Technical Appendix: Can Animal Spirits Solve the Forward Premium Puzzle?

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Abstract

This technical appendix provides the details of modeling, computations, construction of the data set, and estimation.

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1 Elements

- Home Country Utility Function

$$u(c_{xt}, c_{yt}) = \frac{(c_{xt}^{\theta} c_{yt}^{1-\theta})^{1-\gamma}}{1-\gamma} \tag{1}$$

- Foreign Country Utility Function

$$u(c_{xt}^*, c_{yt}^*) = \frac{(c_{xt}^{*\theta} c_{yt}^{*1-\theta})^{1-\gamma}}{1-\gamma} \tag{2}$$

- Home Country Budget Constraint

$$\begin{aligned} c_{xt} + \frac{S_t P_t^*}{P_t} c_{yt} + \omega_{xt} e_t + \omega_{yt} e_t^* + \psi_{Mt} r_t + \psi_{Nt} r_t^* = \frac{P_{t-1}}{P_t} \omega_{xt-1} x_{t-1} \\ + \frac{S_t P_{t-1}^*}{P_t} \omega_{yt-1} y_{t-1} + \frac{\psi_{Mt-1} \Delta M_t}{P_t} + \frac{\psi_{Nt-1} S_t \Delta N_t}{P_t} + \omega_{xt-1} e_t + \omega_{yt-1} e_t^* \\ + \psi_{Mt-1} r_t + \psi_{Nt-1} r_t^* \end{aligned} \tag{3}$$

- Foreign Country Budget Constraint

$$\begin{aligned} c_{xt}^* + \frac{S_t P_t^*}{P_t} c_{yt}^* + \omega_{xt}^* e_t + \omega_{yt}^* e_t^* + \psi_{Mt}^* r_t + \psi_{Nt}^* r_t^* = \frac{P_{t-1}}{P_t} \omega_{xt-1}^* x_{t-1} \\ + \frac{S_t P_{t-1}^*}{P_t} \omega_{yt-1}^* y_{t-1} + \frac{\psi_{Mt-1}^* \Delta M_t}{P_t} + \frac{\psi_{Nt-1}^* S_t \Delta N_t}{P_t} + \omega_{xt-1}^* e_t + \omega_{yt-1}^* e_t^* \\ + \psi_{Mt-1}^* r_t + \psi_{Nt-1}^* r_t^* \end{aligned} \tag{4}$$

- Cash-in-Advance Constraints

$$\begin{aligned} m_t &\geq P_t c_{xt} \\ n_t &\geq P_t^* c_{yt} \qquad (5) \\ m_t^* &\geq P_t c_{xt}^* \qquad (6) \\ n_t^* &\geq P_t^* c_{yt}^* \qquad (7) \end{aligned}$$

- Adding-Up Constraints

$$\begin{aligned}
\omega_{xt} + \omega_{xt}^* &= 1 && (8)\\
\omega_{yt} + \omega_{yt}^* &= 1 && (9)\\
\psi_{Mt} + \psi_{Mt}^* &= 1 && (10)\\
\psi_{Nt} + \psi_{Nt}^* &= 1 && (11)\\
c_{xt} + c_{xt}^* &= x_t && (12)\\
c_{yt} + c_{yt}^* &= y_t && (13)\\
m_t + m_t^* &= M_t && (14)\\
n_t + n_t^* &= N_t && (15)
\end{aligned}$$

- Stochastic Processes

$$\begin{aligned}
x_{t+1} &= (1-\rho_1) + \rho_1 x_t + \varepsilon_{t+1}^x && (16)\\
y_{t+1} &= (1-\rho_2) + \rho_2 y_t + \varepsilon_{t+1}^y && (17)\\
M_{t+1} &= (1-\rho_3) + \rho_3 M_t + \varepsilon_{t+1}^M && (18)\\
N_{t+1} &= (1-\rho_4) + \rho_4 N_t + \varepsilon_{t+1}^N && (19)
\end{aligned}$$

2 Equilibrium

2.1 Centralized Economy

For simplicity, the log utility functions of (1) and (2) respectively can be defined as

$$\begin{aligned}
\log u(c_{xt}, c_{yt}) &= \theta \log(c_{xt}) + (1-\theta)\log(c_{yt}) && (20)\\
\log u(c_{xt}^*, c_{yt}^*) &= \theta \log(c_{xt}^*) + (1-\theta)\log(c_{yt}^*) && (21)
\end{aligned}$$

The central planner maximizes a weighted average of (20) and (21) subject to (12) and (13).

$$\max_{\left\{c_{xt}, c_{yt}, c_{xt}^*, c_{yt}^*\right\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t \left\{\phi\left[\theta \log(c_{xt}) + (1-\theta)\log(c_{yt})\right] + (1-\phi)\left[\theta \log(c_{xt}^*) + (1-\theta)\log(c_{yt}^*)\right]\right\}$$

$$
\begin{aligned}
s.t. \; c_{xt} + c_{xt}^{*} &= x_t \\
c_{yt} + c_{yt}^{*} &= y_t
\end{aligned}
$$

The central planner's problem can be reduced to a static problem (where $t = 0$)

$$
\max_{c_x, c_y, c_x^*, c_y^*} \phi \left[\theta \log(c_x) + (1-\theta) \log(c_y)\right] + (1-\phi) \left[\theta \log(c_x^*) + (1-\theta) \log(c_y^*)\right]
$$

$$
\begin{aligned}
s.t. \; c_x + c_x^{*} &= x \\
c_y + c_y^{*} &= y
\end{aligned}
$$

Insering (13) and (14) into the object function yields an unconstrained maximization problem

$$
\max_{c_x, c_y} \phi \left[\theta \log(c_x) + (1-\theta) \log(c_y)\right] + (1-\phi) \left[\theta \log(c_x - x) + (1-\theta) \log(c_y - y)\right]
$$

Optimizing the unconstrained objective function w.r.t control variables yields F.O.C's

$$
\frac{\partial}{\partial c_x} = 0 : \frac{\phi\theta}{c_x} - \frac{(1-\phi)\,\theta}{x - c_x} = 0 \tag{22}
$$

$$
\frac{\partial}{\partial c_y} = 0 : \frac{\phi(1-\theta)}{c_y} - \frac{(1-\phi)\,(1-\theta)}{y - c_y} = 0 \tag{23}
$$

Using $(22) - (23)$,$(13) - (14)$, and allowing time to evolve the pareto efficient allocations become

$$c_{xt} = (1-\phi)\, x_t \tag{24}$$
$$c_{yt} = (1-\phi)\, y_t \tag{25}$$
$$c^*_{xt} = \phi x_t \tag{26}$$
$$c^*_{yt} = \phi y_t \tag{27}$$

If both countries are weighted equally by the central planner so that $\phi = \frac{1}{2}$ then this results in a perfect sharing pareto efficient allocation

$$c_{xt} = \frac{x_t}{2} \tag{28}$$
$$c_{yt} = \frac{y_t}{2} \tag{29}$$
$$c^*_{xt} = \frac{x_t}{2} \tag{30}$$
$$c^*_{yt} = \frac{y_t}{2} \tag{31}$$

3 Decentralized Economy

3.1 Home Country Decentralized Economy

Home country maximizes (20) subject to (3)

$$\max_{\{c_{xt}, c_{yt}, \omega_{xt}, \omega_{yt}, \psi_{Mt}, \psi_{Nt}\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t \left\{\theta \log(c_{xt}) + (1-\theta)\log(c_{yt})\right\}$$

$$\begin{aligned} s.t. \quad & c_{xt} + \frac{S_t P^*_t}{P_t} c_{yt} + \omega_{xt} e_t + \omega_{yt} e^*_t + \psi_{Mt} r_t + \psi_{Nt} r^*_t = \frac{P_{t-1}}{P_t}\omega_{xt-1} x_{t-1} \\ & + \frac{S_t P^*_{t-1}}{P_t}\omega_{yt-1} y_{t-1} + \frac{\psi_{Mt-1}\Delta M_t}{P_t} + \frac{\psi_{Nt-1} S_t \Delta N_t}{P_t} + \omega_{xt-1} e_t + \omega_{yt-1} e^*_t \\ & + \psi_{Mt-1} r_t + \psi_{Nt-1} r^*_t \end{aligned}$$

The decentralized problem can be reduced to static form (where $t = 0$)

$$\max_{c_x, c_y, \omega_x, \omega_y, \psi_M, \psi_N} \left[\theta \log(c_x) + (1-\theta)\log(c_y)\right]$$

$$s.t. \qquad c_x + \frac{SP^*}{P} c_y + \omega_x e + \omega_y e^* + \psi_M r + \psi_N r^* = \omega_x x + \frac{SP^*}{P}\omega_y y + \omega_x e + \omega_y e^* + \psi_M r + \psi_N r^*$$

To "peg" a decentralized allocation that matches the centralized allocation, set $\omega_x = \omega_y = \psi_M = \psi_N = \frac{1}{2}$ where inserting these values into (3) for $t = 0$, the problem further reduces to

$$\max_{c_x, c_y} [\theta \log(c_x) + (1-\theta)\log(c_y)] \tag{32}$$

$$s.t. \qquad c_x + \frac{SP^*}{P} c_y = \frac{1}{2}\left(x + \frac{SP^*}{P} y\right) \tag{33}$$

Inserting (33) into (32) the unconstrained maximization problem becomes

$$\max_{c_y} \left[\theta \log\left(\frac{1}{2}\left(x + \frac{SP^*}{P} y\right) - \frac{SP^*}{P} c_y\right) + (1-\theta)\log(c_y)\right]$$

$$\frac{\partial}{\partial c_y} = 0 : \frac{-(\frac{SP^*}{P})\theta}{c_x} + \frac{(1-\theta)}{c_y} = 0 \tag{34}$$

Rearranging (34), applying to (33),and redefining $\frac{SP^*}{P} = q$ (real exchange rate) the resulting Home demand functions are

$$c_x = \frac{\theta (x + qy)}{2} \tag{35}$$

$$c_y = \frac{\theta (x + qy)(1-\theta)}{2q} \tag{36}$$

3.2 Foreign Country Decentralized Economy

Foreign country maximizes (21) subject to (4)

$$\max_{\left\{c^*_{xt}, c^*_{yt}, \omega^*_{xt}, \omega^*_{yt}, \psi^*_{Mt}, \psi^*_{Nt}\right\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t \left\{\theta \log(c^*_{xt}) + (1-\theta)\log(c^*_{yt})\right\}$$

$$\begin{aligned} s.t. \quad c^*_{xt} + \frac{S_t P^*_t}{P_t} c^*_{yt} + \omega^*_{xt} e_t + \omega^*_{yt} e^*_t + \psi^*_{Mt} r_t + \psi^*_{Nt} r^*_t = \frac{P_{t-1}}{P_t} \omega^*_{xt-1} x_{t-1} \\ + \frac{S_t P^*_{t-1}}{P_t} \omega^*_{yt-1} y_{t-1} + \frac{\psi^*_{Mt-1} \Delta M_t}{P_t} + \frac{\psi^*_{Nt-1} S_t \Delta N_t}{P_t} + \omega^*_{xt-1} e_t + \omega^*_{yt-1} e^*_t \\ + \psi^*_{Mt-1} r_t + \psi^*_{Nt-1} r^*_t \end{aligned}$$

The decentralized problem can be reduced to static form (where $t = 0$)

$$\max_{c_x^*,c_y^*,\omega_x^*,\omega_y^*,\psi_M^*,\psi_N^*}[\theta\log(c_x^*)+(1-\theta)\log(c_y^*)]$$

$$s.t. \qquad c_x^*+\frac{SP^*}{P}c_y^*+\omega_x^*e+\omega_y^*e^*+\psi_M^*r+\psi_N^*r^*=\omega_x^*x+\frac{SP^*}{P}\omega_y^*y+\omega_x^*e+\omega_y^*e^*+\psi_M^*r+\psi_N^*r^*$$

To "peg" a decentralized allocation that matches the centralized allocation, set $\omega_x^*=\omega_y^*=\frac{1}{2}=\psi_M^*=\frac{1}{2}=\psi_N^*=\frac{1}{2}$ where inserting these values into (4) for $t=0$, the problem further reduces to

$$\max_{c_x^*,c_y^*}[\theta\log(c_x^*)+(1-\theta)\log(c_y^*)] \tag{37}$$

$$s.t. \qquad c_x^*+\frac{SP^*}{P}c_y^*=\frac{1}{2}\left(x+\frac{SP^*}{P}y\right) \tag{38}$$

Inserting (38) into (37) the unconstrained maximization problem becomes

$$\max_{c_y^*}\left[\theta\log\left(\frac{1}{2}\left(x+\frac{SP^*}{P}y\right)-\frac{SP^*}{P}c_y^*\right)+(1-\theta)\log(c_y^*)\right]$$

$$\frac{\partial}{\partial c_y^*}=0:\frac{-(\frac{SP^*}{P})\theta}{c_x^*}+\frac{(1-\theta)}{c_y^*}=0 \tag{39}$$

Rearranging (39), applying to (38), and redefining $\frac{SP^*}{P}=q$ (real exchange rate) the resulting Foreign demand functions are

$$c_x^* = \frac{\theta\,(x+qy)}{2} \tag{40}$$

$$c_y^* = \frac{\theta\,(x+qy)\,(1-\theta)}{2q} \tag{41}$$

3.3 World Decentralized Equilibrium

The static version of (8) – (13) necessary for world economy equilibrium are

$$
\begin{aligned}
\omega_x + \omega_x^* &= 1 && (42)\\
\omega_y + \omega_y^* &= 1 && (43)\\
\psi_M + \psi_M^* &= 1 && (44)\\
\psi_N + \psi_N^* &= 1 && (45)\\
c_x + c_x^* &= x && (46)\\
c_y + c_y^* &= y && (47)
\end{aligned}
$$

using (35) – (36) with (46), or (40) – (41) with (47), and allowing time to evolve yields the equilibrium real exchange rate associated with decentralized economy

$$
\frac{S_t P_t^*}{P_t} = q_t = \frac{x_t (1-\theta)}{y_t \theta} \tag{48}
$$

Finally, Constraints (42) – (47) are satisfied due to $\omega_x = \omega_x^* = \omega_y = \omega_y^* = \psi_M = \psi_M^* = \psi_N = \psi_N^* = \frac{1}{2}$ (perfect risk pooling equilibrium). Using (48), (35) – (36) (40) – (41), (46) – (47), and allowing time to evolve the resulting decentralized world equilibrium allocation is

$$
\begin{aligned}
c_{xt} &= \frac{x_t}{2} && (49)\\
c_{yt} &= \frac{y_t}{2} && (50)\\
c_{xt}^* &= \frac{x_t}{2} && (51)\\
c_{yt}^* &= \frac{y_t}{2} && (52)
\end{aligned}
$$

