Term Paper

Quantum Fourier Transform
&
Quantum Entanglement

Submitted By

ASHMITA PANDA

ROLL NO. 1811042

School of Physical Sciences

National Institute of Science, Education and Research (NISER), Bhubaneswar

Date of Submission : 30th November, 2021

Under the Guidance of

Dr. Anamitra Mukherjee

Reader-F

School of Physical Sciences

National Institute of Science Education and Research (NISER), Bhubaneswar

Contents

I. Quantum Fourier Transform

1.1 Idea and Mathematical Formulation

Quantum Fourier Transform is essentially a change of basis from the computational basis to the fourier basis.
It is analogous to the inverse discrete fourier transform.

1.1.1 Discrete Fourier Transform

Let us consider a set S_n containing $N = 2^n$ integers from 0 to N-1, i.e.,

$$S_n = \{0, 1, 2, \cdots, N-1\}$$

Let x and y be two discrete variables which belong to set S_n.
So the discrete fourier transform of a function $f(x)$ can be written as :

$$\tilde{f}(y) = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} \exp(-i\omega_y x) f(x) \tag{1}$$

ω_y is the the frequency given as $[(2\pi/N)y]$. The cycle is taken to be periodic beyond N-1.
Using this in (1), we obtain the discrete fourier transform as :

$$f\tilde{(y)} = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} \exp\left(-i\frac{2\pi}{N}yx\right) f(x) \tag{2}$$

Given this, we obtain the discrete inverse fourier transform as :

$$f(x) = \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \exp\left(i\frac{2\pi}{N}xy\right) \tilde{f}(y) \tag{3}$$

Now, as x and y are discrete, (3) is essentially a set of N simultaneous equations. So, we can interpret $f(x)$ and $\tilde{f}(y)$ as column matrices, and $\exp\left(i\frac{2\pi}{N}xy\right)$ as an $N \times N$ invertible matrix.

1.1.2 Quantum Fourier Transform

We now wish to extend the concept for discrete fourier transform to an N-dimensional Hilbert space.

Let $|x\rangle$ and $|y\rangle$ be some kets of the computational basis : $\{|0\rangle, |1\rangle, \cdots, |2^n-1\rangle\}\}$.

$$|x\rangle = |x_1 x_2 \cdots x_n\rangle$$
$$|y\rangle = |y_1 y_2 \cdots y_n\rangle$$

Each of x_i and y_i $(i = 0, 1, \cdots N-1)$ can take values 0 or 1.
Let U be a unitary operator. We can then write the state $U|x\rangle$ as :

$$|\tilde{x}\rangle = U|x\rangle = \sum_{y=0}^{N-1} |y\rangle\langle y|U|x\rangle = \sum_{y=0}^{N-1} U(y,x)|y\rangle \tag{4}$$

x and y here are decimal numbers. Comparing (4) to (3), we can identify $U(y,x)$ as :

$$U(y,x) = \frac{1}{\sqrt{N}} \exp\left(i\frac{2\pi}{N}xy\right) \tag{5}$$

where, $N = 2^n$, where n is the number of qubits and $\exp(2\pi i/N)$ are the N^{th} roots of unity.

We have written the ket $|y\rangle$ in the binary representation, we have the decimal value as :

$$y = y_1 2^{n-1} + y_2 2^{n-2} + \cdots + y_n 2^0 = \sum_{k=1}^{n} y_k 2^{n-k} \tag{6}$$

Using (6) in (5) :

$$U(y,x) = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi i}{N} x \sum_{k=1}^{n} y_k 2^{n-k}\right)$$
$$\implies \quad U(y,x) = \frac{1}{\sqrt{N}} \prod_{k=1}^{n} \exp\left(\frac{2\pi i}{N} x y_k 2^{n-k}\right)$$

As $N = 2^n$:

$$\therefore \quad U(y,x) = \frac{1}{\sqrt{N}} \prod_{k=1}^{n} \exp\left(2\pi i x y_k 2^{-k}\right) \tag{7}$$

Using (7) in (4) :

$$|\tilde{x}\rangle = \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \prod_{k=1}^{n} \bigotimes \exp\left(2\pi i x y_k 2^{-k}\right) |y_k\rangle$$

Writing the summation in binary form, i.e., $\sum_{y=0}^{N-1} \longrightarrow \sum_{y_1=0}^{1} \sum_{y_2=0}^{1} \cdots \sum_{y_n=0}^{1}$

$$\implies |\tilde{x}\rangle = \frac{1}{\sqrt{N}} \sum_{y_1=0}^{1} \sum_{y_2=0}^{1} \cdots \sum_{y_n=0}^{1} \prod_{k=1}^{n} \bigotimes \exp\left(2\pi i x y_k 2^{-k}\right) |y_k\rangle$$

$$\implies |\tilde{x}\rangle = \frac{1}{\sqrt{N}} \prod_{k=1}^{n} \bigotimes \left[\sum_{y_k=0}^{1} \exp\left(2\pi i x y_k 2^{-k}\right) |y_k\rangle \right]$$

