VACUOUS TRUTH AND RELATED CONVENTIONS
CSC236 — 2024 FALL

$\sum_{i=s}^{e} a_i$ means the sum of the sequence of numbers $a_s, a_{s+1}, \ldots, a_e$, i.e., $a_s + a_{s+1} + \cdots + a_e$. When $e = s - 1$ the sequence of numbers is empty, so we need to determine if there is a "meaningful" interpretation for that case. More practically we can ask whether there's a convenient interpretation: *useful*, without complicating non-empty cases:

- Are there situations modelled by a sum where an empty sum could arise and have a natural interpretation in the model?
- Are there straightforward implementations (e.g., looping or recursive) of summation that produce a value for empty sums, and/or are empty sums convenient for proofs of correctness of these implementations?
- Does allowing potentially-empty sums to appear temporarily during manipulations of non-empty sums make some calculations easier?
- Are most properties of summation preserved when instantiated with empty sums?

Suppose we're implementing a program to present information about a bank account, and the user can ask for the total amount of all transactions with a certain property. If they ask for the total amount for withdrawls in a certain month, and there were no withdrawls that month, the program should produce zero. If the implementation sums the values of the set of transactions, it's convenient if the sum of an empty set of values is defined, and has been implemented to produce, zero — otherwise we need a conditional statement and a special case in our code.
If we implement summation with a loop or recursively, starting an accumulator as zero (rather than as the value of the first term) or using empty as a base case and returning zero, the implementation automatically produces zero for the empty case.
It turns out that there is a vast array of cases where defining the sum of an empty sequence as zero is useful. And including that case doesn't complicate the properties much: see our "Summation" reference.

Many concepts involving sequences or sets have conventions for the empty case, when there isn't a "directly" natural interpretation, based on considerations like the ones we mentioned for summation. The product of an empty set of numbers is defined to be 1.
In predicate logic, cases where an expression has a value "by convention" instead of being "directly" true or false based on the meaning are called "vacuous". It seems directly true from the meaning of existential quantification that an existential with an empty domain is false, but the case for universal quantification is less clear. To extend many properties involving universal quantification to empty domain domains we are forced to define universal quantification over an empty domain as true. When we're in this case we sometimes say the quantification is "vacuously true", but that is not a different kind of true: "vacuously" is pointing out the *reason* we know (or treat it as) true.
One property of universal quantification that determines this convention is:

$$\forall x \in D, P(x) \wedge \forall x \in E, P(x) \equiv \forall x \in D \cup E, P(x)\,.$$

We'd like the special case

$$\forall x \in D, P(x) \wedge \forall x \in \varnothing, P(x) \equiv \forall x \in D \cup \varnothing, P(x) \equiv \forall x \in D, P(x)$$

to be true, regardless of what P is and whether $\forall x \in D, P(x)$ is true. This requires, and is satisfied by, defining $\forall x \in \varnothing, P(x)$ to be true.
In practice this means, for example, that a proof could start with a non-empty set, break it into two sets based on whether some property is true or false —where we're not sure whether the property is sometimes true and sometimes false— reason about the two sets without knowing whether one of them is empty, and conclude something about the original non-empty set. The conclusion would be correct, not involve vacuousness, but the reasoning might have.
Our perspective in CSC110/111/165/236 is that formal logic is to support us in expression and proof, and there this convention is quite safe and very useful (even though it does complicate some symbolic equivalences that don't directly correspond to standard proof forms, e.g., see Prenex normal form).

It seems directly true that the union of zero sets is empty, and the translation $\bigcup_{i\in\varnothing} s_i = \{x : \exists i \in \varnothing, x \in s_i\}$ confirms it as the set of objects x that make the always-false property $\exists i \in \varnothing, x \in s_i$ true, which is no objects. Translating $\bigcap_{i\in\varnothing} s_i = \{x : \forall i \in \varnothing, x \in s_i\}$ would produce the set of objects x that make an always-true (by convention) property true, which is every object x. But, for reasons outside our scope, the set of every "object" in the universe is not considered a set. For similar reasons, the complement is always relative to another set, possibly a default universe set, and when a universal set is specified and all operations are implicitly relative to that set the intersection of zero sets is defined to be the universe set.