Numerical Methods for PDEs (Spring 2017)

Solutions 1

Problem 1. Derive a finite-difference formula for the mixed derivative

$$\frac{\partial^2 u}{\partial x \partial t}$$

at (x_k, t_j) based on the grid points (x_k, t_j), (x_{k+1}, t_j), (x_k, t_{j+1}) and (x_{k+1}, t_{j+1}), where $t_{j+1} = t_j + \tau$ and $x_{k+1} = x_k + h$.

Solution. Below we use the notation $u_{k,j} \equiv u(x_k, t_j)$. Assuming that function $u(x,t)$ is sufficiently smooth, we expand $u_{k+1,j}$, $u_{k,j+1}$ and $u_{k+1,j+1}$ in Taylor's series

$$\begin{aligned}
u_{k+1,j} &= u(x_k + h, t_j) = u(x_k, t_j) + h\frac{\partial u}{\partial x}(x_k, t_j) + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}(x_k, t_j) + O(h^3), \\
u_{k,j+1} &= u(x_k, t_j + \tau) = u(x_k, t_j) + \tau\frac{\partial u}{\partial t}(x_k, t_j) + \frac{\tau^2}{2}\frac{\partial^2 u}{\partial t^2}(x_k, t_j) + O(\tau^3), \\
u_{k+1,j+1} &= u(x_k + h, t_j + \tau) = u(x_k, t_j) + h\frac{\partial u}{\partial x}(x_k, t_j) + \tau\frac{\partial u}{\partial t}(x_k, t_j) + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}(x_k, t_j) \\
&\qquad + h\,\tau\,\frac{\partial^2 u}{\partial x \partial t}(x_k, t_j) + \frac{\tau^2}{2}\frac{\partial^2 u}{\partial t^2}(x_k, t_j) + O(h^3) + O(h^2\tau) + O(h\tau^2) + O(\tau^3).
\end{aligned}$$

It follows that

$$u_{k+1,j+1} - u_{k+1,j} - u_{k,j+1} + u_{kj} = h\,\tau\,\frac{\partial^2 u}{\partial x \partial t}(x_k, t_j) + O(h^3) + O(h^2\tau) + O(h\tau^2) + O(\tau^3).$$

Hence,

$$\frac{\partial^2 u}{\partial x \partial t}(x_k, t_j) = \frac{u_{k+1,j+1} - u_{k+1,j} - u_{k,j+1} + u_{kj}}{h\,\tau} + O\left(\frac{h^2}{\tau} + h + \tau + \frac{\tau^2}{h}\right).$$

Problem 2. The heat equation

$$\frac{\partial u}{\partial t} - K\frac{\partial^2 u}{\partial x^2} = f(x,t) \quad \text{for} \quad 0 < x < 1, \quad 0 < t < T, \tag{1}$$

subject to the boundary and initial conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \quad u(x,0) = u_0(x),$$

is solved numerically using the Crank-Nicolson finite-difference method:

$$\begin{aligned}
&w_{k0} = u_0(x_k), \quad w_{0j} = 0, \quad w_{Nj} = 0, \\
&w_{k,j+1} - w_{k,j} - \frac{\gamma}{2}\left(w_{k+1,j} - 2w_{kj} + w_{k-1,j} + w_{k+1,j+1} - 2w_{k,j+1} + w_{k-1,j+1}\right) = \tau f(x_k, t_j + \tau/2),
\end{aligned}$$

for $k = 1, 2, \ldots, N-1$ and $j = 1, 2, \ldots, M$. Here w_{kj} is an approximation to $u(x_k, y_j)$ and

$$\gamma = K\tau/h^2, \quad x_k = kh \;\; (k = 0, 1, \ldots, N), \quad t_j = j\tau \;\; (j = 0, 1, \ldots, M), \quad h = \frac{1}{N}, \quad \tau = \frac{T}{M}.$$

Show that the local truncation error, given by

$$\tau_{k,j}(h) = \frac{1}{\tau}\left[u_{k,j+1} - u_{kj} - \frac{\gamma}{2}\left(u_{k+1,j} - 2u_{kj} + u_{k-1,j} + u_{k+1,j+1} - 2u_{k,j+1} + u_{k-1,j+1}\right)\right] - f(x_k, t_j + \tau/2), \tag{2}$$

is $O(\tau^2 + h^2)$. (Here $u_{k,j} = u(x_k, t_j)$.)

