_2))$

The area of the triangle formed by these three points

$$\Delta_2 = \frac{1}{2}\{at_2t_3(at_3-at_2)+at_3t_1(at_1-at_3)+at_1t_2(at_2-at_1)\}$$

$$=\frac{1}{2}a^2(t_2-t_3)(t_3-t_1)(t_1-t_2)$$

Hence $\Delta_1 = 2\Delta_2$

Illustration 27: Prove that the orthocentre of any triangle formed by three tangents to a parabola lies on the directrix.

Solution : Let the equations of the three tangents be

$t_1y = x + at_1^2$ .....(i)

$t_2y = x + at_2^2$ .....(ii)

and $t_3y = x + at_3^2$ .....(iii)

The point of intersection of (ii) and (iii) is found, by solving them, to be $(at_2t_3, a(t_2+t_3))$

The equation of the straight line through this point & perpendicular to (i) is

$$y - a(t_2+t_3) = -t_1(x - at_2t_3)$$

i.e. $y + t_1x = a(t_2+t_3+t_1t_2t_3)$ .....(iv)

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Similarly, the equation of the straight line through the point of intersection of (iii) and (i) & perpendicular to (ii) is

$$y + t_2x = a(t_3 + t_1 + t_1t_2t_3) \qquad \text{.....(v)}$$

and the equation of the straight line through the point of intersection of (i) and (ii) & perpendicular to (iii) is

$$y + t_1x = a(t_1 + t_2 + t_1t_2t_3) \qquad \text{.....(vi)}$$

The point which is common to the straight lines (iv), (v) and (vi)

i.e. the orthocentre of the triangle, is easily seen to be the point whose coordinates are

$$x = -a,\ y = a(t_1 + t_2 + t_3 + t_1t_2t_3)$$

and this point lies on the directrix.

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ANSWERS FOR DO YOURSELF

1 : **(i)** Parabola **(ii)** Vertex : $\left(-\frac{7}{2}, \frac{5}{2}\right)$, Axis : $y = \frac{5}{2}$, Focus : $\left(-\frac{17}{4}, \frac{5}{2}\right)$, Directrix : $x = -\frac{11}{4}$; LR = 3

(iii) $4x^2 + y^2 - 4xy + 8x + 46y - 71 = 0$; Axis : $2x - y = 3$; LR = $4\sqrt{5}$ unit

(iv) $(3x + 4y - 4)^2 = 20(4x - 3y + 7)$

2 : **(i)** $\left(-\infty, -\sqrt{\frac{8}{7}}\right) \cup \left(\sqrt{\frac{8}{7}}, \infty\right)$ **(ii)** $(-7, 8), (9, 8)$ **(iv)** $kb^2 = ac$ **(v)** $8\sqrt{3}$

3 : **(i)** $x - y + 3 = 0, (3, 6)$; $3x - 2y + 4 = 0, \left(\frac{4}{3}, 4\right)$

(ii) $2x - y + 2 = 0$, $(1, 4)$; $x + 2y + 16 = 0$, $(16, -16)$

4 : **(i)** C **(ii)** 1 **(iii)** 48 **(iv)** $(0, 0), (-4, 3)$ and $(16, 8)$

5 : **(i)** $(y + 2)^2 = 28(x - 1)$ **(ii)** $\pi/2$

6 : **(i)** $2x = y + 1$ **(ii)** $(14, 12)$ **(iii)** $y^2 = 2a(x - a)$

7 : **(i)** $1 : 1 : 1$ **(ii)** $y^2 - 4x + 2 = 0$

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EXERCISE (O-1)

[STRAIGHT OBJECTIVE TYPE]

1. The equation of the directrix of the parabola, $y^2 + 4y + 4x + 2 = 0$ is -

(A) $x = -1$ (B) $x = 1$ (C) $x = -3/2$ (D) $x = 3/2$

PR0001

2. Length of the latus rectum of the parabola $25[(x-2)^2 + (y-3)^2] = (3x - 4y + 7)^2$ is-

(A) 4 (B) 2 (C) 1/5 (D) 2/5

PR0002

3. If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$ then one of the values of 'k' is -

(A) 1/8 (B) 8 (C) 4 (D) 1/4

PR0003

4. The length of the intercept on y – axis cut off by the parabola, $y^2 - 5y = 3x - 6$ is

(A) 1 (B) 2 (C) 3 (D) 5

PR0004

5. Maximum number of common chords of a parabola and a circle can be equal to

(A) 2 (B) 4 (C) 6 (D) 8

PR0005

6. A variable circle is drawn to touch the line $3x - 4y = 10$ and also the circle $x^2 + y^2 = 1$ externally then the locus of its centre is -

(A) straight line (B) circle

(C) pair of real, distinct straight lines (D) parabola

PR0006

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7. The locus of the point of trisection of all the double ordinates of the parabola $y^2 = \ell x$ is a parabola whose latus rectum is -

(A) $\frac{\ell}{9}$ (B) $\frac{2\ell}{9}$ (C) $\frac{4\ell}{9}$ (D) $\frac{\ell}{36}$

PR0007

8. The straight line $y = m(x - a)$ will meet the parabola $y^2 = 4ax$ in two distinct real points if

(A) $m \in R$ (B) $m \in [-1, 1]$

(C) $m \in (-\infty, 1] \cup [1, \infty)$ (D) $m \in R - \{0\}$

PR0008

9. The vertex A of the parabola $y^2 = 4ax$ is joined to any point P on it and PQ is drawn at right angles to AP to meet the axis in Q. Projection of PQ on the axis is equal to

(A) twice the length of latus rectum (B) the latus length of rectum

(C) half the length of latus rectum (D) one fourth of the length of latus rectum

PR0009

10. If on a given base, a triangle be described such that the sum of the tangents of the base angles is a constant, then the locus of the vertex is :

(A) a circle (B) a parabola (C) an ellipse (D) a hyperbola

PR0010

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11. The equation of the circle drawn with the focus of the parabola $(x-1)^2 - 8y = 0$ as its centre and touching the parabola at its vertex is :

(A) $x^2 + y^2 - 4y = 0$
(B) $x^2 + y^2 - 4y + 1 = 0$
(C) $x^2 + y^2 - 2x - 4y = 0$
(D) $x^2 + y^2 - 2x - 4y + 1 = 0$

PR0011

12. Which one of the following equations represented parametrically, represents equation to a parabolic profile?

(A) $x = 3\cos t$; $y = 4\sin t$
(B) $x^2 - 2 = -2\cos t$; $y = 4\cos^2\frac{t}{2}$
(C) $\sqrt{x} = \tan t$; $\sqrt{y} = \sec t$
(D) $x = \sqrt{1-\sin t}$; $y = \sin\frac{t}{2} + \cos\frac{t}{2}$

PR0012

13. Angle between the parabolas $y^2 = 4(x-1)$ and $x^2 + 4(y-3) = 0$ at the common end of their latus rectum, is -

(A) $\tan^{-1}(1)$
(B) $\tan^{-1}1 + \cot^{-1}2 + \cot^{-1}3$
(C) $\tan^{-1}\left(\sqrt{3}\right)$
(D) $\tan^{-1}(2) + \tan^{-1}(3)$

PR0013

14. If a focal chord of $y^2 = 4x$ makes an angle $\alpha, \alpha \in \left(0, \frac{\pi}{4}\right]$ with the positive direction of x-axis, then minimum length of this focal chord is -

(A) $2\sqrt{2}$
(B) $4\sqrt{2}$
(C) 8
(D) 16

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PR0014

15. A parabola $y = ax^2 + bx + c$ crosses the x-axis at $(\alpha,0)$ $(\beta,0)$ both to the right of the origin. A circle also passess through these two points. The length of a tangent from the origin to the circle is :

(A) $\sqrt{\frac{bc}{a}}$
(B) ac^2
(C) $\frac{b}{a}$
(D) $\sqrt{\frac{c}{a}}$

PR0015

16. If (2,–8) is one end of a focal chord of the parabola $y^2 = 32x$, then the other end of the focal chord, is-

(A) (32,32)
(B) (32,–32)
(C) (–2,8)
(D) (2,8)

PR0016

17. Minimum distance between the curves $y^2 = x - 1$ and $x^2 = y - 1$ is equal to

(A) $\frac{3\sqrt{2}}{4}$
(B) $\frac{5\sqrt{2}}{4}$
(C) $\frac{7\sqrt{2}}{4}$
(D) $\frac{\sqrt{2}}{4}$

PR0017

18. The length of a focal chord of the parabola $y^2 = 4ax$ at a distance b from the vertex is c, then

(A) $2a^2 = bc$
(B) $a^3 = b^2c$
(C) $ac = b^2$
(D) $b^2c = 4a^3$

PR0018

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19. Locus of trisection point of any arbitrary double ordinate of the parabola $x^2 = 4by$, is -

(A) $9x^2 = by$ (B) $3x^2 = 2by$ (C) $9x^2 = 4by$ (D) $9x^2 = 2by$

PR0019

20. Consider the graphs of $y = Ax^2$ and $y^2 + 3 = x^2 + 4y$, where A is a positive constant and $x,y \in R$. Number of points in which the two graphs intersect, is-

(A) exactly 4

(B) exactly 2

(C) at least 2 but the number of points varies for different positive values of A.

(D) zero for atleast one positive A.

PR0020

21. From an external point P, pair of tangent lines are drawn to the parabola, $y^2 = 4x$. If θ_1 & θ_2 are the inclinations of these tangents with the axis of x such that, $\theta_1 + \theta_2 = \frac{\pi}{4}$, then the locus of P is :

(A) $x - y + 1 = 0$ (B) $x + y - 1 = 0$ (C) $x - y - 1 = 0$ (D) $x + y + 1 = 0$

PR0021

22. y-intercept of the common tangent to the parabola $y^2 = 32x$ and $x^2 = 108y$ is

(A) – 18 (B) – 12 (C) – 9 (D) – 6

PR0022

23. The points of contact Q and R of tangent from the point P (2, 3) on the parabola $y^2 = 4x$ are

(A) (9, 6) and (1, 2) (B) (1, 2) and (4, 4) (C) (4, 4) and (9, 6) (D) (9, 6) and $(\frac{1}{4}, 1)$

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PR0023

24. If the lines $(y - b) = m_1(x + a)$ and $(y - b) = m_2(x + a)$ are the tangents to the parabola $y^2 = 4ax$, then

(A) $m_1 + m_2 = 0$ (B) $m_1m_2 = 1$ (C) $m_1m_2 = -1$ (D) $m_1 + m_2 = 1$

PR0024

25. The equation of a straight line passing through the point (3,6) and cutting the curve $y = \sqrt{x}$ orthogonally is-

(A) $4x + y - 18 = 0$ (B) $x + y - 9 = 0$ (C) $4x - y - 6 = 0$ (D) none

PR0025

26. The equation of the common tangent touching the circle $(x - 3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the x-axis is -

(A) $\sqrt{3}y = 3x + 1$ (B) $\sqrt{3}y = -(x + 3)$ (C) $\sqrt{3}y = x + 3$ (D) $\sqrt{3}y = -(3x + 1)$

PR0026

27. Let BC be the latus rectum of the parabola $y^2 = 4x$ with vertex A. Minimum length of the projection of BC on a tangent drawn in the portion BAC is -

(A) 2 (B) $2\sqrt{3}$ (C) $2\sqrt{2}$ (D) $2 + \sqrt{2}$

PR0027

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28. If $x + y = k$ is normal to $y^2 = 12x$, then 'k' is-

(A) 3 (B) 9 (C) –9 (D) –3

PR0028

29. Equation of the other normal to the parabola $y^2 = 4x$ which passes through the intersection of those at (4,–4) and (9a, –6a) is -

(A) $5x - y + 115 = 0$ (B) $5x + y - 135 = 0$ (C) $5x - y - 115 = 0$ (D) $5x + y + 115 = 0$

PR0029

30. Length of the normal chord of the parabola, $y^2 = 4x$, which makes an angle of $\frac{\pi}{4}$ with the axis of x is:

(A) 8 (B) $8\sqrt{2}$ (C) 4 (D) $4\sqrt{2}$

PR0030

31. Suppose that three points on the parabola $y = x^2$ have the property that their normal lines intersect at a common point (a,b). The sum of their x-coordinates is -

(A) 0 (B) $\frac{2b-1}{2}$ (C) $\frac{a}{2}$ (D) $a + b$

PR0031

32. The normal chord of a parabola $y^2 = 4ax$ at the point whose ordinate is equal to the abscissa, then angle subtended by normal chord at the focus is :

(A) $\frac{\pi}{4}$ (B) $\tan^{-1}\sqrt{2}$ (C) $\tan^{-1}2$ (D) $\frac{\pi}{2}$

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PR0032

33. Which one of the following lines cannot be the normals to $x^2 = 4y$?

(A) $x - y + 3 = 0$ (B) $x + y - 3 = 0$ (C) $x - 2y + 12 = 0$ (D) $x + 2y + 12 = 0$

PR0033

34. Tangents are drawn from the points on the line $x - y + 3 = 0$ to parabola $y^2 = 8x$. Then the variable chords of contact pass through a fixed point whose coordinates are :

(A) (3,2) (B) (2,4) (C) (3,4) (D) (4,1)

PR0034

35. Consider two curves $C_1 : \left(y-\sqrt{3}\right)^2 = 4\left(x-\sqrt{2}\right)$ and $C_2 : x^2 + y^2 = \left(6+2\sqrt{2}\right)x + 2\sqrt{3}y - 6\left(1+\sqrt{2}\right)$, then-

(A) C_1 and C_2 touch each other only at one point.

(B) C_1 and C_2 touch each other exactly at two points.

(C) C_1 and C_2 intersect (but do not touch) at exactly two points.

(D) C_1 and C_2 neither intersect nor touch each other.

PR0035

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36. ABCD and EFGC are squares and the curve $y = k\sqrt{x}$ passes through the origin D and the points B and F. The ratio $\frac{FG}{BC}$ is -

(A) $\frac{\sqrt{5}+1}{2}$ (B) $\frac{\sqrt{3}+1}{2}$

(C) $\frac{\sqrt{5}+1}{4}$ (D) $\frac{\sqrt{3}+1}{4}$

PR0036

37. C is the centre of the circle with centre (0,1) and radius unity. P is parabola $y = ax^2$. The set of values of 'a' for which they meet at a point other than the origin, is-

(A) $a > 0$ (B) $a \in \left(0, \frac{1}{2}\right)$ (C) $\left(\frac{1}{4}, \frac{1}{2}\right)$ (D) $\left(\frac{1}{2}, \infty\right)$

PR0037

38. Tangents are drawn from the point (– 1, 2) on the parabola $y^2 = 4x$. The length, these tangents will intercept on the line x = 2 is :

(A) 6 (B) $6\sqrt{2}$ (C) $2\sqrt{6}$ (D) none of these

PR0038

39. If the locus of the middle points of the chords of the parabola $y^2 = 2x$ which touches the circle $x^2 + y^2 - 2x - 4 = 0$ is given by $(y^2 + 1 - x)^2 = \lambda(1 + y^2)$, then the value of λ is equal to-

(A) 3 (B) 4 (C) 5 (D) 6

PR0039

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[MULTIPLE OBJECTIVE TYPE]

40. A variable circle is described to pass through the point (1,0) and tangent to the curve $y = \tan(\tan^{-1}x)$. The locus of the centre of the circle is a parabola whose -

(A) length of the latus rectum is $2\sqrt{2}$ (B) axis of symmetry has the equation x + y = 1

(C) vertex has the co-ordinates (3/4, 1/4) (D) directrix is x – y = 0

PR0040

41. The locus of the mid point of the focal radii of a variable point moving on the parabola, $y^2 = 8x$ is a parabola whose-

(A) Latus rectum is half the latus rectum of the original parabola

(B) Vertex is (1,0)

(C) Directrix is y-axis

(D) Focus has the co-ordinates (2,0)

PR0041

42. Consider a circle with its centre lying on the focus of the parabola, $y^2 = 2px$ such that it touches the directrix of the parabola. Then a point of intersection of the circle & the parabola is

(A) $\left(\frac{p}{2}, p\right)$ (B) $\left(\frac{p}{2}, -p\right)$ (C) $\left(-\frac{p}{2}, p\right)$ (D) $\left(-\frac{p}{2}, -p\right)$

PR0042

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43. Let $y^2 = 4ax$ be a parabola and $x^2 + y^2 + 2bx = 0$ be a circle. If parabola and circle touch each other externally then :

(A) $a > 0, b > 0$ (B) $a > 0, b < 0$ (C) $a < 0, b > 0$ (D) $a < 0, b < 0$

PR0043

44. The focus of the parabola is (1,1) and the tangent at the vertex has the equation $x + y = 1$. Then :

(A) equation of the parabola is $(x - y)^2 = 2(x + y - 1)$

(B) equation of the parabola is $(x - y)^2 = 4(x + y - 1)$

(C) the co-ordinates of the vertex are $\left(\frac{1}{2}, \frac{1}{2}\right)$

(D) length of the latus rectum is $2\sqrt{2}$

PR0044

45. The straight line $y + x = 1$ touches the parabola

(A) $x^2 + 4y = 0$ (B) $x^2 - x + y = 0$ (C) $4x^2 - 3x + y = 0$ (D) $x^2 - 2x + 2y = 0$

PR0045

46. The parabola $x = y^2 + ay + b$ intersect the parabola $x^2 = y$ at (1,1) at right angle. Which of the following is/are correct ?

(A) $a = 4, b = -4$

(B) $a = 2, b = -2$

(C) Equation of the director circle for the parabola $x = y^2 + ay + b$ is $4x + 1 = 0$.

(D) Area enclosed by the parabola $x = y^2 + ay + b$ and its latus rectum is $\frac{1}{6}$.

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PR0046

[ASSERTION AND REASON]

47. Consider a curve $C : y^2 - 8x - 2y - 15 = 0$ in which two tangents T_1 and T_2 are drawn from P(–4,1).

Statement-1 : T_1 and T_2 are mutually perpendicular tangents.

Statement-2 : Point P lies on the axis of curve C.

(A) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1.

(B) Statement-1 is True, Statement-2 is True ; Statement-2 is NOT a correct explanation for Statement-1.

(C) Statement-1 is True, Statement-2 is False.

(D) Statement-1 is False, Statement-2 is True.

PR0047

[COMPREHENSION TYPE]

Paragraph for question nos. 48 & 49

Consider the parabola $y^2 = 8x$

48. Area of the figure formed by the tangents and normals drawn at the extremities of its latus rectum is

(A) 8 (B) 16 (C) 32 (D) 64

PR0048

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49. Distance between the tangent to the parabola and a parallel normal inclined at 30° with the x-axis, is

(A) $\frac{16}{3}$ (B) $\frac{16\sqrt{3}}{9}$ (C) $\frac{2}{3}$ (D) $\frac{16}{\sqrt{3}}$

PR0048

[MATRIX MATCH TYPE]

50. Identify the conic whose equations are given in column-I.

Column-I (Equation of a conic)	**Column-II (Nature of conic)**
(A) $xy + a^2 = a(x + y)$	(P) Ellipse **PR0049**
(B) $2x^2 - 72xy + 23y^2 - 4x - 28y - 48 = 0$	(Q) Hyperbola **PR0050**
(C) $6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$	(R) Parabola **PR0051**
(D) $14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$	(S) line pair **PR0052**
(E) $4x^2 - 4xy + y^2 - 12x + 6y + 9 = 0$	**PR0053**

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EXERCISE (O-2)

[STRAIGHT OBJECTIVE TYPE]

1. Two unequal parabolas have the same common axis which is the x-axis and have the same vertex which is the origin with their concavities in opposite direction. If a variable line parallel to the common axis meet the parabolas in P and P' the locus of the middle point of PP' is

(A) a parabola (B) a circle (C) an ellipse (D) a hyperbola

PR0054

2. PN is an ordinate of the parabola $y^2 = 4ax$(P on $y^2 = 4ax$ and N on x-axis). A straight line is drawn parallel to the axis to bisect NP and meets the curve in Q. NQ meets the tangent at the vertex in a point T such that AT = kNP, then the value of k is (where A is the vertex)

(A) 3/2 (B) 2/3 (C) 1 (D) none

PR0055

3. Let A and B be two points on a parabola $y^2 = x$ with vertex V such that VA is perpendicular to VB and θ is the angle between the chord VA and the axis of the parabola. The value of $\frac{|VA|}{|VB|}$ is

(A) tan θ (B) $\tan^3\theta$ (C) $\cot^2\theta$ (D) $\cot^3\theta$

PR0056

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4. Locus of the feet of the perpendiculars drawn from vertex of the parabola $y^2 = 4ax$ upon all such chords of the parabola which subtend a right angle at the vertex is

(A) $x^2 + y^2 - 4ax = 0$ (B) $x^2 + y^2 - 2ax = 0$

(C) $x^2 + y^2 + 2ax = 0$ (D) $x^2 + y^2 + 4ax = 0$

PR0057

5. The triangle PQR of area 'A' is inscribed in the parabola $y^2 = 4ax$ such that the vertex P lies at the vertex of the parabola and the base QR is a focal chord. The modulus of the difference of the ordinates of the point Q and R is :

(A) $\frac{A}{2a}$ (B) $\frac{A}{a}$ (C) $\frac{2A}{a}$ (D) $\frac{4A}{a}$

PR0058

6. Through the focus of the parabola $y^2 = 2px (p > 0)$ a line is drawn which intersects the curve at $A(x_1,y_1)$ and $B(x_2,y_2)$. The ratio $\frac{y_1y_2}{x_1x_2}$ equals-

(A) 2 (B) –1 (C) –4 (D) some function of p

PR0059

7. A circle with radius unity has its centre on the positive y-axis. If this circle touches the parabola $y = 2x^2$ tangentially at the point P and Q then the sum of the ordinates of P and Q, is-

(A) 15/4 (B) 15/8 (C) $2\sqrt{15}$ (D) 5

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PR0060

8. The straight line joining any point P on the parabola $y^2 = 4ax$ to the vertex and perpendicular from the focus to the tangent at P, intersect at R, then the equation of the locus of R is

(A) $x^2 + 2y^2 - ax = 0$ (B) $2x^2 + y^2 - 2ax = 0$

(C) $2x^2 + 2y^2 - ay = 0$ (D) $2x^2 + y^2 - 2ay = 0$

PR0061

9. If two normals to a parabola $y^2 = 4ax$ intersect at right angles then the chord joining their feet pass through a fixed point whose co-ordinates are :

(A) (–2a,0) (B) (a,0) (C) (2a,0) (D) none

PR0062

10. If the normal to a parabola $y^2 = 4ax$ at P meets the curve again in Q and if PQ and the normal at Q makes angles α and β respectively with the x-axis then $\tan\alpha(\tan\alpha + \tan\beta)$ has the value equal to

(A) 0 (B) – 2 (C) $-\frac{1}{2}$ (D) – 1

PR0063

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11. If a normal to a parabola $y^2 = 4ax$ makes an angle ϕ with its axis, then it will cut the curve again at an angle

(A) $\tan^{-1}(2\tan\phi)$ (B) $\tan^{-1}\left(\frac{1}{2}\tan\phi\right)$ (C) $\cot^{-1}\left(\frac{1}{2}\tan\phi\right)$ (D) none

PR0064

12. The tangent and normal at P(t), for all real positive t, to the parabola $y^2 = 4ax$ meet the axis of the parabola in T and G respectively, then the angle at which the tangent at P to the parabola is inclined to the tangent at P to the circle passing through the points P, T and G is

(A) $\cot^{-1}t$ (B) $\cot^{-1}t^2$ (C) $\tan^{-1}t$ (D) $\tan^{-1}t^2$

PR0065

13. Normal to the parabola $y^2 = 8x$ at the point P (2, 4) meets the parabola again at the point Q. If C is the centre of the circle described on PQ as diameter then the coordinates of the image of the point C in the line y = x are

(A) (– 4, 10) (B) (– 3, 8) (C) (4, – 10) (D) (– 3, 10)

PR0066

14. PQ is a normal chord of the parabola $y^2 = 4ax$ at P, A being the vertex of the parabola. Through P a line is drawn parallel to AQ meeting the x-axis in R. Then the length of AR is -

(A) equal to the length of the latus rectum.

(B) equal to the focal distance of the point P.

(C) equal to twice the focal distance of the point P.

(D) equal to the distance of the point P from the directrix.

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PR0067

15. Normals are drawn at points A, B, and C on the parabola $y^2 = 4x$ which intersect at P(h, k). The locus of the point P if the slope of the line joining the feet of two of them is 2 , is

(A) x + y = 1 (B) x – y = 3 (C) $y^2 = 2(x - 1)$ (D) $y^2 = 2\left(x - \frac{1}{2}\right)$

PR0068

16. TP & TQ are tangents to the parabola, $y^2 = 4ax$ at P & Q. If the chord PQ passes through the fixed point (– a, b) then the locus of T is :

(A) ay = 2b (x – b) (B) bx = 2a (y – a) (C) by = 2a (x – a) (D) ax = 2b (y – b)

PR0069

17. Through the vertex O of the parabola, $y^2 = 4ax$ two chords OP & OQ are drawn and the circles on OP & OQ as diameters intersect in R. If θ_1, θ_2 & ϕ are the angles made with the axis by the tangents at P & Q on the parabola & by OR then the value of, $\cot\theta_1 + \cot\theta_2 =$

(A) $-2\tan\phi$ (B) $-2\tan(\pi - \phi)$ (C) 0 (D) $2\cot\phi$

PR0070

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18. Length of the intercept on the normal at the point $P(at^2,2at)$ of the parabola $y^2 = 4ax$ made by the circle described on the focal distance of the point P as diameter is :

(A) $a\sqrt{2+t^2}$ (B) $\frac{a}{2}\sqrt{1+t^2}$ (C) $2a\sqrt{1+t^2}$ (D) $a\sqrt{1+t^2}$

PR0071

[MULTIPLE OBJECTIVE TYPE]

19. P is a point on the parabola $y^2 = 4ax(a > 0)$ whose vertex is A. PA is produced to meet the directrix in D and M is the foot of the perpendicular from P on the directrix. If a circle is described on MD as a diameter then it intersects the x-axis at a point whose co-ordinates are :

(A) (–3a,0) (B) (–a,0) (C) (–2a,0) (D) (a,0)

PR0072

20. If from the vertex of a parabola $y^2 = 4x$ a pair of chords be drawn at right angles to one another and with these chords as adjacent sides a rectangle be made, then the locus of the further end of the rectangle is -

(A) an equal parabola
(B) a parabola with focus at (9,0)
(C) a parabola with directrix as $x - 7 = 0$
(D) a parabola having tangent at its vertex $x = 8$

PR0073

21. A circle 'S' is described on the focal chord of the parabola $y^2 = 4x$ as diameter. If the focal chord is inclined at an angle of 45° with axis of x, then which of the following is/are true ?

(A) Radius of the circle is 4.
(B) Centre of the circle is (3,2)
(C) The line $x + 1 = 0$ touches the circle
(D) The circle $x^2 + y^2 + 2x - 6y + 3 = 0$ is orthogonal to 'S'.

PR0074

22. PQ is a double ordinate of the parabola $y^2 = 4ax$. If the normal at P intersect the line passing through Q and parallel to axis of x at G, then locus of G is a parabola with -

(A) length of latus rectum equal to 4a
(B) vertex at (4a,0)
(C) directrix as the line $x - 3a = 0$
(D) focus at (5a,0)

PR0075

23. TP and TQ are tangents to parabola $y^2 = 4x$ and normals at P and Q intersect at a point R on the curve. The locus of the centre of the circle circumscribing ΔTPQ is a parabola whose

(A) vertex is (1, 0).
(B) foot of directrix is $\left(\frac{7}{8}, 0\right)$.
(C) length of latus-rectum is $\frac{1}{4}$.
(D) focus is $\left(\frac{9}{8}, 0\right)$.

PR0076

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24. The locus of the point of intersection of those normals to the parabola $x^2 = 8y$ which are at right angles to each other, is a parabola. Which of the following hold(s) good in respect of the locus ?

(A) Length of the latus rectum is 2.
(B) Coordinates of focus are $\left(0, \frac{11}{2}\right)$
(C) Equation of a director circle is $2y - 11 = 0$
(D) Equation of axis of symmetry is $y = 0$

PR0077

25. Consider the parabola whose equation is $y = x^2 - 4x$ and the line $y = 2x - b$. Then which of the following is/are correct ?

(A) For $b = 9$ the line is a tangent to the parabola.
(B) For $b = 7$ the line cuts the parabola in A and B such that the $\angle AOB$ is a right angle when 'O' is the origin.
(C) For some $b \in R$ the line cuts the parabola in C and D such that x-axis bisects the $\angle COD$.
(D) For $b > 9$ the line passes outside the parabola.

PR0078

26. Through a point P (– 2, 0), tangents PQ and PR are drawn to the parabola $y^2 = 8x$. Two circles each passing through the focus of the parabola and one touching at Q and other at R are drawn. Which of the following point(s) with respect to the triangle PQR lie(s) on the radical axis of the two circles?

(A) centroid
(B) orthocenter
(C) incentre
(D) circumcenter

PR0079

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[COMPREHENSION TYPE]

Paragraph for question nos. 27 to 29

Tangents are drawn to the parabola $y^2 = 4x$ from the point P(6, 5) to touch the parabola at Q and R. C_1 is a circle which touches the parabola at Q and C_2 is a circle which touches the parabola at R. Both the circles C_1 and C_2 pass through the focus of the parabola.

27. Area of the ΔPQR equals

(A) $\frac{1}{2}$
(B) 1
(C) 2
(D) $\frac{1}{4}$

PR0080

28. Radius of the circle C_2 is

(A) $5\sqrt{5}$
(B) $5\sqrt{10}$
(C) $10\sqrt{2}$
(D) $\sqrt{210}$

PR0080

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29. The common chord of the circles C_1 and C_2 passes through the

(A) incentre of the ΔPQR (B) circumcenter of the ΔPQR

(C) centroid of the ΔPQR (D) orthocenter of the ΔPQR

PR0080

[MATRIX MATCH TYPE]

30. Consider the parabola $y^2 = 12x$

	Column-I	**Column-II**
(A)	Tangent and normal at the extremities of the latus rectum intersect the x axis at T and G respectively. The coordinates of the middle point of T and G are	(P) (0, 0)
		PR0081
(B)	Variable chords of the parabola passing through a fixed point K on the axis, such that sum of the squares of the reciprocals of the two parts of the chords through K, is a constant. The coordinate of the point K are	(Q) (3, 0)
		(R) (6, 0)
		PR0082
(C)	All variable chords of the parabola subtending a right angle at the origin are concurrent at the point	
		PR0083
(D)	AB and CD are the chords of the parabola which intersect at a point E on the axis. The radical axis of the two circles described on AB and CD as diameter always passes through	(S) (12, 0)
		PR0084

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EXERCISE (S-1)

1. 'O' is the vertex of the parabola $y^2 = 4ax$ & L is the upper end of the latus rectum. If LH is drawn perpendicular to OL meeting OX in H, prove that the length of the double ordinate through H is $4a\sqrt{5}$.

PR0085

2. A point P on a parabola $y^2 = 4x$, the foot of the perpendicular from it upon the directrix, and the focus are the vertices of an equilateral triangle, find the area of the equilateral triangle.

PR0086

3. Through the vertex O of a parabola $y^2 = 4x$, chords OP & OQ are drawn at right angles to one another. Show that for all positions of P, PQ cuts the axis of the parabola at a fixed point. Also find the locus of the middle point of PQ.

PR0087

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4. Find the equations of the tangents to the parabola $y^2 = 16x$, which are parallel & perpendicular respectively to the line $2x - y + 5 = 0$. Find also the coordinates of their points of contact.

PR0088

5. Find the equations of the tangents of the parabola $y^2 = 12x$, which passes through the point (2,5). Also find the point of contact.

PR0089

6. Through the vertex O of the parabola $y^2 = 4ax$, a perpendicular is drawn to any tangent meeting it at P & the parabola at Q. Show that $OP \cdot OQ$ = constant.

PR0090

7. In the parabola $y^2 = 4ax$, the tangent at the point P, whose abscissa is equal to the latus rectum meets the axis in T & the normal at P cuts the parabola again in Q. Prove that PT : PQ = 4 : 5.

PR0091

8. Show that the normals at the points (4a, 4a) & at the upper end of the latus rectum of the parabola $y^2 = 4ax$ intersect on the same parabola.

PR0092

9. Three normals to $y^2 = 4x$ pass through the point (15, 12). Show that if one of the normals is given by $y = x - 3$ & find the equations of the others.

PR0093

10. The normal at a point P to the parabola $y^2 = 4ax$ meets its axis at G. Q is another point on the parabola such that QG is perpendicular to the axis of the parabola. Prove that $QG^2 - PG^2$ = constant.

PR0094

11. Prove that the locus of the middle point of portion of a normal to $y^2 = 4ax$ intercepted between the curve & the axis is another parabola. Find the vertex & the latus rectum of the second parabola.

PR0095

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12. If the normal at P(18, 12) to the parabola $y^2 = 8x$ cuts it again at Q, show that $9PQ = 80\sqrt{10}$

PR0096

13. Prove that, the normal to $y^2 = 12x$ at (3, 6) meets the parabola again in (27, –18) & circle on this normal chord as diameter is $x^2 + y^2 - 30x + 12y - 27 = 0$.

PR0097

14. From the point P(h, k) three normals are drawn to the parabola $x^2 = 8y$ and m_1, m_2 and m_3 are the slopes of three normals

(a) Find the algebraic sum of the slopes of these three normals.

PR0098

(b) If two of the three normals are at right angles then the locus of point P is a conic, find the latus rectum of conic.

PR0099

(c) If the two normals from P are such that they make complementary angles with the axis then the locus of point P is a conic, find a directrix of conic.

PR0100

15. Show that the normals at two suitable distinct real points on the parabola $y^2 = 4ax$ $(a > 0)$ intersect at a point on the parabola whose abscissa > 8a.

PR0101

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16. The normal to the parabola $y^2 = 4x$ at the point P,Q & R are concurrent at the point(15,12). Find
(a) the equation of the circle circumscribing the triangle PQR
(b) the co-ordinates of the centroid of the triangle PQR.

PR0102

17. From the point (–1, 2) tangent lines are drawn to the parabola $y^2 = 4x$. Find the equation of the chord of contact. Also find the area of the triangle formed by the chord of contact & the tangents.

PR0103

18. Find the equation of the circle which passes through the focus of the parabola $x^2 = 4y$ & touches it at the point (6, 9).

PR0104

19. Find the equations of the chords of the parabola $y^2 = 4ax$ which pass through the point (–6a, 0) and which subtends an angle of 45° at the vertex.

PR0105

EXERCISE (S-2)

1. PC is the normal at P to the parabola $y^2 = 4ax$, C being on the axis. CP is produced outwards to Q so that PQ = CP; show that the locus of Q is a parabola.

PR0106

2. A quadrilateral is inscribed in a parabola $y^2 = 4ax$ and three of its sides pass through fixed points on the axis. Show that the fourth side also passes through fixed point on the axis of the parabola.

PR0107

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3. Tangents are drawn to the parabola $y^2 = 12x$ at the points A,B and C such that the three tangents form a triangle PQR. If θ_1, θ_2 and θ_3 be the inclinations of these tangents with the axis of x such that their cotangents form an arithmetical progression (in the same order) with common difference 2. Find the area of the triangle PQR.

PR0108

4. Two straight lines one being a tangent to $y^2 = 4ax$ and the other to $x^2 = 4by$ are right angles. Find the locus of their point of intersection.

PR0109

5. Let P(a,b) and Q(c,d) are the two points on the parabola $y^2 = 8x$ such that the normals at them meet in (18,12). Find the product (abcd).

PR0110

6. A variable circle passes through the point A(2,1) and touches the x-axis. Locus of the other end of the diameter through A is a parabola.

(a) Find the length of the latus rectum of the parabola.

(b) Find the co-ordinates of the foot of the directrix of the parabola.

(c) The two tangents and two normals at the extremities of the latus rectum of the parabola constitutes a quadrilateral. Find area of quadrilateral.

PR0111

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7. Find the condition on 'a' & 'b' so that the two tangents drawn to the parabola $y^2 = 4ax$ from a point are normals to the parabola $x^2 = 4by$.

PR0112

8. Two tangents to the parabola $y^2 = 8x$ meet the tangent at its vertex in the points P & Q. If PQ = 4 units, prove that the locus of the point of the intersection of the two tangents is $y^2 = 8(x+2)$.

PR0113

9. Let a variable point A be lying on the directrix of parabola $y^2 = 4x$. Tangents AB & AC are drawn to the curve where B & C are points of contact of tangent. If the locus of centroid of ΔABC is a conic then find the length of its latus rectum.

PR0114

10. A variable chords of the parabola $y^2 = 8x$ touches the parabola $y^2 = 2x$. The locus of the point of intersection of the tangent at the end of the chord is a parabola. Find its latus rectum.

PR0115

11. Show that the circle through three points the normals at which to the parabola $y^2 = 4ax$ are concurrent at the point (h,k) is $2(x^2 + y^2) - 2(h + 2a)x - ky = 0$.

PR0116

12. Let $L_1 : x + y = 0$ and $L_2 : x - y = 0$ are tangent to a parabola whose focus is S(1, 2).

If the length of latus-rectum of the parabola can be expressed as $\frac{m}{\sqrt{n}}$ (where m and n are coprime) then find the value of $(m + n)$.

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PR0117

13. PQ, a variable chord of the parabola $y^2 = 4x$ subtends a right angle at the vertex. The tangents at P and Q meet at T and the normals at those points meet at N. If the locus of the mid point of TN is a parabola, then find its latus rectum.

PR0118

EXERCISE (JM)

1. If two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles then the locus of P is :-

[AIEEE-2010]

(1) $x = 1$ (2) $2x + 1 = 0$ (3) $x = -1$ (4) $2x - 1 = 0$

PR0119

2. Given : A circle, $2x^2 + 2y^2 = 5$ and a parabola, $y^2 = 4\sqrt{5}x$.

Statement–I : An equation of a common tangent to these curves is $y = x + \sqrt{5}$.