Thus, we obtain the Quantum Fourier Transform as :

$$\boxed{|\tilde{x}\rangle = \frac{1}{\sqrt{N}} \prod_{k=1}^{n} \bigotimes \left[|0\rangle + \exp\left(\frac{2\pi i}{2^k} x\right) |1\rangle \right]} \tag{8}$$

This takes from the computational basis $|0\rangle$ and $|1\rangle$ to the Fourier basis, which is the $|+\rangle$, $|-\rangle$ in the Bloch sphere (equatorial plane).

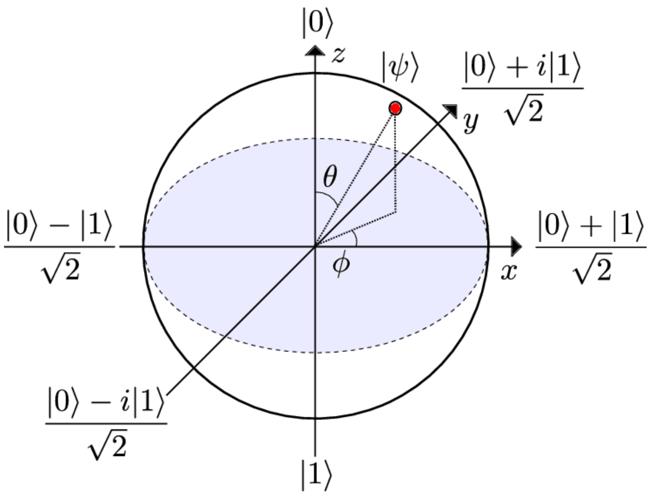

Fig 1. Bloch Sphere

(https://www.researchgate.net/figure/The-Bloch-sphere-representation-of-a-qubit-The-basis-states-are-located-at-the-north_fig2_284259345)

1.1.3 Calculation for a few qubit states

For n=1 : (one qubit state)
For a one qubit state, $|x\rangle = |0\rangle$ or $|1\rangle$. Also, $N = 2$.
Calculating the fourier transform using (8) :

$$|\tilde{x}\rangle = \frac{1}{\sqrt{2}} \left[|0\rangle + \exp(\pi i x) |1\rangle \right]$$

This is the same as the Hadamard gate acting on $|x\rangle$.

$$\text{For } x = 0 : \qquad |\tilde{0}\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle$$
$$\text{For } x = 1 : \qquad |\tilde{1}\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |-\rangle$$

For n=2 : (two qubit state)
For two qubit state, $|x\rangle = |00\rangle, |01\rangle, |10\rangle, |11\rangle$. Also, $N = 2^2 = 4$.

In decimal representation,

$$x = x_1 2^1 + x_2 2^0 = 2x_1 + x_2$$

So, $|x\rangle$ can take values $|0\rangle, |1\rangle, |2\rangle, |3\rangle$. Calculating the fourier transform using (8) :

$$|\tilde{x}\rangle = \frac{1}{\sqrt{4}} \left[|0\rangle + \exp\left(\frac{2\pi i}{2} x \right) |1\rangle \right] \bigotimes \left[|0\rangle + \exp\left(\frac{\pi i}{2} x \right) |1\rangle \right]$$

For different values of x :

$$\begin{aligned}
\text{For x=0 :} \qquad & |\tilde{0}\rangle = \frac{1}{2} \left[|0\rangle + |1\rangle \right] \otimes \left[|0\rangle + |1\rangle \right] \\
\text{For x=1 :} \qquad & |\tilde{1}\rangle = \frac{1}{2} \left[|0\rangle - |1\rangle \right] \otimes \left[|0\rangle + i|1\rangle \right] \\
\text{For x=2 :} \qquad & |\tilde{2}\rangle = \frac{1}{2} \left[|0\rangle + |1\rangle \right] \otimes \left[|0\rangle - |1\rangle \right] \\
\text{For x=3 :} \qquad & |\tilde{3}\rangle = \frac{1}{2} \left[|0\rangle - |1\rangle \right] \otimes \left[|0\rangle - i|1\rangle \right]
\end{aligned}$$

1.2 Implementation in a Quantum Circuit

From the formulation of the Quantum Fourier Transform in (8), we can see we go from the state $|x\rangle$ in the computational basis :

$$|x\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle$$

to that state $|\tilde{x}\rangle$ in the fourier basis (equitorial plane of Bloch sphere) :

$$|\tilde{x}\rangle = \frac{1}{\sqrt{N}} \left[|0\rangle + \exp\left(\frac{2\pi i}{2} x \right) |1\rangle \right] \otimes \left[|0\rangle + \exp\left(\frac{2\pi i}{2^2} x \right) |1\rangle \right] \otimes \cdots \otimes \left[|0\rangle + \exp\left(\frac{2\pi i}{2^n} x \right) |1\rangle \right]$$