4 Model

4.1 Inducing Malevolent Nature

Maximizing player believes state of the world to evolve according to the approximating system of stochastic equations, restated for convenience

$$
\begin{aligned}
x_{t+1} &= (1-\rho_1) + \rho_1 x_t + \varepsilon_{t+1}^x\\
y_{t+1} &= (1-\rho_2) + \rho_2 y_t + \varepsilon_{t+1}^y\\
M_{t+1} &= (1-\rho_3) + \rho_3 M_t + \varepsilon_{t+1}^M\\
N_{t+1} &= (1-\rho_4) + \rho_4 N_t + \varepsilon_{t+1}^N
\end{aligned}
$$

where the maximizing player believes the error terms are distributed according to

$$\varepsilon_t^j \sim N(0, \sigma_j^2) \ for \ j = x, y, M, N \tag{53}$$

where σ_j^2 is constant variance term for $j = x, y, M, N$

The minimizing player perturbs approximating system through error terms

$$\begin{aligned}
x_{t+1} &= (1-\rho_1) + \rho_1 x_t + \widetilde{\varepsilon_{t+1}^x} &(54)\\
y_{t+1} &= (1-\rho_2) + \rho_2 y_t + \widetilde{\varepsilon_{t+1}^y} &(55)\\
M_{t+1} &= (1-\rho_3) + \rho_3 M_t + \widetilde{\varepsilon_{t+1}^M} &(56)\\
N_{t+1} &= (1-\rho_4) + \rho_4 N_t + \widetilde{\varepsilon_{t+1}^N} &(57)
\end{aligned}$$

where $\widetilde{\varepsilon_t^j}$ for $j = x, y, M, N$ represents perturbed error terms distributed as

$$\begin{aligned}
\widetilde{\varepsilon_t^j} &\sim N(w_{t+1}, \sigma_j^2) \ for \ j = x, y, M, N &(58)\\
\widetilde{\varepsilon_t^j} - w_{t+1} &\sim N(0, \sigma_j^2) \ for \ j = x, y, M, N &(59)\\
\varepsilon_t^j &= \widetilde{\varepsilon_t^j} - w_{t+1} &(60)\\
\widetilde{\varepsilon_t^j} &= \varepsilon_t^j + w_{t+1} &(61)
\end{aligned}$$

inserting (61) for $j = x, y, M, N$ into (54) – (57) yields the perturbed stochastic system

$$\begin{aligned}
x_{t+1} &= (1-\rho_1) + \rho_1 x_t + w_{t+1} + \varepsilon_{t+1}^x &(62)\\
y_{t+1} &= (1-\rho_2) + \rho_2 y_t + w_{t+1} + \varepsilon_{t+1}^y &(63)\\
M_{t+1} &= (1-\rho_3) + \rho_3 M_t + w_{t+1} + \varepsilon_{t+1}^M &(64)\\
N_{t+1} &= (1-\rho_4) + \rho_4 N_t + w_{t+1} + \varepsilon_{t+1}^N &(65)
\end{aligned}$$

4.2 Max-Min Optimization Problem

- Objective Function and Budget Constraints

$$\max_{\substack{\{c_{xt}, c^*_{xt}, \\ c_{yt}, c^*_{yt}, \\ \omega_{xt}, \omega^*_{xt}, \\ \omega_{yt}, \omega^*_{yt}, \\ \psi_{Mt}, \psi^*_{Mt}, \\ \psi_{Nt}, \psi_{Nt}\}_{t=0}^{\infty}}} \quad \min_{\{w_{t+1}\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t \left\{ \phi \left[-\frac{(c_{xt}^{\theta} c_{yt}^{1-\theta})^{1-\gamma}}{1-\gamma} + \beta \overline{\theta} w_{t+1}^2 \right] + (1-\phi) \left[-\frac{(c_{xt}^{*\theta} c_{yt}^{*1-\theta})^{1-\gamma}}{1-\gamma} + \beta \overline{\theta^*} w_{t+1}^2 \right] \right\}$$

$$\begin{aligned} s.t. \quad \phi \sum_{t=0}^{\infty} \beta^t \Bigg\{ & c_{xt} + \frac{S_t P^*_t}{P_t} c_{yt} + \omega_{xt} e_t + \omega_{yt} e^*_t + \psi_{Mt} r_t + \psi_{Nt} r^*_t - \frac{P_{t-1}}{P_t} \omega_{xt-1} x_{t-1} - \frac{S_t P^*_{t-1}}{P_t} \omega_{yt-1} y_{t-1} \\ & - \frac{\psi_{Mt-1} \Delta M_t}{P_t} - \frac{\psi_{Nt-1} S_t \Delta N_t}{P_t} - \omega_{xt-1} e_t - \omega_{yt-1} e^*_t - \psi_{Mt-1} r_t - \psi_{Nt-1} r^*_t \Bigg\} = 0 \end{aligned}$$

$$\begin{aligned} (1- \ \phi) \sum_{t=0}^{\infty} \beta^t \Bigg\{ & c^*_{xt} + \frac{S_t P^*_t}{P_t} c^*_{yt} + \omega^*_{xt} e_t + \omega^*_{yt} e^*_t + \psi^*_{Mt} r_t + \psi^*_{Nt} r^*_t - \frac{P_{t-1}}{P_t} \omega^*_{xt-1} x_{t-1} - \frac{S_t P^*_{t-1}}{P_t} \omega^*_{yt-1} y_{t-1} \\ & - \frac{\psi^*_{Mt-1} \Delta M_t}{P_t} - \frac{\psi^*_{Nt-1} S_t \Delta N_t}{P_t} - \omega^*_{xt-1} e_t - \omega^*_{yt-1} e^*_t - \psi^*_{Mt-1} r_t - \psi^*_{Nt-1} r^*_t \Bigg\} = 0 \end{aligned}$$

where $\overline{\theta}$and $\overline{\theta^*}$represent Home and Foreign pessimism respectively, w_{t+1} is malevolent nature's control variable used to perturb $(16)-(19)$, and ϕ represents importance of country in decentralized world economy. Remaining constraints are reproduced below for convenience

- Cash-in-Advance Constraints

$$
\begin{aligned}
m_t &= P_t c_{xt} && (66)\\
n_t &= P_t^* c_{yt} && (67)\\
m_t^* &= P_t c_{xt}^* && (68)\\
n_t^* &= P_t^* c_{yt}^* && (69)
\end{aligned}
$$

note that CIA constraints bind in equilibrium.

- Approximating Stochastic Processes

$$
\begin{aligned}
x_{t+1} &= (1-\rho_1) + \rho_1 x_t + \varepsilon_{t+1}^x\\
y_{t+1} &= (1-\rho_2) + \rho_2 y_t + \varepsilon_{t+1}^y\\
M_{t+1} &= (1-\rho_3) + \rho_3 M_t + \varepsilon_{t+1}^M\\
N_{t+1} &= (1-\rho_4) + \rho_4 N_t + \varepsilon_{t+1}^N
\end{aligned}
$$

- Perturbed Stochastic Processes

$$
\begin{aligned}
x_{t+1} &= (1-\rho_1) + \rho_1 x_t + w_{t+1} + \varepsilon_{t+1}^x\\
y_{t+1} &= (1-\rho_2) + \rho_2 y_t + w_{t+1} + \varepsilon_{t+1}^y\\
M_{t+1} &= (1-\rho_3) + \rho_3 M_t + w_{t+1} + \varepsilon_{t+1}^M\\
N_{t+1} &= (1-\rho_4) + \rho_4 N_t + w_{t+1} + \varepsilon_{t+1}^N
\end{aligned}
$$

- Adding-Up Constraints

$$\begin{aligned}
\omega_{xt} + \omega^*_{xt} &= 1 \\
\omega_{yt} + \omega^*_{yt} &= 1 \\
\psi_{Mt} + \psi^*_{Mt} &= 1 \\
\psi_{Nt} + \psi^*_{Nt} &= 1 \\
c_{xt} + c^*_{xt} &= x_t \\
c_{yt} + c^*_{yt} &= y_t \\
m_t + m^*_t &= M_t \\
n_t + n^*_t &= N_t
\end{aligned}$$

4.2.1 Maximizing Player Chooses First

The optimization problem is broken into two components with maximizing player choosing first

$$\max_{\substack{\{c_{xt}, c^*_{xt}, \\ c_{yt}, c^*_{yt}, \\ \omega_{xt}, \omega^*_{xt}, \\ \omega_{yt}, \omega^*_{yt}, \\ \psi_{Mt}, \psi^*_{Mt}, \\ \psi_{Nt}, \psi_{Nt}\}_{t=0}^{\infty}}} \quad -E_t \sum_{t=0}^{\infty} \beta^t \left\{ \phi \left[\frac{(c^{\theta}_{xt} c^{1-\theta}_{yt})^{1-\gamma}}{1-\gamma} \right] + (1-\phi) \left[\frac{(c^{*\theta}_{xt} c^{*1-\theta}_{yt})^{1-\gamma}}{1-\gamma} \right] \right\}$$

$$\begin{aligned}
s.t. \quad \phi \sum_{t=0}^{\infty} \beta^t \; \Bigg\{ & c_{xt} + \frac{S_t P^*_t}{P_t} c_{yt} + \omega_{xt} e_t + \omega_{yt} e^*_t + \psi_{Mt} r_t + \psi_{Nt} r^*_t - \frac{P_{t-1}}{P_t} \omega_{xt-1} x_{t-1} - \frac{S_t P^*_{t-1}}{P_t} \omega_{yt-1} y_{t-1} \\
& - \frac{\psi_{Mt-1} \Delta M_t}{P_t} - \frac{\psi_{Nt-1} S_t \Delta N_t}{P_t} - \omega_{xt-1} e_t - \omega_{yt-1} e^*_t - \psi_{Mt-1} r_t - \psi_{Nt-1} r^*_t \Bigg\} = 0
\end{aligned}$$

$$(1-\ \phi)\sum_{t=0}^{\infty}\beta^t\left\{c^*_{xt}+\frac{S_tP^*_t}{P_t}c^*_{yt}+\omega^*_{xt}e_t+\omega^*_{yt}e^*_t+\psi^*_{Mt}r_t+\psi^*_{Nt}r^*_t-\frac{P_{t-1}}{P_t}\omega^*_{xt-1}x_{t-1}-\frac{S_tP^*_{t-1}}{P_t}\omega^*_{yt-1}y_{t-1}\right.$$
$$\left.-\frac{\psi^*_{Mt-1}\Delta M_t}{P_t}-\frac{\psi^*_{Nt-1}S_t\Delta N_t}{P_t}-\omega^*_{xt-1}e_t-\omega^*_{yt-1}e^*_t-\psi^*_{Mt-1}r_t-\psi^*_{Nt-1}r^*_t\right\}=0$$

where (66) − (69), (8) − (15), and (16) − (19) completes maximizing player problem specification. Forming the lagrangian,

$$\pounds=E_t\sum_{t=0}^{\infty}\beta^t\left\{\phi\left[\frac{(c^{\theta}_{xt}c^{1-\theta}_{yt})^{1-\gamma}}{1-\gamma}-\lambda^1_t\left(c_{xt}+\frac{S_tP^*_t}{P_t}c_{yt}+\omega_{xt}e_t+\omega_{yt}e^*_t+\psi_{Mt}r_t+\psi_{Nt}r^*_t-\frac{P_{t-1}}{P_t}\omega_{xt-1}x_{t-1}\right.\right.\right.$$
$$-\frac{S_tP^*_{t-1}}{P_t}\omega_{yt-1}y_{t-1}-\frac{\psi_{Mt-1}\Delta M_t}{P_t}-\frac{\psi_{Nt-1}S_t\Delta N_t}{P_t}-\omega_{xt-1}e_t$$
$$\left.\left.-\omega_{yt-1}e^*_t-\psi_{Mt-1}r_t-\psi_{Nt-1}r^*_t\right)\right]$$
$$+(1-\phi)\left[\frac{(c^{*\theta}_{xt}c^{*1-\theta}_{yt})^{1-\gamma}}{1-\gamma}-\lambda^2_t\left(c^*_{xt}+\frac{S_tP^*_t}{P_t}c^*_{yt}+\omega^*_{xt}e_t+\omega^*_{yt}e^*_t+\psi^*_{Mt}r_t+\psi^*_{Nt}r^*_t-\frac{P_{t-1}}{P_t}\omega^*_{xt-1}x_{t-1}\right.\right.$$
$$-\frac{S_tP^*_{t-1}}{P_t}\omega^*_{yt-1}y_{t-1}-\frac{\psi^*_{Mt-1}\Delta M_t}{P_t}-\frac{\psi^*_{Nt-1}S_t\Delta N_t}{P_t}-\omega^*_{xt-1}e_t$$
$$\left.\left.\left.-\omega^*_{yt-1}e^*_t-\psi^*_{Mt-1}r_t-\psi^*_{Nt-1}r^*_t\right)\right]\right\}$$

$$\frac{\partial\pounds}{\partial c_{xt}}=0:\left(c^{\theta}_{xt}c^{1-\theta}_{yt}\right)^{-\gamma}\theta c^{\theta-1}_{xt}c^{1-\theta}_{yt}\ -\lambda^1_t=0 \tag{70}$$
$$\frac{\partial\pounds}{\partial c_{yt}}=0:\left(c^{\theta}_{xt}c^{1-\theta}_{yt}\right)^{-\gamma}(1-\theta)\,c^{\theta}_{xt}c^{-\theta}_{yt}\ -\lambda^1_t\frac{S_tP^*_t}{P_t}=0 \tag{71}$$
$$\frac{\partial\pounds}{\partial\omega_{xt}}=0:-\lambda^1_te_t+\beta E_t\left(\lambda^1_{t+1}e_{t+1}+\lambda^1_{t+1}\frac{P_t}{P_{t+1}}x_t\right)=0 \tag{72}$$
$$\frac{\partial\pounds}{\partial\omega_{yt}}=0:-\lambda^1_te^*_t+\beta E_t\left(\lambda^1_{t+1}e^*_{t+1}+\lambda^1_{t+1}\frac{S_{t+1}P^*_t}{P_{t+1}}x_t\right)=0 \tag{73}$$
$$\frac{\partial\pounds}{\partial\psi_{Mt}}=0:-\lambda^1_tr_t+\beta E_t\left(\lambda^1_{t+1}r_{t+1}+\lambda^1_{t+1}\frac{\Delta M_{t+1}}{P_{t+1}}x_t\right)=0 \tag{74}$$
$$\frac{\partial\pounds}{\partial\psi_{Nt}}=0:-\lambda^1_tr^*_t+\beta E_t\left(\lambda^1_{t+1}r^*_{t+1}+\lambda^1_{t+1}\frac{S_{t+1}\Delta N_{t+1}}{P_{t+1}}x_t\right)=0 \tag{75}$$

Foreign has an analogous set of F.O.C's. Using $(66)-(69)$ and $(14)-(15)$ derive aggregate relative prices

$$\begin{aligned} P_t &= \frac{M_t}{x_t} && (76) \\ P_t^* &= \frac{N_t}{y_t} && (77) \end{aligned}$$

Solving (70) for λ_t^1, insert the result along with $(76)-(77)$ into $(71)-(75)$ and rearranging yields

$$\begin{aligned}
S_t &= \frac{(1-\theta)\,M_t}{\theta N_t} && (78) \\
\frac{e_t}{x_t} &= \beta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\left(\frac{e_{t+1}}{e_t}+\frac{M_t}{M_{t+1}}\right)\right] && (79) \\
\frac{e_t^*}{q_t y_t} &= \beta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\left(\frac{e_{t+1}^*}{q_{t+1}y_{t+1}}+\frac{N_t}{N_{t+1}}\right)\right] && (80) \\
\frac{r_t}{x_t} &= \beta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\left(\frac{r_{t+1}}{x_{t+1}}+\frac{\Delta M_{t+1}}{M_{t+1}}\right)\right] && (81) \\
\frac{r_t^*}{x_t} &= \beta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\left(\frac{r_{t+1}}{x_{t+1}}+\frac{(1-\theta)}{\theta}\frac{\Delta N_{t+1}}{N_{t+1}}\right)\right] && (82)
\end{aligned}$$

where $C_t = c_{xt}^{\theta}c_{yt}^{1-\theta}$. $(78)-(82)$ represent maximizing player's Euler equations in terms of exogenous state variables which evolve according to the approximating system $(16)-(19)$.Although no explicit foreign exchange market exists in the model , the forward exchange rate can be written as

$$F_t = S_t \frac{\beta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\frac{N_t}{N_{t+1}}\right]}{\beta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\frac{M_t}{M_{t+1}}\right]} \qquad (83)$$

The focus will be on (78) and (83).