Statement–II : If the line, $y = mx + \frac{\sqrt{5}}{m}$ $(m \neq 0)$ is their common tangent, then m satisfies $m^4 - 3m^2 + 2 = 0$. **[JEE (Main)-2013]**

(1) Statement-I is true, Statement-II is true; statement-II is a **correct** explanation for Statement-I.

(2) Statement-I is true, Statement-II is true; statement-II is **not** a correct explanation for Statement-I.

(3) Statement-I is true, Statement-II is false.

(4) Statement-I is false, Statement-II is true.

PR0120

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3. Statement 1 : The slope of the tangent at any point P on a parabola, whose axis is the axis of x and vertex is at the origin, is inversely proportional to the ordinate of the point P.

Statement 2 : The system of parabolas $y^2 = 4ax$ satisfies a differential equation of degree 1 and order 1.

(1) Statement 1 is True Statement 2 is True, Statement 2 is a correct explanation for Statement 1.

(2) Statement 1 is True, Statement 2 is False.

(3) Statement 1 is True, Statement 2 is True statement 2 is not a correct explanation for statement 1.

(4) Statement 1 is False, Statement 2 is True **[JEE-Main (On line)-2013]**

PR0121

4. **Statement 1 :** The line $x - 2y = 2$ meets the parabola, $y^2 + 2x = 0$ only at the point $(-2, -2)$

Statement 2 : The line $y = mx - \frac{1}{2m}$ $(m \neq 0)$ is tangent to the parabola, $y^2 = -2x$ at the point $\left(-\frac{1}{2m^2}, -\frac{1}{m}\right)$.

(1) Statement 1 is false; Statement 2 is true.

(2) Statement l is true; Statement 2 is true; Statement 2 is not a correct explanation for Statement 1.

(3) Statement 1 is true; Statement 2 is false.

(4) Statement 1 is true; Statement 2 is true; Statement 2 is a correct explanation for Statement 1.

[JEE-Main (On line)-2013]

PR0122

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5. The point of intersection of the normals tothe parabola $y^2 = 4x$ at the ends of its latus rectum is :

[JEE-Main (On line)-2013]

(1) (0, 3) (2) (2, 0) (3) (3, 0) (4) (0, 2)

PR0123

6. The slope of the line touching both, the parabolas $y^2 = 4x$ and $x^2 = -32y$ is : **[JEE(Main)-2014]**

(1) $\frac{1}{2}$ (2) $\frac{3}{2}$ (3) $\frac{1}{8}$ (4) $\frac{2}{3}$

PR0124

7. Let O be the vertex and Q be any point on the parabola, $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio 1 : 3, then the locus of P is :- **[JEE(Main)-2015]**

(1) $y^2 = 2x$ (2) $x^2 = 2y$ (3) $x^2 = y$ (4) $y^2 = x$

PR0125

8. Let P be the point on the parabola, $y^2 = 8x$ which is at a minimum distance from the cente C of the circle, $x^2 + (y + 6)^2 = 1$. Then the equation of the circle, passing through C and having its centre at P is : **[JEE(Main)-2016]**

(1) $x^2 + y^2 - 4x + 9y + 18 = 0$ (2) $x^2 + y^2 - 4x + 8y + 12 = 0$

(3) $x^2 + y^2 - x + 4y - 12 = 0$ (4) $x^2 + y^2 - \frac{x}{4} + 2y - 24 = 0$

PR0126

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9. The radius of a circle, having minimum area, which touches the curve $y = 4 - x^2$ and the lines, $y = |x|$ is :- **[JEE-Main 2017]**

(1) $4(\sqrt{2}+1)$ (2) $2(\sqrt{2}+1)$ (3) $2(\sqrt{2}-1)$ (4) $4(\sqrt{2}-1)$

PR0127

10. Tangent and normal are drawn at P(16, 16) on the parabola $y^2 = 16x$, which intersect the axis of the parabola at A and B, respectively. If C is the centre of the circle through the points P, A and B and $\angle CPB = \theta$, then a value of $\tan\theta$ is - **[JEE-Main 2018]**

(1) 2 (2) 3 (3) $\frac{4}{3}$ (4) $\frac{1}{2}$

PR0128

11. If the tangent at (1, 7) to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ then the value of c is : **[JEE-Main 2018]**

(1) 185 (2) 85 (3) 95 (4) 195

PR0129

12. Let A(4,–4) and B(9,6) be points on the parabola, $y^2 = 4x$. Let C be chosen on the arc AOB of the parabola, where O is the origin, such that the area of ΔACB is maximum. Then, the area (in sq. units) of ΔACB, is: **[JEE (Main) 2019]**

(1) $31\frac{3}{4}$ (2) 32 (3) $30\frac{1}{2}$ (4) $31\frac{1}{4}$

PR0130

13. If the parabolas $y^2=4b(x-c)$ and $y^2=8ax$ have a common normal, then which one of the following is a valid choice for the ordered triad (a,b,c) **[JEE (Main)-2019]**

(1) (1, 1, 0) (2) $\left(\frac{1}{2},2,3\right)$ (3) $\left(\frac{1}{2},2,0\right)$ (4) (1, 1, 3)

PR0131

14. If the tangent to the parabola $y^2 = x$ at a point (α, β), $(\beta > 0)$ is also a tangent to the ellipse, $x^2 + 2y^2 = 1$, then α is equal to : **[JEE (Main)-2019]**

(1) $2\sqrt{2}+1$ (2) $\sqrt{2}-1$ (3) $\sqrt{2}+1$ (4) $2\sqrt{2}-1$

PR0132

15. The area (in sq. units) of the smaller of the two circles that touch the parabola, $y^2 = 4x$ at the point (1, 2) and the x-axis is :- **[JEE (Main)-2019]**

(1) $4\pi(2-\sqrt{2})$ (2) $8\pi(3-2\sqrt{2})$ (3) $4\pi(3+\sqrt{2})$ (4) $8\pi(2-\sqrt{2})$

PR0133

16. The tangents to the curve $y = (x-2)^2 - 1$ at its points of intersection with the line $x - y = 3$, intersect at the point : **[JEE (Main)-2019]**

(1) $\left(-\frac{5}{2},-1\right)$ (2) $\left(-\frac{5}{2},1\right)$ (3) $\left(\frac{5}{2},-1\right)$ (4) $\left(\frac{5}{2},1\right)$

PR0134

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17. The equation of a common tangent to the curves, $y^2 = 16x$ and $xy = -4$ is : **[JEE (Main)-2019]**

(1) $x + y + 4 = 0$ (2) $x - 2y + 16 = 0$ (3) $2x - y + 2 = 0$ (4) $x - y + 4 = 0$

PR0135

18. If $y = mx + 4$ is a tangent to both the parabolas, $y^2 = 4x$ and $x^2 = 2by$, then b is equal to : **[JEE (Main)-2020]**

(1) 128 (2) –64 (3) –128 (4) –32

PR0136

19. Let a line $y = mx$ $(m > 0)$ intersect the parabola, $y^2 = x$ at a point P, other than the origin. Let the tangent to it at P meet the x-axis at the point Q. If area $(\Delta OPQ) = 4$ sq. units, then m is equal to. **[JEE (Main)-2020]**

PR0137

20. The locus of a point which divides the line segment joining the point (0,–1) and a point on the parabola, $x^2 = 4y$, internally in the ratio 1 : 2, is- **[JEE (Main)-2020]**

(1) $9x^2 - 3y = 2$ (2) $9x^2 - 12y = 8$ (3) $x^2 - 3y = 2$ (4) $4x^2 - 3y = 2$

PR0138

21. If one end of a focal chord AB of the parabola $y^2 = 8x$ is at $A\left(\frac{1}{2}, -2\right)$, then the equation of the tangent to it at B is : **[JEE (Main)-2020]**

(1) $2x + y - 24 = 0$ (2) $x - 2y + 8 = 0$ (3) $2x - y - 24 = 0$ (4) $x + 2y + 8 = 0$

PR0139

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EXERCISE (JA)

1. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at points T and N, respectively. The locus of the centroid of the triangle PTN is a parabola whose

(A) vertex is $\left(\frac{2a}{3}, 0\right)$ (B) directrix is $x = 0$ **[JEE 2009, 4]**

(C) latus rectum is $\frac{2a}{3}$ (D) focus is (a, 0)

PR0140

2. Let A and B be two distinct point on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be -

(A) $\frac{-1}{r}$ (B) $\frac{1}{r}$ (C) $\frac{2}{r}$ (D) $\frac{-2}{r}$ **[JEE 2010,3]**

PR0141

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3. Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latus rectum and the point $P\left(\frac{1}{2}, 2\right)$ on the parabola, and Δ_2 be the area of the triangle formed by drawing tangents at P and at the end points of the latus rectum. Then $\frac{\Delta_1}{\Delta_2}$ is **[JEE 2011,4]**

PR0142

4. Let (x,y) be any point on the parabola $y^2 = 4x$. Let P be the point that divides the line segment from (0, 0) to (x,y) in the ratio 1 : 3. Then the locus of P is- **[JEE 2011,3]**

(A) $x^2 = y$ (B) $y^2 = 2x$ (C) $y^2 = x$ (D) $x^2 = 2y$

PR0143

5. Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point (9,6), then L is given by - **[JEE 2011,4]**

(A) $y - x + 3 = 0$ (B) $y + 3x - 33 = 0$ (C) $y + x - 15 = 0$ (D) $y - 2x + 12 = 0$

PR0144

6. Let S be the focus of the parabola $y^2 = 8x$ & let PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. The area of the triangle PQS is **[JEE 2012, 4M]**

PR0145

Paragraph for Question 7 and 8

Let PQ be a focal chord of the parabolas $y^2 = 4ax$. The tangents to the parabola at P and Q meet at a point lying on the line $y = 2x + a$, $a > 0$.

7. If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, then $\tan\theta =$

[JEE(Advanced) 2013, 3, (–1)]

(A) $\frac{2}{3}\sqrt{7}$ (B) $\frac{-2}{3}\sqrt{7}$ (C) $\frac{2}{3}\sqrt{5}$ (D) $\frac{-2}{3}\sqrt{5}$

PR0146

8. Length of chord PQ is **[JEE(Advanced) 2013, 3, (–1)]**

(A) 7a (B) 5a (C) 2a (D) 3a

PR0146

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9. A line L : y = mx + 3 meets y – axis at E(0,3) and the arc of the parabola $y^2 = 16x$, $0 \le y \le 6$ at the point $F(x_0,y_0)$. The tangent to the parabola at $F(x_0,y_0)$ intersects the y-axis at $G(0,y_1)$. The slope m of the line L is chosen such that the area of the triangle EFG has a local maximum. Match List-I with List-II and select the correct answer using the code given below the lists.

	List-I		**List-II**
P.	m =	1.	$\frac{1}{2}$
Q.	Maximum area of ΔEFG is	2.	4
R.	y_0 =	3.	2
S.	y_1 =	4.	1

Codes :

	P	Q	R	S
(A)	4	1	2	3
(B)	3	4	1	2
(C)	1	3	2	4
(D)	1	3	4	2

[JEE(Advanced) 2013, 3, (–1)]

PR0147

10. The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the point P, Q and the parabola at the points R,S. Then the area of the quadrilateral PQRS is -

(A) 3 (B) 6 (C) 9 (D) 15

[JEE(Advanced)-2014, 3(–1)]

PR0148

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Paragraph For Questions 11 and 12

Let a,r,s,t be nonzero real numbers. Let $P(at^2, 2at)$, Q, $R(ar^2, 2ar)$ and $S(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR and PK are parallel, where K is the point (2a,0).

11. The value of r is- **[JEE(Advanced)-2014, 3(–1)]**

(A) $-\frac{1}{t}$ (B) $\frac{t^2+1}{t}$ (C) $\frac{1}{t}$ (D) $\frac{t^2-1}{t}$

PR0149

12. If st = 1, then the tangent at P and the normal at S to the parabola meet at a point whose ordinate is-

[JEE(Advanced)-2014, 3(–1)]

(A) $\frac{(t^2+1)^2}{2t^3}$ (B) $\frac{a(t^2+1)^2}{2t^3}$ (C) $\frac{a(t^2+1)^2}{t^3}$ (D) $\frac{a(t^2+2)^2}{t^3}$

PR0149

13. If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are tangents to the circle $(x - 3)^2 + (y + 2)^2 = r^2$, then the value of r^2 is **[JEE 2015, 4M, –0M]**

PR0150

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14. Let the curve C be the mirror image of the parabola $y^2 = 4x$ with respect to the line $x + y + 4 = 0$. If A and B are the points of intersection of C with the line $y = -5$, then the distance between A and B is **[JEE 2015, 4M, –0M]**

PR0151

15. Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of the triangle ΔOPQ is $3\sqrt{2}$, then which of the following is(are) the coordinates of P ? **[JEE 2015, 4M, –2M]**

(A) $\left(4, 2\sqrt{2}\right)$ (B) $\left(9, 3\sqrt{2}\right)$ (C) $\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$ (D) $\left(1, \sqrt{2}\right)$

PR0152

16. The circle C_1: $x^2 + y^2 = 3$, with centre at O, intersects the parabola $x^2 = 2y$ at the point P in the first quadrant. Let the tangent to the circle C_1 at P touches other two circles C_2 and C_3 at R_2 and R_3, respectively. Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and centres Q_2 and Q_3, respectively. If Q_2 and Q_3 lie on the y-axis, then - **[JEE(Advanced)-2016, 4(–2)]**

(A) $Q_2Q_3 = 12$ (B) $R_2R_3 = 4\sqrt{6}$

(C) area of the triangle OR_2R_3 is $6\sqrt{2}$ (D) area of the triangle PQ_2Q_3 is $4\sqrt{2}$

PR0153

17. Let P be the point on the parabola $y^2 = 4x$ which is at the shortest distance from the center S of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$. Let Q be the point on the circle dividing the line segment SP internally. Then-

(A) $SP = 2\sqrt{5}$

(B) $SQ : QP = \left(\sqrt{5}+1\right) : 2$

(C) the x-intercept of the normal to the parabola at P is 6

(D) the slope of the tangent to the circle at Q is $\frac{1}{2}$ **[JEE(Advanced)-2016, 4(–2)]**

PR0154

18. If a chord, which is not a tangent, of the parabola $y^2 = 16x$ has the equation $2x + y = p$, and midpoint (h, k), then which of the following is(are) possible value(s) of p, h and k ? **[JEE(Advanced)-2017, 4(–2)]**

(A) $p = 5, h = 4, k = -3$ (B) $p = -1, h = 1, k = -3$

(C) $p = -2, h = 2, k = -4$ (D) $p = 2, h = 3, k = -4$

PR0155

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ANSWER KEY

PARABOLA

EXERCISE (O-1)

1. D	**2.** D	**3.** C	**4.** A	**5.** C	**6.** D	**7.** A	**8.** D
9. B	**10.** B	**11.** D	**12.** B	**13.** B	**14.** C	**15.** D	**16.** A
17. A	**18.** D	**19.** C	**20.** A	**21.** C	**22.** B	**23.** B	**24.** C
25. A	**26.** C	**27.** C	**28.** B	**29.** B	**30.** B	**31.** A	**32.** D
33. D	**34.** C	**35.** B	**36.** A	**37.** D	**38.** B	**39.** C	
40. B,C,D	**41.** A,B,C,D		**42.** A,B	**43.** A,D	**44.** B,C,D	**45.** A,B,C	**46.** C,D
47. B	**48.** C	**49.** A	**50.** (A) S, (B) Q, (C) S, (D) P, (E) S				

EXERCISE (O-2)

1. A	**2.** B	**3.** D	**4.** A	**5.** C	**6.** C	**7.** A	**8.** B
9. B	**10.** B	**11.** B	**12.** C	**13.** A	**14.** C	**15.** B	**16.** C
17. A	**18.** D	**19.** A,D	**20.** A,B,C,D		**21.** A,B,C	**22.** A,B,C,D	
23. A,B,D	**24.** A,C	**25.** A,B,D	**26.** A,B,C,D		**27.** A	**28.** B	**29.** C
30. (A) Q, (B) R, (C) S, (D) P							

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EXERCISE (S-1)

2. $4\sqrt{3}$ **3.** $(4, 0)$; $y^2 = 2a(x - 4a)$ **4.** $2x - y + 2 = 0, (1, 4)$; $x + 2y + 16 = 0, (16, -16)$

5. $3x - 2y + 4 = 0$; $x - y + 3 = 0$ **9.** $y = -4x + 72, y = 3x - 33$ **11.** (a, 0); a

14. (a) $\frac{k-4}{h}$; (b) 2; (c) $2y - 3 = 0$ **16.** (a) $x^2 + y^2 - 17x - 6y = 0$; (b) (26/3, 0)

17. $x - y = 1$; $8\sqrt{2}$ sq. units **18.** $x^2 + y^2 + 18x - 28y + 27 = 0$ **19.** $7y \pm 2(x + 6a) = 0$

EXERCISE (S-2)

3. 72 **4.** $(ax + by)(x^2 + y^2) + (bx - ay)^2 = 0$ **5.** 512 **6.** (a) 4, (b) (2,–1), (c) 8 sq. units

7. $a^2 > 8b^2$ **9.** 3 **10.** 32 **12.** 11 **13.** 25/2

EXERCISE (JM)

1. 3	**2.** 2	**3.** 3	**4.** 4	**5.** 3	**6.** 1	**7.** 2
8. 2	**9.** **Bonus**	**10.** 1	**11.** 3	**12.** 4	**13.** 1,2,3,4	**14.** 3
15. 2	**16.** 3	**17.** 4	**18.** 3	**19.** 0.50	**20.** 2	**21.** 2

EXERCISE (JA)

1. A,D	**2.** C,D	**3.** 2	**4.** C	**5.** A,B,D	**6.** 4	**7.** D	**8.** B	**9.** A
10. D	**11.** D	**12.** B	**13.** 2	**14.** 4	**15.** A,D	**16.** A,B,C	**17.** A,C,D	**18.** D

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ELLIPSE

1. STANDARD EQUATION & DEFINITION :

Standard equation of an ellipse referred to its principal axes along the co-ordinate axes is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

where $a > b$ & $b^2 = a^2(1-e^2) \Rightarrow a^2 - b^2 = a^2e^2$.

where e = eccentricity $(0 < e < 1)$.

FOCI : $S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

(a) Equation of directrices :

$$x = \frac{a}{e} \quad \& \quad x = -\frac{a}{e}.$$

(b) Vertices :

$A' \equiv (-a, 0)$ & $A \equiv (a, 0)$.

(c) Major axis : The line segment A' A in which the foci S' & S lie is of length 2a & is called the **major axis** (a > b) of the ellipse. Point of intersection of major axis with directrix is called **the foot of the directrix (z)** $\left(\pm\frac{a}{e}, 0\right)$.

(d) Minor Axis : The y-axis intersects the ellipse in the points $B' \equiv (0,-b)$ & $B \equiv (0, b)$. The line segment B'B of length 2b (b < a) is called the **Minor Axis** of the ellipse.

(e) Principal Axes : The major & minor axis together are called **Principal Axes** of the ellipse.

(f) Centre : The point which bisects every chord of the conic drawn through it is called the **centre** of the conic. $C \equiv (0,0)$ the origin is the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(g) Diameter : A chord of the conic which passes through the centre is called a **diameter** of the conic.

(h) Focal Chord : A chord which passes through a focus is called a **focal chord.**

(i) Double Ordinate : A chord perpendicular to the major axis is called a **double ordinate.**

(j) Latus Rectum : The focal chord perpendicular to the major axis is called the **latus rectum**.

(i) Length of latus rectum $(LL') = \frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}} = 2a(1-e^2)$

(ii) Equation of latus rectum : $x = \pm ae$.

(iii) Ends of the latus rectum are $L\left(ae, \frac{b^2}{a}\right), L'\left(ae, -\frac{b^2}{a}\right), L_1\left(-ae, \frac{b^2}{a}\right)$

and $L_1'\left(-ae, -\frac{b^2}{a}\right)$.

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(k) Focal radii : SP = a –ex & S'P = a + ex $\Rightarrow$ SP + S 'P = 2a = Major axis.

(l) Eccentricity : $e = \sqrt{1 - \frac{b^2}{a^2}}$

Note :

(i) The sum of the focal distances of any point on the ellipse is equal to the major Axis. Hence distance of focus from the extremity of a minor axis is equal to semi major axis. **i.e BS = CA.**

(ii) If the equation of the ellipse is given as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ & nothing is mentioned, then the rule is to assume that a > b.

Illustration 1 : If LR of an ellipse is half of its minor axis, then its eccentricity is -

(A) $\frac{3}{2}$ (B) $\frac{2}{3}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{\sqrt{2}}{3}$

Solution : As given $\frac{2b^2}{a} = b$ $\Rightarrow$ $2b = a$ $\Rightarrow$ $4b^2 = a^2$

$\Rightarrow$ $4a^2(1 - e^2) = a^2$ $\Rightarrow$ $1 - e^2 = 1/4$

$\therefore$ $e = \sqrt{3}/2$ **Ans. (C)**

Illustration 2 : Find the equation of the ellipse whose foci are (2, 3), (–2, 3) and whose semi minor axis is of length $\sqrt{5}$.

Solution : Here S is (2, 3) & S' is (–2, 3) and $b = \sqrt{5}$ $\Rightarrow$ SS' = 4 = 2ae $\Rightarrow$ ae = 2

but $b^2 = a^2(1 - e^2)$ $\Rightarrow$ $5 = a^2 - 4$ $\Rightarrow$ a = 3.

Hence the equation to major axis is y = 3

Centre of ellipse is midpoint of SS' i.e. (0, 3)

$\therefore$ Equation to ellipse is $\frac{x^2}{a^2} + \frac{(y-3)^2}{b^2} = 1$ or $\frac{x^2}{9} + \frac{(y-3)^2}{5} = 1$ **Ans.**

Illustration 3 : Find the equation of the ellipse having centre at (1, 2), one focus at (6, 2) and passing through the point (4, 6).

Solution : With centre at (1, 2), the equation of the ellipse is $\frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1$. It passes through the point (4, 6)

$\Rightarrow$ $\frac{9}{a^2} + \frac{16}{b^2} = 1$ ..... (i)

Distance between the focus and the centre = (6 – 1) = 5 = ae

$\Rightarrow$ $b^2 = a^2 - a^2e^2 = a^2 - 25$ ..... (ii)

Solving for a^2 and b^2 from the equations (i) and (ii), we get $a^2 = 45$ and $b^2 = 20$.

Hence the equation of the ellipse is $\frac{(x-1)^2}{45} + \frac{(y-2)^2}{20} = 1$ **Ans.**

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Do yourself - 1 :

(i) If LR of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $(a<b)$ is half of its major axis, then find its eccentricity.

(ii) Find the equation of the ellipse whose foci are (4, 6) & (16, 6) and whose semi-minor axis is 4.

(iii) Find the eccentricity, foci and the length of the latus-rectum of the ellipse $x^2+4y^2+8y-2x+1=0$.

2. ANOTHER FORM OF ELLIPSE : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, **(a <b)**

(a) **AA' = Minor axis = 2a**

(b) **BB' = Major axis = 2b**

(c) $a^2 = b^2(1-e^2)$

(d) **Latus rectum LL' = $L_1L_1' = \frac{2a^2}{b}$, equation y = ± be**

(e) **Ends of the latus rectum are :**

$$L\left(\frac{a^2}{b}, be\right), L'\left(-\frac{a^2}{b}, be\right), L_1\left(\frac{a^2}{b}, -be\right), L_1'\left(-\frac{a^2}{b}, -be\right)$$

(f) **Equation of directrix y = ± b/e**

(g) **Eccentricity :** $e=\sqrt{1-\frac{a^2}{b^2}}$

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Illustration 4 : The equation of the ellipse with respect to coordinate axes whose minor axis is equal to the distance between its foci and whose LR = 10, will be-

(A) $2x^2 + y^2 = 100$ (B) $x^2 + 2y^2 = 100$ (C) $2x^2 + 3y^2 = 80$ (D) none of these

Solution : When $a > b$

As given $2b = 2ae \Rightarrow b = ae$ ..... (i)

Also $\frac{2b^2}{a}=10 \Rightarrow b^2 = 5a$ ..... (ii)

Now since $b^2 = a^2 - a^2e^2 \Rightarrow b^2 = a^2 - b^2$ [From (i)]

$\Rightarrow 2b^2 = a^2$ ..... (iii)

(ii), (iii) $\Rightarrow a^2 = 100, b^2 = 50$

Hence equation of the ellipse will be $\frac{x^2}{100}+\frac{y^2}{50}=1 \Rightarrow x^2 + 2y^2 = 100$

Similarly when $a < b$ then required ellipse is $2x^2 + y^2 = 100$ **Ans. (A, B)**

Do yourself - 2 :

(i) The foci of an ellipse are (0, ±2) and its eccentricity is $\frac{1}{\sqrt{2}}$. Find its equation

(ii) Find the centre, the length of the axes, eccentricity and the foci of ellipse

$12x^2 + 4y^2 + 24x - 16y + 25 = 0$

(iii) The equation $\frac{x^2}{8-t}+\frac{y^2}{t-4}=1$, will represent an ellipse if

(A) $t \in (1, 5)$ (B) $t \in (2, 8)$ (C) $t \in (4, 8) - \{6\}$ (D) $t \in (4, 10) - \{6\}$

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3. GENERAL EQUATION OF AN ELLIPSE

Let (a, b) be the focus S, and $lx + my + n = 0$ is the equation of directrix.

Let P(x, y) be any point on the ellipse. Then by definition.

$\Rightarrow$ **SP = e PM** (e is the eccentricity) $\Rightarrow (x-a)^2 + (y-b)^2 = e^2 \dfrac{(lx+my+n)^2}{(l^2+m^2)}$

$\Rightarrow$ $(l^2 + m^2)\{(x-a)^2 + (y-b)^2\} = e^2\{lx + my + n\}^2$

4. POSITION OF A POINT W.R.T. AN ELLIPSE :

The point $\mathbf{P(x_1, y_1)}$ lies **outside**, **inside** or **on** the ellipse according as ; $\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} - 1$ **> < or = 0**.

5. AUXILLIARY CIRCLE/ECCENTRIC ANGLE :

A circle described on major axis as diameter is called the **auxiliary circle.** Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that QP produced is perpendicular to the x-axis then P & Q are called as the CORRESPONDING POINTS on the ellipse & the auxiliary circle respectively. 'θ' is called the **ECCENTRIC ANGLE** of the point P on the ellipse $(0 \le \theta < 2\pi)$.

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Note that $\dfrac{l(PN)}{l(QN)} = \dfrac{b}{a} = \dfrac{\text{Semi minor axis}}{\text{Semi major axis}}$

Hence "If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle".

6. PARAMETRIC REPRESENTATION :

The equations $x = a\cos\theta$ & $y = b\sin\theta$ together represent the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

where θ is a parameter (eccentric angle).

Note that if $P(\theta) \equiv (a\cos\theta, b\sin\theta)$ is on the ellipse then ; $\mathbf{Q(\theta) \equiv (a\cos\theta, a\sin\theta)}$ is on the auxiliary circle.

7. LINE AND AN ELLIPSE :

The line $y = mx + c$ meets the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ in two points real, coincident or imaginary according as **c^2 is < = or > $a^2m^2 + b^2$.**

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Hence y = mx + c is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $\mathbf{c^2 = a^2m^2 + b^2}$.

The equation to the chord of the ellipse joining two points with eccentric angles α & β is given by

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}.$$

Illustration 5 : For what value of λ does the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$.

Solution : $\because$ Equation of ellipse is $9x^2 + 16y^2 = 144$ or $\frac{x^2}{16}+\frac{y^2}{9}=1$

Comparing this with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then we get $a^2 = 16$ and $b^2 = 9$

and comparing the line $y = x + \lambda$ with $y = mx + c$ $\therefore$ $m = 1$ and $c = \lambda$

If the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$, then $c^2 = a^2m^2 + b^2$

$\Rightarrow$ $\lambda^2 = 16 \times 1^2 + 9$ $\Rightarrow$ $\lambda^2 = 25$ $\therefore \lambda = \pm 5$ **Ans.**

Illustration 6 : If α, β are eccentric angles of end points of a focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $\tan \alpha/2 . \tan \beta/2$ is equal to -

(A) $\frac{e-1}{e+1}$ (B) $\frac{1-e}{1+e}$ (C) $\frac{e+1}{e-1}$ (D) $\frac{e-1}{e+1}$

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Solution : Equation of line joining points 'α' and 'β' is $\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$

If it is a focal chord, then it passes through focus (ae, 0), so $e\cos\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$

$$\Rightarrow \frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}=\frac{e}{1} \quad \Rightarrow \quad \frac{\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha+\beta}{2}}=\frac{e-1}{e+1}$$

$$\Rightarrow \frac{2\sin\alpha/2\ \sin\beta/2}{2\cos\alpha/2\ \cos\beta/2}=\frac{e-1}{e+1} \quad \Rightarrow \quad \tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{e-1}{e+1}$$

using (–ae, 0) , we get $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{e+1}{e-1}$ **Ans. (A,C)**

Do yourself - 3 :

(i) Find the position of the point (4, 3) relative to the ellipse $2x^2 + 9y^2 = 113$.

(ii) A tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $(a>b)$ having slope –1 intersects the axis of x & y in point A & B respectively. If O is the origin then find the area of triangle OAB.

(iii) Find the condition for the line $x\cos\theta + y\sin\theta = P$ to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

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8. TANGENT TO THE ELLIPSE $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:

(a) Point form : Equation of tangent to the given ellipse at its point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

Note : For general ellipse replace x^2 by (xx_1), y^2 by (yy_1), $2x$ by $(x + x_1)$, $2y$ by $(y + y_1)$, $2xy$ by $(xy_1 + yx_1)$ & c by (c).

(b) Slope form : Equation of tangent to the given ellipse whose slope is 'm', is $\mathbf{y = mx \pm \sqrt{a^2m^2+b^2}}$

Point of contact are $\left(\frac{\mp a^2m}{\sqrt{a^2m^2+b^2}}, \frac{\pm b^2}{\sqrt{a^2m^2+b^2}}\right)$

Note that there are two tangents to the ellipse having the same m, i.e. there are two tangents parallel to any given direction.

(c) Parametric form : Equation of tangent to the given ellipse at its point $(a\cos\theta, b\sin\theta)$, is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

Note :

(i) The eccentric angles of point of contact of two parallel tangents differ by π.

(ii) Point of intersection of the tangents at the point α & β is $\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

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Illustration 7 : Find the equations of the tangents to the ellipse $3x^2 + 4y^2 = 12$ which are perpendicular to the line $y + 2x = 4$.

Solution : Let m be the slope of the tangent, since the tangent is perpendicular to the line $y + 2x = 4$.

$\therefore \quad mx - 2 = -1 \Rightarrow \quad m = \frac{1}{2}$

Since $3x^2 + 4y^2 = 12$ or $\frac{x^2}{4}+\frac{y^2}{3}=1$

Comparing this with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\therefore \quad a^2 = 4$ and $b^2 = 3$

So the equation of the tangent are $y=\frac{1}{2}x\pm\sqrt{4\times\frac{1}{4}+3}$

$\Rightarrow \quad y = \frac{1}{2}x \pm 2$ or $\quad x - 2y \pm 4 = 0.$ **Ans.**

Illustration 8 : The tangent at a point P on an ellipse intersects the major axis in T and N is the foot of the perpendicular from P to the same axis. Show that the circle drawn on NT as diameter intersects the auxiliary circle orthogonally.

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Solution : Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Let P(acosθ, bsinθ) be a point on the ellipse.

The equation of the tangent at P is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. It meets the major axis at T ≡ (a secθ, 0).

The coordinates of N are (a cosθ, 0). The equation of the circle with NT as its diameter is $(x - a\sec\theta)(x - a\cos\theta) + y^2 = 0$.

$\Rightarrow \quad x^2 + y^2 - ax(\sec\theta + \cos\theta) + a^2 = 0$

It cuts the auxiliary circle $x^2 + y^2 - a^2 = 0$ orthogonally if

$2g \cdot 0 + 2f \cdot 0 = a^2 - a^2 = 0$, which is true. **Ans.**

Do yourself - 4 :

(i) Find the equation of the tangents to the ellipse $9x^2 + 16y^2 = 144$ which are parallel to the line $x + 3y + k = 0$.

(ii) Find the equation of the tangent to the ellipse $7x^2 + 8y^2 = 100$ at the point (2, –3).

9. NORMAL TO THE ELLIPSE $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:

(a) **Point form :** Equation of the normal to the given ellipse at (x_1, y_1) is

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2=a^2e^2.$$

(b) **Slope form :** Equation of a normal to the given ellipse whose slope is 'm' is

$$y = mx \mp \frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}.$$

(c) **Parametric form :** Equation of the normal to the given ellipse at the point **(acosθ, bsinθ)** is **axsecθ – bycosecθ = (a² – b²)**.

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Illustration 9 : Find the condition that the line $\ell x + my = n$ may be a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution : Equation of normal to the given ellipse at (a cos θ, b sin θ) is $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$ ...(i)

If the line $\ell x + my = n$ is also normal to the ellipse then there must be a value of θ for which line (i) and line $\ell x + my = n$ are identical. For that value of θ we have

$$\frac{\ell}{\left(\frac{a}{\cos\theta}\right)}=\frac{m}{-\left(\frac{b}{\sin\theta}\right)}=\frac{n}{(a^2-b^2)} \quad \text{or} \quad \cos\theta=\frac{an}{\ell(a^2-b^2)} \qquad \text{..... (iii)}$$

$$\text{and} \quad \sin\theta=\frac{-bn}{m(a^2-b^2)} \qquad \text{..... (iv)}$$

Squaring and adding (iii) and (iv), we get $1=\frac{n^2}{(a^2-b^2)^2}\left(\frac{a^2}{\ell^2}+\frac{b^2}{m^2}\right)$ which is the required condition.

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Illustration 10 : If the normal at an end of a latus-rectum of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through one extremity of the minor axis, show that the eccentricity of the ellipse is given by $e=\sqrt{\frac{\sqrt{5}-1}{2}}$

Solution : The co-ordinates of an end of the latus-rectum are $(ae, b^2/a)$.

The equation of normal at $P(ae, b^2/a)$ is

$$\frac{a^2x}{ae}-\frac{b^2(y)}{b^2/a}=a^2-b^2 \quad \text{or} \quad \frac{ax}{e}-ay=a^2-b^2$$

It passes through one extremity of the minor axis whose co-ordinates are $(0, -b)$

$\therefore \quad 0+ab=a^2-b^2 \quad \Rightarrow \quad (a^2b^2)=(a^2-b^2)^2$

$\Rightarrow \quad a^2.a^2(1-e^2)=(a^2e^2)^2 \quad \Rightarrow \quad 1-e^2=e^4$

$\Rightarrow \quad e^4+e^2-1=0 \quad \Rightarrow \quad (e^2)^2+e^2-1=0$

$\therefore \quad e^2=\frac{-1\pm\sqrt{1+4}}{2} \quad \Rightarrow \quad e=\sqrt{\frac{\sqrt{5}-1}{2}}$ (taking positive sign) **Ans.**

Illustration 11 : P and Q are corresponding points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the auxiliary circles respectively. The normal at P to the ellipse meets CQ in R, where C is the centre of the ellipse. Prove that CR = a + b

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Solution : Let $P \equiv (a\cos\theta, b\sin\theta)$

$\therefore \quad Q \equiv (a\cos\theta, a\sin\theta)$

Equation of normal at P is

$(a\sec\theta)x-(b\operatorname{cosec}\theta)y=a^2-b^2$ ..... (i)

equation of CQ is $y=\tan\theta\,.\,x$ ..... (ii)

Solving equation (i) & (ii), we get $(a-b)x=(a^2-b^2)\cos\theta$

$x=(a+b)\cos\theta$, & $y=(a+b)\sin\theta$

$\therefore \quad R \equiv ((a+b)\cos\theta, (a+b)\sin\theta$

$\therefore \quad CR = a + b$ **Ans.**

Do yourself - 5 :

(i) Find the equation of the normal to the ellipse $9x^2+16y^2=288$ at the point (4, 3)

(ii) Let P be a variable point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with foci F_1 and F_2. If A is the area of the triangle PF_1F_2, then find maximum value of A.

(iii) If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{3}+\frac{y^2}{2}=1$ intersects it again at the point $Q(2\theta)$, then find $\cos\theta$.

(iv) Show that for all real values of 't' the line $2tx+y\sqrt{1-t^2}=1$ touches a fixed ellipse. Find the eccentricity of the ellipse.

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10. CHORD OF CONTACT :

If PA and PB be the tangents from point $P(x_1,y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The equation of the chord of contact AB is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ **or T = 0 (at x_1,y_1).**

Illustration 12 : If tangents to the parabola $y^2 = 4ax$ intersect the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at A and B, the find the locus of point of intersection of tangents at A and B.

Solution : Let P ≡ (h, k) be the point of intersection of tangents at A & B

$\therefore$ equation of chord of contact AB is $\frac{xh}{a^2}+\frac{yk}{b^2}=1$ ..... (i)

which touches the parabola.

Equation of tangent to parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$

$\Rightarrow$ $mx - y = -\frac{a}{m}$ ..... (ii)

equation (i) & (ii) as must be same

$\therefore$ $\frac{m}{\left(\frac{h}{a^2}\right)}=\frac{-1}{\left(\frac{k}{b^2}\right)}=\frac{-\frac{a}{m}}{1} \Rightarrow m=-\frac{h}{k}\frac{b^2}{a^2}$ & $m=\frac{ak}{b^2}$

$\therefore$ $-\frac{hb^2}{ka^2}=\frac{ak}{b^2}$ $\Rightarrow$ locus of P is $y^2=-\frac{b^4}{a^3}.x$ **Ans.**

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Do yourself - 6 :

(i) Find the equation of chord of contact to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ at the point (1, 3).

(ii) If the chord of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are at right angles, then find $\frac{x_1x_2}{y_1y_2}$.