Each qubit goes from :

$$|x_k\rangle \longrightarrow \left[|0\rangle + \exp\left(\frac{2\pi i}{2^k} x \right) |1\rangle \right]$$

So to implement the quantum fourier transform in a quantum circuit we first need a Hadamard gate (H) to go from $|x_k\rangle$ to $\dfrac{|0\rangle \pm |1\rangle}{\sqrt{2}}$, then a Unitary rotation gate ($URot_k$) for applying a phase.

$$H|x_k\rangle = \begin{cases} \dfrac{|0\rangle + |1\rangle}{\sqrt{2}} & \text{if } x_k = 0 \\ \dfrac{|0\rangle - |1\rangle}{\sqrt{2}} & \text{if } x_k = 1 \end{cases}$$

$URot_k$ is essentially a controlled gate.

$$URot_k|x_j\rangle = \exp\left(\frac{2\pi i}{2^k} \right) |x_j\rangle = \begin{cases} |0\rangle & \text{if } x_j = 0 \\ \exp\left(\dfrac{2\pi i}{2^k} \right) |1\rangle & \text{if } x_j = 1 \end{cases}$$

The circuit to implement the function is as follows : The output of the above gate is obtained

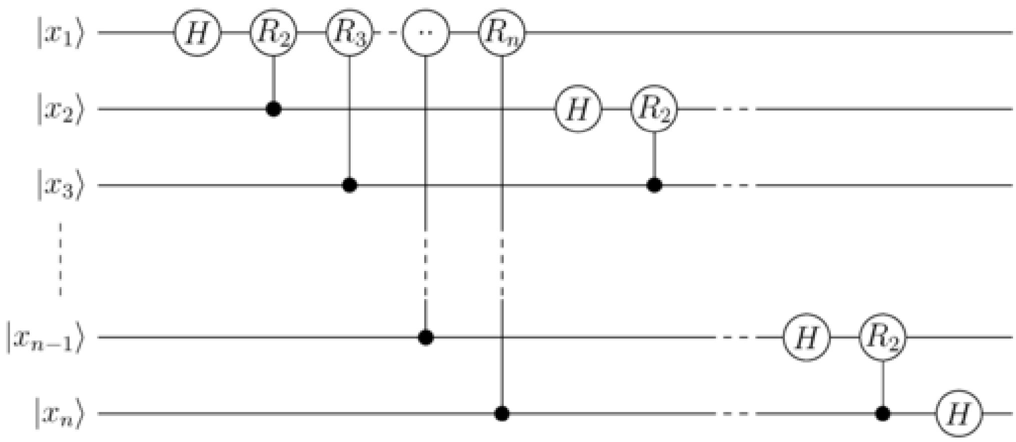

Fig 2. Quantum Circuit Implementation of Fourier Transform without SWAP gates

https://en.wikipedia.org/wiki/Quantum_Fourier_transform/media/File:Q_fourier_nqubits.png

as :

$$\boxed{|x_{OUT}\rangle = \frac{1}{\sqrt{N}}\left[|0\rangle + \exp\left(\frac{2\pi i}{2^n}x\right)|1\rangle\right]\bigotimes\cdots\bigotimes\left[|0\rangle + \exp\left(\frac{2\pi i}{2^1}x\right)|1\rangle\right]} \tag{9}$$

The order here is reversed from the derived equation (8) but the structure is essentially the same. We can just apply swap gates at the end of the circuit to obtain the required form.

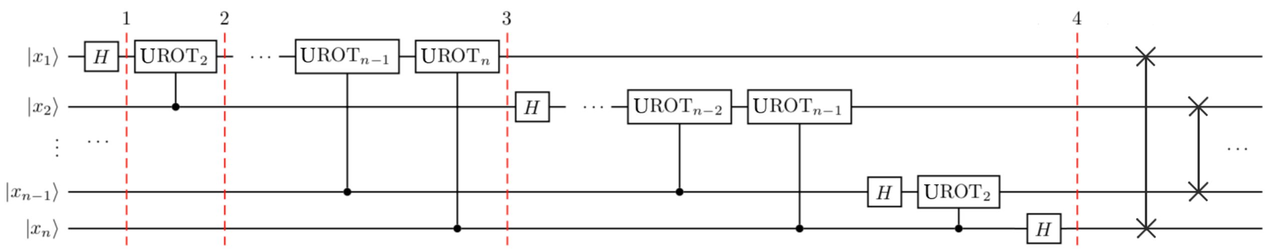

Fig 3. Quantum Circuit Implementation of Fourier Transform with SWAP gates

https://qiskit.org/textbook/ch-algorithms/quantum-fourier-transform.html

1.3 Application - Quantum Phase Estimation

Quantum Phase Estimation is one of the very useful applications of Quantum Fourier Transform.

