Handout on Mixed Strategies

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The main lesson of the last class was the following:

Main Lesson *If a mixed strategy is a best response then each of the pure strategies involved in the mix must itself be a best response. In particular, each must yield the same expected payoff.*

Before explaining why this must be true, let's just try to rewrite this lesson formally, using our new notation:

More Formal statement of the Same Lesson. *If player i's mixed strategy p_i is a best response to the (mixed) strategies of the other players, p_{-i}, then, for each pure strategy s_i such that $p_i(s_i) > 0$, it must be the case that s_i is itself a best response to p_{-i}. In particular, $Eu_i(s_i, p_{-i})$ must be the same for all such strategies.*

Why is this true? Suppose it were *not* true. Then there must be at least one pure strategy s_i that is assigned positive probability by my best-response mix and that yields a lower expected payoff against p_{-i}. If there is more than one, focus on the one that yields the lowest expected payoff. Suppose I drop that (low-yield) pure strategy from my mix, assigning the weight I used to give it to one of the other (higher-yield) strategies in the mix. This must raise my expected payoff (just as dropping the player with the lowest batting average on a team must raise the team average). But then the original mixed strategy cannot have been a best response: it does not do as well as the new mixed strategy. This is a contradiction. ■

So what? An immediate implication of this lesson is that if a mixed strategy forms part of a Nash Equilibrium then each pure strategy in the mix must itself be a best response. Hence all the strategies in the mix must yield the same expected payoff. We will use this fact to find mixed-strategy Nash Equilibria.

Finding Mixed-Strategy Nash Equilibria.

Let's look at some examples and use our lesson to find the mixed-strategy NE.

Example 1 Battle of the Sexes

	a	b
A	$2,1$	$0,0$
B	$0,0$	$1,2$

In this game, we know that there are two pure-strategy NE at (A, a) and (B, b). Let's see if there are any other mixed-strategy NE. Suppose that there was another equilibrium in which the row mixed on both A and B. By our lesson of the day, we know that in this case both A and B must be best responses to whatever the column player is doing. But for them both to be best responses, they must both yield the same expected payoff for the row player. We will use this fact about *row*'s expected payoffs to find what *column* must be playing!

Suppose that column's mixed strategy assigns probability weight q to a and probability weight $(1-q)$ to b. Then,

$$\text{row's expected payoff from } A \text{ against } (q, 1-q) = q\,[2] + (1-q)\,[0] = 2q$$
$$\text{row's expected payoff from } B \text{ against } (q, 1-q) = q\,[0] + (1-q)\,[1] = 1-q$$

But if these expected payoff are to be equal, we must have $2q = 1 - q$ or $q = \frac{1}{3}$.

To summarize so far, if *row* is mixing on both her strategies in a NE then both must yield the same expected payoff, in which case *column* must be mixing with weights $\left(\frac{1}{3}, \frac{2}{3}\right)$.

Notice the trick here: *we used the fact that, in equilibrium,* ***row*** *must be indifferent between the strategies involved in her mix to solve for* ***column****'s equilibrium mixed strategy.*

Now let's reverse the trick for find row's equilibrium mix. If there is an equilibrium in which the column mixes on both a and b, then (by our lesson of the day) we know that both a and b must be best responses to whatever row is doing. But for them both to be best responses, they must both yield column the same expected payoff. We will use this fact about *column*'s expected payoffs to find what *row* must be playing. Suppose that row's mixed strategy assigns probability weight p to A and probability weight $(1-p)$ to B. Then,

$$\begin{array}{l} \text{column's expected payoff from } a \text{ against } (p, 1-p) = p\,[1] + (1-p)\,[0] = p \\ \text{Row's expected payoff from } b \text{ against } (p, 1-p) = p\,[0] + (1-p)\,[2] = 2\,(1-p) \end{array}$$

But if these are to be equal, we have $2\,(1-p) = p$ or $p = \frac{2}{3}$.

To summarize, if *column* is mixing on both her strategies in a NE then both must yield the same expected payoff, in which case *row* must be mixing with weights $\left(\frac{2}{3}, \frac{1}{3}\right)$. We *used the fact that, in equilibrium,* ***column*** *must be indifferent between the strategies involved in her mix to solve for* ***row****'s equilibrium mixed strategy.*

I claim that the mixed-strategy profile $\left[\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right]$ is a NE. To show this I still need to check that neither player has a strictly profitable deviation but this turns out to be easy. We constructed the equilibrium so that, given column's mix, $\left(\frac{1}{3}, \frac{2}{3}\right)$, each of row's pure strategies, A and B yields the same expected payoff. But, in this case, any mix of those pure strategies (including the equilibrium mix itself) will yield the same expected payoff. So all potential deviations yield the same expected payoff: none are *strictly* profitable. The same argument applies to column.

