Question 1

→ for condition (a), to perfectly reconstructed the signal, we need to fulfill Nyquist sampling theorem.

if $|X_s(w)| = 0$ for all $|w| > w_s/2$, then $x(t)$ can be exactly reconstructed from sampled signal.
so minimum sample rate f:
$|X_s(w)| = 0$ when $|w| \geq 250\pi \times 10^3$
$\frac{w_s}{2} = 250\pi \times 10^3$, $w_s = 500\pi \times 10^3$ rad/s

→ for condition (b), DTFT of the filter & sample noise signal for $|w| < 250\pi \times 10^3$ is at least 20 times lower than desire signal at $w=0$

$$H(0) = \frac{1}{1 + j0/w_0} = \frac{1}{1} = 1$$

we need to ensure $|H(0)| \times |N(0)| \leq \frac{1}{20}$
$N(0) = 0$ for $w=0$, which is $|w| < 360\pi \times 10^3$ category.
↳ $|H(0)| \times |N(0)| = 1 \times 0 = 0 \leq \frac{1}{20}$

so the minimum sample rate is $500\pi \times 10^3$ rad/s and satisfy (a) & (b) *

Question 2.

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since $x(t)$ is reconstruct by ZOH method, $x(t) \triangleq x_d(n)$, $n\Delta t \leq t < (n+1)\Delta t$
which is obtain by holding each sample of discrete-time signal constant over Δt , $\Delta t = 1/f_s = 1/30\text{kHz}$
given $x_d(n) = \sin(2\pi n f_0/f_s)$ → express $x(t)$ as $x(t) = \sin(2\pi f_0 t)$ for $0 \leq t < \Delta t$, since $n\Delta t = n(1/f_s)$

to obtain the magnitude of fundamental component, apply FT to $x(t)$
$X(w) = \int [x(t) \times e^{-jwt}] dt = \int [\sin(2\pi f_0 t) \times e^{-jwt}] dt$
by Euler's formula, $e^{jx} = \cos x + j\sin x$

$$X(w) = \int [\sin(2\pi f_0 t) \times (\cos wt - j\sin(wt))] dt$$
$$= \int \{[\sin(2\pi f_0 t) \times \cos(wt)] - [\sin(2\pi f_0 t) \times (-j\sin(wt))]\} dt$$

⇒ ↳ equal to zero since it's an odd function, integrat odd function over $0 \leq t < \Delta t$ → 0
↳ this can be evaluate to determine the magnitude of frequency component at $w = 2\pi f_0$

$$\Rightarrow \int [\sin(2\pi f_0 t) \times \sin(wt)] dt = (\Delta t/2) \times \text{sinc}[(w - 2\pi f_0)\Delta t/2]$$

$w = 2\pi f_0$, $X(2\pi f_0) = (\Delta t/2)\text{sinc}[(2\pi f_0 - 2\pi f_0)\Delta t/2] = \Delta t/2 \,(\text{sinc } 0) = \frac{\Delta t}{2}$
↳ fundamental frequency
$w = 2\pi f_0$ fundamental angular frequency.

$|X(2\pi f_0)| = |\Delta t \times \frac{1}{2}| = \frac{1}{2}\Delta t$ *

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the 3 harmonic will be $2f_0$, $3f_0$, $4f_0$

$\frac{5k}{30k} = \frac{1}{6}$

$2f_0 \to w = 4\pi f_0$, $X(4\pi f_0) = (\Delta t/2)(\mathrm{sinc}[(4\pi f_0 - 2\pi f_0)\Delta t/2]$

$= \Delta t/2 \ \mathrm{sinc}(2\pi f_0 \Delta t/2) = \Delta t/2 \ \mathrm{sinc}(\pi f_0 \Delta t)$

$= \frac{\Delta t}{2} \mathrm{sinc}(\frac{1}{6}\pi) = \frac{\Delta t}{2} \times \frac{3}{\pi} = \frac{3}{2\pi}\Delta t$ ☆

$3f_0 \to w = 6\pi f_0$, $X(6\pi f_0) = (\Delta t/2)(\mathrm{sinc}[(6\pi f_0 - 2\pi f_0)\Delta t/2]$

$= \Delta t/2 \ \mathrm{sinc}(4\pi f_0 \Delta t/2) = \Delta t/2 \ \mathrm{sinc}(2\pi f_0 \Delta t)$

$= \frac{\Delta t}{2} \mathrm{sinc}(\frac{1}{3}\pi) = \frac{\Delta t}{2} \times \frac{3\sqrt{3}}{2\pi} = \frac{3\sqrt{3}\,\Delta t}{4\pi}$ ☆

$4f_0 \to w = 8\pi f_0$, $X(8\pi f_0) = (\Delta t/2)(\mathrm{sinc}[(8\pi f_0 - 2\pi f_0)\Delta t/2]$

$= \Delta t/2 \ \mathrm{sinc}(6\pi f_0 \Delta t/2) = \Delta t/2 \ \mathrm{sinc}(3\pi f_0 \Delta t)$

$= \frac{\Delta t}{2} \mathrm{sinc}(\frac{1}{2}\pi) = \frac{\Delta t}{2} \times \frac{2}{\pi} = \frac{\Delta t}{\pi}$ ☆