Now You See it, Now You Don't: Obfuscation of Online Third-Party Information Sharing (Online Supplement)

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Proof of Lemma 1

We first describe the details for deriving the equilibrium. As discussed, The Bayesian Nash equilibria of this game are vectors $(\eta_t^*, K_{v,r}^*)$ such that:

$$\begin{aligned} B_{v,r}(\eta_L^*, \eta_H^*) &= K_{v,r}^* \quad , \qquad \forall \ (v \in [\underline{v}, \overline{v}] \, , \, r \in [0,1]) \\ B_t(\{K_{v,r}^* | \, \underline{v} \leqslant v \leqslant \overline{v} \, , \, 0 \leqslant r \leqslant 1\}) &= \eta_t^* \quad , \qquad \forall \ t \in \{L, H\} \end{aligned} \tag{1}$$

We first focus on the low-type website's decision. Demand for a low-type website given the best response function of the users and the high-type website is $\frac{\int_0^{\min\{\tilde{r},1\}} \hat{V} \, dr + \int_{\min\{\tilde{r},1\}}^1 \tilde{v} \, dr - \underline{v}}{\overline{v} - \underline{v}}$. We can re-write this piecewise function as:

$$D_L(\eta_L, \eta_H) = \begin{cases} \frac{2\theta[X - sm_L\underline{v}] - [\theta\eta_L + [1-\theta]\eta_H]}{2sm_L\theta[\overline{v} - \underline{v}]} & \text{if } \eta_L \leqslant \frac{\tilde{\eta} - [1-\theta]\eta_H}{\theta} \\ \frac{\frac{[m_H - m_L]^2[1-\theta]^2\theta X^2 - 2Xm_L[\theta\eta_L + [1-\theta]\eta_H][\theta m_L + [1-\theta]m_H]}{2sm_L[\theta\eta_L + [1-\theta]\eta_H][\theta m_L + [1-\theta]m_H]^2} - \underline{v}}{\overline{v} - \underline{v}} & \text{if } \eta_L > \frac{\tilde{\eta} - [1-\theta]\eta_H}{\theta} \end{cases} \tag{2}$$

We can calculate $\frac{\partial D_L}{\partial \eta_L}$ as:

$$\frac{\partial D_L}{\partial \eta_L} = \begin{cases} -\frac{1}{2sm_L[\overline{v}-\underline{v}]} < 0 & \text{if } \eta_L \leqslant \frac{\tilde{\eta}-[1-\theta]\eta_H}{\theta} \\ -\frac{[m_H-m_L]^2[1-\theta]^2\theta^2X^2}{2sm_L[\overline{v}-\underline{v}][\theta\eta_L+[1-\theta]\eta_H]^2[\theta m_L+[1-\theta]m_H]^2} < 0 & \text{if } \eta_L > \frac{\tilde{\eta}-[1-\theta]\eta_H}{\theta} \end{cases} \tag{3}$$

Therefore $\frac{\partial D_L}{\partial \eta_L} < 0$. The profit of a low-type website is given as $\Pi_L = D_L(\eta_L, \eta_H)m_L - C(\eta_L)$, thus, $\frac{\partial \Pi_L}{\partial \eta_L} = \frac{\partial D_L}{\partial \eta_L}m_L - C'(\eta_L)$. We know that $C'(\eta_L) > 0$, therefore $\frac{\partial \Pi_L}{\partial \eta_L} < 0$. For a low-type website, there is no incentive to obfuscate, thus $\eta_L^* = 0$.

Now we focus on a high-type website's decision. Demand for a high-type website given the best response function of the users is $\frac{\int_0^{\min\{\tilde{r},1\}} \check{V}\, dr + \int_{\min\{\tilde{r},1\}}^1 \tilde{v}\, dr - \underline{v}}{\overline{v}-\underline{v}}$. We can rewrite this given $\eta_L^* = 0$ as:

$$D_H(\eta_L = 0, \eta_H) = \begin{cases} \frac{\eta_H + 2X - 2sm_H\underline{v}}{2sm_H[\overline{v}-\underline{v}]} & \text{if } \eta_H \leqslant \frac{\tilde{\eta}}{1-\theta} \\ \frac{\frac{[m_H-m_L]^2\theta^2X + 2\eta_H m_H[\theta m_L + [1-\theta]m_H]}{2s[\theta\eta_L+[1-\theta]\eta_H]^2} - \underline{v}}{\overline{v}-\underline{v}} & \text{if } \eta_H > \frac{\tilde{\eta}}{1-\theta} \end{cases} \tag{4}$$

We can calculate $\frac{\partial D_H}{\partial \eta_H}$ as:

$$\frac{\partial D_H}{\partial \eta_H} = \begin{cases} \frac{1}{2sm_H[\overline{v}-\underline{v}]} > 0 & \text{if } \eta_H \leqslant \frac{\tilde{\eta}}{1-\theta} \\ \frac{[m_H-m_L]^2\theta^2X^2}{2sm_H[\overline{v}-\underline{v}]\eta_H^2[\theta m_L+[1-\theta]m_H]^2} > 0 & \text{if } \eta_H > \frac{\tilde{\eta}}{1-\theta} \end{cases} \tag{5}$$