Steady-State Beginning with (78) and eliminating time subscripts,

$$\overline{S} = \frac{(1-\theta)\,\overline{M}}{\theta\overline{N}} \tag{84}$$

where the bar above each variable represents its steady-state counterpart and $\overline{M}$ aswell as $\overline{N}$ are steady-state Home and Foreign money respectively whose values are developed in minimizing player's section.

Next given (83) and removing time subscripts in addition to expectations operators results in

$$\overline{F} = \overline{S}\frac{\beta\left[\left(\frac{\overline{C}}{\overline{C}}\right)^{1-\gamma}\frac{\overline{N}}{\overline{N}}\right]}{\beta\left[\left(\frac{\overline{C}}{\overline{C}}\right)^{1-\gamma}\frac{\overline{M}}{\overline{M}}\right]}$$

$$\overline{F} = \overline{S} \tag{85}$$

where $\overline{C} = \bar{c}_x^{\theta}\bar{c}_y^{1-\theta}$ and both $\bar{c}_x$ aswell as $\bar{c}_y$ values are developed in minimizing player's section.

Log-Linearlization

Spot Exchange Rate Beginning with and applying Uhlig's Method to log-linearlize (78),

$$\begin{aligned}
S_t &= \frac{(1-\theta)\,M_t}{\theta N_t} \\
\overline{S}e^{\widetilde{S}_t} &= \frac{(1-\theta)\,\overline{M}e^{\widetilde{M}_t}}{\theta\overline{N}e^{\widetilde{N}_t}} \\
\frac{\theta\overline{SN}}{(1-\theta)\,\overline{M}} &= e^{\widetilde{M}_t-\widetilde{N}_t-\widetilde{S}_t} \\
1 &= e^{\widetilde{M}_t-\widetilde{N}_t-\widetilde{S}_t} \qquad (86)
\end{aligned}$$

where (84) was used.Taking the first-order taylor expansion to natural exponential in (86)

$$e^{\widetilde{M}_t-\widetilde{N}_t-\widetilde{S}_t} \simeq e^{\widetilde{M}-\widetilde{N}-\widetilde{S}} + e^{\widetilde{M}-\widetilde{N}-\widetilde{S}}\left(\widetilde{M}_t-\widetilde{M}\right) - e^{\widetilde{M}-\widetilde{N}-\widetilde{S}}\left(\widetilde{N}_t-\widetilde{N}\right) - e^{\widetilde{M}-\widetilde{N}-\widetilde{S}}\left(\widetilde{S}_t-\widetilde{S}\right)$$

$$e^{\widetilde{M}_t-\widetilde{N}_t-\widetilde{S}_t} \simeq 1+\widetilde{M}_t-\widetilde{N}_t-\widetilde{S}_t \tag{87}$$

where $\widetilde{M}=\widetilde{N}=\widetilde{S}=0$ was used. Inserting (87) into (86) and simplifying results in

$$\widetilde{S}_t=\widetilde{M}_t-\widetilde{N}_t \tag{88}$$

Forward Exchange Rate Next, simplifying (83)

$$\begin{aligned}
F_t &= S_t\frac{\beta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\frac{N_t}{N_{t+1}}\right]}{\beta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\frac{M_t}{M_{t+1}}\right]}\\
F_t &= S_t\frac{N_t}{M_t}E_t\left[\frac{M_{t+1}}{N_{t+1}}\right]\\
F_t &= \frac{(1-\theta)}{\theta}E_t\left[\frac{M_{t+1}}{N_{t+1}}\right] \qquad (89)
\end{aligned}$$

where (78) was used. Applying Uhlig's method to (89) and first order taylor expansion results in

$$\begin{aligned}
F_t &= \frac{(1-\theta)}{\theta}E_t\left[\frac{M_{t+1}}{N_{t+1}}\right]\\
\overline{F}e^{\widetilde{F}_t} &= \frac{(1-\theta)}{\theta}E_t\left[\frac{\overline{M}e^{\widetilde{M}_{t+1}}}{\overline{N}e^{\widetilde{N}_{t+1}}}\right]\\
1 &= e^{E_t\left[\widetilde{M}_{t+1}\right]-E_t\left[\widetilde{N}_{t+1}\right]-\widetilde{F}_t}
\end{aligned}$$

$$\widetilde{F_t} = E_t\left[\widetilde{M_{t+1}}\right] - E_t\left[\widetilde{N_{t+1}}\right] \tag{90}$$

where (84) and (85) were used. Thus (88) and (90) are expressed in terms of log-linearlized exogenous stochastic state variables.

Approximating Stochastic Processes Finally, the log-linearlized counterparts to (16) – (19) are

$$\begin{aligned}
\widetilde{x_{t+1}} &= \rho_1 \widetilde{x_t} + \varepsilon^x_{t+1} && (91)\\
\widetilde{y_{t+1}} &= \rho_2 \widetilde{y_t} + \varepsilon^y_{t+1} && (92)\\
\widetilde{M_{t+1}} &= \rho_3 \widetilde{M_t} + \varepsilon^M_{t+1} && (93)\\
\widetilde{N_{t+1}} &= \rho_4 \widetilde{N_t} + \varepsilon^N_{t+1} && (94)
\end{aligned}$$

System (91) – (94) can be converted into linear algebra

$$X_{t+1} = AX_t + C\varepsilon_{t+1} \tag{95}$$

where

$$X_t = \begin{bmatrix} \widetilde{x_t} \\ \widetilde{y_t} \\ \widetilde{M_t} \\ \widetilde{N_t} \end{bmatrix}, \qquad \varepsilon_{t+1} = \begin{bmatrix} \varepsilon^x_{t+1} \\ \varepsilon^y_{t+1} \\ \varepsilon^M_{t+1} \\ \varepsilon^N_{t+1} \end{bmatrix}$$

$$A = \begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_3 & 0 \\ 0 & 0 & 0 & \rho_4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4.2.2 Minimizing Player Chooses Second

The malevolent nature assumes the maximizing player is both optimizing and in equilibrium so that perfect risk sharking equilibrium, (49) – (52), (78) – (82) all hold, and (3) – (4) bind so that the minimizing player's problem is presented as

$$\min_{\{w_{t+1}\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t \left\{ \phi \left[-\frac{(c_{xt}^{\theta} c_{yt}^{1-\theta})^{1-\gamma}}{1-\gamma} + \beta \overline{\theta} w_{t+1}^2 \right] + (1-\phi) \left[-\frac{(c_{xt}^{*\theta} c_{yt}^{*1-\theta})^{1-\gamma}}{1-\gamma} + \beta \overline{\theta^*} w_{t+1}^2 \right] \right\}$$

$$\begin{aligned} s.t. \quad x_{t+1} &= (1-\rho_1) + \rho_1 x_t + w_{t+1} + \varepsilon_{t+1}^x \\ y_{t+1} &= (1-\rho_2) + \rho_2 y_t + w_{t+1} + \varepsilon_{t+1}^y \\ M_{t+1} &= (1-\rho_3) + \rho_3 M_t + w_{t+1} + \varepsilon_{t+1}^M \\ N_{t+1} &= (1-\rho_4) + \rho_4 N_t + w_{t+1} + \varepsilon_{t+1}^N \end{aligned}$$

(12) – (15), (66) – (69), aswell as aforementioned conditions complete specification of minimizing player's problem.

Lemma 1 *Home and Foreign have homogenous pessimism so that* $\overline{\theta} = \overline{\theta^*}$

The resulting minimization problem with homogenous pessimism is

$$\min_{\{w_{t+1}\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t \left\{ -\left[\phi \frac{(c_{xt}^{\theta} c_{yt}^{1-\theta})^{1-\gamma}}{1-\gamma} + (1-\phi) \frac{(c_{xt}^{*\theta} c_{yt}^{*1-\theta})^{1-\gamma}}{1-\gamma} \right] + \beta \overline{\overline{\theta}} w_{t+1}^2 \right\}$$

$$\begin{aligned} s.t. \quad x_{t+1} &= (1-\rho_1) + \rho_1 x_t + w_{t+1} + \varepsilon_{t+1}^x \\ y_{t+1} &= (1-\rho_2) + \rho_2 y_t + w_{t+1} + \varepsilon_{t+1}^y \\ M_{t+1} &= (1-\rho_3) + \rho_3 M_t + w_{t+1} + \varepsilon_{t+1}^M \\ N_{t+1} &= (1-\rho_4) + \rho_4 N_t + w_{t+1} + \varepsilon_{t+1}^N \end{aligned}$$

where $\overline{\overline{\theta}} = \phi\overline{\theta} + (1-\phi)\overline{\theta^*}$, in other words world pessimism $\overline{\overline{\theta}}$ is an implicit weighted average of $\overline{\theta}$ and $\overline{\theta^*}$ so since $\overline{\theta} = \overline{\theta^*}$ and $\phi = \frac{1}{2}$, $\overline{\overline{\theta}}$ becomes a parameter used by both Home and Foreign as a concern for model misspecification.To solve the minization problem, Linear-Quadratic Approximation Control is used

Steady-State The steady-states needed for log-linearlization of minimizing player problem involve (49) − (52), (62) − (65), and (76) − (77). Initially, eliminating time subscripts and turning off both perturbation and error terms of (62) − (65) yields

$$\begin{aligned} x &= (1-\rho_1) + \rho_1 x \\ y &= (1-\rho_2) + \rho_2 y \\ M &= (1-\rho_3) + \rho_3 M \\ N &= (1-\rho_4) + \rho_4 N \end{aligned}$$

simplifying results in

$$\begin{aligned} \overline{x} &= 1 && (96) \\ \overline{y} &= 1 && (97) \\ \overline{M} &= 1 && (98) \\ \overline{N} &= 1 && (99) \end{aligned}$$

where (96) − (99) are steady-state versions of their time-varying stochastic counterparts. Next, eliminating time subscripts from (49)−(52) and inserting (96)−(97) where appropriate results in

$$\begin{aligned} \overline{c_x} &= \frac{1}{2} && (100) \\ \overline{c_y} &= \frac{1}{2} && (101) \\ \overline{c_x^*} &= \frac{1}{2} && (102) \\ \overline{c_y^*} &= \frac{1}{2} && (103) \end{aligned}$$

where (100)−(103) are steady-state versions of their time-varying stochastic counterparts. Finally, following similar steps for (76) − (77) and using (96) − (99) results in

$$\begin{aligned} \overline{P} &= 1 && (104)\\ \overline{P^*} &= 1 && (105) \end{aligned}$$

where (104) − (105) represent steady-state home price and foreign price respectively.

Log-Linearlization To properly use LQA-Control techniques, the minimizing agents problem must be transfored using log-linearlization techniques .

Minimizing Agent's Weighted Average Component Beginning with (1) and using Uhlig's Method

$$\begin{aligned} \frac{(c_{xt}^{\theta}c_{yt}^{1-\theta})^{1-\gamma}}{1-\gamma} &= \frac{\left[\left(\overline{c_x^{\theta}}e^{\theta\widetilde{c_{xt}}}\right)\left(\overline{c_y^{(1-\theta)}}e^{(1-\theta)\widetilde{c_{yt}}}\right)\right]^{1-\gamma}}{1-\gamma} \\ &= \frac{\left(\overline{c_x^{\theta}c_y^{(1-\theta)}}\right)^{(1-\gamma)}e^{(1-\gamma)(\theta\widetilde{c_{xt}}+(1-\theta)\widetilde{c_{yt}})}}{1-\gamma} \end{aligned} \tag{106}$$

Performing a second order taylor expansion (Kim and Kim 2007) on the natural exponential component of (106)

$$\begin{aligned} e^{(1-\gamma)(\theta\widetilde{c_{xt}}+(1-\theta)\widetilde{c_{yt}})} \simeq{}& e^{(1-\gamma)(\theta\widetilde{c_x}+(1-\theta)\widetilde{c_y})} + e^{(1-\gamma)(\theta\widetilde{c_x}+(1-\theta)\widetilde{c_y})}(1-\gamma)\,\theta\left(\widetilde{c_{xt}}-\widetilde{c_x}\right) \\ &+ e^{(1-\gamma)(\theta\widetilde{c_x}+(1-\theta)\widetilde{c_y})}(1-\gamma)(1-\theta)\left(\widetilde{c_{yt}}-\widetilde{c_y}\right) \\ &+ \frac{1}{2}\left[e^{(1-\gamma)(\theta\widetilde{c_x}+(1-\theta)\widetilde{c_y})}(1-\gamma)^2\theta^2\left(\widetilde{c_{xt}}-\widetilde{c_x}\right)^2\right. \\ &+2e^{(1-\gamma)(\theta\widetilde{c_x}+(1-\theta)\widetilde{c_y})}(1-\gamma)^2(1-\theta)\,\theta\left(\widetilde{c_{xt}}-\widetilde{c_x}\right)\left(\widetilde{c_{yt}}-\widetilde{c_y}\right) \\ &\left.+e^{(1-\gamma)(\theta\widetilde{c_x}+(1-\theta)\widetilde{c_y})}(1-\gamma)^2(1-\theta)^2\left(\widetilde{c_{yt}}-\widetilde{c_y}\right)^2\right] \end{aligned} \tag{107}$$

