Parameter estimation for the oscillating systems

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Experiment performed Friday 18th Januarty 2019

Thursday 3rd October 2019

Experiment details

Apparatus

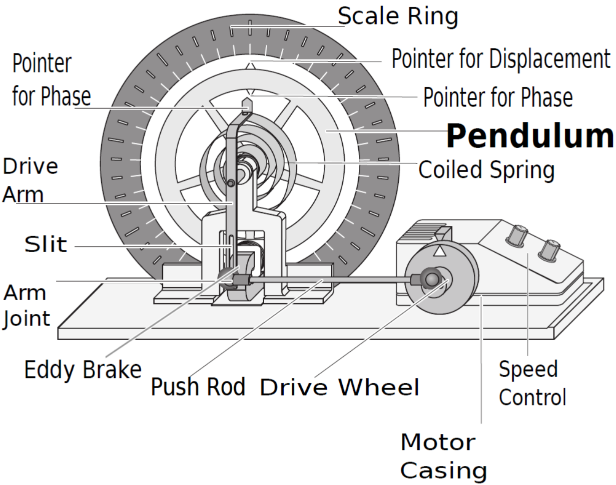
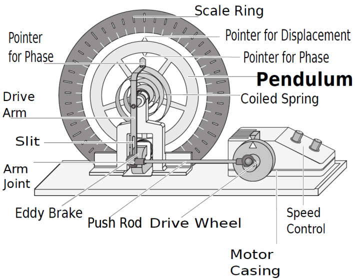


Figure: The diagram of the oscillating system in our experiment.

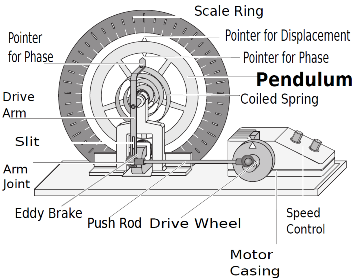
Experiment details

- Measure Amplitude by the ruler around the pendulum



Experiment details

- Measure Amplitude by the ruler around the pendulum
- Measure period of oscillation by timer



Experiment details

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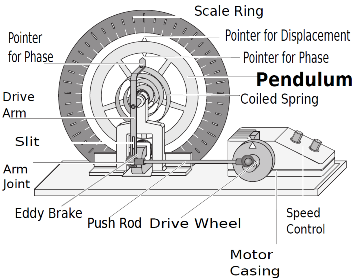
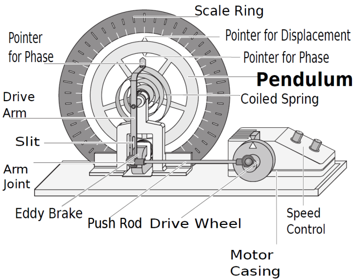
- Measure and adjust frequency of the **driving force**



Experiment details

- Measure Amplitude by the ruler around the pendulum
- Measure period of oscillation by timer
- Measure and adjust frequency of the **driving force**
- Adjust **dumping force** by Eddy brake current



Theoretical background

Damped harmonic oscillator

Equation of motion:

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega_0^2\theta = 0 \tag{1}$$

γ measure of dumping

ω_0 natural frequency

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Solution:

$$\theta = \theta_0\, e^{-\gamma t} \cos(\omega_1 t)$$
$$\text{where} \quad \omega_1 = \sqrt{\omega_0^2 - \gamma^2} \tag{2}$$

Damped harmonic oscillator

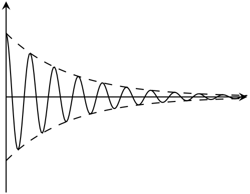

Figure: Underdamped case, which has oscillatory motion.

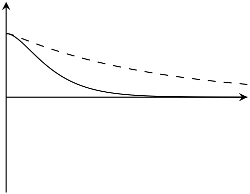

Figure: *Critically damped* (solid), *overdamped* (dashed) at the same natural frequency.

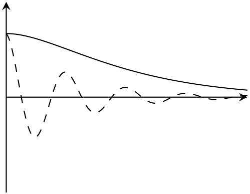

Figure: *Critically damped* (dashed), *underdamped* (doted) at the same damping constant.

