Numerical Methods for PDEs (Spring 2017)

Problems 1

Attempt each problem as soon as the material comes up in your assigned reading. The deadline for handing in the written solutions to all problems is the start of the lecture on 7 February.

Problem 1. Derive a finite-difference formula for the mixed derivative

$$\frac{\partial^2 u}{\partial x \partial t}$$

at (x_k, t_j) based on the grid points (x_k, t_j), (x_{k+1}, t_j), (x_k, t_{j+1}) and (x_{k+1}, t_{j+1}), where $t_{j+1} = t_j + \tau$ and $x_{k+1} = x_k + h$.

Problem 2. The heat equation

$$\frac{\partial u}{\partial t} - K\frac{\partial^2 u}{\partial x^2} = f(x,t) \quad \text{for} \quad 0 < x < 1, \quad 0 < t < T, \tag{1}$$

subject to the boundary and initial conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \quad u(x,0) = u_0(x), \tag{2}$$

is solved numerically using the Crank-Nicolson finite-difference method:

$$\begin{aligned} &w_{k0} = u_0(x_k), \quad w_{0j} = 0, \quad w_{Nj} = 0, \\ &w_{k,j+1} - w_{k,j} - \frac{\gamma}{2}\left(w_{k+1,j} - 2w_{kj} + w_{k-1,j} + w_{k+1,j+1} - 2w_{k,j+1} + w_{k-1,j+1}\right) = \tau f(x_k, t_j + \tau/2), \end{aligned} \tag{3}$$

for $k = 1, 2, \ldots, N-1$ and $j = 1, 2, \ldots, M$. Here w_{kj} is an approximation to $u(x_k, y_j)$ and

$$\gamma = K\tau/h^2, \quad x_k = kh \ \ (k = 0, 1, \ldots, N), \quad t_j = j\tau \ \ (j = 0, 1, \ldots, M), \quad h = \frac{1}{N}, \quad \tau = \frac{T}{M}.$$

Show that the local truncation error, given by

$$\tau_{k,j}(h) = \frac{1}{\tau}\left[u_{k,j+1} - u_{kj} - \frac{\gamma}{2}\left(u_{k+1,j} - 2u_{kj} + u_{k-1,j} + u_{k+1,j+1} - 2u_{k,j+1} + u_{k-1,j+1}\right)\right] - f(x_k, t_j + \tau/2),$$

is $O(\tau^2 + h^2)$. (Here $u_{k,j} = u(x_k, t_j)$.)

Problem 3. The equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha\frac{\partial u}{\partial x} \quad \text{for} \quad 0 < x < 1, \quad t > 0, \tag{4}$$

where α is a real constant, subject to the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \tag{5}$$

and the initial condition

$$u(x,0) = u_0(x), \tag{6}$$

is solved numerically using the finite-difference method:

$$\begin{aligned} &w_{k0} = u_0(x_k), \quad w_{0j} = 0, \quad w_{Nj} = 0, \\ &\frac{w_{k,j} - w_{k,j-1}}{\tau} - \frac{w_{k+1,j} - 2w_{kj} + w_{k-1,j}}{h^2} - \alpha\frac{w_{k+1,j} - w_{k-1,j}}{2h} = 0, \end{aligned} \tag{7}$$

for $k = 1, 2, \ldots, N-1$ and $j = 1, 2, \ldots$. Here w_{kj} is an approximation to $u(x_k, y_j)$ and $x_k = kh$ $(k = 0, 1, \ldots, N)$, $t_j = j\tau$ $(j = 0, 1, \ldots)$, $h = \frac{1}{N}$.

(a) Find the local truncation error of this finite-difference scheme.

(b) Investigate the stability of the scheme (using the Fourier method).