Pricing Options with Monte Carlo Simulation

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1 Summary of the Problem

First, I will begin by stating some motivation for the project by answering the question "Why use Monte Carlo simulation?" The Black-Scholes formula can be used to analytically price European options, which will give an answer using much less computational effort. However, the formula assumes that the log-returns of the underlying are normally distributed. In reality, this could not be further from the truth. The distribution of log-returns often has fat tails, and a normal distribution will vastly underestimate the probability of a large move, which will obviously lead to a mispricing of the option. We see evidence of fat tailed log returns via phenomena like implied volatility smile, which means that options that are further out-of-the-money are priced at a higher implied volatility, and steep term structure, where options that are nearer to expiration are priced at a higher implied volatility than those further to expiration. The proposed solution to this problem is to use Monte Carlo simulation, for which we can use any distribution of log-returns, to price the option.

2 Finding a Distribution

I chose to look at Tesla stock for this example. Referring to figure 1, on the top left is the histogram for Tesla log returns, and on the top right a plot over time of the returns. The histogram is very peaked, indicating kurtosis in excess of a normal distribution and there are also some large deviations, we see log returns in excess of $\pm$ 20%. Both of these would suggest that the

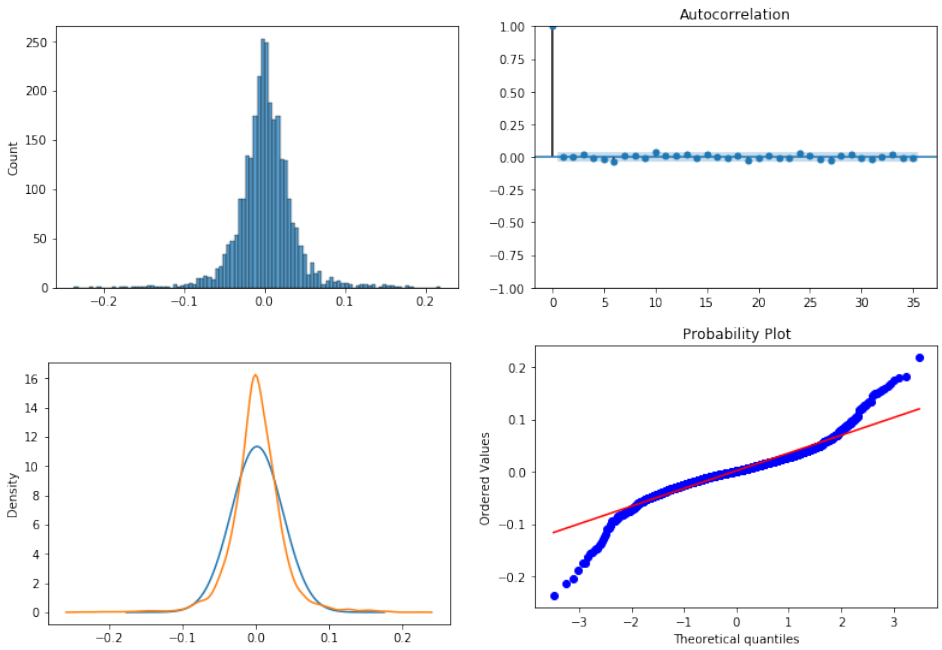

Figure 1: Histogram. noise plot, kernel density estimation and probability plot for Tesla log returns

actual distribution has tails that are fatter than a normal distribution. We also see that the auto correlation function does not suggest that there is any correlation between successive days returns.

If we perform a kernel density estimation on the distribution of log-returns and plot it against a normally distributed probability density function(shown in the bottom row of figure 1) fitted with the mean and standard deviation of the log returns, we see quite the divergence in the probability density functions, especially in the peakedness of the distribution. The sample kurtosis of the log-returns was 5.88 whereas the kurtosis of a normal distribution is always 3, so we have excess kurtosis. I've also included a probability plot, from which we can see that the distributions differ largely in the tails. This suggests a distribution with fatter tails than a normal distribution would be a good fit.

Judging only by the eye suggests a Student t distribution might be a good fit. This can be justified theoretically as well by considering the following argument, which is highlighted in this paper. [1] If we assume that returns

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AndersonResult(statistic=41.8756893337013, critical_values=array([0.575, 0.6
55, 0.786, 0.917, 1.09 ]), significance_level=array([15. , 10. ,  5. ,  2.5,
1. ]))
CramerVonMisesResult(statistic=0.09347324862089226, pvalue=0.617860191080073
2)
CramerVonMisesResult(statistic=0.21246292004295075, pvalue=0.244580407086617
66)
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Figure 2: Showing the Anderson-Darling test, and Cramer-VonMises test for Student-t and Laplace distributions respectively.

follow a normal distribution with mean μ but randomly distributed variance according to a chi-squared distribution, then the resulting distribution will be Student-t.