Quality factor

Definition

Quality factor is a dimensionless parameter that describes how underdamped the oscillator is, how well it oscillates. Quality factor Q is defined as

$$Q = \frac{\omega_0}{2\gamma} \tag{3}$$

For a *good* oscillators $Q \gg 1$.

Forced oscillation

Equation of motion:

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega_0^2\theta = f\cos(\omega t) \tag{4}$$

- γ measure of dumping
- ω_0 natural frequency
- ω angular frequency of the driving force
- f measure of driving amplitude

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Solution:

$$\begin{aligned} \theta(\omega) &= X(\omega)\cos(\omega t - \phi(\omega)) \\ X(\omega) &= \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}} \\ \tan\phi(\omega) &= \frac{2\gamma\omega}{\omega_0^2 - \omega^2} \end{aligned} \tag{5}$$

Forced oscillation

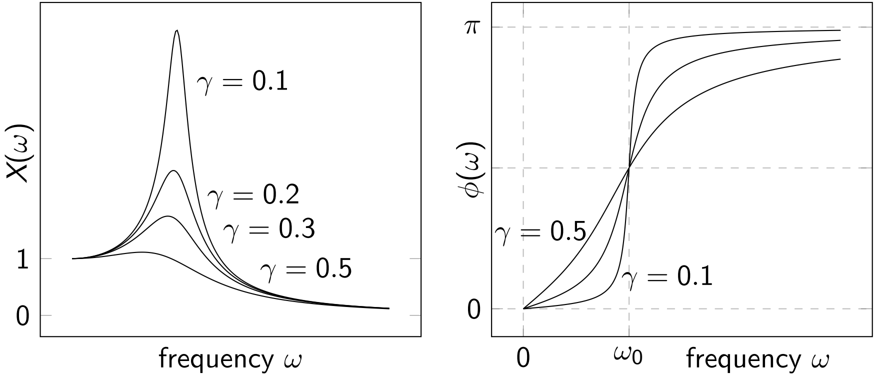

Figure: Dependence of *amplitude* (on the left) and *phase difference* (on the right) from frequency of the driving force

$$\omega_{max} = \sqrt{\omega_0^2 - 2\gamma^2} \qquad X_{max} \equiv X(\omega_{max}) = \frac{f}{2\gamma\sqrt{\omega_0^2 - \gamma^2}}$$

Measurements and Analysis

Measurement of the Amplitude

$$\theta = \theta_0 \, e^{-\gamma t} \cos(\omega_1 t)$$

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$$\Rightarrow \ln(a_n) = ln(a_0) - n\gamma T$$

Measurement of the Amplitude

results

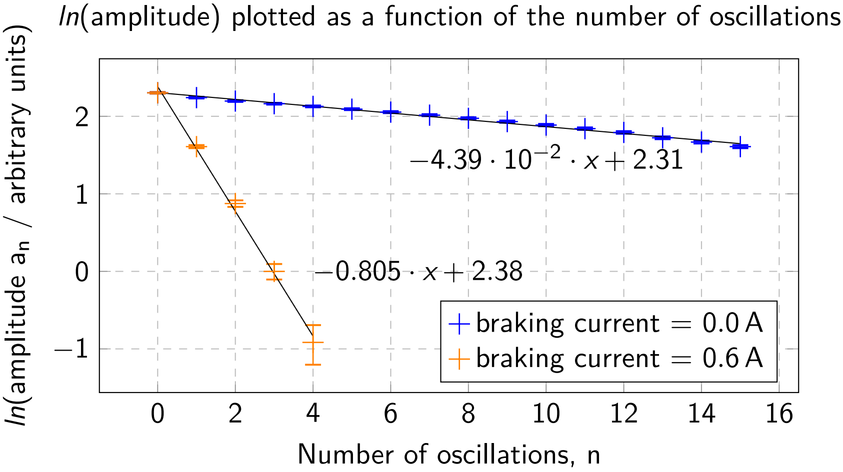

Figure: ln(amplitude) versus oscillation number n. Dependence is linear as we expected.

