6th Semester Project

on

Derivation of Vacuum Polarization Tensor

Submitted By

ASHMITA PANDA

School of Physical Sciences

National Institute of Science, Education and Research (NISER), Bhubaneswar

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Under the Guidance of

Dr. Najmul Haque

Reader-F

School of Physical Sciences

National Institute of Science Education and Research (NISER), Bhubaneswar

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Ashmita Panda
3rd Year Integrated M.Sc.
School of Physical Sciences
NISER, Bhubaneswar

Notations

$g_{\mu\nu}$: Components of flat space-time metric tensor Diag=$(1, -1, -1, -1)$

p^μ : Contravariant 4-vector

p_μ : Covariant 4-vector

$\boldsymbol{p}$: 3-vector

$\mathscr{L}$: Lagrangian Density

L : Total Lagrangian $\left(\int d^3x\, \mathscr{L}\right)$

$\mathscr{H}$: Hamiltonian Density

H : Total Hamiltonian $\left(\int d^3x \mathscr{H}\right)$

$\mathscr{T}\ [\ldots]$: Time-ordered Product

$:[\ldots]:$: Normal-ordered Product

$\Delta_F(p)$: Feynman propagator for Scalar field in momentum space

$S_F(p)$: Feynman propagator for Dirac field in momentum space

$D_{\mu\nu}(p)$: Feynman propagator for vector boson field in momentum space

σ^i : Pauli Matrices

ϵ_r^μ : Polarization vector for a vector boson

Γ_μ : General Electromagnetic Vertex

D_μ : Gauge Covariant Derivative

Abstract

Vacuum Polarization is a phenomenon in which a propagating photon forms virtual electron-positron pairs, which disturbs or changes the distribution of charges and currents of the source. The process is quantified by the Vacuum Polarization Tensor.
The process was first discussed by Dirac and Heisenberg in 1934.

The process cannot be described by particle quantum mechanics, as it cannot explain particle creation and annihilation. We need to study Quantum Field Theory.

In this project, the theory of Fermion Fields and Electromagnetic Fields is discussed in detail, along with the concepts of S-Matrix Expansion and Wick's Theorem. The results from these discussions are used to derive the Vacuum Polarisation Tensor with the help of Feynman Diagrams.

Contents

Chapter 1

Quantisation of Dirac Fields

The one particle solution of Klein-Gordan equation gives us negative values of energy of the particle, which cannot be explained physically. It implies that the particle has no ground state, which does not make sense.
The problem arises due to the fact that the hamiltonian of the Klein-Gordan equation is not linear in time, but instead has double derivative of time.

1.1 Dirac Hamiltonian

Dirac tried to find a solution to this problem by attempting to construct a Hamiltonian which would be linear in time and also satisfy the equation :

$$H^2 = \boldsymbol{p}^2 + m^2 \tag{1.1}$$

where,

$$H = i\frac{\partial}{\partial t}$$
$$\boldsymbol{p} = -i\boldsymbol{\nabla}$$

He wrote the Hamiltonian to be of the form :

$$H = \gamma^0 \left(\boldsymbol{\gamma}.\boldsymbol{p} + m\right) \tag{1.2}$$

where, γ^μs are 4×4 matrices.

1.1.1 γ-matrices and their properties

The properties of γ matrices are :

i.) $[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}$

ii.) $\mathrm{Tr}\left(\gamma^{\mu}\right)=0 \quad \forall \mu$

iii.) $\left(\gamma^{\mu}\right)^{\dagger}=\gamma^{0} \gamma^{\mu} \gamma^{0}$

They can be proved as follows :

a.) Proof of $[\boldsymbol{\gamma^{\mu}}, \boldsymbol{\gamma^{\nu}}]_{+}=\boldsymbol{2 g^{\mu \nu}}$

We know :

$$H=\gamma^{0}(\boldsymbol{\gamma} . \boldsymbol{p}+m)=\gamma^{0}\left(\gamma^{i} p_{i}+m\right)$$

and :

$$H^{2}=\boldsymbol{p}^{2}+m^{2}=\left(p_{i}\right)^{2}+m^{2}$$

Squaring the H equation and equating :

$$\begin{aligned}
&\left(p_{i}\right)^{2}+m^{2}=\left(\gamma^{0} \gamma^{i} p_{i}+\gamma^{0} m\right)^{2} \\
\Longrightarrow\ &\left(p_{i}\right)^{2}+m^{2}=\left(\gamma^{0} \gamma^{i} p_{i}+\gamma^{0} m\right)\left(\gamma^{0} \gamma^{j} p_{j}+\gamma^{0} \gamma^{j} p_{j}+\gamma^{0} m\right) \\
\Longrightarrow\ &\left(p_{i}\right)^{2}+m^{2}=\gamma^{0} \gamma^{i} p_{i} \gamma^{0} \gamma^{j} p_{j}+\gamma^{0} \gamma^{i} p_{i} \gamma^{0} m+\gamma^{0} m \gamma^{0} \gamma^{j} p_{j}+\left(\gamma^{0} m\right)^{2} \\
\Longrightarrow\ &\left(p_{i}\right)^{2}+m^{2}=\gamma^{0} \gamma^{i} \gamma^{0} \gamma^{j} p_{i} p_{j}+\gamma^{0} \gamma^{i} \gamma^{0} m p_{i}+\gamma^{0} \gamma^{0} \gamma^{j} m p_{j}+\left(\gamma^{0} m\right)^{2}
\end{aligned}$$

As i, j are just dummy indices, we can write :

$$\begin{aligned}
&\gamma^{0} \gamma^{j} \gamma^{0} \gamma^{i} p_{i} p_{j}=\gamma^{0} \gamma^{i} \gamma^{0} \gamma^{j} p_{i} p_{j} \\
\Longrightarrow\ & 2 \gamma^{0} \gamma^{i} \gamma^{0} \gamma^{j} p_{i} p_{j}=\gamma^{0} \gamma^{i} \gamma^{0} \gamma^{j} p_{i} p_{j}+\gamma^{0} \gamma^{j} \gamma^{0} \gamma^{i} p_{j} p_{i} \\
\therefore\ & \gamma^{0} \gamma^{i} \gamma^{0} \gamma^{j} p_{i} p_{j}=\frac{1}{2}\left[\gamma^{0} \gamma^{i}, \gamma^{0} \gamma^{j}\right]_{+} \qquad (1.3)
\end{aligned}$$

We can also write :

$$\gamma^{0} \gamma^{0} \gamma^{i} m p_{i}=\gamma^{0} \gamma^{0} \gamma^{j} m p_{j} \qquad (1.4)$$

Putting these values back into the equation, we get :

$$\Longrightarrow\left(p_{i}\right)^{2}+m^{2}=\frac{1}{2}\left[\gamma^{0} \gamma^{i}, \gamma^{0} \gamma^{j}\right]_{+} p_{i} p_{j}+\left[\gamma^{0} \gamma^{i}, \gamma^{0}\right]_{+} m p_{i}+\left(\gamma^{0}\right)^{2} m^{2}$$

Comparing term by term :

$$\begin{aligned}
\left(\gamma^{0}\right)^{2}&=1 \qquad (1.5)\\
\left[\gamma^{0} \gamma^{i}, \gamma^{0} \gamma^{j}\right]_{+}&=2 \delta^{i j} \qquad (1.6)\\
\left[\gamma^{0} \gamma^{i}, \gamma^{0}\right]_{+}&=0 \qquad (1.7)
\end{aligned}$$

Expanding (1.7) and using (1.5), we get :

$$\begin{aligned} \gamma^0\gamma^i\gamma^0 + \left(\gamma^0\right)^2\gamma^i &= 0 \\ \implies \gamma^0\gamma^i\gamma^0 + \gamma^i &= 0 \end{aligned}$$

Multiplying on the left by γ^0 :

$$\begin{aligned} \implies \gamma^0\gamma^i\left(\gamma^0\right)^2 + \gamma^i\gamma^0 &= 0 \\ \implies \gamma^0\gamma^i + \gamma^i\gamma^0 &= 0 \\ \therefore \left[\gamma^0, \gamma^i\right]_+ &= 0 \end{aligned} \tag{1.8}$$

Expanding (1.6) we obtain :

$$\begin{aligned} \gamma^0\gamma^i\gamma^0\gamma^i + \gamma^0\gamma^i\gamma^0\gamma^i &= 2 \\ \implies \gamma^0\gamma^i\gamma^0\gamma^i &= 1 \end{aligned}$$