(iii) If a line $3x - y = 2$ intersects ellipse $\frac{x^2}{8}+\frac{y^2}{4}=1$ at points A & B, then find co-ordinates of point of intersection of tangents at points A & B.

11. PAIR OF TANGENTS :

If $P(x_1,y_1)$ be any point lies outside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,
and a pair of tangents PA, PB can be drawn to it from P.
Then the equation of pair of tangents of PA and PB is $SS_1 = T^2$

where $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$, $T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$

i.e. $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$

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12. DIRECTOR CIRCLE :

Locus of the point of intersection of the tangents which meet at right angles is called the **Director Circle.** The equation to this locus is $\mathbf{x^2 + y^2 = a^2 + b^2}$ i.e. a circle whose centre is the centre of the ellipse & whose radius is the length of the line joining the ends of the major & minor axis.

Illustration 13 : A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles.

Solution : Given ellipse are $\frac{x^2}{4}+\frac{y^2}{1}=1$ ..... (i)

and, $\frac{x^2}{6}+\frac{y^2}{3}=1$ ..... (ii)

any tangent to (i) is $\frac{x\cos\theta}{2}+\frac{y\sin\theta}{1}=1$ ..... (iii)

It cuts (ii) at P and Q, and suppose tangent at P and Q meet at (h, k) Then equation of chord of contact of (h, k) with respect to ellipse (ii) is $\frac{hx}{6}+\frac{ky}{3}=1$ ..... (iv)

comparing (iii) and (iv), we get $\frac{\cos\theta}{h/3}=\frac{\sin\theta}{k/3}=1$

$\Rightarrow \cos\theta=\frac{h}{3}$ and $\sin\theta=\frac{k}{3}$ $\Rightarrow h^2+k^2=9$

locus of the point (h, k) is $x^2 + y^2 = 9$ $\Rightarrow$ $x^2 + y^2 = 6 + 3 = a^2 + b^2$

i.e. director circle of second ellipse. Hence the tangents are at right angles.

13. EQUATION OF CHORD WITH MID POINT (x_1,y_1) :

The equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose mid-point be (x_1,y_1) is $\mathbf{T = S_1}$

where $T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$, $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$, i.e. $\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)=\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)$

Illustration 14 : Find the locus of the mid-point of focal chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution : Let P ≡ (h, k) be the mid-point

$\therefore$ equation of chord whose mid-point is given $\frac{xh}{a^2}+\frac{yk}{b^2}-1=\frac{h^2}{a^2}+\frac{k^2}{b^2}-1$

since it is a focal chord,

$\therefore$ It passes through focus, either (ae, 0) or (–ae, 0)

If it passes through (ae, 0)

$\therefore$ locus is $\frac{ex}{a}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

If it passes through (–ae, 0)

$\therefore$ locus is $-\frac{ex}{a}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ **Ans.**

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Do yourself - 7 :

(i) Find the equation of chord of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ whose mid point be (–1, 1).

14. IMPORTANT POINTS :

Referring to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(a) If P be any point on the ellipse with S & S' as its foci then $\ell(SP)+\ell(S'P)=2a$.

(b) The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice versa .

(c) **The product of the length's of the perpendicular segments from the foci on any tangent to the ellipse is b^2** and the feet of these perpendiculars lie on its auxiliary circle and the tangents at these feet to the auxiliary circle meet on the ordinate of P and that the locus of their point of intersection is a similar ellipse as that of the original one.

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(d) The portion of the tangent to an ellipse between the point of contact & the directrix subtends a **right angle** at the corresponding focus.

(e) If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively, & if CF be perpendicular upon this normal, then

(i) **PF . PG = b^2** **(ii)** **PF . Pg = a^2**

(iii) **PG . Pg = SP . S' P** **(iv)** **CG . CT = CS^2**

(v) locus of the mid point of Gg is another ellipse having the same eccentricity as that of the original ellipse.

[where S and S' are the focii of the ellipse and T is the point where tangent at P meet the major axis]

(f) Atmost four normals & two tangents can be drawn from any point to an ellipse.

(g) The circle on any focal distance as diameter touches the auxiliary circle.

(h) Perpendiculars from the centre upon all chords which join the ends of any perpendicular diameters of the ellipse are of constant length.

(i) If the tangent at the point P of a standard ellipse meets the axes in T and t and CY is the perpendicular on it from the centre then,

(i) **Tt . PY = $a^2 - b^2$** **and** **(ii)** **least value of Tt is a + b.**

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Do yourself - 8 :

(i) A man running round a racecourse note that the sum of the distance of two flag-posts from him is always 20 meters and distance between the flag-posts is 16 meters. Find the area of the path be encloses in square meters

(ii) If chord of contact of the tangent drawn from the point (α, β) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches the circle $x^2+y^2=k^2$, then find the locus of the point (α, β).

Miscellaneous Illustration :

Illustration 15 : A point moves so that the sum of the squares of its distances from two intersecting straight lines is constant. Prove that its locus is an ellipse.

Solution : Let two intersecting lines OA and OB, intersect at origin O and let both lines OA and OB makes equal angles with x axis.

i.e., $\angle XOA = \angle XOB = \theta$.

$\therefore$ Equations of straight lines OA and OB are

$y = x\tan\theta$ and $y = -x\tan\theta$

or $x\sin\theta - y\cos\theta = 0$ ..... (i)

and $x\sin\theta + y\cos\theta = 0$ ..... (ii)

Let $P(\alpha, \beta)$ is the point whose locus is to be determine.

According to the example $(PM)^2 + (PN)^2 = 2\lambda^2$ (say)

$\therefore$ $(\alpha\sin\theta + \beta\cos\theta)^2 + (\alpha\sin\theta - \beta\cos\theta)^2 = 2\lambda^2 \Rightarrow 2\alpha^2\sin^2\theta + 2\beta^2\cos^2\theta = 2\lambda^2$

or $\alpha^2\sin^2\theta + \beta^2\cos^2\theta = \lambda^2 \Rightarrow \frac{\alpha^2}{\lambda^2\text{cosec}^2\theta} + \frac{\beta^2}{\lambda^2\sec^2\theta} = 1 \Rightarrow \frac{\alpha^2}{(\lambda\ \text{cosec}\ \theta)^2} + \frac{\beta^2}{(\lambda\sec\theta)^2} = 1$

Hence required locus is $\frac{x^2}{(\lambda\ \text{cosec}\ \theta)^2} + \frac{y^2}{(\lambda\sec\theta)^2} = 1$ **Ans.**

Illustration 16 : Find the condition on 'a' and 'b' for which two distinct chords of the ellipse $\frac{x^2}{2a^2}+\frac{y^2}{2b^2}=1$ passing through $(a, -b)$ are bisected by the line $x + y = b$.

Solution : Let $(t, b - t)$ be a point on the line $x + y = b$.

Then equation of chord whose mid point $(t, b - t)$ is

$$\frac{tx}{2a^2}+\frac{y(b-t)}{2b^2}-1=\frac{t^2}{2a^2}+\frac{(b-t)^2}{2b^2}-1 \quad \text{..... (i)}$$

$(a, -b)$ lies on (i) then $\frac{ta}{2a^2}-\frac{b(b-t)}{2b^2}=\frac{t^2}{2a^2}+\frac{(b-t)^2}{2b^2} \Rightarrow t^2(a^2+b^2)-ab(3a+b)t+2a^2b^2=0$

Since t is real $B^2 - 4AC \ge 0 \Rightarrow a^2b^2(3a + b)^2 - 4(a^2 + b^2)2a^2b^2 \ge 0$

$\Rightarrow a^2 + 6ab - 7b^2 \ge 0 \Rightarrow a^2 + 6ab \ge 7b^2$, which is the required condition.

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Illustration 17 : Any tangent to an ellipse is cut by the tangents at the ends of the major axis in T and T '. Prove that circle on TT' as diameter passes through foci.

Solution : Let ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

and let P(acosϕ, bsinϕ) be any point on this ellipse

∴ Equation of tangent at P(acosϕ, bsinϕ) is

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1 \qquad .....(i)$$

The two tangents drawn at the ends of the major axis are $x = a$ and $x = -a$

Solving (i) and x = a we get $T=\left\{a, \frac{b(1-\cos\phi)}{\sin\phi}\right\}\equiv\left\{a, b\tan\left(\frac{\phi}{2}\right)\right\}$

and solving (i) and x = – a we get $T'=\left\{-a, \frac{b(1+\cos\phi)}{\sin\phi}\right\}\equiv\left\{-a, b\cot\left(\frac{\phi}{2}\right)\right\}$

Equation of circle on TT' as diameter is $(x-a)(x+a)+(y-b\tan(\phi/2))(y-b\cot(\phi/2))=0$

or $x^2+y^2-by(\tan(\phi/2)+\cot(\phi/2))-a^2+b^2=0$ ..... (ii)

Now put $x=\pm ae$ and $y=0$ in LHS of (ii), we get

$a^2e^2+0-0-a^2+b^2=a^2-b^2-a^2+b^2=0=$ RHS

Hence foci lie on this circle

ANSWERS FOR DO YOURSELF

1 : **(i)** $e=\frac{1}{\sqrt{2}}$ **(ii)** $\frac{(x-10)^2}{52}+\frac{(y-6)^2}{16}=1$ **(iii)** $e=\frac{\sqrt{3}}{2}$; foci $=(1\pm\sqrt{3}, -1)$; LR = 1

2 : **(i)** $\frac{x^2}{4}+\frac{y^2}{8}=1$

(ii) $C\equiv(-1, 2)$, length of major axis $=2b=\sqrt{3}$, length of minor axis $=2a=1$; $e=\sqrt{\frac{2}{3}}$;

$f\left(-1,\ 2\pm\frac{1}{\sqrt{2}}\right)$

(iii) C

3 : **(i)** On the ellipse **(ii)** $\frac{1}{2}(a^2+b^2)$ **(iii)** $P^2=a^2\cos^2\theta+b^2\sin^2\theta$

4 : **(i)** $3y+x\pm\sqrt{97}=0$ **(ii)** $7x-12y=50$

5 : **(i)** $4x-3y=7$ **(ii)** abe **(iii)** –1 **(iv)** $\frac{\sqrt{3}}{2}$

6 : **(i)** $\frac{x}{16}+\frac{y}{3}=1$ **(ii)** $-\frac{a^4}{b^4}$ **(iii)** (12, –2)

7 : **(i)** $-9x+16y=25$ **8 :** **(i)** 60π **(ii)** $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{k^2}$

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EXERCISE (O-1)

[STRAIGHT OBJECTIVE TYPE]

1. Let 'E' be the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ & 'C' be the circle $x^2+y^2=9$. Let P & Q be the points (1 , 2) and (2, 1) respectively. Then :

(A) Q lies inside C but outside E
(B) Q lies outside both C & E
(C) P lies inside both C & E
(D) P lies inside C but outside E.

EL0001

2. The eccentricity of the ellipse $(x-3)^2+(y-4)^2=\frac{y^2}{9}$ is

(A) $\frac{\sqrt{3}}{2}$
(B) $\frac{1}{3}$
(C) $\frac{1}{3\sqrt{2}}$
(D) $\frac{1}{\sqrt{3}}$

EL0002

3. The equation, $2x^2+3y^2-8x-18y+35=K$ represents

(A) no locus if $K>0$
(B) an ellipse if $K<0$
(C) a point if $K=0$
(D) a hyperbola if $K>0$

EL0003

4. If the ellipse $\frac{(x-h)^2}{M}+\frac{(y-k)^2}{N}=1$ has major axis on the line $y=2$, minor axis on the line $x=-1$, major axis has length 10 and minor axis has length 4. The number h,k,M,N (in this order only) are -

(A) –1,2,5,2
(B) –1,2,10,4
(C) 1,–2,25,4
(D) –1,2,25,4

EL0004

5. The y-axis is the directrix of the ellipse with eccentricity $e=1/2$ and the corresponding focus is at (3, 0), equation to its auxiliary circle is

(A) $x^2+y^2-8x+12=0$
(B) $x^2+y^2-8x-12=0$
(C) $x^2+y^2-8x+9=0$
(D) $x^2+y^2=4$

EL0005

6. Imagine that you have two thumbtacks placed at two points, A and B. If the ends of a fixed length of string are fastened to the thumtacks and the string is drawn taut with a pencil, the path traced by the pencil will be an ellipse. The best way to maximise the area surrounded by the ellipse with a fixed length of string occurs when

I the two points A and B have the maximum distance between them.
II two points A and B coincide.
III A and B are placed vertically.
IV The area is always same regardless of the location of A and B .

(A) I
(B) II
(C) III
(D) IV

EL0006

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7. The latus rectum of a conic section is the width of the function through the focus. The positive difference between the length of the latus rectum of $3y = x^2 + 4x - 9$ and $x^2 + 4y^2 - 6x + 16y = 24$ is-

(A) $\frac{1}{2}$ (B) 2 (C) $\frac{3}{2}$ (D) $\frac{5}{2}$

EL0007

8. Let S(5,12) and S'(–12,5) are the foci of an ellipse passing through the origin. The eccentricity of ellipse equals -

(A) $\frac{1}{2}$ (B) $\frac{1}{\sqrt{3}}$ (C) $\frac{1}{\sqrt{2}}$ (D) $\frac{2}{3}$

EL0008

9. A circle has the same centre as an ellipse & passes through the foci F_1 & F_2 of the ellipse, such that the two curves intersect in 4 points. Let 'P' be any one of their point of intersection. If the major axis of the ellipse is 17 & the area of the triangle PF_1F_2 is 30, then the distance between the foci is :

(A) 11 (B) 12 (C)13 (D) none

EL0009

10. An ellipse is inscribed in a circle and a point within the circle is chosen at random. If the probability that this point lies outside the ellipse is 2/3 then the eccentricity of the ellipse is :

(A) $\frac{2\sqrt{2}}{3}$ (B) $\frac{\sqrt{5}}{3}$ (C) $\frac{8}{9}$ (D) $\frac{2}{3}$

EL0010

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11. (a) Which of the following is an equation of the ellipse with centre (–2,1), major axis running from (–2,6) to (–2,–4) and focus at (–2,5) ?

(A) $\frac{(x-2)^2}{25}+\frac{(y+1)^2}{16}=1$ (B) $\frac{(x+2)^2}{25}+\frac{(y-1)^2}{9}=1$

(C) $\frac{(x-2)^2}{9}+\frac{(y+1)^2}{25}=1$ (D) $\frac{(x+2)^2}{9}+\frac{(y-1)^2}{25}=1$

(b) Which of the following statement(s) is/are correct for the ellipse of 11(a) ?

(A) auxiliary circle is $(x + 2)^2 + (y - 1)^2 = 25$

(B) director circle is $(x + 2)^2 + (y - 1)^2 = 34$

(C) Latus rectum $= \frac{18}{5}$

(D) eccentricity $= \frac{4}{5}$

EL0011

12. $x - 2y + 4 = 0$ is a common tangent to $y^2 = 4x$ & $\frac{x^2}{4}+\frac{y^2}{b^2} = 1$. Then the value of b and the other common tangent are given by :

(A) $b = \sqrt{3}$; $x + 2y + 4 = 0$ (B) $b = 3$; $x + 2y + 4 = 0$

(C) $b = \sqrt{3}$; $x + 2y - 4 = 0$ (D) $b = \sqrt{3}$; $x - 2y - 4 = 0$

EL0012

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13. Consider the particle travelling clockwise on the elliptical path $\frac{x^2}{100}+\frac{y^2}{25}=1$. The particle leaves the orbit at the point (–8, 3) and travels in a straight line tangent to the ellipse. At what point will the particle cross the y-axis?

(A) $\left(0, \frac{25}{3}\right)$ (B) $\left(0, \frac{23}{3}\right)$ (C) (0, 9) (D) $\left(0, \frac{26}{3}\right)$

EL0013

[MULTIPLE OBJECTIVE TYPE]

14. Consider the ellipse $\frac{x^2}{\tan^2\alpha}+\frac{y^2}{\sec^2\alpha}=1$ where $\alpha \in (0,\pi/2)$.

Which of the following quantities would vary as α varies ?

(A) degree of flatness (B) ordinate of the vertex

(C) coordinates of the foci (D) length of the latus rectum

EL0014

15. The equation of the common tangents of the parabola $y^2 = 4x$ and an ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ are -

(A) $x - 2y + 4 = 0$ (B) $x + 2y + 4 = 0$

(C) $2x - y + 1 = 0$ (D) $2x + y + 1 = 0$

EL0015

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16. Consider an ellipse $\frac{x^2}{a}+\frac{y^2}{b}=1$. If A_1 and A_2 denotes area of ellipse and its director circle respectively, then-

(A) $A_2 \geq 2A_1$ (B) $A_2 < 2A_1$

(C) if a = 4b, then $A_1 = \frac{2}{5}A_2$ (D) if a = 4b, then $A_2 = \frac{2}{5}A_1$

EL0016

17. If length of perpendicular drawn from origin to any normal of the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ is ℓ, then ℓ cannot be -

(A) 4 (B) $\frac{5}{2}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$

EL0017

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[REASONING TYPE]

18. **Statement-1 :** The ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ and $\frac{x^2}{9}+\frac{y^2}{16}=1$ are congruent.

and

Statement-2 : The ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ and $\frac{x^2}{9}+\frac{y^2}{16}=1$ have the same eccentricity.

(A) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True ; Statement-2 is NOT a correct explanation for Statement-1.
(C) Statement-1 is True, Statement-2 is False.
(D) Statement-1 is False, Statement-2 is True.

EL0018

19. Tangents are drawn from the point $P\left(-\sqrt{3},\sqrt{2}\right)$ to an ellipse $4x^2 + y^2 = 4$.

Statement-1 : The tangents are mutually perpendicular.

and

Statement-2 : The locus of the points from which mutually perpendicular tangents can be drawn to given ellipse is $x^2 + y^2 = 5$.

(A) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True ; Statement-2 is NOT a correct explanation for Statement-1.
(C) Statement-1 is True, Statement-2 is False.
(D) Statement-1 is False, Statement-2 is True.

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EL0019

[COMPREHENSION TYPE]

Paragraph for question nos. 20 to 22

Consider the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the parabola $y^2 = 2x$. They intersect at P and Q in the first and fourth quadrants respectively. Tangents to the ellipse at P and Q intersect the x-axis at R and tangents to the parabola at P and Q intersect the x-axis at S.

20. The ratio of the areas of the triangles PQS and PQR, is

(A) 1 : 3 (B) 1 : 2 (C) 2 : 3 (D) 3 : 4

EL0020

21. The area of quadrilateral PRQS, is

(A) $\frac{3\sqrt{15}}{2}$ (B) $\frac{15\sqrt{3}}{2}$ (C) $\frac{5\sqrt{3}}{2}$ (D) $\frac{5\sqrt{15}}{2}$

EL0020

22. The equation of circle touching the parabola at upper end of its latus rectum and passing through its vertex, is

(A) $2x^2 + 2y^2 - x - 2y = 0$
(B) $2x^2 + 2y^2 + 4x - \frac{9}{2}y = 0$
(C) $2x^2 + 2y^2 + x - 3y = 0$
(D) $2x^2 + 2y^2 - 7x + y = 0$

EL0020

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[MATRIX MATCH TYPE]

23. **Column-I** **Column-II**

(A) The eccentricity of the ellipse which meets the straight line $2x - 3y = 6$ on the X-axis and the straight line $4x + 5y = 20$ on the Y-axis and whose principal axes lie along the coordinate axes, is (P) $\frac{1}{2}$

EL0021

(B) A bar of length 20 units moves with its ends on two fixed straight lines at right angles. A point P marked on the bar at a distance of 8 units from one end describes a conic whose eccentricity is (Q) $\frac{1}{\sqrt{2}}$

EL0022

(C) If one extremity of the minor axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the foci form an equilateral triangle, then its eccentricity, is (R) $\frac{\sqrt{5}}{3}$

EL0023

(D) There are exactly two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose distance from the centre of the ellipse are greatest and equal to $\sqrt{\frac{a^2+2b^2}{2}}$. Eccentricity of this ellipse is equal to (S) $\frac{\sqrt{7}}{4}$

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EL0024

EXERCISE (O-2)

[STRAIGHT OBJECTIVE TYPE]

1. If α & β are the eccentric angles of the extremities of a focal chord of an standard ellipse, then the eccentricity of the ellipse is :

(A) $\frac{\cos\alpha + \cos\beta}{\cos(\alpha+\beta)}$ (B) $\frac{\sin\alpha - \sin\beta}{\sin(\alpha-\beta)}$

(C) $\frac{\cos\alpha - \cos\beta}{\cos(\alpha-\beta)}$ (D) $\frac{\sin\alpha + \sin\beta}{\sin(\alpha+\beta)}$

EL0025

2. If P is any point on ellipse with foci S_1 & S_2 and eccentricity is $\frac{1}{2}$ such that $\angle PS_1S_2 = \alpha$, $\angle PS_2S_1 = \beta$, $\angle S_1PS_2 = \gamma$, then $\cot\frac{\alpha}{2}, \cot\frac{\gamma}{2}, \cot\frac{\beta}{2}$ are in

(A) A.P. (B) G.P.

(C) H.P. (D) ***NOT*** A.P., G.P. & H.P.

EL0026

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3. Equation of the common tangent to the ellipses, $\frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{a^2}+\frac{y^2}{a^2+b^2}=1$ is -

(A) $ay = bx + \sqrt{a^4 - a^2b^2 + b^4}$
(B) $by = ax - \sqrt{a^4 + a^2b^2 + b^4}$
(C) $ay = bx - \sqrt{a^4 + a^2b^2 + b^4}$
(D) $by = ax + \sqrt{a^4 - a^2b^2 + b^4}$

EL0027

4. The normal at a variable point P on an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ of eccentricity e meets the axes of the ellipse in Q and R then the locus of the mid-point of QR is a conic with an eccentricity e' such that :

(A) e' is independent of e
(B) $e' = 1$
(C) $e' = e$
(D) $e' = 1/e$

EL0028

5. The area of the rectangle formed by the perpendiculars from the centre of the standard ellipse to the tangent and normal at its point whose eccentric angle is $\pi/4$, is :

(A) $\frac{(a^2-b^2)ab}{a^2+b^2}$
(B) $\frac{(a^2-b^2)}{(a^2+b^2)ab}$
(C) $\frac{(a^2-b^2)}{ab(a^2+b^2)}$
(D) $\frac{a^2+b^2}{(a^2-b^2)ab}$

EL0029

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6. The locus of the middle point of chords of an ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ passing through P(0, 5) is another ellipse E. The coordinates of the foci of the ellipse E, is

(A) $\left(0, \frac{3}{5}\right)$ and $\left(0, \frac{-3}{5}\right)$
(B) (0, – 4) and (0, 1)
(C) (0, 4) and (0, 1)
(D) $\left(0, \frac{11}{2}\right)$ and $\left(0, \frac{-1}{2}\right)$

EL0030

[MULTIPLE OBJECTIVE TYPE]

7. Extremities of the latus rectum of the ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ (a>b)$ having a given major axis 2a lies on-

(A) $x^2 = a(a - y)$
(B) $x^2 = a(a + y)$
(C) $y^2 = a(a + x)$
(D) $y^2 = a(a - x)$

EL0031

8. If a number of ellipse (whose axes are x & y axes) be described having the same major axis 2a but a variable minor axis then the tangents at the ends of their latus rectum pass through fixed points which can be -

(A) (0,a)
(B) (0,0)
(C) (0,–a)
(D) (a,a)

EL0032

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9. Tangents are drawn from any point on the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ to the circle $x^2+y^2=1$ and respective chord of contact always touches a conic 'C', then -

(A) minimum distance between 'C' & ellipse is $\frac{3}{2}$

(B) maximum distance between 'C' & ellipse is $\frac{10}{3}$

(C) eccentricity of 'C' is $\frac{\sqrt{5}}{3}$

(D) product of eccentricity of 'C' & ellipse is 1

EL0033

10. Two lines are drawn from point P(α, β) which touches $y^2 = 8x$ at A, B and touches $\frac{x^2}{4}+\frac{y^2}{6}=1$ at C, D, then -

(A) $\alpha + \beta = -4$

(B) $\alpha\beta = 4$

(C) Area of triangle PAB is $128\sqrt{2}$

(D) Area of triangle PAB is $32\sqrt{2}$

EL0034

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11. Let LMNP be a rectangle (which is not a square) inscribed in the ellipse as shown in the figure. Let 'λ' denotes the number of ways in which we can select four points out of A,A',B,B', L,M,P and N such that normal at these four points are concurrent, then 'λ' is less than or equal to-

(A) 5

(B) 6

(C) 7

(D) 4

EL0035

12. Let P(r, s), (r < 0) is a variable point on ellipse $C_1 : \frac{x^2}{16}+\frac{y^2}{9}=1$. PA and PB are two tangents of parabola $C_2 : y^2 - 4x = 0$, where A and B are point of contacts, then -

(A) number of common tangents of curves C_1 and C_2 is 4.

(B) minimum integral value of r + s is –5

(C) mid point of chord AB lie on curve $\frac{y^2}{9}+\frac{(y^2-2x)^2}{64}=1$

(D) mid point of chord AB lie on curve $\frac{x^2}{9}+\frac{(y^2-2x)^2}{64}=1$

EL0036

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[COMPREHENSION TYPE]

Paragraph for question nos. 13 to 16

Let the two foci of an ellipse be $(-1, 0)$ and $(3, 4)$ and the foot of perpendicular from the focus $(3, 4)$ upon a tangent to the ellipse be $(4, 6)$.

13. The foot of perpendicular from the focus $(-1, 0)$ upon the same tangent to the ellipse is

(A) $\left(\frac{12}{5}, \frac{34}{5}\right)$ (B) $\left(\frac{7}{3}, \frac{11}{3}\right)$ (C) $\left(2, \frac{17}{4}\right)$ (D) $(-1, 2)$

EL0037

14. The equation of auxiliary circle of the ellipse is

(A) $x^2 + y^2 - 2x - 4y - 5 = 0$ (B) $x^2 + y^2 - 2x - 4y - 20 = 0$

(C) $x^2 + y^2 + 2x + 4y - 20 = 0$ (D) $x^2 + y^2 + 2x + 4y - 5 = 0$

EL0037

15. The length of semi-minor axis of the ellipse is

(A) 1 (B) $2\sqrt{2}$ (C) $\sqrt{17}$ (D) $\sqrt{19}$

EL0037

16. The equations of directrices of the ellipse are

(A) $x - y + 2 = 0,\ x - y - 5 = 0$ (B) $x + y - \frac{21}{2} = 0,\ x + y + \frac{17}{2} = 0$

(C) $x - y + \frac{3}{2} = 0,\ x - y - \frac{5}{2} = 0$ (D) $x + y - \frac{31}{2} = 0,\ x + y + \frac{19}{2} = 0$

EL0037

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EXERCISE (S-1)

1. **(a)** Find the equation of the ellipse with its centre (1, 2), focus at (6, 2) and passing through the point (4, 6).

EL0038

(b) An ellipse passes through the points $(-3, 1)$ & $(2, -2)$ & its principal axis are along the coordinate axes in order. Find its equation.

EL0039

2. If any two chords be drawn through two points on the major axis of an ellipse equidistant from the centre, show that $\tan\frac{\alpha}{2}\cdot\tan\frac{\beta}{2}\cdot\tan\frac{\gamma}{2}\cdot\tan\frac{\delta}{2} = 1$, where α, β, γ, δ are the eccentric angles of the extremities of the chords.

EL0040

3. Suppose x and y are real numbers and that $x^2 + 9y^2 - 4x + 6y + 4 = 0$ then find the maximum value of $(4x - 9y)$.

EL0041

4. Point 'O' is the centre of the ellipse with major axis AB & minor axis CD. Point F is one focus of t ellipse. If OF = 6 & the diameter of the inscribed circle of triangle OCF is 2, then find the product (AB) (CD).

EL0042

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5. 'O' is the origin & also the centre of two concentric circles having radii of the inner & the outer circle as 'a' & 'b' respectively. A line OPQ is drawn to cut the inner circle in P & the outer circle in Q. PR is drawn parallel to the y-axis & QR is drawn parallel to the x-axis. Prove that the locus of R is an ellipse touching the two circles. If the focii of this ellipse lie on the inner circle, find the ratio of inner : outer radii & find also the eccentricity of the ellipse.

EL0043

6. Let d be the perpendicular distance from the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to the tangent drawn at a point P on the ellipse. If F_1 & F_2 are the two foci of the ellipse, then show that $(PF_1 - PF_2)^2 = 4a^2\left[1-\frac{b^2}{d^2}\right]$.

EL0044

7. Find the condition so that the line px + qy = r intersects the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in points whose eccentric angles differ by $\pi/4$.

EL0045

8. A circle intersects an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ precisely at three points A, B, C as shown in the figure. AB is a diameter of the circle and is perpendicular to the major axis of the ellipse. If the eccentricity of the ellipse is 4/5, find the length of the diameter AB in terms of a.

EL0046

9. The tangent at any point P of a circle $x^2 + y^2 = a^2$ meets the tangent at a fixed point A (a, 0) in T and T is joined to B, the other end of the diameter through A, prove that the locus of the intersection of AP and BT is an ellipse whose eccentricity is $1/\sqrt{2}$.

EL0047

10. Find the equations of the lines with equal intercepts on the axes & which touch the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$.

EL0048

11. A tangent having slope $-\frac{4}{3}$ to the ellipse $\frac{x^2}{18}+\frac{y^2}{32}=1$, intersects the axis of x & y in points A & B respectively. If O is the origin, find the area of triangle OAB.

EL0049

12. Tangents drawn from the point P(2,3) to the circle $x^2 + y^2 - 8x + 6y + 1 = 0$ touch the circle at the points A and B. The circumcircle of the ΔPAB cuts the director circle of ellipse $\frac{(x+5)^2}{9}+\frac{(y-3)^2}{b^2}=1$ orthogonally. Find the value of b^2.

EL0050

13. The tangent at the point α on a standard ellipse meets the auxiliary circle in two points which subtends a right angle at the centre. Show that the eccentricity of the ellipse is $(1 + \sin^2\alpha)^{-1/2}$.

EL0051

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14. A tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches at the point P on it in the first quadrant & meets the coordinate axes in A & B respectively. If P divides AB in the ratio 3 : 1 reckoning from the x-axis find the equation of the tangent.

EL0052

15. Consider the family of circles, $x^2 + y^2 = r^2$, $2 < r < 5$. If in the first quadrant, the common tangent to a circle of the family and the ellipse $4x^2 + 25y^2 = 100$ meets the co–ordinate axes at A & B, then find the equation of the locus of the mid–point of AB.

EL0053

16. If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{14}+\frac{y^2}{5}=1$, intersects it again at the point $Q(2\theta)$, show that $\cos\theta = -(2/3)$.

EL0054

17. Find the equation of the largest circle with centre (1, 0) that can be inscribed in the ellipse $x^2 + 4y^2 = 16$.

EL0055

18. Let P_i and P_i' be the feet of the perpendiculars drawn from foci S,S' on a tangent T_i to an ellipse whose length of semi-major axis is 20. If $\sum_{i=1}^{10}(SP_i)(S'P_i') = 2560$, then find the value of 100e.

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EL0056

19. A ray emanating from the point (– 4, 0) is incident on the ellipse $9x^2 + 25y^2 = 225$ at the point P with abscissa 3. Find the equation of the reflected ray after first reflection.

EL0057

20. If s, s' are the length of the perpendicular on a tangent from the foci, a, a' are those from the vertices, c is that from the centre and e is the eccentricity of the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then prove that $\frac{ss'-c^2}{aa'-c^2}=e^2$

EL0058

21. An ellipse has foci at $F_1(9, 20)$ and $F_2(49, 55)$ in the xy-plane and is tangent to the x-axis. Find the length of its major axis.

EL0059

22. Prove that, in an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the perpendicular from a focus upon any tangent and the line joining the centre of the ellipse to the point of contact meet on the corresponding directrix.

EL0060

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EXERCISE (S-2)

1. Rectangle ABCD has area 200. An ellipse with area 200π passes through A and C and has foci at B and D. Find the perimeter of the rectangle.

EL0061

2. A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P & Q. Prove that the tangents at P & Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles.

EL0062

3. A straight line AB touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ & the circle $x^2 + y^2 = r^2$; where $a > r > b$. A focal chord of the ellipse, parallel to AB intersects the circle in P & Q, find the length of the perpendicular drawn from the centre of the ellipse to PQ. Hence show that PQ = 2b.

EL0063

4. If the tangent at any point of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ makes an angle α with the major axis and an angle β with the focal radius of the point of contact then show that the eccentricity 'e' of the ellipse is given by the absolute value of $\frac{\cos\beta}{\cos\alpha}$.

EL0064

5. A normal inclined at 45° to the axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is drawn. It meets the x-axis & the y-axis in P & Q respectively. If C is the centre of the ellipse, show that the area of triangle CPQ is $\frac{(a^2-b^2)^2}{2(a^2+b^2)}$ sq. units.

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EL0065

6. PG is the normal to a standard ellipse at P, G being on the major axis. GP is produced outwards to Q so that PQ = GP. Show that the locus of Q is an ellipse whose eccentricity is $\frac{a^2-b^2}{a^2+b^2}$.

EL0066

7. Consider the parabola $y^2 = 4x$ and the ellipse $2x^2 + y^2 = 6$, intersecting at P and Q.

(a) Prove that the two curves are orthogonal.

EL0067

(b) Find the area enclosed by the parabola and the common chord of the ellipse and parabola.

EL0068

(c) If tangent and normal at the point P on the ellipse intersect the x-axis at T and G respectively then find the area of the triangle PTG.

EL0069

8. The tangents from (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intersect at right angles. Show that the normals at the points of contact meet on the line $\frac{y}{y_1}=\frac{x}{x_1}$.

EL0070

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9. Find the number of integral values of parameter 'a' for which three chords of the ellipse $\frac{x^2}{2a^2}+\frac{y^2}{a^2}=1$ (other than its diameter) passing through the point $P\left(11a,-\frac{a^2}{4}\right)$ are bisected by the parabola $y^2 = 4ax$.

EL0071

10. Prove that the equation to the circle, having double contact with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (with eccentricity e) at the ends of a latus rectum, is $x^2 + y^2 - 2ae^3x = a^2(1 - e^2 - e^4)$.

EL0072

11. Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with centre 'O' where $a > b > 0$. Tangent at any point P on the ellipse meets the coordinate axes at X and Y and N is the foot of the perpendicular from the origin on the tangent at P. Minimum length of XY is 36 and maximum length of PN is 4.

(a) Find the eccentricity of the ellipse.

(b) Find the maximum area of an isosceles triangle inscribed in the ellipse if one of its vertex coincides with one end of the major axis of the ellipse.

(c) Find the maximum area of the triangle OPN.

EL0073

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12. Consider an ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ with centre C and a point P on it with eccentric angle $\frac{\pi}{4}$. Normal drawn at P intersects the major and minor axes in A and B respectively. N_1 and N_2 are the feet of the perpendiculars from the foci S_1 and S_2 respectively on the tangent at P and N is the foot of the perpendicular from the centre of the ellipse on the normal at P. Tangent at P intersects the axis of x at T.

Match the entries of **Column-I** with the entries of **Column-II**.

	Column-I		**Column-II**
(A)	(CA)(CT) is equal to	(P)	9
(B)	(PN)(PB) is equal to	(Q)	16
(C)	$(S_1N_1)(S_2N_2)$ is equal to	(R)	17
(D)	$(S_1P)(S_2P)$ is equal to	(S)	25

EL0074

EXERCISE (JM)

1. The ellipse $x^2 + 4y^2 = 4$ is inscribed in a rectangle aligned with the coordinate axes, which in turn is inscribed in another ellipse that passes through the point (4, 0). Then the equation of the ellipse is :- **[AIEEE-2009]**

(1) $4x^2 + 48y^2 = 48$ (2) $4x^2 + 64y^2 = 48$ (3) $x^2 + 16y^2 = 16$ (4) $x^2 + 12y^2 = 16$

EL0075

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2. Equation of the ellipse whose axes are the axes of coordinates and which passes through the point (–3, 1) and has eccentricity $\sqrt{2/5}$ is :- **[AIEEE-2011]**

(1) $3x^2 + 5y^2 - 15 = 0$ (2) $5x^2 + 3y^2 - 32 = 0$

(3) $3x^2 + 5y^2 - 32 = 0$ (4) $5x^2 + 3y^2 - 48 = 0$

EL0076

3. An ellipse is drawn by taking a diameter of the circle $(x - 1)^2 + y^2 = 1$ as its semi-minor axis and a diameter of the circle $x^2 + (y - 2)^2 = 4$ as its semi-major axis. If the centre of the ellipse is at the origin and its axes are the coordinate axes, then the equation of the ellipse is : **[AIEEE-2012]**

(1) $x^2 + 4y^2 = 16$ (2) $4x^2 + y^2 = 4$ (3) $x^2 + 4y^2 = 8$ (4) $4x^2 + y^2 = 8$

EL0077

4. **Statement–1 :** An equation of a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$ is $y = 2x + 2\sqrt{3}$.

Statement–2 : If the line $y = mx + \frac{4\sqrt{3}}{m}$, $(m \neq 0)$ is a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$, then m satisfies $m^4 + 2m^2 = 24$. **[AIEEE-2012]**

(1) Statement–1 is true, Statement–2 is false.

(2) Statement–1 is false, Statement–2 is true.

(3) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.

(4) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for Statement–1.

