A Refined Macroscopic Traffic Model of Herty-Illner Type

Ash Richardson, Reinhard Illner

University of Victoria

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What it's all about

Overview

Context, Objectives

The Herty-Illner Models
- The Herty-Illner Vlasov Model
- The Herty-Illner Macroscopic Model

Refinements To The Herty-Illner Macroscopic Model
- Refinements To The Acceleration Force

Simulations Of The Refined Macroscopic Model
- Speed Limit Scenario

Discussion

Traffic flow studied at three scales

- Microscopic models: follow individual cars (ODE, DDE, Cellular Automata).
- Mescoscopic (Kinetic) models: probability $f(x, v, t)$ of finding car in a given state (Integro-differential evolution equations).
- Macroscopic models: follow (averaged) quantities of interest (Partial Differential Equations).

In principle all three represent the same information.

Derivations allow us to move between the three scales of observation...

We'll derive a macroscopic model from a kinetic model...

Why Macroscopic Models?

- Simpler: two equations, two variables: (x, t).
- Averaged quantities: artefacts less of a concern...
- Continuum description $\rightarrow$ **qualitative predictions!!!**
- Qualitative, quantitative features e.g. **stop & go waves.**
- Derive wave speeds (not so clear from microscopic models).
- **Use predictions to address traffic control.**
- Control density & speed $\rightarrow$ limit destructive phenomena, optimize flux.

Note: Direct derivation from microscopic models possible (e.g. the accepted **Aw-Rascle** macroscopic model[1]).

For derivation of **Aw-Rascle** model from Kinetic (Vlasov) model see Illner-Kirschner-Pinnau (QAM 2009).

[1] A. Aw, A. Klar, T. Materne, and M. Rascle, "Derivation of continuum traffic flow models from microscopic follow-the-leader models," *SIAM Journal on Applied Mathematics*, vol. 63, no. 1, pp. 259–278, 2002

Objectives (this talk)

- Summarize the Herty-Illner Kinetic (Vlasov) traffic model[2].
- Derive from the above, the Herty-Illner Macroscopic model (which generalizes the model of Aw & Rascle).
- **Refine the Herty-Illner Macroscopic model.**
- **Exhibit simulations for the refined model presented.**

The central assumption of the Herty-Illner models is that drivers should change speed based on what they see ahead. So nonlocal behaviour should be considered for a realistic understanding of the structure of jams and how they arise.

Q: Can we identify parameter regimes for which jams do not amplify?

[2] M. Herty and R. Illner, "Analytical and numerical investigations of refined macroscopic traffic flow models," *Kinetic and Related Models*, 2009

Why further refine the Herty-Illner models?

Herty-Illner (KRM 2008,2009) macroscopic models focused on the role of nonlocal braking force in braking wave (moving jam) formation.

Travelling wave $\mathrm{sol}^{\mathrm{n}}\mathrm{s}$ found for $2^{\text{nd-order}}$ approx. model; existence for unabridged (nonlocal) models: difficult & interesting problem! Presented by R. Illner: *ICIAM 2011 (MS477) Fri. Jul. 22, 3pm (Rm. 121)*.

(*) Would also like to see stop-and-go waves (concatenation of acceleration, braking waves).

(**) Max. principles show: traffic not already moving won't already move!

Prescription for (*),(**): more detailed modelling of nonlocal acceleration force.

The Kinetic (Vlasov) Model

Traffic on a single lane (or lane-homogenized) highway:

$$\partial_t f + v\partial_x f + \partial_v(B(\rho, v - u^X)f) = 0 \tag{1}$$

$$\rho = \int_0^\infty f(x,v,t)\ dv; \qquad \rho u = \int_0^\infty v\ f(x,v,t)\ dv;$$

$$u^X = u(x + H + Tv, t - \tau)$$

H - minimal safety distance,
T - look-ahead time multiplying the driver's speed,
τ - individual reaction time,
$B(\rho, v - u^X)$ - braking/acceleration force dependent on the local density ρ, and the relative speed v of observer w.r.t the "nonlocal delay-observed average speed" u^X.

The Braking/Acceleration Force

- Model origin: charged particle motion w. long-range interaction.
- Assumptions used: the same as for micro. models (next slide).
- Should see the same qualitative & quantitative features!
- Model characteristics.. the microscopic rules for cars:

$$x'(t) = v$$
$$v' = B(\rho, v - u^X)$$

The Braking/Acceleration Force

Assumptions..