Multiplying with γ^0 from left :

$$\begin{aligned} \implies \left(\gamma^0\right)^2\gamma^i\gamma^0\gamma^i &= \gamma^0 \\ \implies \gamma^i\gamma^0\gamma^i &= \gamma^0 \end{aligned}$$

Multiplying with γ^i from left :

$$\begin{aligned} \implies \left(\gamma^i\right)^2\gamma^0\gamma^i &= \gamma^i\gamma^0 \\ \implies \left(\gamma^i\right)^2\gamma^0\gamma^i &= -\gamma^i\gamma^0 \qquad \text{from (1.8)} \\ \therefore \left(\gamma^i\right)^2 &= -1 \end{aligned} \tag{1.9}$$

Now, for $[\gamma^\mu, \gamma^\nu]_+ = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$:

$$\begin{aligned} \text{For } \mu = 0,\, \nu = 0 : \left[\gamma^0, \gamma^0\right]_+ &= 2\left(\gamma^0\right)^2 = 1 \\ \text{For } \mu = 0,\, \nu = i : \left[\gamma^0, \gamma^i\right]_+ &= 0 \\ \text{For } \mu = i,\, \nu = i : \left[\gamma^i, \gamma^i\right]_+ &= 2\left(\gamma^i\right)^2 = -1 \\ \text{For } \mu = i,\, \nu = 0 : \left[\gamma^i, \gamma^0\right]_+ &= 0 \end{aligned}$$

Thus, proved :

$$\boxed{[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}} \tag{1.10}$$

b.) Proof of $\mathrm{Tr}(\gamma^\mu) = 0 \quad \forall \mu$

Using (1.7), we can write :

$$\gamma^0\gamma^i\gamma^0 + \left(\gamma^0\right)^2\gamma^i = 0$$

$$\implies \gamma^0\gamma^i\gamma^0 + \gamma^i = 0$$

Multiplying first by γ^0 then γ^i from left :

$$\begin{aligned}
\implies \left(\gamma^0\right)^2\gamma^i\gamma^0 + \gamma^0\gamma^i &= 0 \\
\implies \gamma^i\gamma^0 + \gamma^0\gamma^i &= 0 \\
\implies \left(\gamma^i\right)^2\gamma^0 + \gamma^i\gamma^0\gamma^0 &= 0 \\
\implies -\gamma^0 + \gamma^i\gamma^0\gamma^i &= 0
\end{aligned}$$

Thus, we obtain :

$$\therefore \gamma^0 = \gamma^i\gamma^0\gamma^i \tag{1.11}$$
$$\therefore \text{Tr}\left(\gamma^0\right) = \text{Tr}\left(\gamma^i\gamma^0\gamma^i\right) \tag{1.12}$$

Using the properties of Trace :

$$\text{Tr}(ABC) = \text{Tr}(CAB)$$
$$\text{Tr}(cA) = c\text{Tr}(A)$$

We can write :

$$\text{Tr}(\gamma^0) = \text{Tr}\left(\left(\gamma^i\right)^2\gamma^0\right)$$

Using (1.9) :

$$\implies \text{Tr}(\gamma^0) = \text{Tr}(-\gamma^0) = -\text{Tr}(\gamma^0)$$
$$\therefore \text{Tr}(\gamma^0) = 0 \tag{1.13}$$

We can rewrite (1.11) as :

$$\gamma^i = -\gamma^0\gamma^i\gamma^0 \tag{1.14}$$

Using (1.14) and properties of Trace :

$$\text{Tr}\left(\gamma^i\right) = -\text{Tr}\left(\gamma^0\gamma^i\gamma^0\right) = -\text{Tr}\left(\left(\gamma^0\right)^2\gamma^i\right) = \text{Tr}\left(\gamma^i\right)$$
$$\therefore \text{Tr}(\gamma^i) = 0 \tag{1.15}$$

Thus, using (1.13) and (1.15), we can write :

$$\boxed{\text{Tr}\left(\gamma^\mu\right) = 0 \quad \forall\mu} \tag{1.16}$$

c.) Proof of $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$

As the Hamiltonian H is Hermitian, and as

$$H = \gamma^0 \left(\boldsymbol{\gamma}.\boldsymbol{p} + m\right)$$

We know that γ^0 and $\gamma^0\gamma^i$ are Hermitian matrices. So we can write :

$$\begin{aligned} \left(\gamma^i\right)^\dagger &= \left[\gamma^i \left(\gamma^0\right)^2\right]^\dagger = \left[\left(\gamma^i\gamma^0\right)\gamma^0\right]^\dagger \\ \implies \left(\gamma^i\right)^\dagger &= \left(\gamma^0\right)^\dagger \left(\gamma^i\gamma^0\right)^\dagger = \gamma^0\gamma^i\gamma^0 = -\gamma^i \\ \therefore \left(\gamma^i\right)^\dagger &= -\gamma^i \end{aligned} \tag{1.17}$$

Thus, using (1.14) and (1.17), we can write :

$$\boxed{\left(\gamma^\mu\right)^\dagger = \gamma^0\gamma^\mu\gamma^0} \tag{1.18}$$

1.1.2 Defining new matrices using γ matrices

We can define two new matrices using the γ matrices and their properties.

$$\boxed{\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right] = -\sigma^{\nu\mu}} \tag{1.19}$$

There are 6 σ matrices in 4 dimensions due to anti-symmetry in the indices of the matrix.

$$\boxed{\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\varepsilon_{\mu\nu\lambda\rho}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho} \tag{1.20}$$

$\varepsilon_{\mu\nu\lambda\rho}$ is a completely anti-symmetric matrix.
Interchange of any two indices brings in a factor of (-1). So, it has only one independent component. We have the freedom to normalize this to 1.

$$\varepsilon_{0123} = 1$$

We can construct a set of 16 linearly independent matrices using γ^5 and $\sigma^{\mu\nu}$:

$$\boxed{\Gamma = \{1, \gamma^\mu, \sigma^{\mu\nu}, \gamma^5\gamma^\mu, \gamma^5\}} \tag{1.21}$$

We can also think of Γ as a basis which spans a 16-dimensional space. If this has to be valid, each matrix must be 4×4.

1.1.3 Non-uniqueness of γ matrices

The choice of γ matrices is not uinque. If a set of γ^μ matrices satisfy the Dirac equation and all the properties of γ matrices, then another set of matrices given by :

$$\widetilde{\gamma}^\mu = U\gamma^\mu U^\dagger$$

where, U is a unitary matrix, also satisfies the same set of equations and properties.
We can prove that :

i.) $[\widetilde{\gamma}^\mu, \widetilde{\gamma}^\nu]_+ = 2\widetilde{g}^{\mu\nu}$

ii.) $\mathrm{Tr}\,(\widetilde{\gamma}^\mu) = 0$

iii.) $(\widetilde{\gamma}^\mu)^\dagger = \widetilde{\gamma}^0\widetilde{\gamma}^\mu\widetilde{\gamma}^\nu$

a.) Proof that $[\widetilde{\gamma}^\mu, \widetilde{\gamma}^\nu]_+ = 2\widetilde{g}^{\mu\nu}$
We know

$$\widetilde{\gamma}^\mu = U\gamma^\mu U^\dagger$$

So, calculating the anti-commutator :

$$\begin{aligned}
[\widetilde{\gamma}^\mu, \widetilde{\gamma}^\nu]_+ &= \widetilde{\gamma}^\mu\widetilde{\gamma}^\nu + \widetilde{\gamma}^\nu\widetilde{\gamma}^\mu \\
&= U\gamma^\mu U^\dagger U\gamma^\nu U^\dagger + U\gamma^\nu U^\dagger U\gamma^\mu U^\dagger \\
&= U\gamma^\mu\gamma^\nu U^\dagger + U\gamma^\nu\gamma^\mu U^\dagger \\
&= U\left[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right]U^\dagger \\
[\widetilde{\gamma}^\mu, \widetilde{\gamma}^\nu]_+ &= U\left[\gamma^\mu, \gamma^\nu\right]_+ U^\dagger = U\left(2g^{\mu\nu}\right) = 2Ug^{\mu\nu}U^\dagger
\end{aligned}$$

Here, $g^{\mu\nu}$ represents each element of the $g^{\mu\nu}$ matrix, so we can shift it.