Solution. Assuming that the exact solution $u(x,t)$ is smooth enough, we expand $u_{k\pm1,j}$ and $u_{k\pm1,j+1}$ in Taylor's series

$$u_{k\pm1,j} = u(x_{k\pm1},t_j) = u(x_k,t_j) \pm h\frac{\partial u}{\partial x}(x_k,t_j) + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}(x_k,t_j) \pm \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}(x_k,t_j) + O(h^4),$$
$$u_{k\pm1,j+1} = u(x_{k\pm1},t_{j+1}) = u(x_k,t_{j+1}) \pm h\frac{\partial u}{\partial x}(x_k,t_{j+1}) + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}(x_k,t_{j+1}) \pm \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}(x_k,t_{j+1}) + O(h^4)$$

It follows that

$$u_{k+1,j} - 2u_{kj} + u_{k-1,j} = 2\frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}(x_k,t_j) + O(h^4) = h^2\left(\frac{\partial^2 u}{\partial x^2}(x_k,t_j) + O(h^2)\right). \tag{3}$$

Similarly,

$$\begin{aligned} u_{k+1,j+1} - 2u_{k,j+1} + u_{k-1,j+1} &= h^2\left(\frac{\partial^2 u}{\partial x^2}(x_k,t_{j+1}) + O(h^2)\right) \\ &= h^2\left(\frac{\partial^2 u}{\partial x^2}(x_k,t_j) + \tau\frac{\partial^3 u}{\partial x^2\partial t}(x_k,t_j) + O(\tau^2) + O(h^2)\right) \end{aligned} \tag{4}$$

Also, we have

$$\begin{aligned} u_{k,j+1} - u_{kj} &= u(x_k,t_j) + \tau\frac{\partial u}{\partial t}(x_k,t_j) + \frac{\tau^2}{2}\frac{\partial^2 u}{\partial t^2}(x_k,t_j) + O(\tau^3) - u_{kj} \\ &= \tau\left(\frac{\partial u}{\partial t}(x_k,t_j) + \frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}(x_k,t_j) + O(\tau^2)\right), \end{aligned} \tag{5}$$

Substituting (3)–(5) into the formula for τ_{kj}, we obtain

$$\begin{aligned} \tau_{k,j} &= \frac{\partial u}{\partial t}(x_k,t_j) + \frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}(x_k,t_j) + O(\tau^2) \\ &\quad -\frac{\gamma h^2}{2\tau}\left(2\frac{\partial^2 u}{\partial x^2}(x_k,t_j) + \tau\frac{\partial^3 u}{\partial x^2\partial t}(x_k,t_j) + O(\tau^2) + O(h^2)\right) - \left(f(x_k,t_j) + \frac{\tau}{2}\frac{\partial f}{\partial t}(x_k,t_j) + O(\tau^2)\right) \\ &= \frac{\partial u}{\partial t}(x_k,t_j) - K\frac{\partial^2 u}{\partial x^2}(x_k,t_j) - f(x_k,t_j) + \frac{\tau}{2}\left(\frac{\partial^2 u}{\partial t^2}(x_k,t_j) - K\frac{\partial^3 u}{\partial x^2\partial t}(x_k,t_j) - \frac{\partial f}{\partial t}(x_k,t_j)\right) + O(\tau^2+h^2). \end{aligned}$$

It follows from Eq. (1) that

$$\frac{\partial^2 u}{\partial t^2} - K\frac{\partial^3 u}{\partial x^2\partial t} = \frac{\partial f}{\partial t}.$$

Hence, $\tau_{k,j} = O(\tau^2+h^2)$.

Problem 3. The equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha\,\frac{\partial u}{\partial x} \quad \text{for} \quad 0<x<1, \quad t>0, \tag{6}$$

where α is a real constant, subject to the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \tag{7}$$

and the initial condition

$$u(x,0) = u_0(x), \tag{8}$$

is solved numerically using the finite-difference method:

$$\begin{aligned} &w_{k0} = u_0(x_k), \quad w_{0j} = 0, \quad w_{Nj} = 0, \\ &\frac{w_{k,j} - w_{k,j-1}}{\tau} - \frac{w_{k+1,j} - 2w_{kj} + w_{k-1,j}}{h^2} - \alpha\,\frac{w_{k+1,j} - w_{k-1,j}}{2h} = 0, \end{aligned} \tag{9}$$

for $k = 1,2,\ldots,N-1$ and $j = 1,2,\ldots$. Here w_{kj} is an approximation to $u(x_k,y_j)$ and $x_k = kh$ $(k=0,1,\ldots,N)$, $t_j = j\tau$ $(j=0,1,\ldots)$, $h = \frac{1}{N}$.

