NATURALS, INTEGERS, RATIONALS, REALS
CSC236 — 2024 FALL

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of Natural numbers, aka the Naturals.
Whether zero is a natural number depends on context, but our course always considers zero to be a natural number.
$\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, \ldots\}$ is the set of Integral numbers, aka the Integers.
$\mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z} \wedge q \neq 0\right\}$ is the set of Rational numbers, aka the Rationals.
$\mathbb{R}$ is the set of Real numbers, aka the Reals.
A "*" / "$^+$" / "$^{\geq 0}$" superscript on these sets means the non-zero / (strictly) positive / non-negative elements.

Closure Properties

$\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are "closed" under addition and multiplication:

$$\begin{aligned}\forall x, y \in \mathbb{R}, \quad & (x \in \mathbb{N} \wedge y \in \mathbb{N} \Rightarrow x + y \in \mathbb{N} \wedge x \cdot y \in \mathbb{N}) \\ & \wedge (x \in \mathbb{Z} \wedge y \in \mathbb{Z} \Rightarrow x + y \in \mathbb{Z} \wedge x \cdot y \in \mathbb{Z}) \\ & \wedge (x \in \mathbb{Q} \wedge y \in \mathbb{Q} \Rightarrow x + y \in \mathbb{Q} \wedge x \cdot y \in \mathbb{Q})\end{aligned}$$

$\mathbb{Z}$ and $\mathbb{Q}$ are closed under taking the negative and subtraction:

$$\begin{aligned}\forall x, y \in \mathbb{R}, \quad & (x \in \mathbb{Z} \wedge y \in \mathbb{Z} \Rightarrow -x \in \mathbb{Z} \wedge x - y \in \mathbb{Z}) \\ & \wedge (x \in \mathbb{Q} \wedge y \in \mathbb{Q} \Rightarrow -x \in \mathbb{Q} \wedge x - y \in \mathbb{Q})\end{aligned}$$

$\mathbb{Q}$ is closed under taking the reciprocal and division (when defined):

$$\forall x \in \mathbb{Q}, y \in \mathbb{Q}^*, \, \frac{1}{y} \in \mathbb{Q} \wedge \frac{x}{y} \in \mathbb{Q}$$

$\mathbb{N}$ and $\mathbb{Z}$ are closed under raising to a natural number power, and the rationals are furthermore closed under raising to an integer power (when defined):

$$\begin{aligned}\forall x \in \mathbb{R}, c \in \mathbb{N}, \quad & (x \in \mathbb{N} \Rightarrow x^c \in \mathbb{N}) \\ & \wedge (x \in \mathbb{Z} \Rightarrow x^c \in \mathbb{Z}) \\ & \wedge (x \in \mathbb{Q} \Rightarrow x^c \in \mathbb{Q})\end{aligned}$$

$$\forall x \in \mathbb{R}^*, c \in \mathbb{Z}, \, x^c \in \mathbb{Q}$$

For 0^0 those assume a context that defines 0^0 to be 1.
$\mathbb{Z}$ is closed under taking the absolute value, and furthermore produces a natural number hence $\mathbb{N}$ is also closed under absolute value:

$$\forall x \in \mathbb{Z}, \, |x| \in \mathbb{N}$$

$$\begin{aligned}\forall x, y \in \mathbb{R}, \quad & (x \in \mathbb{N} \wedge y \in \mathbb{N} \Rightarrow \min(x, y) \in \mathbb{N} \wedge \max(x, y) \in \mathbb{N}) \\ & \wedge (x \in \mathbb{Z} \wedge y \in \mathbb{Z} \Rightarrow \min(x, y) \in \mathbb{Z} \wedge \max(x, y) \in \mathbb{Z}) \\ & \wedge (x \in \mathbb{Q} \wedge y \in \mathbb{Q} \Rightarrow \min(x, y) \in \mathbb{Q} \wedge \max(x, y) \in \mathbb{Q})\end{aligned}$$

To use an instance of a closure property one refers to the closed set, e.g., "since $\mathbb{Z}$ is closed under addition / $+$", or to elements of the set, e.g., "since the difference of (any) two integers is an integer".
These properties are all universally quantified implications, so when it's clear that the conclusion of an instance of one of these properties is being used the instance can be referred to by just its hypotheses / pre-conditions. For example, if x and y are clearly integers and it's clear that one wants to conclude that $x - y$ is an integer one often just says "since x and y are integers".