$$\therefore [\widetilde{\gamma}^\mu, \widetilde{\gamma}^\nu]_+ = 2g^{\mu\nu}UU^\dagger = 2g^{\mu\nu}$$

Thus,

$$[\widetilde{\gamma}^\mu, \widetilde{\gamma}^\nu]_+ = 2g^{\mu\nu}$$

b.) Proof of Tr $(\widetilde{\gamma}^\mu) = 0$
As we know,

$$\widetilde{\gamma}^\mu = U\gamma^\mu U^\dagger$$

So, using the above relation and properties of Trace :

$$\begin{aligned}
\mathrm{Tr}\,(\widetilde{\gamma}^\mu) &= \mathrm{Tr}\left(U\gamma^\mu U^\dagger\right) \\
\implies \mathrm{Tr}\,(\widetilde{\gamma}^\mu) &= \mathrm{Tr}\left(U^\dagger U\gamma^\mu\right) = \mathrm{Tr}(\gamma^\mu) \\
\therefore \mathrm{Tr}\,(\widetilde{\gamma}^\mu) &= \mathrm{Tr}(\gamma^\mu) = 0
\end{aligned}$$

c.) Proof of $(\widetilde{\gamma}^{\mu})^{\dagger} = \widetilde{\gamma}^{0}\widetilde{\gamma}^{\mu}\widetilde{\gamma}^{0}$

Taking transpose of $\widetilde{\gamma}^{\mu}$ and using its definition :

$$(\widetilde{\gamma}^{\mu})^{\dagger} = \left(U\gamma^{\mu}U^{\dagger}\right)^{\dagger} = \left(U^{\dagger}\right)^{\dagger}(\gamma^{\mu})^{\dagger}\,U^{\dagger} = U\,(\gamma^{\mu})\,U^{\dagger}$$

Using (1.18)

$$\begin{aligned} \implies (\widetilde{\gamma}^{\mu})^{\dagger} &= U\left(\gamma^{0}\gamma^{\mu}\gamma^{0}\right) = \left(U\gamma^{0}U^{\dagger}\right)\left(U\gamma^{\mu}U^{\dagger}\right)\left(U\gamma^{0}U^{\dagger}\right) \\ \therefore (\widetilde{\gamma}^{\mu})^{\dagger} &= \widetilde{\gamma}^{0}\widetilde{\gamma}^{\mu}\widetilde{\gamma}^{0} \end{aligned}$$

So, $\widetilde{\gamma}^{\mu}$ satisfies all the properties of γ matrices.

1.2 Dirac Equation

We know the Dirac Hamiltonian can be written as :

$$H = \gamma^{0}\left(\boldsymbol{\gamma}.\boldsymbol{p} + m\right)$$

Putting in the values of H and $\boldsymbol{p}$ as :

$$\boldsymbol{p} = -i\boldsymbol{\nabla} \qquad H = -i\frac{\partial}{\partial t}$$

We get the equation as :

$$\begin{aligned} i\frac{\partial\psi}{\partial t} &= \gamma^{0}\left(-i\boldsymbol{\gamma}.\boldsymbol{\nabla} + m\right)\psi \\ \implies i\frac{\partial\psi}{\partial t} &= -\gamma^{0}\gamma^{i}\partial_{i}\psi + \gamma^{0}m\psi \end{aligned}$$

Multiplying with γ^{0} from the left on both sides :

$$\begin{aligned} \implies i\gamma^{0}\frac{\partial\psi}{\partial t} &= -i\left(\gamma^{0}\right)^{2}\gamma^{i}\partial_{i}\psi + \left(\gamma^{0}\right)^{2}m\psi \\ \implies \gamma^{0}\partial_{0}\psi &= -i\gamma^{i}\partial_{i}\psi + m\psi \end{aligned}$$

Thus, we can write the Dirac equation as :

$$\boxed{\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x) = 0} \tag{1.22}$$

We can use a more compact notation as :

$$\not{a} \equiv \gamma^{\mu}a_{\mu} \equiv a_{\mu}\gamma^{\mu}$$

In this notation, the Dirac Equation can be written as :

$$(i\partial\!\!\!/ - m)\,\psi(x) = 0$$

The γ matrices are 4×4 matrices. So, for the above to make sense, $\psi(x)$ must be a column matrix with 4 entries. We cannot say it is a 4-vector because we have not looked into the transformation property of $\psi(x)$ yet.

Important: As the choice of γ matrices is not unique, the solution, $\psi(x)$ is also not unique. If the Dirac equation involves $\widetilde{\gamma}^\mu$ of the form :

$$\widetilde{\gamma}^\mu = U\gamma^\mu U^\dagger$$

then, $\widetilde{\psi}(x)$ of the form :

$$\widetilde{\psi}(x) = U\psi(x)$$

also solves the equation.
U is a unitary matrix.

a.) Proof that $\boldsymbol{\widetilde{\psi}(x) = U\psi(x)}$ is a solution of Dirac equation if $\boldsymbol{\gamma^\mu \longrightarrow \widetilde{\gamma}^\mu = U\gamma^\mu U^\dagger}$
U is a unitary matrix.

$$\begin{aligned}\widetilde{\gamma}^\mu &= U\gamma^\mu U^\dagger \\ \widetilde{\psi}(x) &= U\psi(x)\end{aligned}$$

Putting these values in the LHS of the Dirac equation (1.22) :

$$\begin{aligned}\left(i\widetilde{\partial\!\!\!/} - m\right)\widetilde{\psi}(x) &= i\left(\widetilde{\gamma}^\mu\partial_\mu - m\right)\widetilde{\psi}(x) \\ &= \left[iU\gamma^\mu U^\dagger\partial_\mu - m\right]U\psi(x) \\ &= \left[iU\gamma^\mu U^\dagger U\partial_\mu - mU\right]\psi(x) \\ &= \left[iU\gamma^\mu\partial_\mu - Um\right]\psi(x) \\ &= U\left[i\gamma^\mu\partial_\mu - m\right]\psi(x) \\ \therefore \left(i\widetilde{\partial\!\!\!/} - m\right)\widetilde{\psi}(x) &= 0\end{aligned}$$

Thus, $\widetilde{\psi}(x)$ solves the equation.

1.2.1 Relativistic Covariance of Dirac Equation and Spinors

We expect the Dirac equation to be covariant under relativistic transformation. To check if this is true, we need to consider the equation in some other frame of reference :

$$\{x^\mu\} \longrightarrow \{x'^\mu\} = \Lambda^\mu_\nu x^\nu \ ; \quad \partial_\mu \longrightarrow \partial'_\mu = \Lambda^\alpha_\mu \partial_\alpha$$

The Dirac Equation in the new frame is :

$$(i\partial\!\!\!/' - m)\,\psi'(x') = \left(i\gamma^\mu\partial'_\mu - m\right)\psi'(x') = 0$$

The γ matrices are constant matrices, so they would remain invariant.

We will assume a linear relation between $\psi'(x')$ and $\psi(x)$:

$$\psi'(x') = S(\Lambda)\psi(x) \tag{1.23}$$

Putting in the values of x'^μ, ∂'_μ and $\psi'(x')$ in the Dirac equation in new frame :

$$i\gamma^\mu\Lambda^\alpha_\mu\partial_\alpha S(\Lambda)\psi(x) - mS(\Lambda)\psi(x) = 0$$

Multiplying by $S^{-1}(\Lambda)$:

$$\Longrightarrow iS^{-1}(\Lambda)\gamma^\mu\Lambda^\alpha_\mu S(\Lambda)\partial_\alpha\psi(x) - mS^{-1}(\Lambda)S(\Lambda)\psi(x) = 0$$
$$\Longrightarrow i\left[S^{-1}(\Lambda)\gamma^\mu\Lambda^\nu_\mu S(\Lambda)\right]\partial_\nu\psi(x) - m\psi(x) = 0$$

Comparing the above equation with the Dirac equation :

$$i\gamma^\nu\partial_\nu\psi(x) - m\psi(x) = 0$$

We get :

$$S^{-1}(\Lambda)\gamma^\mu\Lambda^\nu_\mu S(\Lambda) = \gamma^\nu \tag{1.24}$$

Now, we are considering infinitesimal transformation given by :

$$\Lambda_{\mu\nu} = g_{\mu\nu} + w_{\mu\nu} \tag{1.25}$$
$$\Longrightarrow \quad \Lambda^\nu_\mu = g^{\nu\alpha}g_{\mu\alpha} + g^{\nu\alpha}w_{\mu\alpha} = \delta^\nu_\mu + g^{\nu\alpha}w_{\mu\alpha}$$
$$\therefore \Lambda^\nu_\mu = \delta^\nu_\mu + g^{\alpha\nu}w_{\mu\alpha} \tag{1.26}$$

