Q1)

sample mean of variable X

$$m(X) = \frac{1}{N}\sum_{i=1}^{N} x_i$$

sample covariance of 2 variables X & Y

$$cov(X,Y) = \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))(y_i - m(Y))$$

sample variance of X

$$s^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))^2$$

1) $m(a+bX) = a + b\cdot m(X)$

$$m(a+bX) = \frac{1}{N}\sum_{i=1}^{N}(a+bx_i)$$

split into 2 summations

$$= \frac{1}{N}\sum_{i=1}^{N}(a) + \frac{1}{N}\sum_{i=1}^{N}(bx_i)$$

a is a constant

$$= a + \frac{1}{N}\sum_{i=1}^{N}(bx_i)$$

$$= a + b\left[\frac{1}{N}\sum_{i=1}^{N}(x_i)\right]$$

$$\therefore m(a+bX) = a + b\cdot m(X)$$

2) $cov(X, a+bY) = b \times cov(X,Y)$

$$cov(X, a+bY) = \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))\left[\underbrace{(a+bY)}_{y_i} - m\underbrace{(a+bY)}_{Y}\right]$$

let $Z = a+bY$

$$cov(X,Z) = \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))(z_i - m(Z))$$

since $z_i = a+by_i$ and $m(Z) = m(a+bY) = a + b\cdot m(Y)$

$$z_i - m(Z) = (\cancel{a}+by_i) - (\cancel{a}+bm(Y))$$

$$z_i - m(Z) = by_i - bm(Y)$$

$$z_i - m(Z) = b(y_i - m(Y))$$

$$= \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))\cdot b(y_i - m(Y))$$

$$= b \times \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))(y_i - m(Y))$$

$$cov(X,Z) = b \times cov(X,Y)$$

$$\therefore cov(X, a+bY) = b \times cov(X,Y)$$

3) $cov(a+bX, a+bX) = b^2 cov(X,X)$, & in particular that $cov(X,X) = s^2$

$$cov(a+bX, a+bX) = \frac{1}{N}\sum_{i=1}^{N}\left[(a+bx_i) - m(a+bX)\right]^2$$

since $m(a+bX) = a + b\,m(X)$

$$= \frac{1}{N}\sum_{i=1}^{N}\left[(\cancel{a}+bx_i) - (\cancel{a}-bm(X))\right]^2$$

$$= \frac{1}{N}\sum_{i=1}^{N}\left[b(x_i - m(X))\right]^2$$

$cov(x,x) = s^2$

$$= b^2\cdot \underbrace{\frac{1}{N}\sum_{i=1}^{N}(x_i - m(x))^2}$$

$$\therefore cov(a+bX, a+bX) = b^2\cdot cov(X,X) = b^2\cdot s^2$$

4) instead of mean, consider median. consider the transformations that are non-decreasing (if $x \geq x'$, then $g(x) \geq g(x')$), like $2+5\cdot X$ or arcsinh(X). Is a non-decreasing transformation of the median the median of the transformed variable? Explain. Does your answer apply to any quantile. The IQR? The range?

$Median(g(X)) = g(Median(X))$ → yes b/c non-decreasing transformation doesn't change the order of the points, but rather shifts or scales it

$Quantile_q(g(X)) = g(Quantile_q(X))$ → yes b/c doesn't change order of values

$IQR(g(X)) = g(IQR(X))$ → yes b/c will preserve the spread b/n values

$range(g(X)) \neq g(range(X))$ → no b/c max & min will be diff

5) consider a non-decreasing transformation g(). Is it always true that $m(g(X)) = g(m(X))$?

↓ no if g = non-linear transformation

$$m(X) = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$m(g(X)) = \frac{1}{N}\sum_{i=1}^{N} g(x_i)$$

→ for $m(g(X)) = g(m(X))$

$$\frac{1}{N}\sum_{i=1}^{N} g(x_i) = g\left[\frac{1}{N}\sum_{i=1}^{N} x_i\right]$$