Results

$$\ln(a_n) = ln(a_0) - n\gamma T$$

$$\text{slope of the line} = -\gamma T = -\gamma \frac{2\pi}{\omega_1}$$

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slope $[I_b = 0.6A]$	-0.80 ± 0.03

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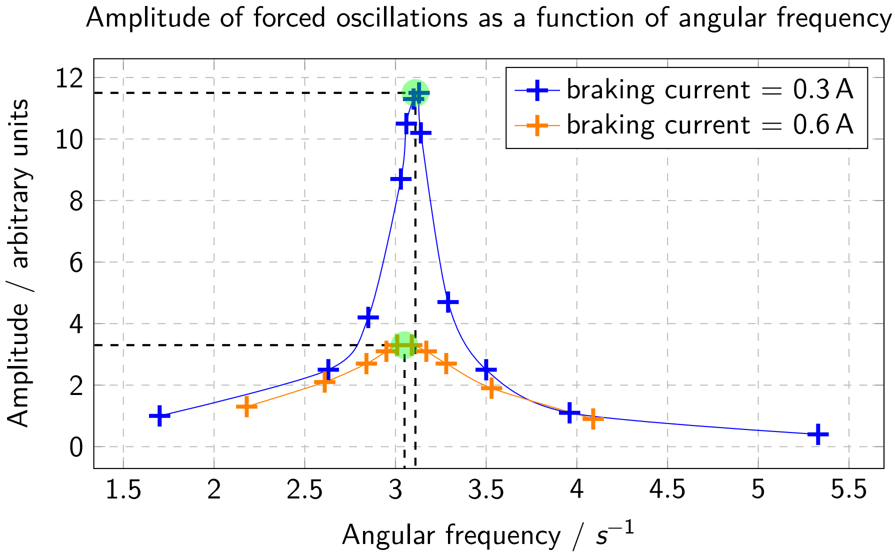
And by assumption slope $\approx -\frac{\pi}{Q}$

Q $[I_b = 0.0A]$	71 ± 1
Q $[I_b = 0.6A]$	3.9 ± 0.1

Forced oscillation

Resonance curve



Forced oscillation

Analysis

$$X(\omega) = \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}} \quad \Rightarrow \quad X(0) = \frac{f}{\omega_0} \tag{6}$$

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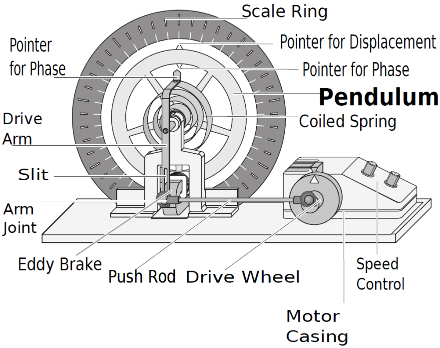
When $\gamma \ll \omega_0$

$$\begin{aligned} X_{max} \approx \frac{f}{2\gamma\omega_0} &= \frac{f}{\omega_0^2} \cdot \frac{\omega_0}{2\gamma} \\ &= X(0) \cdot Q \end{aligned} \tag{8}$$

Forced oscillation

Amplitude at zero frequency

To measure the amplitude at $\omega = 0$ the drive wheel was rotated slowly by hand and the maximum displacement of the pendulum on either side of zero was read.



Amplitude at $\omega = 0$ estimated to be $X(0) = 0.7 \pm 0.1$.

Forced oscillation
Results

Resonance frequency and the amplitude at resonance from the plot.

	X_{max}	ω_{max}
$I_b = 0.3\text{A}$	11.5 ± 0.1	$3.11 \pm 0.02\ \text{s}^{-1}$
$I_b = 0.6\text{A}$	3.3 ± 0.1	$3.05 \pm 0.05\ \text{s}^{-1}$

Thus, Quality factors are estimated to be:

$Q^{2nd}\ [I_b = 0.3A]$	16 ± 2
$Q^{2nd}\ [I_b = 0.6A]$	4.7 ± 0.7

Bayesian Approach

Mean and std estimation

Problem setup

Problem

Given i.i.d. samples from Normal distribution:

$$x_i \stackrel{iid}{\sim} \mathcal{N}(\,\cdot\,|\,\mu, \sigma^2) \quad \textit{for i=1,2,3 ..., N}$$

Suppose σ is known. Find estimation for μ.

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This solution does not satisfy!

Sometimes we need to know **uncertainty** in our measurements. How much confident are we in out estimation?