Also, $w_{\mu\nu}$ is an anti-symmetric matrix, so $w_{\mu\nu} = -w_{\nu\mu}$.
Putting (1.26) in (1.24):

$$S^{-1}(\Lambda)\gamma^\mu\left(\delta^\nu_\mu + g^{\alpha\nu}w_{\mu\alpha}\right)S(\Lambda) = \gamma^\nu$$
$$\therefore S^{-1}(\Lambda)\gamma^\mu\delta^\nu_\mu S(\Lambda) + S^{-1}(\Lambda)\gamma^\mu g^{\alpha\nu}w_{\mu\alpha}S(\Lambda) = \gamma^\nu \tag{1.27}$$

We can immediately see from above equation (1.27) that if $w_{\mu\nu} = 0$, then equation reduces to :

$$S^{-1}(\Lambda)\gamma^{\mu}\delta^{\nu}_{\mu}S(\Lambda) = \gamma^{\nu}$$

This can only be true of $\mu = \nu$ and $S^{-1}(\Lambda) = S(\Lambda) = 1$.

Our goal is to have a $S(\Lambda)$ which is linear in $w_{\mu\nu}$ and reduces to the identity whenever $w_{\mu\nu} = 0$.
So, we can define $S(\Lambda)$ as :

$$S(\Lambda) = 1 - \frac{i}{4}\beta_{\mu\nu}w^{\mu\nu} \tag{1.28}$$

$\beta_{\mu\nu}$ is some 4×4 anti-symmetric matrix. $w^{\mu\nu}$ is the infinitesimal anti-symmetric matrix for Lorentz transformation.

To solve (1.27) and find the conditions on β matrices, we first need to find $S^{-1}(\Lambda)$.

a.) Finding $S^{-1}(\Lambda)$
As the transformation is infinitesimal and $w^{\mu\nu}$ is an infinitesimal matrix, we can find the inverse by Taylor expansion. We will ignore terms with $\mathcal{O}\left((w^{\mu\nu})^2\right)$.
So,

$$\begin{aligned}
S(\Lambda) &= 1 - \frac{i}{4}\beta_{\mu\nu}w^{\mu\nu}; \qquad \beta_{\mu\nu}w^{\mu\nu} \ll 1 \\
\implies S^{-1}(\Lambda) &= \left[1 - \frac{i}{4}\beta_{\mu\nu}w^{\mu\nu}\right]^{-1} \\
\therefore S^{-1}(\Lambda) &= 1 + \frac{i}{4}\beta_{\mu\nu}w^{\mu\nu}
\end{aligned} \tag{1.29}$$

b.) Finding the conditions to be satisfied by $\beta_{\mu\nu}$ matrices.
Putting in the values of $S(\Lambda)$ and $S^{-1}(\Lambda)$ in (1.27) :

$$\left(1 + \frac{i}{4}\beta_{ab}w^{ab}\right)\gamma^{\mu}\left(\delta^{\nu}_{\mu} + g^{\alpha\nu}w_{\mu\alpha}\right)\left(1 - \frac{i}{4}\beta_{cd}w^{cd}\right) = \gamma^{\nu}$$

To make calculations easier, we wish to convert all $w_{\mu\alpha}$ terms to $w^{\mu\nu}$.

$$\begin{aligned}
w^{\nu}_{\mu} &= g^{\alpha\nu}w_{\mu\alpha} \\
w^{ab} &= g^{\lambda a}w^{b}_{\lambda} \\
\therefore w^{\nu}_{\mu} &= g_{\lambda\mu}w^{\lambda\nu}
\end{aligned}$$

Putting this value in the equation and only keeping terms first order in $w^{\mu\nu}$:

$$\begin{aligned}\left(1+\frac{i}{4}\beta_{ab}w^{ab}\right)\gamma^{\mu}\left(\delta^{\nu}_{\mu}+g_{\lambda\mu}w^{\lambda\nu}\right)\left(1-\frac{i}{4}\beta_{cd}w^{cd}\right)&=\gamma^{\nu}\\ \implies\left(\gamma^{\mu}\delta^{\nu}_{\mu}+\gamma^{\mu}g_{\lambda\mu}w^{\lambda\nu}+\frac{i}{4}\beta_{ab}w^{ab}\gamma^{\mu}\delta^{\nu}_{\mu}\right)\left(1-\frac{i}{4}\beta_{cd}w^{cd}\right)&=\gamma^{\nu}\\ \implies \gamma^{\mu}\delta^{\nu}_{\mu}-\frac{i}{4}\gamma^{\mu}\delta^{\nu}_{\mu}\beta_{cd}w^{cd}+\gamma^{\mu}g_{\lambda\mu}w^{\lambda\nu}+\frac{i}{4}\beta_{ab}w^{ab}\gamma^{\mu}\delta^{\nu}_{\mu}&=\gamma^{\nu}\end{aligned}$$

Changing the dummy indices : a,b,c,d

$$\implies \gamma^{\mu}\delta^{\nu}_{\mu}-\frac{i}{4}\gamma^{\mu}\delta^{\nu}_{\mu}\beta_{\lambda\rho}w^{\lambda\rho}+\gamma^{\mu}g_{\lambda\mu}w^{\lambda\mu}+\frac{i}{4}\beta_{\lambda\rho}w^{\lambda\rho}\gamma^{\mu}\delta^{\nu}_{\mu}=\gamma^{\nu}$$

For $\nu=\mu$:

$$\begin{aligned}\implies \gamma^{\nu}-\frac{i}{4}\gamma^{\nu}\beta_{\lambda\rho}w^{\lambda\rho}+\gamma^{\nu}g_{\lambda\nu}w^{\lambda\nu}+\frac{i}{4}\beta_{\lambda\rho}w^{\lambda\rho}\gamma^{\nu}&=\gamma^{\nu}\\ \implies -\frac{i}{4}\gamma^{\nu}\beta_{\lambda\rho}w^{\lambda\rho}+\gamma_{\lambda}w^{\lambda\nu}+\frac{i}{4}\beta_{\lambda\rho}w^{\lambda\rho}\gamma^{\nu}&=0\\ \implies \left[\gamma^{\nu}\beta_{\lambda\rho}-\beta_{\lambda\rho}\gamma^{\nu}\right]w^{\lambda\rho}+4i\gamma_{\lambda}w^{\lambda\nu}&=0\end{aligned}$$

Using the transformation law $\gamma^{\nu}=\gamma_{\mu}g^{\mu\nu}$, we can write the equation as :

$$\left(g^{\mu\nu}\gamma_{\mu}\beta_{\lambda\rho}-\beta_{\lambda\rho}\gamma_{\mu}g^{\mu\nu}\right)w^{\lambda\rho}=-4i\gamma_{\lambda}w^{\lambda\nu}$$

Multiplying with $g^{\mu\rho}$ from left, non-zero for $\rho=\nu$:

$$\implies \left(\gamma_{\mu\rho}g^{\mu\nu}\gamma_{\mu}\beta_{\lambda\rho}-\beta_{\lambda\rho}\gamma_{\mu}g_{\mu\rho}g^{\mu\nu}\right)w^{\lambda\rho}=-4ig_{\mu\rho}\gamma_{\lambda}w^{\lambda\nu}$$

Setting $\rho=\nu$:

$$\therefore \left(\gamma_{\mu}\beta_{\lambda\rho}-\beta_{\lambda\rho}\gamma_{\mu}\right)w^{\lambda\rho}=-4ig_{\mu\rho}\gamma_{\lambda}w^{\lambda\rho} \tag{1.30}$$

Now, on the RHS, we can interchange the dummy indices.

$$g_{\mu\rho}\gamma_{\lambda}w^{\lambda\rho}=g_{\mu\lambda}\gamma_{\rho}w^{\rho\lambda}$$

Now, using the anit-symmetry of $w^{\lambda\rho}$:

$$\therefore g_{\mu\rho}\gamma_{\lambda}w^{\lambda\rho}=-g_{\mu\lambda}\gamma_{\rho}w^{\lambda\rho} \tag{1.31}$$