EL0078

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5. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and having centre at (0, 3) is : **[JEE (Main)-2013]**

(1) $x^2 + y^2 - 6y - 7 = 0$ (2) $x^2 + y^2 - 6y + 7 = 0$

(3) $x^2 + y^2 - 6y - 5 = 0$ (4) $x^2 + y^2 - 6y + 5 = 0$

EL0079

6. If a and c are positive real number and the ellipse $\frac{x^2}{4c^2} + \frac{y^2}{c^2} = 1$ has four distinct points in common with the circle $x^2 + y^2 = 9a^2$, then **[JEE-Main (On line)-2013]**

(1) $6ac + 9a^2 - 2c^2 > 0$ (2) $6ac + 9a^2 - 2c^2 < 0$

(3) $9ac - 9a^2 - 2c^2 < 0$ (4) $9ac - 9a^2 - 2c^2 > 0$

EL0080

7. Equation of the line passing through the points of intersection of the parabola $x^2 = 8y$ and the ellipse $\frac{x^2}{3} + y^2 = 1$ is - **[JEE-Main (On line)-2013]**

(1) $y + 3 = 0$ (2) $3y + 1 = 0$ (3) $3y - 1 = 0$ (4) $y - 3 = 0$

EL0081

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8. Let the equations of two ellipses be $E_1 : \frac{x^2}{3}+\frac{y^2}{2}=1$ and $E_2 : \frac{x^2}{16}+\frac{y^2}{b^2}=1$. If the product of their eccentricities is $\frac{1}{2}$, then the length of the minor axis of ellipse E_2 is :- **[JEE-Main (On line)-2013]**

(1) 9 (2) 8 (3) 2 (4) 4

EL0082

9. If the curves $\frac{x^2}{\alpha}+\frac{y^2}{4}=1$ and $y^3 = 16x$ intersect at right angles, then a value of α is :

[JEE-Main (On line)-2013]

(1) $\frac{4}{3}$ (2) $\frac{3}{4}$ (3) $\frac{1}{2}$ (4) 2

EL0083

10. A point on the ellipse, $4x^2 + 9y^2 = 36$, where the normal is parallel to the line, $4x - 2y - 5 = 0$, is :

[JEE-Main (On line)-2013]

(1) $\left(\frac{8}{5},-\frac{9}{5}\right)$ (2) $\left(-\frac{9}{5},\frac{8}{5}\right)$ (3) $\left(\frac{8}{5},\frac{9}{5}\right)$ (4) $\left(\frac{9}{5},\frac{8}{5}\right)$

EL0084

11. The locus of the foot of perpendicular drawn from the centre of the ellipse $x^2 + 3y^2 = 6$ on any tangent to it is : **[JEE(Main)-2014]**

(1) $(x^2 - y^2)^2 = 6x^2 + 2y^2$ (2) $(x^2 - y^2)^2 = 6x^2 - 2y^2$

(3) $(x^2 + y^2)^2 = 6x^2 + 2y^2$ (4) $(x^2 + y^2)^2 = 6x^2 - 2y^2$

EL0085

12. The area (in sq. units) of the quadrilateral formed by the tangents at the end points of the latera recta to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ is : **[JEE (Main)-2015]**

(1) $\frac{27}{2}$ (2) 27 (3) $\frac{27}{4}$ (4) 18

EL0086

13. The eccentricity of an ellipse whose centre is at the origin is $\frac{1}{2}$. If one of its directices is $x = -4$, then the equation of the normal to it at $\left(1,\frac{3}{2}\right)$ is :- **[JEE(Main) 2017]**

(1) $x + 2y = 4$ (2) $2y - x = 2$ (3) $4x - 2y = 1$ (4) $4x + 2y = 7$

EL0087

14. If tangents are drawn to the ellipse $x^2 + 2y^2 = 2$ at all points on the ellipse other than its four vertices then the mid points of the tangents intercepted betwen the coordinate axes lie on the curve :

[JEE (Main)-2019]

(1) $\frac{x^2}{2}+\frac{y^2}{4}=1$ (2) $\frac{x^2}{4}+\frac{y^2}{2}=1$ (3) $\frac{1}{2x^2}+\frac{1}{4y^2}=1$ (4) $\frac{1}{4x^2}+\frac{1}{2y^2}=1$

EL0088

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15. Let S and S' be the foci of the ellipse and B be any one of the extremities of its minor axis. If $\Delta S'BS$ is a right angled triangle with right angle at B and area ($\Delta S'BS$) = 8 sq. units, then the length of a latus rectum of the ellipse is : **[JEE (Main)-2019]**

(1) $2\sqrt{2}$ (2) 2 (3) 4 (4) $4\sqrt{2}$

EL0089

16. If the line x – 2y = 12 is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $\left(3,\frac{-9}{2}\right)$, then the length of the latus recturm of the ellipse is : **[JEE (Main)-2019]**

(1) 9 (2) $8\sqrt{3}$ (3) $12\sqrt{2}$ (4) 5

EL0090

17. If the normal to the ellipse $3x^2 + 4y^2 = 12$ at a point P on it is parallel to the line, 2x + y = 4 and the tangent to the ellipse at P passes through Q(4, 4) then PQ is equal to : **[JEE (Main)-2019]**

(1) $\frac{\sqrt{221}}{2}$ (2) $\frac{\sqrt{157}}{2}$ (3) $\frac{\sqrt{61}}{2}$ (4) $\frac{5\sqrt{5}}{2}$

EL0091

18. If $3x + 4y = 12\sqrt{2}$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{9}=1$ for some $a \in R$, then the distance between the foci of the ellipse is : **[JEE (Main)-2020]**

(1) 4 (2) $2\sqrt{7}$ (3) $2\sqrt{5}$ (4) $2\sqrt{2}$

EL0092

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19. If the distance between the foci of an ellipse is 6 and the distance between its directrices is 12, then the length of its latus rectum is : **[JEE (Main)-2020]**

(1) $\sqrt{3}$ (2) $2\sqrt{3}$ (3) $3\sqrt{2}$ (4) $\frac{3}{\sqrt{2}}$

EL0093

20. Let the line y = mx and the ellipse $2x^2 + y^2 = 1$ intersect at a ponit P in the first quadrant. If the normal to this ellipse at P meets the co-ordinate axes at $\left(-\frac{1}{3\sqrt{2}},0\right)$ and $(0,\beta)$, then β is equal to

[JEE (Main)-2020]

(1) $\frac{2}{\sqrt{3}}$ (2) $\frac{2\sqrt{2}}{3}$ (3) $\frac{2}{3}$ (4) $\frac{\sqrt{2}}{3}$

EL0094

21. The length of the minor axis (along y-axis) of an ellipse in the standard form is $\frac{4}{\sqrt{3}}$. If this ellipse touches the line, x + 6y = 8; then its eccentricity is : **[JEE (Main)-2020]**

(1) $\sqrt{\frac{5}{6}}$ (2) $\frac{1}{2}\sqrt{\frac{11}{3}}$ (3) $\frac{1}{3}\sqrt{\frac{11}{3}}$ (4) $\frac{1}{2}\sqrt{\frac{5}{3}}$

EL0095

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EXERCISE (JA)

PARAGRAPH :

Tangents are drawn from the point P(3,4) to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ touching the ellipse at points A and B. **[JEE 2010, 3+3+3]**

1. The coordinates of A and B are

(A) (3,0) and (0,2)

(B) $\left(-\frac{8}{5},\frac{2\sqrt{261}}{15}\right)$ and $\left(-\frac{9}{5},\frac{8}{5}\right)$

(C) $\left(-\frac{8}{5},\frac{2\sqrt{161}}{15}\right)$ and (0,2)

(D) (3,0) and $\left(-\frac{9}{5},\frac{8}{5}\right)$

EL0096

2. The orthocenter of the triangle PAB is

(A) $\left(5,\frac{8}{7}\right)$

(B) $\left(\frac{7}{5},\frac{25}{8}\right)$

(C) $\left(\frac{11}{5},\frac{8}{5}\right)$

(D) $\left(\frac{8}{25},\frac{7}{5}\right)$

EL0096

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3. The equation of the locus of the point whose distances from the point P and the line AB are equal, is -

(A) $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

(B) $x^2 + 9y^2 + 6xy - 54x + 62y - 241 = 0$

(C) $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$

(D) $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$

EL0096

4. The ellipse $E_1 : \frac{x^2}{9}+\frac{x^2}{4}=1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point (0,4) circumscribes the rectangle R. The eccentricity of the ellipse E_2 is - **[JEE 2012, 3M, –1M]**

(A) $\frac{\sqrt{2}}{2}$

(B) $\frac{\sqrt{3}}{2}$

(C) $\frac{1}{2}$

(D) $\frac{3}{4}$

EL0097

5. A vertical line passing through the point (h,0) intersects the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of the triangle PQR, $\Delta_1 = \max\limits_{1/2\le h\le 1}\Delta(h)$ and $\Delta_2 = \min\limits_{1/2\le h\le 1}\Delta(h)$, then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 =$ **[JEE-Advanced 2013, 4, (–1)]**

EL0098

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6. **List-I** **List-II**

P. Let $y(x) = \cos(3\cos^{-1}x)$, $x \in [-1, 1]$, $x \neq \pm\frac{\sqrt{3}}{2}$. **1.** 1

Then $\frac{1}{y(x)}\left\{(x^2-1)\frac{d^2y(x)}{dx^2}+x\frac{dy(x)}{dx}\right\}$ equals

EL0099

Q. Let $A_1, A_2, \ldots\ldots, A_n$ $(n > 2)$ be the vertices of a regular polygon of n sides with its centre at the origin. Let $\vec{a_k}$ be the position vector of the point A_k, $k = 1, 2, \ldots\ldots n$. If $\left|\sum_{k=1}^{n-1}(\vec{a_k}\times\vec{a_{k+1}})\right| = \left|\sum_{k=1}^{n-1}(\vec{a_k}.\vec{a_{k+1}})\right|$, then the minimum value of n is **2.** 2

EL0100

R. If the normal from the point P(h, 1) on the ellipse $\frac{x^2}{6}+\frac{y^2}{3}=1$ is perpendicular to the line $x + y = 8$, then the value of h is **3.** 8

EL0101

S. Number of positive solutions satisfying the equation $\tan^{-1}\left(\frac{1}{2x+1}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$ is **4.** 9

EL0102

Codes :

	P	Q	R	S
(A)	4	3	2	1
(B)	2	4	3	1
(C)	4	3	1	2
(D)	2	4	1	3

[JEE(Advanced)-2014, 3(–1)]

7. Suppose that the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ are $(f_1,0)$ and $(f_2,0)$ where $f_1 > 0$ and $f_2 < 0$. Let P_1 and P_2 be two parabolas with a common vertex at (0,0) and with foci at $(f_1,0)$ and $(2f_2,0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2,0)$ and T_2 be a tangent to P_2 which passes through $(f_1,0)$. If m_1 is the slope of T_1 and m_2 is the slope of T_2, then the value of $\left(\frac{1}{m_1^2}+m_2^2\right)$ is

[JEE 2015, 4M, –0M]
EL0103

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8. Let E_1 and E_2 be two ellipses whose centers are at the origin. The major axes of E_1 and E_2 lie along the x-axis and the y-axis, respectively. Let S be the circle $x^2 + (y - 1)^2 = 2$. The straight line $x + y = 3$ touches the curves S, E_1 and E_2 at P,Q and R, respectively. Suppose that $PQ = PR = \frac{2\sqrt{2}}{3}$.

If e_1 and e_2 are the eccentricities of E_1 and E_2, respectively, then the correct expression(s) is(are) **[JEE 2015, 4M, –0M]**

(A) $e_1^2 + e_2^2 = \frac{43}{40}$ (B) $e_1e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$ (C) $\left|e_1^2 - e_2^2\right| = \frac{5}{8}$ (D) $e_1e_2 = \frac{\sqrt{3}}{4}$

EL0104

PARAGRAPH :

Let $F_1(x_1, 0)$ and $F_2(x_2, 0)$ for $x_1 < 0$ and $x_2 > 0$, be the foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{8} = 1$. Suppose a parabola having vertex at the origin and focus at F_2 intersects the ellipse at point M in the first quadrant and at point N in the fourth quadrant.

9. The orthocentre of the triangle F_1MN is- **[JEE(Advanced)-2016, 4(–2)]**

(A) $\left(-\frac{9}{10}, 0\right)$ (B) $\left(\frac{2}{3}, 0\right)$ (C) $\left(\frac{9}{10}, 0\right)$ (D) $\left(\frac{2}{3}, \sqrt{6}\right)$

EL0105

10. If the tangents to the ellipse at M and N meet at R and the normal to the parabola at M meets the x-axis at Q, then the ratio of area of the triangle MQR to area of the quadrilateral MF_1NF_2 is - **[JEE(Advanced)-2016, 3(0)]**

(A) 3 : 4 (B) 4 : 5 (C) 5 : 8 (D) 2 : 3

EL0105

11. Consider two straight lines, each of which is tangent to both the circle $x^2 + y^2 = \frac{1}{2}$ and the parabola $y^2 = 4x$. Let these lines intersect at the point Q. Consider the ellipse whose center is at the origin O(0, 0) and whose semi-major axis is OQ. If the length of the minor axis of this ellipse is $\sqrt{2}$, then the which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018, 4(–2)]**

(A) For the ellipse, the eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is 1

(B) For the ellipse, the eccentricity is $\frac{1}{2}$ and the length of the latus rectum is $\frac{1}{2}$

(C) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{4\sqrt{2}}(\pi - 2)$

(D) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{16}(\pi - 2)$

EL0106

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12. Define the collections $\{E_1, E_2, E_3, ...\}$ of ellipses and $\{R_1, R_2, R_3, ...\}$ of rectangles as follows :

$E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1$;

R_1 : rectangle of largest area, with sides parallel to the axes, inscribed in E_1 ;

E_n : ellipse $\frac{x^2}{a_n^2} + \frac{y^2}{b_n^2} = 1$ of largest area inscribed in R_{n-1}, $n > 1$;

R_n : rectangle of largest area, with sides parallel to the axes, inscribed in E_n, $n > 1$.

Then which of the following options is/are correct ? **[JEE(Advanced)-2019, 4(–1)]**

(1) The eccentricities of E_{18} and E_{19} are NOT equal

(2) The distance of a focus from the centre in E_9 is $\frac{\sqrt{5}}{32}$

(3) The length of latus rectum of E_9 is $\frac{1}{6}$

(4) $\sum_{n=1}^{N} (\text{area of } R_n) < 24$, for each positive integer N

EL0107

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ANSWER KEY

ELLIPSE

EXERCISE (O-1)

1. D **2.** B **3.** C **4.** D **5.** A **6.** B **7.** A **8.** C

9. C **10.** A **11.** (a) D; (b) A,B,C,D **12.** A **13.** A **14.** A,B,D **15.** A,B

16. A,C **17.** A,B **18.** B **19.** A **20.** C **21.** B **22.** D

23. (A) S, (B) R, (C) P, (D) Q

EXERCISE (O-2)

1. D **2.** A **3.** B **4.** C **5.** A **6.** C **7.** A,B **8.** A,C

9. A,B,C **10.** A,D **11.** A,B,C **12.** B,C **13.** A **14.** B **15.** C **16.** D

EXERCISE (S-1)

1. (a) $20x^2 + 45y^2 - 40x - 180y - 700 = 0$; (b) $3x^2 + 5y^2 = 32$ **3.** 16 **4.** 65

5. $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ **7.** $a^2p^2 + b^2q^2 = r^2\sec^2\frac{\pi}{8} = (4 - 2\sqrt{2})r^2$ **8.** $\frac{18a}{17}$

10. $x + y - 5 = 0$, $x + y + 5 = 0$ **11.** 24 sq. units **12.** 54 **14.** $bx + a\sqrt{3}\,y = 2ab$

15. $25y^2 + 4x^2 = 4x^2y^2$ **17.** $(x - 1)^2 + y^2 = \frac{11}{3}$ **18.** 60 **19.** $12x + 5y = 48$; $12x - 5y = 48$

21. 85

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EXERCISE (S-2)

1. 80 **3.** $\sqrt{r^2 - b^2}$ **7.** (b) 8/3, (c) 4 **9.** 2 **11.** (a) $\frac{3}{5}$; (b) $240\sqrt{3}$; (c) 36

12. (A) Q; (B) S; (C) P; (D) R

EXERCISE (JM)

1. 4 **2.** 3 **3.** 1 **4.** 3 **5.** 1 **6.** 4 **7.** 3 **8.** 4

9. 1 **10.** 4 **11.** 3 **12.** 2 **13.** 3 **14.** 3 **15.** 3 **16.** 1

17. 4 **18.** 2 **19.** 3 **20.** 4 **21.** 2

EXERCISE (JA)

1. D **2.** C **3.** A **4.** C **5.** 9 **6.** A **7.** 4 **8.** A,B

9. A **10.** C **11.** A,C **12.** 3,4

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HYPERBOLA

The **Hyperbola** is a conic whose eccentricity is greater than unity. **(e > 1)**.

1. STANDARD EQUATION & DEFINITION(S) :

Standard equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ where } b^2 = a^2(e^2 - 1)$$

or $a^2e^2 = a^2 + b^2$ i.e. $e^2 = 1 + \dfrac{b^2}{a^2}$

$$= 1 + \left(\frac{\text{Conjugate Axis}}{\text{Transverse Axis}}\right)^2$$

(a) Foci :

$S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

(b) Equations of directrices :

$x = \dfrac{a}{e}$ & $x = -\dfrac{a}{e}$.

(c) Vertices :

$A \equiv (a, 0)$ & $A' \equiv (-a, 0)$.

(d) Latus rectum :

(i) Equation : $x = \pm ae$

(ii) Length $= \dfrac{2b^2}{a} = \dfrac{(\text{Conjugate Axis})^2}{(\text{Transverse Axis})} = 2a(e^2 - 1) = 2e$ (distance from focus to directrix)

(iii) Ends : $\left(ae, \dfrac{b^2}{a}\right), \left(ae, \dfrac{-b^2}{a}\right)$; $\left(-ae, \dfrac{b^2}{a}\right), \left(-ae, \dfrac{-b^2}{a}\right)$

(e) (i) Transverse Axis :

The line segment A′A of length 2a in which the foci S′ & S both lie is called the **Transverse Axis of the Hyperbola.**

(ii) Conjugate Axis :

The line segment B′B between the two points $B' \equiv (0, -b)$ & $B \equiv (0, b)$ is called as the **Conjugate Axis of the Hyperbola.**

The Transverse Axis & the Conjugate Axis of the hyperbola are together called the **Principal axes of the hyperbola.**

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(f) Focal Property :

The difference of the focal distances of any point on the hyperbola is constant and equal to transverse axis i.e. $\big||PS| - |PS'|\big| = 2a$. The distance SS′ = focal length.

(g) Focal distance :

Distance of any point P(x, y) on Hyperbola from foci **PS = ex – a** & **PS′ = ex + a.**

Illustration 1 : Find the equation of the hyperbola whose directrix is 2x + y = 1, focus (1, 2) and eccentricity $\sqrt{3}$.

Solution : Let P(x, y) be any point on the hyperbola and PM is perpendicular from P on the directrix.

Then by definition SP = e PM

$\Rightarrow$ $(SP)^2 = e^2 (PM)^2$ $\Rightarrow$ $(x-1)^2 + (y-2)^2 = 3\left\{\dfrac{2x+y-1}{\sqrt{4+1}}\right\}^2$

$\Rightarrow$ $5(x^2 + y^2 - 2x - 4y + 5) = 3(4x^2 + y^2 + 1 + 4xy - 2y - 4x)$

$\Rightarrow$ $7x^2 - 2y^2 + 12\,xy - 2x + 14y - 22 = 0$

which is the required hyperbola.

Illustration 2 : The eccentricity of the hyperbola $4x^2 - 9y^2 - 8x = 32$ is -

(A) $\dfrac{\sqrt{5}}{3}$ (B) $\dfrac{\sqrt{13}}{3}$ (C) $\dfrac{\sqrt{13}}{2}$ (D) $\dfrac{3}{2}$

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Solution : $4x^2 - 9y^2 - 8x = 32 \Rightarrow 4(x-1)^2 - 9y^2 = 36 \Rightarrow \dfrac{(x-1)^2}{9} - \dfrac{y^2}{4} = 1$

Here $a^2 = 9, b^2 = 4$

$\therefore$ eccentricity $e = \sqrt{1+\dfrac{b^2}{a^2}} = \sqrt{1+\dfrac{4}{9}} = \dfrac{\sqrt{13}}{3}$ **Ans.(B)**

Illustration 3 : If foci of a hyperbola are foci of the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$. If the eccentricity of the hyperbola be 2, then its equation is -

(A) $\dfrac{x^2}{4} - \dfrac{y^2}{12} = 1$ (B) $\dfrac{x^2}{12} - \dfrac{y^2}{4} = 1$ (C) $\dfrac{x^2}{12} + \dfrac{y^2}{4} = 1$ (D) none of these

Solution : For ellipse $e = \dfrac{4}{5}$, so foci = (±4, 0)

For hyperbola e = 2, so $a = \dfrac{ae}{e} = \dfrac{4}{2} = 2$, $b = 2\sqrt{4-1} = 2\sqrt{3}$

Hence equation of the hyperbola is $\dfrac{x^2}{4} - \dfrac{y^2}{12} = 1$ **Ans.(A)**

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Illustration 4 : Find the coordinates of foci, the eccentricity and latus-rectum, equations of directrices for the hyperbola $9x^2 - 16y^2 - 72x + 96y - 144 = 0$.

Solution : Equation can be rewritten as $\frac{(x-4)^2}{4^2} - \frac{(y-3)^2}{3^2} = 1$ so a = 4, b = 3

$b^2 = a^2(e^2 - 1)$ given $e = \frac{5}{4}$

Foci : $X = \pm ae$, $Y = 0$ gives the foci as (9, 3), (–1, 3)

Centre : $X = 0$, $Y = 0$ i.e. (4, 3)

Directrices : $X = \pm \frac{a}{e}$ i.e. $x - 4 = \pm \frac{16}{5}$ $\therefore$ directrices are $5x - 36 = 0$; $5x - 4 = 0$

Latus-rectum $= \frac{2b^2}{a} = 2.\frac{9}{4} = \frac{9}{2}$

Do yourself - 1 :

(i) Find the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which passes through (4, 0) & $(3\sqrt{2}, 2)$

(ii) Find the equation to the hyperbola, whose eccentricity is $\frac{5}{4}$, focus is (a, 0) and whose directrix is $4x - 3y = a$.

(iii) In the hyperbola $4x^2 - 9y^2 = 36$, find length of the axes, the co-ordinates of the foci, the eccentricity, and the latus rectum.

(iv) Find the equation to the hyperbola, the distance between whose foci is 16 and whose eccentricity is $\sqrt{2}$.

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2. CONJUGATE HYPERBOLA :

Two hyperbolas such that transverse & conjugate axes of one hyperbola are respectively the conjugate & the transverse axes of the other are called **Conjugate Hyperbolas** of each other. eg. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are conjugate hyperbolas of each other .

Note that :

(i) If e_1 & e_2 are the eccentricities of the hyperbola & its conjugate then $e_1^{-2} + e_2^{-2} = 1$.

(ii) The foci of a **hyperbola** and its **conjugate** are **concyclic and form the vertices of a square.**

(iii) Two hyperbolas are said to be **similar** if they have the **same eccentricity**.

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Illustration 5 : The eccentricity of the conjugate hyperbola to the hyperbola $x^2 - 3y^2 = 1$ is-

(A) 2 (B) $2/\sqrt{3}$ (C) 4 (D) 4/3

Solution : Equation of the conjugate hyperbola to the hyperbola $x^2 - 3y^2 = 1$ is

$-x^2 + 3y^2 = 1 \quad \Rightarrow \quad -\frac{x^2}{1} + \frac{y^2}{1/3} = 1$

Here $a^2 = 1, b^2 = 1/3$

$\therefore$ eccentricity $e = \sqrt{1 + a^2/b^2} = \sqrt{1+3} = 2$ **Ans. (A)**

Do yourself - 2 :

(i) Find eccentricity of conjugate hyperbola of hyperbola $4x^2 - 16y^2 = 64$, also find area of quadrilateral formed by foci of hyperbola & its conjugate hyperbola

3. RECTANGULAR OR EQUILATERAL HYPERBOLA :

The particular kind of hyperbola in which the lengths of the transverse & conjugate axis are equal is called an **Equilateral Hyperbola**. Note that the eccentricity of the rectangular hyperbola is $\sqrt{2}$ and **the length of it's latus rectum is equal to it's transverse or conjugate axis.**

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4. AUXILIARY CIRCLE :

A circle drawn with centre C & T.A. as a diameter is called the **Auxiliary Circle** of the hyperbola. Equation of the auxiliary circle is $\mathbf{x^2 + y^2 = a^2}$.

Note from the figure that P & Q are called the **"Corresponding Points"** on the hyperbola & the auxiliary circle. 'θ' is called the **eccentric angle** of the point 'P' on the hyperbola. **($0 \le \theta < 2\pi$)**.

Parametric Equation :

The equations $\mathbf{x = a \sec\theta}$ & $\mathbf{y = b \tan\theta}$ together represents the hyperbola $\mathbf{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$ where θ is a parameter. The parametric equations ; $\mathbf{x = a \cos h\,\phi, y = b \sin h\,\phi}$ also represents the same hyperbola.

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General Note :

Since the fundamental equation to the hyperbola only differs from that to the ellipse in having $-b^2$ instead of b^2 it will be found that many propositions for the hyperbola are derived from those for the ellipse by simply changing the sign of b^2.

5. POSITION OF A POINT 'P' w.r.t. A HYPERBOLA :

The quantity $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ is **positive , zero** or **negative** according as the point (x_1, y_1) lies within , upon or outside the curve.

6. LINE AND A HYPERBOLA :

The straight line $y = mx + c$ is a secant , a tangent or passes outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as : $c^2 >= < a^2m^2 - b^2$.

Equation of a chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ joining its two points $P(\alpha)$ & $Q(\beta)$ is

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2}$$

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Illustration 6 : Show that the line $x\cos\alpha + y\sin\alpha = p$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $a^2\cos^2\alpha - b^2\sin^2\alpha = p^2$.

Solution : The given line is $x\cos\alpha + y\sin\alpha = p \Rightarrow y\sin\alpha = -x\cos\alpha + p$

$\Rightarrow y = -x\cot\alpha + p\,\text{cosec}\,\alpha$

Comparing this line with $y = mx + c$

$m = -\cot\alpha,\ c = p\,\text{cosec}\,\alpha$

Since the given line touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then

$c^2 = a^2m^2 - b^2 \Rightarrow p^2\,\text{cosec}^2\alpha = a^2\cot^2\alpha - b^2$ or $p^2 = a^2\cos^2\alpha - b^2\sin^2\alpha$

Illustration 7 : If $(a\sec\theta, b\tan\theta)$ and $(a\sec\phi, b\tan\phi)$ are the ends of a focal chord of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\tan\frac{\theta}{2}\tan\frac{\phi}{2}$ equal to -

(A) $\frac{e-1}{e+1}$ (B) $\frac{1-e}{1+e}$ (C) $\frac{1+e}{1-e}$ (D) $\frac{e+1}{e-1}$

Solution : Equation of chord connecting the points $(a\sec\theta, b\tan\theta)$ and $(a\sec\phi, b\tan\phi)$ is

$$\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right) \quad \text{..... (i)}$$

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If it passes through (ae, 0); we have, $e\cos\left(\frac{\theta-\phi}{2}\right)=\cos\left(\frac{\theta+\phi}{2}\right)$

$$\Rightarrow \quad e=\frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}=\frac{1-\tan\frac{\theta}{2}.\tan\frac{\phi}{2}}{1+\tan\frac{\theta}{2}.\tan\frac{\phi}{2}} \quad \Rightarrow \quad \tan\frac{\theta}{2}.\tan\frac{\phi}{2}=\frac{1-e}{1+e}$$

Similarly if (i) passes through (–ae, 0), $\tan\frac{\theta}{2}.\tan\frac{\phi}{2}=\frac{1+e}{1-e}$ **Ans. (B, C)**

Do yourself - 3 :

(i) Find the condition for the line $\ell x + my + n = 0$ to touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

(ii) If the line $y = 5x + 1$ touch the hyperbola $\frac{x^2}{4}-\frac{y^2}{b^2}=1$ $\{b>4\}$, then -

(A) $b^2=\frac{1}{5}$ (B) $b^2 = 99$ (C) $b^2 = 4$ (D) $b^2 = 100$

7. TANGENT TO THE HYPERBOLA $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$:

(a) Point form : Equation of the tangent to the given hyperbola at the point (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

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Note : In general two tangents can be drawn from an external point $(x_1 y_1)$ to the hyperbola and they are $y-y_1=m_1(x-x_1)$ & $y-y_1=m_2(x-x_1)$, where m_1 & m_2 are roots of the equation $(x_1^2-a^2)m^2-2x_1y_1m+y_1^2+b^2=0$. If $D<0$, then **no tangent** can be drawn from $(x_1 y_1)$ to **the hyperbola.**

(b) Slope form : The equation of tangents of slope m to the given hyperbola is $y = mx \pm\sqrt{a^2m^2-b^2}$. Point of contact are $\left(\mp\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \frac{\mp b^2}{\sqrt{a^2m^2-b^2}}\right)$

Note that there **are two parallel tangents having the same slope m.**

(c) Parametric form : Equation of the tangent to the given hyperbola at the point $(a\sec\theta, b\tan\theta)$ is $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$.

Note : Point of intersection of the tangents at θ_1 & θ_2 is $x=a\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}$, $y=b\tan\left(\frac{\theta_1+\theta_2}{2}\right)$

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Illustration 8 : Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

Solution : Let m be the slope of the tangent. Since the tangent is perpendicular to the line $x - y = 0$

$\therefore \quad m \times 1 = -1 \Rightarrow m = -1$

Since $x^2 - 4y^2 = 36$ or $\frac{x^2}{36} - \frac{y^2}{9} = 1$

Comparing this with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore \quad a^2 = 36$ and $b^2 = 9$

So the equation of tangents are $y = (-1)x \pm \sqrt{36 \times (-1)^2 - 9}$

$y = -x \pm \sqrt{27} \Rightarrow x + y \pm 3\sqrt{3} = 0$ **Ans.**

Illustration 9 : The locus of the point of intersection of two tangents of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if the product of their slopes is c^2, will be -

(A) $y^2 - b^2 = c^2(x^2 + a^2)$ (B) $y^2 + b^2 = c^2(x^2 - a^2)$

(C) $y^2 + a^2 = c^2(x^2 - b^2)$ (D) $y^2 - a^2 = c^2(x^2 + b^2)$

Solution : Equation of any tangent of the hyperbola with slope m is $y = mx \pm \sqrt{a^2m^2 - b^2}$

If it passes through (x_1, y_1) then

$(y_1 - mx_1)^2 = a^2m^2 - b^2 \Rightarrow (x_1^2 - a^2)m^2 - 2x_1y_1m + (y_1^2 + b^2) = 0$

If $m = m_1, m_2$ then as given $m_1m_2 = c^2 \Rightarrow \frac{y_1^2 + b^2}{x_1^2 - a^2} = c^2$

Hence required locus will be : $y^2 + b^2 = c^2(x^2 - a^2)$ **Ans.(B)**

Illustration 10 : A common tangent to $9x^2 - 16y^2 = 144$ and $x^2 + y^2 = 9$ is -

(A) $y = 3\sqrt{\frac{2}{7}}x - \frac{15}{\sqrt{7}}$ (B) $y = 3\sqrt{\frac{2}{7}}x + \frac{15}{\sqrt{7}}$ (C) $y = -3\sqrt{\frac{2}{7}}x + \frac{15}{\sqrt{7}}$ (D) $y = -3\sqrt{\frac{2}{7}}x - \frac{15}{\sqrt{7}}$

Solution : $\frac{x^2}{16} - \frac{y^2}{9} = 1$, $x^2 + y^2 = 9$

Equation of tangent $y = mx + \sqrt{16m^2 - 9}$ (for hyperbola)

Equation of tangent $y = m'x + 3\sqrt{1 + m'^2}$ (circle)

For common tangent $m = m'$ and $3\sqrt{1 + m'^2} = \sqrt{16m^2 - 9}$

or $9 + 9m^2 = 16m^2 - 9$

or $7m^2 = 18 \Rightarrow m = \pm 3\sqrt{\frac{2}{7}}$

$\therefore$ required equation is $y = \pm 3\sqrt{\frac{2}{7}}\, x \pm 3\sqrt{1 + \frac{18}{7}}$

or $y = \pm 3\sqrt{\frac{2}{7}}x \pm \frac{15}{\sqrt{7}}$ **Ans. (A,B,C,D)**

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Do yourself - 4 :

(i) Find the equation of the tangent to the hyperbola $4x^2 - 9y^2 = 1$, which is parallel to the line $4y = 5x + 7$.

(ii) Find the equation of the tangent to the hyperbola $16x^2 - 9y^2 = 144$ at $\left(5, \frac{16}{3}\right)$.

(iii) Find the common tangent to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and an ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$.

8. NORMAL TO THE HYPERBOLA $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

(a) Point form : The equation of the normal to the given hyperbola at the point P (x_1, y_1) on it is $\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2 = a^2 e^2$.

(b) **Slope form :** The equation of normal of slope m to the given hyperbola is $y = mx \mp \frac{m(a^2 + b^2)}{\sqrt{(a^2 - m^2 b^2)}}$ foot of normal are $\left(\pm \frac{a^2}{\sqrt{(a^2 - m^2 b^2)}}, \mp \frac{mb^2}{\sqrt{(a^2 - m^2 b^2)}}\right)$

(c) **Parametric form :** The equation of the normal at the point **P (a secθ , b tanθ)** to the given hyperbola is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2 e^2$.

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Illustration 11 : Line $x \cos\alpha + y \sin\alpha = p$ is a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if -

(A) $a^2 \sec^2\alpha - b^2 \operatorname{cosec}^2\alpha = \frac{(a^2 + b^2)^2}{p^2}$ (C) $a^2 \sec^2\alpha + b^2 \operatorname{cosec}^2\alpha = \frac{(a^2 + b^2)^2}{p^2}$

(C) $a^2 \cos^2\alpha - b^2 \sin^2\alpha = \frac{(a^2 + b^2)^2}{p^2}$ (D) $a^2 \cos^2\alpha + b^2 \sin^2\alpha = \frac{(a^2 + b^2)^2}{p^2}$

Solution : Equation of a normal to the hyperbola is $ax \cos\theta + by \cot\theta = a^2 + b^2$
comparing it with the given line equation

$$\frac{a\cos\theta}{\cos\alpha} = \frac{b\cot\theta}{\sin\alpha} = \frac{a^2 + b^2}{p} \quad \Rightarrow \quad \sec\theta = \frac{ap}{\cos\alpha(a^2 + b^2)}, \ \tan\theta = \frac{bp}{\sin\alpha(a^2 + b^2)}$$

Eliminating θ, we get

$$\frac{a^2 p^2}{\cos^2\alpha(a^2 + b^2)^2} - \frac{b^2 p^2}{\sin^2\alpha(a^2 + b^2)^2} = 1 \Rightarrow a^2 \sec^2\alpha - b^2 \operatorname{cosec}^2\alpha = \frac{(a^2 + b^2)^2}{p^2}$$

Ans.(A)

Illustration 12 : The normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the axes in M and N, and lines MP and NP are drawn at right angles to the axes. Prove that the locus of P is hyperbola $(a^2x^2 - b^2y^2) = (a^2 + b^2)^2$.

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Solution : Equation of normal at any point Q is $ax\cos\theta + by\cot\theta = a^2 + b^2$

$\therefore \quad M \equiv \left(\frac{a^2+b^2}{a}\sec\theta, 0\right), \; N \equiv \left(0, \frac{a^2+b^2}{b}\tan\theta\right)$

$\therefore$ Let $P \equiv (h, k)$

$\Rightarrow \quad h = \frac{a^2+b^2}{a}\sec\theta, \qquad k = \frac{a^2+b^2}{b}\tan\theta$

$\Rightarrow \quad \frac{a^2h^2}{(a^2+b^2)} - \frac{b^2k^2}{(a^2+b^2)^2} = \sec^2\theta - \tan^2\theta = 1$

$\therefore$ locus of P is $(a^2x^2 - b^2y^2) = (a^2 + b^2)$.

Do yourself - 5 :

(i) Find the equation of normal to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ at (5, 0).

(ii) Find the equation of normal to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at the point $\left(6, \frac{3}{2}\sqrt{5}\right)$.

(iii) Find the condition for the line $\ell x + my + n = 0$ is normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

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9. HIGHLIGHTS ON TANGENT AND NORMAL :

(a) Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ upon any tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ & the product of lengths to these perpendiculars is b^2 **(semi Conjugate Axis)**^{2}

(b) The portion of the tangent between the **point of contact** & **the directrix** subtends a **right angle** at the corresponding **focus**.

(c) The tangent & normal at any point of a hyperbola **bisect** the angle between the **focal radii**. This spells the **reflection property of the hyperbola** as **"An incoming light ray"** aimed towards one **focus is reflected from the outer surface of the hyperbola towards the other focus.** It follows that if an ellipse and a hyperbola have the same foci, they cut at **right angles** at any of their **common point.**

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Note that the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **& the hyperbola** $\frac{x^2}{a^2-k^2}-\frac{y^2}{k^2-b^2}=1$ **(a > k > b > 0) are confocal and therefore orthogonal.**

(d) The **foci of the hyperbola** and the **points P** and **Q** in which any tangent meets the tangents at the vertices are **concyclic** with **PQ** as **diameter of the circle.**

10. DIRECTOR CIRCLE :

The locus of the intersection of tangents which are at **right angles** is known as the **Director Circle** of the hyperbola. The equation to the **director circle is** : $x^2+y^2=a^2-b^2$.

If $b^2<a^2$, this circle is **real ;** if $b^2=a^2$ the **radius** of the **circle is zero** & it reduces to a **point circle at the origin.** In this case the **centre is the only point** from which the **tangents at right angles** can be drawn to the **curve**.

If $b^2>a^2$, the **radius** of the **circle is imaginary**, so that there is **no such circle** & so **no tangents at right angle** can be drawn to the curve.

Note : Equations of chord of contact, chord with a given middle point, pair of tangents from an external point are to be interpreted in the similar way as in ellipse.

11. ASYMPTOTES :

Definition : If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the **Asymptote of the Hyperbola**.

To find the asymptote of the hyperbola :

Let $y=mx+c$ is the **asymptote** of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Solving these two we get the quadratic as $(b^2-a^2m^2)x^2-2a^2mcx-a^2(b^2+c^2)=0$ .....(1)

In order that $y=mx+c$ **be an asymptote**, both **roots** of equation (1) must approach infinity, the conditions for which are : **coefficient of** $x^2=0$ & **coefficient of** $x=0$.

