Discrete Probability Distribution

PART I

Learning Outcomes

Learning Outcomes

At the end of this lesson, you will be able to:

1. Identify characteristics of a probability distribution.
2. Calculate the mean, variance and expected value of a discrete probability distribution.
3. Identify characteristics of a Binomial and Poisson distribution through the assumptions for each distribution.
4. Calculate probabilities, mean and standard deviation of the Binomial and Poisson distribution using relevant formulas.
5. Solve real-life business problems by applying concepts of discrete, Binomial and Poisson distribution.

Introduction to Discrete Probability Distribution

- Your company's network keeps getting disruptions on a particular day. Based on past incidents, you have collected data on number of network disruptions and possible outcomes:

Disruptions/day	Probability
0	0.35
1	0.25
	……

- Your boss wants to know what is the average network disruptions per day. How would you find out?

- This case study is an example of a discrete probability distribution.
- In this topic, you will apply concepts of discrete probability distribution to solve real life problems.

What is a Discrete Probability Distribution?

- A Discrete Random Variable is one that takes on countable values.
- A **Discrete Probability Distribution** is a table that shows each value of random variable and its probability of occurrence.
- Characteristics of a Discrete Probability Distribution
  - ✓ The probability of a particular outcome is between 0 and 1 inclusive.
  - ✓ The outcomes are mutually exclusive.
  - ✓ Sum of the probabilities of the outcomes is equal to 1.

What is a Probability Distribution?

Example: The Probability Distribution of x,number of network interruptions per day is shown below:

x=k	0	1	2	3	4	5
P(x=k)	0.35	0.25	0.20	0.10	0.05	0.05

where

- x= number of network interruptions per day
- P(x=k) is the probability of outcome

Notice that:

- Probability of a particular outcome is between 0 and 1 inclusive.
- Sum of the probabilities of the outcomes is equal to 1.

What is a Probability Distribution?

- Example 1: Explain whether the following is a discrete probability distribution function.

X=k	1	2	3	4
P(X=k)	0.09	0.36	0.49	0.05

$0 \leq P(X=k) \leq 1$

$$\sum_{k=1}^{4} P(X=k) = 0.09 + 0.36 + 0.49 + 0.05 = 0.99 \neq 1$$

Since sum of probability is not 1, it is NOT a discrete probability distribution function.

Mean and Variance of a Probability Distribution

Mean, μ

- A typical value used to represent the central location of the data.
- Also referred to as the expected value

Mean of a Probability Distribution, $\mu = \Sigma[k.P(X = k)]$

Variance, σ^2

- Amount of spread (or variation)

Variance of a Probability Distribution, $\sigma^2 = [\Sigma k^2.P(X = k)] - \mu^2$

Standard deviation

- Positive square root of the variance = $\sqrt{\sigma^2} = \sigma$

Mean and Variance of a Probability Distribution

Example 2: Find the mean, variance and standard deviation of the random variable in the following probability distribution:

$X=k$	1	2	3	4	5
$P(X=k)$	0.16	0.22	0.28	0.20	0.14

Mean, $\mu=\sum kP(X=k)=1(0.16)+2(0.22)+3(0.28)+4(0.20)+5(0.14)=$ 2.94

$\sum k^2P(X=k)=1^2(0.16)+2^2(0.22)+3^2(0.28)+4^2(0.20)+5^2(0.14)=$ 10.26

Variance, $\sigma^2=\sum k^2P(X=k)-\mu^2=10.26-2.94^2=1.6164$

Standard deviation, σ = $\sqrt{1.6164}=1.27$

Types of Discrete Probability Distribution

- There are a variety of discrete probability distributions that you can use to model different types of data. The correct discrete distribution depends on the properties of your data. For example, use

  - ✓ Binomial distribution to model binary data, such as coin tosses.
  - ✓ Poisson distribution to model count data, such as the count of library book checkouts per hour.

Binomial Distribution

PART II

Introduction to Binomial Distribution

- You work for an online store selling shoes.
- Recent statistics suggest that 15% of those who visit your company website will make a purchase.
- A sample of 16 ‘hits’ were selected, and it was found that 4 purchases were made.
- What is the probability of exactly 4 purchases are made?