Bayesian Inference

Introduction

Theorem (Bayes' Theorem)

$$P(A \mid B) = \frac{P(B \mid A)\, P(A)}{P(B)}$$

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$P(Parameters)$ Prior

$P(Parametes \mid Data)$ Posterior

$P(Data \mid Parameters)$ Likelihood

$P(Data)$ Marginal

Bayesian Inference

Introduction

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Example

We are given **Data**, we want to find **distribution over parameters**:

$$P(\mu \mid \{x_i\}) = \frac{P(\{x_i\} \mid \mu)\, P(\mu)}{P(\{x_i\})}$$

$P(\mu)$ Prior

$P(\mu \mid \{x_i\})$ Posterior

$P(\{x_i\} \mid \mu)$ Likelihood

$P(\{x_i\})$ Marginal

Modeling The Problem

Likelihood

Likelihood for each sample is

$$p\left(x_i \mid \mu\right) = \left(2\pi\sigma^2\right)^{-1/2} \exp\left(-\frac{1}{2}\frac{\left(x_i - \mu\right)^2}{\sigma^2}\right)$$

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and as samples are i.i.d., the *Likelihood of the Data* is

$$p(x \mid \mu) = \prod_{i=1}^{n} p\left(x_i \mid \mu\right) = \left(2\pi\sigma^2\right)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(x_i - \mu\right)^2\right)$$

Modeling The Problem

Prior and Marginal

Prior (belief) for μ we take

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Note that this is the hardest part of the model. Sometimes his integral could be intractable.
Also note that this is normalisation constant and

$$p(\mu \mid x) \sim p(x \mid \mu)p(\mu)$$

Sample Mean estimation

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Solution (Bayesian inference [1])

$$p(\mu \mid x) = \mathcal{N}\left(\mu \mid \mu_N, \sigma_N^2\right)$$

$$\textit{where} \quad \mu_N = \frac{\sigma^2}{N\sigma_0^2 + \sigma^2}\mu_0 + \frac{N\sigma_0^2}{N\sigma_0^2 + \sigma^2}\mu_{\mathrm{ML}} \quad \textit{and} \quad \frac{1}{\sigma_N^2} = \frac{1}{\sigma_0^2} + \frac{N}{\sigma^2}$$

[1] Christopher M. Bishop, *Pattern Recognition and Machine Learning*, Chapter 2.3.6

Model for oscillating system

From our experiment we have (ω_i, X_i) $i = 1, ..., N$ pairs, where N is number of measurements. We can *assume* that our measurements come from distribution

$$X_i \overset{\text{iid}}{\sim} \mathcal{P}(\,\cdot\,|\,\omega_i, \omega_0, X_0, \gamma) \equiv \mathcal{N}(X(\omega_i;\, \omega_0, X_0, \gamma),\, \sigma^2) \qquad \text{for } i = 1, ..., n$$

$$\text{where } X(\omega;\, \omega_0, X_0, \gamma) = \frac{X_0\,\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}}$$

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$$\text{where } X(\omega;\, \omega_0, X_0, \gamma) = \frac{X_0\, \omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}}$$

As precision of our measurements is 0.1, we take $\sigma = 0.1$.
Our task is to estimate parameters ω_0, γ and X_0 from the resonance curve $(\omega_i, X_i)_{1...n}$. More formally, find **posterior distribution**

$$\omega_0,\, \gamma,\, X_0 \sim \mathcal{P}(\,\cdot\,|\,\omega_1...\omega_n, X_1...X_n) \tag{9}$$

Using Bayes' Theorem

For Posterior distribution we have

$$\mathcal{P}(\omega_0, \gamma, X_0 \,|\, X_1 ... X_n, \omega_1 ... \omega_n) = \frac{\mathcal{P}(X_1 ... X_n \,|\, \omega_0, \gamma, X_0, \omega_1 ... \omega_n) \cdot \mathcal{P}(\omega_0, \gamma, X_0)}{margnal}$$

$$\text{i.i.d. assumption} \quad = \frac{\prod_{i=1}^{n} \mathcal{P}(X_i \,|\, \omega_0, \gamma, X_0, \omega_i) \cdot \mathcal{P}(\omega_0, \gamma, X_0)}{margnal}$$

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We take *Gamma distributions* for priors of ω_0, γ and X_0.

