1 Mixed Strategies

Recall the game of Matching Pennies from the previous lecture.

	Heads	Tails
Heads	$1, -1$	$-1, 1$
Tails	$-1, 1$	$1, -1$

Matching Pennies

It is easy to see that this game does not have a pure Nash Equilibrium (for every pure strategy in this game, one of the players has an incentive to deviate). However, if we allow the players to *randomize* over their choice of actions we can find a Nash equilibrium. Assume that player 1 picks "Head" with probability p and "Tail" with probability $1-p$, and that player 2 picks both "Head" and "Tail" with probability $\frac{1}{2}$. Then, with probability

$$\frac{1}{2}p + \frac{1}{2}(1-p) = \frac{1}{2}$$

player 1 will receive 1. Likewise she will receive -1 with the same probability. This means that if player 2 plays the strategy $(\frac{1}{2}, \frac{1}{2})$ then no matter what strategy player 1 chooses, she will get the same output. Due to the symmetry of the game, we conclude that the randomized strategy $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$ is stable. We call such a state, a mixed strategy Nash Equilibrium. In order to formalize this notion we will require some new notation.

1.1 Notation

1. Let $\Delta(A_i)$ denote the set of probability distributions over A_i. We identify this set with the simplex: $\alpha_i \in \Delta(A_i)$ implies $\sum_{a_i \in A_i} \alpha_i(a_i) = 1$ and $\alpha_i(a_i) \geq 0$.

2. We refer to α_i as a mixed strategy, while the members of A_i are pure strategies (or actions).

3. The *support* of a probability measure μ is defined as $x : \mu(x) > 0$. In the context of $\alpha \in \Delta(A_i)$ the support of α_i is all a_i such that $\alpha_i(a_i) > 0$.

Assume that a player's mixed strategies are independent randomizations—in other words, all of the players use independent random coins when they sample from their mixed strategies. In our notation, we say that the probability of obtaining a specific (pure) action profile $a = (a_j)_{j \in N}$ is

$$\prod_{j \in N} \alpha_j(a_j).$$

The payoff of a mixed strategy corresponds to the expected value of the pure strategy profiles in its support. More precisely, we can say

$$U_i(\alpha) = \sum_{a \in A} \left(\prod_{j \in N} \alpha_j(a_j) \right) u_i(a). \tag{1}$$

The mixed extension of strategic games is defined similarly to strategic games replacing the action set and the utility function.

Definition 1 (Mixed Extension of a Strategic Game) *The* **mixed extension of a strategic game** $\langle N, (A_i), (u_i)_{i \in \mathcal{N}} \rangle$ *is a strategic game* $\langle N, (\Delta A_i), (U_i) \rangle$ *where*

i. (ΔA_i) *is a set of probability distributions on* A_i *is*

ii. $U_i : \prod_{j \in N} \Delta(A_i) \to \mathbb{R}$ *satisfies Eq. (1)*

A mixed strategy Nash equilibrium is simply an equilibrium of the mixed extension of the Nash equilibrium. In many books you will find the following definition:

Definition 2 *A mixed strategy profile* α^* *is a* **mixed strategy Nash Equilibrium** *if for each player* i *and for each* $\alpha_i' \in \Delta(A_i)$*,*

$$U_i(\alpha_i^*, \alpha_{-i}^*) \geq U_i(\alpha_i', \alpha_{-i}^*).$$

Claim 1 *A Nash equilibrium of the original strategic game is also an equilibrium (as a probability mass function focused on the pure actions) of the mixed extension.*

1.2 On the existence of NE

Theorem 1 *Every finite strategic game has a mixed strategy NE.*

*Proof:*We identify $\Delta(A_i)$ with the set of player i's mixed strategies. We check that the conditions for Kakutani's fixed point theorem hold. Each $\Delta(A_i)$ is non-empty, convex, and compact. The expected payoff (Eq. (1)) is linear in probabilities hence quasi-concave (and continuous). □

Example: Unit circle game

Two players pick points a_1 and a_2 on the unit circle. The payoffs for the two players are

$$u_1(a_1, a_2) = d(a_1, a_2)$$

$$u_2(a_1, a_2) = -d(a_1, a_2)$$

where d is the Euclidean distance metric.