- The above case study is an example of Binomial Distribution.
- One of its characteristics is that there are only 2 outcomes :

  Probability (purchase) = 0.15

  Probability (No purchase made) = 1 - 0.15 = 0.85
- In this topic, you will apply concepts of Binomial Distribution to solve real life problems.

Binomial Distribution

- Binomial distribution is a common type of discrete probability distribution.
- What type of random experiment qualifies to be a binomial experiment?

A binomial experiment whose intent is to know 'the number of trials which is successful' has to satisfy the following conditions:

- ✓ A trial is repeated for a fixed number of times in the experiment.
- ✓ Each trial has exactly 2 outcomes: success or failure.
- ✓ The probability of success (or failure) is the same for each trial.
- ✓ The trials are independent.

Binomial Distribution

Let *X* be the random variable ‘the number of successes in n trials’ in a binomial experiment $X \sim Bin\ (n, p)$

The probability of exactly *x* successes in *n* trials is given by:

$$P(X = x) = nCx\, p^x\, (1 - p)^{n-x} \quad x = 0,1,2\ \ldots n$$

n : number of times a trial is repeated
p : probability of success in a single trial
x : the random variable that represents the number of successes in *n* trials.

Binomial Distribution

Example 3: Recent statistics suggest that 15% of those who visit your company website will make a purchase. A sample of 16 'hits' were selected, and it was found that 4 purchases were made.

What is the probability of exactly 4 purchases are made?

X ~ B(n, p)
X ~ B(16, 0.15)

$\mathrm{n} = 16, \ \ p = 0.15, \ x = 4$

$P(x) = {}_{n}C_{x}\, p^{x}(1-p)^{n-x}$

$P(4) = {}_{16}C_{4}\, 0.15^{4}(1-0.15)^{16-4}$ = 0.1311

P (Exactly 4 purchases made) = 0.1311

n : number of times a trial is repeated

p : probability of success in a single trial

x : the random variable that represents the number of successes in n trials.

Binomial Distribution

Example 4: Microfracture knee surgery has a 75 % chance of success on patients with degenerative knees. The surgery is performed on three patients. Find the probability of the surgery being successful on exactly two patients.

X ~ B(n, p)

X ~ B(3, 0.75)

$\mathrm{n} = 3, \;\; p = 0.75, \; x = 2$

$P(x) = {}_nC_x \, p^x (1-p)^{n-x}$

$P(2) = {}_3C_2 \, 0.75^2 (1-0.75)^{3-2} = 0.422$

P (Successful on exactly two patients) = 0.422

n : number of times a trial is repeated

p : probability of success in a single trial

x : the random variable that represents the number of successes in n trials.

Mean and Variance of a Binomial Distribution

For Binomial Distribution,

Expectation or population mean, μ is given by

$$\mu = \text{np}$$

Population variance, σ^2 is given by

$$\sigma^2 = \text{np(1-p)}$$

n : number of times a trial is repeated
p : probability of success in a single trial
q : probability of failure in a single trial

Mean and Variance of a Binomial Distribution

Example 5: 5% of workers at construction sites are known to suffer from hearing impaired problem due to the unhealthy noise level. If we randomly select 28 workers from construction sites, find

a) probability that exactly 4 of them suffer from hearing impaired problem

b) mean and standard deviation of the number of workers suffering from hearing impaired problem.

X ~ B(n, p)
X ~ B(28, 0.05)

a) $P(k) = {}_{n}C_{k}\, p^{k}(1-p)^{n-k}$

$P(x=4) = {}_{28}C_{4} 0.05^{4}(1-0.05)^{28-4}$

$= (20{,}475)\ 0.05^{4}(0.95)^{24} = 0.0374$

P(Exactly 4 has hearing impaired problem) = 0.0374

Mean and Variance of a Binomial Distribution

b) $\mu = np = (28)(0.05) = 1.4$

$\sigma^2 = np(1 - p) = (1.4)(0.95) = 1.33$

$\sigma = 1.1532$

Poisson Distribution

PART III

Introduction to Poisson Distribution

- You are working in the Traffic Police Department.
- Based on known data, you know the average number of accidents per month at traffic junction at Sin Min road is 3.
- What is the probability in any given month that 4 accidents will happen at that junction?