Inserting $\widetilde{c_x} = \widetilde{c_y} = 0$ into (107) and simplifying results in

$$e^{(1-\gamma)(\theta\widetilde{c_{xt}}+(1-\theta)\widetilde{c_{yt}})} \simeq 1 + (1-\gamma)\,\theta\widetilde{c_{xt}} + (1-\gamma)\,(1-\theta)\,\widetilde{c_{yt}} + \frac{1}{2}\left[(1-\gamma)^2\,\theta^2\widetilde{c_{xt}}^2 \right.$$
$$\left. +2\,(1-\gamma)^2\,(1-\theta)\,\theta\widetilde{c_{xt}}\widetilde{c_{yt}} + (1-\gamma)^2\,(1-\theta)^2\,\widetilde{c_{yt}}^2\right] \quad (108)$$

Inserting $\widetilde{c_{xt}}\widetilde{c_{yt}} = 0$ into (108) for the cross-product term

$$e^{(1-\gamma)(\theta\widetilde{c_{xt}}+(1-\theta)\widetilde{c_{yt}})} \simeq 1 + (1-\gamma)\,\theta\widetilde{c_{xt}} + (1-\gamma)\,(1-\theta)\,\widetilde{c_{yt}} + \frac{1}{2}\left[(1-\gamma)^2\,\theta^2\widetilde{c_{xt}}^2 \right.$$
$$\left. +\,(1-\gamma)^2\,(1-\theta)^2\,\widetilde{c_{yt}}^2\right] \quad (109)$$

$$\therefore \frac{(c_{xt}^{\theta}c_{yt}^{1-\theta})^{1-\gamma}}{1-\gamma} \simeq \frac{\left(\overline{c_x^{\theta}}\overline{c_y^{(1-\theta)}}\right)^{(1-\gamma)}}{1-\gamma} + \left(\overline{c_x^{\theta}}\overline{c_y^{(1-\theta)}}\right)^{(1-\gamma)}\left[\theta\widetilde{c_{xt}} + (1-\theta)\,\widetilde{c_{yt}}\right.$$
$$\left. +\frac{1}{2}\left((1-\gamma)\,\theta^2\widetilde{c_{xt}}^2 + (1-\gamma)\,(1-\theta)^2\,\widetilde{c_{yt}}^2\right)\right] \quad (110)$$

where (109) is inserted into (106) for the natural exponential term, and simplified resulting in (110). The constant term implies that (110) is expressed in terms of levels but the expression must be in terms of log-deviations so subtract the constant term from both sides to yield

$$\widetilde{U_H} \simeq \left(\overline{c_x^{\theta}}\overline{c_y^{(1-\theta)}}\right)^{(1-\gamma)}\left[\theta\widetilde{c_{xt}} + (1-\theta)\,\widetilde{c_{yt}} + \frac{1}{2}\left((1-\gamma)\,\theta^2\widetilde{c_{xt}}^2 + (1-\gamma)\,(1-\theta)^2\,\widetilde{c_{yt}}^2\right)\right] \quad (111)$$

where $\widetilde{U}_H = \frac{(c_{xt}^{\theta} c_{yt}^{1-\theta})^{1-\gamma}}{1-\gamma} - \frac{\left(\overline{c_x^{\theta}} \overline{c_y^{(1-\theta)}}\right)^{(1-\gamma)}}{1-\gamma}$. Analogously for Foreign,

$$\widetilde{U}_F \simeq \left(\overline{c_x^{*\theta}} \overline{c_y^{*(1-\theta)}}\right)^{(1-\gamma)} \left[\theta \widetilde{c_{xt}^*} + (1-\theta)\widetilde{c_{yt}^*} + \frac{1}{2}\left((1-\gamma)\theta^2 \widetilde{c_{xt}^*}^2 + (1-\gamma)(1-\theta)^2 \widetilde{c_{yt}^*}^2\right)\right] \tag{112}$$

where $\widetilde{U}_F = \frac{(c_{xt}^{*\theta} c_{yt}^{*1-\theta})^{1-\gamma}}{1-\gamma} - \frac{\left(\overline{c_x^{*\theta}} \overline{c_y^{*(1-\theta)}}\right)^{(1-\gamma)}}{1-\gamma}$.

The minimizing player's objective function contains a weighted average component of (1) and (2) or

$$\phi \frac{(c_{xt}^{\theta} c_{yt}^{1-\theta})^{1-\gamma}}{1-\gamma} + (1-\phi)\frac{(c_{xt}^{*\theta} c_{yt}^{*1-\theta})^{1-\gamma}}{1-\gamma} \tag{113}$$

Inserting (111) for (1) and (112) for (2) in (113)

$$\phi \widetilde{U_H} + (1-\phi)\widetilde{U_F} \tag{114}$$

Setting $\phi = \frac{1}{2}$, expanding, and simplifying (114)

$$\alpha\left[\theta\left(\widetilde{c_{xt}} + \widetilde{c_{xt}^*}\right) + (1-\theta)\left(\widetilde{c_{yt}} + \widetilde{c_{yt}^*}\right) + \frac{1}{2}(1-\gamma)\theta^2\left(\widetilde{c_{xt}^2} + \widetilde{c_{xt}^{2*}}\right) + \frac{1}{2}(1-\gamma)(1-\theta)^2\left(\widetilde{c_{yt}^2} + \widetilde{c_{yt}^{2*}}\right)\right] \tag{115}$$

where $\alpha = \frac{\left(\overline{c_x^{\theta}} \overline{c_y^{(1-\theta)}}\right)^{(1-\gamma)}}{2}$ and (115) ultimately is the log-linearlized (in terms of deviations froms steady-state) form of (113). Now in order to express (115) in terms of exogenous variables x_t, y_t, M_t, and N_t the remainder of minimizing player's specification is used namely $(12)-(15)$, and $(66)-(69)$. Log-linearlize (12) using Uhlig's method and second-order taylor approximation,

$$
\begin{aligned}
c_{xt} + c_{xt}^{*} &= x_t \\
\overline{c_x} e^{\widetilde{c_{xt}}} + \overline{c_x} e^{\widetilde{c_{xt}^{*}}} &= \overline{x} e^{\widetilde{x}_t} \\
\overline{c_x}\left(1 + \widetilde{c_{xt}} + \frac{1}{2}\widetilde{c_{xt}}^2\right) + \overline{c_x^{*}}\left(1 + \widetilde{c_{xt}^{*}} + \frac{1}{2}\widetilde{c_{xt}^{*}}^2\right) &= \overline{x}\left(1 + \widetilde{x}_t + \frac{1}{2}\widetilde{x}_t^2\right) \\
\overline{c_x} + \overline{c_x}\widetilde{c_{xt}} + \frac{\overline{c_x}}{2}\widetilde{c_{xt}}^2 + \overline{c_x^{*}} + \overline{c_x^{*}}\widetilde{c_{xt}}^{*} + \frac{\overline{c_x^{*}}}{2}\widetilde{c_{xt}^{*}}^2 &= \overline{x} + \overline{x}\widetilde{x}_t + \frac{\overline{x}}{2}\widetilde{x}_t^2 \\
\overline{c_x}\widetilde{c_{xt}} + + \overline{c_x^{*}}\widetilde{c_{xt}}^{*} &= \overline{x}\widetilde{x}_t + \frac{\overline{x}}{2}\widetilde{x}_t^2 - \frac{\overline{c_x}}{2}\widetilde{c_{xt}}^2 - \frac{\overline{c_x^{*}}}{2}\widetilde{c_{xt}^{*}}^2 \\
\widetilde{c_{xt}} + \widetilde{c_{xt}}^{*} &= \frac{\overline{x}}{\overline{c_x}}\widetilde{x}_t + \frac{\overline{x}}{2\overline{c_x}}\widetilde{x}_t^2 - \frac{1}{2}\widetilde{c_{xt}}^2 - \frac{1}{2}\widetilde{c_{xt}^{*}}^2
\end{aligned}
$$

$$
\widetilde{c_{xt}} + \widetilde{c_{xt}}^{*} = \frac{\overline{x}}{\overline{c_x}}\widetilde{x}_t + \frac{\overline{x}}{2\overline{c_x}}\widetilde{x}_t^2 - \frac{1}{2}\left(\widetilde{c_{xt}}^2 + \widetilde{c_{xt}^{*}}^2\right) \tag{116}
$$

where $\overline{c_x} = \overline{c_x^{*}}$ is used. Following a similar procedure for (13) results in

$$
\widetilde{c_{yt}} + \widetilde{c_{yt}}^{*} = \frac{\overline{y}}{\overline{c_y}}\widetilde{y}_t + \frac{\overline{y}}{2\overline{c_y}}\widetilde{y}_t^2 - \frac{1}{2}\left(\widetilde{c_{yt}}^2 + \widetilde{c_{yt}^{*}}^2\right) \tag{117}
$$

where $\overline{c_y} = \overline{c_y^{*}}$ is used. Plug (116) and (117) into (115) for linear consumptions

$$
\begin{aligned}
\alpha\Bigg[\theta\left(\frac{\overline{x}}{\overline{c_x}}\widetilde{x}_t + \frac{\overline{x}}{2\overline{c_x}}\widetilde{x}_t^2\right) - \frac{\theta}{2}\left(\widetilde{c_{xt}}^2 + \widetilde{c_{xt}^{*}}^2\right) + (1-\theta)\left(\frac{\overline{y}}{\overline{c_y}}\widetilde{y}_t + \frac{\overline{y}}{2\overline{c_y}}\widetilde{y}_t^2\right) - \frac{(1-\theta)}{2}\left(\widetilde{c_{yt}}^2 + \widetilde{c_{yt}^{*}}^2\right) \\
+\frac{1}{2}(1-\gamma)\theta^2\left(\widetilde{c_{xt}^2} + \widetilde{c_{xt}^{2*}}\right) + \frac{1}{2}(1-\gamma)(1-\theta)^2\left(\widetilde{c_{yt}^2} + \widetilde{c_{yt}^{2*}}\right)\Bigg]
\end{aligned}
$$

further simplification yields

$$
\begin{aligned}
\alpha\Bigg[\theta\left(\frac{\overline{x}}{\overline{c_x}}\widetilde{x}_t + \frac{\overline{x}}{2\overline{c_x}}\widetilde{x}_t^2\right) + (1-\theta)\left(\frac{\overline{y}}{\overline{c_y}}\widetilde{y}_t + \frac{\overline{y}}{2\overline{c_y}}\widetilde{y}_t^2\right) + \left(\frac{1}{2}(1-\gamma)\theta^2 - \frac{\theta}{2}\right)\left(\widetilde{c_{xt}^2} + \widetilde{c_{xt}^{2*}}\right) \\
+\left(\frac{1}{2}(1-\gamma)(1-\theta)^2 - \frac{(1-\theta)}{2}\right)\left(\widetilde{c_{yt}^2} + \widetilde{c_{yt}^{2*}}\right)\Bigg]
\end{aligned} \tag{118}
$$

Next, combining (14) – (15) with (66) – (69), log-linearlizing using Uhlig's method with second-order taylor expansion so that a similar process to (116) and (117) is followed results in

$$\widetilde{c_{xt}^2}+\widetilde{c_{xt}^{*2}}=\frac{2\overline{M}}{\overline{P}\overline{c_x}}\widetilde{M_t}+\frac{\overline{M}}{\overline{P}\overline{c_x}}\widetilde{M_t^2}-2\left(\widetilde{c_{xt}}+\widetilde{c_{xt}^*}\right) \tag{119}$$

and analogously,

$$\widetilde{c_{yt}^2}+\widetilde{c_{yt}^{*2}}=\frac{2\overline{N}}{\overline{P^*}\overline{c_y}}\widetilde{N_t}+\frac{\overline{N}}{\overline{P^*}\overline{c_y}}\widetilde{N_t^2}-2\left(\widetilde{c_{yt}}+\widetilde{c_{yt}^*}\right) \tag{120}$$

Inserting (119) and (120) into (118) for quadratic consumption terms yields

$$\begin{aligned}\alpha\Bigg[\theta\left(\frac{\overline{x}}{\overline{c_x}}\widetilde{x_t}+\frac{\overline{x}}{2\overline{c_x}}\widetilde{x_t}^2\right)+(1-\theta)\left(\frac{\overline{y}}{\overline{c_y}}\widetilde{y_t}+\frac{\overline{y}}{2\overline{c_y}}\widetilde{y_t}^2\right)&\\
+\left(\frac{1}{2}(1-\gamma)\theta^2-\frac{\theta}{2}\right)\left(\frac{2\overline{M}}{\overline{P}\overline{c_x}}\widetilde{M_t}+\frac{\overline{M}}{\overline{P}\overline{c_x}}\widetilde{M_t^2}-2\left(\widetilde{c_{xt}}+\widetilde{c_{xt}^*}\right)\right)&\\
+\left(\frac{1}{2}(1-\gamma)(1-\theta)^2-\frac{(1-\theta)}{2}\right)\left(\frac{2\overline{N}}{\overline{P^*}\overline{c_y}}\widetilde{N_t}+\frac{\overline{N}}{\overline{P^*}\overline{c_y}}\widetilde{N_t^2}-2\left(\widetilde{c_{yt}}+\widetilde{c_{yt}^*}\right)\right)\Bigg]&\end{aligned} \tag{121}$$