The eigen value equation of any unitary matrix (U) can be written as :

$$U|\psi\rangle = \exp(i\theta)|\psi\rangle$$

$\exp(i\theta)$ are the eigen values, and eigen vectors $|\psi\rangle$ form an orthonormal basis. θ gives us the global phase of the state.

Determining this global phase is not straight forward, as measurements on a states $|\psi\rangle$ and $\exp(i\theta)|\psi\rangle$ yield same results.
For example, let :

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

Both states $|\psi\rangle$ and $\exp(i\theta)|\psi\rangle$ yield states $|0\rangle$ and $|1\rangle$ with probability 1/2.
If we wish to determine the state $|\psi\rangle$ with more precision, we need a way to estimate its global phase.

Considering the following quantum circuit : The output state is obtained as :

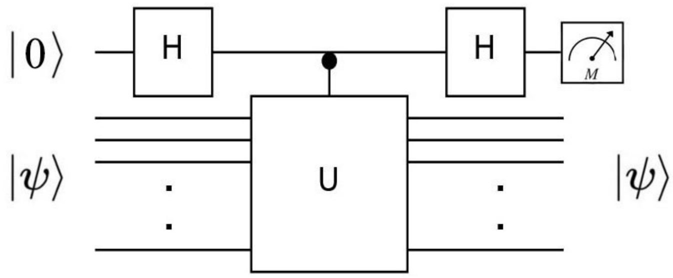

Fig 4. Phase Estimation for a single qubit

https://jonathan-hui.medium.com/qc-phase-estimation-in-shors-algorithm-acef265ebe50

$$|\psi_{out}\rangle = \frac{1}{2}\left[|0\rangle\left\{1+\exp(i\theta)\right\} + |1\rangle\left\{1-\exp(i\theta)\right\}\right]|\psi\rangle$$

Now the probability of measuring $|0\rangle$ or $|1\rangle$ depends on the global phase θ.

$$Prob(0) = \left|\frac{1+\exp(i\theta)}{2}\right|^2$$
$$Prob(1) = \left|\frac{1-\exp(i\theta)}{2}\right|^2$$

To improve the precision of measuring this global phase, we can use multiple qubits instead of just 1. Stepwise breakdown of the circuit :

$$\underline{\text{Step 0}}: \quad |\Psi_0\rangle = |0\rangle^{\otimes n}|\psi\rangle$$
$$\underline{\text{Step 1}}: \quad |\Psi_1\rangle = \frac{1}{(\sqrt{2})^n}\left(|0\rangle + |1\rangle\right)^{\otimes n}|\psi\rangle$$

In the next 2^n step, we need to act multiple U^{2^x} operators on the state $|\psi\rangle$, when $x = n-1, n-2, \ldots, 0$. For any general x :

$$U^{2^x}|\psi\rangle = U^{2^x-1}U|\psi\rangle = U^{2^x-1}\exp(i\theta)|\psi\rangle = \cdots = \exp(i\theta 2^x)|\psi\rangle$$

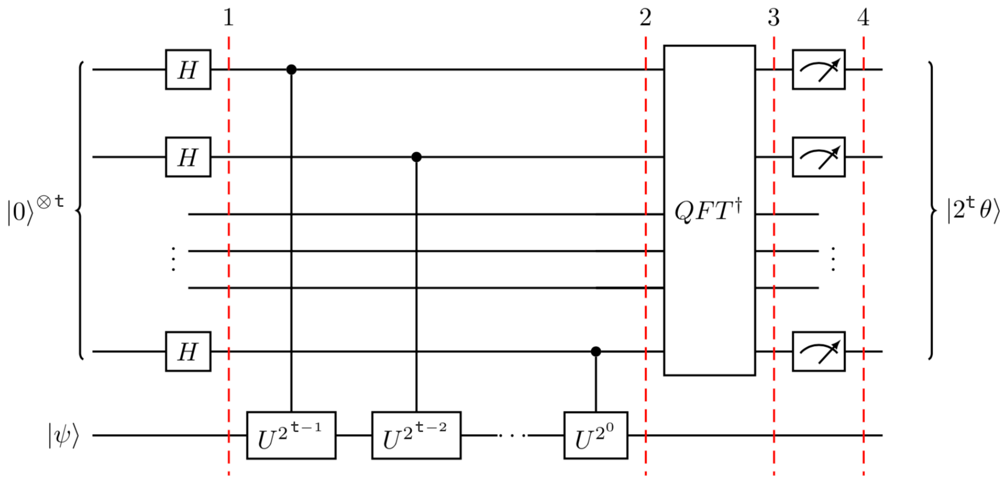

Fig 5. Quantum Phase Estimation

https://qiskit.org/textbook/ch-algorithms/quantum-phase-estimation.html

Thus, the final state becomes :

