# Numerical modeling of a toroidal inclusion in a linear elastic torus

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December 11, 2016

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**Math 6640, Numerical Methods for PDEs final project**

**Abstract**

In this short note we study the stress field generated by a toroidal inclusion embedded in an incompressible elastic torus such that the generating circles of the inclusion and the torus are concentric. We derive the governing equations of the problem in the setting of linear elasticity (small strains) and then discretize the equations. We then utilize finite difference and differential quadrature methods based on differential operators and present some numerical results for the induced displacement and pressure fields.

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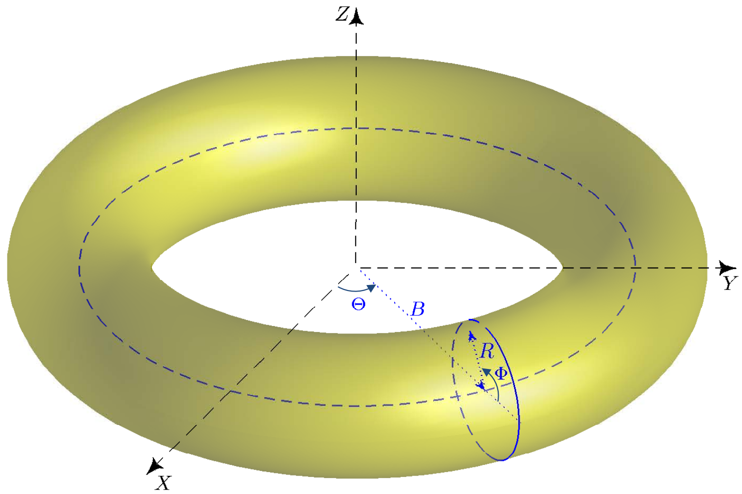


Figure 1: *A solid torus containing a concentric toroidal inclusion.*

**Keywords:** Linear elasticity; inclusion problem; elastic torus, growth mechanics, inherent strains.

# 1 Introduction

Toroidal inclusions and inhomogeneities naturally exist in the microstructures of many materials. Therefore, understanding the stress and deformation fields generated by toroidal inclusions is of a great importance. Onaka [2003] studied the strain field caused by uniform and purely dilatational inherent strains for a toroidal inclusion. In particular, they observed two strain-free points, where all the components of strain vanish. The deformation of a torus made of a homogeneous, isotropic, incompressible elastic material under inflation by uniform internal pressure and rotation were explored by Kydoniefs and Spencer [1965] and Kydoniefs [1966]. They assumed that in the deformed state the torus is generated by rotating a region surrounded between two concentric circles around an axis in their planes. Based on the assumption that the radii of the circles are small compared to the torus overall radius, approximate solutions for the stress and deformation fields in the torus were found. Hill [1980] extended their work further to tori inflated from a torus in their undeformed state. He also used the same assumption that the ratio of the radius of the generating circles to the overall radius of the torus is small and found the solutions to the first order of the small parameter. Kydoniefs and Spencer [1967], under the same assumption, investigated the finite inflation of an elastic toroidal membrane due to uniform internal pressure, where the membrane has a circular cross section in the reference configuration. The solutions to the second order of the small parameter were found. Some numerical results for a toroidal membrane of a Mooney-Rivlin material were presented, which describes the dependence of the deformation and the induced stresses on the internal pressure. Recently, Argatov and Sevostianov [2011] studied the reinforcing effects of toroidal inhomogeneities on the overall properties of an elastic medium. Their results suggest that there is no significant difference in the reinforcing effects of toroidal and spherical inhomogeneities with equal volume and diameter. Here, we study the effects of a linear toroidal inclusion placed in a solid torus.

This note is organized as follows. In section 2 we explain the toroidal inclusion problem and derive the governing equations. We include the MATLAB code in a separate file, where we use mixed finite difference and differential quadrature methods to solve the equations numerically. In section 2.1, we present the numerical results. Conclusions are given in section 3.

# 2 Linear toroidal inclusion in an elastic neo-Hookean torus

In this section we consider a solid torus with radius $R_o$ containing an inclusion with radius $R_i$ generated by rotating two concentric circles with radii $R_o$ and $R_i$ about an axis in their planes, where the distance from the origin to the center of the circle is $B$. Let $(R, \Theta, \Phi)$ and $(r, \theta, \phi)$ be the initial and deformed toroidal coordinates, respectively, as illustrated in Figure 1. Let us consider the following distribution of inherent strains in the torus

$$\Omega(R) = \begin{cases} \delta\Omega\,, & 0 \leq R < R_i \\ 0\,, & R_i < R \leq R_o \end{cases} . \tag{2.1}$$

Let us consider a symmetric class of deformations of the following form

$$r = r(R, \Phi)\,, \qquad \theta = \Theta\,, \qquad \phi = \phi(R, \Phi)\,. \tag{2.2}$$