Since the minimizing player assumes the maximizing player is in equilibrium, use log-linear versions of (49) – (52) or

$$\widetilde{c_{xt}} = \frac{\widetilde{x_t}}{2}$$

$$\widetilde{c_{yt}} = \frac{\widetilde{y_t}}{2} \tag{122}$$

$$\widetilde{c_{xt}^*} = \frac{\widetilde{x_t}}{2} \tag{123}$$

$$\widetilde{c_{yt}^*} = \frac{\widetilde{y_t}}{2} \tag{124}$$

Insert (122) – (124) into (121) for remaining linear consumption terms and expanding results in

$$\alpha\left[\frac{\theta\overline{x}}{\overline{c_x}}\widetilde{x}_t+\frac{\theta\overline{x}}{2\overline{c_x}}\widetilde{x}_t^{\,2}+\frac{(1-\theta)\,\overline{y}}{\overline{c_y}}\widetilde{y}_t+\frac{(1-\theta)\,\overline{y}}{2\overline{c_y}}\widetilde{y}_t^{\,2}\right.$$
$$+\frac{2\overline{M}}{\overline{P}\overline{c_x}}\left(\frac{1}{2}(1-\gamma)\,\theta^2-\frac{\theta}{2}\right)\widetilde{M}_t+\frac{\overline{M}}{\overline{P}\overline{c_x}}\left(\frac{1}{2}(1-\gamma)\,\theta^2-\frac{\theta}{2}\right)\widetilde{M_t^2}-2\left(\frac{1}{2}(1-\gamma)\,\theta^2-\frac{\theta}{2}\right)\widetilde{x}_t$$
$$+\frac{2\overline{N}}{\overline{P^*}\overline{c_y}}\left(\frac{1}{2}(1-\gamma)(1-\theta)^2-\frac{(1-\theta)}{2}\right)\widetilde{N}_t+\frac{\overline{N}}{\overline{P^*}\overline{c_y}}\left(\frac{1}{2}(1-\gamma)(1-\theta)^2-\frac{(1-\theta)}{2}\right)\widetilde{N_t^2}$$
$$\left.-2\left(\frac{1}{2}(1-\gamma)(1-\theta)^2-\frac{(1-\theta)}{2}\right)\widetilde{y}_t\right] \quad (125)$$

consolidating linear and quadratic terms in (125)

$$\alpha\left\{\left[\frac{\theta\overline{x}}{\overline{c_x}}-2\left(\frac{1}{2}(1-\gamma)\,\theta^2-\frac{\theta}{2}\right)\right]\widetilde{x}_t+\left[\frac{(1-\theta)\,\overline{y}}{\overline{c_y}}-2\left(\frac{1}{2}(1-\gamma)(1-\theta)^2-\frac{(1-\theta)}{2}\right)\right]\widetilde{y}_t\right.$$
$$+\frac{2\overline{M}}{\overline{P}\overline{c_x}}\left(\frac{1}{2}(1-\gamma)\,\theta^2-\frac{\theta}{2}\right)\widetilde{M}_t+\frac{2\overline{N}}{\overline{P^*}\overline{c_y}}\left(\frac{1}{2}(1-\gamma)(1-\theta)^2-\frac{(1-\theta)}{2}\right)\widetilde{N}_t$$
$$\left.+\frac{\theta\overline{x}}{2\overline{c_x}}\widetilde{x}_t^{\,2}+\frac{(1-\theta)\,\overline{y}}{2\overline{c_y}}\widetilde{y}_t^{\,2}+\frac{\overline{M}}{\overline{P}\overline{c_x}}\left(\frac{1}{2}(1-\gamma)\,\theta^2-\frac{\theta}{2}\right)\widetilde{M_t^2}+\frac{\overline{N}}{\overline{P^*}\overline{c_y}}\left(\frac{1}{2}(1-\gamma)(1-\theta)^2-\frac{(1-\theta)}{2}\right)\widetilde{N_t^2}\right\} \quad (126)$$

For small deviations in each state variable, the linear terms in (126) can be ignored (Levine and Pearlman 2006) yielding

$$\alpha\left[\frac{\theta\overline{x}}{2\overline{c_x}}\widetilde{x}_t^{\,2}+\frac{(1-\theta)\,\overline{y}}{2\overline{c_y}}\widetilde{y}_t^{\,2}+\frac{\overline{M}}{\overline{P}\overline{c_x}}\left(\frac{1}{2}(1-\gamma)\,\theta^2-\frac{\theta}{2}\right)\widetilde{M_t^2}+\frac{\overline{N}}{\overline{P^*}\overline{c_y}}\left(\frac{1}{2}(1-\gamma)(1-\theta)^2-\frac{(1-\theta)}{2}\right)\widetilde{N_t^2}\right] \quad (127)$$

Distributing α and redefining each coefficient where

$$
\begin{aligned}
r_1 &= \alpha \frac{\theta \overline{x}}{2\overline{c_x}} \\
r_2 &= \alpha \frac{(1-\theta)\,\overline{y}}{2\overline{c_y}} \\
r_3 &= \alpha \frac{\overline{M}}{\overline{P}\overline{c_x}} \left(\frac{1}{2} (1-\gamma)\,\theta^2 - \frac{\theta}{2} \right) \\
r_4 &= \alpha \frac{\overline{N}}{\overline{P^*}\overline{c_y}} \left(\frac{1}{2} (1-\gamma)\,(1-\theta)^2 - \frac{(1-\theta)}{2} \right)
\end{aligned}
$$

(127) is reduced to

$$
r_1 \widetilde{x_t}^2 + r_2 \widetilde{y_t}^2 + r_3 \widetilde{M_t^2} + r_4 \widetilde{N_t^2} \tag{128}
$$

(128) expresses the weighted average component of minizing agent's objective fuction as a quadratic in terms exogenous state variables.

Perturbed Stochastic Processes The log-linearlization of (62) – (65) yields

$$
\begin{aligned}
\widetilde{x_{t+1}} &= \rho_1 \widetilde{x_t} + w_{t+1} + \varepsilon_{t+1}^x && (129) \\
\widetilde{y_{t+1}} &= \rho_2 \widetilde{y_t} + w_{t+1} + \varepsilon_{t+1}^y && (130) \\
\widetilde{M_{t+1}} &= \rho_3 \widetilde{M_t} + w_{t+1} + \varepsilon_{t+1}^M && (131) \\
\widetilde{N_{t+1}} &= \rho_4 \widetilde{N_t} + w_{t+1} + \varepsilon_{t+1}^N && (132)
\end{aligned}
$$

Replacing the weighted average component of minimizing player's problem with (128) and using (129) – (132) as constraints instead, the malevolent nature's problem becomes

$$
\min_{\{w_{t+1}\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t \left\{ -\left(r_1 \widetilde{x_t}^2 + r_2 \widetilde{y_t}^2 + r_3 \widetilde{M_t^2} + r_4 \widetilde{N_t^2} \right) + \beta \overline{\overline{\theta}} w_{t+1}^2 \right\}
$$

$$
\begin{aligned}
s.t. \quad \widetilde{x_{t+1}} &= \rho_1 \widetilde{x}_t + w_{t+1} + \varepsilon_{t+1}^x \\
\widetilde{y_{t+1}} &= \rho_2 \widetilde{y}_t + w_{t+1} + \varepsilon_{t+1}^y \\
\widetilde{M_{t+1}} &= \rho_3 \widetilde{M}_t + w_{t+1} + \varepsilon_{t+1}^M \\
\widetilde{N_{t+1}} &= \rho_4 \widetilde{N}_t + w_{t+1} + \varepsilon_{t+1}^N
\end{aligned}
$$

Linear Quadratic Approximation Control To cast in LQA framework, transform the above minimization problem in terms of matrix algebra

$$
\min_{\{w_{t+1}\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t \left\{ - \begin{bmatrix} \widetilde{x}_t \\ \widetilde{y}_t \\ \widetilde{M}_t \\ \widetilde{N}_t \end{bmatrix}' \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{bmatrix} \begin{bmatrix} \widetilde{x}_t \\ \widetilde{y}_t \\ \widetilde{M}_t \\ \widetilde{N}_t \end{bmatrix} + \beta \bar{\bar{\theta}} I_4 w_{t+1}' w_{t+1} \right\}
$$

$$
s.t. \quad \begin{bmatrix} \widetilde{x_{t+1}} \\ \widetilde{y_{t+1}} \\ \widetilde{M_{t+1}} \\ \widetilde{N_{t+1}} \end{bmatrix} = \begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_3 & 0 \\ 0 & 0 & 0 & \rho_4 \end{bmatrix} \begin{bmatrix} \widetilde{x}_t \\ \widetilde{y}_t \\ \widetilde{M}_t \\ \widetilde{N}_t \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} w_{t+1} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{t+1}^x \\ \varepsilon_{t+1}^y \\ \varepsilon_{t+1}^M \\ \varepsilon_{t+1}^N \end{bmatrix}
$$

or

$$
\min_{\{w_{t+1}\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t \left\{ -X_t' R X_t + \beta \bar{\bar{\theta}} I_4 w_{t+1}' w_{t+1} \right\}
$$

$$
s.t. \quad X_{t+1} = A X_t + C w_{t+1} + C \varepsilon_{t+1}
$$

where

$$X_t = \begin{bmatrix} \widetilde{x}_t \\ \widetilde{y}_t \\ \widetilde{M}_t \\ \widetilde{N}_t \end{bmatrix}, \qquad \varepsilon_{t+1} = \begin{bmatrix} \varepsilon^x_{t+1} \\ \varepsilon^y_{t+1} \\ \varepsilon^M_{t+1} \\ \varepsilon^N_{t+1} \end{bmatrix}, \qquad R = \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{bmatrix},$$

$$A = \begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_3 & 0 \\ 0 & 0 & 0 & \rho_4 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, and \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where the minimizing player's LQA-C problem is expressed in terms of exogenous state variables (i.e. excludes maximizing player's control variables), which is natural to purely exogenous endowment economies, and malevolent nature's control variable w_{t+1}.The problem can be translated into a Bellman equation,

$$-X_t' P X_t - p = \min_{w_{t+1}} \left\{ -X_t' R X_t + \beta \overline{\overline{\theta}} I_4 w_{t+1}' w_{t+1} + \beta E_t \left(-X_{t+1}' P X_{t+1} - p \right) \right\}$$

$$s.t. \quad X_{t+1} = A X_t + C w_{t+1} + C \varepsilon_{t+1}$$

where $V(X_t) = -X_t' P X_t - p$ is a specific quadratic structure of the value function and the symmetric matrix P is a fixed point used to find a stable solution that minimizes malevolent nature's problem . Using certainty equivalence, all uncertainty in the problem is eliminated so that $\varepsilon_{t+1} = 0$ which in turn eliminates p,

$$-X_t' P X_t = \min_{w_{t+1}} \left\{ -X_t' R X_t + \beta \overline{\overline{\theta}} I_4 w_{t+1}' w_{t+1} + \beta \left(-X_{t+1}' P X_{t+1} \right) \right\}$$

$$s.t. \quad X_{t+1} = A X_t + C w_{t+1}$$

Inserting state-evolution constraint into the objective function yields the unconstrained minimization problem,

$$-X_t^{'}PX_t = \min_{w_{t+1}} \left\{ -X_t^{'}RX_t + \beta\bar{\bar{\theta}} I_4 w_{t+1}^{'} w_{t+1} + \beta \left[-\left(AX_t + Cw_{t+1}\right)^{'} P \left(AX_t + Cw_{t+1}\right) \right] \right\}$$

$$\frac{\partial V}{\partial w_{t+1}} = 0 : 2\beta\bar{\bar{\theta}} I_4 w_{t+1} - 2\beta C'PAX_t - 2\beta C'PCw_{t+1} = 0 \tag{133}$$

$$\frac{\partial V}{\partial X_t} = 0 : -2RX_t - 2\beta A'PAX_t - 2A'PCw_{t+1} = 0 \tag{134}$$

Beginning with (133),

$$\begin{aligned} 2\beta\bar{\bar{\theta}} I_4 w_{t+1} - 2\beta C'PAX_t - 2\beta C'PCw_{t+1} &= 0 \\ \bar{\bar{\theta}} I_4 w_{t+1} - C'PAX_t - C'PCw_{t+1} &= 0 \\ w_{t+1} &= \left(\bar{\bar{\theta}} I_4 - C'PC\right)^{-1} C'PAX_t \end{aligned}$$

$$w_{t+1} = KX_t \tag{135}$$

where $K = \left(\bar{\bar{\theta}} I_4 - C'PC\right)^{-1} C'PA$. (135) represents movement of malevolent nature's perturbation w_{t+1} as a function of the state vector X_t. Next, utilizing (134) and inserting (135) for expanded K,

$$\begin{aligned} -2RX_t - 2\beta A'PAX_t - 2A'PC\left(\bar{\bar{\theta}} I_4 - C'PC\right)^{-1} C'PAX_t &= 0 \\ R + \beta A'PA + A'PC\left(\bar{\bar{\theta}} I_4 - C'PC\right)^{-1} C'PA &= 0 \end{aligned}$$

$$P = R + \beta A'PA + A'PC\left(\bar{\bar{\theta}} I_4 - C'PC\right)^{-1} C'PA \tag{136}$$

(136) represents the Riccati equation whose properites are well-known in linear-quadratic control theory. The fixed-point matrix P is used to find a stable solution so that (136) becomes

$$P_{k+1} = R + \beta A' P_k A + A' P_k C \left(\bar{\bar{\theta}} I_4 - C' P_k C \right)^{-1} C' P_k A \tag{137}$$

Setting $P_0 = 0$ as an initial condition and iterating (137) until convergence yields

$$\widehat{P} = R + \beta A' \widehat{P} A + A' \widehat{P} C \left(\bar{\bar{\theta}} I_4 - C' \widehat{P} C \right)^{-1} C' \widehat{P} A \tag{138}$$

where $\widehat{P}$ is a steady-state fixed point that when inserted into (135) for expanded K,

$$w_{t+1} = \left(\bar{\bar{\theta}} I_4 - C' \widehat{P} C \right)^{-1} C' \widehat{P} A X_t \tag{139}$$

$$w_{t+1} = K(\widehat{P}) X_t \tag{140}$$

where $K(\widehat{P}) = \left(\bar{\bar{\theta}} I_4 - C' \widehat{P} C \right)^{-1} C' \widehat{P} A$ and (140) is malevolent nature's perturbation encoded with fixed-point matrix $\widehat{P}$.

(140) feedsback into the perturbed state-evolution system

$$\begin{aligned} X_{t+1} &= A X_t + C w_{t+1} + C \varepsilon_{t+1} \\ X_{t+1} &= A X_t + C K X_t + C \varepsilon_{t+1} \end{aligned}$$

$$X_{t+1} = A^0 X_t + C \varepsilon_{t+1} \tag{141}$$

where $A^0 = (A + CK)$ .Ultimately, (141) is the perturbed state-evolution system embedded with malevolent natures feedback rule w_{t+1}.

4.3 Spot & Forward Exchange Rates

Generating inference involves (88), (90), and (141) but recall that both (88) and (90) are in terms of two exogenous state variables and (141) is in terms of four exogenous state variables. To remedy the issue, convert (88) into linear-algebra form

$$
\begin{aligned}
\widetilde{S}_t &= \widetilde{M}_t - \widetilde{N}_t \\
\widetilde{S}_t &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \widetilde{M}_t \\ \widetilde{N}_t \end{bmatrix} \\
\widetilde{S}_t &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{x}_t \\ \widetilde{y}_t \\ \widetilde{M}_t \\ \widetilde{N}_t \end{bmatrix} \\
\widetilde{S}_t &= \begin{bmatrix} 1 & -1 \end{bmatrix} U_s X_t
\end{aligned}
$$

$$
\widetilde{S}_t = \Pi X_t \tag{142}
$$

where $\Pi = \begin{bmatrix} 1 & -1 \end{bmatrix} U_s$ and $U_s = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is known as a selection matrix that creates dependence between $\widetilde{S}_t$ and $\widetilde{M}_t, \widetilde{N}_t$.