Finally, to confirm this statistically, I performed a Anderson-Darling normality test and a Cramer-VonMises test according to a t distribution. The Anderson-Darling test rejected normality at significance levels, and the Cramer-VonMises test could failed to reject the null hypothesis that the distribution is Student$-t$. I also performed a Cramer-VonMises test for a Laplace distribution which could not be rejected.

The better fit distribution now comes down to the tails. If the tail is pro-

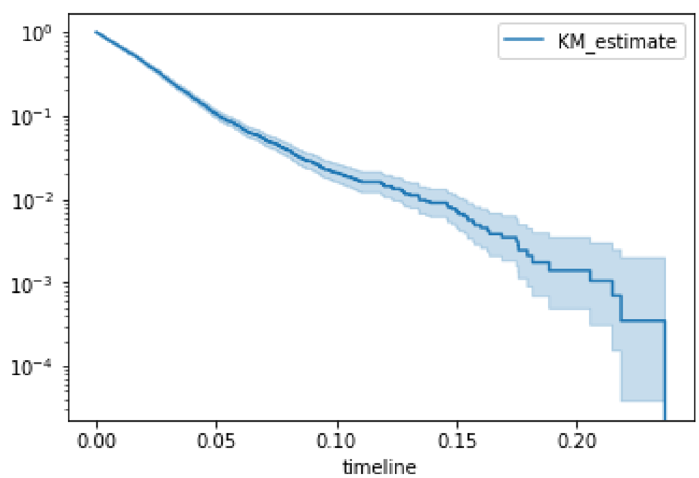

Figure 3: Log Survival function for Tesla log-returns

portional to an exponential distribution, then the survival function on a log scale will be straight. If the tail is proportional to $x^{-\alpha}$ as in a $t-$Distribution, then the survival function will bow. Referring to figure 3, we see a slight bow of the survival function, where is then decays rapidly because there is no more data. This is evidence for a $t-$Distribution. However, the tail may be fat up only to a point and then decay rapidly. It is hard to say for sure, but if the tail did decay rapidly at some point that suggests truncating the distribution may yield a better fit to the empirical data.

3 Simulating the Evolution of a Stock Price

The log returns for period i are defined as

$$r_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

where S_i is the stock price at time i. From this equation, we can get the stock price at time t as

$$S_t = \exp\left(\sum_{i=1}^{t} r_i\right).$$

Using this equation we can simulate the trajectory of a stock price by generating a sequence of i.i.d random variables representing the log-returns, and then find the distribution of the stock price at time t, and then use this to price the option.

The Black-Scholes model assumes that a stock trajectory follows a geo-

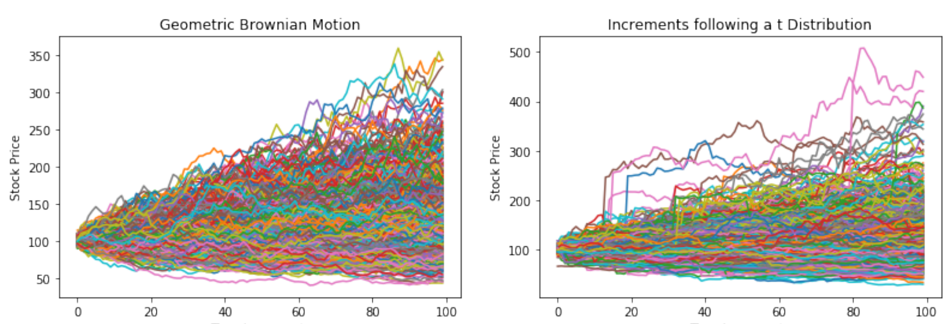

Figure 4: Geometric Brownian Motion and the same process with $t-$Distributed Increments

metric brownian motion, which mean that log-returns are i.i.d normally distributed random variables. Figure 4 highlights the difference between a geometric brownian motion and a process whose increments are $t-$Distributed. We can see that the paths for a geometric brownian motion are smooth, whereas with t increments, the process exhibits large jumps.

4 Pricing an Option

A European style call option gives the holder the right but not the obligation to purchase 100 shares of the underlying stock on the day of the options expiration. For a call option the price of European style option is theoretically the same as an American style option, for which the exercise can be done at any time before the option expires.[2] We can express the value of an option at expiration as

$$V_e = \max\{S_t - K, 0\}.$$

This makes the value of the option, and the quantity we want to estimate,

$$\mathbb{E}\left[\max\{S_t - K, 0\}\right].$$

From here, the estimation of this quantity involves generating the trajectories of the Stock price, and applying that formula and taking the sample average.