Example 2. Rock, Scissors, Paper.

	r	s	p
R	$0,0$	$1,-1$	$-1,1$
S	$-1,1$	$0,0$	$1,-1$
P	$1,-1$	$-1,1$	$0,0$

In this game, we 'know' that the mixed-strategy NE is $\left[\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right]$, but let's use the method from the last example to 'find' the equilibrium as if we did not know. Once again, suppose that there is an equilibrium in which row is mixing on all of R, S and P. By our lesson of the day, we know that in this case R, S and P must each be a best response to whatever the column player is doing. But for them each to be best response, each must yield the same expected payoff for the row player. We will use this fact about *row*'s expected payoffs to find what *column* must be playing.

Suppose that column's mixed strategy assigns probability weight q_r to R, q_s to S and $(1-q_s-q_r)$ to P. Then,

$$\begin{array}{l} \text{row's expected payoff from } R \text{ against } (q_r, q_s, 1-q_r-q_s) = q_r\,[0] + q_s\,[1] + (1-q_r-q_s)\,[-1] \\ \text{row's expected payoff from } S \text{ against } (q_r, q_s, 1-q_r-q_s) = q_r\,[-1] + q_s\,[0] + (1-q_r-q_s)\,[1] \\ \text{row's expected payoff from } P \text{ against } (q_r, q_s, 1-q_r-q_s) = q_r\,[1] + q_s\,[-1] + (1-q_r-q_s)\,[0]. \end{array}$$

Setting these three expected payoffs equal to one another (and using a little basic algebra) solves to $q_r = q_s = (1 - q_r - q_s) = \frac{1}{3}$.

To summarize, if *row* is mixing on all of her strategies in a NE then each must yield the same expected payoff, in which case *column* must be mixing with weights $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Once again, *we used the fact that, in equilibrium, **row** must be indifferent between the strategies involved in her mix to solve for **column**'s equilibrium mixed strategy.*

We could do the same to find row's equilibrium mix. That is, we could use the fact, *in equilibrium, **column** must be indifferent between the strategies involved in her mix to solve for **row**'s equilibrium mixed strategy.* However, since the argument is symmetric, let's skip it. As in the previous example, checking that neither player has a strictly profitable deviation is easy. We constructed the equilibrium so that, given column's mix, each of row's pure strategies yields the same expected payoff. But, in this case, any mix of those pure strategies (including the equilibrium mix itself) will yield the same expected payoff. So all potential deviations yield the same expected payoff: none are *strictly* profitable. The same argument applies to column. So we have shown that this is an equilibrium.

For nerds only. In this example, it is a little more involved to show that there are no other mixed-strategy equilibria. We have shown that in the only equilibrium in which each player mixes on all of her strategies, each player mixes $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. But, in principle, there could be other mixed-strategy equilibria in which one player only mixes on two of her three strategies. Let me sketch the argument why no such equilibrium exists. Suppose there was such an equilibrium. Without loss of generality, let column be the player who mixes on two of her strategies, and without loss of generality, assume that column's mixes only on r and s; that is, column's mixed strategy assigns probability zero to r. Given this, row's expected return from R is strictly greater than that from S. Thus, row's best response must assign probability zero to S. But, given this, column's expected return from p is strictly greater than that from r. Thus, column's best response must assign probability zero to r. But if column assigns zero probability to p (by assumption) and zero probability to r (as we have just shown), then he must be playing the pure strategy s. But we already know that neither can be playing a pure strategy in any equilibrium of rock, scissors paper. ■

What did we learn here? We have found a general method to find mixed-strategy Nash Equilibria.

Method to find mixed-strategies NE *Suppose we conjecture that there is an equilibrium in which row mixes between several of her strategies. If there is such an equilibrium then each of these strategies must yield the same expected payoff given column's equilibrium strategy. If we write down these payoffs [just as we did in the examples above] we can solve for column's equilibrium mix. In other words, row's indifference among her strategies implies column's equilibrium mix. Now we reverse. Look at the strategies that column is mixing on, write down column's indifference condition, and solve for row's equilibrium mix.*

We are almost done but we still need to check a few (easy) things. First, the equilibrium mix we have found for row must indeed involve those strategies from which we started our conjecture! Second, each of the mixing probabilities we have constructed must indeed be probabilities: they must lie between zero and one! Third, as always, we need to check that neither player has a strictly profitable deviation. But, [as we saw in the examples above] if the mix involved all strategies then this last check is for free!