$$|\psi_{out}\rangle = \left(\frac{1}{\sqrt{2}}\right)^n \left[|0\rangle + \exp\left(i\theta 2^{n-1}\right)|1\rangle\right] \otimes \left[|0\rangle + \exp\left(i\theta 2^{n-2}\right)|1\rangle\right] \otimes \cdots \otimes \left[|0\rangle + \exp\left(i\theta 2^{0}\right)|1\rangle\right] \tag{10}$$

Comparing (10) with (8), we observe that $|\psi_{out}\rangle$ is the same as the fourier transform of a variable θ_ψ, related to θ as :

$$\theta = 2\pi \frac{\theta_\psi}{2^n} \tag{11}$$

We can apply an inverse Quantum Fourier Transform at the end of the circuit, and obtain θ_ψ. Thus, we can then find the global phase from (11).

II. Entanglement and Measures of Entanglement

2.1 Basics

Quantum entanglement is a physical phenomenon two or more states where each state cannot be described independent of the states.

LOCC (Local Operations and Classical Communications) is a method in which some local operations are performed on some part of the complete system, and the results of the operations are communicated to the other parts of the system classically. Subsequently, some local operations may be performed on the other part of the system based on the received information.

In quantum communication experiments, we may want to distribute qubits over a long distance. But the channels we use for the same could be noisy. We try to overcome the affect of the noise using Local Operations, as they are performed in controlled local environments and are almost ideal.

2.1.1 LOCC based definition on entanglement

When we perform LOCC on some part of the system, classical correlations can be generated. Entanglement requires the implementation of non-local quantum operations. LOCC operations are not sufficient to achieve these transformations.

Thus, entanglement can be defined as the correlations that cannot be created by LOCC alone. These correlations are present in the initial state itself, irrespective of how noisy it is.

2.1.2 Entanglement does not increase under LOCC

As we know from the previous subsection, given LOCC we can only create clasical correlations, not quantum correlations. Thus, we cannot create entanglement just using LOCC.

Looking at it from the context of teleportation, if Alice and Bob initially donot share any entangled state, then they cannot teleport any state between them, as teleportation requires an entangled state.
Now, if we could have created entanglement from LOCC, however small, then Alice and Bob can teleport a state. This would essentially imply they teleported a state even when they did not have any entangled state to start with. This is not possible, and thus entanglement cannot be created from purely LOCC.

Considering now, we already have an entangled state, ρ. We can convert it to another entangled state σ, via LOCC. Then, any property that we can get from σ, we can get it from ρ also. This essentially means, the utility of the state does not increase. Thus, we can say the entanglement of the state does not increase.

2.1.3 Entanglement is preserved under local unitary transformations

Any operation we perform on a state using local unitary operators is invertible. And we also know that the entanglement cannot increase under LOCC.

Now, considering we go from ρ to σ using unitary transformations, and entanglement decreases in the process. Thus, σ is less entangled than ρ. Now, as unitary operations are invertible, we should be able to reach ρ from σ. But in this process, entanglement must increase, which is not possible.

Thus, entanglement must be preserved under unitary transformations.

2.1.4 Positive and Completely Positive Maps

Positive Map :
A linear map $\Lambda : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ maps operators from $\mathcal{A}_1$ to $\mathcal{A}_2$. It is called a positive map if it maps positive operators to positive operators, i.e.,

$$\Lambda(A) \geq 0 \qquad \forall A \geq 0 \tag{12}$$

We can interpret $\mathcal{A}_1$ and $\mathcal{A}_2$ as some linear space of density matrix operators acting on some Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively.

An example of a positive map is the transpose map.

$$T : \rho \longrightarrow \rho^T$$

The defining properties of density matrices, i.e.,

i.) $\rho^\dagger = \rho$

ii.) $Tr(\rho) = 1$

iii.) ρ is positive semidefinite

are left unchanged by the transpose map.

Completely Positive Map :
A positive map is called completely positive if the map :

$$\Lambda \otimes \mathbb{1}_d : \mathcal{A}_1 \otimes \mathcal{M}_d \longrightarrow \mathcal{A}_2 \otimes \mathcal{M}_d \tag{13}$$

is a positive map for all values of $d = 2, 3, \ldots$.
$\mathbb{1}_d$ is the matrix space $\mathcal{M}_d$ of all $d \times d$ matrices.

An example of completely positive maps is the Unitary transformation.
If we have a state $|\psi\rangle$ which evolves to a state $|\psi'\rangle = U|\psi\rangle$ under a unitary operator U, the map :

$$U : \rho \longrightarrow \rho' = U\rho U^\dagger$$

is a completely positive map.