Show that there is no pure strategy Nash equilibrium and find the mixed strategy Nash equilibrium. (Hint: If both players pick the same location, player 1 has incentive to deviate. If they pick different locations, player 2 has incentive to deviate).

2 More on the Existence of Nash Equilibria for continuous strategy spaces

In last lecture, we proved the existence of a pure strategy NE for continuous strategy spaces (Debreu, Glicksberg, Fan Theorem) under assumptions of:

- nonempty, convex, compact strategy spaces A_i.
- payoff functions $u_i(a)$ continuous in a.
- $u_i(a_i, a_{-i})$ concave in a_i (quasi-concave is sufficient).

Question: What if we relax the quasi concavity assumption? Think about approximating finite strategy spaces by continuous spaces: the corresponding payoff functions will not be quasi concave. We use mixed strategies to obtain convex-valued best responses:

Theorem 2 (Glicksberg) *Consider a strategic form game $\langle \mathcal{N}, (A_i), (u_i) \rangle$. If the utility functions $u_i(a)$ are continuous in a and if A_i is compact and convex (for all $i \in \mathcal{N}$) then there exists a mixed strategy Nash Equilibrium.*

But continuity of the payoffs is still a strong assumption. For discontinuous payoffs, use Dasgupta and Maskin Theorem (requires a weaker form of continuity).

2.1 A Useful Lemma in Calculating the Mixed Strategy NE

Lemma 1 *α^* is a Nash Equilibrium if and only if, for each player $i \in N$, the following two conditions hold:*

i. The expected payoff given α^*_{-i} to every $a_i \in supp(\alpha^*_i)$ is the same.

ii. The expected payoff given α^*_{-i} to the actions a_i which are not in the support of α^*_i must be less than or equal to the expected payoff described in (i.).

Remark: We note that the lemma also extends to the continuous case.

Intuitively, the lemma means that for a player i, every action in the support of a Nash equilibrium is a best response to α^*_{-i}. This lemma follows from the fact that if the strategies in the support have different payoffs, then it would be better to just take the pure strategy with the highest expected payoff. This would contradict the assumption that α^* is a Nash equilibrium. Using the same argument, it follows that the pure strategies which are not in the support must have lower (or equal) expected payoffs. More formally,

Proof: Let α^* be a mixed strategy Nash equilibrium, and let $E^*_i = U_i(\alpha^*_i, \alpha^*_{-i})$ denote the expected utility for player i. It follows that $E^*_i \geq U_i(a_i, \alpha^*_{-i})$ for all $a_i \in A_i$ otherwise i can get a higher payoff by switching to another action (which already proves the second part of the lemma). We proceed to show the first part of the lemma, namely that $E^*_i = U_i(a_i, \alpha^*_{-i})$ for all $a_i \in \text{supp}(\alpha^*_i)$. Assume, for contradiction that this is not the case, i.e., that there exist an action $a'_i \in \text{supp}(\alpha^*_i)$ such that $U_i(a'_i, \alpha^*_{-i}) > E^*_i$. This means that

$$E^*_i = \sum_{a_i \in A_i} \alpha^*_i(a_i) u_i(a_i, \alpha^*_{-i}) > E^*_i,$$

since $u_i(a_i, \alpha^*_{-i}) \geq E^*_i$ for all $a_i \in A_i$, except for a'_i where the the inequality is strict. This is a contradiction. $\square$

Application of the Lemma Let us return to the Battle of the Sexes Game.

	BALLET	SOCCER
BALLET	$2, 1$	$0, 0$
SOCCER	$0, 0$	$1, 2$

Battle of the Sexes

Recall that this game has 2 pure Nash Equilibria. We show that it has one *unique* mixed strategy Nash Equilibrium. First, note that by using the Lemma (and inspecting the game tableau) it is easy to see that there are no Nash Equilibria where only one of the players randomizes over its actions. Now, assume instead that player 1 chooses the action "Ballet" with probability p and "Soccer" with probability $1 - p$, and that player 2 picks "Ballet" with probability q and "Soccer" with probability $1 - q$. Using the Lemma on player 1's actions, we obtain the following equation:

$$2q + 0 * (1 - q) = 0 * q + 1 * (1 - q)$$

Next, applying the lemma on player 2's actions, we obtain:

$$1 * p + 0 * (1 - p) = 0 * p + 2 * (1 - p)$$

We conclude that the only possible mixed strategy Nash Equilibrium is when $q = \frac{1}{3}$ and $p = \frac{2}{3}$.