Next, insert (93) and (94) into (90) ,distribute expectation operator, and convert into linear algebra

$$\begin{aligned}
\widetilde{F}_t &= E_t\left[\widetilde{M}_{t+1}\right] - E_t\left[\widetilde{N}_{t+1}\right] \\
\widetilde{F}_t &= \rho_3 \widetilde{M}_t - \rho_4 \widetilde{N}_t \\
\widetilde{F}_t &= \begin{bmatrix} \rho_3 & -\rho_4 \end{bmatrix} \begin{bmatrix} \widetilde{M}_t \\ \widetilde{N}_t \end{bmatrix} \\
\widetilde{F}_t &= \begin{bmatrix} \rho_3 & -\rho_4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{x}_t \\ \widetilde{y}_t \\ \widetilde{M}_t \\ \widetilde{N}_t \end{bmatrix} \\
\widetilde{F}_t &= \begin{bmatrix} \rho_3 & -\rho_4 \end{bmatrix} U_s X_t
\end{aligned}$$

$$\widetilde{F}_t = \Gamma X_t \tag{143}$$

where $\Gamma = \begin{bmatrix} \rho_3 & -\rho_4 \end{bmatrix} U_s$ and U_s is defined as above.

4.4 Gross Rate of Depreciation & Forward Premium

The *Gross Rate of Depreciation* can be constructed using (142) and a lead of (142) or

$$\widetilde{S}_{t+1} - \widetilde{S}_t \tag{144}$$

where $\widetilde{S}_{t+1}$ is *ex-post.*

and the *Forward Premium* can be constructed using both (142) and (143)

$$\widetilde{F}_t - \widetilde{S}_t \tag{145}$$

Again, the motivator for dynamics of (142) − (145) is perturbed stochastic system

$$X_{t+1} = A^0 X_t + C\varepsilon_{t+1} \tag{146}$$

5 Detection Error Probabilities

To discipline the choice of $\overline{\overline{\theta}}$, detection error probability method is enforced so state the approximating state-evolution system (95)

$$X_{t+1} = AX_t + C\widehat{\varepsilon_{t+1}} \tag{147}$$

which will be known as "Model A." State the perturbed state-evolution system (141)

$$X_{t+1} = A^0 X_t + C\varepsilon_{t+1} \tag{148}$$

which will be known as "Model B."

5.1 Log-Liklihood Ratio Test with Model A

Assume the worst-case shock is generated under Model A,

$$w^A_{t+1} = KX^A_{t+1} \tag{149}$$

Define Model A innovations,

$$\begin{aligned} X_{t+1} &= AX_t + C\widehat{\varepsilon_{t+1}} && (150) \\ C\widehat{\varepsilon_{t+1}} &= X_{t+1} - AX_t && (151) \\ \widehat{\varepsilon_{t+1}} &= \left(C'C\right)^{-1} C' \left(X_{t+1} - AX_t\right) && (152) \end{aligned}$$

Define Model B innovations,

$$\begin{aligned}
X_{t+1} &= A^0 X_t + C\varepsilon_{t+1} && (153)\\
C\varepsilon_{t+1} &= X_{t+1} - A^0 X_t && (154)\\
\varepsilon_{t+1} &= \left(C'C\right)^{-1} C' \left(X_{t+1} - A^0 X_t\right) && (155)\\
\varepsilon_{t+1} &= \left(C'C\right)^{-1} C' \left(X_{t+1} - AX_t\right) - \left(C'C\right)^{-1} C' K X_t && (156)\\
\varepsilon_{t+1} &= \widehat{\varepsilon_{t+1}} - w_{t+1} && (157)
\end{aligned}$$

where $A^0 = (A + CK)$ was used. Next, define both

$$\begin{aligned}
Log\ L_A &= -\frac{1}{T}\sum_{t=0}^{T-1}\left\{n \log\sqrt{2\pi} + \frac{1}{2}\left(\widehat{\varepsilon_{t+1}}'\widehat{\varepsilon_{t+1}}\right)\right\} && (158)\\
Log\ L_B &= -\frac{1}{T}\sum_{t=0}^{T-1}\left\{n \log\sqrt{2\pi} + \frac{1}{2}\left(\varepsilon_{t+1}'\varepsilon_{t+1}\right)\right\} && (159)
\end{aligned}$$

where (158) is generated under innovations from (152) and (159) is generated under innovations from (157). To construct log-liklihood ratio test w.r.t Model A define,

$$\begin{aligned}
r|A &= Log\ L_A - Log\ L_B\\
&= -\frac{1}{T}\sum_{t=0}^{T-1}\left\{n \log\sqrt{2\pi} + \frac{1}{2}\left(\widehat{\varepsilon_{t+1}}'\widehat{\varepsilon_{t+1}}\right)\right\} + \frac{1}{T}\sum_{t=0}^{T-1}\left\{n \log\sqrt{2\pi} + \frac{1}{2}\left(\varepsilon_{t+1}'\varepsilon_{t+1}\right)\right\}\\
&= \frac{1}{T}\sum_{t=0}^{T-1}\left\{\frac{1}{2}\left(\varepsilon_{t+1}'\varepsilon_{t+1}\right) - \frac{1}{2}\left(\widehat{\varepsilon_{t+1}}'\widehat{\varepsilon_{t+1}}\right)\right\}\\
&= \frac{1}{T}\sum_{t=0}^{T-1}\left\{\frac{1}{2}\left(\widehat{\varepsilon_{t+1}} - w_{t+1}^A\right)'\left(\widehat{\varepsilon_{t+1}} - w_{t+1}^A\right) - \frac{1}{2}\left(\widehat{\varepsilon_{t+1}}'\widehat{\varepsilon_{t+1}}\right)\right\}\\
&= \frac{1}{T}\sum_{t=0}^{T-1}\left\{\frac{1}{2}\left(\widehat{\varepsilon_{t+1}}'\widehat{\varepsilon_{t+1}} - \widehat{\varepsilon_{t+1}}'w_{t+1}^A - w_{t+1}^{A\prime}\widehat{\varepsilon_{t+1}} + w_{t+1}^{A\prime}w_{t+1}^A\right) - \frac{1}{2}\left(\widehat{\varepsilon_{t+1}}'\widehat{\varepsilon_{t+1}}\right)\right\}
\end{aligned}$$

$$r|A = \frac{1}{T}\sum_{t=0}^{T-1}\left\{\frac{1}{2}w_{t+1}^{A\prime}w_{t+1}^{A} - \frac{1}{2}w_{t+1}^{A\prime}\widehat{\varepsilon_{t+1}}\right\} \tag{160}$$

The objective of this section is to give the probability associated with incorrectly choosing Model B when the true data driving process is Model A or

$$p_A = \Pr\left(r|A < 0\right) \tag{161}$$

5.2 Log-Liklihood Ratio Test with Model B

Assume the worst-case shock is generated under Model B,

$$w_{t+1}^{B} = KX_{t+1}^{B} \tag{162}$$

(158) and (159) can be used to construct the log-liklihood ratio test under Model B,

$$r|B = \frac{1}{T}\sum_{t=0}^{T-1}\left\{\frac{1}{2}w_{t+1}^{B\prime}w_{t+1}^{B} - \frac{1}{2}w_{t+1}^{B\prime}\widehat{\varepsilon_{t+1}}\right\} \tag{163}$$

where (163) is produced under a similar process to (160) except that the worst-case shocks are generated by (162). The objective of this section is to give the probability associated with incorrectly choosing Model A when the true data driving process is Model B or

$$p_B = \Pr\left(r|B > 0\right) \tag{164}$$

5.3 Probability of Detection Error

Using (161) and (164) yields the formula

$$p = \frac{1}{2}(p_A + p_B) \tag{165}$$

where p is the probability of error in choosing the correct model which implies that $1-p$ is the probability of success in choosing the correct model.

Remark 1 *There lies a positive relationship between p and $\overline{\overline{\theta}}$*

Thus, as $\overline{\overline{\theta}}$ decreases its associated p will decrease as well so that the desired level of detection error will implicitly discipline the value of $\overline{\overline{\theta}}$.

5.4 Probability of Detection Error Estimation

The following steps denote detection error probability estimation:

For p_A

Step 1) Set prior about $\overline{\overline{\theta}}$ and generate $\{\widehat{\varepsilon_{t+1}}\}_{t=0}^{T}$ from $\widehat{\varepsilon}_t\tilde{\ }N(0_4, I_4)$ for data observations of length T where 0_4 is a $4x4$ matrix of zero's and I_4 is $4x4$ identity matrix.

Step 2) Use pseudo $\{\widehat{\varepsilon_{t+1}}\}_{t=0}^{T}$ to iterate on (147) resulting in $\left\{X_t^A\right\}_{t=0}^{T}$

Step 3) Use $\left\{X_t^A\right\}_{t=0}^{T}$ to iterate on (149) resulting in $\left\{w_{t+1}^A\right\}_{t=0}^{T}$

Step 4) Insert pseudo-generated $\{\widehat{\varepsilon_{t+1}}\}_{t=0}^{T}$ and $\left\{w_{t+1}^A\right\}_{t=0}^{T}$ into (160) and sum

Step 5) Simulate (Repeat steps 1-4) for large sample size

Step 6) Count number of times $r|A < 0$ from simulation and average by number of times simulated: The result is p_A

For p_B

Step 1) Using same prior $\overline{\overline{\theta}}$ generate $\{\varepsilon_{t+1}\}_{t=0}^{T}$ from $\varepsilon_t \tilde{} N(0_4, I_4)$ for data observations of length T where 0_4 is a $4x4$ matrix of zero's and I_4 is $4x4$ identity matrix.

Step 2) Use pseudo $\{\varepsilon_{t+1}\}_{t=0}^{T}$ to iterate on (148) resulting in $\left\{X_t^B\right\}_{t=0}^{T}$

Step 3) Use $\left\{X_t^B\right\}_{t=0}^{T}$ to iterate on (162) resulting in $\left\{w_{t+1}^B\right\}_{t=0}^{T}$

Step 4) Insert pseudo-generated $\{\varepsilon_{t+1}\}_{t=0}^{T}$ and $\left\{w_{t+1}^B\right\}_{t=0}^{T}$ into (163) and sum

Step 5) Simulate (Repeat steps 1-4) for large sample size

Step 6) Count number of times $r|B > 0$ from simulation and average by number of times simulated: The result is p_B

Once p_A and p_B are obtained insert into (165) to obtain p. When estimating p using the above process, begin with prior about $\overline{\overline{\theta}}$ and drive down $\overline{\overline{\theta}}$ until desired level of p is achieved. Reaching this desired level of p in turn disciplines the choice of $\overline{\overline{\theta}}$.

6 Forward Premium Puzzle

6.1 Downward Bias of Estimator

To derive the downward bias of the forward premium estimator, we begin by using an OLS regression of (144) on to (145) so that,

$$\widetilde{S}_{t+1} - \widetilde{S}_t = \widehat{\beta_0} + \widehat{\beta_1}\left(\widetilde{F}_t - \widetilde{S}_t\right) + \epsilon_{t+1} \tag{166}$$

where $e_{t+1} \sim N(0, \sigma^2)$. Inserting (142) and (143) into (166) results in,

$$\begin{aligned}
\Pi X_{t+1} - \Pi X_t &= \widehat{\beta_0} + \widehat{\beta_1}\left(\widetilde{F}_t - \widetilde{S}_t\right) + \epsilon_{t+1} \\
\Pi A^0 X_t + \Pi C \epsilon_{t+1} - \Pi X_t &= \widehat{\beta_0} + \widehat{\beta_1}\left(\widetilde{F}_t - \widetilde{S}_t\right) + \epsilon_{t+1} \\
\Pi (A - I) X_t + \Pi C \epsilon_{t+1} &= \widehat{\beta_0} + \widehat{\beta_1}(\Gamma - \Pi) X_t + \epsilon_{t+1}
\end{aligned}$$

Next, expand each matrix and vector by the number of observations from $t = 1, ..., T$ so that,

$$\underbrace{\begin{bmatrix} \Pi(A-I) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Pi(A-I) \end{bmatrix}}_{Q} \begin{bmatrix} X_1 \\ \vdots \\ X_T \end{bmatrix} + \underbrace{\begin{bmatrix} \Pi C & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Pi C \end{bmatrix}}_{R} \begin{bmatrix} \epsilon_2 \\ \vdots \\ \epsilon_{T+1} \end{bmatrix} =$$

$$\widehat{\beta_0} + \widehat{\beta_1} \overbrace{\underbrace{\begin{bmatrix} (\Gamma-\Pi) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (\Gamma-\Pi) \end{bmatrix}}_{V} \begin{bmatrix} X_1 \\ \vdots \\ X_T \end{bmatrix}}^{z} + \begin{bmatrix} \epsilon_2 \\ \vdots \\ \epsilon_{T+1} \end{bmatrix}$$

$$\underbrace{QX + R\epsilon}_{Y} = \underbrace{\begin{bmatrix} i & z \end{bmatrix}}_{W} \underbrace{\begin{bmatrix} \widehat{\beta_0} \\ \widehat{\beta_1} \end{bmatrix}}_{\widehat{\beta}} + \epsilon$$

Where subscripts have been eliminated from X and ϵ which imply current time-period. Utilizing ordinary least squares optimal slope estimator formula,

$$\begin{aligned} \widehat{\beta} &= (W'W)^{-1} W'Y \\ \widehat{\beta} &= \left\{ \begin{bmatrix} i' \\ z' \end{bmatrix} \begin{bmatrix} i & z \end{bmatrix} \right\}^{-1} \begin{bmatrix} i' \\ z' \end{bmatrix} (QX + R\epsilon) \\ \widehat{\beta} &= \begin{bmatrix} i'i & i'z \\ z'i & z'z \end{bmatrix}^{-1} \begin{bmatrix} i' \\ z' \end{bmatrix} (QX + R\epsilon) \end{aligned}$$

$$\widehat{\beta} = \begin{bmatrix} i'i & i'z \\ z'i & z'z \end{bmatrix}^{-1} \begin{bmatrix} i'QX + i'R\epsilon \\ z'QX + z'R\epsilon \end{bmatrix} \tag{167}$$