$\Rightarrow$ $b^2-a^2m^2=0$ or $m=\pm\frac{b}{a}$ & $a^2mc=0 \Rightarrow c=0$.

$\therefore$ equations of asymptote are $\frac{x}{a}+\frac{y}{b}=0$ and $\frac{x}{a}-\frac{y}{b}=0$.

combined equation to the asymptotes $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$.

Particular Case :

When $b=a$ the asymptotes of the rectangular hyperbola.

$x^2-y^2=a^2$ are $y=\pm x$ which are at **right angles**.

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Note :

(i) **Equilateral hyperbola ⇔ rectangular hyperbola.**

(ii) If a **hyperbola is equilateral** then the **conjugate hyperbola is also equilateral**.

(iii) A **hyperbola** and its **conjugate** have the **same asymptote.**

(iv) The equation of the **pair of asymptotes** differ the **hyperbola** & **the conjugate hyperbola** by the **same constant** only.

(v) The asymptotes pass through the **centre of the hyperbola** & the bisectors of the angles between the asymptotes are the axes of the hyperbola.

(vi) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.

(vii) Asymptotes are the tangent to the hyperbola from the centre.

(viii) A simple method to find the co–ordinates of the centre of the hyperbola expressed as a general equation of degree 2 should be remembered as : Let $\mathbf{f(x, y) = 0}$ represents a hyperbola.

Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$. Then the point of intersection of $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ gives the **centre** of the **hyperbola**.

Illustration 13 : Find the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$. Find also the general equation of all the hyperbolas having the same set of asymptotes.

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Solution : Let $2x^2 + 5xy + 2y^2 + 4x + 5y + \lambda = 0$ be asymptotes. This will represent two straight line

so $4\lambda + 25 - \frac{25}{2} - 8 - \frac{25}{4}\lambda = 0$

$\Rightarrow \quad \lambda = 2$

$\Rightarrow \quad 2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$ are asymptotes

$\Rightarrow \quad (2x + y + 2) = 0$ and $(x + 2y + 1) = 0$ are asymptotes

and $\quad 2x^2 + 5xy + 2y^2 + 4x + 5y + c = 0$ is general equation of hyperbola.

Illustration 14 : Find the hyperbola whose asymptotes are $2x - y = 3$ and $3x + y - 7 = 0$ and which passes through the point (1, 1).

Solution : The equation of the hyperbola differs from the equation of the asymptotes by a constant

$\Rightarrow$ The equation of the hyperbola with asymptotes $3x + y - 7 = 0$ and $2x - y = 3$ is $(3x + y - 7)(2x - y - 3) + k = 0$

It passes through (1, 1)

$\Rightarrow \quad k = -6.$

Hence the equation of the hyperbola is $(2x - y - 3)(3x + y - 7) = 6$.

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Do yourself - 6 :

(i) Find the equation to the chords of the hyperbola $x^2 - y^2 = 9$ which is bisected at (5, –3)

(ii) If m_1 and m_2 are the slopes of the tangents to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ which pass through the point (6, 2), then find the value of $11m_1m_2$ and $11(m_1 + m_2)$.

(iii) Find the locus of the mid points of the chords of the circle $x^2 + y^2 = 16$ which are tangents to the hyperbola $9x^2 - 16y^2 = 144$.

(iv) The asymptotes of a hyperbola are parallel to lines $2x + 3y = 0$ and $3x + 2y = 0$. The hyperbola has its centre at (1, 2) and it passes through (5, 3). Find its equation.

12. HIGHLIGHTS ON ASYMPTOTES

(a) If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point & the curve is always equal to the square of the semi conjugate axis.

(b) Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix & the common points of intersection lie on the auxiliary circle.

(c) The tangent at any point P on a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with **centre C**, meets the **asymptotes in Q** and **R** and cuts off a **Δ CQR** of **constant area** equal to **ab** from the asymptotes & the portion of the tangent intercepted between the asymptote is bisected at the point of contact . This implies that locus of the centre of the circle circumscribing the **Δ CQR** in case of a rectangular hyperbola is the **hyperbola** itself.

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(d) If the angle between the asymptote of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 2θ then the eccentricity of the **hyperbola** is $\sec\theta$.

13. RECTANGULAR HYPERBOLA :

Rectangular hyperbola referred to its asymptotes as axis of coordinates.

(a) Equation is $xy = c^2$ with parametric representation $x = ct,\ \ y = c/t,\ \ t \in R - \{0\}$.

(b) Equation of a chord joining the points $P(t_1)$ & $Q(t_2)$ is $x + t_1t_2y = c(t_1 + t_2)$ with slope, $m = \frac{-1}{t_1t_2}$

(c) Equation of the tangent at $P(x_1, y_1)$ is $\frac{x}{x_1} + \frac{y}{y_1} = 2$

& at P(t) is $\frac{x}{t} + ty = 2c$.

(d) Equation of normal is $y - \frac{c}{t} = t^2(x - ct)$

(e) Chord with a given middle point as **(h, k)** is $kx + hy = 2hk$.

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Note :

For the hyperbola, $\mathbf{xy = c^2}$

(i) Vertices : **(c, c)** & **(–c, –c)**.

(ii) Foci : $(\sqrt{2}c, \sqrt{2}c)$ & $(-\sqrt{2}c, -\sqrt{2}c)$

(iii) Directrices : $x + y = \pm\sqrt{2}c$

(iv) Latus rectum : $\ell = 2\sqrt{2}c$ = **T . A = C . A**

Illustration 15 : A triangle has its vertices on a rectangular hyperbola. Prove that the orthocentre of the triangle also lies on the same hyperbola.

Solution : Let t_1, t_2 and t_3 are the vertices of the triangle ABC, described on the rectangular hyperbola $xy = c^2$.

$\therefore$ co-ordinates of A, B and C are $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ respectively

Now slope of BC is $\dfrac{\frac{c}{t_3} - \frac{c}{t_2}}{ct_3 - ct_2} = -\dfrac{1}{t_2t_3}$

$\therefore$ Slope of AD is t_2t_3

Equation of altitude AD is $y - \dfrac{c}{t_1} = t_2t_3(x - ct_1)$

or $t_1y - c = xt_1t_2t_3 - ct_1^2t_2t_3$ ..... (i)

Similarly equation of altitude BE is

$t_2y - c = xt_1t_2t_3 - ct_1t_2^2t_3$ ..... (ii)

Solving (i) and (ii), we get the orthocentre $\left(-\dfrac{c}{t_1t_2t_3}, -ct_1t_2t_3\right)$ which lies on $xy = c^2$.

Do yourself - 7 :

(i) If equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a rectangular hyperbola then write required conditions.

(ii) Find the equation of tangent at the point (1, 2) to the rectangular hyperbola $xy = 2$.

(iii) Prove that the locus of point, tangents from where to hyperbola $x^2 - y^2 = a^2$ inclined at an angle α & β with x-axis such that $\tan\alpha \tan\beta = 2$ is also a hyperbola. Find the eccentricity of this hyperbola.

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Miscellaneous Illustrations :

Illustration 16 : Chords of the circle $x^2 + y^2 = a^2$ touch the hyperbola $x^2/a^2 - y^2/b^2 = 1$. Prove that locus of their middle point is the curve $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$.

Solution : Let (h, k) be the mid-point of the chord of the circle $x^2 + y^2 = a^2$, so that its equation by $T = S_1$ is $hx + ky = h^2 + k^2$

or $y = -\frac{h}{k}x + \frac{h^2+k^2}{k}$ i.e. of the form $y = mx + c$

It will touch the hyperbola if $c^2 = a^2m^2 - b^2$

$\therefore \left(\frac{h^2+k^2}{k}\right)^2 = a^2\left(-\frac{h}{k}\right)^2 - b^2$ or $(h^2 + k^2)^2 = a^2h^2 - b^2k^2$

Generalising, the locus of mid-point (h, k) is $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$

Illustration 17 : C is the centre of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The tangent at any point P on this hyperbola meets the straight lines $bx - ay = 0$ and $bx + ay = 0$ in the points Q and R respectively. Show that $CQ \cdot CR = a^2 + b^2$.

Solution : P is $(a\sec\theta, b\tan\theta)$

Tangent at P is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

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It meets $bx - ay = 0$ i.e. $\frac{x}{a} = \frac{y}{b}$ in Q

$\therefore$ Q is $\left(\frac{a}{\sec\theta - \tan\theta}, \frac{b}{\sec\theta - \tan\theta}\right)$

It meets $bx + ay = 0$ i.e. $\frac{x}{a} = -\frac{y}{b}$ in R.

$\therefore$ R is $\left(\frac{a}{\sec\theta + \tan\theta}, \frac{-b}{\sec\theta + \tan\theta}\right)$

$\therefore$ $CQ.CR = \frac{\sqrt{(a^2+b^2)}}{\sec\theta - \tan\theta} \cdot \frac{\sqrt{(a^2+b^2)}}{\sec\theta + \tan\theta} = a^2 + b^2$ ($\because$ $\sec^2\theta - \tan^2\theta = 1$) **Ans.**

Illustration 18 : A circle of variable radius cuts the rectangular hyperbola $x^2 - y^2 = 9a^2$ in points P, Q, R and S. Determine the equation of the locus of the centroid of triangle PQR.

Solution : Let the circle be $(x - h)^2 + (y - k)^2 = r^2$ where r is variable. Its intersection with $x^2 - y^2 = 9a^2$ is obtained by putting $y^2 = x^2 - 9a^2$.

$x^2 + x^2 - 9a^2 - 2hx + h^2 + k^2 - r^2 = 2k\sqrt{(x^2 - 9a^2)}$

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or $[2x^2 - 2hx + (h^2 + k^2 - r^2 - 9a^2)]^2 = 4k^2(x^2 - 9a^2)$

or $4x^4 - 8hx^3 + ... = 0$

$\therefore$ Above gives the abscissas of the four points of intersection.

$$\therefore \quad \Sigma x_1 = \frac{8h}{4} = 2h$$

$$x_1 + x_2 + x_3 + x_4 = 2h$$

Similarly $y_1 + y_2 + y_3 + y_4 = 2k$.

Now if (α, β) be the centroid of ΔPQR, then $3\alpha = x_1 + x_2 + x_3$, $3\beta = y_1 + y_2 + y_3$

$\therefore$ $x_4 = 2h - 3\alpha$, $y_4 = 2k - 3\beta$

But (x_4, y_4) lies on $x^2 - y^2 = 9a^2$

$\therefore$ $(2h - 3\alpha)^2 + (2k - 3\beta)^2 = 9a^2$

Hence the locus of centroid (α, β) is $(2h - 3x)^2 + (2k - 3y)^2 = 9a^2$

$$\text{or} \quad \left(x - \frac{2h}{3}\right)^2 + \left(y - \frac{2k}{3}\right)^2 = a^2$$

Illustration 19 : If a circle cuts a rectangular hyperbola $xy = c^2$ in A, B, C, D and the parameters of these four points be t_1, t_2, t_3 and t_4 respectively, then prove that :

(a) $t_1t_2t_3t_4 = 1$

(b) The centre of mean position of the four points bisects the distance between the centres of the two curves.

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Solution : (a) Let the equation of the hyperbola referred to rectangular asymptotes as axes be $xy = c^2$ or its parametric equation be

$x = ct$, $y = c/t$ ..... (i)

and that of the circle be

$x^2 + y^2 + 2gx + 2fy + k = 0$ ..... (ii)

Solving (i) and (ii), we get

$$c^2t^2 + \frac{c^2}{t^2} + 2gct + 2f\frac{c}{t} + k = 0$$

or $c^2t^4 + 2gct^3 + kt^2 + 2fct + c^2 = 0$ ..... (iii)

Above equation being of fourth degree in t gives us the four parameters t_1, t_2, t_3, t_4 of the points of intersection.

$$\therefore \quad t_1 + t_2 + t_3 + t_4 = -\frac{2gc}{c^2} = -\frac{2g}{c} \quad \text{..... (iv)}$$

$$t_1t_2t_3 + t_1t_2t_4 + t_3t_4t_1 + t_3t_4t_2$$

$$= -\frac{2fc}{c^2} = -\frac{2f}{c} \quad \text{..... (v)}$$

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$$t_1t_2t_3t_4 = \frac{c^2}{c^2} = 1 \text{. It proves (a)} \qquad \text{..... (vi)}$$

Dividing (v) by (vi), we get

$$\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+\frac{1}{t_4} = -\frac{2f}{c} \qquad \text{..... (vii)}$$

(b) The centre of mean position of the four points of intersection is

$$\left[\frac{c}{4}(t_1+t_2+t_3+t_4),\ \frac{c}{4}\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+\frac{1}{t_4}\right)\right] = \left[\frac{c}{4}\left(-\frac{2g}{c}\right),\ \frac{c}{4}\left(-\frac{2f}{c}\right)\right], \text{ by (iv) and}$$

(vii)

$= (-g/2, -f/2)$

Above is clearly the mid-point of (0, 0) and $(-g, -f)$ i.e. the join of the centres of the two curves.

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ANSWERS FOR DO YOURSELF

1 : **(i)** $\sqrt{3}$ **(ii)** $7y^2 + 24xy - 24ax - 6ay + 15a^2 = 0$

(iii) 6, 4; $(\pm\sqrt{13}, 0)$; $\sqrt{13}/3$; 8/3 **(iv)** $x^2 - y^2 = 32$

2 : **(i)** $\sqrt{5}$ & 40 sq. units

3 : **(i)** $n^2 = a^2\ell^2 - b^2m^2$ **(ii)** B

4 : **(i)** $24y = 30x \pm \sqrt{161}$ **(ii)** $5x - 3y = 9$ **(iii)** $y = \pm x \pm \sqrt{7}$

5 : **(i)** $y = 0$; **(ii)** $8\sqrt{5}x + 18y = 75\sqrt{5}$ **(iii)** $\frac{a^2}{\ell^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$

6 : **(i)** $5x + 3y = 16$ **(ii)** 20 & 24 **(iii)** $(x^2 + y^2)^2 = 16x^2 - 9y^2$

(iv) $(2x + 3y - 8)(3x + 2y - 7) = 154$

7 : **(i)** $\Delta \neq 0, h^2 > ab, a + b = 0$ **(ii)** $2x + y = 4$ **(iii)** $e = \sqrt{3}$

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EXERCISE (O-1)

1. Consider the hyperbola $9x^2 - 16y^2 + 72x - 32y - 16 = 0$. Find the following:

(a) centre **(b)** eccentricity

(c) focii **(d)** equation of directrix

(e) length of the latus rectum **(f)** equation of auxilary circle

(g) equation of director circle

HB0001

[STRAIGHT OBJECTIVE TYPE]

2. The area of the quadrilateral with its vertices at the foci of the conics

$$9x^2 - 16y^2 - 18x + 32y - 23 = 0 \text{ and}$$
$$25x^2 + 9y^2 - 50x - 18y + 33 = 0, \text{ is}$$

(A) 5/6 (B) 8/9 (C) 5/3 (D) 16/9

HB0002

3. Eccentricity of the hyperbola conjugate to the hyperbola $\frac{x^2}{4} - \frac{y^2}{12} = 1$ is

(A) $\frac{2}{\sqrt{3}}$ (B) 2 (C) $\sqrt{3}$ (D) $\frac{4}{3}$

HB0003

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4. The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. Then the equation of the hyperbola with eccentricity 2 is

(A) $\frac{x^2}{12} - \frac{y^2}{4} = 1$ (B) $\frac{x^2}{4} - \frac{y^2}{12} = 1$ (C) $3x^2 - y^2 + 12 = 0$ (D) $9x^2 - 25y^2 - 225 = 0$

HB0004

5. If the eccentricity of the hyperbola $x^2 - y^2 \sec^2\alpha = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2 \sec^2\alpha + y^2 = 25$, then a value of α is :

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $\pi/2$

HB0005

6. The foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide. Then the value of b^2 is-

(A) 5 (B) 7 (C) 9 (D) 4

HB0006

7. The graph of the equation $x + y = x^3 + y^3$ is the union of -

(A) line and an ellipse (B) line and a parabola

(C) line and hyperbola (D) line and a point

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HB0007

8. The focal length of the hyperbola $x^2 - 3y^2 - 4x - 6y - 11 = 0$, is-

(A) 4 (B) 6 (C) 8 (D) 10

HB0008

9. The equation $\frac{x^2}{29-p} + \frac{y^2}{4-p} = 1$ $(p \neq 4, 29)$ represents -

(A) an ellipse if p is any constant greater than 4

(B) a hyperbola if p is any constant between 4 and 29.

(C) a rectangular hyperbola if p is any constant greater than 29.

(D) no real curve is p is less than 29.

HB0009

10. If $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$ represents family of hyperbolas where 'α' varies then -

(A) distance between the foci is constant

(B) distance between the two directrices is constant

(C) distance between the vertices is constant

(D) distances between focus and the corresponding directrix is constant

HB0010

11. The locus of the point of intersection of the lines $\sqrt{3}x - y - 4\sqrt{3}\,t = 0$ & $\sqrt{3}tx + ty - 4\sqrt{3} = 0$ (where t is a parameter) is a hyperbola whose eccentricity is

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(A) $\sqrt{3}$ (B) 2 (C) $\frac{2}{\sqrt{3}}$ (D) $\frac{4}{3}$

HB0011

12. Latus rectum of the conic satisfying the differential equation, $x\,dy + y\,dx = 0$ and passing through the point (2, 8) is :

(A) $4\sqrt{2}$ (B) 8 (C) $8\sqrt{2}$ (D) 16

HB0012

13. The magnitude of the gradient of the tangent at an extremity of latera recta of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is equal to (where e is the eccentricity of the hyperbola)

(A) be (B) e (C) ab (D) ae

HB0013

14. The number of possible tangents which can be drawn to the curve $4x^2 - 9y^2 = 36$, which are perpendicular to the straight line $5x + 2y - 10 = 0$ is :

(A) zero (B) 1 (C) 2 (D) 4

HB0014

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15. Locus of the point of intersection of the tangents at the points with eccentric angles ϕ and $\frac{\pi}{2} - \phi$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is :

(A) $x = a$ (B) $y = b$ (C) $x = ab$ (D) $y = ab$

HB0015

16. Locus of the feet of the perpendiculars drawn from either foci on a variable tangent to the hyperbola $16y^2 - 9x^2 = 1$ is

(A) $x^2 + y^2 = 9$ (B) $x^2 + y^2 = 1/9$ (C) $x^2 + y^2 = 7/144$ (D) $x^2 + y^2 = 1/16$

HB0016

17. A tangent to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ with centre C meets its director circle at P and Q. Then the product of the slopes of CP and CQ, is -

(A) $\frac{9}{4}$ (B) $\frac{-4}{9}$ (C) $\frac{2}{9}$ (D) $-\frac{1}{4}$

HB0017

18. The asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ form with any tangent to the hyperbola a triangle whose area is $a^2\tan\lambda$ in magnitude then its eccentricity is :

(A) $\sec\lambda$ (B) $\operatorname{cosec}\lambda$ (C) $\sec^2\lambda$ (D) $\operatorname{cosec}^2\lambda$

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HB0018

19. In which of the following cases maximum number of normals can be drawn from a point P lying in the same plane

(A) circle (B) parabola (C) ellipse (D) hyperbola

HB0019

20. PQ is a double ordinate of the ellipse $x^2 + 9y^2 = 9$, the normal at P meets the diameter through Q at R, then the locus of the mid point of PR is

(A) a circle (B) a parabola (C) an ellipse (D) a hyperbola

HB0020

21. With one focus of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ as the centre , a circle is drawn which is tangent to the hyperbola with no part of the circle being outside the hyperbola. The radius of the circle is

(A) less than 2 (B) 2 (C) $\frac{11}{3}$ (D) none

HB0021

22. If the normal to the rectangular hyperbola $xy = c^2$ at the point 't' meets the curve again at 't_1' then $t^3 t_1$ has the value equal to

(A) 1 (B) – 1 (C) 0 (D) none

HB0022

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23. Number of common tangent with finite slope to the curves $xy = c^2$ & $y^2 = 4ax$ is :
(A) 0 (B) 1 (C) 2 (D) 4

HB0023

24. Locus of the middle points of the parallel chords with gradient m of the rectangular hyperbola $xy = c^2$ is
(A) $y + mx = 0$ (B) $y - mx = 0$ (C) $my - x = 0$ (D) $my + x = 0$

HB0024

[MULTIPLE OBJECTIVE TYPE]

25. Let p and q be non-zero real numbers. Then the equation $(px^2 + qy^2 + r)(4x^2 + 4y^2 - 8x - 4) = 0$ represents
(A) two straight lines and a circle, when $r = 0$ and p, q are of the opposite sign.
(B) two circles, when $p = q$ and r is of sign opposite to that of p.
(C) a hyperbola and a circle, when p and q are of opposite sign and $r \neq 0$.
(D) a circle and an ellipse, when p and q are unequal but of same sign and r is of sign opposite to that of p.

HB0025

26. Solutions of the differential equation $(1 - x^2)\dfrac{dy}{dx} + xy = ax$ where $a \in R$, is
(A) a conic which is an ellipse or a hyperbola with principal axes parallel to coordinates axes.
(B) centre of the conic is (0, a)
(C) length of one of the principal axes is 1.
(D) length of one of the principal axes is equal to 2.

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HB0026

27. If θ is eliminated from the equations $a \sec\theta - x\tan\theta = y$ and $b\sec\theta + y\tan\theta = x$ (a and b are constant), then the eliminant denotes the equation of
(A) the director circle of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$
(B) auxiliary circle of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
(C) Director circle of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
(D) Director circle of the circle $x^2 + y^2 = \dfrac{a^2 + b^2}{2}$.

HB0027

28. The tangent to the hyperbola, $x^2 - 3y^2 = 3$ at the point $\left(\sqrt{3}, 0\right)$ when associated with two asymptotes constitutes -
(A) isosceles triangle which is not equilateral (B) an equilateral triangle
(C) a triangles whose area is $\sqrt{3}$ sq. units (D) a right isosceles triangle.

HB0028

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29. If latus rectum of a hyperbola subtends a right angle at other focus of hyperbola, then eccentricity is equal to-

(A) $1-\sqrt{2}$ (B) $\tan\frac{\pi}{8}$ (C) $\cot\frac{\pi}{8}$ (D) $\left(\frac{1}{\sqrt{2}-1}\right)$

HB0029

30. If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ in four points $P(x_1,y_1)$, $Q(x_2,y_2)$, $R(x_3,y_3)$, $S(x_4,y_4)$, then -

(A) $x_1 + x_2 + x_3 + x_4 = 0$ (B) $y_1 + y_2 + y_3 + y_4 = 0$
(C) $x_1x_2x_3x_4 = c^4$ (D) $y_1y_2y_3y_4 = c^4$

HB0030

[COMPREHENSION TYPE]

Paragraph for question nos. 31 to 33

The graph of the conic $x^2 - (y-1)^2 = 1$ has one tangent line with positive slope that passes through the origin. the point of tangency being (a, b). Then

31. The value of $\sin^{-1}\left(\frac{a}{b}\right)$ is

(A) $\frac{5\pi}{12}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$

HB0031

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32. Length of the latus rectum of the conic is

(A) 1 (B) $\sqrt{2}$ (C) 2 (D) none

HB0031

33. Eccentricity of the conic is

(A) $\frac{4}{3}$ (B) $\sqrt{3}$ (C) 2 (D) none

HB0031

Paragraph for question nos. 34 to 36

Consider the conic C : $\frac{x^2}{16}+\frac{y^2}{12}=1$

34. Equation of circle touching C at one extremity of latus-rectum and passing through centre of C is/are-

(A) $8x^2 + 8y^2 - 19x - 22y = 0$ (B) $8x^2 + 8y^2 - 19x + 22y = 0$
(C) $8x^2 + 8y^2 + 19x - 22y = 0$ (D) $8x^2 + 8y^2 + 19x + 22y = 0$

HB0032

35. The equation of parabolas with same latus-rectum as conic C, is/are-

(A) $y^2 - 6x + 3 = 0$ (B) $y^2 + 6x - 21 = 0$
(C) $y^2 - 6x - 21 = 0$ (D) $y^2 + 6x + 3 = 0$

HB0032

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36. If a hyperbola passes through extremities of minor axis of conic C and its transverse and conjugate axis coincides with the minor and major axis of conic C respectively, and the product of eccentricity of hyperbola and conic C is 1, then -

(A) equation of hyperbola is $\frac{x^2}{36}-\frac{y^2}{12}=-1$.

(B) equation of hyperbola is $\frac{x^2}{9}-\frac{y^2}{3}=-1$.

(C) focus of hyperbola is $(0, 2\sqrt{3})$

(D) focus of hyperbola is $(0, -4\sqrt{3})$

HB0032

[REASONING TYPE]

37. **Statement-1 :** Consider two hyperbola $S \equiv 2x^2 - 4y^2 - 8 = 0$ and $S' \equiv 2x^2 - 4y^2 + 8 = 0$. S and S' are conjugate of each other.

and

Statement-2 : Length of transverse axis and conjugate axis of one of the given hyperbolas are respectively equals to length of conjugate axis and transverse axis of other hyperbola.

(A) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1.

(B) Statement-1 is True, Statement-2 is True ; Statement-2 is NOT a correct explanation for Statement-1.

(C) Statement-1 is True, Statement-2 is False.

(D) Statement-1 is False, Statement-2 is True.

HB0033

38. **Statement-1:** The points of intersection of the tangents at three distinct points A, B, C on the parabola $y^2 = 4x$ can be collinear.

Statement-2: If a line L does not intersect the parabola $y^2 = 4x$, then from every point of the line two tangents can be drawn to the parabola.

(A) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is True, Statement-2 is True ; Statement-2 is NOT a correct explanation for Statement-1

(C) Statement -1 is True, Statement -2 is False

(D) Statement -1 is False, Statement -2 is True

HB0034

39. **Statement-1:** The latus rectum is the shortest focal chord in a parabola of length 4a.

because

Statement-2: As the length of a focal chord of the parabola $y^2 = 4ax$ is $a\left(t+\frac{1}{t}\right)^2$, which is minimum when t = 1.

(A) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is True, Statement-2 is True ; Statement-2 is NOT a correct explanation for Statement-1

(C) Statement -1 is True, Statement -2 is False

(D) Statement -1 is False, Statement -2 is True

HB0035

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40. **Statement-1:** The quadrilateral formed by the pair of tangents drawn from the point (0, 2) to the parabola $y^2 - 2y + 4x + 5 = 0$ and the normals at the point of contact of tangents is a square.

Statement-2: The angle between tangents drawn from the given point to the parabola is 90°.

(A) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is True, Statement-2 is True ; Statement-2 is NOT a correct explanation for Statement-1

(C) Statement -1 is True, Statement -2 is False

(D) Statement -1 is False, Statement -2 is True

HB0036

[MATRIX MATCH TYPE]

41. Match the properties given in **column-I** with the corresponding curves given in the **column-II**.

	Column-I	**Column-II**
(A)	The curve such that product of the distances of any of its tangent from two given points is constant, can be	(P) Circle
		HB0037
(B)	A curve for which the length of the subnormal at any of its point is equal to 2 and the curve passes through (1, 2), can be	(Q) Parabola
		HB0038
(C)	A curve passes through (1, 4) and is such that the segment joining any point P on the curve and the point of intersection of the normal at P with the x-axis is bisected by the y-axis. The curve can be	(R) Ellipse (S) Hyperbola
		HB0039
(D)	A curve passes through (1, 2) is such that the length of the normal at any of its point is equal to 2. The curve can be	
		HB0040

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EXERCISE (O-2)

[STRAIGHT OBJECTIVE TYPE]

1. Let F_1, F_2 are the foci of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and F_3, F_4 are the foci of its conjugate hyperbola. If e_H and e_C are their eccentricities respectively then the statement which holds true is

(A) Their equations of the asymptotes are different.

(B) $e_H > e_C$

(C) Area of the quadrilateral formed by their foci is 50 sq. units.

(D) Their auxiliary circles will have the same equation.

HB0041

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2. For each positive integer n, consider the point P with abscissa n on the curve $y^2 - x^2 = 1$. If d_n represents the shortest distance from the point P to the line y = x then $\lim_{n\to\infty}(n.d_n)$ has the value equal to-

(A) $\frac{1}{2\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{2}}$ (D) 0

HB0042

3. AB is a double ordinate of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ such that ΔAOB (where 'O' is the origin) is an equilateral triangle, then the eccentricity e of the hyperbola satisfies

(A) $e > \sqrt{3}$ (B) $1 < e < \frac{2}{\sqrt{3}}$ (C) $e = \frac{2}{\sqrt{3}}$ (D) $e > \frac{2}{\sqrt{3}}$

HB0043

4. P is a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, N is the foot of the perpendicular from P on the transverse axis. The tangent to the hyperbola at P meets the transverse axis at T . If O is the centre of the hyperbola, then OT.ON is equal to :

(A) e^2 (B) a^2 (C) b^2 (D) b^2/a^2

HB0044

5. Let the major axis of a standard ellipse equals the transverse axis of a standard hyperbola and their director circles have radius equal to 2R and R respectively. If e_1 and e_2 are the eccentricities of the ellipse and hyperbola then the correct relation is

(A) $4e_1^2 - e_2^2 = 6$ (B) $e_1^2 - 4e_2^2 = 2$ (C) $4e_2^2 - e_1^2 = 6$ (D) $2e_1^2 - e_2^2 = 4$

HB0045

6. The tangent to the hyperbola $xy = c^2$ at the point P intersects the x-axis at T and the y-axis at T′. The normal to the hyperbola at P intersects the x-axis at N and the y-axis at N′. The areas of the triangles PNT and PN'T' are Δ and Δ' respectively, then $\frac{1}{\Delta}+\frac{1}{\Delta'}$ is

(A) equal to 1 (B) depends on t (C) depends on c (D) equal to 2

HB0046

7. The chord PQ of the rectangular hyperbola $xy = a^2$ meets the axis of x at A ; C is the mid point of PQ & 'O' is the origin. Then the Δ ACO is :

(A) equilateral (B) isosceles (C) right angled (D) right isosceles.

HB0047

8. The locus of the foot of the perpendicular from the centre of the hyperbola $xy = c^2$ on a variable tangent is :

(A) $(x^2 - y^2)^2 = 4c^2 xy$ (B) $(x^2 + y^2)^2 = 2c^2 xy$

(C) $(x^2 + y^2) = 4c^2 xy$ (D) $(x^2 + y^2)^2 = 4c^2 xy$

HB0048

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9. The equation to the chord joining two points (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy = c^2$ is :

(A) $\frac{x}{x_1 + x_2} + \frac{y}{y_1 + y_2} = 1$

(B) $\frac{x}{x_1 - x_2} + \frac{y}{y_1 - y_2} = 1$

(C) $\frac{x}{y_1 + y_2} + \frac{y}{x_1 + x_2} = 1$

(D) $\frac{x}{y_1 - y_2} + \frac{y}{x_1 - x_2} = 1$

HB0049

10. At the point of intersection of the rectangular hyperbola $xy = c^2$ and the parabola $y^2 = 4ax$ tangents to the rectangular hyperbola and the parabola make an angle θ and ϕ respectively with the axis of X, then

(A) $\theta = \tan^{-1}(-2\tan\phi)$

(B) $\phi = \tan^{-1}(-2\tan\theta)$

(C) $\theta = \frac{1}{2}\tan^{-1}(-\tan\phi)$

(D) $\phi = \frac{1}{2}\tan^{-1}(-\tan\theta)$

HB0050

[MULTIPLE OBJECTIVE TYPE]

11. Which of the following equations in parametric form can represent a hyperbolic profile, where 't' is a parameter.

(A) $x = \frac{a}{2}\left(t + \frac{1}{t}\right)$ & $y = \frac{b}{2}\left(t - \frac{1}{t}\right)$

(B) $\frac{tx}{a} - \frac{y}{b} + t = 0$ & $\frac{x}{a} + \frac{ty}{b} - 1 = 0$

(C) $x = e^t + e^{-t}$ & $y = e^t - e^{-t}$

(D) $x^2 - 6 = 2\cos t$ & $y^2 + 2 = 4\cos^2\frac{t}{2}$

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HB0051

12. Let A(–1,0) and B(2,0) be two points on the x - axis . A point M is moving in xy-plane (other than x - axis) in such a way that $\angle MBA = 2\angle MAB$, then the point M moves along a conic whose

(A) coordinate of vertices are $(\pm 3, 0)$.

(B) length of latus-rectum equals 6.

(C) eccentricity equals 2.

(D) equation of directrices are $x = \pm\frac{1}{2}$.

HB0052

13. Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{3} = 1$ of eccentricity e is confocal with the ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$. Let A,B,C & D are points of intersection of hyperbola & ellipse, then-

(A) $e = \frac{5}{2}$

(B) $e = 2$

(C) A,B,C,D are concyclic points

(D) Number of common tangents of hyperbola & ellipse is 2

HB0053

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14. If the ellipse $4x^2 + 9y^2 + 12x + 12y + 5 = 0$ is confocal with a hyperbola having same principal axes, then -

(A) angle between normals at their each point of intersection is 90º.

(B) centre of the ellipse is $\left(-\frac{3}{2}, -\frac{2}{3}\right)$

(C) distance between foci of the hyperbola is $\frac{2\sqrt{10}}{3}$

(D) ellipse and hyperbola has same length of latus rectum

HB0054

15. If the eccentricity of the ellipse $\frac{x^2}{(\log a)^2} + \frac{y^2}{(\log b)^2} = 1$ $(a > b > 0,\ a,\ b \neq 1)$ is $\frac{1}{\sqrt{2}}$ and 'e' be the eccentricity of the hyperbola $\frac{x^2}{(\log_b a)^2} - y^2 = 1$, then e^2 is greater than (where $\log x = \ell nx$)-

(A) $\frac{3}{2}$ (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) $\frac{5}{4}$

HB0055

[COMPREHENSION TYPE]

Paragraph for question nos. 16 to 18

From a point 'P' three normals are drawn to the parabola $y^2 = 4x$ such that two of them make angles with the abscissa axis, the product of whose tangents is 2. Suppose the locus of the point 'P' is a part of a conic 'C'. Now a circle S = 0 is described on the chord of the conic 'C' as diameter passing through the point (1, 0) and with gradient unity. Suppose (a, b) are the coordinates of the centre of this circle. If L_1 and L_2 are the two asymptotes of the hyperbola with length of its transverse axis 2a and conjugate axis 2b (principal axes of the hyperbola along the coordinate axes) then answer the following questions.

16. Locus of P is a

(A) circle (B) parabola (C) ellipse (D) hyperbola

HB0056

17. Radius of the circle S = 0 is

(A) 4 (B) 5 (C) $\sqrt{17}$ (D) $\sqrt{23}$

HB0056

18. The angle $\alpha \in (0, \pi/2)$ between the two asymptotes of the hyperbola lies in the interval

(A) (0, 15°) (B) (30°, 45°) (C) (45°, 60°) (D) (60°, 75°)

HB0056

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Paragraph for question nos. 19 to 21

A conic C passes through the point (2, 4) and is such that the segment of any of its tangents at any point contained between the co-ordinate axes is bisected at the point of tangency. Let S denotes circle described on the foci F_1 and F_2 of the conic C as diameter.

19. Vertex of the conic C is

(A) (2, 2), (–2, – 2)
(B) $\left(2\sqrt{2}, 2\sqrt{2}\right), \left(-2\sqrt{2}, -2\sqrt{2}\right)$
(C) (4, 4), (–4, – 4)
(D) $\left(\sqrt{2}, \sqrt{2}\right), \left(-\sqrt{2}, -\sqrt{2}\right)$

HB0057

20. Director circle of the conic is

(A) $x^2 + y^2 = 4$
(B) $x^2 + y^2 = 8$
(C) $x^2 + y^2 = 2$
(D) None

HB0057

21. Equation of the circle S is

(A) $x^2 + y^2 = 16$
(B) $x^2 + y^2 = 8$
(C) $x^2 + y^2 = 32$
(D) $x^2 + y^2 = 4$

HB0057

[REASONING TYPE]

22. **Statement-1:** Diagonals of any parallelogram inscribed in an ellipse always intersect at the centre of the ellipse.

Statement-2: Centre of the ellipse is the only point at which two chords can bisect each other and every chord passing through the centre of the ellipse gets bisected at the centre.

(A) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1
(B) Statement-1 is True, Statement-2 is True ; Statement-2 is NOT a correct explanation for Statement-1
(C) Statement -1 is True, Statement -2 is False
(D) Statement -1 is False, Statement -2 is True

HB0058

23. **Statement-1:** If P(2a, 0) be any point on the axis of parabola $y^2 = 4ax$, then the chord QPR, satisfy

$$\frac{1}{(PQ)^2} + \frac{1}{(PR)^2} = \frac{1}{4a^2}.$$

Statement-2 : There exists a point P on the axis of the parabola $y^2 = 4ax$ (other than vertex), such that

$$\frac{1}{(PQ)^2} + \frac{1}{(PR)^2} = \text{constant for all chord QPR of the parabola.}$$

(A) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1
(B) Statement-1 is True, Statement-2 is True ; Statement-2 is NOT a correct explanation for Statement-1
(C) Statement -1 is True, Statement -2 is False
(D) Statement -1 is False, Statement -2 is True

HB0059

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EXERCISE (S-1)

1. Find the equation to the hyperbola whose directrix is $2x + y = 1$, focus (1, 1) & eccentricity $\sqrt{3}$. Find also the length of its latus rectum.

HB0060

2. The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passes through the point of intersection of the lines, $7x + 13y - 87 = 0$ and $5x - 8y + 7 = 0$ & the latus rectum is $32\sqrt{2}/5$. Find 'a' & 'b'.

HB0061

3. For the hyperbola $\frac{x^2}{100} - \frac{y^2}{25} = 1$, prove that

(i) eccentricity $= \sqrt{5}/2$

(ii) $SA.\ S'A = 25$, where S & S′ are the foci & A is the vertex.