We assume that the torus is made of an incompressible neo-Hookean solid with the strain energy function $W = \frac{\mu}{2}(I_1 - 3)$, where $\mu$ is the shear modulus of the material in the ground state. Let $u(R, \Theta)$ and $w(R, \Theta)$ and be the non-zero displacement components in the radial and circumferential directions, respectively. Incompressibility implies that

$$u\,(B + R\cos\Phi) + R\,[\delta b + u\cos\Phi - wR\sin\Phi] + R(B + R\cos\Phi)(u_{,R} + w_{,\Phi}) = \frac{3}{2}R(B + R\cos\Phi)\delta\Omega, \tag{2.3}$$

where $\delta b$ is a constant describing the change in the overall radius of the torus after the deformations induced by the inclusion. We assume that the material is piecewise homogeneous and find the components of the stress tensor as

$$\delta\sigma^{rr} = -\delta p + \mu\,(2u_{,R} - \delta\Omega)\ , \tag{2.4a}$$

$$\delta\sigma^{r\phi} = \mu\left(w_{,R} + \frac{u_{,\Phi}}{R^2}\right)\ , \tag{2.4b}$$

$$\delta\sigma^{\theta\theta} = -\frac{\delta p}{(B + R\cos\Phi)^2} + \frac{2\mu}{(B + R\cos\Phi)^3}\,[\delta b + u\cos\Phi - wR\sin\Phi] - \frac{\mu\,\delta\Omega}{(B + R\cos\Phi)^2}\,, \tag{2.4c}$$

$$\delta\sigma^{\phi\phi} = -\frac{\delta p}{R^2} + \frac{2\mu}{R^2}\left(\frac{u}{R} + w_{,\Phi} - \frac{\delta\Omega}{2}\right)\ , \tag{2.4d}$$

where $\delta p(R, \Theta)$ is the pressure arises due to the incompressibility constraint. After some simplifications, the nontrivial equilibrium equations are written as

$$\begin{aligned} \frac{\delta p_{,R}}{\mu} + \frac{2}{R}\left[\frac{u}{R} - u_{,R} - Ru_{,RR} - \frac{u_{,\Phi\Phi}}{2R} - \frac{Rw_{,R\Phi}}{2} + w_{,\Phi}\right] - \frac{2u_{,R}\cos\Phi}{B + R\cos\Phi} + \frac{R\sin\Phi}{B + R\cos\Phi}\left(\frac{u_{,\Phi}}{R^2} + w_{,R}\right) \\ + \frac{2\cos\Phi}{(B + R\cos\Phi)^2}\,(\delta b + u\cos\Phi - Rw\sin\Phi) = 0\,, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \frac{1}{R^2}\frac{\delta p_{,\Phi}}{\mu} + \frac{2\sin\Phi}{B + R\cos\Phi}\left(\frac{u}{R^2} + \frac{w_{,\Phi}}{R}\right) + \frac{2\sin\Phi}{(B + R\cos\Phi)^2}\left[w\sin\Phi - \frac{u}{R}\cos\Phi - \frac{\delta b}{R}\right] - \frac{3B + 4R\cos\Phi}{B + R\cos\Phi}\left[\frac{u_{,\Phi}}{R^3} + \frac{w_{,R}}{R}\right] \\ - \frac{u_{,R\Phi}}{R^2} - w_{,RR} - \frac{2w_{,\Phi\Phi}}{R^2} = 0\,. \end{aligned} \tag{2.6}$$

As we are interested in finding the residual stress field, we assume that the boundary of the torus is traction-free, i.e.,

$$2u_{,R} - \frac{\delta p}{\mu} = 0\,, \qquad R = R_o\,, \quad -\pi \leq \Phi \leq \pi\,, \tag{2.7a}$$

$$w_{,R} + \frac{u_{,\Phi}}{R^2} = 0\,, \qquad R = R_o\,, \quad -\pi \leq \Phi \leq \pi\,. \tag{2.7b}$$

Also, we eliminate the rigid body motion by setting $u_i\,(0, \Phi) = 0$ and $w\,(R, 0) = 0$. It follows from the symmetry of the problem and the fact that $u$ and $w$ are $C^2$ everywhere, except maybe at $R = 0$

$$\begin{aligned} w(R, \pm\pi) = 0\,, \quad u_{,\Phi}(R, \Phi)|_{\Phi=0,\pm\pi} = 0\,, \quad & \frac{\delta p_{,\Phi}(R, \Phi)}{\mu}|_{\Phi=0,\pm\pi} = 0\,, \\ & w_{,\Phi\Phi}(R, \Phi)|_{\Phi=0,\pm\pi} = 0\,, \end{aligned} \qquad 0 < R \leq R_o\,. \tag{2.8}$$

Next, we use a finite difference method to find the solution of the boundary value problem (2.3), (2.5), and (2.6) numerically.