5 Pricing a 1400 Strike Tesla Call

I chose to look at pricing a 1400 strike Tesla call option expiring on December 31st, which at the time gave 22 trading days until expiration. The stock price was at 1095.00 when I performed the analysis. I simulated both a geometric brownian motion(GBM) and t process to compare the results. The value under a GBM was 13.32, whereas using the t process is was 17.24. Figure 5 shows the convergence of the sample mean for the sample means. We see that the under a GBM the convergence was smooth, but under a t process the convergence had large jumps, which indicates that the sample mean may not be defined because the tails are too fat. Even though the mean and variance of the fitted t-Distribution are defined, the mean of S_t may not be defined because $S_t = \exp\left(\sum_{i=1}^{t} r_i\right)$. Truncating the distribution may solve

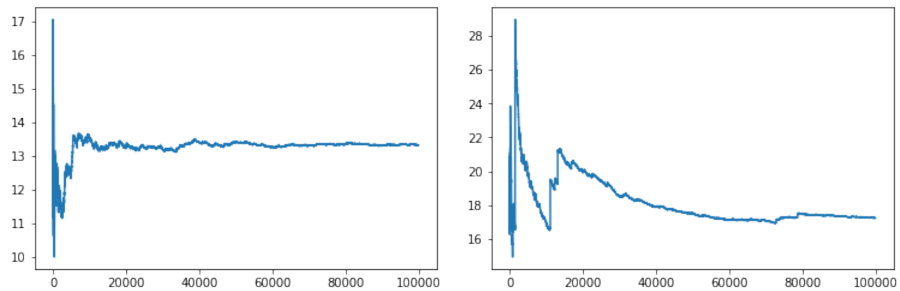

Figure 5: Convergence of the sample mean for a geometric brownian motion and t process.

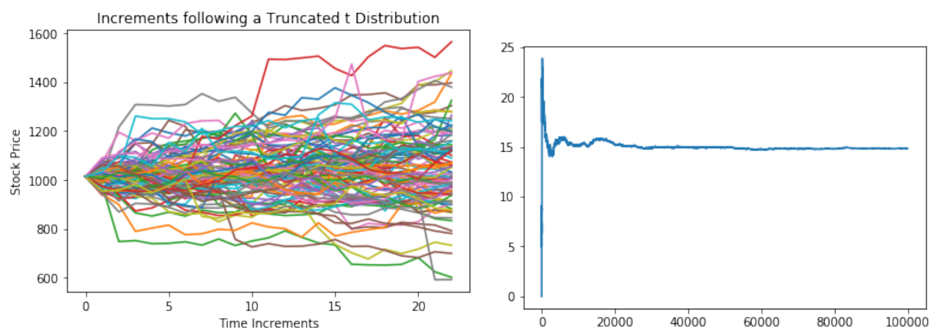

Figure 6: Process with truncated t increments and convergence of the mean under that process.

this problem.

When truncating the distribution we get a process where the tails aren't quite as fat, but still fatter than a GBM. We see from figure 6 that we retain the large jumps one would expect, but the mean converges when the distribution is truncated. The value found under the truncated t process was still higher than the Black-Scholes price at 14.85, but notably much less than under the non-truncated t distribution. However, the convergence is much nicer which suggests that we don't have the problem of an undefined mean of S_t.

6 Conclusion

Truncating the distribution seems to solve many of the convergence problems, but presents problems all of its own, namely at what point should the distribution be truncated. I think this method has potential, although further work can be done in the areas of variance reduction in order to achieve faster pricing. If importance sampling could be implemented, that would greatly help because the price of the option is influenced heavily by small probability events. The problem with importance sampling is finding the best probability density function to use for S_t. Since $S_t = \exp\left(\sum_{i=1}^{t} r_i\right)$ it wont be log normally distributed when log-returns are non-normal except for very large values of t. A better fit would be the Log Student-t Distribution, which is discussed in[1]. Using this distribution could allow for importance sampling to be done, but as previously stated more work is required.

References

[1] D. T. Cassidy, M. J. Hamp, and R. Ouyed, *Pricing european options with a log student's t-distribution: A gosset formula*, Physica A: Statistical Mechanics and its Applications, 389 (2010), p. 5736–5748.

[2] H. Wang, *Monte Carlo Simuation with Applications to Finance*, Chapman and Hall/CRC, Boca Raton, FL, 2012.