2.1.5 Requirements for bipartite entanglement measures

An entanglement measure is a mathematical quantity which quantifies how much entanglement is present in a state and it must be related to some operational procedure. For example, we could measure how much a state (in a bipartite system) is entangled by quantifying how well can an arbitrary state be teleported between the two parties.

A bipartite entanglement measure must satisfy the following conditions :

i.) A bipartite entanglement measure $E(\rho)$ is a mapping from density matrices to positive real numbers.

$$E : \rho \longrightarrow E(\rho) \in \mathbb{R}^+$$

A normalisation factor can also be added such that the maximally entangled state :

$$|\psi_d^+\rangle = \frac{|0,0\rangle + |1,1\rangle + \ldots + |d-1, d-1\rangle}{\sqrt{d}}$$

has the value of measure as $E(|\psi_d^+\rangle) = \log d$. This is the largest value of measure.

ii.) If the states are separable, then measure of entanglement must be 0.

$$E(\rho) = 0$$

iii.) The measure of entanglement should not increase on average under LOCC.

iv.) For pure states, the entanglement measure should reduce to the entropy of entanglement.

2.2 Detection

Separability criterion for detecting whether a state is entangled or not, can be broadly distinguished into two types : *Operational* and *Non-operational* criteria.

A criterion is called *non-operational* if there does not exist any definite procedure, or sequence of operations to perform the criterion on a given state.

If such a sequnce exists, it is called as an *operational* criterion.

Also, a separability criterion can either be *necessary* or both, *necessary and sufficient.*
A necessary condition for separability must be satisfied by all seperable states, but fulfilment of this does not guarantee separability. If, on the other hand, this condition is not satisfied by a state, then the state has to be entangled.
A necessary and sufficient condition for separability can be satidfied by only separable states.

2.2.1 Peres Separability Criterion

The Peres Separability Criterion is an operational criterion to ascertain whether a state is separable or entangled, which is valid only for 2×2 or 2×3 dimensional Hilbert Spaces. In these dimensions, it is a necessary and sufficient condition for separability.

Peres Separability Criterion :
A state ρ acting on $\mathcal{H}^2 \times \mathcal{H}^2$, $\mathcal{H}^3 \times \mathcal{H}^2$ or $\mathcal{H}^2 \times \mathcal{H}^3$ is separable if and only if its partial transpose is a positive operator.

$$\rho^{T_B} = (\mathbb{1} \otimes T)\, \rho \geq 0 \tag{14}$$

Thus, if the partial transpose of a state is negative, then the state is entangled.

For higher dimensional Hilbert spaces, the condition is called Positive Partial Transpose (PPT) criterion and it is no longer a sufficient condition for separability. It is only a necessary condition for separability (in higher dimensions). There could be states whose partial traces are positive but they still turn our to be non-separable.

Relation to Positive Maps :
The Positive Map Theorem (PMT) states that a bipartite state ρ is separable if and only if

$$(\mathbb{1} \otimes \Lambda)\, \rho \geq 0 \qquad \forall \text{ positive maps } \Lambda \tag{15}$$

This essentially implies that a state is entangled if and only if there exists a positive map Λ, such that :

$$(\mathbb{1} \otimes \Lambda)\, \rho_{ent} < 0 \tag{16}$$

Now, partial transpose operator (T) is just one example of a map which is positive, and not completely positive. The Peres Separability criterion is just a special case of the Positive Map Theorem with only one map $\Lambda = T$.

Example :
We can consider the Werner state as an example.

The 2-qubit Werner state is given as :

$$\rho_\alpha = \alpha|\psi_-\rangle\langle\psi_-| + \frac{1-\alpha}{4}\mathbb{1}\otimes\mathbb{1}\,, \qquad -\frac{1}{3} \leq \alpha \leq 1 \tag{17}$$

The state $|\psi_-\rangle$ is one of the Bell basis states :

$$|\psi_-\rangle = \frac{|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B}{\sqrt{2}}$$

The second term in (17) corresponds to some noise. The value of α must lie between (-1/3) and 1 so that $Tr(\rho_\alpha) = 1$.
In matrix form, we can write the Werner state as :

$$\rho_\alpha = \begin{pmatrix} \frac{1-\alpha}{4} & 0 & 0 & 0 \\ 0 & \frac{1-\alpha}{4} & \frac{-\alpha}{2} & 0 \\ 0 & \frac{-\alpha}{2} & \frac{1-\alpha}{4} & 0 \\ 0 & 0 & 0 & \frac{1-\alpha}{4} \end{pmatrix} \tag{18}$$

Now taking a partial transpose on the second qubit space.