- The above case study is an example of Poisson Distribution.
- A Poisson distribution is a statistical distribution showing the likely number of times that an event will occur within a <u>specified period of time</u>.
- In this topic, you will apply concepts of Poisson Distribution to solve real life problems.

Poisson Distribution

A Poisson experiment whose intent is to know ‘the number of occurrences in an interval’ has to satisfy the following conditions:

- ✓ The experiment counts the number of times an event occurs in a given interval.
- ✓ The interval can be an interval of time, area, or volume.
- ✓ The probability of the event occurring is the same for each interval.
- ✓ The number of occurrences in one interval is independent of the number of occurrences in other intervals.

Poisson Distribution

Let X be the random variable 'the number of occurrences in an interval".

$$X \sim Po(\mu)$$

The probability of exactly x occurrences in an interval is given by:

$$P(X = k) = e^{-\mu} \ \frac{\mu^k}{k!} \quad \mathrm{k} = 0,1,2....$$

Where X = number of events occurring per unit interval

μ = mean rate of occurrences per unit interval

e = Euler's number, *e* approx. equal to **2.718**

Poisson Distribution

Example 6: Mean number of accidents per month at traffic junction X is 3. What is the probability in any given month

a) 4 accidents will happen at the junction

b) more than 2 accidents will happen at the junction

$$P(x) = \frac{\mu^{x} e^{-\mu}}{x!}$$

where

X = number of events occurring in an interval

μ = mean rate

e = constant equal to 2.718

a) $P(4) = \frac{3^{4} e^{-3}}{4!} = 0.1680$

P(4 accidents will happen at the junction) = 0.1680

Poisson Distribution

b) $P(\mathrm{x} > 2) = 1 \ - (\ \mathrm{P}(x = 0) + \mathrm{P}(x = 1) + \mathrm{P}(x = 2)\ )$

$$= 1 \ - (\frac{3^0 \mathrm{e}^{-3}}{0!} + \frac{3^1 \mathrm{e}^{-3}}{1!} + \frac{3^2 \mathrm{e}^{-3}}{2!})$$

$$= 1 \ - (\ 0.0498 + 0.1494 + 0.2240) = \ 0.5768$$

P(More than 2 accidents will happen at the junction) = 0.5768

Poisson Distribution

Example 7: A car salesman sells on the average 3 cars per week (assume 5 working days per week). What is the probability that in a given week, he will sell some cars?

$$P(x) = \frac{\mu^x e^{-\mu}}{x!}$$

$$P(x > 0) = 1 - P(x = 0)$$

$$= 1 - \frac{3^0 e^{-3}}{0!} = 0.9502$$

P(Salesman will sell some cars) = 0.9502

Mean and Variance of Poisson Distribution

For a Poisson random variable,

- ✓ Expectation or population mean = μ
- ✓ Population Variance, $\sigma^2 = \mu$

Mean and Variance of Poisson Distribution

Example 8: On average, 2 accidents take place every hour in a factory.

a) What is the probability that at most 2 accidents take place in three hours.
b) Compute the expected number of accidents and the standard deviation in 4 hours

a) Average number of accidents per hour, μ = 2

Average number of accidents per 3 hours = 6

$$P(x) = \frac{\mu^x e^{-\mu}}{x!}$$

$$P(x \leq 2) = P(x = 0) + P(x = 1) + P(x = 2)$$

$$= \frac{6^0 e^{-6}}{0!} + \frac{6^1 e^{-6}}{1!} + \frac{6^2 e^{-6}}{2!} = 0.0620$$

P(at most 2 accidents take place in three hours) = 0.0620

Mean and Variance of Poisson Distribution

b) Expected number of accidents = 8 hours

$\sigma = \sqrt{8}$ = 2.8248