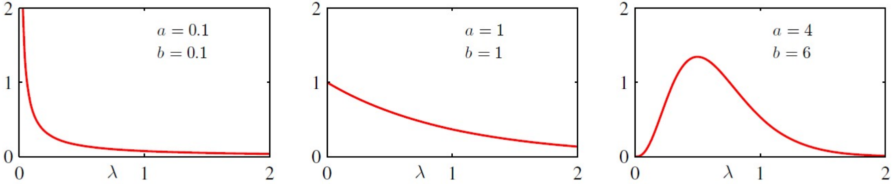

Figure: Gamma distribution pdf for different parameter values.

Method

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There are different sampling algorithms and frameworks. We have used **Stan**[2]which uses *Markov chain Monte Carlo* (MCMC) sampling algorithm.

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Now we sample from our posterior distribution. Our samples are triplets $(\omega_0, \gamma, X_0)_i$.
Sample mean and **sample std** for each parameter are **estimation** and **uncertainty** for that parameter respectively.

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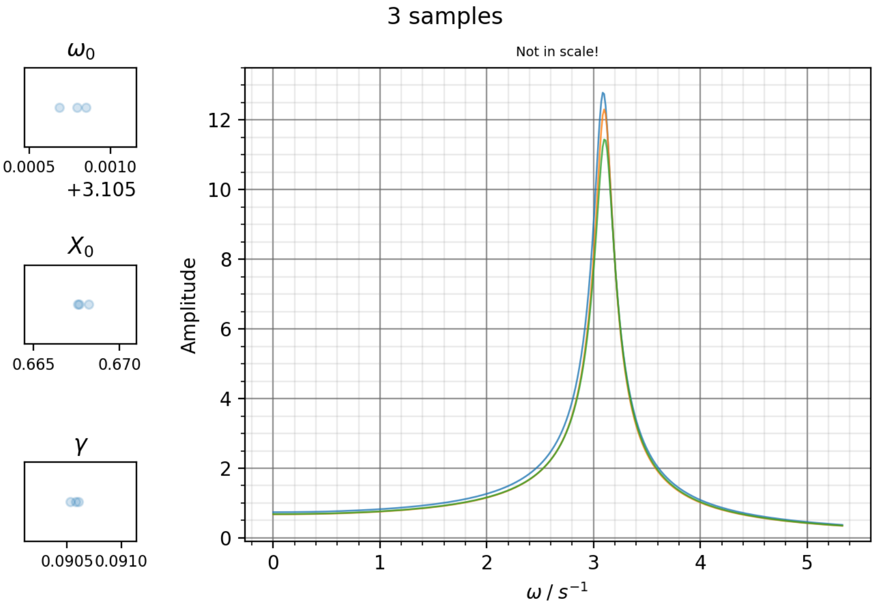
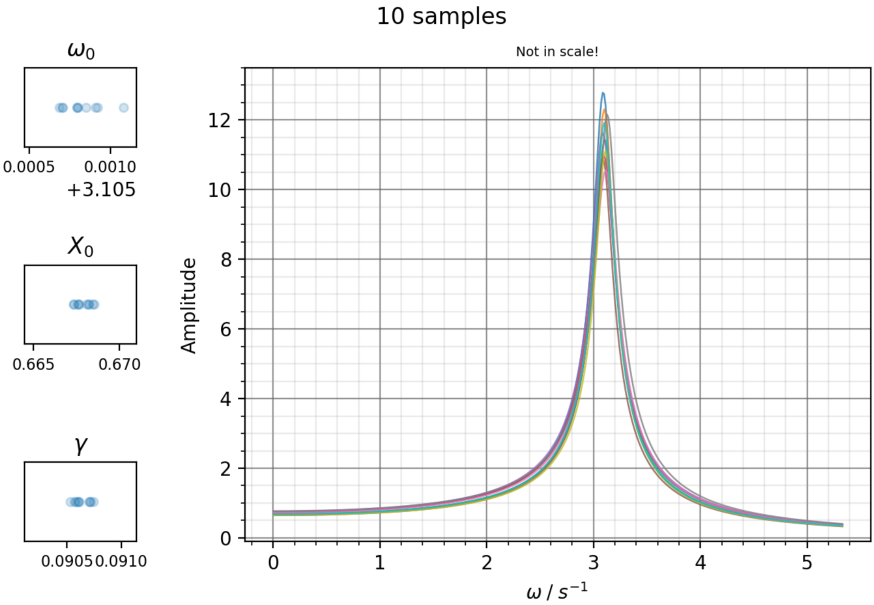
Results