HB0062

4. Find the centre, the foci, the directrices, the length of the latus rectum, the length & the equations of the axes of the hyperbola $16x^2 - 9y^2 + 32x + 36y - 164 = 0$.

HB0063

5. If θ_1 & θ_2 are the parameters of the extremities of a chord through (ae, 0) of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then show that $\tan\frac{\theta_1}{2} \cdot \tan\frac{\theta_2}{2} + \frac{e-1}{e+1} = 0$.

HB0064

6. Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

HB0065

7. Tangents are drawn to the hyperbola $3x^2 - 2y^2 = 25$ from the point (0, 5/2). Find their equations.

HB0066

8. A conic C satisfies the differential equation, $(1 + y^2)dx - xy\,dy = 0$ and passes through the point (1,0). An ellipse E which is confocal with C having its eccentricity equal to $\sqrt{2/3}$.

(a) Find the length of the latus rectum of the conic C

(b) Find the equation of the ellipse E.

(c) Find the locus of the point of intersection of the perpendicular tangents to the ellipse E.

HB0067

9. A hyperbola has one focus at the origin and its eccentricity $= \sqrt{2}$ and one of its directrix is $x + y + 1 = 0$. Find the equation to its asymptotes.

HB0068

10. If the lines $x + y + 1 = 0$ and $2x - y + 2 = 0$ are the asymptotes of a hyperbola. If the line $x - 2 = 0$ touches the hyperbola then the equation of the hyperbola is $4(x + y + 1)(2x - y + 2) = \lambda$. Find the value of λ.

HB0069

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11. If C is the centre of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, S, S′ its foci and P a point on it. Prove that SP. S′P = $CP^2 - a^2 + b^2$.

HB0070

12. Chords of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are tangents to the circle drawn on the line joining the foci as diameter. Find the locus of the point of intersection of tangents at the extremities of the chords.

HB0071

13. Let 'p' be the perpendicular distance from the centre C of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ to the tangent drawn at a point R on the hyperbola. If S & S′ are the two foci of the hyperbola, then show that $(RS + RS')^2 = 4a^2\left(1+\frac{b^2}{p^2}\right)$.

HB0072

14. If two points P & Q on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ whose centre is C be such that CP is perpendicular to CQ & a < b, then prove that $\frac{1}{CP^2}+\frac{1}{CQ^2}=\frac{1}{a^2}-\frac{1}{b^2}$.

HB0073

15. Locus of the feet of the perpendicular from centre of the hyperbola $x^2 - 4y^2 = 4$ upon a variable normal to it has the equation, $(x^2 + y^2)^2 (4y^2 - x^2) = \lambda x^2y^2$, find λ.

HB0074

16. Let P (a sec θ, b tan θ) and Q (a sec ϕ, b tan ϕ), where $\theta + \phi = \frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If (h, k) is the point of intersection of the normals at P & Q, then find k.

HB0075

EXERCISE (S-2)

1. Tangent and normal are drawn at the upper end (x_1,y_1) of the latus rectum P with $x_1 > 0$ and $y_1 > 0$, of the hyperbola $\frac{x^2}{4}-\frac{y^2}{12}=1$, intersecting the transverse axis at T and G respectively. Find the area of the triangle PTG.

HB0076

2. Find the equations of the tangents to the hyperbola $x^2 - 9y^2 = 9$ that are drawn from (3, 2). Find the area of the triangle that these tangents form with their chord of contact.

HB0077

3. The normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ drawn at an extremity of its latus rectum is parallel to an asymptote. Show that the eccentricity is equal to the square root of $(1+\sqrt{5})/2$.

HB0078

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4. An ellipse and a hyperbola have their principal axes along the coordinate axes and have a common foci separated by a distance $2\sqrt{13}$, the difference of their focal semi axes is equal to 4. If the ratio of their eccentricities is 3/7. Find the equation of these curves.

HB0079

5. From the centre C of the hyperbola $x^2 - y^2 = 9$, CM is drawn perpendicular to the tangent at any point of the curve, meeting the tangent at M and the curve at N. Find the value of the product (CM)(CN).

HB0080

6. Prove that the part of the tangent at any point of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ intercepted between the point of contact and the transverse axis is a harmonic mean between the lengths of the perpendicular drawn from the foci on the normal at the same point.

HB0081

7. Ascertain the co-ordinates of the two points Q & R, where the tangent to the hyperbola $\frac{x^2}{45} - \frac{y^2}{20} = 1$ at the point P(9, 4) intersects the two asymptotes. Finally prove that P is the middle point of QR. Also compute the area of the triangle CQR where C is the centre of the hyperbola.

HB0082

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8. The tangent at P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets one of the asymptote in Q. If the locus of the mid point of PQ is a hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \lambda$, find the value of 4λ.

HB0083

9. Through any point P of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ a line QPR is drawn with a fixed gradient m, meeting the asymptotes in Q & R. Show that the product, (QP) (PR) $= \frac{a^2b^2(1+m^2)}{b^2 - a^2m^2}$.

HB0084

10. Tangents are drawn from the point (α, β) to the hyperbola $3x^2 - 2y^2 = 6$ and are inclined at angles θ and ϕ to the x –axis. If $\tan\theta . \tan\phi = 2$, prove that $\beta^2 = 2\alpha^2 - 7$.

HB0085

EXERCISE (JM)

1. The equation of the hyperbola whose foci are (–2,0) and (2, 0) and eccentricity is 2 is given by :

[AIEEE-2011]

(1) $-3x^2 + y^2 = 3$ (2) $x^2 - 3y^2 = 3$ (3) $3x^2 - y^2 = 3$ (4) $-x^2 + 3y^2 = 3$

HB0086

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2. A tangent to the hyperbola $\frac{x^2}{4}-\frac{y^2}{2}=1$ meets x-axis at P and y-axis at Q. Lines PR and QR are drawn such that OPRQ is a rectangle (where O is the origin). Then R lies on : **[JEE-Main (On line)-2013]**

(1) $\frac{2}{x^2}-\frac{4}{y^2}=1$ (2) $\frac{4}{x^2}-\frac{2}{y^2}=1$ (3) $\frac{4}{x^2}+\frac{2}{y^2}=1$ (4) $\frac{2}{x^2}+\frac{4}{y^2}=1$

HB0087

3. A common tangent to the conics $x^2 = 6y$ and $2x^2 - 4y^2 = 9$ is : **[JEE-Main (On line)-2013]**

(1) $x + y = \frac{9}{2}$ (2) $x + y = 1$ (3) $x - y = \frac{3}{2}$ (4) $x - y = 1$

HB0088

4. The eccentricity of the hyperbola whose length of the latus rectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is : **[JEE (Main) 2016]**

(1) $\sqrt{3}$ (2) $\frac{4}{3}$ (3) $\frac{4}{\sqrt{3}}$ (4) $\frac{2}{\sqrt{3}}$

HB0089

5. A hyperbola passes through the point $P(\sqrt{2},\sqrt{3})$ and has foci at (± 2, 0). Then the tangent to this hyperbola at P also passes through the point : **[JEE (Main) 2017]**

(1) $(-\sqrt{2},-\sqrt{3})$ (2) $(3\sqrt{2},2\sqrt{3})$ (3) $(2\sqrt{2},3\sqrt{3})$ (4) $(\sqrt{3},\sqrt{2})$

HB0090

6. Tangents are drawn to the hyperbola $4x^2 - y^2 = 36$ at the point P and Q. If these tangents intersect at the point T(0, 3) then the area (in sq. units) of ΔPTQ is - **[JEE (Main) 2018]**

(1) $54\sqrt{3}$ (2) $60\sqrt{3}$ (3) $36\sqrt{5}$ (4) $45\sqrt{5}$

HB0091

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7. Let $S=\left\{(x,y)\in R^2 : \frac{y^2}{1+r}-\frac{x^2}{1-r}=1\right\}$, where $r \neq \pm 1$. Then S represents : **[JEE (Main)-2019]**

(1) A hyperbola whose eccentricity is $\frac{2}{\sqrt{r+1}}$, where $0 < r < 1$.

(2) An ellipse whose eccentricity is $\frac{1}{\sqrt{r+1}}$, where $r > 1$

(3) A hyperbola whose eccentricity is $\frac{2}{\sqrt{1-r}}$, when $0 < r < 1$.

(4) An ellipse whose eccentricity is $\sqrt{\frac{2}{r+1}}$, when $r > 1$

HB0092

8. Equation of a common tangent to the parabola $y^2 = 4x$ and the hyperbole $xy = 2$ is : **[JEE (Main)-2019]**

(1) $x + 2y + 4 = 0$ (2) $x - 2y + 4 = 0$

(3) $x + y + 1 = 0$ (4) $4x + 2y + 1 = 0$

HB0093

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9. If a hyperbola has length of its conjugate axis equal to 5 and the distance between its foci is 13, then the eccentricity of the hyperbola is :- **[JEE (Main)-2019]**

(1) 2 (2) $\frac{13}{6}$ (3) $\frac{13}{8}$ (4) $\frac{13}{12}$

HB0094

10. If the line $y = mx + 7\sqrt{3}$ is normal to the hyperbola $\frac{x^2}{24} - \frac{y^2}{18} = 1$, then a value of m is

[JEE (Main)-2019]

(1) $\frac{\sqrt{5}}{2}$ (2) $\frac{3}{\sqrt{5}}$ (3) $\frac{2}{\sqrt{5}}$ (4) $\frac{\sqrt{15}}{2}$

HB0095

11. If a directrix of a hyperbola centred at the origin and passing through the point $\left(4, -2\sqrt{3}\right)$ is $5x = 4\sqrt{5}$ and its eccentricity is e, then : **[JEE (Main)-2019]**

(1) $4e^4 - 24e^2 + 35 = 0$ (2) $4e^4 + 8e^2 - 35 = 0$

(3) $4e^4 - 12e^2 - 27 = 0$ (4) $4e^4 - 24e^2 + 27 = 0$

HB0096

12. Let P be the point of intersection of the common tangents to the parabola $y^2 = 12x$ and the hyperbola $8x^2 - y^2 = 8$. If S and S' denote the foci of the hyperbola where S lies on the positive x-axis then P divides SS' in a ratio : **[JEE (Main)-2019]**

(1) 5:4 (2) 14:13 (3) 2:1 (4) 13:11

HB0097

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13. If a hyperbola passes through the point P(10,16) and it has vertices at (±6,0), then the equation of the normal to it at P is **[JEE (Main)-2020]**

(1) $x + 2y = 42$ (2) $3x + 4y = 94$ (3) $2x + 5y = 100$ (4) $x + 3y = 58$

HB0098

14. If e_1 and e_2 are the eccentricities of the ellipse, $\frac{x^2}{18} + \frac{y^2}{4} = 1$ and the hyperbola, $\frac{x^2}{9} - \frac{y^2}{4} = 1$ respectively and (e_1, e_2) is a point on the ellipse, $15x^2 + 3y^2 = k$, then k is equal to : **[JEE (Main)-2020]**

(1) 15 (2) 14 (3) 17 (4) 16

HB0099

EXERCISE (JA)

1. Consider a branch of the hyperbola, $x^2 - 2y^2 - 2\sqrt{2}\,x - 4\sqrt{2}\,y - 6 = 0$ with vertex at the point A. Let B be one of the end points of its latus rectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is **[JEE 2008, 3]**

(A) $1 - \sqrt{\frac{2}{3}}$ (B) $\sqrt{\frac{3}{2}} - 1$ (C) $1 + \sqrt{\frac{2}{3}}$ (D) $\sqrt{\frac{3}{2}} + 1$

HB0100

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2. The locus of the orthocentre of the triangle formed by the lines $(1 + p)x - py + p(1 + p) = 0$, $(1 + q)x - qy + q(1 + q) = 0$ and $y = 0$, where $p \neq q$, is **[JEE 2009, 3]**

(A) a hyperbola (B) a parabola (C) an ellipse (D) a straight line

HB0101

3. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then **[JEE 2009, 4]**

(A) Equation of ellipse is $x^2 + 2y^2 = 2$ (B) The foci of ellipse are $(\pm 1, 0)$

(C) Equation of ellipse is $x^2 + 2y^2 = 4$ (D) The foci of ellipse are $(\pm \sqrt{2}, 0)$

HB0102

4. Match the conics in **Column I** with the statements/expressions in **Column II.** **[JEE 2009, 8]**

	Column I		**Column II**
(A)	Circle	(p)	The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$ **HB0103**
(B)	Parabola	(q)	Points z in the complex plane satisfying $\lvert z + 2\rvert - \lvert z - 2\rvert = \pm 3$ **HB0104**
(C)	Ellipse	(r)	Points of the conic have parametric representation $x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right)$, $y = \frac{2t}{1+t^2}$ **HB0105**
(D)	Hyperbola	(s)	The eccentricity of the conic lies in the interval $1 \le x < \infty$ **HB0106**
		(t)	Points z in the complex plane satisfying $\text{Re}(z + 1)^2 = \lvert z\rvert^2 + 1$ **HB0107**

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Comprehension : **[JEE 2010, 3+3]**

The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B.

5. Equation of a common tangent with positive slope to the circle as well as to the hyperbola is -

(A) $2x - \sqrt{5}y - 20 = 0$ (B) $2x - \sqrt{5}y + 4 = 0$ (C) $3x - 4y + 8 = 0$ (D) $4x - 3y + 4 = 0$

HB0108

6. Equation of the circle with AB as its diameter is -

(A) $x^2 + y^2 - 12x + 24 = 0$ (B) $x^2 + y^2 + 12x + 24 = 0$

(C) $x^2 + y^2 + 24x - 12 = 0$ (D) $x^2 + y^2 - 24x - 12 = 0$

HB0108

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7. The line 2x + y =1 is tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is **[JEE 2010, 3]**

HB0109

8. Let the eccentricity of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then - **[JEE 2011, 4]**

(A) the equation of the hyperbola is $\frac{x^2}{3}-\frac{y^2}{2}=1$ (B) a focus of the hyperbola is (2,0)

(C) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$ (D) the equation of the hyperbola is $x^2-3y^2=3$

HB0110

9. Let P(6, 3) be a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If the normal at the point P intersects the x-axis at (9,0), then the eccentricity of the hyperbola is - **[JEE 2011, 3]**

(A) $\sqrt{\frac{5}{2}}$ (B) $\sqrt{\frac{3}{2}}$ (C) $\sqrt{2}$ (D) $\sqrt{3}$

HB0111

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10. Tangents are drawn to the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$, parallel to the straight line 2x – y = 1. The points of contact of the tangents on the hyperbola are **[JEE 2012, 4M]**

(A) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (B) $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

(C) $\left(3\sqrt{3}, -2\sqrt{2}\right)$ (D) $\left(-3\sqrt{3}, 2\sqrt{2}\right)$

HB0112

11. Consider the hyperbola H : $x^2 - y^2 = 1$ and a circle S with center $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the x-axis at point M. If (l, m) is the centroid of the triangle ΔPMN, then the correct expression(s) is(are) **[JEE 2015, 4M, –0M]**

(A) $\frac{dl}{dx_1}=1-\frac{1}{3x_1^2}$ for $x_1 > 1$ (B) $\frac{dm}{dx_1}=\frac{x_1}{3\left(\sqrt{x_1^2-1}\right)}$ for $x_1 > 1$

(C) $\frac{dl}{dx_1}=1+\frac{1}{3x_1^2}$ for $x_1 > 1$ (D) $\frac{dm}{dy_1}=\frac{1}{3}$ for $y_1 > 0$

HB0113

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12. If 2x – y + 1 = 0 is tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{16}=1$, then which of the following CANNOT be sides of a right angled triangle ? **[JEE(Advanced)-2017, 4(–2)]**

(A) 2a, 4, 1 (B) 2a, 8, 1 (C) a, 4, 1 (D) a, 4, 2

Column 1, 2 and 3 contain conics, equation of tangents to the conics and points of contact, respectively.

Column 1	Column 2	Column 3
(I) $x^2 + y^2 = a^2$	(i) $my = m^2x + a$	(P) $\left(\frac{a}{m^2},\frac{2a}{m}\right)$
(II) $x^2 + a^2y^2 = a^2$	(ii) $y = mx + a\sqrt{m^2+1}$	(Q) $\left(\frac{-ma}{\sqrt{m^2+1}},\frac{a}{\sqrt{m^2+1}}\right)$
(III) $y^2 = 4ax$	(iii) $y = mx + \sqrt{a^2m^2-1}$	(R) $\left(\frac{-a^2m}{\sqrt{a^2m^2+1}},\frac{1}{\sqrt{a^2m^2+1}}\right)$
(IV) $x^2 - a^2y^2 = a^2$	(iv) $y = mx + \sqrt{a^2m^2+1}$	(S) $\left(\frac{-a^2m}{\sqrt{a^2m^2-1}},\frac{-1}{\sqrt{a^2m^2-1}}\right)$

HB0114

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13. The tangent to a suitable conic (Column 1) at $\left(\sqrt{3},\frac{1}{2}\right)$ is found to be $\sqrt{3}x+2y=4$, then which of the following options is the only **CORRECT** combination ? **[JEE(Advanced)-2017, 3(–1)]**

(A) (II) (iii) (R) (B) (IV) (iv) (S) (C) (IV) (iii) (S) (D) (II) (iv) (R)

HB0115

14. If a tangent to a suitable conic (Column 1) is found to be y = x + 8 and its point of contact is (8,16), then which of the following options is the only **CORRECT** combination ? **[JEE(Advanced)-2017, 3(–1)]**

(A) (III) (i) (P) (B) (III) (ii) (Q) (C) (II) (iv) (R) (D) (I) (ii) (Q)

HB0115

15. For $a=\sqrt{2}$, if a tangent is drawn to a suitable conic (Column 1) at the point of contact (–1,1), then which of the following options is the only **CORRECT** combination for obtaining its equation ? **[JEE(Advanced)-2017, 3(–1)]**

(A) (II) (ii) (Q) (B) (III) (i) (P) (C) (I) (i) (P) (D) (I) (ii) (Q)

HB0115

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16. Let H : $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where a > b > 0, be a hyperbola in the xy-plane whose conjugate axis LM subtends an angle of 60° at one of its vertices N. Let the area of the triangle LMN be $4\sqrt{3}$.

LIST-I	LIST-II
P. The length of the conjugate axis of H is	**1.** 8
Q. The eccentricity of H is	**2.** $\frac{4}{\sqrt{3}}$
R. The distance between the foci of H is	**3.** $\frac{2}{\sqrt{3}}$
S. The length of the latus rectum of H is	**4.** 4

The correct option is :

(A) **P → 4; Q → 2, R → 1; S → 3**
(B) **P → 4; Q → 3; R → 1; S → 2**
(C) **P → 4; Q → 1, R → 3; S → 2**
(D) **P → 3; Q → 4; R → 2; S → 1**

[JEE(Advanced)-2018, 3(–1)]

HB0116

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ANSWER KEY

HYPERBOLA

EXERCISE (O-1)

1. **(a)** (–4,–1) ; **(b)** $\frac{5}{4}$; **(c)** (1,–1), (–9,–1); **(d)** 5x + 4 = 0, 5x + 36 = 0, **(e)** $\frac{9}{2}$; **(f)** $(x+4)^2+(y+1)^2=16$; **(g)** $(x+4)^2+(y+1)^2=7$

2. B	**3.** A	**4.** B	**5.** B	**6.** B	**7.** A	**8.** C	**9.** B
10. A	**11.** B	**12.** C	**13.** B	**14.** A	**15.** B	**16.** D	**17.** B
18. A	**19.** A	**20.** C	**21.** B	**22.** B	**23.** B	**24.** A	
25. A,B,C,D		**26.** A,B,D	**27.** C,D	**28.** B,C	**29.** C,D	**30.** A,B,C,D	
31. D	**32.** C	**33.** D	**34.** A,B,C,D		**35.** A,B,C,D		**36.** A,D
37. B	**38.** D	**39.** A	**40.** D	**41.** (A) R,S, (B) Q, (C) R, (D) P			

EXERCISE (O-2)

1. C	**2.** A	**3.** D	**4.** B	**5.** C	**6.** C	**7.** B	**8.** D
9. A	**10.** A	**11.** A,C,D	**12.** B,C,D	**13.** B,C	**14.** A,B,C	**15.** B,C,D	**16.** B
17. A	**18.** D	**19.** B	**20.** D	**21.** C	**22.** A	**23.** A	

EXERCISE (S-1)

1. $\sqrt{\frac{48}{5}}$ **2.** $a^2 = 25/2$; $b^2 = 16$ **4.** (–1, 2) ; (4, 2) & (–6, 2) ; 5x – 4 = 0 & 5x + 14 = 0; $\frac{32}{3}$; 6 ; 8; y – 2 = 0 ; x + 1 = 0 ; 4x – 3y + 10 = 0 ; 4x + 3y – 2 = 0.

6. $x + y \pm 3\sqrt{3} = 0$ **7.** 3x + 2y – 5 = 0 ; 3x – 2y + 5 = 0 **8.** (a) 2; (b) $\frac{x^2}{3}+\frac{y^2}{1}=1$; (c) $x^2+y^2=4$

9. x + 1 = 0 only y + 1 = 0 **10.** 81 **12.** $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}$ **15.** 25 **16.** $-\left(\frac{a^2+b^2}{b}\right)$

EXERCISE (S-2)

1. 45 **2.** $y=\frac{5}{12}x+\frac{3}{4}$; x – 3 = 0 ; 8 sq. unit **4.** $\frac{x^2}{49}+\frac{y^2}{36}=1$; $\frac{x^2}{9}-\frac{y^2}{4}=1$

5. 9 **7.** (15, 10) and (3, –2) and 30 sq. units **8.** 3

EXERCISE (JM)

1. 3	**2.** 2	**3.** 3	**4.** 4	**5.** 3	**6.** 4	**7.** 4	**8.** 1
9. 4	**10.** 3	**11.** 1	**12.** 1	**13.** 3	**14.** 4		

EXERCISE (JA)

1. B	**2.** D	**3.** A,B	**4.** (A) p, (B) s,t; (C) r; (D) q,s			**5.** B
6. A	**7.** 2	**8.** B,D	**9.** B	**10.** A,B	**11.** A,B,D	**12.** B,C,D
13. D	**14.** A	**15.** D	**16.** B			

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HEIGHT AND DISTANCES

1. INTRODUCTION :

One of the important application of trigonometry is in finding the height and distance of the point which are not directly measurable. This is done with the help of trigonometric ratios.

2. ANGLES OF ELEVATION AND DEPRESSION :

Let OP be a horizontal line in the vertical plane in which an object R is given and let OR be joined.

Fig. (a) Fig. (b)

In Fig. (a), where the object R is above the horizontal line OP, the angle POR is called the angle of elevation of the object R as seen from the point O. In Fig. (b) where the object R is below the horizontal line OP, the angle POR is called the angle of depression of the object R as seen from the point O.

Remark :

Unless stated to the contrary, it is assumed that the height of the observer is neglected, and that the angles of elevation are measured from the ground.

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Ex.1 Find the angle of elevation of the sum when the length of shadow of a vertical pole is equal to its height.

Sol. Let height of the pole AB = h and

length of the shadow of the Pole (AC) = h

In ΔABC $\tan\theta = \frac{AB}{AC} = \frac{h}{h} = 1 \Rightarrow \tan\theta = 1$

$\Rightarrow \tan\theta = \tan 45° \Rightarrow \theta = 45°$

Ex.2 The shadow of the tower standing on a level ground is found to be 60 metres longer when the sun's altitude is 30° than when it is 45°. The height of the tower is-

(1) 60 m (2) $30(\sqrt{3}-1)$m (3) $60\sqrt{3}$ m (4) $30(\sqrt{3}+1)$ m.

Sol.(4) AC = h cot 30° = $\sqrt{3}$ h

AB = h cot 45° = h

$\therefore$ BC = AC – AB = h $(\sqrt{3}-1) \Rightarrow 60 = h(\sqrt{3}-1)$

$\therefore$ $h = \frac{60}{\sqrt{3}-1} = \frac{60(\sqrt{3}+1)}{3-1} = 30(\sqrt{3}+1)$

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Ex.3 The angle of elevation of the tower observed from each of the three point A,B,C on the ground, forming a triangle is the same angle α . If R is the circum - radius of the triangle ABC, then the height of the tower is -

(1) R sin α (2) R cos α (3) R cot α (4) R tan α

Sol.(4) The tower makes equal angles at the vertices of the triangle, therefore foot of the tower is at the circumcentre.

From Δ OCP, OP is perpendicular to OC.

$\angle OCP = \alpha$

so $\tan\alpha = \dfrac{OP}{OA} \Rightarrow OP = OA \tan\alpha$

$OP = R \tan\alpha$

3. SOME USEFUL RESULTS :

- In a triangle ABC,

$\sin\theta = \dfrac{p}{h}$, $\cos\theta = \dfrac{b}{h}$, $\tan\theta = \dfrac{P}{b}$

- In any triangle ABC,

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ [By sine rule]

or cosine formula

i.e. $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$; $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$, $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

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- In any triangle ABC

if BD : DC = m : n and $\angle BAD = \alpha$

$\angle CAD = \beta$ and $\angle ADC = \theta$,

then $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

- In a triangle ABC, if DE | | AB

then, $\dfrac{AB}{DE} = \dfrac{BC}{DC}$

- In a triangle the internal bisector of an angle divides the opposite side in the ratio of the arms of the angle

$\therefore \dfrac{BD}{DC} = \dfrac{AB}{AC}$

- In an isosceles triangle the median is perpendicular to the base

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SOLVED EXAMPLES

Ex.1 A tower subtends an angle of 30° at a point on the same level as its foot, and at a second point h m above the first, the depression of the foot of tower is 60°. The height of the tower is.

(1) h m (2) 3h m (3) $\sqrt{3}$ h m (4) $\frac{h}{3}$ m.

Sol.(4) Let OP be the tower of height x.,A the point on the same level as the foot O of the tower and B be the point h m above A (see Fig.) Then $\angle$ AOB = 60° and $\angle$ PAO = 30°. From right-angled triangle AOP, we have

$$OA = x \cot 30°$$

and from right-angled triangle OAB, we have

$$OA = h \cot 60°$$

Therefore, from (1) and (2), we get

$$x \cot 30° = h \cot 60°$$

$$\sqrt{3}\,x = \frac{1}{\sqrt{3}}h \Rightarrow x = \frac{1}{3}h$$

Ex.2 At a point on level ground, the angle of elevation of a vertical tower is found to be such that its tangent is $\frac{5}{12}$. On walking 192 metres towards the tower, the tangent of the angle of elevation is $\frac{3}{4}$. Find the height of the tower.

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Sol. Let AB be the tower and let the angle of elevation of its top at C be α. Let D be a point at a distance of 192 metres from C such that the angle of elevation of the top of the tower at D be β.

Let h be the height of the tower and AD = x,

It is given that $\tan\alpha = \frac{5}{12}$ and $\tan\beta = \frac{3}{4}$.

In ΔABC, we have

$$\tan\alpha = \frac{AB}{AC} \Rightarrow \tan\alpha = \frac{h}{192+x} \Rightarrow \frac{5}{12} = \frac{h}{192+x} \quad \text{..... (i)}$$

In ΔABD, we have

$$\tan\beta = \frac{AB}{AD} \Rightarrow \tan\beta = \frac{h}{x} \Rightarrow \frac{3}{4} = \frac{h}{x} \quad \text{..... (ii)}$$

We have to find h. This means that we have to eliminate x from (i) and (ii).

From (ii), we have $3x = 4h \Rightarrow x = \frac{4h}{3}$

Substituting this value of x in (i), we get

$$\frac{5}{12} = \frac{h}{192 + 4h/3} \Rightarrow 5\left(192 + \frac{4h}{3}\right) = 12h$$

$$\Rightarrow 5(576 + 4h) = 36h \Rightarrow 2880 + 20h = 36h$$

$$\Rightarrow 16h = 2880 \Rightarrow h = \frac{2880}{16} = 180$$

Hence, height of tower = 180 metres.

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Ex.3 Let α be the solution of $16^{\sin^2\theta}+16^{\cos^2\theta}=10$ in $(0, \pi/4)$. If the shadow of a vertical pole is $\frac{1}{\sqrt{3}}$ of its height, then then the altitude of the sun is-

(1) α (2) $\frac{\alpha}{2}$ (3) 2α (4) $\frac{\alpha}{3}$

Sol. We have $16^{\sin^2\theta}+16^{\cos^2\theta}=10$

$\Rightarrow\ 16^{\sin^2\theta}+16^{1-\sin^2\theta}=10 \Rightarrow x+\frac{16}{x}=10,$ where $x=16^{\sin^2\theta}$

$\Rightarrow\ x=2, 8 \Rightarrow 16^{\sin^2\theta}=2, 8$

$\Rightarrow\ 2^{4\sin^2\theta}=2, 2^3$

$\Rightarrow\ 4\sin^2\theta=2, 3$

$\Rightarrow\ \sin^2\theta=\frac{1}{2}, \left(\frac{\sqrt{3}}{2}\right)^2 \Rightarrow \theta=\frac{\pi}{6}, \frac{\pi}{3}$

$\therefore\ \alpha=\frac{\pi}{6}$

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Let the altitude of the sun be θ. Then,

$\tan\theta=\frac{h}{\frac{h}{\sqrt{3}}} \Rightarrow \tan\theta=\sqrt{3} \Rightarrow \theta=\frac{\pi}{3} \Rightarrow \theta=2\alpha$

Ex.4 A vertical lamp-post of height 9 metres stands at the corner of a rectangular field. The angle of elevation of its top from the farthest corner is 30°, while from another corner it is 45°. The area of the field is-

(1) $81\sqrt{2}\ m^2$ (2) $9\sqrt{2}\ m^2$ (3) $81\sqrt{3}\ m^2$ (4) $9\sqrt{3}\ m^2$

Sol. Let AP be the lamp-post of 9 m standing at corner A of the rectangular field ABCD.

In Δ's BAP and CAP, we have

$\tan 45° = \frac{PA}{BA}$ and $\tan 30° = \frac{PA}{AC}$

$\Rightarrow\ BA = 9\ m$ and $AC = 9\sqrt{3}\ m$

$\therefore\ BC=\sqrt{AC^2-AB^2}=\sqrt{243-81}=\sqrt{162}=9\sqrt{2}\ m$

Hence, area of the field = $AB \times BC = 9\times 9\sqrt{2}\ m^2 = 81\sqrt{2}\ m^2$

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Ex.5 A vertical tower stands on a horizontal plane and is surmounted by a vertical flag staff of height h. At a point on the plane, the angle of elevation of the bottom and the top of the flag staff are α and β respectively. Prove that the height of tower is $\frac{h\tan\alpha}{\tan\beta-\tan\alpha}$.

Sol. Let AB be the tower and BC be the flag staff. Let O be a point on the plane containing the foot of the tower such that the angles of elevation of the bottom B and top C of the flagstaff at O are α and β respectively. Let OA = x metres, AB = y metres and BC = h metres.

In ΔOAB, we have

$$\cot\alpha = \frac{OA}{AB} \Rightarrow \cot\alpha = \frac{x}{y}$$

$$\Rightarrow x = y\cot\alpha \quad \text{...(i)}$$

In ΔOAC, we have

$$\cot\beta = \frac{x}{y+h}$$

$$\Rightarrow x = (y+h)\cot\beta \quad \text{...(ii)}$$

Equating the values of x from (i) and (ii), we get

$y\cot\alpha = (y+h)\cot\beta$

$\Rightarrow y\cot\alpha - y\cot\beta = h\cot\beta$

$\Rightarrow y(\cot\alpha - \cot\beta) = h\cot\beta$

$$\Rightarrow y = \frac{h\cot\beta}{\cot\alpha-\cot\beta} \Rightarrow y = \frac{h/\tan\beta}{\frac{1}{\tan\alpha}-\frac{1}{\tan\beta}} = \frac{h\tan\alpha}{\tan\beta-\tan\alpha}$$

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Ex.6 A spherical ball of diameter-δ subtends an angle α at the eye of an observer when the elevation of its centre is β. Prove that the height of the centre of the ball is $\frac{1}{2}\delta\sin\beta\,\text{cosec}\left(\frac{\alpha}{2}\right)$.

Sol. O is the position of eye.

As is clear from figure, from ΔODC,

$$OC = \frac{h}{\sin\beta}$$

From ΔOAC,

$$\sin\frac{\alpha}{2} = \frac{CA}{OC} = \frac{\frac{\delta}{2}}{h/\sin\beta}$$

$$\Rightarrow h = \frac{1}{2}\delta\sin\beta.\,\text{cosec}\,\frac{\alpha}{2}.$$

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CHECK YOUR GRASP — HEIGHTS AND DISTANCE — EXERCISE-I

1. The angle of elevation of the top of a tower from a point 20 metre away from its base is 45°. Then height of the tower is-

(1) 10 m (2) 20 m (3) 40 m (4) $20\sqrt{3}$ m

HD0001

2. At a point 15 metre away from the base of a 15 metre high house, the angle of elevation of the top is-

(1) 45° (2) 30° (3) 60° (4) 90°

HD0002

3. An aeroplane flying at a height 300 metre above the ground passes vertically above another plane at an instant when the angles of elevation of the two planes from the same point on the ground are 60° and 45° respectively. Then the height of the lower plane from the ground in metres is-

(1) $100\sqrt{3}$ (2) $100/\sqrt{3}$

(3) 50 (4) $150(\sqrt{3}+1)$.

HD0003

4. If the elevation of the sun is 30° then the length of the shadow cast by a tower of 150 metres height is-

(1) $75\sqrt{3}$m (2) $200\sqrt{3}$m

(3) $150\sqrt{3}$m (4) None of these

HD0004

5. From the top of the cliff 300 metres heigh, the top of a tower was observed at an angle of depression 30° and from the foot of the tower the top of the cliff was observed at an angle of elevation 45°. The height of the tower is -

(1) $50(3-\sqrt{3})$m (2) $200(3-\sqrt{3})$m

(3) $100(3-\sqrt{3})$m (4) None of these

HD0005

6. The angles of elevation of the top of a tower at the top and the foot of a pole of height 10 m are 30° and 60° respectively. The height of the tower is-

(1) 10 m (2) 15 m

(3) 20 m (4) None of these

HD0006

7. A person, standing on the bank of a river, observes that the angle subtended by a tree on the opposite bank is 60°; when he retreates 20 m from the bank, he finds the angle to be 30°. The height of the tree and the breadth of the river are -

(1) $10\sqrt{3}$ m, 10 m (2) 10 m ; $10\sqrt{3}$ m

(3) 20 m, 30m (4) None of these

HD0007

8. From the top of a light house 60 m high with its base at sea level the angle of depression of a boat is 15°. The distance of the boat from the light house is-

(1) $60\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$m (2) $60\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$m

(3) $30\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$m (4) $30\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$m

HD0008

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9. The angle of elevation of the sun when the length of the shadow of a pole is $\sqrt{3}$ times the height of the pole will be-

(1) 30° (2) 60° (3) 90° (4) 45°

HD0009

10. If the length of the shadow of a vertical pole on the horizontal ground is equal to its height, find the angle of elevation of the sun -

(1) 60° (2) 30° (3) 45° (4) 90°

HD0010

11. The angle of elevation of the top of a tower from a point 20m away from its base is 60°. The height of the tower is -

(1) 10 m (2) $20/\sqrt{3}$ m (3) 40 m (4) $20\sqrt{3}$ m

HD0011

12. A flag staff on the top of the tower 80 meter high, subtends an angle $\tan^{-1}\left(\frac{1}{9}\right)$ at a point on the ground 100 meters away from the foot of the tower. Find the height of the flag-staff -

(1) 20 m (2) 30 m (3) 25 m (4) 35 m

HD0012

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13. A person walking along a straight road observes that a two points 1 km apart, the angles of elevation of a pole in front of him are 30° and 75°. The height of the pole is -

(1) $250\left(\sqrt{3}+1\right)$ m

(2) $250\left(\sqrt{3}-1\right)$ m

(3) $225\left(\sqrt{2}-1\right)$ m

(4) $225\left(\sqrt{2}+1\right)$ m

HD0013

14. An observer in a boat finds that the angle of elevation of a tower standing on the top of a cliff is 60° and that of the top of cliff is 30°. If the height of the tower be 60 meters, then the height of the cliff is -

(1) 30 m (2) $60\sqrt{3}$ m

(3) $20\sqrt{3}$ m (4) None of these

HD0014

15. ABCD is a square plot. The angle of elevation of the top of a pole standing at D from A and C is 30° and that from B is θ, then tan θ is equal to -

(1) $\sqrt{6}$ (2) $1/\sqrt{6}$

(3) $\sqrt{3}/\sqrt{2}$ (4) $\sqrt{2}/\sqrt{3}$

HD0015

16. The angle of elevation of a ladder against a wall is 58° and the length of foot of the ladder is 9.6 m from the wall. Then the length of the ladder is -

[cos 58° = 0.5299]

(1) 18.11 m (2) 16.11 m

(3) 17.11 m (4) 19.11 m

HD0016

17. From the top of a tower, the angle of depression of a point P on the ground is 30°. If the distance of the point P from the tower be 24 meters then height of the tower is.