Using (1.31) in (1.30) :

$$\begin{aligned}\implies \left(\gamma_{\mu}\beta_{\lambda\rho}-\beta_{\lambda\rho}\gamma_{\mu}\right)w^{\lambda\rho}&=-4ig_{\mu\rho}\gamma_{\lambda}w^{\lambda\rho}=4ig_{\mu\lambda}\gamma_{\rho}w^{\lambda\rho}\\ \therefore \gamma_{\mu}\beta_{\lambda\rho}-\beta_{\lambda\rho}&=-4ig_{\mu\rho}\gamma_{\lambda}=4ig_{\mu\lambda}\gamma_{\rho}\end{aligned}$$

Adding the two equations :

$$2\left[\gamma_\mu, \beta_{\lambda\rho}\right] = 4\left(g_{\mu\lambda}\gamma_\rho - g_{\mu\rho}\gamma_\lambda\right)$$

Thus, finally we obtain :

$$\boxed{\left[\gamma_\mu, \beta_{\lambda\rho}\right] = 2i\left(g_{\mu\lambda}\gamma_\rho - g_{\mu\rho}\gamma_\lambda\right)} \tag{1.32}$$

One valid candidate for β matrix is $\sigma_{\mu\nu}$. It satisfies equation (1.32). So, we can set :

$$\beta_{\mu\nu} = \sigma_{\mu\nu} = \frac{1}{2}\left[\gamma_\mu, \gamma_\nu\right] \tag{1.33}$$

So, the Dirac equation is covariant under infinitesimal relativistic transformation of $\psi(x)$ if it transforms as $\psi'(x') = S(\Lambda)\psi(x)$. We can obtain the condition for covariance for finite transformation by exponentiating $S(\Lambda)$.

Thus, the Dirac equation is covariant if $\psi(x)$ transforms as :

$$\boxed{\psi'(x') = \exp\left(-\frac{i}{4}\sigma_{\mu\nu}w^{\mu\nu}\right)\psi(x)} \tag{1.34}$$

Objects that transform according to the transformation law in (1.34) are called **Spinors**.

1.2.2 Angular Momentum Operator

If we consider angular momentum operators $J_{\mu\nu}$, the transformation law is generated as :

$$\psi'(x) = \left(1 - \frac{i}{2}J_{\mu\nu}w^{\mu\nu}\right)\psi(x) \tag{1.35}$$

Considering infinitesimal transformation parametrized by $w_{\mu\nu}$, we can write :

$$\begin{aligned}
\psi'(x'^\mu) &= \psi'\left(\Lambda^{\mu\nu}x_\nu\right) = \psi'\left[\left(g^{\mu\nu} + w^{\mu\nu}\right)x_\nu\right] \\
&= \psi'\left[g_{\mu\nu}x_\nu\right] + w^{\mu\nu}x_\nu\frac{\partial\psi'(x^\mu)}{\partial x^\mu} + \mathcal{O}\left(\left(w^{\mu\nu}\right)^2\right) \\
\therefore \psi'(x') &= \psi'(x) + w^{\mu\nu}x_\nu\partial_\mu\psi'(x)
\end{aligned} \tag{1.36}$$

We will now to attempt to find the relation between $J_{\mu\nu}$ and $\sigma_{\mu\nu}$.

a.) Finding $J_{\mu\nu}$ in terms of $\sigma_{\mu\nu}$.
Starting from equation (1.36), we will put the value of $\psi'(x)$ in it from (1.35) and use the infinitesimal equation of (1.34).

$$\psi'(x') = \psi'(x) + w^{\mu\nu}x_\nu\partial_\mu\psi'(x)$$

$$\implies \psi'(x') = \left(1 - \frac{i}{2}J_{\mu\nu}w^{\mu\nu}\right)\psi(x) + w^{\mu\nu}x_\nu\partial_\mu\left[\left(1 - \frac{i}{2}J_{\mu\nu}w^{\mu\nu}\right)\psi(x)\right]$$
$$\implies \psi'(x') = \psi(x) - \frac{i}{2}J_{\mu\nu}w^{\mu\nu}\psi(x) + w^{\mu\nu}x_\nu\partial_\mu\psi(x) - \frac{i}{2}w^{\mu\nu}x_\nu J_{\mu\nu}w^{\mu\nu}\partial_\mu\psi(x)$$

Ignoring terms with $(w^{\mu\nu})^2$ and up :

$$\implies \psi'(x') = \left[1 - \frac{i}{2}J_{\mu\nu}w^{\mu\nu} + w^{\mu\nu}x_\nu\partial_\mu\right]\psi(x)$$

Using (1.34), we can replace $\psi'(x')$:

$$\begin{aligned}\left[1 - \frac{i}{4}\sigma_{\mu\nu}w^{\mu\nu}\right]\psi(x) &= \left[1 - \frac{i}{2}J_{\mu\nu}w^{\mu\nu} + w^{\mu\nu}x_\nu\partial_\mu\right]\psi(x)\\ \implies \frac{i}{4}\sigma_{\mu\nu}w^{\mu\nu} &= \left[\frac{i}{2}J_{\mu\nu} - x_\nu\partial_\mu\right]w^{\mu\nu}\\ \implies \frac{i}{4}\sigma_{\mu\nu} &= \frac{i}{2}J_{\mu\nu} - x_\nu\partial_\mu\\ \implies \frac{1}{2}\sigma_{\mu\nu} &= J_{\mu\nu} - \frac{2}{i}x_\nu\partial_\mu = J_{\mu\nu} + 2ix_\nu\partial_\mu\end{aligned} \tag{1.37}$$

As σ and J are anti-symmetric :

$$\begin{aligned}\frac{1}{2}\sigma_{\nu\mu} &= J_{\nu\mu} + 2ix_\mu\partial_\nu = -J_{\mu\nu} + 2ix_\mu\partial_\nu\\ \implies -\frac{\sigma_{\mu\nu}}{2} &= -J_{\mu\nu} + 2ix_\mu\partial_\nu\\ \implies \frac{\sigma_{\mu\nu}}{2} &= J_{\mu\nu} - 2ix_\mu\partial_\nu\end{aligned} \tag{1.38}$$

Adding (1.37) and (1.38), we get the final result as :

$$\boxed{J_{\mu\nu} = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) + \frac{\sigma_{\mu\nu}}{2}} \tag{1.39}$$

The first term in $J_{\mu\nu}$ represents the orbital angular momentum.

We also have an additional second term, which shows that there is an intrinsic angular momentum carried by the solutions of the Dirac equation. We usually refer to this intrinsic angular momentum as **spin**.

1.3 Plane Wave Solutions of Dirac Equation

We would first attempt to solve the Dirac equation for a single particle interpretation.

1.3.1 Positive and Negative Energy Spinors

Starting from the Dirac equation :

$$(i\not{\partial} - m)\,\psi(x) = 0$$

For the single particle interpretation, $\psi(x)$ would be a wave-function.

$$\begin{aligned}(i\gamma^{\mu}\partial_{\mu} - m)\,\psi(x) = 0\\ \implies i\gamma^{0}\partial_{0}\psi(x) + i\gamma^{i}\partial_{i}\psi(x) - m\psi(x) = 0\end{aligned}$$

Dividing throughout by $\psi(x)$:

$$\implies i\frac{\gamma^{0}}{\psi(x)}\partial_{0}\psi(x) + i\frac{\gamma^{i}}{\psi(x)}\partial_{i}\psi(x) - m = 0$$

We have a time derivative and a spatial derivative with a constant. If they must add up to zero, they must both be a constant, with opposite signs. Let E be the eigen value of the equation. It represents the energy of the particle at rest, i.e., $\boldsymbol{p} = 0$.
Let :

$$\begin{aligned}i\frac{\gamma^{0}}{\psi(x)}\partial_{0}\psi(x) = \gamma^{0}E\\ i\frac{\gamma^{i}}{\psi(x)}\partial_{i}\psi(x) - m = -\gamma^{0}E\end{aligned}$$

The time derivative equation solves to give the time dependence as $\exp(-iEt)$. For the spatial derivative part, we get :

$$\begin{aligned}i\gamma^{i}\partial_{i}\psi(x) - m\psi(x) = -\gamma^{0}E\psi(x)\\ i\gamma^{i}\partial_{i}\psi(x) + \left(\gamma^{0}E - m\right)\psi(x) = 0\end{aligned}$$

The first term is a function of $\partial_{i}\psi(x)$ and second term is a function of $\psi(x)$. For the entire term to be zero, each term must go to zero independently. So,

$$\left(\gamma^{0}E - m\right)\psi(x) = 0 \tag{1.40}$$

Now, trying to solve for the eigen values of γ^{0} :

$$\gamma^{0}E\psi(x) = m\psi(x)$$

$$\implies E\gamma^0\psi(x) = m\psi(x)$$

Multiplying on both sides by γ^0 :

$$\implies E\psi(x) = m\gamma^0\psi(x)$$

So, from the above two equations we get :

$$\frac{E}{m} = \frac{m}{E}$$
$$\implies E^2 = m^2$$

Thus, we can write :

$$E = \pm m \tag{1.41}$$

So, the eigen values of γ^0 are ± 1
We can also see that for particle at rest, even the Dirac equation has negative energy solutions.