ω_0
0.0005
0.0010
+3.105
X_0
0.665
0.670
γ
0.0905
0.0910
3 samples
Not in scale!
Amplitude
0
2
4
6
8
10
12
0
1
2
3
4
5
ω / s^{-1}

Results



10 samples
ω_0
0.0005
0.0010
+3.105
X_0
0.665
0.670
γ
0.0905
0.0910
Not in scale!
Amplitude
0
2
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2
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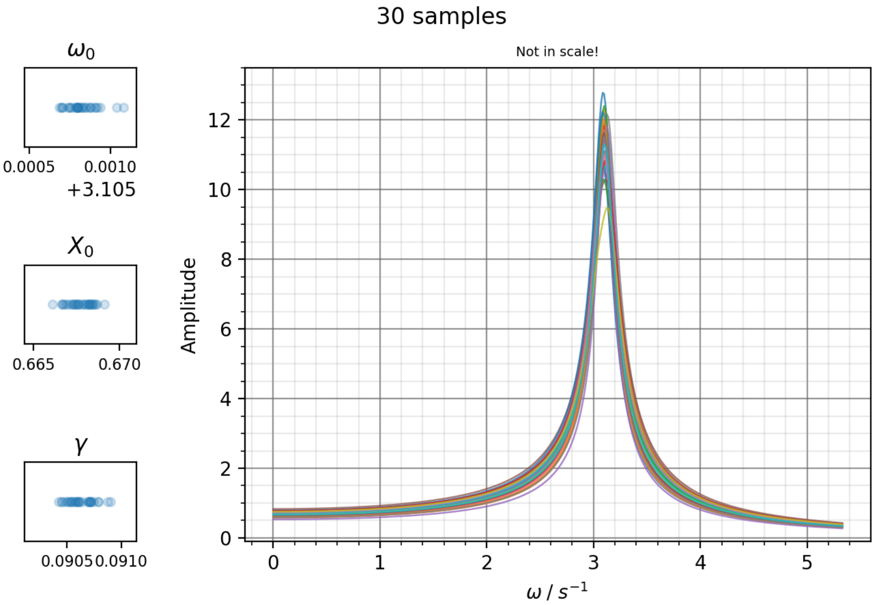
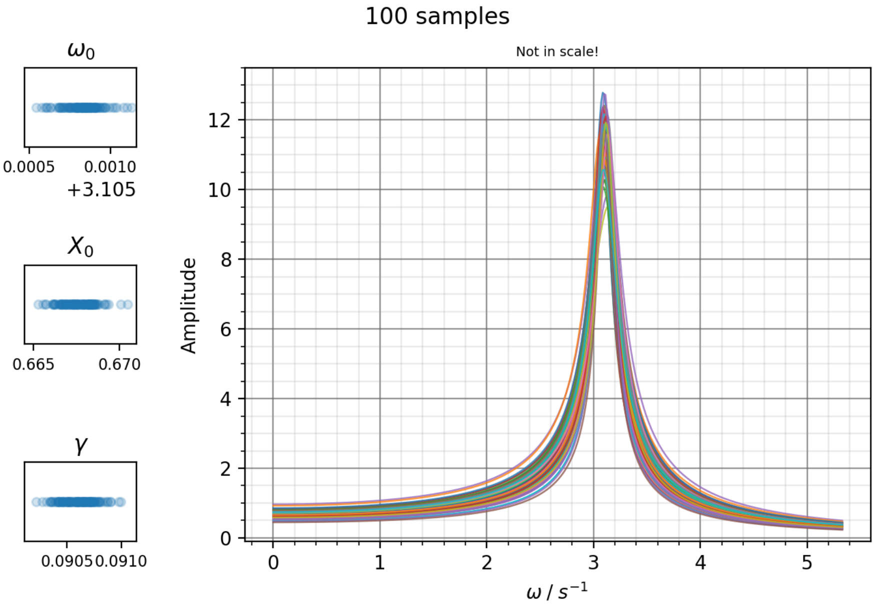
4
5
$\omega \,/\, s^{-1}$