- Adjust our speed to match those ahead
- Brake harder in higher densities
- Accelerate to reach higher densities

$$B(\rho, v - u^X) = \begin{cases} -c_1\rho(v - u^X) & \text{if } v - u^X > 0 \\ -c_2(\rho_{max} - \rho)(v - u^X) & \text{if } v - u^X < 0 \end{cases}$$

Macroscopic dependence realistic and needed for consistency.

Deriving A Macroscopic Model From The Kinetic Model

Assuming limited speed variations about (x, t), make the ansatz:

$$f(x, v, t) = \rho(x, t)\delta(v - u(x, t)).$$

This f is a weak $\mathrm{sol}^{\mathrm{n}}$ of the Kinetic model cf. (2) iff (3).

$$T_f(\phi) := \int\int\int \phi_t f + \phi_x v f + \phi_v (B(\rho, v - u^X) f) dx\, dv\, dt = 0 \quad (2)$$

$$\begin{aligned} \rho_t + (\rho u)_x &= 0 \\ \rho\big(u_t + u_x u - B(\rho, u - u^X)\big) &= 0 \end{aligned} \quad (3)$$

Here the test function $\phi(x, v, t)$ is bounded in v and compactly supported in x and t.

Note: for (3) u^X has a slightly different meaning:

$$u^X = u(x + H + Tu(x, t), t - \tau).$$

Proof: Macroscopic Model as Weak $\mathrm{Sol}^{\mathrm{n}}$ of Kinetic Model

Idea: use the independence of v with respect to the other variables.

We'll show only the derivation of the system of equations assuming the weak $\mathrm{sol}^{\mathrm{n}}$ ("$\Rightarrow$").

Proof: Macroscopic Model as Weak $\mathrm{Sol}^{\mathrm{n}}$ of Kinetic Model

("$\Rightarrow$"): The assumption is, for arbitrary $\phi(x, v, t)$, the f given is a weak solution (2) of the Kinetic model (1). Consider ϕ of form

$$\phi(x, v, t) = \varphi(x, t)h(v).$$

Setting $\psi(x, t) := \phi(x, u(x, t), t) = \varphi(x, t)h(u(x, t))$ yields:

$$\psi_t = \phi_t + \phi_v u_t = \varphi_t h + \varphi h' u_t$$
$$\psi_x = \phi_x + \phi_v u_x = \varphi_x h + \varphi h' u_x$$

$$\implies \qquad \begin{aligned} \phi_t &= \psi_t - \partial_v \phi u_t \\ \phi_x &= \psi_x - \partial_v \phi u_x. \end{aligned}$$

We substitute these into the weak solution expression (2)...

Proof: Macroscopic Model as Weak $\mathrm{Sol}^{\mathrm{n}}$ of Kinetic Model

$$\int\int\int [\psi_t f + \psi_x v f - \varphi h'|_v u_t f - \varphi h'|_v u_x v f + \varphi h'|_v B(\ldots) f] dv\, dx\, dt \;= 0$$

$f = \rho\delta(v-u) \Longrightarrow$ (for h bounded away from 0)

$$\int\int [\psi_t \rho + \psi_x \rho u] - \int\int \psi \frac{h'|_u}{h|_u} \rho[u_t + u u_x - B(\rho, u - u^X)] = 0.$$

Since the first integral is independent of $\frac{h'(u)}{h(u)}$, both expressions inside the integrals are 0.

The conclusion is the macroscopic traffic model (3), a system of partial differential equations. $\square$

Macroscopic Model as Weak $\mathrm{Sol}^{\mathrm{n}}$ of Kinetic Model

Note: for the Kinetic model:

$$B(\rho, v - u^X) = \begin{cases} -c_1 \rho (v - u^X) & \text{if } v - u^X > 0 \\ -c_2(\rho_{max} - \rho)(v - u^X) & \text{if } v - u^X < 0 \end{cases}$$

where $u^X = u(x + H + Tv, t - \tau)$.

For the Macroscopic model:

$$B(\rho, u - u^X) = \begin{cases} -c_1 \rho (u - u^X) & \text{if } u - u^X > 0 \\ -c_2(\rho_{max} - \rho)(u - u^X) & \text{if } u - u^X < 0 \end{cases}$$

where $u^X = u(x + H + Tu(x,t), t - \tau)$.