For a general value of 3-momentum, we can write the solution as :

$$\psi(x) \sim \begin{cases} u_s(\boldsymbol{p}) \exp(-ip.x) \\ v_s(\boldsymbol{p}) \exp(ip.x) \end{cases} \tag{1.42}$$

$u_s(\boldsymbol{p})$ solutions correspond to particles with positive momentum $\boldsymbol{p}$ and with energy $E_p = \sqrt{\boldsymbol{p}^2 + m^2}$.

$v_s(\boldsymbol{p})$ solutions correspond to particles with negative momentum $-\boldsymbol{p}$ and with energy $-E_p = -\sqrt{\boldsymbol{p}^2 + m^2}$.

Now, $u_s(\boldsymbol{p})$ and $v_s(\boldsymbol{p})$ satisfy the relations :

i.) $(\not{p} - m)u_s(\boldsymbol{p}) = 0$

ii.) $(\not{p} + m)v_s(\boldsymbol{p}) = 0$

We can prove them as follows :

a.) Proof of $(\not{p} - m)u_s(p) = 0$ and $(\not{p} + m)v_s(p) = 0$

Putting the value of $\psi(x)$:

$$\psi(x) \sim u_s(\boldsymbol{p}) \exp(-ip.x)$$

in the Dirac equation.

$$
\begin{aligned}
(i\gamma^\mu \partial_\mu - m)\, u_s(\boldsymbol{p}) \exp(-ip.x) &= 0 \\
\implies \{i\gamma^\mu \partial_\mu [\exp(-ip.x)] - m \exp(-ip.x)\}\, u_s(\boldsymbol{p}) &= 0 \\
\implies \{i\gamma^\mu \partial_\mu [\exp(-ip_\mu x^\mu)] - m \exp(-ip_\mu x^\mu)\}\, u_s(\boldsymbol{p}) &= 0 \\
\implies \{i\gamma^\mu (-i) p_\mu \exp(-ip_\mu x^\mu) - m \exp(-ip_\mu x^\mu)\}\, u_s(\boldsymbol{p}) &= 0 \\
\implies (\gamma^\mu p_\mu - m) \exp(-ip.x) u_s(\boldsymbol{p}) &= 0 \\
\therefore (\not{p} - m)\, u_s(\boldsymbol{p}) &= 0
\end{aligned}
$$

We can repeat the same procedure for $v_s(\boldsymbol{p})$, and thus we get :

$$
\boxed{
\begin{aligned}
(\not{p} - m)\, u_s(\boldsymbol{p}) = 0 \qquad (1.43)\\
(\not{p} + m)\, v_s(\boldsymbol{p}) = 0 \qquad (1.44)
\end{aligned}
}
$$

For the spinors, the Dirac Conjugation is a more useful concept of normal conjugation, as we cannot form a Lorentz scalar with $\psi^\dagger \psi$. The Dirac Conjugation is defined as :

$$\overline{\psi} \equiv \psi^\dagger \gamma^0 \tag{1.45}$$

Using the definition above in (1.46), we can see that $\overline{u}_s(\boldsymbol{p})$ and $\overline{v}_s(\boldsymbol{p})$ satisfy :

i.) $\overline{u}_s(\boldsymbol{p})\, (\not{p} - m) = 0$

ii.) $\overline{v}_s(\boldsymbol{p})\, (\not{p} + m) = 0$

b.) Proof of $\overline{u}_s(p)\, (\not{p} - m) = 0$

For $u_s(\boldsymbol{p})$:

$$(\not{p} - m)\, u_s(\boldsymbol{p}) = 0$$

Taking Dirac Conjugation of the above equation :

$$
\begin{aligned}
\implies [(\not{p} - m)\, u_s(\boldsymbol{p})]^\dagger \gamma^0 &= 0 \\
\implies u_s^\dagger(\boldsymbol{p})\, (\not{p} - m)^\dagger \gamma^0 &= 0
\end{aligned}
$$

Evaluating $(\not{p} - m)^\dagger \gamma^0$:

$$
\begin{aligned}
(\not{p} - m)^\dagger \gamma^0 &= (p_\mu \gamma^\mu)^\dagger \gamma^0 - m\gamma^0 \\
&= \gamma^0 \gamma^\mu \gamma^0 p_\mu \gamma^0 - m\gamma^0 \\
&= \gamma^0 \gamma^\mu p_\mu (\gamma^0)^2 - m\gamma^0 \\
\therefore (\not{p} - m)^\dagger \gamma^0 &= \gamma^0 \gamma^\mu p_\mu - \gamma^0 m = \gamma^0 (\not{p} - m)
\end{aligned}
$$

Thus,

$$u_s^\dagger(\boldsymbol{p})\gamma^0\left(\not{p}-m\right)=0$$

We can do the similar calculation for $v_s(\boldsymbol{p})$, and we obtain :

$$\boxed{\begin{aligned}\overline{u}_s(\boldsymbol{p})\left(\not{p}-m\right)=0\\ \overline{v}_s(\boldsymbol{p})\left(\not{p}+m\right)=0\end{aligned}} \qquad \begin{aligned}(1.46)\\(1.47)\end{aligned}$$

The normalisation condition of spinors are given as :

$$\boxed{\begin{aligned}u_r^\dagger(\boldsymbol{p})u_s(\boldsymbol{p})=v_r^\dagger(\boldsymbol{p})v_s(\boldsymbol{p})=2E_p\delta_{rs}\\ v_r^\dagger(\boldsymbol{p})u_s(-\boldsymbol{p})=u_r^\dagger(\boldsymbol{p})v_s(-\boldsymbol{p})=0\end{aligned}} \qquad \begin{aligned}(1.48)\\(1.49)\end{aligned}$$

1.3.2 Explicit solutions in Dirac-Pauli representation

Even though the Dirac equation can be solved in a representation independent way, but we can also solve for plane wave solutions of a single particle state in a particular representation. Here, we have chosen the Dirac-Pauli representation given by :

$$\gamma^0=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix} \quad ; \quad \gamma^i=\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix} \tag{1.50}$$

σ matrices are the Pauli matrices :

$$\sigma^1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad ; \quad \sigma^2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \quad ; \quad \sigma^3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{1.51}$$

In this representation, we can calculate $(\not{p}-m)$:

$$\begin{aligned}(\not{p}-m)&=\gamma^0E_p-\gamma^ip_i-m\\ \therefore(\not{p}-m)&=\begin{pmatrix}E_p-m & -\sigma^ip_i\\ \sigma^ip_i & -E_p-m\end{pmatrix}\end{aligned}$$

Before we proceed further, we need to check if solutions to the above equation exist. To check the same, we take the determinant of the above matrix.

$$\begin{aligned}M&=\begin{pmatrix}E_p-m & -\sigma^ip_i\\ \sigma^ip_i & -E_p-m\end{pmatrix}\\ \det(M)&=-\left(E_p-m\right)\left(E_p+m\right)+\left(\sigma^ip_i\right)^2\end{aligned}$$

$$\implies \det(M) = -\left(E_p^2 - m^2\right) + \left(\sigma^i p_i\right)^2$$

As $E_p^2 = \boldsymbol{p}^2 + m^2$

$$\therefore \det(M) = -\left(\boldsymbol{p}\right)^2 + \left(\sigma^i p_i\right)^2$$

Calculating the value of the second term :

$$\sigma^i p_i = \begin{pmatrix} 0 & p_1 \\ p_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -ip_2 \\ ip_2 & 0 \end{pmatrix} + \begin{pmatrix} p_3 & 0 \\ 0 & -p_3 \end{pmatrix} = \begin{pmatrix} p_3 & p_1 - ip_2 \\ ip_2 + p_1 & -p_3 \end{pmatrix}$$

$$\left(\sigma^i p_i\right)^2 = \begin{pmatrix} p_1^2 + p_2^2 + p_3^2 & p_3p_1 - ip_2p_3 - p_1p_3 + ip_2p_3 \\ ip_3p_2 + p_1p_3 - ip_2p_3 - p_1p_3 & p_1^2 + p_2^2 + p_3^2 \end{pmatrix}$$

$$\therefore \left(\sigma^i p_i\right)^2 = \boldsymbol{p}^2 I$$

Thus,

$$\det(M) = -\boldsymbol{p}^2 + \boldsymbol{p}^2 = 0$$

So, solution to above equation exists, and we can proceed with the calculations.