Therefore $\frac{\partial D_H}{\partial \eta_H} > 0$. The profit of a high-type website is given as $\Pi_H = D_H(\eta_L, \eta_H)m_H - C(\eta_H)$, thus, $\frac{\partial \Pi_H}{\partial \eta_H} = \frac{\partial D_H}{\partial \eta_H}m_H - C'(\eta_H)$. To find the optimal level of obfuscation for the high-type website, we need to derive the optimal solution for each of the demand function intervals and then compare these optimal solutions. The first-order and second-order conditions are as follows:

$$\text{FOC}: \frac{\partial \Pi_H}{\partial \eta_H} = 0 \Rightarrow \begin{cases} (a) \quad \frac{1}{2s[\overline{v}-\underline{v}]} - C'(\eta_H) = 0 & \text{if } \eta_H \leqslant \frac{\tilde{\eta}}{1-\theta} \\ (b) \quad \frac{[m_H-m_L]^2\theta^2X^2}{2s[\overline{v}-\underline{v}]\eta_H^2[\theta m_L+[1-\theta]m_H]^2} - C'(\eta_H) = 0 & \text{if } \eta_H > \frac{\tilde{\eta}}{1-\theta} \end{cases} \tag{6}$$

$$\text{SOC}: \frac{\partial^2 \Pi_H}{\partial \eta_H^2} < 0 \Rightarrow \begin{cases} (a) \quad -C''(\eta_H) < 0 & \text{if } \eta_H \leqslant \frac{\tilde{\eta}}{1-\theta} \\ (b) \quad -\frac{[m_H - m_L]^2 \theta^2 X^2}{s[\overline{v}-\underline{v}]\eta_H^3[\theta m_L + [1-\theta]m_H]^2} - C''(\eta_H) = 0 & \text{if } \eta_H > \frac{\tilde{\eta}}{1-\theta} \end{cases} \tag{7}$$

From FOC(a) in (6) and the assumption $C''(\eta_t) \geqslant 0$, we obtain that if $C'(\frac{\tilde{\eta}}{1-\theta}) < \frac{1}{2s[\overline{v}-\underline{v}]}$, then there is no optimal solution where $\eta_H^* \leqslant \frac{\tilde{\eta}}{1-\theta}$. From FOC(b) in (6), SOC(b) in (7) and the assumption $C''(\eta_t) \geqslant 0$ we obtain that if $C'(\frac{\tilde{\eta}}{1-\theta}) < \frac{1}{2s[\overline{v}-\underline{v}]}$, then there is an interior solution where $\eta_H^* > \frac{\tilde{\eta}}{1-\theta}$. From FOC(b) in (6) and the assumption $C''(\eta_t) \geqslant 0$ we obtain that if $C'(\frac{\tilde{\eta}}{1-\theta}) \geqslant \frac{1}{2s[\overline{v}-\underline{v}]}$, then there is no interior solution where $\eta_H^* > \frac{\tilde{\eta}}{1-\theta}$. From FOC(a) in (6) and SOC(a) in (7) we obtain that if $C'(\frac{\tilde{\eta}}{1-\theta}) \geqslant \frac{1}{2s[\overline{v}-\underline{v}]}$ and $C''(\eta_t) > 0$, then there is an interior solution where $\eta_H^* \leqslant \frac{\tilde{\eta}}{1-\theta}$. If $C'(\frac{\tilde{\eta}}{1-\theta}) \geqslant \frac{1}{2s[\overline{v}-\underline{v}]}$ and $C''(\eta_t) = 0$, then from SOC(a) in (7) and the assumption $C'(\eta_t) > 0$, we obtain that there is a corner solution where $\eta_H^* = 0$. Thus, the characteristics of the high-type website's level of obfuscation are as follows:

$$\begin{aligned} \eta_H^* &>, \frac{\tilde{\eta}}{1-\theta} && \text{if } \ C'(\frac{\tilde{\eta}}{1-\theta}) < \frac{1}{2s[\overline{v}-\underline{v}]} \\ \eta_H^* &\leqslant \frac{\tilde{\eta}}{1-\theta}, && \text{if } \ C'(\frac{\tilde{\eta}}{1-\theta}) \geqslant \frac{1}{2s[\overline{v}-\underline{v}]} \quad \& \quad C''(\eta) > 0 \\ \eta_H^* &= 0, && \text{if } \ C'(\frac{\tilde{\eta}}{1-\theta}) \geqslant \frac{1}{2s[\overline{v}-\underline{v}]} \quad \& \quad C''(\eta) = 0 \qquad \blacksquare \end{aligned} \tag{8}$$

Proof of Proposition 1

From Lemma 1(a) we know that $\eta_L^* = 0$. Therefore a low-type website does not obfuscate, irrespective of content sensitivity.