$$\rho_\alpha^{T_B} = \begin{pmatrix} \frac{1-\alpha}{4} & 0 & 0 & \frac{-\alpha}{2} \\ 0 & \frac{1-\alpha}{4} & 0 & 0 \\ 0 & 0 & \frac{1-\alpha}{4} & 0 \\ \frac{-\alpha}{2} & 0 & 0 & \frac{1-\alpha}{4} \end{pmatrix} \tag{19}$$

We can find the eigen values of the above matrix as :

$$\lambda = 0, 0, \frac{1+\alpha}{4}, \frac{1-3\alpha}{4} \tag{20}$$

Now this tells that for values of α greater than 1/3, the states are entangled.
Summarising,

$$\begin{aligned} -\frac{1}{3} \leq \alpha \leq \frac{1}{3} &\implies \text{states are separable} \\ \frac{1}{3} < \alpha \leq 1 &\implies \text{states are entangled} \end{aligned}$$

2.2.2 Entanglement Witness Criterion

Entanglement Witness criterion is a non-operational separability criteria.

Statement :
A state ρ_{ent} is entangled if and only if there exists a Hermitian Operator $A \in \mathcal{A}$, called entanglement witness, such that it satifies :

$$\begin{aligned} \langle \rho_{ent} | A \rangle = Tr(A\rho_{ent}) < 0 \\ \langle \rho | A \rangle = Tr(A\rho) \geq 0 \end{aligned} \tag{21}$$

where, ρ is the density matrix of some separable state.

Example 1 :
Let us consider a swap operation :

$$SWAP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let us now consider an arbitrary separable state $|\chi\rangle$.

$$|\chi\rangle = |x\rangle \otimes |y\rangle$$

Applying $SWAP$ operator on $|\chi\rangle$:

$$SWAP|\chi\rangle = SWAP\left(|x\rangle \otimes |y\rangle\right) = |y\rangle \otimes |x\rangle$$

Taking inner product :

$$\langle \rho | SWAP \rangle = \langle x | \langle y | SWAP | x \rangle | y \rangle$$

As $SWAP$ is a hermitian operator :

$$\begin{aligned} \langle \rho | SWAP \rangle = \left(\langle x| \otimes \langle y|\right)\left(|y\rangle \otimes |x\rangle\right) \\ \therefore \langle \rho | SWAP \rangle = \langle x|y\rangle\langle y|x\rangle = |\langle x|y\rangle|^2 \geq 0 \end{aligned}$$

Thus, the Entanglement Witness Criterion is satisfied for product states.

Example 2 :
Let us consider an entangled state :

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Acting the $SWAP$ operator :

$$SWAP|\psi\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}} = -|\psi\rangle$$

Taking inner product :

$$\langle\rho|SWAP\rangle = \langle\psi|SWAP|\psi\rangle = -\langle\psi|\psi\rangle = -1 < 0$$

Thus, entanglement witness criterion is satisfied for entangled states.

Geometric Interpretation of EWT A bipartite system is separable, if and only if its density matrix can be written as a an ensemble of product states :

$$\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i| \,, \quad 0 \leq p_i \leq 1 \,, \quad \sum_i p_i = 1$$

ρ is a convex combination of product states.

A set is called convex if, given any points in the set, it containts in it the whole line segment joining the points.
Thus, any density matrix which can be written in the form :

$$\rho = \lambda\sigma_1 + (1-\lambda)\sigma_2 \quad 0 \leq \lambda \leq 1$$

where, σ_1, σ_2 are separable states, is also a separable state. If there is any ρ_{ent} which does not

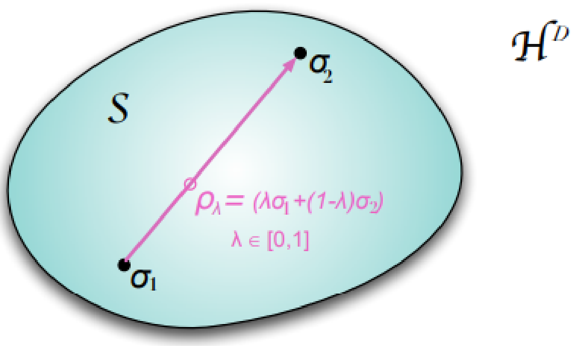

Fig 6. Convex Plane

belong to this set $\mathcal{S}$ of separable matrices, it must be separated from $\mathcal{S}$ by a hyperplane.

Let A be an entanglement witness operator. So, any ρ satisfying $\langle\rho|A\rangle = 0$ defines the hyperplane. It separates the entangled states ρ_{ent} for which $\langle\rho_{ent}|A\rangle < 0$ and the separable states ρ_S for which $\langle\rho_S|A\rangle > 0$.