Let :

$$u \equiv \begin{pmatrix} \phi_t \\ \phi_b \end{pmatrix}$$

Now, calculating :

$$(\not{p} - m)\, u(\boldsymbol{p}) = \begin{pmatrix} E_p - m & -\sigma^i p_i \\ \sigma^i p_i & -E_p - m \end{pmatrix} \begin{pmatrix} \phi_t \\ \phi_b \end{pmatrix}$$

We get a set of two equations :

$$\begin{aligned} (E_p - m)\, \phi_t - \sigma^i p_i \phi_b &= 0 \\ \sigma^i p_i \phi_t - (E_p + m)\, \phi_b &= 0 \end{aligned}$$

As the solutions exist, we can write it as :

$$\phi_b = \frac{\boldsymbol{\sigma.p}}{E_p + m} \phi_t$$

ϕ_t is a two component column vector. It has two solutions. We can choose it to be proportional to :

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ; \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

ϕ_b also has two independent solutions, which can be written as :

$$\phi_b = \frac{\boldsymbol{\sigma.p}}{E_p + m}\chi_\pm$$

So, the four independent solutions of u-spinor can be written as :

$$\boxed{u_\pm(\boldsymbol{p}) = \sqrt{E_p + m}\begin{pmatrix} \chi_\pm \\ \frac{\boldsymbol{\sigma.p}}{E_p + m}\chi_\pm \end{pmatrix}} \tag{1.52}$$

Similarly for v-spinors :

$$\boxed{u_\pm(\boldsymbol{p}) = \pm\sqrt{E_p + m}\begin{pmatrix} \frac{\boldsymbol{\sigma.p}}{E_p + m}\chi_\mp \\ \chi_\mp \end{pmatrix}} \tag{1.53}$$

1.4 Lagrangian for the Dirac Field

We saw in the previous section that the single particle interpretation of Dirac's equation also has negative energy states. Now, we wish to go over the field interpretation to solve the issue.

The Dirac equation can be derived from the Lagrangian :

$$\mathcal{L} = \overline{\psi}\,(i\partial\!\!\!/ - m)\,\psi \tag{1.54}$$

$\overline{\psi}$ and ψ are spinors.

Putting in spinor component indices explicitly, we get the Lagrangian as :

$$\mathcal{L} = \overline{\psi}_i\left[i(\gamma^\mu)^{ij}\partial_\mu - m\delta^{ij}\right]\psi_j$$

The components of $\overline{\psi}$ are linear combinations of components of ψ. So, we can say that ψ and $\overline{\psi}$ are independent.

1.4.1 Euler-Lagrange equations

We can write the Euler-Lagrange equations for $\overline{\psi}$ as :

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\overline{\psi}\right)}\right] = \frac{\partial\mathcal{L}}{\partial\overline{\psi}}$$

LHS=0, as $\mathcal{L}$ has no dependency on $\partial_\mu \overline{\psi}$:

$$\implies (i\partial\!\!\!/ - m)\,\psi = 0$$

We recover the Dirac's equation. We can also recover Dirac's equation from Euler-Lagrange equations for ψ.

a.) Dirac's equation from Euler-Lagrange equations for ψ.

Euler-Lagrange equations for ψ are given as :

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \psi\right)}\right] = \frac{\partial \mathcal{L}}{\partial \psi}$$

$$\mathcal{L} = \overline{\psi}\left[i\left(\gamma^\mu\right)\partial_\mu \psi - m\delta\psi\right]$$

Calculating LHS and RHS separately :

$$\begin{aligned} \text{LHS} &= \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \psi\right)}\right] = \partial_\mu \left[\overline{\psi} i(\gamma^\mu)\right] \\ \text{RHS} &= \frac{\partial \mathcal{L}}{\partial \psi} = -\overline{\psi} m\delta \end{aligned}$$

So, we can write the equation now as :

$$\begin{aligned} i\gamma^\mu \partial_\mu \overline{\psi} + \overline{\psi} m &= 0 \\ \therefore \overline{\psi}\left(i\overleftarrow{\partial\!\!\!/} + m\right) &= 0 \end{aligned} \tag{1.55}$$

Taking Hermitian conjugate of the above equation (1.55) :

$$\begin{aligned} \left[\overline{\psi}\left(i\overleftarrow{\partial\!\!\!/} + m\right)\right]^\dagger &= 0 \\ \implies \left[\overline{\psi}\left(i\gamma^\mu \overleftarrow{\partial}_\mu + m\right)\right]^\dagger &= 0 \\ \implies \left(i\gamma^\mu \overleftarrow{\partial}_\mu + m\right)^\dagger \left(\overline{\psi}\right)^\dagger &= 0 \\ \implies \left[-i\overleftarrow{\partial}^\dagger_\mu (\gamma^\mu)^\dagger + m\right]\gamma^0\psi &= 0 \\ \implies \left[-i\partial_\mu \gamma^0 \gamma^\dagger \gamma^0 + m\right]\gamma^0\psi &= 0 \\ \implies \left[-i\partial_\mu \gamma^0 \gamma^\mu + m\gamma^0\right]\psi &= 0 \\ \implies \left[-i\gamma^0 \partial\!\!\!/ + \gamma^0 m\right]\psi &= 0 \end{aligned}$$

Multiplying from the left by γ^0, we obtain the final equation :

$$(i\partial\!\!\!/ - m)\,\psi = 0$$

1.4.2 Hermiticity of the Lagrangian

As we are trying to solve the Dirac equation for quantum fields, the fields themselves are elevated to the level of operators. Thus, the Lagrangian must be Hermitian.

But, the Dirac lagrangian that we are using is not Hermitian.

a.) Checking if Dirac lagrangian is Hermitian.
Taking the Hermitian conjugate of the Dirac lagrangian :

$$\begin{aligned} \left[i\overline{\psi}\gamma^\mu\partial_\mu\psi + \overline{\psi}m\psi\right]^\dagger &= 0 \\ \implies \left(i\overline{\psi}\gamma^\mu\partial_\mu\psi\right)^\dagger + \left(\overline{\psi}m\psi\right)^\dagger &= 0 \\ \implies \left[-i\left(\partial_\mu\psi\right)^\dagger\left(\gamma^\mu\right)^\dagger\left(\overline{\psi}\right)^\dagger\right] + \psi^\dagger m\overline{\psi}^\dagger &= 0 \end{aligned}$$

Using the relations $\overline{\psi} = \psi^\dagger\gamma^0$ and $\left(\overline{\psi}\right)^\dagger = \gamma^0\psi$:

$$\begin{aligned} \implies -i\left(\partial_\mu\psi\right)^\dagger\gamma^0\gamma^\mu\gamma^0\gamma^0\psi + \psi^\dagger m\gamma^0\psi &= 0 \\ \therefore -i\left(\overline{\psi}\overleftarrow{\partial}_\mu\right)\gamma^\mu\psi + \overline{\psi}m\psi &= 0 \end{aligned}$$

This shows that the Lagrangian is not hermitian.
The Hermitian Lagrangian is given us :

$$\mathcal{L}' = \frac{i}{2}\overline{\psi}\gamma^\mu\partial_\mu\psi - \frac{i}{2}\left(\partial_\mu\overline{\psi}\right)\gamma^\mu\psi - m\overline{\psi}\psi$$

But taking the difference of the non-hermitian and hermitian lagrangian we can see that :

$$\begin{aligned} \mathcal{L} - \mathcal{L}' &= \frac{i}{2}\overline{\psi}\gamma^\mu\partial_\mu\psi - \frac{i}{2}\left(\partial_\mu\overline{\psi}\right)\gamma^\mu\psi \\ \therefore \mathcal{L} - \mathcal{L}' &= \partial_\mu\left(\frac{i}{2}\overline{\psi}\gamma^\mu\psi\right) \end{aligned}$$

The difference of the two lagrangians is a total divergence term. Thus, the two lagrangians are equivalent and we can use $\mathcal{L}$ without any issues.