From Lemma 1(c) we know that if $C''(\eta_t) = 0$ and $C'(\eta_t) \geqslant \frac{1}{2s[\overline{v}-\underline{v}]}$, then $\eta_H^* = 0$. Therefore, where the cost of obfuscation is linear, a high-type website does not obfuscate if $s \geqslant \frac{1}{2C'(\eta_t)[\overline{v}-\underline{v}]}$. In other conditions, to find the impact of content sensitivity on a high-type website's level of obfuscation, we need to determine the sign of $\frac{\partial \eta_H^*}{\partial s}$ in all possible equilibrium solutions. By the implicit function theorem applied to FOC in (6), we obtain:

$$\frac{\partial \eta_H^*}{\partial s} = -\frac{\frac{\partial^2 \Pi_H}{\partial \eta_H^* \partial s}}{\frac{\partial^2 \Pi_H}{\partial {\eta_H^*}^2}} = \begin{cases} -\frac{-\frac{1}{2s^2[\overline{v}-\underline{v}]}}{\frac{\partial^2 \Pi_H}{\partial {\eta_H^*}^2}} < 0 & \text{if } 0 < \eta_H^* \leqslant \frac{\tilde{\eta}}{1-\theta} \\ -\frac{-\frac{[m_H - m_L]^2 \theta^2 X^2}{2s^2 {\eta_H^*}^2 [\theta m_L + [1-\theta] m_H]^2 [\overline{v}-\underline{v}]}}{\frac{\partial^2 \Pi_H}{\partial {\eta_H^*}^2}} < 0 & \text{if } \eta_H^* > \frac{\tilde{\eta}}{1-\theta} \end{cases} \tag{9}$$

From (9) we have $\frac{\partial \eta_H^*}{\partial s} < 0$ for all $\eta_H^* > 0$, thus, if a high-type website obfuscates, then the level of obfuscation is decreasing in content sensitivity. ■

Proof of Proposition 2

From Lemma 1(a) we know that $\eta_L^* = 0$. Therefore a low-type website does not obfuscate irrespective of website value.

From Lemma 1(c) we know that if $C''(\eta_t) = 0$ and $C'(\eta_t) \geqslant \frac{1}{2s[\overline{v}-\underline{v}]}$, then $\eta_H^* = 0$ irrespective of website value. In other conditions, to find the impact of website value on a high-type website's level of obfuscation, we need to determine the sign of $\frac{\partial \eta_H^*}{\partial X}$ in all possible interior equilibrium solutions. However, the condition threshold $(\frac{\tilde{\eta}}{1-\theta})$ in deriving equilibrium solutions depends on X. Given the assumption $C'(\eta) > 0$ we can rewrite the equilibrium solutions as:

$$\begin{aligned} \eta_H^* &> \frac{\tilde{\eta}}{1-\theta}, && \text{if} \quad X < \frac{C'^{-1}(\frac{1}{2s[\overline{v}-\underline{v}]})[\theta m_L + [1-\theta] m_H]}{\theta[m_H - m_L]} \\ 0 < \eta_H^* &\leqslant \frac{\tilde{\eta}}{1-\theta}, && \text{if} \quad X \geqslant \frac{C'^{-1}(\frac{1}{2s[\overline{v}-\underline{v}]})[\theta m_L + [1-\theta] m_H]}{\theta[m_H - m_L]} \quad \& \quad C''(\eta_t) > 0 \\ \eta_H^* &= 0, && \text{if} \quad X \geqslant \frac{C'^{-1}(\frac{1}{2s[\overline{v}-\underline{v}]})[\theta m_L + [1-\theta] m_H]}{\theta[m_H - m_L]} \quad \& \quad C''(\eta_t) = 0 \end{aligned} \tag{10}$$

By the implicit function theorem applied to FOC in (6), we obtain:

$$\frac{\partial \eta_H^*}{\partial X} = -\frac{\frac{\partial^2 \Pi_H}{\partial \eta_H^* \partial X}}{\frac{\partial^2 \Pi_H}{\partial {\eta_H^*}^2}} = \begin{cases} 0 & \text{if } X \geqslant \frac{C'^{-1}(\frac{1}{2s[\overline{v}-\underline{v}]})[\theta m_L + [1-\theta] m_H]}{\theta[m_H - m_L]} \\ -\frac{\frac{[m_H - m_L]^2 \theta^2 X}{s {\eta_H^*}^2 [\theta m_L + [1-\theta] m_H]^2 [\overline{v}-\underline{v}]}}{\frac{\partial^2 \Pi_H}{\partial {\eta_H^*}^2}} > 0 & \text{if } X < \frac{C'^{-1}(\frac{1}{2s[\overline{v}-\underline{v}]})[\theta m_L + [1-\theta] m_H]}{\theta[m_H - m_L]} \end{cases} \tag{11}$$

From 11 we can conclude that as far as $X < \frac{C'^{-1}(\frac{1}{2s[\overline{v}-\underline{v}]})[\theta m_L+[1-\theta]m_H]}{\theta[m_H-m_L]}$ the optimal level of obfuscation is increasing in X ($\frac{\partial \eta_H^*}{\partial X} > 0$). When $X \geqslant \frac{C'^{-1}(\frac{1}{2s[\overline{v}-\underline{v}]})[\theta m_L+[1-\theta]m_H]}{\theta[m_H-m_L]}$, changing X does not impact η_H^*. ■