(1) 12 m (2) $8\sqrt{3}$ m

(3) $24\sqrt{3}$ m (4) $12\sqrt{3}$ m

HD0017

18. A tower subtends an angle of 30° at a point on the same level as the foot of the tower. At a second point, h metre above first, point the depression of the foot of the tower is 60°, the horizontal distance of the tower from the points is

(1) h cos 60° (2)(h/3) cot 30°

(3) (h/3) cot 60° (4) h cot 30°

HD0018

19. From a point on the ground 100 m away from the base of a building, the angle of elevation of the top of the building is 60°. Which of the following is the best approximation for the height of the building-

(1) 172 m (2) 173 m

(3) 174 m (4) 175 m

HD0019

20. A kite is flying with the string inclined at 75° to the horizon. If the length of the string is 25 m, the height of the kite is-

(1) (25/2) $(\sqrt{3}-1)$

(2) (25/4) $(\sqrt{3}+1)$

(3) (25/4) $(\sqrt{3}+1)^2$

(4) (25/4) $(\sqrt{6}+\sqrt{2})$

HD0020

21. If a flagstaff 6 metres high placed on the top of a tower throws a shadow of $2\sqrt{3}$ metres along the ground then the angle (in degrees) that the sun makes with the ground is-

(1) 15° (2) 30°

(3) 60° (4) $\tan^{-1}2\sqrt{3}$

HD0021

22. A 6-ft tall man finds that the angle of elevation of the top of a 24-ft-high pillar and the angle of depression of its base are complementary angles. The distance of the man from the pillar is-

(1) $2\sqrt{3}$ ft (2) $8\sqrt{3}$ ft

(3) $6\sqrt{3}$ ft (4) None of these

HD0022

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23. A flagstaff stands vertically on a pillar, the height of the flagstaff being double the height of the pillar. A man on the ground at a distance finds that both the pillar and the flagstaff subtend equal angles at his eyes. The ratio of the height of the pillar and the distance of the man from the pillar is-

(1) $\sqrt{3}:1$ (2) $1:3$

(3) $1:\sqrt{3}$ (4) $\sqrt{3}:2$

HD0023

24. The shadow of a tower of height $(1+\sqrt{3})$ metre standing on the ground is found to be 2 metre longer when the sun's elevation is 30°, than when the sun's elevation was -

(1) 30° (2) 45° (3) 60° (4) 75°

HD0024

25. The angle of elevation of the top of an incomplete vertical pillar at a horizontal distance of 50 m from its base is 45°. If the angle of elevation of the top of the complete pillar the same point is to be 60°, then the height of the incomplete pillar is to be increased by-

(1) $50(\sqrt{3}-1)$ m (2) $50(\sqrt{3}+1)$ m

(3) 50 m (4) $25\sqrt{2}$ m.

HD0025

26. A vertical tower stands on a declivity which is inclined at 15° to the horizon. From the foot of the tower a man ascends the declivity for 80 feet and then finds that tower subtends an angle of 30°. The height of the tower is-

(1) $20(\sqrt{6}-\sqrt{2})$

(2) $40(\sqrt{6}-\sqrt{2})$

(3) $40(\sqrt{6}+\sqrt{2})$

(4) None of these

HD0026

27. AB is a vertical pole. The point A of pole AB is on the level ground. C is the middle point of AB. P is a point on the level ground. The portion BC substends an angle β at P. If AP = n AB, then tan β is equal to-

(1) $\frac{n}{2n^2+1}$ (2) $\frac{n}{n^2-1}$

(3) $\frac{n}{n^2+1}$ (4) None of these

HD0027

28. The top of a hill observed from the top and bottom of a building of height h is at angles of elevation p and q respectively. The height of the hill is -

(1) $\frac{h\cot q}{\cot q-\cot p}$ (2) $\frac{h\cot p}{\cot p-\cot q}$

(3) $\frac{h\tan p}{\tan p-\tan q}$ (4) None of these

HD0028

29. A and B are two points 30 m apart in a line on the horizontal plane through the foot of a tower lying on opposite sides of the tower. If the distance of the top of the tower from A and B are 20 m and 15 m respectively, the angle of elevation of the top of the tower at A is-

(1) $\cos^{-1}(43/48)$ (2) $\sin^{-1}(43/48)$

(3) $\cos^{-1}(29/36)$ (4) $\sin^{-1}(29/36)$

HD0029

30. A vertical pole subtends an angle $\tan^{-1}(1/2)$ at a point P on the ground. The angle subtended by the upper half of the pole at the point P is-

(1) $\tan^{-1}(1/4)$ (2) $\tan^{-1}(2/9)$

(3) $\tan^{-1}(1/8)$ (4) $\tan^{-1}(2/3)$

HD0030

31. The angle of elevation of the top of a tower standing on a horizontal plane from a point A is α. After walking a distance d towards the foot of the tower, the angle of elevation is found to be β. The height of the tower is-

(1) $\frac{d\sin\alpha\sin\beta}{\sin(\beta-\alpha)}$ (2) $\frac{d\sin\alpha\sin\beta}{\sin(\alpha-\beta)}$

(3) $\frac{d\sin(\beta-\alpha)}{\sin\alpha\sin\beta}$ (4) $\frac{d\sin(\alpha-\beta)}{\sin\alpha\sin\beta}$

HD0031

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32. The angle of elevation of the top of two vertical towers as seen from the middle point of the line joining the foot of the towers are 60° and 30° respectively. The ratio of the height of the towers is-

(1) 2 : 1 (2) $\sqrt{3}:1$

(3) 3 : 2 (4) 3 : 1

HD0032

33. A person walking along a st. road towards a hill observes at two points, distance $\sqrt{3}$ km, the angles of elevation of the hill to be 30° and 60°. The height of the hill is-

(1) $\frac{3}{2}$ km (2) $\sqrt{\frac{2}{3}}$ km

(3) $\frac{\sqrt{3}+1}{2}$ km (4) $\sqrt{3}$ km

HD0033

34. The length of the shadow of a vertical pole of height h, thrown by the sun's rays at three different moments are h, 2h and 3h. The sum of the angles of elevation of the rays at these three moments is equal to-

(1) $\frac{\pi}{2}$ (2) $\frac{\pi}{3}$

(3) $\frac{\pi}{4}$ (4) $\frac{\pi}{6}$

HD0034

35. The upper $\frac{3}{4}$th portion of a vertical pole subtends an an angle $\tan^{-1}\frac{3}{5}$ at a point in the horizontal plane through its foot and at a distance 40 m from the foot. A possible height of the vertical pole is-

[AIEEE-2002]

(1) 80 m (2) 20 m

(3) 40 m (4) 60 m

HD0035

36. A person standing on the bank of a river observes that the angle of elevation of the top of a tree on the opposite bank of the river is 60° and when he retires 40 meters away from the tree the angle of elevation becomes 30°. The breadth of the river is-

[AIEEE-2004]

(1) 20 m (2) 30 m

(3) 40 m (4) 60 m

HD0036

37. A tower stands at the centre of a circular park. A and B are two points on the boundary of the park such that AB (=a) subtends an angle of 60° at the foot of the tower, and the angle of elevation of the top of the tower from A or B is 30°. The height of the tower is- **[AIEEE-2007]**

(1) $2a/\sqrt{3}$ (2) $2a\sqrt{3}$

(3) $a/\sqrt{3}$ (4) $a\sqrt{3}$

HD0037

38. AB is a vertical pole with B at the ground level and A at the top. A man finds that the angle of elevation of the point A from a certain point C on the ground is 60°. He moves away from the pole along the line BC to a point D such that CD = 7 m. From D the angle of elevation of the point A is 45°. Then the height of the pole is- **[AIEEE-2008]**

(1) $\frac{7\sqrt{3}}{2}\frac{1}{\sqrt{3}-1}$m

(2) $\frac{7\sqrt{3}}{2}(\sqrt{3}+1)$m

(3) $\frac{7\sqrt{3}}{2}(\sqrt{3}-1)$m

(4) $\frac{7\sqrt{3}}{2}\frac{1}{\sqrt{3}+1}$m

HD0038

39. A man standing on a horizontal plane, observes the angle of elevation of the top of a tower to be α. After walking a distance equal to double the height of the tower, the angle of elevation becomes 2α, then α is -

(1) $\frac{\pi}{18}$ (2) $\frac{\pi}{12}$

(3) $\frac{\pi}{6}$ (4) $\frac{\pi}{2}$

HD0039

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40. ABCD is a trapezium such that AB and CD are parallel and BC ⊥ CD. If ∠ADB = θ, BC = p and CD = q, then AB is equal to **[JEE-MAINS-2013]**

(1) $\frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$ (2) $\frac{p^2+q^2\cos\theta}{p\cos\theta+q\sin\theta}$

(3) $\frac{p^2+q^2}{p^2\cos\theta+q^2\sin\theta}$ (4) $\frac{(p^2+q^2)\sin\theta}{(p\cos\theta+q\sin\theta)^2}$

HD0040

41. If the angles of elevation of the top of a tower from three collinear points A, B and C, on a line leading to the foot of the tower, are 30°, 45° and 60° respectively, then the ratio, AB : BC, is : **[JEE(Main)-2015]**

(1) $1:\sqrt{3}$ (2) 2 : 3

(3) $\sqrt{3}:1$ (4) $\sqrt{3}:\sqrt{2}$

HD0041

42. A man is walking towards a vertical pillar in a straight path, at a uniform speed. At a certain point A on the path, he observes that the angle of elevation of the top of the pillar is 30°. After walking for 10 minutes from A in the same direction, at a point B, he observes that the angle of elevation of the top of the pillar is 60°. Then the time taken (in minutes) by him, form B to reach the pillar, is : **[JEE(Main)-2016]**

(1) 5 (2) 6 (3) 10 (4) 20

HD0042

43. Let a vertical tower AB have its end A on the level ground. Let C be the mid-point of AB and P be a point on the ground such that AP = 2AB. If ∠BPC = β, then tanβ is equal to :- **[JEE(Main)-2017]**

(1) $\frac{4}{9}$ (2) $\frac{6}{7}$ (3) $\frac{1}{4}$ (4) $\frac{2}{9}$

HD0043

44. PQR is a triangular park with PQ = PR = 200 m. A T.V. tower stands at the mid-point of QR. If the angles of elevation of the top of the tower at P,Q and R are respectively 45°, 30° and 30°, then the height of the tower (in m) is- **[JEE(Main)-2018]**

(1) 50 (2) $100\sqrt{3}$

(3) $50\sqrt{2}$ (4) 100

HD0044

45. Consider a triangular plot ABC with sides AB=7m, BC=5m and CA=6m. A vertical lamp-post at the mid point D of AC subtends an angle 30° at B. The height (in m) of the lamp-post is : **[JEE(Main) Jan-19]**

(1) $7\sqrt{3}$ (2) $\frac{2}{3}\sqrt{21}$

(3) $\frac{3}{2}\sqrt{21}$ (4) $2\sqrt{21}$

HD0045

46. If the angle of elevation of a cloud from a point P which is 25 m above a lake be 30° and the angle of depression of reflection of the cloud in the lake from P be 60°, then the height of the cloud (in meters) from the surface of the lake is : **[JEE(Main) Jan-19]**

(1) 42 (2) 50 (3) 45 (4) 60

HD0046

47. Two vertical poles of heights, 20m and 80m stand a part on a horizontal plane. The height (in meters) of the point of intersection of the lines joining the top of each pole to the foot of the other, from this horizontal plane is : **[JEE(Main) Apr-19]**

(1) 12 (2) 15 (3) 16 (4) 18

HD0047

48. Two poles standing on a horizontal ground are of heights 5m and 10 m respectively. The line joining their tops makes an angle of 15° with ground. Then the distance (in m) between the poles, is :- **[JEE(Main) Apr-19]**

(1) $\frac{5}{2}(2+\sqrt{3})$ (2) $5(\sqrt{3}+1)$

(3) $5(2+\sqrt{3})$ (4) $10(\sqrt{3}-1)$

HD0048

49. ABC is a triangular park with AB = AC = 100 metres. A vertical tower is situated at the mid-point of BC. If the angles of elevation of the top of the tower at A and B are $\cot^{-1}\left(3\sqrt{2}\right)$ and $\text{cosec}^{-1}\left(2\sqrt{2}\right)$ respectively, then the height of the tower (in metres) is : **[JEE(Main) Apr-19]**

(1) $10\sqrt{5}$ (2) $\frac{100}{3\sqrt{3}}$

(3) 20 (4) 25

HD0049

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50. A 2 m ladder leans against a vertical wall. If the top of the ladder begins to slide down the wall at the rate 25 cm/ sec., then the rate (in cm/sec.) at which the bottom of the ladder slides away from the wall on the horizontal ground when the top of the ladder is 1 m above the ground is : **[JEE(Main) Apr-19]**

(1) $25\sqrt{3}$ (2) 25

(3) $\frac{25}{\sqrt{3}}$ (4) $\frac{25}{3}$

HD0050

51. The angle of elevation of the top of vertical tower standing on a horizontal plane is observed to be 45° from a point A on the plane. Let B be the point 30 m vertically above the point A. If the angle of elevation of the top of the tower from B be 30°, then the distance (in m) of the foot of the tower from the point A is: **[JEE(Main) Apr-19]**

(1) $15\left(3-\sqrt{3}\right)$ (2) $15\left(3+\sqrt{3}\right)$

(3) $15\left(1+\sqrt{3}\right)$ (4) $15\left(5-\sqrt{3}\right)$

HD0051

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ANSWER KEY

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	**2**	**1**	**1**	**3**	**3**	**2**	**1**	**2**	**1**	**3**	**4**	**1**	**1**	**1**	**2**
Que.	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Ans.	**1**	**2**	**2**	**2**	**4**	**3**	**3**	**3**	**2**	**1**	**2**	**1**	**2**	**1**	**2**
Que.	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
Ans.	**1**	**4**	**1**	**1**	**3**	**1**	**3**	**2**	**2**	**1**	**3**	**1**	**4**	**4**	**2**
Que.	46	47	48	49	50	51									
Ans.	**2**	**3**	**3**	**3**	**3**	**2**									

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PRINCIPLE OF MATHEMATICAL INDUCTION

1. THEOREM-I

If P(n) is a statement depending upon n, then to prove P(n) by induction, we procced as follows :

(i) Verify the validity of P(n) for n = 1

(ii) Assume that P(n) is true for any positive integer m and then using it establish the validity of P(n) for n = m + 1.

Then P(n) is true for each $n \in N$

2. THEOREM-II

If P(n) is a statement depending upon n but begining with any positive integer k, then to prove P(n) by Induction, we procced as follows :

(i) Verify the validity of P(n) for n = k.

(ii) Assume that the P(n) is true for $n = m \geq k$.

Then using it estabish the validity of P(n) for n = m + 1.

Then P(n) is true for each $n \geq k$

3. SOME USEFUL RESULT BASED ON PRINCIPLE OF MATHEMATICAL INDUCTION :

For any natural number n

(i) $1 + 2 + 3 + \ldots + n = \Sigma n = \frac{n(n+1)}{2}$

(ii) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \Sigma n^3 = (\Sigma n)^2 = \left\{\frac{n(n+1)}{2}\right\}^2$

(iv) $2 + 4 + 6 + \ldots + 2n = \Sigma 2n = n(n + 1)$

(v) $1 + 3 + 5 + \ldots + (2n - 1) = \Sigma(2n - 1) = n^2$

(vi) $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + xy^{n-2} + y^{n-1})$

(vii) $x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 + \ldots - xy^{n-2} + y^{n-1})$

when n is odd positive integer

4. IMPORTANT TIPS :

(i) Product of r successive integers is divisible by r!

(ii) For $x \neq y$, $x^n - y^n$ is divisible by

(a) x + y, if n is even (b) x – y, if n is even or odd

(iii) $x^n + y^n$ is divisible by

x + y, If n is odd

(iv) For solving objective question related to natural numbers we find out the correct alternative by negative examination of this principle. If the given statement is P(n), then by putting n = 1, 2, 3 .... in P(n) we decide the correct answer. We also use the above formulae established by this principle to find the sum of n terms of a given series. For this we first express T_n as a polynomial in n and then for finding S_n, we put Σ before each term of this polynomial and then use above results of Σn, Σn^2, Σn^3 etc.

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SOLVED EXAMPLES

Ex.1 Use the principle of mathematical induction to show that $5^{2n+1} + 3^{n+2}.2^{n-1}$ divisible by 19 for all natural numbers n.

Sol. Let $P(n) = 5^{2n+1} + 3^{n+2}.2^{n-1}$

Step I : For n = 1

$P(1) = 5^{2+1} + 3^{1+2}.2^{1-1}$

$= 125 + 27$

$= 152$, which is divisible by 19.

Therefore, the result is true for n = 1.

Step II : Assume that the result is true for n = k, i.e. $P(k) = 5^{2k+1} + 3^{k+2}, 2^{k-1}$ is divisible by 19.

$\Rightarrow P(k) = 19r$, where r is an integer.

Step III : For n = k + 1

$P(k + 1) = 5^{2(k+1)+1} + 3^{k+1+2}. 2^{k+1-1}$

$= 5^{2k+3} + 3^{k+3}.2^{k}$

$= 25.5^{2k+1} + 3.3^{k+2}.2.2^{k-1}$

$= 25.5^{2k} + 6.3^{k+2}.2^{k-1}$

Now $25.5^{2k+1} + 6.3^{k+2}2^{k-1} = 25.(5^{2k+1} + 3^{k-2}.2^{k-1}) - 19.3^{k+2}.2^{k-1}$

i.e. $P(k + 1) = 25\,P(k) - 19.3^{k+2}.2^{k-1}$

But we know that P(k) is divisible by 19. Also $19.3^{k+2}.2^{k-1}$ is clearly divisible by 19.

Hence P(k + 1) is divisible by 19. This shows that the result is true for n = k + 1. Hence by the priciniple of mathematical induction, the result is true for all $n \in N$.

Ex.2. Use the principle of mathematical induction to show that $1.3 + 2.4 + \ldots\ldots + n.(n+2) = \frac{1}{6}n(n+1)(2n+7)$.

Sol. Let P(n) : $1.3 + 2.4 + \ldots\ldots + n.(n+2) = \frac{1}{6}n(n + 1)(2n + 7)$

Step I : For n = 1

LHS of $P(1) = 1.3 = 3 = \frac{1}{6}.1.2.9 = \frac{1}{6}.1(1+1)(2.1+7) =$ RHS of P(1)

So P(1) is true

Step II : Now assume P(k) is true, for some natural number k, i.e

$1.3 + 2.4 + \ldots\ldots + k.(k + 2) = \frac{1}{6}k(k + 1)(2k + 7)$.

Now deduce P(k + 1).

LHS of $P(k + 1) = 1.3 + 2.4 + \ldots\ldots + k.(k+2) + (k + 1).(k + 1 + 2)$

= (LHS of P(k)) + (k + 1) (k + 3)

= (RHS of P(k)) + (k + 1) (k + 3), (by inductive assumption)

$$= \frac{1}{6}k(k+1)(2k+7) + (k+1)(k+3)$$

$$= \frac{1}{6}(k+1)\left(k(2k+7) + 6(k+3)\right)$$

$$= \frac{1}{6}(k+1)\left(2k^2 + 13k + 18\right)$$

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$$= \frac{1}{6}(k+1)(k+2)(2k+9)$$

$$= \frac{1}{6}(k+1)(k+1+1)(2(k+1)+7)$$

= RHS of P(k + 1).

So P(k + 1) is true, if P(k) is true.

Hence by induction P(n) is true for all natural numbers n.

Ex.3 Use the principle of mathematical induction to show that for any positive integer number n, $n^3 + 2n$, is divisible by 3.

Sol. Statement P(n) is defined by $n^3 + 2n$ is divisible 3

Step 1 : We first show that P(1) is true. Let n = 1 and calculate $n^3 + 2n$

$1^3 + 2(1) = 3$

Hence P(1) is true.

Step 2 : We now assume that P(k) is true $k^3 + 2k$ is divisible by 3. is equivalent to

$k^3 + 2k = 3M$, where M is a positive integer.

We now consider the algebraic expression $(k + 1)^3 + 2(k + 1)$; expand it and group like terms.

$(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 5k + 3$

$= [k^3 + 2k] + [3k^2 + 3k + 3]$

$= 3M + 3[k^2 + k + 1] = 3[M + k^2 + k + 1]$

Hence $(k + 1)^3 + 2(k + 1)$ is also divisible by 3 and therefore statement P(k + 1) is true.

Ex.4 Prove that $3^n > n^2$ for n = 1, n = 2 and use the mathematical induction to prove that $3^n > n^2$ for n, a positive integer greater than 2.

Sol Statement P(n) is defined by

$3^n > n^2$

Step 1 : We first show that P(1) is true. Let n =1 and calculate 3^1 and 1^2 and compare them

$3^1 = 3$

$1^2 = 1$

3 is greater than 1 and hence P(1) is true.

Let us also show that P(2) is true.

$3^2 = 9$

$2^2 = 4$

Hence P(2) is also true.

Step 2 : We now assume that P(k) is true

$3^k > k^2$

Multiply both sides of the above inequality by 3.

$3 * 3^k > 3 * k^2$

The left side is equal to 3^{k+1}. For k > 2, we can write

$k^2 > 2k$ and $k^2 > 1$

We now combine the above inequalities by adding the left hand sides and the right hand sides of the two inequalities.

$2k^2 > 2k + 1$

We now add k^2 to both sides of the above inequality to obtain the inequality

$3k^2 > k^2 + 2k + 1$

Factor the right side we can write

$3 * k^2 > (k + 1)^2$

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If $3 * 3^k > 3*k^2$ and $3*k^2 > (k + 1)^2$ then

$$3 * 3^k > (k + 1)^2$$

Rewrite the left side as 3^{k+1}

$$3^{k+1} > (k + 1)^2$$

* Which proves that P(k + 1) is true.

Ex.5. Use mathematical induction to prove De Moiver's theorem

$[R (\cos t + i \sin t)]^n = R^n(\cos x\, nt + i \sin nt)$ for n a positive integer.

Sol. Step 1 : For n = 1

$[R (\cos t + i \sin t)]^1 = R^1(\cos 1.t + i \sin 1.t)$

It can be easily be seen that the two sides are equal.

Step 2 : We now assume that the theorem is true for n = k, hence $[R(\cos t + i\sin t)]^k = R^k(\cos kt + i \sin kt)$

Multiply both sides of the above equation by $R(\cos t + i \sin t)$

$[R (\cos t + i \sin t)]^k R (\cos t + i \sin t) = R^k(\cos kt + i \sin kt) R(\cos t + i \sin t)$

Rewrite the above as follows

$[R (\cos t + i \sin t)]^{k+1} = R^{k+1}[(\cos kt \cos t - \sin kt \sin t) + i(\sin kt \cos t + \cos kt \sin t)]$

Trigonometric identities can be used to write the trigonometric expressions $(\cos kt \cos t - \sin kt \sin t)$ and $(\sin kt \cos t + \cos kt \sin t)$ as follows

$(\cos kt \cos t - \sin kt \sin t) = \cos(kt + t) = \cos(k + 1)t$

$(\sin kt \cos t + \cos kt \sin t) = \sin(kt + t) = \sin(k + 1) t$

Substitute the above into the last equation to obtain.

$[R (\cos t + i \sin t)]^{k+1} = R^{k+1}[\cos (k + 1) t + \sin(k + 1)t]$

It has been established that the theorem is true for n = 1 and that if it assumed true for n = k it is true for n = k + 1.

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CHECK YOUR GRASP PRINCIPLE OF MATHEMATICAL INDUCTION EXERCISE-I

1. Let P(n) : $n^2 + n$ is an odd integer. It is seen that truth of P(n) $\Rightarrow$ the truth of P(n + 1). Therefore, P(n) is true for all-
(1) $n > 1$ (2) n
(3) $n > 2$ (4) None of these
MI0001

2. If $n \in N$, then $x^{2n-1} + y^{2n-1}$ is divisible by-
(1) $x + y$ (2) $x - y$ (3) $x^2 + y^2$ (4) $x^2 + xy$
MI0002

3. If $n \in N$, then $11^{n+2} + 12^{2n+1}$ is divisible by-
(1) 113 (2) 123
(3) 133 (4) None of these
MI0003

4. If $n \in N$, then $3^{4n+2} + 5^{2n+1}$ is a multiple of-
(1) 14 (2) 16 (3) 18 (4) 20
MI0004

5. For every positive integer

$$n, \frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105} \text{ is-}$$

(1) an integer
(2) a rational number which is not an integer
(3) a negative real number
(4) an odd integer
MI0005

6. Sum of the infinite seriese $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \ldots$ equals-
(1) 1/3 (2) 3 (3) 1/4 (4) 2
MI0006

7. If $a_k = \frac{1}{k(k+1)}$; then $\left(\sum_{k=1}^{n} a_k\right)^2$ is equal to-

(1) $\frac{n^2}{(n+1)^2}$ (2) $\frac{n^4}{(n+1)^4}$ (3) $\frac{n^2}{n^2+1}$ (4) $\frac{n}{n+1}$
MI0007

8. The sum of n terms of the series

$$\frac{\frac{1}{2}\cdot\frac{2}{2}}{1^3} + \frac{\frac{2}{2}\cdot\frac{3}{2}}{1^3+2^3} + \frac{\frac{3}{2}\cdot\frac{4}{2}}{1^3+2^3+3^3} + \ldots\ldots \text{ is-}$$

(1) $\frac{1}{n(n+1)}$ (2) $\frac{n}{n+1}$ (3) $\frac{n+1}{n}$ (4) $\frac{n+1}{n+2}$
MI0008

9. For all $n \in N$, $7^{2n} - 48n - 1$ is divisible by-
(1) 25 (2) 26 (3) 1234 (4) 2304
MI0009

10. For all positive integral values of n, $3^{2n} - 2n + 1$ is divisible by-
(1) 2 (2) 4 (3) 8 (4) 12
MI0010

11. The smallest positive integer for which the statement $3^{n+1} < 4^n$ holds is-
(1) 1 (2) 2 (3) 3 (4) 4
MI0011

12. For positive integer n, $10^{n-2} > 81n$ when-
(1) $n < 5$ (2) $n > 5$ (3) $n \geq 5$ (4) $n > 6$
MI0012

13. If P is a prime number then $n^p - n$ is divisible by p when n is a
(1) natural number greater than 1
(2) odd number
(3) even number
(4) None of these **MI0013**

14. A student was asked to prove a statement by induction. He proved
(i) P(5) is true and
(ii) Truth of P(n) $\Rightarrow$ truth of p(n + 1), $n \in N$
On the basis of this, he could conclude that P(n) is true for
(1) no $n \in N$ (2) all $n \in N$
(3) all $n \geq 5$ (4) None of these
MI0014

ANSWER KEY

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Ans.	4	1	3	1	1	1	1	2	4	1	4	3	1	3

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BRAIN TEASERS PRINCIPLE OF MATHEMATICAL INDUCTION EXERCISE-II

1. If $x > -1$, then the statement $P(n) : (1 + x)^n > 1 + nx$ is true for-
(1) all $n \in N$ (2) all $n > 1$
(3) all $n > 1$ and $x \neq 0$ (4) None of these
MI0015

2. For every positive integral value of n, $3^n, > n^3$ when-
(1) $n > 2$ (2) $n \geq 3$ (3) $n \geq 4$ (4) $n < 4$
MI0016

3. $P(n) : 3^{2n+2} - 8n - 9$ is divisible by 64, is true for-
(1) all $n \in N \cup \{0\}$ (2) $n \geq 2, n \in N$
(3) $n \in N, n > 2$ (4) None of these
MI0017

4. If m, n are any two odd positive integer wih $n < m$, then the largest positive integers which divides all the numbers of the type $m^2 - n^2$ is-
(1) 4 (2) 6 (3) 8 (4) 9
MI0018

5. For all $n \in N$, $\cos\theta \cos 2\theta \cos 4\theta \ldots \cos 2^{n-1}\theta$ equals to
(1) $\frac{\sin 2^n\theta}{2^n \sin\theta}$ (2) $\frac{\sin 2^n\theta}{\sin\theta}$
(3) $\frac{\cos 2^n\theta}{2^n \cos 2\theta}$ (4) $\frac{\cos 2^n\theta}{2^n \sin\theta}$ **MI0019**

6. $x(x^{n-1} - na^{n-1}) + a^n(n - 1)$ is divisible by $(x - a)^2$ for-
(1) $n > 1$ (2) $n > 2$
(3) all $n \in N$ (4) None of these
MI0020

7. For any odd integer $n \geq 1$
$n^3 - (n - 1)^3 + \ldots + (-1)^{n-1}.1^3$ is equal to-
(1) $\frac{1}{4}(n + 1)^2(2n - 1)$ (2) $\frac{1}{4}(n - 1)^2(2n - 1)$
(3) $\frac{1}{2}(n - 1)^2(2n - 1)$ (4) $\frac{1}{2}(n + 1)^2(2n - 1)$
MI0021

8. If p and q are respectively. The sum and the sum of squares of n successive integers beginning with a, then $nq - p^2$ is-
(1) independent of a (2) independent of n
(3) dependent on a (4) None of these
MI0022

9. The sum of first n terms of the given series $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + \ldots$ is $\frac{n(n+1)^2}{2}$, when n is even. When n is odd, then sum will be- **[AIEEE-2004]**
(1) $\frac{n(n+1)^2}{2}$ (2) $\frac{1}{2}n^2(n + 1)$
(3) $n(n + 1)^2$ (4) None
MI0023

10. Let $S(k) = 1 + 3 + 5 + \ldots\ldots + (2k - 1) = 3 + k^2$, then which of the following is true ? **[AIEEE-2004]**
(1) S(1) is true
(2) $S(k) \Rightarrow S(k + 1)$
(3) $S(k) \not\Rightarrow S(k + 1)$
(4) Principle of mathematical Induction can be used to prove that formula **MI0024**

11. The sum of n terms of the series
$1 + (1 + a) + (1 + a + a^2) + (1 + a + a^2 + a^3) + \ldots\ldots$, is-
(1) $\frac{n}{1-a} - \frac{a(1-a^n)}{(1-a)^2}$ (2) $\frac{n}{1-a} + \frac{a(1-a^n)}{(1-a)^2}$
(3) $\frac{n}{1-a} + \frac{a(1+a^n)}{(1-a)^2}$ (4) $-\frac{n}{1-a} + \frac{a(1-a^n)}{(1-a)^2}$
MI0025

12. **Statement :1** For every natural number $n \geq 2$
$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$$
Statement –2 : For every natural number $n \geq 2$, $\sqrt{n(n+1)} < n+1$. **[AIEEE-2008]**
(1) Statement –1 is false, Statement –2 is true
(2) Statement–1 is true, Statement–2 is false
(3) Statement–1 is true, Statement–2 is true; Statement–2 is a correct explanation for Statement–1
(4) Statement–1 is true, Statement–2 is true; Statement–2 is not a correct explanation for Statement–1 **MI0026**

13. **Statement - 1:** For each natural number n, $(n + 1)^7 - n^7 - 1$ is divisible by 7.
Statement - 2: For each natural number n, $n^7 - n$ is divisible by 7. **[AIEEE-2011]**
(1) Statement-1 is false, statement-2 is true.
(2) Statement-1 is true, statement-2 is true; Statement-2 is correct explanation for statement-1.
(3) Statement-1 is true, statement-2 is true; Statement-2 is not a correct explanation for statement-1.
(4) Statement-1 is true, statement-2 is false.
MI0027

14. Consider the statement : "$P(n): n^2 - n + 41$ is prime." Then which one of the following is true? **[JEE(Main) Jan-19]**
(1) P(5) is false but P(3) is true
(2) Both P(3) and P(5) are false
(3) P(3) is false but P(5) is true
(4) Both P(3) and P(5) are true **MI0028**

ANSWER KEY

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Ans.	3	3	1	3	1	3	1	1	2	2	1	3	2	4

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SETS

SET : A set is a collection of well defined objects which are distinct from each other

Set are generally denoted by capital letters A, B, C, ... etc. and the elements of the set by a, b, c .... etc.

If a is an element of a set A, then we write $a \in A$ and say a belongs to A.

If a does not belong to A then we write $a \notin A$,

Ex. The collection of first five prime natural numbers is a set containing the elements 2, 3, 5, 7, 11.

SOME IMPORTANT NUMBER SETS :

N = Set of all natural numbers
= {1, 2, 3, 4, ...}

W = Set of all whole numbers
= {0, 1, 2, 3, ...}

Z or I set of all integers
= {.... –3, –2, –1, 0, 1, 2, 3, ...}

Z^+ = Set of all +ve integers
= {1, 2, 3, ...} = N.

Z^- = Set of all –ve integers
= (–1, –2, –3, ...}

Z_0 = The set of all non-zero integers.
= {±1, ±2, ±3, ...}

Q = The set of all rational numbers.

$= \left\{\frac{p}{q} : p, q \in I, q \neq 0\right\}$

R = the set of all real numbers.

R–Q = The set of all irrational numbers

e.g. $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, ... π, e, log2 etc. are all irrational numbers.

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METHODS TO WRITE A SET :

(i) Roster Method : In this method a set is described by listing elements, separated by commas and enclose then by curly brackets

Ex. The set of vowels of English Alphabet may be described as {a, e, i, o, u}

(ii) Set Builder From : In this case we write down a property or rule p Which gives us all the element of the set

A = {x : P(x)}

Ex. A = {x : x ∈ N and x = 2n for n ∈ N}

i.e. A = {2, 4, 6, ...}

Ex. B = {x^2 : x ∈ z}

i.e. B = {0, 1, 4, 9, ...}

TYPES OF SETS :

Null set or Empty set : A set having no element in it is called an Empty set or a null set or void set it is denoted by ϕ or { }

Ex. A = {x ∈ N : 5 < x < 6} = ϕ

A set consisting of at least one element is called a non-empty set or a non-void set.

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Singleton : A set consisting of a single element is called a singleton set.

Ex. Then set {0}, is a singleton set

Finite Set : A set which has only finite number of elements is called a finite set.

Ex. A = {a, b, c}

Order of a finite set : The number of elements in a finite set is called the order of the set A and is denoted O(A) or n(A). It is also called cardinal number of the set.

Ex. A = {a, b, c, d} $\Rightarrow$ n(A) = 4

Infinite set : A set which has an infinite number of elements is called an infinite set.

Ex. A = {1, 2, 3, 4, ...} is an infinite set

Equal sets : Two sets A and B are said to be equal if every element of A is a member of B, and every element of B is a member of A.

If sets A and B are equal. We write A = B and A and B are not equal then $A \neq B$

Ex. A = {1, 2, 6, 7} and B = {6, 1, 2, 7} $\Rightarrow$ A = B

Equivalent sets : Two finite sets A and B are equivalent if their number of elements are same ie. n(A) = n(B)

Ex. A = {1, 3, 5, 7}, B = {a, b, c, d}

n(A) = 4 and n(B) = 4 $\Rightarrow$ n(A) = n(B)

Note : Equal set always equivalent but equivalent sets may not be equal

Subsets : Let A and B be two sets if every element of A is an element B, then A is called a subset of B if A is a subset of B. we write $A \subseteq B$

Example : A = {1, 2, 3, 4} and B = {1, 2, 3, 4, 5, 6, 7} $\Rightarrow A \subseteq B$

The symbol "$\Rightarrow$" stands for "implies"

Proper subset : If A is a subset of B and $A \neq B$ then A is a proper subset of B. and we write $A \subset B$

Note-1 : Every set is a subset of itself i.e. $A \subseteq A$ for all A

Note-2 : Empty set ϕ is a subset of every set

Note-3 : Clearly $N \subset W \subset Z \subset Q \subset R \subset C$

Note-4 : The total number of subsets of a finite set containing n elements is 2^n

Universal set : A set consisting of all possible elements which occur in the discussion is called a Universal set and is denoted by U

Note : All sets are contained in the universal set

Ex. If A = {1, 2, 3}, B = {2, 4, 5, 6}, C = {1, 3, 5, 7} then U = {1, 2, 3, 4, 5, 6, 7} can be taken as the Universal set.

Power set : Let A be any set. The set of all subsets of A is called power set of A and is denoted by P(A)

Ex.1 Let A = {1, 2} then P(A) = {ϕ, {1}, {2}, {1, 2}}

Ex.2 Let P(ϕ) = {ϕ}

$\because$ P(P(ϕ)) = {ϕ, {ϕ}}

$\because$ P(P(P(ϕ)) = {ϕ, {ϕ}, {{ϕ}}, {ϕ, {ϕ}}

Note-1 : If A = ϕ then P(A) has one element

Note-2 : Power set of a given set is always non empty

Some Operation on Sets :

(i) **Union of two sets :** $A \cup B = \{x : x \in A \text{ or } x \in B\}$

e.g. A = {1, 2, 3}, B = {2, 3, 4} then $A \cup B$ = {1, 2, 3, 4}

(ii) **Intersection of two sets :** $A \cap B = \{x : x \in A \text{ and } x \in B\}$

e.g. A = {1, 2, 3, }, B = {2, 3, 4} then $A \cap B$ = {2, 3}

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(iii) **Difference of two sets :** $A - B = \{x : x \in A \text{ and } x \notin B\}$

e.g. $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$; $A - B = \{1\}$

(iv) **Complement of a set :** $A' = \{x : x \notin A \text{ but } x \in U\} = U - A$

e.g. $U = \{1, 2, ..., 10\}$, $A = \{1, 2, 3, 4, 5\}$ then $A' = \{6, 7, 8, 9, 10\}$

(v) **De-Morgan Laws :** $(A \cup B)' = A' \cap B'$; $(A \cap B)' = A' \cup B'$

(vi) $A - (B \cup C) = (A - B) \cap (A - C)$; $A - (B \cap C) = (A - B) \cup (A - C)$

(vii) **Distributive Laws :** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(viii) **Commutative Laws :** $A \cup B = B \cup A$; $A \cap B = B \cap A$

(ix) **Associative Laws :** $(A \cup B) \cup C = A \cup (B \cup C)$; $(A \cap B) \cap C = A \cap (B \cap C)$

(x) $A \cap \phi = \phi$; $A \cap U = A$

$A \cup \phi = A$; $A \cup U = U$

(xi) $A \cap B \subseteq A$; $A \cap B \subseteq B$

(xii) $A \subseteq A \cup B$; $B \subseteq A \cup B$

(xiii) $A \subseteq B \Rightarrow A \cap B = A$

(xiv) $A \subseteq B \Rightarrow A \cup B = B$

Disjoint Sets :

IF $A \cap B = \phi$, then A, B are disjoint.

e.g. if $A = \{1, 2, 3\}$, $B = \{7, 8, 9\}$ then $A \cap B = \phi$

Note : $A \cap A' = \phi$ $\therefore$ A, A' are disjoint.

