BIOSTAT 701

Introduction to Statistical Theory and Methods I

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Moments

- Let X be a random variable and $c \in \mathbb{R}$ a scalar. Then, the k^{th} moment of X is $E(X^k)$ and the k^{th} moment of X (about c) is $E((X - c)^k)$.

Moments

- The first four moments of a distribution/RV are commonly used (though we only talked about the first two)

  1. The first moment of X is the mean $\mu = E(X)$, describing the center or average

  2. The second moment of X about μ is the variance $\sigma^2 = Var(X) = E((X - \mu)^2)$, describing the spread of a distribution

Moments

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3. The third standardized moment is skewness $E((\frac{X-\mu}{\sigma})^3)$, describing the asymmetry of a distribution about its peak. If skewness is positive, then the mean is larger than the median and there are lots of extremely high values. If skewness is negative, then the median is larger than the mean and there are lots of extremely low values.

Q: What is the skewness value for a normal distribution?

Moments

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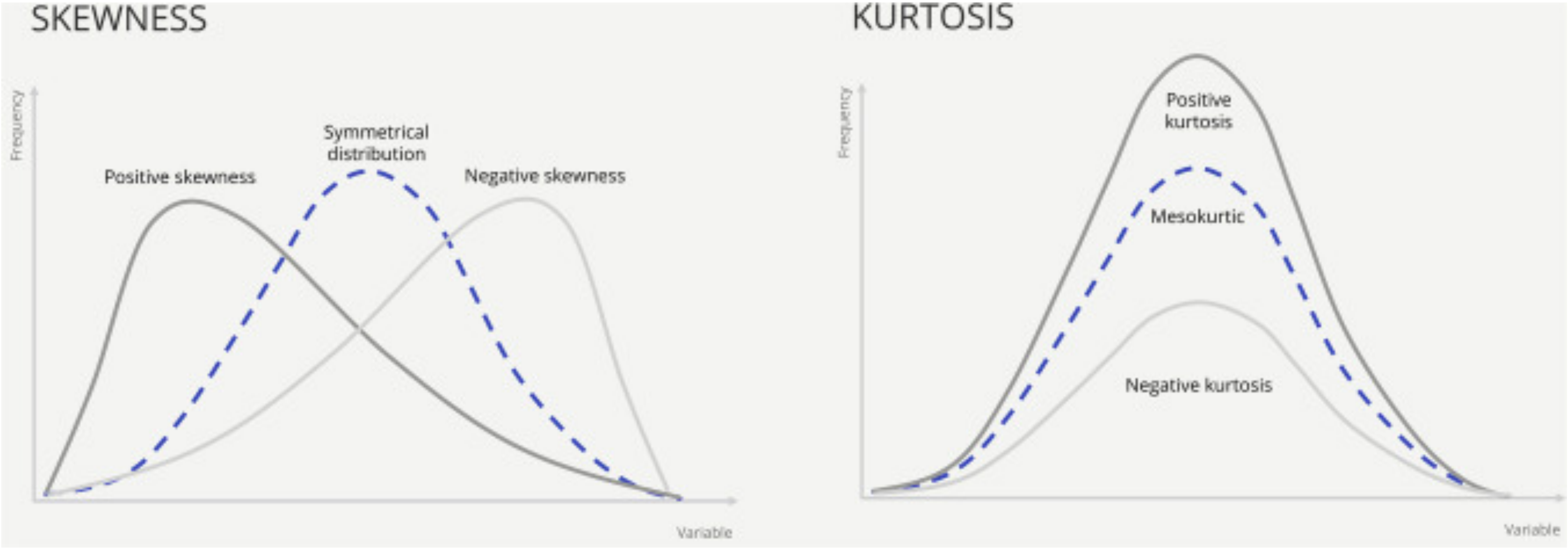
  4. The fourth standardized moment is kurtosis $E((\frac{X-\mu}{\sigma})^4)$, measuring how "tailedness" a distribution is. It is a measure of whether the data are heavy-tailed or light-tailed relative to a normal distribution. If the kurtosis is positive, then the distribution is thin and pointy, and if the kurtosis is negative, the distribution is flat and wide.

Skewness and Kurtosis



Moment Generating Functions (MGFs)

- The MGF of X: $M_X(t) = E(e^{tX})$
  - If X is discrete, $M_X(t) = \sum_x e^{tx} P_X(x)$
  - If X is continuous, $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx$
- We say that the MGF of X exists, if there is a $\epsilon > 0$ such that the MGF is finite for all $t \in (-\epsilon, \epsilon)$, since it is possible that the sum or integral diverges.

Example

- Find the MGF of the following random variables:

1. X is a discrete random variable with PMF: $p_X(k) = 1/3$ if $k = 1$ and $p_X(k) = 2/3$ if $k = 2$

2. Y is a Unif(0,1) continuous random variable.

Properties of MGFs

- Let X, Y be independent random variables, and $a, b \in \mathbb{R}$ be scalars.

1. We can compute the MGF of $aX + b$ if we know the MGF of X (Computing MGF of linear transformations)
   $M_{aX+b}(t) = E(e^{t(aX+b)}) = e^{tb}E(e^{(at)X}) = e^{tb}M_X(at)$

2. We can compute the MGF of the sum of independent RVs given the individual MGFs (Computing MGF of sums)
   $M_{X+Y}(t) = E(e^{t(X+Y)}) = E(e^{tX}e^{tY}) = E(e^{tX})E(e^{tY}) = M_X(t)M_Y(t)$

Properties of MGFs

- Let X, Y be independent random variables, and $a, b \in \mathbb{R}$ be scalars.

    3. Generating moments with MGFs: they can be used to compute $E(X), E(X^2), E(X^3)$ and so on. Let's take the derivate of an MGF w.r.t. t:
    $$M_X'(t) = \frac{d}{dt}E(e^{tX}) = \frac{d}{dt}\sum_x e^{tx}p_X(x) = \sum_x \frac{d}{dt}(e^{tx}p_X(x)) = \sum_x xe^{tx}p_X(x)$$
    if evaluate the derivative at $t = 0$, we get $E(X)$ since $e^0 = 1$.

Properties of MGFs

- Now consider the second derivative:
$$M''_X(t) = \frac{d}{dt}M'_X(t) = \frac{d}{dt}\sum_x xe^{tx}p_X(x) = \sum_x \frac{d}{dt}xe^{tx}p_X(x) = \sum_x x^2e^{tx}p_X(x)$$
If evaluate the second derivative at $t = 0$, we get $E(X^2)$.

- Following the pattern, if we take the n^{th} derivative of $M_X(t)$, then we will generate the n^{th} moment $E(X^n)$.

Uniqueness of MGFs

- The followings are equivalent:

  A. X and Y have the same distribution.

  B. $f_X(z) = f_Y(z)$ for all $z \in \mathbb{R}$.

  C. $F_X(z) = F_Y(z)$ for all $z \in \mathbb{R}$.

  D. There is an $\epsilon > 0$ such that $M_X(t) = M_Y(t)$ for all $t \in (-\epsilon, \epsilon)$ (they match on a small interval around $t = 0$).

That is M_X uniquely identifies a distribution, just like PDFs or CDFs do.

Example

- Suppose $X \sim Poi(\lambda)$, which means that X has range $\{0,1,2,...\}$ and PMF $P_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}$. (1) Compute $M_X(t)$. (2) Compute $E(X)$ using its MGF.

- Hint: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ (Taylor series)

Example

- If $Y \sim Poi(\gamma)$ and $Z \sim Poi(\mu)$, and Y and Z are independent, show that $Y + Z \sim Poi(\lambda + \mu)$.