We describe now the best response graph. For the row player, the best response is to choose the action Ballet if player 2 chooses "Ballet" with probability more than 1/3 and "Soccer" if the other player chooses "Ballet" with probability less than 1/3. Similarly, we can compute the best response of player 2. The Nash equilibria point are the intersections between the best response of the two players.

3 Interpretation of mixed Nash equilibrium

Issues:

1. Is randomness natural way to choose the outcome or does it serve other purposes:
   (a) Poker - hide information.
   (b) Tax auditing - a threat. But then the game model is different - first the government chooses probability of audit, and then each player optimizes his/her risk.
   (c) Promotions - improves customer's payoff.
2. Randomness implies indifference. This is strange because we expect decision makers to have preference in such problems.
3. Randomness can be viewed as long-term steady state behavior. This interpretation is problematic since as we will see repeated games have their own special rules. An alternative is to look at games where players are constrained to stick to the original plan.

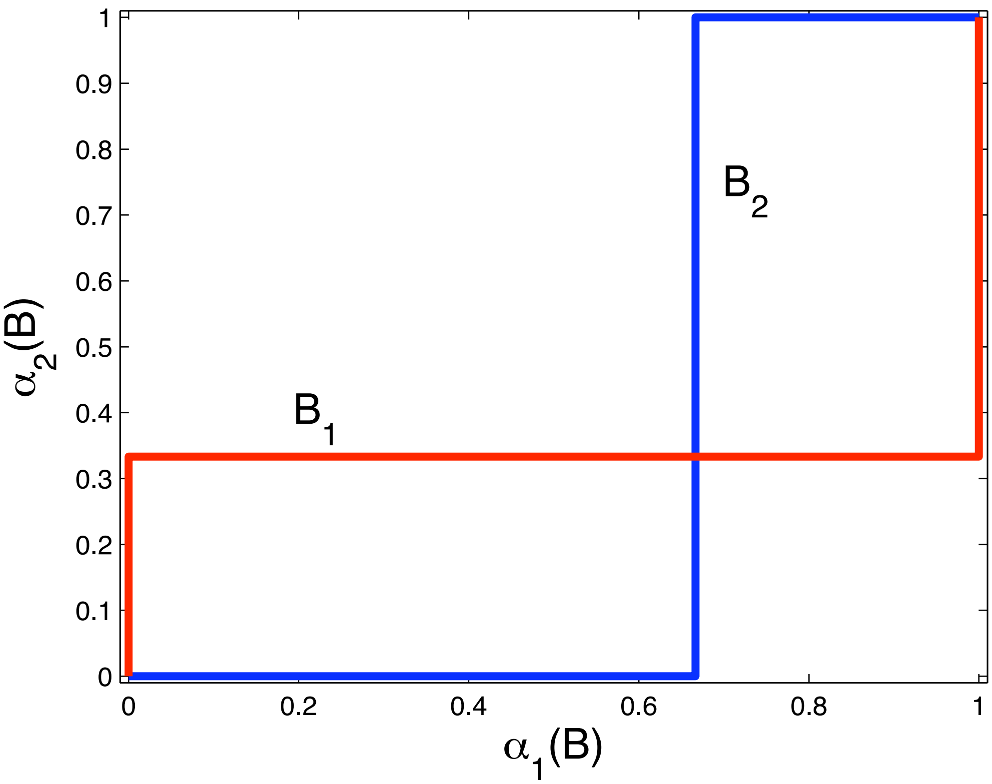

Figure 1: Best response graph for the "Battle of the Sexes" game.

4. Mixed strategies are pure strategies in an extended game. The player's choice depends on external and inconsequential event ("mood" in the BoS).