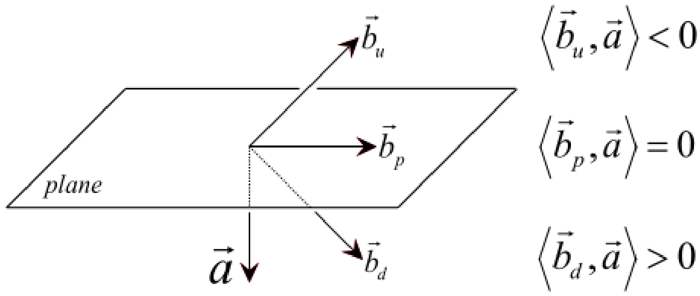

Fig 7. Hyperplane

2.2.3 Negativity

Negativity is a measure of entanglement, which is defined as :

$$\mathcal{N}(\rho) = \frac{\left|\left|\rho^{T_A}\right|\right|_1 - 1}{2} \tag{22}$$

where,

$$\rho^{T_A} \implies \text{ partial transpose of } \rho \text{ with respect to A}$$
$$||X||_1 = Tr|X| = Tr\sqrt{X^\dagger X}$$

This definition in (22) tells us that negativity corresponds to the absolute value of the sum of the negative eigen values of ρ^{T_A}. Thus,

$$\mathcal{N}(\rho) = \sum_i \frac{|\lambda_i| - \lambda_i}{2} \tag{23}$$

λ_i are all the eigen values of ρ^{T_A}.

Example :
Taking an entangled state $|\psi_-\rangle$.

$$\rho = |\psi_-\rangle\langle\psi_-| = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now taking a partial transpose over A :

$$\rho^{T_A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We obtain the eigen values :

$$\lambda = -1, 1, 1, 1$$

Using (23), we get negativity as :

$$\mathcal{N}(\rho) = 1$$

The state is maximally entangled.

2.3 Classification

2.3.1 Distillation

Let us consider a situation in which Alice and Bob wish to set up some quantum communication like teleportation. Alice prepares the an entangled state $|\psi\rangle$ and sends it to Bob. In practical cases, the channel they use for this is noisy, and the state received by Bob is no longer pure, but some mixed state.
If they wish to set-up a quantum communication channel using this entangled state, they would ideally want it to be a pure state. They can achieve it via the process of distillation.

The distillation of entanglement states says that if we have a mixed state which has some inherent entanglement in it, then it is possible to filter out a maximally entangled state using LOCC, with certain finite probability.

Every 2 qubit entangled state can be distilled :
Let us consider a 2 qubit entangled state :

$$|\psi\rangle = \alpha|00\rangle + \beta|11\rangle$$

$\alpha \neq \beta$, so the state is not maximally entangled.
Now Alice can prepare an ancillary qubit $|\phi\rangle$ and add it to the entangled state. She then performs a CNOT operation to obtain :

$$|\phi_{tot}\rangle = \alpha^2|00\rangle_A|0\rangle_B + \alpha\beta|01\rangle_A|1\rangle_B + \alpha\beta|11\rangle_A|0\rangle_B + \beta^2|10\rangle_A|1\rangle_B$$

Now if Alice applies a SWAP Gate, the final state becomes :

$$|\phi_{tot}\rangle = |0\rangle \otimes \left(\alpha^2|00\rangle + \beta^2|11\rangle\right) + \sqrt{2\alpha^2\beta^2}|1\rangle \otimes \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

Now, if Alice does a projective measurement on the ancilla, Bob's state will collapse to a maximally entangled state with probability $\sqrt{2\alpha^2\beta^2}$.

2.3.2 Free and Bound Entanglement

An entangled state that can be distilled is called a Free Entanglement.

An entangled state that cannot be distilled is called a Bound Entanglement.

References

1. Quantum Entanglement: Detection, Classification, and Quantification (Krammer-Diplomarbeit Thesis)

2. An introduction to entanglement measures (Paper by Martin B. Plenio and Shashank Virmani)

3. Entanglement Witness and Geometry of Qubits Bachelorarbeit Verschränkung und Relativistik, Dekohärenz und Geometrie in der Quantenphysik 1 (https://homepage.univie.ac.at/reinhold.bertlmann/pdfs/dipl_diss/Muellner_Bachelorenglish.pdf)

4. Qiskit Website (https://qiskit.org/learn/intro-qc-qh/)

5. Quantiki Website (https://www.quantiki.org/search/node/Category%3AHandbook%2Bof%2BQuantum%2BInformation)

6. A computable measure of entanglement G. Vidal Institut f¨ur Theoretische Physik, Universit¨at Innsbruck, Technikerstraße 25 A-6020 Innsbruck, Austria and R.F. Werner Institut f¨ur Mathematische Physik, TU Braunschweig, Mendelssohnstr. 3, 38304 Braunschweig, Germany (https://arxiv.org/pdf/quant-ph/0102117.pdf)