Appendix I

Set theory

In this appendix, we present some of the basic ideas and concepts of set theory that are essential for a modern introduction to probability and statistics. The origin of set theory is credited to Georg Cantor, when he proved the uncountability of the real line in 1873. A ***set*** is defined as a collection of well-defined distinct objects. These objects of a set are called ***elements*** or ***members***. The elements of a set can be anything: the alphabet, numbers, people, other sets, and so forth. Sets are conventionally denoted with capital letters, A, B, C, and so on. A ***universal set***, denoted by S, is the collection of all possible elements under consideration. If a is an element of a set A, we write $a \in A$. If a is not an element of A, we write $a \notin A$.

A set is described either by listing its elements or by stating the properties that characterize the elements of the set. For example, to specify the set A of all positive integers less than 12, we may write

$$A = \begin{cases} \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} \\ \{\text{all positive integers less than } 12\} \\ \{x: x < 12,\ a \text{ a positive integer}\}. \end{cases}$$

Sets are classified as finite or infinite. A set is ***finite*** if it contains exactly n objects, where n is a nonnegative integer. A set is ***infinite*** if it is not finite. For example, if A is a set containing all positive integers less than or equal to 50, then A is a finite set. If B is a set containing all the positive integers, it is an infinite set.

Describing a set by stating its properties is the practical way to represent a set with a large or infinite number of elements.

A set B is a ***subset*** of a set A if every element of B is also an element of A. We denote this by writing $B \subseteq A$, which is read "A contains B" or "B is contained in A." For example, if A is the set of real numbers and

$$B = \{x: x \leq 5,\ x \text{ a positive integer}\},$$

it is clear that B is a subset of A. Also, every subset is a subset of itself. Two sets A and B are ***equal***, $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$. Thus, two sets A and B are said to be equal if they have the same members. A set B is a ***proper subset*** of a set A if every element of B is an element of A and A contains at least one element that is not an element of B. We denote this relationship by $B \subset A$. In the previous example, we have $B \subset A$. The set, which contains no elements, is called the ***empty set*** (or ***null set***) and is denoted by ϕ. The null set ϕ is a subset of every set.

A ***Venn diagram*** is used for visual representation of sets. In the Venn diagram, the universal set, S, is represented by a rectangle. The subsets are represented by circles inside this rectangle (Fig. AI.1).

AI.1 Set operations

Union, $\cup$: The union of two sets A and B is the *set* of all elements that belong to A or B (or both; elements that belong to both sets are included only once) and is denoted by $A \cup B$ (Fig. AI.2).

$$A \cup B = \{x: x \in A \text{ or } x \in B\}$$

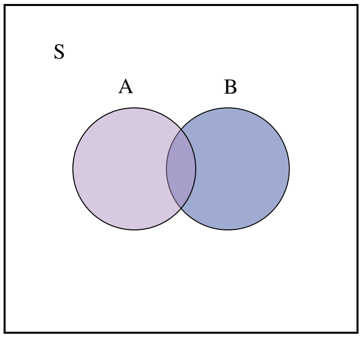

FIGURE AI.1 A Venn diagram.

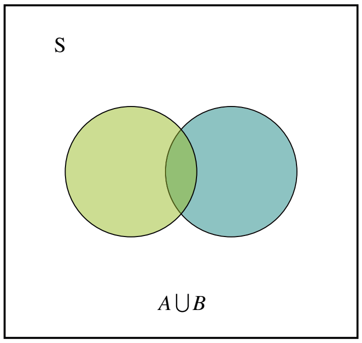

FIGURE AI.2 Union of two sets.

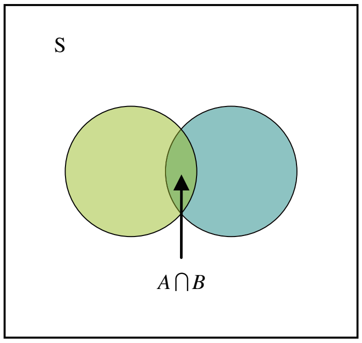

FIGURE AI.3 Intersection of two sets.

Intersection, $\cap$: The intersection of two sets A and B is the set of all elements that belong to both A and B and is denoted by $A \cap B$ (Fig. AI.3).

$$A \cap B = \{x \in S:\ x \in A \text{ and } x \in B\}$$

If $A \cap B = \varnothing$, then the sets A and B are said to be ***disjoint*** or ***mutually exclusive*** sets.

Complement: The complement of a set A is the set of all elements that belong to S but not to A (Fig. AI.4).

$$A^c = \{x: x \in S; x \notin A\}$$

The ***difference*** of any two sets, A and B, denoted by $A \setminus B$, is equal to $A \cap B^c$. Thus, $A^c = S \setminus A$. It should be noted that $(A^c)^c = A$. The ***symmetric difference*** between any two sets, A and B, denoted by $A \Delta B$, is the set of elements in A or B, but not both, that is, $(A \setminus B) \cup (B \setminus A)$.

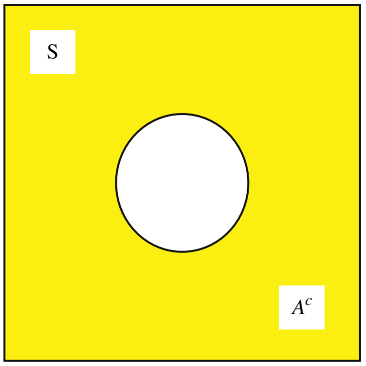

FIGURE AI.4 Complement of a set.

Properties of sets

If A, B, and C are the subsets of the universal set S, then they satisfy the following properties.

Commutative law

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Associative law

$$A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Idempotent law

$$A \cup A = A, \quad A \cap A = A$$

Identity law

$$A \cup S = S, \quad A \cap S = A;$$
$$A \cup \varnothing = A, \quad A \cap \varnothing = \varnothing$$

Complement law

$$A \cup A^c = S, \quad A \cap A^c = \varnothing$$

De Morgan's laws

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

The two sets A and B are said to be in ***one-to-one correspondence*** (denoted by 1:1) if each element $a \in A$ is paired with one and only one element $b \in B$ in such a manner that each element of B is paired with exactly one element of A. For example, if $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{1, 2, 3, 4\}$, then A and B are in a 1:1 correspondence.

A set whose elements can be put into a one-to-one correspondence with the set of all positive integers is referred to as being a ***countably infinite*** set. Also, a set is said to be ***countable***, ***denumerable***, or ***enumerable*** if it is finite or countably infinite. The product or Cartesian product of sets A and B is denoted by $A \times B$ and consists of all ordered pairs (a, b), where $a \in A$ and $b \in B$, that is,

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

For example, if $A = \{a_1, a_2, a_3\}$ and $B = \{1, 2\}$, then

$$A \times B = \{(a_1, 1), (a_1, 2), (a_2, 1), (a_2, 2), (a_3, 1), (a_3, 2)\}.$$

The notion of a Cartesian product can be extended to any finite number of sets; that is, $A_1 \times A_2 \times \cdots \times A_n$ is the set of all ordered n-tuples, $(a_1, a_2, ..., a_n)$, where

$$a_1 \in A_1, a_2 \in A_2, ..., a_n \in A_n$$