Symmetric Difference of Sets :

$A \Delta B = (A - B) \cup (B - A)$

- $(A')' = A$
- $A \subseteq B \Leftrightarrow B' \subseteq A'$

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If A and B are any two sets, then

(i) $A - B = A \cap B'$

(ii) $B - A = B \cap A'$

(iii) $A - B = A \Leftrightarrow A \cap B = \phi$

(iv) $(A - B) \cup B = A \cup B$

(v) $(A - B) \cap B = \phi$

(vi) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

Venn Diagrame :

$A \cup B$ $A \cap B$ $A - B$ $B - A$

Clearly $(A - B) \cup (B - A) \cup (A \cup B) = A \cup B$

A' $(A\Delta B)=(A-B) \cup (B-A)$ Disjoint Sets

Note : $A \cap A' = \phi$, $A \cup A' = U$

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SOME IMPORTANT RESULTS ON NUMBER OF ELEMENTS IN SETS :

If A, B and C are finite sets, and U be the finite universal set, then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A \cup B) = n(A) + n(B) \Leftrightarrow$ A, B are disjoint non-void sets.

(iii) $n(A - B) = n(A) - n(A \cap B)$ i.e. $n(A - B) + n(A \cap B) = n(A)$

(iv) $n(A \Delta B)$ = No. of elements which belong to exactly one of A or B

$= n((A - B) \cup (B - A))$

$= n(A - B) + n(B - A)$ [$\because$ (A – B) and (B – A) are disjoint]

$= n(A) - n(A \cap B) + n(B) - n(A \cap B)$

$= n(A) + n(B) - 2n(A \cap B)$

$= n(A) + n(B) - 2n(A \cap B)$

(v) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(vi) Number of elements in exactly two of the sets A, B, C

$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$

(vii) number of elements in exactly one of the sets A, B, C

$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$

(viii) $n(A' \cup B') = n((A \cap B)') = n(U) - n(A \cap B)$

(ix) $n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$

Ex. In a group of 1000 people, there are 750 who can speak Hindi and 400 who can speak Bengali. How many can speak Hindi only ?How many can spak Bengali ? How many can spak both Hindi and Bengali?

Sol. Let A and B be the sets of persons who can speak Hindi and Bengali respectively.

then $n(A \cap B) = 1000$, $n(A) = 750$, $n(B) = 400$.

Number of persons whos can speak both Hindi and Bengali

$= n(A \cap B) = n(A) + n(B) - n(A \cup B)$

$= 750 + 400 - 1000$

Number of persons who can speak Hindi only = 150

$= n(A - B) = n(A) - n(A \cap B) = 750 - 150 = 600$

Number of persons Whos can speak Bengali only

$= n(B - A) = n(B) - n(A \cap B) = 400 - 150 = 250$

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SOLVED EXAMPLES

Ex.1 The set A = [x : x ∈ R, x^2 = 16 and 2x = 6] equal-

(1) ϕ (2) [14, 3, 4] (3) [3] (4) [4]

Sol.(1) $x^2 = 16 \Rightarrow x = \pm 4$

$2x = 6 \Rightarrow x = 3$

There is no value of x which satisfies both the above equations.

Thus, $A = \phi$

Hence (1) is the correct answer

Ex.2 Let A = {x : x ∈ R, $|x| < 1$] ; B = [x : x ∈ R, $|x - 1| \geq 1$] and $A \cup B = R - D$, then the set D is-

(1) [x : $1 < x \leq 2$] (2) [x : $1 \leq x < 2$] (3) [x : $1 \leq x \leq 2$] (4) none of these

Sol.(2) A = [x : x ∈ R, $-1 < x < 1$]

B = [x : x ∈ R : $x - 1 \leq -1$ or $x - 1 \geq 1$]

= [x : x ∈ R : $x \leq 0$ or $x \geq 2$]

$\therefore A \cup B = R - D$

where D = [x : x ∈ R, $1 \leq x < 2$]

Thus (2) is the correct answer.

Ex.3 If aN = {ax : x ∈ N}, then the set $6N \cap 8N$ is equal to-

(1) 8N (2) 48N (3) 12N (4) 24N

Sol.(4) 6N = {6, 12, 18, 24, 30, ...}

8N = {8, 16, 24, 32, ...}

$\therefore$ $6N \cap 8N$ = {24, 48, ...} = 24N

Short cut Method

$6N \cap 8N = 24N$ [24 is the L.C.M. of 6 and 8]

Ex.4 If P, Q and R subsets of a set A, then $R \times (P' \cup Q')'$ =

(1) $(R \times P) \cap (R \times Q)$ (2) $(R \times Q) \cap (R \times P)$ (3) $(R \times P) \cup (R \times Q)$ (4) none of these

Sol.(1,2)

$R \times (P' \cup Q')' = R \times [(P')' \cap (Q')'] = R \times (P \cap Q) = (R \times P) \cap (R \times Q)$

Hence (1) is the correct answer.

Ex.5 If A = {x, y}, then the power set of A is-

(1) {x^y, y^x} (2) {ϕ, x, y} (3) {ϕ, {x} {2y}} (4) {ϕ, {x}, {y}, {x, y}}

Sol.(4) Clearly P(A) = Power set of A

= set of all subsets of A

= {ϕ, {x}, {y}, {x, y}}

$\therefore$ (4) holds.

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CHECK YOUR GRASP — SETS — EXERCISE-I

1. If A and B are two sets, then $A \cap (A \cup B)'$ is equal to-
(1) A (2) B
(3) ϕ (4) none of these
ST0001

2. If A is any set, then-
(1) $A \cup A' = \phi$ (2) $A \cup A' = U$
(3) $A \cap A' = U$ (4) none of these
ST0002

3. If A, B be any two sets, then $(A \cup B)'$ is equal to-
(1) $A' \cup B'$ (2) $A' \cap B'$
(3) $A \cap B$ (4) $A \cup B$ **ST0003**

4. If A and B be any two sets, then $(A \cap B)'$ is equal to-
(1) $A' \cap B'$ (2) $A' \cup B'$ (3) $A \cap B$ (4) $A \cup B$
ST0004

5. Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, A = {1, 2, 5}, B = {6, 7} then $A \cap B'$ is-
(1) B' (2) A (3) A' (4) B.
ST0005

6. If A and B are two sets, then $A \cup B = A \cap B$ iff-
(1) $A \subseteq B$ (2) $B \subseteq A$
(3) A = B (4) none of these
ST0006

7. Let A and B be two sets in the universal set. Then A – B equals-
(1) $A \cap B'$ (2) $A' \cap B$
(3) $A \cap B$ (4) none of these
ST0007

8. Two sets A, B are disjoint iff-
(1) $A \cup B = \phi$ (2) $A \cap B \neq \phi$
(3) $A \cap B = \phi$ (4) None of these
ST0008

9. Which of the following is a null set ?
(1) {0}
(2) {x : x > 0 or x < 0}
(3) $\{x : x^2 = 4 \text{ or } x = 3\}$
(4) $\{x : x^2 + 1 = 0, x \in R\}$ **ST0009**

10. If $A \subseteq B$, then $A \cap B$ is equal to-
(1) A (2) B (3) A' (4) B'
ST0010

11. If A and B are any two sets, then $A \cup (A \cap B)$ is equal to-
(1) A (2) B (3) A' (4) B'
ST0011

12. If A and B are not disjoint, then $n(A \cup B)$ is equal to-
(1) n(A) + n(B)
(2) $n(A) + n(B) - n(A \cap B)$
(3) $n(A) + n(B) + n(A \cap B)$
(4) n(A).n(B) **ST0012**

13. If A = {2, 4, 5}, B = {7, 8, 9} then $n(A \times B)$ is equal to-
(1) 6 (2) 9 (3) 3 (4) 0
ST0013

14. Let A and B be two sets such that n(A) = 70, n(B) = 60 and $n(A \cup B) = 110$. Then $n(A \cap B)$ is equal to-
(1) 240 (2) 20 (3) 100 (4) 120
ST0014

15. Which set is the subset of all given sets ?
(1) {1, 2, 3, 4, ...} (2) {1}
(3) {0} (4) { } **ST0015**

16. If $Q = \left\{x : x = \frac{1}{y}, \text{ where } y \in N\right\}$, then-
(1) $0 \in Q$ (2) $1 \in Q$ (3) $2 \in Q$ (4) $\frac{2}{3} \in Q$
ST0016

ANSWER KEY

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	3	2	2	2	2	3	1	3	4	1	1	2	2	2	4
Que.	16	17	18	19	20										
Ans.	2	2	2	1	3										

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17. A = {x : x ≠ x} represents-

(1) {0} (2) { } (3) {1} (4) {x}

ST0017

18. Which of the following statements is true ?

(1) $3 \subseteq \{1, 3, 5\}$ (2) $3 \in \{1, 3, 5\}$

(3) $\{3\} \in \{1, 3, 5\}$ (4) $\{3, 5\} \in \{1, 3, 5\}$

ST0018

19. Which of the following is a null set ?

(1) A = {x : x > 1 and x < 1]

(2) B = {x : x + 3 = 3}

(3) C = {ϕ}

(4) D = {x : x ≥ 1 and x ≤ 1} **ST0019**

20. P(A) = P(B) ⇒

(1) $A \subseteq B$ (2) $B \subseteq A$

(3) A = B (4) none of these

ST0020

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PREVIOUS YEAR QUESTIONS SETS EXERCISE-II

1. If A, B, C be three sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then - **[Roorkee 1991]**

(1) A = B (2) B = C
(3) A = C (4) A = B = C

ST0021

2. Sets A and B have 3 and 6 elements respectively. What can be the minimum number of elements in $A \cup B$? **[Roorkee 1991]**

(1) 3 (2) 6 (3) 9 (4) 18

ST0022

3. In a college of 300 students, every student reads 5 new spapers and every newspaper is read by 60 students. The number of newspapers is- **[IIT -1998]**

(1) at least 30 (2) at most 20
(3) exactly 25 (4) none of these

ST0023

4. The set of intelligent students in a class is- **[A.M.U.-1998]**

(1) a null set
(2) a singleton set
(3) a finite set
(4) not a will defined collection **ST0024**

5. The shaded region in the given figure is-

(1) $A \cap (B \cup C)$ (2) $A \cup (B \cap C)$
(3) $A \cap (B - C)$ (4) $A - (B \cup C)$

ST0025

6. Let n(U) = 700, n(A) = 200, n(B) = 300 and $n(A \cap B) = 100$, then $n(A' \cap B') =$

(1) 400 (2) 600
(3) 300 (4) 200 **ST0026**

7. If A = {1, 2, 3, 4, 5}, then the number of proper subsets of A is-

(1) 120 (2) 30
(3) 31 (4) 32 **ST0027**

8. Let A and B be two sets such that n(A) = 0.16, n(B) = 0.14, $n(A \cup B) = 0.25$. Then $n(A \cap B)$ is equal to- **[Jamia Milia Entrance Exam. 2001]**

(1) 0.3 (2) 0.5
(3) 0.05 (4) none of these

ST0028

9. If $A = \{x : x^2 - 5x + 6 = 0\}$, B = {2, 4}, C = {4, 5}, then $A \times (B \cap C)$ is- **[Kerala P.E.T. 2002]**

(1) {(2, 4), (3, 4)}
(2) {(4, 2), (4, 3)}
(3) {(2, 4), (3, 4), (4, 4)}
(4) {(2, 2), (3, 3), (4, 4), (5, 5)} **ST0029**

10. If $A = \{(x, y) : x^2 + y^2 = 25\}$ and $B = \{(x, y) : x^2 + 9y^2 = 144\}$ then $A \cap B$ contains-

(1) one point (2) three points
(3) two points (4) four points

ST0030

11. A class has 175 students. The following data shows the number of students obtaining one or more subjects. Mathematics 100; Physics 70; Chemistry 40; Mathematics and Physics 30; Mathematics and Chemistry 28; Physics and Chemitry 23; Mathematics, Physics and Chemistry 18. How many students have offered Mathematics alone ?

(1) 35 (2) 48 (3) 60 (4) 22

ST0031

12. The set S : {1, 2, 3, ..., 12} is to be partitioned into three sets A, B, C of equal size. Thus $A \cup B \cup C = S$, $A \cap B = B \cap C = A \cap C = \phi$. The number of ways to partition S is- **[AIEEE - 2007]**

(1) $12!/3!(4!)^3$ (2) $12!/3!(3!)^4$
(3) $12!/(4!)^3$ (4) $12!/(3!)^4$ **ST0032**

13. If A, B and C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then :- **[AIEEE- 2009]**

(1) B = C (2) $A \cap B = \phi$
(3) A = B (4) A = C **ST0033**

14. Two sets A and B are as under

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$A = \{(a, b) \in R \times R : |a - 5| < 1$ and $|b - 5| < 1\}$;

$B = \{(a, b) \in R \times R : 4(a-6)^2 + 9(b-5)^2 \le 36\}$. Then :-

[AIEEE-2018]

(1) $A \subset B$

(2) $A \cap B = \phi$ (an empty set)

(3) neither $A \subset B$ nor $B \subset A$

(4) $B \subset A$ **ST0034**

15. In a class of 140 students numbered 1 to 140, all even numbered students opted mathematics course, those whose number is divisible by 3 opted Physics course and theose whose number is divisible by 5 opted Chemistry course. Then the number of students who did not opt for any of the three courses is : **[JEE(Main) Jan-19]**

(1) 102 (2) 42 (3) 1 (4) 38

ST0035

16. Two newspapers A and B are published in a city. It is known that 25% of the city populations reads A and 20% reads B while 8% reads both A and B. Further, 30% of those who read A but not B look into advertisements and 40% of those who read B but not A also look into advertisements, while 50% of those who read both A and B look into advertisements. Then the percentage of the population who look into advertisement is :-

[JEE(Main) Apr-19]

(1) 12.8 (2) 13.5 (3) 13.9 (4) 13

ST0036

17. Let A, B and C be sets such that $\phi \ne A \cap B \subseteq C$. Then which of the following statements is not true?

[JEE(Main) Apr-19]

(1) If $(A - C) \subseteq B$, then $A \subseteq B$

(2) $(C \cup A) \cap (C \cup B) = C$

(3) If $(A - B) \subseteq C$, then $A \subseteq C$

(4) $B \cap C \ne \phi$

ST0037

18. Let $X = \{n \in N : 1 \le n \le 50\}$. If

$A = \{n \in X : n$ is a multiple of $2\}$ and

$B = \{n \in X : n$ is a multiple of $7\}$, then the number of elements in the smallest subset of X containing both A and B is_____ **[JEE(Main) Jan-20]**

ST0038

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ANSWER KEY

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	2	2	3	4	4	3	3	3	1	4	3	3	1	1	4
Que.	16	17	18												
Ans.	3	1	29.00												

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RELATIONS

INTRODUCTION :

Let A and B be two sets. Then a relation R from A to B is a subset of $A \times B$.
thus, R is a relation from A to B $\Leftrightarrow R \subseteq A \times B$.

Ex. If A = {1, 2, 3} and B = {a, b, c}, then R = {(1, b), (2, c), (1, a), (3, a)} being a subset of $A \times B$, is a relation from A to B. Here (1, b), (2, c), (1, a) and (3, a) $\in$ R, so we write 1 Rb, 2Rc, 1Ra and 3Ra. But (2, b) $\notin$ R, so we write $2 \not R\, b$

Total Number of Realtions : Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then $A \times B$ consists of mn ordered pairs. So, total number of subsets of $A \times B$ is 2^{mn}.

Domain and Range of a relation : Let R be a relation from a set A to a set B. Then the set of all first components or coordinates of the ordered pairs belonging to R is called to domain of R, while the set of all second components or coordinates of the ordered pairs in R is called the range of R.

Thus, Dom (R) = {a : (a, b) $\in$ R}

and, Range (R) = {b : (a, b) $\in$ R}

It is evident from the definition that the domain of a relation from A to B is a subset of A and its range is a subset of B.

Ex. Let A = {1, 3, 5, 7} and B = {2, 4, 6, 8} be two sets and let R be a relation from A to B defined by the phrase "(x, y) $\in$ R $\Leftrightarrow$ x > y". Under this relation R, we have

3R2, 5R2, 5R4, 7R2, 7R4 and 7R6

i.e. R = {(3, 2), (5, 2), (5, 4), (7, 2), (7, 4), (7, 6)}

$\therefore$ Dom (R) = {3, 5, 7} and Range (R) = {2, 4, 6}

Inverse Relation : Let A, B be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$

Also, Dom(R) = Range(R^{-1}) and Range (R) = Dom (R^{-1})

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Ex.1 Let A be the set of first ten natural numbers and let R be a relation on A defined by (x, y) $\in$ R $\Leftrightarrow$ x + 2y = 10, i.e. R = {(x, y) : x $\in$ A, y $\in$ A and x + 2y = 10}. Express R and R^{-1} as sets of ordered pairs. Determine also (i) domain of R and R^{-1} (ii) range of R and R^{-1}

Sol. We have $(x, y) \in R \Leftrightarrow x + 2y = 10 \Leftrightarrow y = \dfrac{10-x}{2}$, x, y $\in$ A

where A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Now, $x = 1 \Rightarrow y = \dfrac{10-1}{2} = \dfrac{9}{2} \notin A.$

This shows that 1 is not related to any element in A. Similarly we can observe. that 3, 5, 7, 9 and 10 are not related to any element of A under the defined relation

Further we find that :

For $x = 2,\ y = \dfrac{10-2}{2} = 4 \in A$ $\quad\therefore (2, 4) \in R$

For $x = 4,\ y = \dfrac{10-4}{2} = 3 \in A$ $\quad\therefore (4, 3) \in R$

For $x = 6,\ y = \dfrac{10-6}{2} = 2 \in A$ $\quad\therefore (6, 2) \in R$

For $x = 8,\ y = \dfrac{10-8}{2} = 1 \in A$ $\quad\therefore (8, 1) \in R$

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Thus, $R = \{(2, 4), (4, 3), (6, 2), (8, 1)\}$

$\Rightarrow \quad R^{-1} = \{(4, 2), (3, 4), (2, 6), (1, 8)\}$

Clearly, $Dom(R) = \{2, 4, 6, 8\} = Range(R^{-1})$

and, $Range(R) = \{4, 3, 2, 1\} = Dom(R^{-1})$

TYPES OF RELATIONS :

In this section we intend to define various types of relations on a given set A.

Void Relation : Let A be a set. Then $\phi \subseteq A \times A$ and so it is a relation on A. This relation is called the void or empty relation on A.

Universal Relation : Let A be a set. Then $A \times A \subseteq A \times A$ and so it is a relation on A. This relation is called the universal relation on A.

Identity Relation : Let A be a set. Then the relation $I_A = \{(a, a) : a \in A\}$ on A is called the identity relation on A.

In other words, a relation I_A on A is called the identity relation if every element of A is related to itself only.

Ex. The relation $I_A = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on set $A = \{1, 2, 3\}$. But relations $R_1 = \{(1, 1), (2, 2)\}$ and $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$ are not identity relations on A, because $(3, 3) \notin R_1$ and in R_2 element 1 is related to elements 1 and 3.

Reflexive Relation : A relation R on a set A is said to be reflexive if every element of A is related to itself.

Thus, R on a set A is not reflexive if there exists an element $A \in A$ such that $(a , a) \notin R$.

Ex. Let $A = \{1, 2, 3\}$ be a set. Then $R = \{(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)\}$ is a reflexive relation on A. But $R_1 = \{(1, 1), (3, 3), (2, 1), (3, 2)\}$ is not a reflexive relation on A, because $2 \in A$ but $(2, 2) \notin R_1$.

Note : Every Identity relation is reflexive but every reflexive ralation is not identity.

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Symmetric Relation : A relation R on a set A is said to be a symmetric relation iff

$(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

i.e. $a R b \Rightarrow bRa$ for all $a, b, \in A$.

Ex. Let L be the set of all lines in a plane and let R be a relation defined on L by the rule $(x, y) \in R \Leftrightarrow x$ is perpendicular to y. Then R is a symmetric relation on L, because $L_1 \perp L_2 \Rightarrow L_2 \perp L_1$

i.e. $(L_1, L_2) \in R \Rightarrow (L_2, L_1) \in R$.

Ex. Let $A = \{1, 2, 3, 4\}$ and Let R_1 and R_2 be realtion on A given by $R_1 = \{(1, 3), (1, 4), (3, 1), (2, 2), (4, 1)\}$ and $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$. Clearly, R_1 is a symmetric relation on A. However, R_2 is not so, because $(1, 3) \in R_2$ but $(3, 1) \notin R_2$

Transitive Relation : Let A be any set. A relation R on A is said to be a transitive relation iff

$(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$

i.e. $a R b$ and $b R c \Rightarrow a R c$ for all $a, b, c \in A$

Ex. On the set N of natural numbers, the relation R defined by $x R y \Rightarrow x$ is less than y is transitive, because for any $x, y, z \in N$

$x < y$ and $y < z \Rightarrow x < z \Rightarrow x R y$ and $y R z \Rightarrow x R z$

Ex. Let L be the set of all straight lines in a plane. Then the realtion 'is parallel to' on L is a transitive relation, because from any $\ell_1, \ell_2, \ell_3 \in L$.

$$\ell_1 \parallel \ell_2 \text{ and } \ell_2 \parallel \ell_3 \Rightarrow \ell_1 \parallel \ell_3$$

Antisymmetric Relation : Let A be any set. A relation R on set A is said to be an antisymmetric relation iff

$(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$

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Ex. Let R be a relation on the set N of natural numbers defined by

$$x\ R\ y \Leftrightarrow \text{'x divides y' for all } x, y \in N$$

This relation is an antisymmetric relation on N. Since for any two numbers a, b $\in$ N

$$a \mid b \text{ and } b \mid a \Rightarrow a = b \quad \text{i.e. } a\ R\ b \text{ and } b\ R\ a \Rightarrow a = b$$

Equivalence Relation : A relation R on a set A is said to be an equivalence relation on A iff

(i) it is reflexive i.e. $(a, a) \in R$ for all $a \in A$

(ii) it is symmetric i.e. $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

(iii) it is transitive i.e. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

Ex. Let R be a relation on the set of all lines in a plane defined by $(\ell_1, \ell_2) \in R \Leftrightarrow$ line ℓ_1 is parallel to line ℓ_2. R is an equivalence relation.

Note : It is not neccessary that every relation which is symmetric and transitive is also reflexive.

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SOLVED EXAMPLES

Ex.1 Three relation R_1, R_2 and R_3 are defined on set A = {a, b, c} as follows :

(i) R_1 {(a, a), (a, b), (a, c), (b, b), (b, c), (c, a), (c, b), (c, c)} (ii) R_2 {(a, b), (b, a), (a, c), (c, a)}

(iii) R_3 {(a, b), (b, c), (c, a)}

Find whether each of R_1, R_2 and R_3 is reflexive, symmetric and transitive.

Sol. (i) Reflexive : Clearly, (a, a), (b, b), (c, c) $\in R_1$. So, R_1 is reflexive on A.

Symmetric : We observe that (a, b) $\in R_1$ but (b, a) $\notin R_1$. So, R_1 is not symmetric on A.

Transitive : We find that (b, c) $\in R_1$ and (c, a) $\in R_1$ but (b, a) $\notin R_1$. So, R is not transitive on A.

(ii) Reflexive : Since (a, a), (b, b) and (c, c) are not in R_2. So, it is not a reflexive realtion on A.

Symmetric : We find that the ordered pairs obtained by interchanging the components of ordered pairs in R_2 are also in R_2. So, R_2 is a symmetric relation on A.

Transitive : Clearly (c, a) $\in R_2$ and (a, b) $\in R_2$ but (c, b) $\notin R_2$. So, it is not a transitive relation on R_2.

(iii) Reflexive : Since non of (a, a), (b, b) and (c, c) is an element of R_3. So, R_3 is not reflexive on A.

Symmetric : Clearly, (b, c) $\in R_3$ but (c, b) $\notin R_3$. so, is not symmetric on A.

Transitive : Clearly, (b, c) $\in R_3$ and (c, a) $\in R_3$ but (b, a) $\notin R_3$. So, R_3 is not transitive on A.

Ex.2 Prove that therelation R on the set Z of all integers defined by

$$(x, y) \in R \Leftrightarrow x - y \text{ is divisible by } n$$

is an equivalence relation on Z.

Sol. We observe the following properties

Reflexivity : For any a $\in$ N, we have

$$a - a = 0 = 0 \times n \Rightarrow a - a \text{ is divisible by } n \Rightarrow (a, a) \in R$$

Thus, (a, a) $\in$ R for all a $\in$ Z

So, R is reflexive on Z

symmetry : Let (a, b) $\in$ R. Then,

(a, b) $\in$ R $\Rightarrow$ (a – b) is divisible by n

$\Rightarrow$ a – b = np for some p $\in$ Z

$\Rightarrow$ b – a = n(–p)

$\Rightarrow$ b – a is divisible by n [$\because$ p $\in$ Z $\Rightarrow$ –p $\in$ Z]

$\Rightarrow$ (b, a) $\in$ R

Thus, (a , b) $\in$ R $\Rightarrow$ (b, a) $\in$ R for all a, b, $\in$ Z

So, R is symmetric on Z.

Transitivity : Let a, b, c $\in$ Z such that (a, b) $\in$ R and (b, c) $\in$ R. Then,

(a , b) $\in$ R $\Rightarrow$ (a – b) is divisible by n

$\Rightarrow$ a – b = np for some p $\in$ Z

(b, c) $\in$ R $\Rightarrow$ (b – c) is divisible by n

$\Rightarrow$ b – c = nq for some q $\in$ Z

$\therefore$ (a, b) $\in$ R and (b, c) $\in$ R

$\Rightarrow$ a – b = np and b – c – nq

$\Rightarrow$ (a – b) + (b – c) = np + nq

$\Rightarrow$ a – c = n(p + q)

$\Rightarrow$ a – c is divisible by n [$\because$ p, q $\in$ Z $\Rightarrow$ p + q = Z]

$\Rightarrow$ (a, c) $\in$ R

thus, (a, b) $\in$ R and (b, c) $\in$ R $\Rightarrow$ (a, c) $\in$ R for all a, b, c $\in$ Z. so, R is transitive realtion in Z.

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Ex.3 Shw that the relation is congruent to' on the set of all triangles in a plane is an equivalence relation.

Sol. Let S be the set of all triangles in a plane and let R be the relation on S defined by $(\Delta_1, \Delta_2) \in R \Leftrightarrow$ triangle Δ_1 is congruent to triangle Δ_2. We observe the following properties.

Reflexivity : For each triangle $\Delta \in S$, we have

$\Delta \cong \Delta \Rightarrow (\Delta, \Delta) \in R$ for all $\Delta \in S \Rightarrow R$ is reflexive on S

Symmetry : Let $\Delta_1, \Delta_2 \in S$ such that $(\Delta_1, \Delta_2) \in R$. Then, $(\Delta_1, \Delta_2) \in R \Rightarrow \Delta_1 \cong \Delta_2 \Rightarrow \Delta_2 \cong \Delta_1 \Rightarrow (\Delta_2, \Delta_1) \in R$

So, R is symmetric on S

Transitivity : Let $\Delta_1, \Delta_2, \Delta_3 \in S$ such that $(\Delta_1, \Delta_2) \in R$ and $(\Delta_2, \Delta_3) \in R$. Then,

$(\Delta_1, \Delta_2) \in R$ and $(\Delta_2, \Delta_3) \in R \Rightarrow \Delta_1 \cong \Delta_2$ and $\Delta_2 \cong \Delta_3 \Rightarrow \Delta_1 \cong \Delta_3 \Rightarrow (\Delta_1, \Delta_3) \in R$

So, R is transitive on S.

Hence, R being reflexive, symmetric and transitive, is an equivalence relation on S.

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CHECK YOUR GRASP RELATIONS EXERCISE-I

1. If R is a relation from a finite set A having m elements to a finite set B having n elements, then the number of relations from A to B is-
(1) 2^{mn} (2) $2^{mn}-1$ (3) 2mn (4) m^n
RT0001

2. In the set A = {1, 2, 3, 4, 5}, a relation R is defined by R = {(x, y) | x, y ∈ A and x < y}. Then R is-
(1) Reflexive (2) Symmetric
(3) Transitive (4) None of these
RT0002

3. For real numbers x and y, we write $x\,R\,y \Leftrightarrow x - y + \sqrt{2}$ is an irrational number. Then the relation R is-
(1) Reflexive (2) Symmetric
(3) Transitive (4) none of these
RT0003

4. Let X = {1, 2, 3, 4} and Y = {1, 3, 5, 7, 9}. Which of the following is relations from X to Y-
(1) R_1 = {(x, y) | y = 2 + x, x ∈ X, y ∈ Y}
(2) R_2 = {(1, 1), (2, 1), (3, 3), (4, 3), (5, 5)}
(3) R_3 = {(1, 1), (1, 3), (3, 5), (3, 7), (5, 7)}
(4) R_4 = {(1, 3), (2, 5), (2, 4), (7, 9)}
RT0004

5. Let L denote the set of all straight lines in a plane. Let a relation R be defined by $\alpha\,R\,\beta \Leftrightarrow \alpha \perp \beta$, α, β ∈ L. Then R is-
(1) Reflexive (2) Symmetric
(3) Transitive (4) none of these
RT0005

6. Let R be a relation defined in the set of real numbers by $a\,R\,b \Leftrightarrow 1 + ab > 0$. Then R is-
(1) Equivalence relation (2) Transitive
(3) Symmetric (4) Anti-symmetric
RT0006

7. Which one of the following relations on R is equivalence relation-
(1) $x\,R_1 y \Leftrightarrow |x| = |y|$ (2) $x\,R_2 y \Leftrightarrow x \ge y$
(3) $x\,R_3 y \Leftrightarrow x \mid y$ (4) $x\,R_4 y \Leftrightarrow x < y$
RT0007

8. Two points P and Q in a plane are related if OP = OQ, where O is a fixed point. This relation is-
(1) Reflexive but not symmetric
(2) Symmetric but not transitive
(3) An equivalence relation
(4) none of these **RT0008**

9. The relation R defined in A = {1, 2, 3} by a R b if $|a^2 - b^2| \le 5$. Which of the following is false-
(1) R ={(1, 1), (2, 2), (3, 3), (2, 1), (1, 2), (2, 3), (3, 2)
(2) $R^{-1} = R$
(3) Domain of R = {1, 2, 3}
(4) Range of R = {5} **RT0009**

10. Let a relation R is the set N of natural numbers be defined as (x, y) ∈ R if and only if $x^2 - 4xy + 3y^2 = 0$ for all x, y ∈ N. The relation R is-
(1) Reflexive
(2) Symmetric
(3) Transitive
(4) An equivalence relation **RT0010**

11. Let A = {2, 3, 4, 5} and let R = {(2, 2), (3, 3), (4, 4), (5, 5), (2, 3), (3, 2), (3, 5), (5, 3)} be a relation in A. Then R is-
(1) Reflexive and transitive
(2) Reflexive and symmetric
(3) Reflexive and antisymmetric
(4) none of these **RT0011**

12. If A = {2, 3} and B = {1, 2}, then A × B is equal to-
(1) {(2, 1), (2, 2), (3, 1), (3, 2)}
(2) {(1, 2), (1, 3), (2, 2), (2, 3)}
(3) {(2, 1), (3, 2)}
(4) {(1, 2), (2, 3)} **RT0012**

13. Let R be a relation over the set N × N and it is defined by (a, b) R (c, d) ⇒ a + d = b + c. Then R is-
(1) Reflexive only
(2) Symmetric only
(3) Transitive only
(4) An equivalence relation **RT0013**

14. Let N denote the set of all natural numbers and R be the relation on N × N defined by (a, b) R (c, d) if ad (b + c) = bc(a + d), then R is-
(1) Symmetric only
(2) Reflexive only
(3) Transitive only
(4) An equivalence relation **RT0014**

15. If A = {1, 2, 3}, B = {1, 4, 6, 9} and R is a relation from A to B defined by 'x is greater than y'. Then range of R is-
(1) {1, 4, 6, 9} (2) {4, 6, 9}
(3) {1} (4) none of these
RT0015

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16. Let L be the set of all straight lines in the Euclidean plane. Two lines ℓ_1 and ℓ_2 are said to be related by the relation R if ℓ_1 is parallel to ℓ_2. Then the relation R is-
(1) Reflexive (2) Symmetric
(3) Transitive (4) Equivalence
RT0016

17. A and B are two sets having 3 and 4 elements respectively and having 2 elements in common. The number of relations which can be defined from A to B is-
(1) 2^5 (2) $2^{10} - 1$
(3) $2^{12} - 1$ (4) 2^{12} **RT0017**

18. For n, m $\in$ N, n | m means that n is a factor of m, the relation | is-
(1) reflexive and symmetric
(2) transitive and symmetric
(3) reflexive, transitive and symmetric
(4) reflexive, transitive and not symmetric **RT0018**

19. Let R = {(x, y) : x, y $\in$ A, x + y = 5} where A = {1, 2, 3, 4, 5} then
(1) R is not reflexive, symmetric and not transitive
(2) R is an equivalence relation
(3) R is reflexive, symmetric but not transitive
(4) R is not reflexive, not symmetric but transitive
RT0019

20. Let R be a relation on a set A such that R = R^{-1} then R is-
(1) reflexive
(2) symmetric
(3) transitive
(4) none of these **RT0020**

21. Let x, y $\in$ I and suppose that a relation R on I is defined by x R y if and only if x $\le$ y then
(1) R is partial order ralation
(2) R is an equivalence relation
(3) R is reflexive and symmetric
(4) R is symmetric and transitive **RT0021**

22. Let R be a relation from a set A to a set B, then-
(1) R = A $\cup$ B (2) R = A $\cap$ B
(3) R $\subseteq$ A $\times$B (4) R $\subseteq$ B $\times$A
RT0022

23. Given the relation R = = {(1, 2), (2, 3)} on the set A = {1, 2, 3}, the minimum number of ordered pairs which when added to R make it an equivalence relation is-
(1) 5 (2) 6 (3) 7 (4) 8
RT0023

24. Let P = {(x, y) | $x^2 + y^2 = 1$, x, y $\in$ R} Then P is-
(1) reflexive (2) symmetric
(3) transitive (4) anti-symmetric
RT0024

25. Let X be a family of sets and R be a relation on X defined by 'A is disjoint from B'. Then R is-
(1) reflexive (2) symmetric
(3) anti-symmetric (4) transitive
RT0025

26. In order that a relation R defined in a non-empty set A is an equivalence relation, it is sufficient that R
(1) is reflexive
(2) is symmetric
(3) is transitive
(4) possesses all the above three properties
RT0026

27. If R is an equivalence relation in a set A, then R^{-1} is-
(1) reflexive but not symmetric
(2) symmetric but not transitive
(3) an equivalence relation
(4) none of these **RT0027**

28. Let A = {p, q, r}. Which of the following is an equivalence relation in A ?
(1) R_1 = {(p, q), (q, r), (p, r), (p, p)}
(2) R_2 = {(r, q) (r, p), (r, r), (q, q)}
(3) R_3 = {(p, p), (q, q), (r, r), (p, q)}
(4) none of these **RT0028**

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ANSWER KEY

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	1	3	1	1	2	3	1	3	4	1	2	1	4	4	3
Que.	16	17	18	19	20	21	22	23	24	25	26	27	28		
Ans.	4	4	4	1	2	1	3	3	2	2	4	3	4		

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PREVIOUS YEAR QUESTIONS RELATIONS EXERCISE-II

1. Let R = {(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)} be a releation on the set A = {1, 2, 3, 4}. The relation R is- **[AIEEE - 2004]**

(1) transitive (2) not symmetric

(3) reflexive (4) a function

RT0029

2. Let R = {(3, 3), (6, 6), (9, 9), (12, 12), (6, 12), (3, 9), (3, 12), (3, 6)} be relation on the set A = {3, 6, 9, 12). The relation is- **[AIEEE - 2005]**

(1) rflexive and transitive only

(2) reflexive only

(3) an equilvalence relation

(4) reflexive and symmetric only **RT0030**

3. Let W denote the words in the English dictionary. Define the relation R by : R = {(x, y) ∈ W × W | the words x and y have at least one letter in common}. Then R is- **[AIEEE - 2006]**

(1) reflexive, symmetric and not transitive

(2) reflexive, symmetric and transitive

(3) reflexive, not symmetric and transtive

(4) not reflexive, symmetric and transitive

RT0031

4. Consider the following relations :-

R = {(x, y) | x, y are real numbers and x = wy for some rational number w} ;

S = {$(\frac{m}{n}, \frac{p}{q})$ | m, n, p and q are integers such that n, q ≠ 0 and qm = pn}.

Then : **[AIEEE - 2010]**

(1) R is an equivalence relation but S is not an equivalence relation

(2) Neither R nor S is an equivalence relation

(3) S is an equivalence relation but R is not an equivalence relation

(4) R and S both are equivalence relations

RT0032

5. Let R be the set of real numbers. **[AIEEE - 2011]**

Statement-1:

A = {(x, y) ∈ R × R : y – x is an integer} is an equivalence relation on R.

Statement-2:

B = {(x, y)∈ R ×R : x = αy for some rational number α} is an equivalence relation on R.

(1) Statement-1 is true, Statement-2 is false.

(2) Statement-1 is false, Statement-2 is true

(3) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(4) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1. **RT0033**

6. Let Z be the set of integers.

If $A = \left\{x \in Z : 2^{(x+2)(x^2-5x+6)} = 1\right\}$

and $B = \{x \in Z : -3 < 2x - 1 < 9\}$,

then the number of subsets of the set A × B, is:

[JEE(Main) Jan-19]

(1) 2^{18} (2) 2^{10}

(3) 2^{15} (4) 2^{12} **RT0034**

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ANSWER KEY

Que.	1	2	3	4	5	6
Ans.	2	1	1	3	1	3

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Important Notes

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MATHEMATICS

NOTICE

QUESTION CORRECTION SHEET : LIMIT (EXERCISE-O2)

Consider following diagram for Q.15

15. Which of the following statement are true for the function f defined for $-1 \le x \le 3$ in the figure shown.

(A) $\lim\limits_{x \to -1^+} f(x) = 1$

(B) $\lim\limits_{x \to 2} f(x)$ does not exist

(C) $\lim\limits_{x \to 1^-} f(x) = 2$

(D) $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x)$

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