Illner-Herty Macroscopic Model: Maximum Principle

For the macroscopic system (3) with $B(\rho, u - u^X)$ as indicated, the relative speed terms disallow acceleration above the highest speed present and disallow braking below the lowest speed present:

Proposition Suppose that for all x and all $s \in [0, \tau]$:

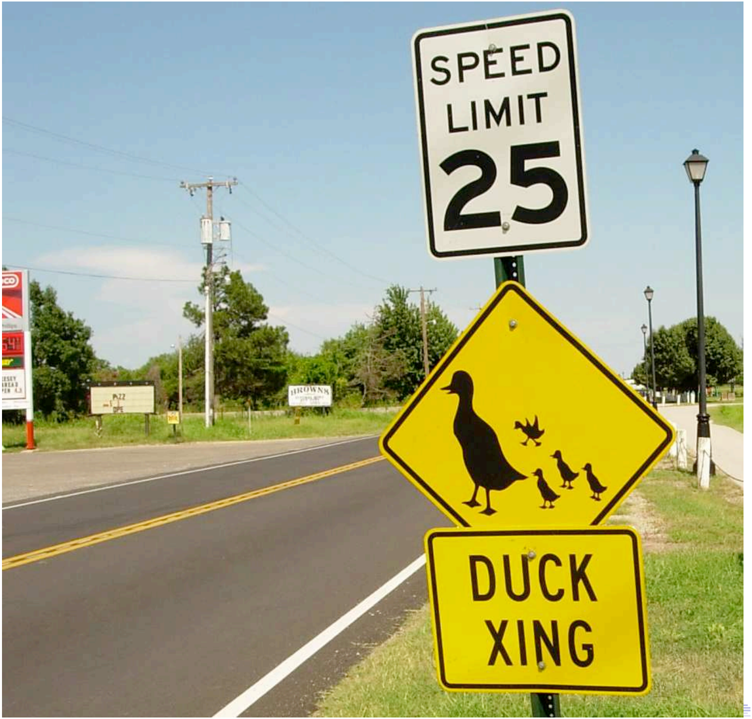
$$0 \le a \le u(x,s) \le b$$

then for any smooth solution of the macroscopic equations (3),

$$0 \le a \le u(x,t) \le b$$

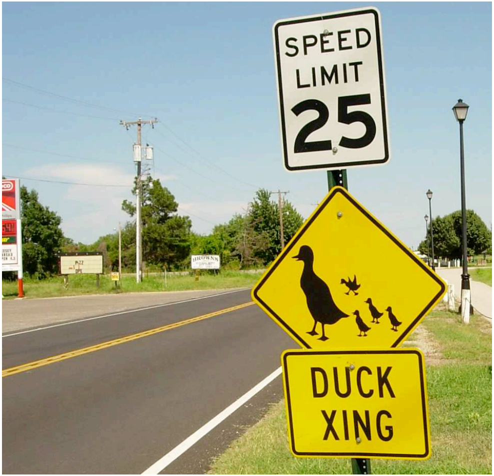
for all x and for all $t \ge 0$.

Question: how do jams form?


SPEED
LIMIT
25
DUCK
XING

Triggers!!!!

Not arising spontaneously, jams have a causal relationship with some perturbing stimulus (from initial conditions or otherwise).



Example Braking Wave

Loading movie...

Aside: Relationship with Aw-Rascle Model

The Aw-Rascle system[3] is the mass law with a momentum law:

$$u_t + u_x(u - \rho\frac{\partial p}{\partial \rho}) = 0. \tag{4}$$

The Illner-Herty Macroscopic model in the braking case $u - u^X > 0$ has momentum law:

$$u_t + uu_x - g_1(\rho)(u - u^X) = 0.$$

Using a 1^{st}-order truncation $u - u^X \simeq -(H + Tu)u_x$:

$$u_t + (u - g_1(\rho)(H + Tu))u_x = 0$$

which is (4) with $\frac{\partial p}{\partial \rho} = \frac{g_1(\rho)}{\rho}(H + Tu)$.

[3] A. Aw and M. Rascle, "Resurrection of "second order" models of traffic flow," *SIAM Journal on Applied Mathematics*, vol. 60, no. 3, pp. 916–938, 2000

Refinements To The Herty-Illner Macro. Models

- Problem: modelling should allow spontaneous acceleration at various densities.
- Specifically, acceleration behaviour should vary with both speed & density.

Two ingredients...

- (I) Mediate between spontaneous vs. synchronized acceleration behaviours at different densities
- (II) Mediate between braking and the new (potentially spontaneous) acceleration

For (I) allow cars to inch ahead at high density & low speed..
For (II) we add a limited degree of driver nonreactivity (mild degree of urgency $\longrightarrow$ brake!)