Results


30 samples
Not in scale!
Amplitude
ω / s^{-1}
0
1
2
3
4
5
2
4
6
8
10
12
ω_0
0.0005
0.0010
+3.105
X_0
0.665
0.670
γ
0.0905
0.0910

Results



100 samples
Not in scale!
Amplitude
ω / s^{-1}
0
1
2
3
4
5
0
2
4
6
8
10
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ω_0
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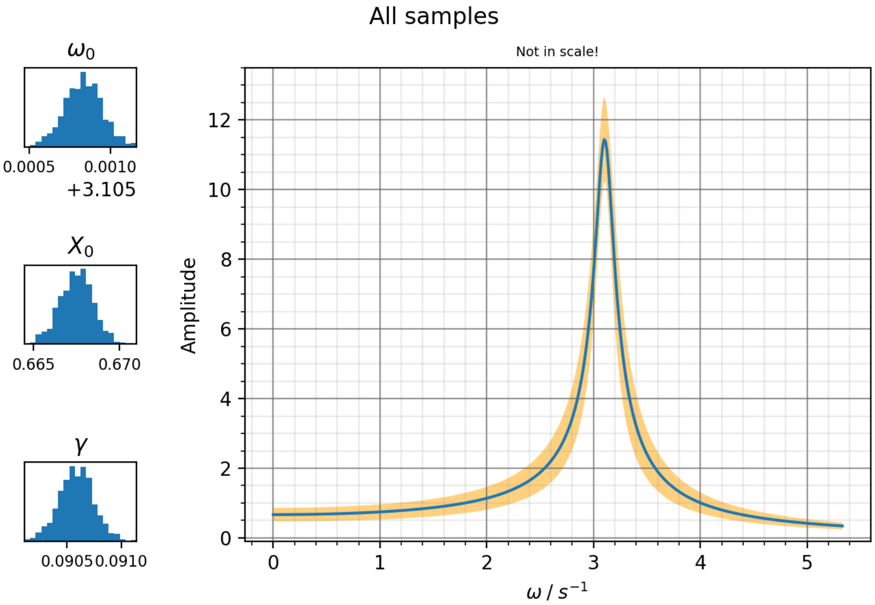
ω_0
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0.0910
All samples
Not in scale!
Amplitude
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2
4
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10
12
0
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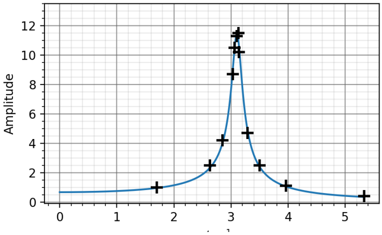
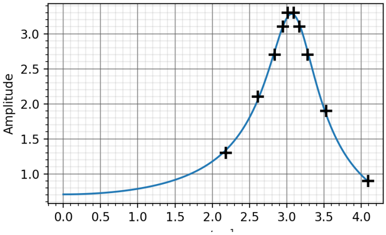
2
3
4
5
$\omega \,/\, s^{-1}$

Results

Braking current = 0.3A

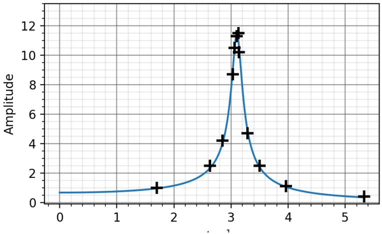


Braking current = 0.6A

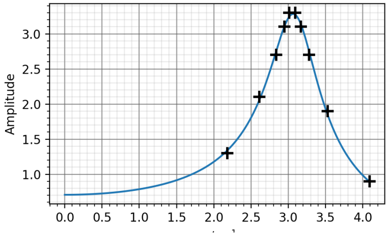


Results

Braking current $= 0.3$A



Braking current $= 0.6$A



	$I_b = 0.3$ A	$I_b = 0.6$ A
ω_0 / s^{-1}	$3.105\,84 \pm 0.000\,12$	3.1025 ± 0.0013
γ / s^{-1}	$0.090\,59 \pm 0.000\,16$	0.3363 ± 0.0018
$X_0 \equiv X(0)$	0.6675 ± 0.0010	0.708 ± 0.003
Q^{3rd} / s^{-1}	17.14 ± 0.03	4.61 ± 0.02

Table: Resonance frequency and the amplitude at resonance.

Thank You