5. Mixed strategies as pure strategies in a perturbed game. The game is repeated, but the payoffs are (slightly) perturbed. In such a game each player knows his own utility but has a posterior over the utilities of other players (this is a Bayesian game). Harsanyi showed that as we take the perturbation to be smaller, the mixed Nash equilibrium becomes a limit of a sequence of pure Nash equilibria of the Bayesian games. This is known as the purification theorem.

4 Zero-sum games

Zero-sum games have a particularly simple interpretation for their Nash equilibria. Consider the space of mixed actions of player 2, $\Delta(A_2)$, The best response of player 1 against any mixed action $\alpha_2 \in \Delta(A_2)$ is

$$r^*(\alpha_2) = \max_{\alpha_1 \in \Delta(A_1)} \sum_{a_2 \in A_2} \sum_{a_1 \in A_1} u_1(a_1, a_2)\alpha_2(a_2)\alpha_1(a_1) = \max_{a_1 \in A_1} \sum_{a_2 \in A_2} u_1(a_1, a_2)\alpha_2(a_2).$$

This is a piecewise linear convex function as the maximum of linear functions. The utility of player 1 at a Nash equilibrium is the minimum of $r^*(\alpha_2)$. This is often referred to as a *value* of the game.

Conclusion: The number of NE is a 0-sum game is either one or infinitely many.

Remark: another consequence is that the set of NE for zero-sum games is convex. This property does not hold for matrix games in general.

5 Strict Dominance by a Mixed Strategy

Consider the following game.

	LEFT	RIGHT
UP	$2,0$	$-1,0$
MIDDLE	$0,0$	$0,0$
DOWN	$-1,0$	$2,0$

Game 4: Strict Dominance by Mixed Strategies

Note that Player 1 has no pure strategies that strictly dominate MIDDLE. However, the expected outcome of the strategy $(\frac{1}{2}, 0, \frac{1}{2})$ is strictly higher than the outcome of playing MIDDLE. We thus extend the notion of strict domination also to mixed strategies, and say that MIDDLE is strictly dominated by the strategy $(\frac{1}{2}, 0, \frac{1}{2})$. More formally,

Definition 3 (Strict Domination by Mixed Strategies) *An action a_i is* **strictly dominated** *if there exists a mixed strategy $\alpha_i' \in \Delta(A_i)$ such that $U_i(\alpha_i', \alpha_{-i}) > U_i(a_i, \alpha_{-i})$, for all $\alpha_{-i} \in \prod_{j \neq i} \Delta(A_i)$.*

6 Correlated Equilibrium

In Nash equilibrium, players choose strategies (or randomize over strategies) independently.

Example 3 (Game of Chicken - Traffic Intersection Game) *Consider a game where two cars arrive at an intersection simultaneously. Each car has the option to stop or to go, with payoff as follows.*

	Stop	*Go*
Stop	*4,4*	*1,5*
Go	*5,1*	*0,0*

There are two pure strategy Nash equilibria: (Go, Stop) and (Stop, Go). To find the mixed strategy Nash equilibria, assume Player 1 stops with probability p and Player 2 stops with probability q. Then we have

$$4q + (1-q) = 5q \Rightarrow q = \frac{1}{2}$$
$$4p + (1-p) = 5p \Rightarrow p = \frac{1}{2}$$

There is a unique mixed strategy equilibrium with expected payoff (2.5,2.5).

If there is a publicly observable random variable (such as a fair coin or a traffic light) such that with probability 1/2 (H) $\rightarrow$ Player 1 goes and Player 2 stops with probability 1/2 (T) $\rightarrow$ Player 1 stops and Player 2 goes. Then the expected payoff increases to (3,3).

The above example suggests the players may get higher expected payoff if, for example, the game can be transformed to a game with a signaling device. Then, the strategies in the game would be functions from signals to pure strategies. Now, we want to ask: Would any player have an incentive to deviate? In the example, if the signal is H, Player 1 goes and Player 2 has no incentive to deviate. This is similar when the signal is T.

We have moved from the question 'Can I do better?' to 'Observe correlated signals'. The signals is represented as an arbitrary probability distribution on strategy profiles.