Refinements To The Herty-Illner Macro. Models

Original (2008) model (braking, acceleration):

$$B(\rho, u - u^X) = \begin{cases} -c_1 \rho (u - u^X) & \text{if } u - u^X > 0 \\ -c_2(\rho_{max} - \rho)(u - u^X) & \text{if } u - u^X < 0 \end{cases}$$

New model (density dependent switching by density quantity ρ_{rule}):

$$B = \begin{cases} -c_1 \cdot \rho \cdot u & \text{if } u_{rule} > c_4,\ \rho_{rule} \geq c_3 \\ -c_1 \cdot \rho \cdot (u - u^X) & \text{if } u_{rule} > c_4,\ \rho_{rule} < c_3 \\ c_2 \cdot \mathcal{S}[u, u_{\max}, \overline{\rho}^X, \rho_{\max}] \cdot [u_{max} - u]_+ & \text{if } u_{rule} \leq c_4,\ \overline{\rho}_{rule} < c_3 \\ c_2 \cdot \mathcal{S}[u, u_{\max}, \overline{\rho}^X, \rho_{\max}] \cdot [\overline{u}^X - u]_+ & \text{if } u_{rule} \leq c_4,\ \overline{\rho}_{rule} \geq c_3 \\ 0 & \text{otherwise} \end{cases}$$

Rules for **(braking,acceleration) x (high density, low density)**.
We'll focus on the low and high density acceleration forces...

Refinements To The Herty-Illner Macro. Models

$$B = \begin{cases} c_2 \cdot \mathcal{S}[u, u_{\max}, \overline{\rho}^X, \rho_{\max}] \cdot [u_{max} - u]_+ & \text{if } u_{rule} \leq c_4,\ \overline{\rho}_{rule} < c_3 \\ c_2 \cdot \mathcal{S}[u, u_{\max}, \overline{\rho}^X, \rho_{\max}] \cdot [\overline{u}^X - u]_+ & \text{if } u_{rule} \leq c_4,\ \overline{\rho}_{rule} \geq c_3 \end{cases}$$

- For maximal density $(\rho_{\max} - \rho) = 0$ (orig. model: stopped cars can't move).
- Allow slight accel. above max. density at low speed (crude example): $|\rho_{\max} - \rho|^2$.
- Don't accel. at high speed & max. density: $[\rho_{\max} - \rho]_+^2$.
- Choose γ so the "inching ahead" term: $|\rho_{\max} - \rho|^2$ falls quickly with u.

$$\mathcal{S} = \left(\frac{u}{u_{\max}}\right)^{\frac{1}{\gamma}} \cdot \left[\frac{\rho_{\max} - \rho}{\rho_{\max}}\right]_+^2 + \left[1 - \left(\frac{u}{u_{\max}}\right)^{\frac{1}{\gamma}}\right] \cdot \left|\frac{\rho_{\max} - \rho}{\rho_{\max}}\right|^2$$

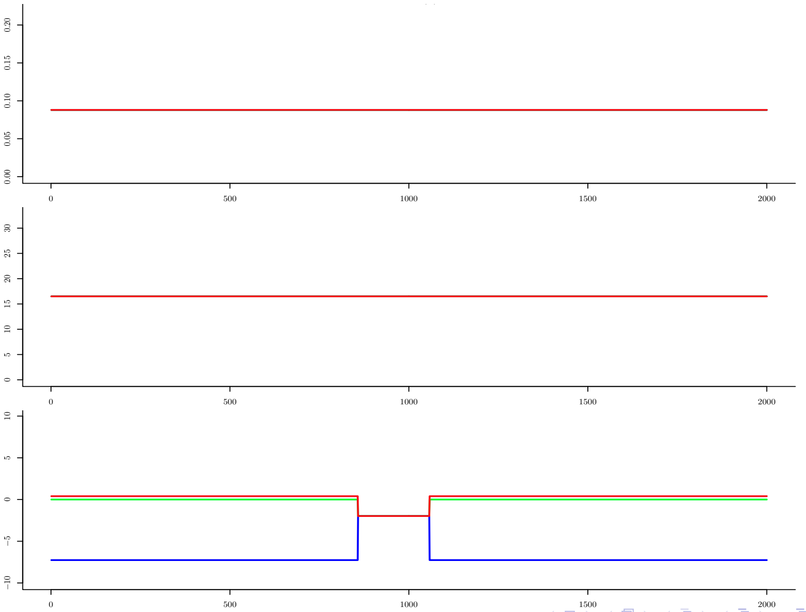
Simulations

- Periodic boundary conditions (ring road).
- Update (ρ, u) by a Godunov scheme (neglecting the source term, treat the rest as a system of conservation laws).
- Add in the source term by updating u with a Euler step.