To rid the X term on the RHS of (167), exploit equation (148) where if $t = 0$,

$$\begin{aligned} X_1 &= A^0 X_0 + C\epsilon_1 \\ X_1 - A^0 X_0 &= C\epsilon_1 \end{aligned} \tag{168}$$

$$X_1 = C\epsilon_1 \tag{169}$$

where it's assumed $X_0 = 0$ in (169). "Pushing" (168) forward in time results in,

$$\begin{aligned} X_2 &= A^0 X_1 + C\epsilon_2 \\ X_2 - A^0 X_1 &= C\epsilon_2 \end{aligned}$$

continuing in this manner results from $t = 1, ..., T$ results in a system of equations,

$$\begin{aligned} X_1 &= C\epsilon_1 \\ X_2 - A^0 X_1 &= C\epsilon_2 \\ X_3 - A^0 X_2 &= C\epsilon_3 \\ &\vdots \\ X_T - A^0 X_{T-1} &= C\epsilon_T \end{aligned}$$

converting to matrix form,

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ -A^0 & I & 0 & \ddots & \vdots \\ 0 & -A^0 & I & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -A^0 & I \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_T \end{bmatrix} = \begin{bmatrix} C & 0 & 0 & \cdots & 0 \\ 0 & C & 0 & \ddots & \vdots \\ 0 & 0 & C & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & C \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \vdots \\ \epsilon_T \end{bmatrix}$$

$$
\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_T \end{bmatrix} = \underbrace{\begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ -A^0 & I & 0 & \ddots & \vdots \\ 0 & -A^0 & I & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -A^0 & I \end{bmatrix}^{-1} \begin{bmatrix} C & 0 & 0 & \cdots & 0 \\ 0 & C & 0 & \ddots & \vdots \\ 0 & 0 & C & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & C \end{bmatrix}}_{\Psi} \underbrace{\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \vdots \\ \epsilon_T \end{bmatrix}}_{\epsilon_{-1}}
$$

or,

$$
X = \Psi \epsilon_{-1} \tag{170}
$$

Inserting (170) into (167) results in,

$$\widehat{\beta} = \begin{bmatrix} i'i & i'z \\ z'i & z'z \end{bmatrix}^{-1} \begin{bmatrix} i'Q\Psi\epsilon_{-1} + i'R\epsilon \\ z'Q\Psi\epsilon_{-1} + z'R\epsilon \end{bmatrix}$$

$$\widehat{\beta} = \begin{bmatrix} i'i & i'z \\ z'i & z'z \end{bmatrix}^{-1} \begin{bmatrix} i'Q\Psi\epsilon_{-1} + i'R\epsilon \\ (VX)'\,Q\Psi\epsilon_{-1} + (VX)'\,R\epsilon \end{bmatrix}$$

$$\widehat{\beta} = \begin{bmatrix} i'i & i'z \\ z'i & z'z \end{bmatrix}^{-1} \begin{bmatrix} i'Q\Psi\epsilon_{-1} + i'R\epsilon \\ (V\Psi\epsilon_{-1})'\,Q\Psi\epsilon_{-1} + (V\Psi\epsilon_{-1})'\,R\epsilon \end{bmatrix}$$

$$\widehat{\beta} = \begin{bmatrix} i'i & i'z \\ z'i & z'z \end{bmatrix}^{-1} \begin{bmatrix} i'Q\Psi\epsilon_{-1} + i'R\epsilon \\ \epsilon_{-1}'\Psi'V'Q\Psi\epsilon_{-1} + \epsilon_{-1}'\Psi'V'R\epsilon \end{bmatrix}$$

$$\widehat{\beta} = \begin{bmatrix} i'i & i'z \\ z'i & z'z \end{bmatrix}^{-1} \begin{bmatrix} i'Q\Psi\epsilon_{-1} + i'R\epsilon \\ \epsilon_{-1}'\Psi'V'Q\Psi\epsilon_{-1} + \epsilon_{-1}'\Psi'V'R\epsilon \end{bmatrix}$$

$$\widehat{\beta} = \frac{1}{(i'i)\,(z'z) - (i'z)\,(z'i)} \begin{bmatrix} z'z & -i'z \\ -z'i & i'i \end{bmatrix} \begin{bmatrix} i'Q\Psi\epsilon_{-1} + i'R\epsilon \\ \epsilon_{-1}'\Psi'V'Q\Psi\epsilon_{-1} + \epsilon_{-1}'\Psi'V'R\epsilon \end{bmatrix}$$

$$\widehat{\beta} = \frac{1}{T\left[(VX)'\,VX\right] - (i'VX)\left[(VX)'\,i\right]} \begin{bmatrix} (VX)'\,VX & -i'VX \\ -(VX)'\,i & i'i \end{bmatrix} \begin{bmatrix} i'Q\Psi\epsilon_{-1} + i'R\epsilon \\ \epsilon_{-1}'\Psi'V'Q\Psi\epsilon_{-1} + \epsilon_{-1}'\Psi'V'R\epsilon \end{bmatrix}$$

$$\widehat{\beta} = \frac{1}{TX'V'VX - i'VXX'V'i} \begin{bmatrix} X'V'VX & -i'VX \\ -X'V'i & i'i \end{bmatrix} \begin{bmatrix} i'Q\Psi\epsilon_{-1} + i'R\epsilon \\ \epsilon_{-1}'\Psi'V'Q\Psi\epsilon_{-1} + \epsilon_{-1}'\Psi'V'R\epsilon \end{bmatrix}$$

$$\widehat{\beta} = \frac{1}{T\epsilon_{-1}'\Psi'V'V\Psi\epsilon_{-1} - i'V\Psi\epsilon_{-1}\epsilon_{-1}'\Psi'V'i} \begin{bmatrix} \epsilon_{-1}'\Psi'V'V\Psi\epsilon_{-1} & -i'V\Psi\epsilon_{-1} \\ -\epsilon_{-1}'\Psi'V'i & T \end{bmatrix} \begin{bmatrix} i'Q\Psi\epsilon_{-1} + i'R\epsilon \\ \epsilon_{-1}'\Psi'V'Q\Psi\epsilon_{-1} + \epsilon_{-1}'\Psi'V'R\epsilon \end{bmatrix} \tag{171}$$

where extensive use of $z = VX$ and $X = \Psi\epsilon_{-1}$ was made. Taking the probability limit of (171) results in,

$$p\lim\left(\widehat{\beta}\right) = E\left\{\cdot\right\}$$

where the $\cdot$ inside the brackets represents RHS terms of (171) .Distributing the expectations term,

$$p\lim\left(\widehat{\beta}\right)=\Omega E\left\{\begin{bmatrix}\epsilon_{-1}'\Psi'V'V\Psi\epsilon_{-1} & -i'V\Psi\epsilon_{-1}\\ -\epsilon_{-1}'\Psi'V'i & T\end{bmatrix}\begin{bmatrix}i'Q\Psi\epsilon_{-1}+i'R\epsilon\\ \epsilon_{-1}'\Psi'V'Q\Psi\epsilon_{-1}+\epsilon_{-1}'\Psi'V'R\epsilon\end{bmatrix}\right\}$$

where $\Omega=1/\left[Ttr(\Psi'V'V\Psi)\sigma^2+\epsilon_{-1}^{m\prime}\Psi'V'V\Psi\epsilon_{-1}^{m}-tr\left(i'V\Psi\Psi'V'i\right)\sigma^2+i'V\Psi\epsilon_{-1}^{m}\epsilon_{-1}^{m\prime}\Psi'V'i\right].$ Further consolidation yields,

$$p\lim\left(\widehat{\beta}\right)=\Omega^{*}E\left\{\begin{bmatrix}\epsilon_{-1}'\Psi'V'V\Psi\epsilon_{-1}\left(i'Q\Psi\epsilon_{-1}+i'R\epsilon\right)-i'V\Psi\epsilon_{-1}\left(\epsilon_{-1}'\Psi'V'Q\Psi\epsilon_{-1}+\epsilon_{-1}'\Psi'V'R\epsilon\right)\\ -\epsilon_{-1}'\Psi'V'i\left(i'Q\Psi\epsilon_{-1}+i'R\epsilon\right)+T\left(\epsilon_{-1}'\Psi'V'Q\Psi\epsilon_{-1}+\epsilon_{-1}'\Psi'V'R\epsilon\right)\end{bmatrix}\right\}$$

where $\Omega^{*}=1/\sigma^2\left[Ttr(\Psi'V'V\Psi)-tr\left(i'V\Psi\Psi'V'i\right)\right]$.Distributing E operator through the matrix results in,

$$p\lim\left(\widehat{\beta}\right)=\Omega^{*}\begin{bmatrix}0\\ E\left\{-\epsilon_{-1}'\Psi'V'i\left(i'Q\Psi\epsilon_{-1}+i'R\epsilon\right)+T\left(\epsilon_{-1}'\Psi'V'Q\Psi\epsilon_{-1}+\epsilon_{-1}'\Psi'V'R\epsilon\right)\right\}\end{bmatrix}$$

where further simplification finally results in,

$$p\lim\left(\beta_1\right)=\frac{E\left\{-\epsilon_{-1}'\Psi'V'i\left(i'Q\Psi\epsilon_{-1}+i'R\epsilon\right)+T\left(\epsilon_{-1}'\Psi'V'Q\Psi\epsilon_{-1}+\epsilon_{-1}'\Psi'V'R\epsilon\right)\right\}}{Ttr(\Psi'V'V\Psi)\sigma^2-tr\left(i'V\Psi\Psi'V'i\right)\sigma^2}\tag{172}$$

where the superscript m in $\epsilon_{-1}^{m\prime}$ denotes the mean of $\epsilon_{-1}^{'}$ so that $\epsilon_{-1}^{m\prime}=0$. Additionally, manipulations using $E(\epsilon_{-1})=E(\epsilon)=E(\epsilon_{-1}\epsilon')=0$ aided in eliminating $\widehat{\beta_0}$ from the $\widehat{\beta}$ vector.Further simplifaction of (172) results in,

$$
\begin{aligned}
p\lim\left(\widehat{\beta_1}\right) &= \frac{E\left\{-\epsilon_{-1}'\Psi'V'ii'Q\Psi\epsilon_{-1} - \epsilon_{-1}'\Psi'V'ii'R\epsilon + T\left(\epsilon_{-1}'\Psi'V'Q\Psi\epsilon_{-1} + \epsilon_{-1}'\Psi'V'R\epsilon\right)\right\}}{\sigma^2\left[Ttr(\Psi'V'V\Psi) - tr\left(i'V\Psi\Psi'V'i\right)\right]} \\
p\lim\left(\widehat{\beta_1}\right) &= \frac{\sigma^2\left[-tr\left(\Psi'V'ii'Q\Psi\right) + Ttr\left(\Psi'V'Q\Psi\right)\right]}{\sigma^2\left[Ttr(\Psi'V'V\Psi) - tr\left(i'V\Psi\Psi'V'i\right)\right]} \\
p\lim\left(\widehat{\beta_1}\right) &= \frac{tr\left[T\Psi'V'Q\Psi - \Psi'V'ii'Q\Psi\right]}{tr\left[T\Psi'V'V\Psi - i'V\Psi\Psi'V'i\right]} \\
p\lim\left(\widehat{\beta_1}\right) &= \frac{tr\left[T\Psi'V'Q\Psi - \Psi'V'ii'Q\Psi\right]}{tr\left[T\Psi'V'V\Psi - \Psi'V'ii'V\Psi\right]} \\
p\lim\left(\widehat{\beta_1}\right) &= \frac{tr\left[\Psi'V'\left(TI - ii'\right)Q\Psi\right]}{tr\left[\Psi'V'\left(TI - ii'\right)V\Psi\right]} \\
p\lim\left(\beta_1\right) &= \frac{Ttr\left[\Psi'V'\left(I - \frac{ii'}{T}\right)Q\Psi\right]}{Ttr\left[\Psi'V'\left(I - \frac{ii'}{T}\right)V\Psi\right]} \\
p\lim\left(\widehat{\beta_1}\right) &= \frac{tr\left(\Psi'V'M_iQ\Psi\right)}{tr\left(\Psi'V'M_iV\Psi\right)} \\
p\lim\left(\widehat{\beta_1}\right) &= 1 - 1 + \frac{tr\left(\Psi'V'M_iQ\Psi\right)}{tr\left(\Psi'V'M_iV\Psi\right)} \\
p\lim\left(\widehat{\beta_1}\right) &= 1 - 1 + \frac{tr\left(\Psi'V'M_iQ\Psi\right)}{tr\left(\Psi'V'M_iV\Psi\right)} \\
p\lim\left(\widehat{\beta_1}\right) &= 1 - \left[\frac{tr\left(\Psi'V'M_iV\Psi\right) - tr\left(\Psi'V'M_iQ\Psi\right)}{tr\left(\Psi'V'M_iV\Psi\right)}\right]
\end{aligned}
$$

$$
\therefore \quad p\lim\left(\widehat{\beta_1}\right) = 1 - \frac{tr\left[\Psi'V'M_i\left(V-Q\right)\Psi\right]}{tr\left(\Psi'V'M_iV\Psi\right)} \tag{173}
$$

6.2 Consistency of Estimator

To show consistency of the slope estimator, take the limit of $\overline{\overline{\theta}} \to \infty$ to $p\lim\left(\widehat{\beta_1}\right)$ or,

$$
\lim_{\overline{\overline{\theta}}\to\infty}\left\{p\lim\left(\widehat{\beta_1}\right)\right\} = \lim_{\overline{\overline{\theta}}\to\infty}\left\{1 - \frac{tr\left[\Psi'V'M_i\left(V-Q\right)\Psi\right]}{tr\left(\Psi'V'M_iV\Psi\right)}\right\} \tag{174}
$$

Focusing on the $(V-Q)$ matrix,

$$(V-Q) = \begin{bmatrix} (\Gamma-\Pi) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (\Gamma-\Pi) \end{bmatrix} - \begin{bmatrix} \Pi\left(A^0-I\right) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Pi\left(A^0-I\right) \end{bmatrix}$$

$$(V-Q) = \begin{bmatrix} \Gamma-\Pi A^0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Gamma-\Pi A^0 \end{bmatrix}$$

$$\lim_{\overline{\overline{\theta}}\to\infty}(V-Q) = \begin{bmatrix} \lim\limits_{\overline{\overline{\theta}}\to\infty}\{\Gamma-\Pi A^0\} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lim\limits_{\overline{\overline{\theta}}\to\infty}\{\Gamma-\Pi A^0\} \end{bmatrix}$$

$$\lim_{\overline{\overline{\theta}}\to\infty}(V-Q) = \begin{bmatrix} \Gamma-\Pi A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Gamma-\Pi A \end{bmatrix} \tag{175}$$

where we recall that $A^0 = A + C\left[\overline{\overline{\theta}}I_{4x4} - C'PC\right]^{-1}C'PA$ and $C\left[\overline{\overline{\theta}}I_{4x4} - C'PC\right]^{-1}C'PA \to 0$ as $\overline{\overline{\theta}} \to \infty$. Now, to show that (175) is essentially a null matrix, we show that $\Gamma = \Pi A$ where recall,

$$\Gamma = \begin{bmatrix} \rho_3 & -\rho_4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} 0 & 0 & \rho_3 & -\rho_4 \end{bmatrix}$$

and

$$\Pi A = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_3 & 0 \\ 0 & 0 & 0 & \rho_4 \end{bmatrix}$$

$$\Pi A = \begin{bmatrix} 0 & 0 & \rho_3 & -\rho_4 \end{bmatrix}$$

Implementing this null matrix back in (174) eliminates the bias term reducing (174) to

$$\lim_{\bar{\bar{\theta}} \to \infty} \{p \lim (\beta_1)\} = 1 \tag{176}$$

Hence an unbiased slope estimator.