Definition 4 (First Attempt) Correlated equilibrium *is a probability distribution function $p(\cdot)$ over the pure strategies $A = \prod_{i=1}^{n} A_i$ such that for all $i \in N$ and every $a_i, a_i' \in A_i$,*

$$\sum_{a_{-i}} p(a_i, a_{-i}) u_i(a_i, a_{-i}) \geq \sum_{a_{-i}} p(a_i, a_{-i}) u_i(a_i', a_{-i}),$$

In other words,

$$\sum_{a_{-i}} p(a_i, a_{-i})[u_i(a_i, a_{-i}) - u_i(a_i', a_{-i})] \geq 0$$

for all $a_i, a_i' \in A_i$.

Intuition: A "universal device" chooses an action profile $a \in A$ at random according to distribution $p(\cdot)$. Each player then holds only his own coordinate $a_i \in A_i$, and the original game is played. A correlated equilibrium results if each player playing the recommended strategy is a best result.

7 Correlated Equilibrium (again)

The previous definition was somewhat unnatural, as the correlation mechanism was very crude. We now consider a definition that focuses on the signalling mechanism.

Definition 5 (Second Attempt) *Correlated equilibrium of a strategic game $\langle \mathcal{N}, (A_i), (u_i) \rangle$ consists of:*

1. *A set of states ("signals") Ω.*
2. *A probability measure π over Ω (probability over signals)*
3. *For every player $i \in \mathcal{N}$ a partition $\mathcal{P}_i$ of Ω (player i's information partition).*
4. *For each player $i \in \mathcal{N}$ a function $\sigma_i : \Omega \to A_i$ with $\sigma_i(w) = \sigma_i(w')$ for w, w' in the same partition $P_i \in \mathcal{P}_i$ (player i's strategy).*

Such that for every $i \in \mathcal{N}$ and every function $\tau_i : \Omega \to A_i$ with $\tau_i(w) = \tau_i(w')$ whenever $w, w' \in P_i$ are in the same partition $P_i \in \mathcal{P}_i$ we have:

$$\sum_{w \in \Omega} \pi(w) u_i(\sigma_i(w), \sigma_{-i}(w)) \geq \sum_{w \in \Omega} \pi(w) u_i(\tau_i(w), \sigma_{-i}(w)). \tag{2}$$

Remarks:

1. The states and signalling structure are part of the equilibrium definition and are not exogenous.
2. The first attempt definition is equivalent to this definition (exercise).

A game of chicken

Consider the following game:

	STOP	GO
STOP	$4, 4$	$1, 5$
GO	$5, 1$	$0, 0$

Chicken

The following are correlated equilibria corresponding to Nash equilibria:

	STOP	GO
STOP	0	1
GO	0	0

	STOP	GO
STOP	0	0
GO	1	0

	STOP	GO
STOP	1/4	1/4
GO	1/4	1/4

And the following two are correlated equilibria:

	STOP	GO
STOP	0	1/2
GO	1/2	0

	STOP	GO
STOP	1/3	1/3
GO	1/3	0

The first one representing a "traffic signal" and the second one being the correlated equilibrium that maximizes the sum of rewards.

Proposition 1 *For every mixed Nash equilibrium α of a finite strategic game $\langle \mathcal{N}, (A_i), (u_i) \rangle$ there is a correlated equilibrium $\langle \Omega, \pi, (\mathcal{P}_i), (\sigma_i) \rangle$ in which for each $i \in \mathcal{N}$ the distribution on A_i induced by σ_i is α_i.*

*Proof:*We need to specify the parameters of the correlated equilibrium. We let $\Omega = A$, and π be defined by $\pi(a) = \prod_j \alpha_i(a_j)$. The partition of Ω to information sets is such that for every player i and action $b_i \in A_i$ we let $P_i(b_i) = \{a : a_i = b_i\}$. The strategy of player i is to play $\sigma_i(a) = a_i$. Then $\langle \Omega, \pi, (\mathcal{P}_i), (\sigma_i) \rangle$ is a correlated equilibrium since Eq. (2) is satisfied for every τ_i (this is exactly the condition for a Nash equilibrium). $\square$