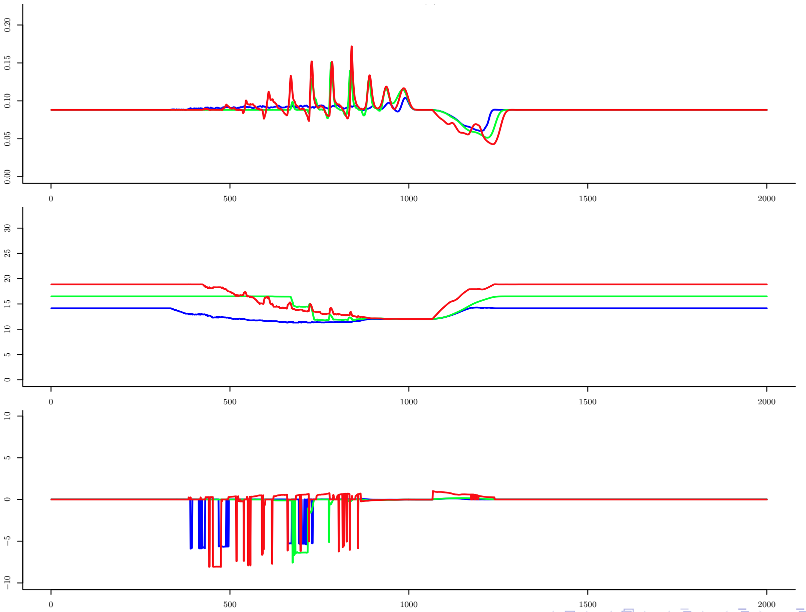
Simulations: parameters

Simulation Parameters			
General	N_x	2000	Number of grid points
	$X_{\max}$	2000	Length of road (m)
	$T_{\max}$	25	Time horizon (s)
	$\tau_{reaction}$	1	Reaction time (s)
	$g_{\max}$	10	Maximum acceleration $(m/s$
Non-local	H	6	Safety distance
	T	2	Reaction time multiplying dr
	$\rho_{\max}$	$\frac{2}{H}$	Maximum density
	$u_{\max}$	30	Maximum speed
Braking	c_1	$\frac{30}{u_{\max}\rho_{\max}}$	Braking coefficient
Acceleration	c_2	$c_1/6$	Acceleration coefficient

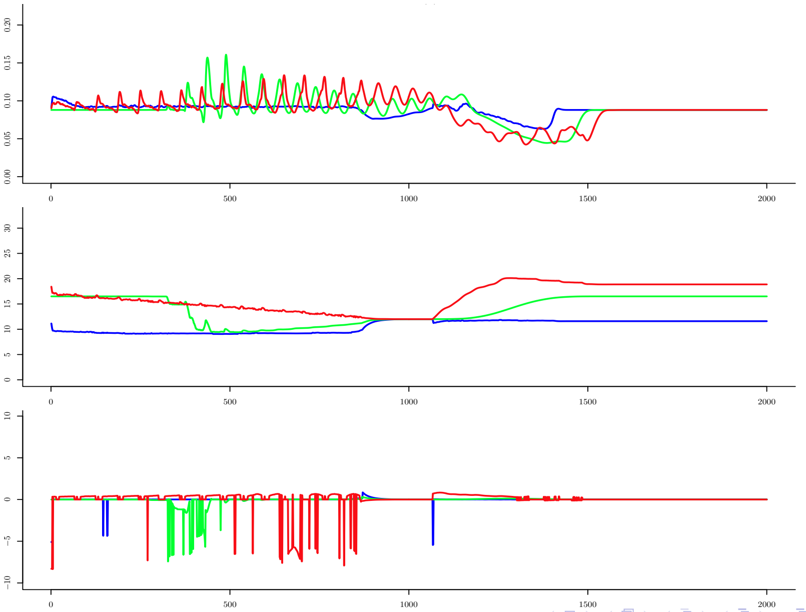
Speed Limit Scenario ($t = t_0$)



Speed Limit Scenario ($t = t_0 + 12\frac{1}{2}s$)



Speed Limit Scenario ($t = t_0 + 25s$)



Discussion

- Summarized the derivation of a Macroscopic model from a Kinetic model
- Noted the Macroscopic model generalizes that of Aw & Rascle
- Added model refinements treating spontaneous acceleration and the dependence of acceleration on speed and density.
- Simulated the refined model yielding numerical solutions featuring low-intensity oscillations consistent with stop-and-go waves

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