**Due**: Thursday 8 February

## General instructions

Please read the following instructions carefully before starting each problem set. They contain important information about general problem set expectations, problem set submission instructions, and reminders of course policies.

- Your problem sets are **not** graded, but we will provide feedback on both correctness and clarity of communication.
- Solutions must be submitted as PDFs or photos. **Illegible submissions will receive no feedback.**
- Problem sets must be submitted online through MarkUs. If you haven't used MarkUs before, give yourself plenty of time to figure it out, and ask for help if you need it!
- You may work with whoever you want, and consult any source you want, but *you must cite all your sources*. If you are working with one or more partner(s), please form a group on MarkUs and **make a single submission for the entire group**. This saves time for everyone: it allows us to give feedback only once per submission, which frees up more time to give more detailed feedback. please consult `https://github.com/MarkUsProject/Markus/wiki/Student-Guide` for a brief explanation of how to create a group on MarkUs.
- Your submitted file(s) should not be larger than **19MB**. You might exceed this limit if you use a word processor like Microsoft Word to create a PDF, or if you combine multiple photos into a single document without compression, or if you export work from OneNote; in that case, you should look into compression tools to make your files smaller, but please make sure that the result is still legible! (Another solution is to split your work into multiple documents and submit each one separately.)

**1. Understanding Statements, and Writing Direct Proofs.**

**(a)** For each subset of real numbers $S \subseteq \mathbb{R}$ and real number $\ell \in \mathbb{R}$, define $\underline{\ell \text{ is a lower bound for } S}$ (also stated $\underline{S \text{ is bounded below by } \ell}$) as: "every element of $S$ is at least (i.e., greater than or equal to) $\ell$".

**i.** Give three sets of real numbers with the property that 165 is a lower bound, to illustrate a variety of ways in which the property can be true. Give two sets of real numbers for which 165 is *not* a lower bound, to illustrate a variety of ways in which the property can be false.

**ii.** Write this statement as a symbolic expression, where each quantifier is over domain $S$.

**iii.** Rewrite your expression so that each quantifier is over domain $\mathbb{R}$.
Then, write a *natural* English translation of your rewritten expression, making this logical form somewhat more explicit than in the original statement.

**iv.** Rewrite your expression from the previous sub-part, by replacing the main implication with its contrapositive.
Then, write a *natural* English translation of your rewritten expression.

**v.** Write a symbolic expression that expresses the statement: "$\ell$ is *not* a lower bound for $S$". Quantify only over domain $\mathbb{R}$, and do not use any explicit negation in your (final) expression (i.e., do *not* use the symbols $\neg$, $\notin$, $\not\leq$, etc.)
Then, write a *natural* English translation of your rewritten predicate expression.

**(b)** Consider the following statement:

"If $\ell$ is a lower bound for $S$, then every real number less than $\ell$ is also a lower bound for $S$."

**i.** Express the statement symbolically, where each quantifier is over domain $\mathbb{R}$ or $\mathcal{P}(\mathbb{R})$.
You will need to think about (and understand) various issues (that are not particularly unique to this question):

- What is the "logical type" of this statement (universal, existential, negation, conjunction, disjunction, implication, bi-implication)?
- How to express "is a lower bound"? You may define a predicate $LB$ and then use it to shorten your answer. But be precise with your definition: use the same approach we did in class and on previous worksheets to introduce all parameters and their domains, followed by the meaning of $LB$ in terms of its parameters.
- How to quantify the subset $S$?

**ii.** Rewrite your expression using the contrapositive of the main implication. "Move" negation "inwards", but leave sub-expressions representing "not a lower bound" as-is.
Then, write a *natural* English translation of your rewritten predicate expression.

**(c)** Define predicate $UB(u, S)$ to express that $u$ is an *upper bound* for set $S$, by analogy with previous parts. Be explicit and precise.

State your definition in English, and also write it as a symbolic expression, where each quantifier is over domain $\mathbb{R}$. There is **no** requirement to give multiple equivalent expressions; just pick the one you feel is simplest / most natural.

**(d)** Consider the following statement:

"Each lower bound for a set of real numbers is less than or equal to each upper bound."

**i.** Give a counter-example to the statement, with a brief explanation of why it is a counter-example. Then state the weakest condition to add to make the statement true.

**ii.** Write a symbolic expression for the modified statement (with your condition from the previous sub-part added in), where each quantifier is over domain $\mathbb{R}$ or $\mathcal{P}(\mathbb{R})$. Use the *unexpanded* predicates $LB$ and $UB$ in your answer. You may use the symbol $\varnothing$ for the empty set directly in your expression, as well as the predicates $=$ and/or $\neq$.

**iii.** Rewrite your expression from the previous sub-part, by expanding the definitions of the predicates $LB$ and $UB$.

**iv.** Write a direct proof of the statement from the previous two sub-parts.
Make sure to properly introduce (in programming terminology: declare) each variable before it's used, and that it is clear which boolean expressions are assumptions and which are deductions.

**(e)** Write a direct proof of the statement:

If $S \subseteq \mathbb{R}$ and $\{1 - 2x : x \in S\}$ is bounded below then $S$ is bounded above.

Make sure to properly introduce each variable before it's used, and that it is clear which boolean expressions are assumptions and which are deductions.