6.3 Estimation

Step 1) Generate sequence of pseudo errors $\{\varepsilon_t\}_{t=1}^{T}$ where $\varepsilon_t \sim N(0_4, I_4)$ and T is equivalent to length of observational data

Step 2) Insert generated $\{\varepsilon_t\}_{t=1}^{T}$ into (146) and iterate to generate $\{X_t\}_{t=1}^{T}$

Step 3) Insert $\{X_t\}_{t=1}^{T}$ into (142) and (143) to generate $\left\{\widetilde{S}_t\right\}_{t=1}^{T}$ and $\left\{\widetilde{F}_t\right\}_{t=1}^{T}$ respectively

Step 4) Construct $\left\{\widetilde{S}_{t+1} - \widetilde{S}_t\right\}_{t=1}^{T-1} = \left\{\widetilde{S}_{t+1}\right\}_{t=1}^{T-1} - \left\{\widetilde{S}_t\right\}_{t=1}^{T-1}$ and $\left\{\widetilde{F}_t - \widetilde{S}_t\right\}_{t=1}^{T-1} = \left\{\widetilde{F}_t\right\}_{t=1}^{T-1} - \left\{\widetilde{S}_t\right\}_{t=1}^{T-1}$

Step 5) Convert into vector so that

$$\mathbf{S}_{t+1} - \mathbf{S}_t = \begin{bmatrix} \widetilde{S}_2 - \widetilde{S}_1 \\ \vdots \\ \widetilde{S}_T - \widetilde{S}_{T-1} \end{bmatrix}$$

$$\mathbf{F}_t - \mathbf{S}_t = \begin{bmatrix} \widetilde{F}_1 - \widetilde{S}_1 \\ \vdots \\ \widetilde{F}_{T-1} - \widetilde{S}_{T-1} \end{bmatrix}$$

From Step 5, utilize vectors and perform the following regression

$$\mathbf{S}_{t+1}-\mathbf{S}_t = \widehat{\beta_0} + \widehat{\beta_1}\left(\mathbf{F}_t-\mathbf{S}_t\right) + e_{t+1} \tag{177}$$

where (180) uses standard ordinary least squares. If $\mathbf{F}_t-\mathbf{S}_t$ and $\mathbf{S}_{t+1}-\mathbf{S}_t$ are temporarily defined as $\mathbf{X}$ and $\mathbf{Y}$ respectively then the estimators $\widehat{\beta_0}$ and $\widehat{\beta_1}$ are

$$\begin{aligned}
\widehat{\beta_1} &= \left(\mathbf{X}'\mathbf{X}\right)^{-1}\left(\mathbf{X}'\mathbf{Y}\right) && (178)\\
\widehat{\beta_0} &= \overline{\mathbf{Y}} - \widehat{\beta_1}\overline{\mathbf{X}} && (179)
\end{aligned}$$

where $\overline{\mathbf{X}}$ and $\overline{\mathbf{Y}}$ represent the mean of $\mathbf{X}$ and $\mathbf{Y}$ respectively. $\overline{\overline{\theta}}$ which correspond to detection error probabilities of $p(\overline{\overline{\theta}}) = 0.100$ and $p(\overline{\overline{\theta}}) = 0.000$ can be inserted within Step 2) in order to yield $\widehat{\beta_1} > 0$ and $\widehat{\beta_1} < 0$ respectively.

6.4 Simulation

Parameter Values Model paramters used for simulation are,

$$\begin{aligned}
\beta &= 0.99\\
\gamma &= 10\\
\theta &= 0.5\\
\phi &= 0.5
\end{aligned}$$

Step 1) Generate sequence of pseudo errors $\{\varepsilon_t\}_{t=1}^{T}$ where $\varepsilon_t \sim N\left(0_4, I_4\right)$ and T is equivalent to length of observational data

Step 2) Insert generated $\{\varepsilon_t\}_{t=1}^{T}$ into (146) and iterate to generate $\{X_t\}_{t=1}^{T}$

Step 3) Insert $\{X_t\}_{t=1}^{T}$ into (142) and (143) to generate $\left\{\widetilde{S}_t\right\}_{t=1}^{T}$ and $\left\{\widetilde{F}_t\right\}_{t=1}^{T}$ respectively

Step 4) Construct $\left\{\widetilde{S}_{t+1}-\widetilde{S}_t\right\}_{t=1}^{T-1}=\left\{\widetilde{S}_{t+1}\right\}_{t=1}^{T-1}-\left\{\widetilde{S}_t\right\}_{t=1}^{T-1}$ and $\left\{\widetilde{F}_t-\widetilde{S}_t\right\}_{t=1}^{T-1}=\left\{\widetilde{F}_t\right\}_{t=1}^{T-1}-\left\{\widetilde{S}_t\right\}_{t=1}^{T-1}$

Step 5) Convert into vector so that

$$\mathbf{S}_{t+1}-\mathbf{S}_t=\begin{bmatrix}\widetilde{S}_2-\widetilde{S}_1\\ \vdots\\ \widetilde{S}_T-\widetilde{S}_{T-1}\end{bmatrix}$$

$$\mathbf{F}_t-\mathbf{S}_t=\begin{bmatrix}\widetilde{F}_1-\widetilde{S}_1\\ \vdots\\ \widetilde{F}_{T-1}-\widetilde{S}_{T-1}\end{bmatrix}$$

From Step 5, utilize vectors and perform the following regression

$$\mathbf{S}_{t+1}-\mathbf{S}_t=\widehat{\beta_0}+\widehat{\beta_1}\left(\mathbf{F}_t-\mathbf{S}_t\right)+e_{t+1} \tag{180}$$

where (180) uses standard ordinary least squares. If $\mathbf{F}_t-\mathbf{S}_t$ and $\mathbf{S}_{t+1}-\mathbf{S}_t$ are temporarily defined as $\mathbf{X}$ and $\mathbf{Y}$ respectively then the estimators $\widehat{\beta_0}$ and $\widehat{\beta_1}$ are

$$\widehat{\beta_1} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\left(\mathbf{X}'\mathbf{Y}\right) \tag{181}$$

$$\widehat{\beta_0} = \overline{\mathbf{Y}}-\widehat{\beta_1}\overline{\mathbf{X}} \tag{182}$$

where $\overline{\mathbf{X}}$ and $\overline{\mathbf{Y}}$ represent the mean of $\mathbf{X}$ and $\mathbf{Y}$ respectively. Repeat Step 1)-Step 5) for large simulation size N where the simulation size is conducted under a specific $\overline{\overline{\theta}}$ corresponding to detection error probability $p(\overline{\overline{\theta}})$. This will generate N number of $\widehat{\beta_1}$ under a specific $\overline{\overline{\theta}}$ regime, ultimately yielding desired PDF.

7 Data

7.1 Description

1. Home Output: Defined as

$$\mathbf{x}_t^D = \begin{bmatrix} x_{1986:III} \\ \vdots \\ x_{2013:III} \end{bmatrix} \tag{183}$$

where $\mathbf{x}_t^D$ represents a time-series data vector containing aggregate U.S. real G.D.P observations in millions of chained 2009 dollars (from Federal Reserve Bank of St.Louis) and quarterly frequency from 3rd quarter of 1986 to 3rd quarter of 2013.

2. Foreign Output: This variable is constructed using

$$y_t^D = \frac{y_t^n}{p_t} \tag{184}$$

where y_t represents U.K. real G.D.P constructed from y_t^n which is aggregate U.K. nominal G.D.P observations in millions of pounds (from Federal Reserve Bank of St. Louis) and p_t which is U.K. C.P.I (with 2010=100 and from Federal Reserve Bank of St. Louis). Using (184) the following can be formed

$$\mathbf{y}_t^D = \begin{bmatrix} y_{1986:III} \\ \vdots \\ y_{2013:III} \end{bmatrix} \tag{185}$$

where $\mathbf{y}_t^D$ represents a time-series data vector containing constructed aggregate U.K. real G.D.P observations in quarterly frequency from 3rd quarter of 1986 to 3rd quarter of 2013.

3. Home Money: Defined as

$$
\mathbf{M}_t^D = \begin{bmatrix} M_{1986:III} \\ \vdots \\ M_{2013:III} \end{bmatrix} \tag{186}
$$

where $\mathbf{M}_t^D$ is a time-series data vector containing aggregate U.S. Household Financial Assests and Currency observations (from Federal Reserve Bank of St.Louis) in quarterly frequency from 3rd quarter of 1986 to 3rd quarter of 2013.

4. Foreign Money: Defined as

$$
\mathbf{N}_t^D = \begin{bmatrix} N_{1986:III} \\ \vdots \\ N_{2013:III} \end{bmatrix} \tag{187}
$$

where $\mathbf{N}_t$ is a time-series data vector containing aggregate U.K. Household Outstanding Holdings of Notes/Coin observations in millions of pounds (from Bank of England) and quarterly frequency from 3rd quarter of 1986 to 3rd quarter of 2013.

5. Spot Exchange Rate: Defined as

$$
\mathbf{S}_t^D = \begin{bmatrix} S_{1986:III} \\ \vdots \\ S_{2013:III} \end{bmatrix} \tag{188}
$$

where $\mathbf{S}_t^D$ is a time-series data vector containing $\$/\pounds$ spot exchange rate observations (from Federal Reserve Bank of St.Louis) in quarterly frequency from 3rd quarter of 1986 to 3rd quarter of 2013.

6. Forward Exchange Rate: Defined as

$$\mathbf{F}_t^D = \begin{bmatrix} F_{1986:III} \\ \vdots \\ F_{2013:III} \end{bmatrix} \tag{189}$$

where $\mathbf{F}_t^D$ is a time-series data vector containing \$/£ 1-month forward exchange rate observations (from Bank of England) in quarterly frequency from 3rd quarter of 1986 to 3rd quarter of 2013.

7.2 Stochastic Processes Estimation with Data

In this paper's model, the stochastic process are assumed to evolve according to (16) − (19) restated below for convenience

$$\begin{aligned} x_{t+1} &= (1-\rho_1) + \rho_1 x_t + \varepsilon_{t+1}^x \\ y_{t+1} &= (1-\rho_2) + \rho_2 y_t + \varepsilon_{t+1}^y \\ M_{t+1} &= (1-\rho_3) + \rho_3 M_t + \varepsilon_{t+1}^M \\ N_{t+1} &= (1-\rho_4) + \rho_4 N_t + \varepsilon_{t+1}^N \end{aligned}$$

The following steps indicate how to process data:

Step 1) Take log of each element in (183), (185), (186), and (187) producing $\log(\mathbf{x}_t^D), \log(\mathbf{y}_t^D), \log(\mathbf{M}_t^D)$, and $\log(\mathbf{N}_t^D)$ respectively.

Step 2) HP-Filter each logged data vectors which seperates trend and cyclical components producing $\log(\mathbf{x}_t^D)^C, \log(\mathbf{y}_t^D)^C, \log(\mathbf{M}_t^D)^C$, and $\log(\mathbf{N}_t^D)^C$ where each superscript C denotes the logged cyclical component of the superscripts respective data vector.

Step 3) Insert $\log(\mathbf{x}_t^D)^C, \log(\mathbf{y}_t^D)^C, \log(\mathbf{M}_t^D)^C$, and $\log(\mathbf{N}_t^D)^C$ in place of x_t, y_t, M_t, and N_t respectively yielding

$$
\begin{aligned}
\log(\mathbf{x}_{t+1}^D)^C &= (1-\rho_1) + \rho_1 \log(\mathbf{x}_t^D)^C + \varepsilon_{t+1}^x && (190)\\
\log(\mathbf{y}_{t+1}^D)^C &= (1-\rho_2) + \rho_2 \log(\mathbf{y}_t^D)^C + \varepsilon_{t+1}^y && (191)\\
\log(\mathbf{M}_{t+1}^D)^C &= (1-\rho_3) + \rho_3 \log(\mathbf{M}_t^D)^C + \varepsilon_{t+1}^M && (192)\\
\log(\mathbf{N}_{t+1}^D)^C &= (1-\rho_4) + \rho_4 \log(\mathbf{N}_t^D)^C + \varepsilon_{t+1}^N && (193)
\end{aligned}
$$

Step 4) Convert system (190) – (193) into VAR form

$$\mathbb{X}_{t+1} = \rho_C + \rho_S \mathbb{X}_t + \varepsilon_{t+1} \tag{194}$$

where

$$\mathbb{X}_t = \begin{bmatrix} \log(\mathbf{x}_t^D)^C \\ \log(\mathbf{y}_t^D)^C \\ \log(\mathbf{M}_t^D)^C \\ \log(\mathbf{N}_t^D)^C \end{bmatrix}, \quad \varepsilon_{t+1} = \begin{bmatrix} \varepsilon_{t+1}^x \\ \varepsilon_{t+1}^y \\ \varepsilon_{t+1}^M \\ \varepsilon_{t+1}^N \end{bmatrix}, \quad \rho_C = \begin{bmatrix} (1-\rho_1) \\ (1-\rho_2) \\ (1-\rho_3) \\ (1-\rho_4) \end{bmatrix}, \quad \rho_S = \begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_3 & 0 \\ 0 & 0 & 0 & \rho_4 \end{bmatrix}$$

Step 5) Estimate (194) via ordinary least squares resulting in

$$\widehat{\rho_S} = \begin{bmatrix} \widehat{\rho_1} & \widehat{\rho_2} & \widehat{\rho_3} & \widehat{\rho_4} \end{bmatrix} \tag{195}$$

$$\widehat{\rho_C} = (1 - \widehat{\rho_S})\,\overline{\mathbb{X}} \tag{196}$$

where the elements of (195) and (196) are used in place of their true counterparts in system (16) – (19)

7.3 Forward Premium Regression with Data

Above, pseudo observations were generated for the regression of forward premium on gross rate of depreciation so for comparison to data-driven regression, the following steps are performed:

Step 1) Take the log of each element in (188)and (189) producing $\log(\mathbf{S}_t)$ and $\log(\mathbf{F}_t)$ respectively

Step 2) Construct $\log(\mathbf{F}_t^D) - \log(\mathbf{S}_t^D)$ which is forward premium

Step 3) Construct $\log(\mathbf{S}_{t+1}^D) - \log(\mathbf{S}_t^D)$ which is gross rate of depreciation

Step 4) Define $\mathbb{F}_t -$ $\mathrm{S}_t = \log(\mathbf{F}_t^D) - \log(\mathbf{S}_t^D)$, $\mathbb{S}_{t+1} - \mathbb{S}_t = \log(\mathbf{S}_{t+1}^D) - \log(\mathbf{S}_t^D)$, and perform the following regression via ordinary least squares

$$\mathbb{S}_{t+1} - \mathbb{S}_t = \alpha_0 + \alpha_1 \left(\mathbb{F}_t - \mathbb{S}_t\right) + e_{t+1} \tag{197}$$

where $\widehat{\alpha_0}$ and $\widehat{\alpha_1}$ are estimators to α_0 and α_1 respectively.