Pensées 1: The Calculus of n-Ball Volume

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Volume and Surface Area of n-balls (review)

An n-ball is a ball in n dimensions, for the cases of $n = 2$ and $n = 3$ being the points inside a circle and a sphere, respectively.

$$\mathbb{B}_n(r) = \{x \in \mathbb{R}^n \mid |x| \leq r\}$$

It is known that this expression evaluates to the volume of a unit n-ball. For example, for $n = 2$ and $n = 3$ the area in a circle and the volume of a sphere.

$$V_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)}$$

Fractional Calculus

These are the Riemann–Liouville fractional integral and derivative, it is not necessary you know them.

$$D^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau) \, \mathrm{d}\tau$$

$$D^{\alpha} = \frac{\partial^n}{\partial t^n} D^{-(n-a)} f(t)$$

Polynomial and Constant Case

This is the νth fractional integral of a polynomial, t^{λ}, and a constant, μ.

$$D^{-\nu} t^{\lambda} = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \nu + 1)} t^{\lambda + \nu}$$

$$\mathrm{d}x = 0$$

$$D^{-\nu} \mu = \mu \lim_{\lambda \to 0} \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \nu + 1)} t^{\lambda + \nu} = \mu \frac{t^{\nu}}{\Gamma(\nu + 1)}$$

The Relationship Between Fractional Integration of Constants and n-Ball Volume

You can see that the volume of an n-ball is a special case of this formula, here is the equivalence. The dimension of n-ball volume may be increased to $n+\rho$ by integration $\frac{\rho}{2}$ times.

$$V_n = D^{-\frac{n}{2}}1|_{t=\pi}$$

In addition, the sum of the volumes of all even-dimensional n-balls is e^{π}.

$$\nu \in \mathbb{Z} \Rightarrow \frac{t^{\nu}}{\Gamma(\nu+1)} = \frac{t^{\nu}}{\nu!}$$

$$\sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!} = e^{t} = \sum_{\nu=0}^{\infty} D^{-\nu}1$$

Here is the ratio between n-ball volumes (obviously substitute π for t). From the limits you can see the volume goes to zero, but has no roots (duh).

$$\frac{D^{-\nu-\rho}1}{D^{-\nu}1} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\rho+1)}t^{\rho}$$

$$\lim_{\rho\to\infty} \frac{D^{-\nu-\rho}1}{D^{-\nu}1} = \lim_{\rho\to\infty} \frac{\Gamma(\nu+1)}{\Gamma(\nu+\rho+1)}t^{\rho} = 0$$

$$\lim_{\rho\to\infty} \frac{\partial}{\partial\rho}\frac{D^{-\nu-\rho}1}{D^{-\nu}1} = \lim_{\rho\to\infty} \frac{t^{\rho}\Gamma(\nu+1)\left(\log(t) - \psi^{0}(\rho+\nu+1)\right)}{\Gamma(\rho+\nu+1)} = 0$$

Better Volume Constants

Just ignore the first part of the λ manifesto where it talks about λ being a good volume constant, it is not. An alternative for shapes which do not have every orthant like a regular n-ball (β is number of orthants) is κ, the area of one quadrant, otherwise π is the best.

$$\kappa = \frac{\pi}{4}$$

$$V_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$$

$$V_n = \frac{\beta\kappa^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$$

$$n \in \mathbb{Z} \Rightarrow V_n = \frac{2^{n-\lfloor\frac{n}{2}\rfloor}\pi^{\lfloor\frac{n}{2}\rfloor}}{n!!} = \frac{2^{\frac{n}{2}+\frac{n}{2}\%1}\pi^{\lfloor\frac{n}{2}\rfloor}}{n!!}$$

The Rational Answer to π vs τ

In a partially subjective way, the surface area is subordinate to the volume constant, in that it is nV_n. Thus calculating the volume constant from the surface area constant is backwards. τ is the 2-dimensional element of the family of surface area constants, while π is the volume constant. I would argue then that it is nicer and makes more sense to have π be the symbol in use, if only one can be chosen. It is possible, although inconvenient, to use both. The problems introduced by using τ in notation are much greater than π, which at most introduces a 2π where you could have τ. The largest and most common problem of using τ is $\sqrt{\pi}$, which becomes $\sqrt{\frac{\tau}{2}} = \frac{\sqrt{\tau}}{\sqrt{2}}$, much uglier and more annoying.

Various Elements of The Case for π

Besides, obviously, being the volume constant (pretty important), π has the advantage when talking about linear dependence, in some cases in complex analysis, and others.

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-x^2} \, \mathrm{d}x = \sqrt{\pi}$$

The principal branch is $(-\pi, \pi]$, and

$$\beta, \omega \in \mathbb{C}, \alpha \in \mathbb{R}, \kappa \in \mathbb{Z} \quad \beta = \alpha\omega \Leftarrow e^{i \arg \beta} = e^{i(\arg \omega + \kappa\pi)}$$

$$e^{i\kappa\pi} = (-1)^{\kappa}$$

The primary point in favor of π is not that it is great or elegant in most situations, but that it is adequate. τ has many advantages and equations which it eliminates a factor of 2 from, but it also makes others much uglier, like $\sqrt{\pi}$, or the formula for "parallel" complex numbers, while π is a good middle ground between possible "circle constants".