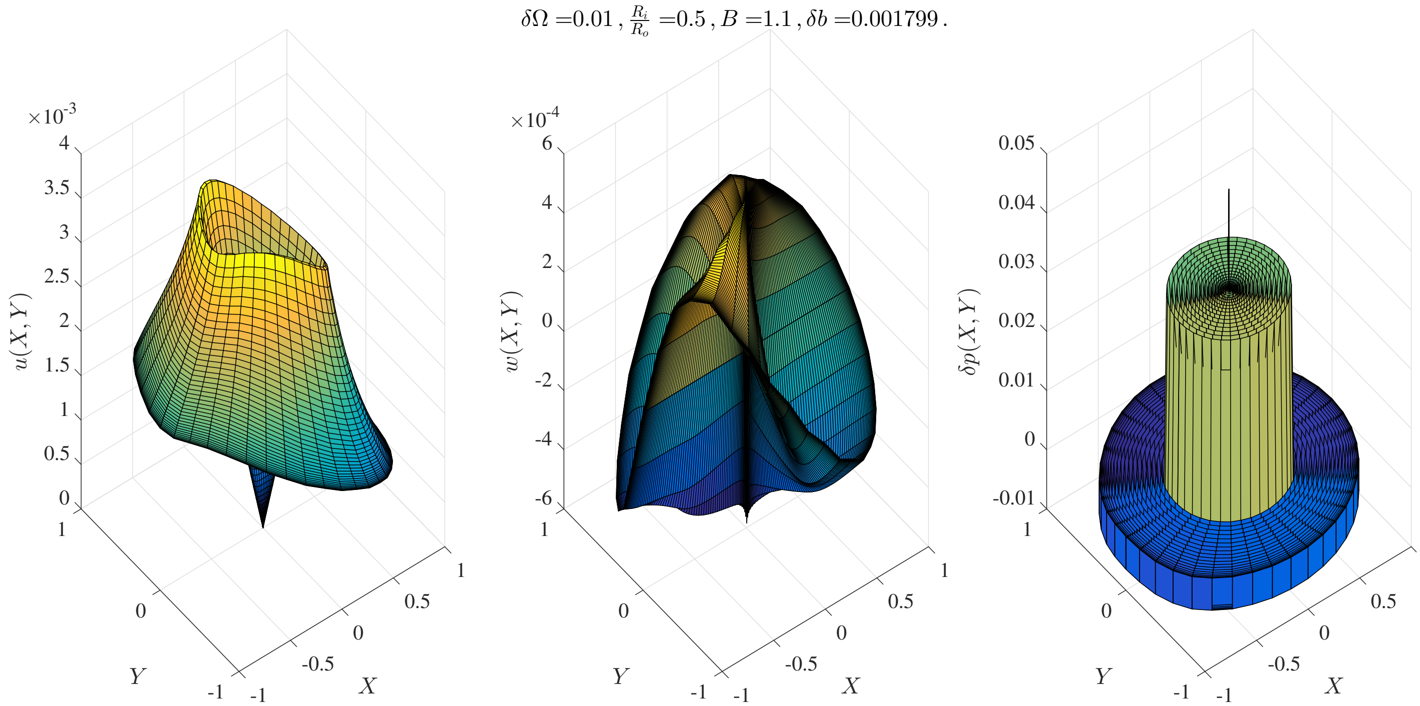


Figure 2: *Contours of displacement and pressure fields for a solid torus containing a toroidal inclusion with inherent strains* $\delta\Omega = 0.01$ *such that* $\frac{R_i}{R_o} = 0.5$ *and* $B = 1.1$.

### 2.1 Numerical Implementation and Results.

To implement the step function (2.1) in the computational scheme, we use a $C^\infty$ error function as follows

$$\delta\Omega(R) = \frac{\delta\Omega}{2}\left(1 - \operatorname{erf}\left(\frac{R - R_i}{\sigma}\right)\right), \tag{2.9}$$

where $\sigma$ is a constant representing how close the constructed function is with respect to the step function. The smaller the constant $\sigma$ is, the closer the function would be to the step function. We then use differential operators to discretize the BVP. Using a FDM in the circumferential direction and a DQM, along with Chebyshev grid in the radial direction, we obtain the solutions numerically. Figure 2 shows the contours of the displacement and pressure fields for a torus with a toroidal inclusion. For this case, where $\frac{R_i}{R_o} = 0.5$, $B = 1.1$, and $\delta\Omega = 0.01$, we calculated $\delta b$ as 0.0018 . Note that in the inclusion the radial displacement field tends to be varying linearly with respect to $R$. The pressure field in the inclusion tends to be uniform, whereas in the matrix it exhibits some variations.

## 3 Conclusions

In this note, we numerically studied the displacement and stress fields induced by a toroidal inclusion in a neo-Hookean solid in the setting of linear elasticity. We derived the governing equations of the problem and discretized the equations by using differential operators. To have an optimal computational scheme, we used a FDM in the circumferential direction, while in the radial direction, a DQM was used on Chebyshev grid. In particular, we found that for small values of inherent strains, the radial displacement field tends to be linear in the inclusion. It was observed that the pressure field tends to be uniform in the inclusion.

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