(a) Find the local truncation error of this finite-difference scheme.

(b) Investigate the stability of the scheme (using the Fourier method).

Solution. (a) Let $u(x,t)$ be sufficiently smooth solution of the initial boundary-value problem (6)–(8). The truncation error of the method (9) is

$$\tau_{kj} = \frac{u_{k,j} - u_{k,j-1}}{\tau} - \frac{u_{k+1,j} - 2u_{kj} + u_{k-1,j}}{h^2} - \alpha\,\frac{u_{k+1,j} - u_{k-1,j}}{2h}. \tag{10}$$

We have

$$\frac{u_{k,j} - u_{k,j-1}}{\tau} = \frac{1}{\tau}\left(u_{k,j} - u(x_k,t_j) + \tau\frac{\partial u}{\partial t}(x_k,t_j) + O(\tau^2)\right) = \frac{\partial u}{\partial t}(x_k,t_j) + O(\tau). \tag{11}$$

Also,

$$\frac{u_{k+1,j} - u_{k-1,j}}{2h} = \frac{1}{2h}\left(2h\frac{\partial u}{\partial x}(x_k,t_j) + O(h^3)\right) = \frac{\partial u}{\partial x}(x_k,t_j) + O(h^2). \tag{12}$$

Substituting (11), (12) and (3) into (10) yields

$$\begin{aligned}
\tau_{kj} &= \frac{\partial u}{\partial t}(x_k,t_j) + O(\tau) - \frac{\partial^2 u}{\partial x^2}(x_k,t_j) - \alpha\,\frac{\partial u}{\partial x}(x_k,t_j) + O(h^2) \\
&= \frac{\partial u}{\partial t}(x_k,t_j) - \frac{\partial^2 u}{\partial x^2}(x_k,t_j) - \alpha\,\frac{\partial u}{\partial x}(x_k,t_j) + O(\tau) + O(h^2) = O(\tau + h^2).
\end{aligned}$$

(b) Let w_{k0} and $\tilde{w}_{k0}$ be two solutions of Eq. (9) corresponding to slightly different initial conditions and let $z_{kj} = \tilde{w}_{kj} - w_{kj}$ be the perturbation at the mesh point (x_k,t_j) for each $k = 0,1,2,\ldots,N$ and $j = 0,1,\ldots$. It follows from (9) that z_{kj} satisfies the difference equation

$$\frac{z_{k,j} - z_{k,j-1}}{\tau} - \frac{z_{k+1,j} - 2z_{kj} + z_{k-1,j}}{h^2} - \alpha\,\frac{z_{k+1,j} - z_{k-1,j}}{2h} = 0 \tag{13}$$

for $k = 1,2,\ldots,N-1$ and $j = 1,2,\ldots$. We will seek a particular solution of (13) in the form

$$z_{k,j} = \rho_q^j e^{iqx_k}, \quad q \in \mathbb{R}. \tag{14}$$

Here $i = \sqrt{-1}$.

The finite-difference method (9) is stable with respect to initial condition, if

$$|\rho_q| \le 1 \quad \text{for all} \quad q \in \mathbb{R}.$$

Substitution of (14) into (13) yields

$$\frac{e^{iqx_k}}{\tau}\left(\rho_q^j - \rho^{j-1}\right) - \frac{\rho_q^j}{h^2}\left(e^{iqx_{k+1}} - 2e^{iqx_k} + e^{iqx_{k-1}}\right) - \alpha\,\frac{\rho_q^j}{2h}\left(e^{iqx_{k+1}} - e^{iqx_{k-1}}\right) = 0$$

or

$$1 - \frac{1}{\rho_q} - \frac{\tau}{h^2}\left(e^{iqh} - 2 + e^{-iqh}\right) - \alpha\,\frac{\tau}{2h}\left(e^{iqh} - e^{-iqh}\right) = 0.$$

It follows that

$$\rho_q = \frac{1}{1 + \frac{4\tau}{h^2}\sin^2\frac{qh}{2} - \alpha\,\frac{i\tau}{h}\sin qh} \quad\Rightarrow\quad |\rho_q|^2 = \frac{1}{\left(1 + \frac{4\tau}{h^2}\sin^2\frac{qh}{2}\right)^2 + \alpha^2\,\frac{\tau^2}{h^2}\sin^2 qh}.$$

Evidently, $|\rho_q|^2 \le 1$ for all q, and therefore it is (unconditionally) stable.