1.4.3 Conserved quantities

The Dirac Larangian remains invariant under the transformation :

$$\psi \longrightarrow \exp(-iq\theta)\psi$$

θ is a global parameter which does not depend on the space-time.
We can derive the conserved current using Noether's theorem.

a.) Derivation of Noether's current for the given symmetry.

The Lagrangian for the Dirac field is conserved under the transformations :

$$\begin{aligned}\psi &\longrightarrow \exp(-iq\theta)\psi \\ \overline{\psi} &\longrightarrow \exp(iq\theta)\overline{\psi}\end{aligned}$$

For infinitesimal θ :

$$\begin{aligned}\delta\psi &= -iq\theta\psi \\ \delta\overline{\psi} &= iq\theta\overline{\psi}\end{aligned}$$

So, we can find the conserved current as :

$$J^{\mu} = \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\psi_{\alpha}\right)}\delta\psi_{\alpha} \tag{1.56}$$

We know :

$$\mathcal{L} = \overline{\psi}\left(i\gamma^{\mu}\partial_{\mu}\psi - m\psi\right)$$

So, computing :

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\overline{\psi}\right)}\delta\overline{\psi} &= 0 \\ \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\psi\right)}\delta\psi &= i\overline{\psi}\gamma^{\mu}\end{aligned}$$

Thus, we get the conserved current as :

$$\begin{aligned}J^{\mu} &= \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\overline{\psi}\right)}\delta\overline{\psi} + \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\psi\right)}\delta\psi \\ &= 0 + i\overline{\psi}\gamma^{\mu}\left(-iq\psi\right) \\ \therefore J^{\mu} &= q\overline{\psi}\gamma^{\mu}\psi \end{aligned}\tag{1.57}$$

The conserved charge is given as :

$$Q = \int d^3x J^0 \tag{1.58}$$

Thus, we get conserved charge as :

$$\begin{aligned}Q &= \int d^3x\, q\overline{\psi}\gamma^0\psi = q\int d^3x\, \overline{\psi}\gamma^0\psi \\ &= q\int d^3x\, \psi^{\dagger}\gamma^0\gamma^0\psi = q\int d^3x \psi^{\dagger}\psi\end{aligned}$$

$$\therefore Q = q \int d^3x\, \psi^\dagger \psi \tag{1.59}$$

1.5 Fourier Decomposition of the Field

1.5.1 Fourier decomposition of the scalar field

To find the fourier decomposition of the Dirac field, we will use the same form of the decomposition as for scalar fields, given by :

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left[a(p)\exp(-ip.x) + a^\dagger \exp(ip.x)\right] \tag{1.60}$$

But before using the form, we wish to derive it.

a.) Derivation of the fourier decomposition of Scalar fields

For classical free fields, we can write the fourier decomposition as :

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int d^4p\, \delta\left(p^2 - m^2\right) A(p)\exp(-ip.x) \tag{1.61}$$

where,

$$\begin{aligned} A(p) &\implies \text{Fourier components} \\ \frac{1}{(2\pi)^{3/2}} &\implies \text{Normalisation constant} \\ \delta\left(p^2 - m^2\right) &\implies \text{Ensures } \phi(x) \text{ satisfies the Klein-Gordan equation} \end{aligned}$$

If we now consider quantum fields, then $\phi(x)$ is elevated to the level of operators, and it must be Hermitian, i.e., $\phi(x) = \phi^\dagger(x)$. So,

$$\phi^\dagger(x) = \frac{1}{(2\pi)^{3/2}} \int d^4p\, \delta\left(p^2 - m^2\right) A^\dagger(p)\exp(ip.x)$$

Now, using the Hermiticity condition, and the fact that (1.61) is also true for $-p$, we get :

$$A(-p) = A^\dagger(p) \tag{1.62}$$

We will now define the step-function Θ as :

$$\Theta(z) = \begin{cases} 1 & \text{if } z > 0 \\ \frac{1}{2} & \text{if } z = 0 \\ 0 & \text{if } z < 0 \end{cases} \tag{1.63}$$

Now, from the above defintion we can see :

$$\Theta(p^0) + \Theta(-p^0) = 1$$

Using this, we can rewrite (1.61) as :

$$\begin{aligned}
\phi(x) &= \frac{1}{(2\pi)^{3/2}} \int d^4p\, \delta\left(p^2 - m^2\right) .1.A(p) \exp(-ip.x) \\
&= \frac{1}{(2\pi)^{3/2}} \int d^4p\, \delta\left(p^2 - m^2\right) \left[\Theta(p^0) + \Theta(-p^0)\right] A(p) \exp(-ip.x) \\
&= \frac{1}{(2\pi)^{3/2}} \int d^4p\, \delta\left(p^2 - m^2\right) \left[\Theta(p^0)A(p) \exp(-ip.x) + \Theta(-p^0)A(-p) \exp(ip.x)\right]
\end{aligned}$$

As we want to avoid states with negative energy, writing the form of $\phi(x)$ for $p^0 > 0$, and using (1.62), we get :

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int d^4p\, \delta\left(p^2 - m^2\right) \Theta(p^0) \left[A(p) \exp(-ip.x) + A^{\dagger}(p) \exp(ip.x)\right] \tag{1.64}$$

We now need to use a result :

$$\delta\left(f(z)\right) = \sum_n \frac{\delta\left(z - z_n\right)}{|df/dz|_{z=z_n}} \tag{1.65}$$

provided, $\dfrac{df}{dz} \neq 0$ at $z = z_n$.

Now as $E_p^2 = \boldsymbol{p}^2 + m^2$, we can write using (1.65) :

$$\begin{aligned}
\delta\left(p^2 - m^2\right) &= \delta\left((p^0)^2 - E_p^2\right) \\
\delta\left(p^2 - m^2\right) &= \frac{\delta\left(p^0 - E_p\right)}{\dfrac{d\left((p^0)^2\right)}{dp}} + \frac{\delta\left(p^0 + E_p\right)}{\dfrac{d\left((p^0)^2\right)}{dp}} \\
\therefore \delta\left(p^2 - m^2\right) &= \frac{1}{2|p^0|}\left[\delta\left(p^0 - E_p\right) + \delta\left(p^0 + E_p\right)\right]
\end{aligned} \tag{1.66}$$

Now, on substituting (1.66) in (1.64), we would see that the second δ term of (1.66) is always zero, as $p^0 > 0$ always. E_p is also always positive. Thus, the equation becomes :

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int d^4p. \frac{1}{2|p^0|}.\Theta(p^0)\delta\left(p^0 - E_p\right) \left[A(p) \exp(-ip.x) + A^{\dagger}(p) \exp(ip.x)\right]$$

We obtain the final equation as :

$$\boxed{\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2E_p}} \left[a(p) \exp(-ip.x) + a^{\dagger}(p) \exp(ip.x)\right]} \tag{1.67}$$

where,

$$a(p) = \frac{A(p)}{\sqrt{2E_p}}$$

1.5.2 Fourier decomposition of the Dirac field

From (1.67), we can write the Dirac fields as :

$$\boxed{\begin{aligned}\psi(x) &= \frac{1}{(2\pi)^{3/2}}\int \frac{d^3p}{\sqrt{2E_p}}\sum_{s=1,2}\left[f_s(\boldsymbol{p})u_s(\boldsymbol{p})\exp(-ip.x) + \hat{f}_s^\dagger(\boldsymbol{p})v_s(\boldsymbol{p})\exp(ip.x)\right] \quad (1.68)\\ \overline{\psi}(x) &= \frac{1}{(2\pi)^{3/2}}\int \frac{d^3p}{\sqrt{2E_p}}\sum_{s=1,2}\left[f_s^\dagger(\boldsymbol{p})\overline{u}_s(\boldsymbol{p})\exp(ip.x) + \hat{f}_s(\boldsymbol{p})\overline{v}_s(\boldsymbol{p})\exp(-ip.x)\right] \quad (1.69)\end{aligned}}$$

where,

$s = 1, 2$: as each u_s and v_s has two independent solutions

$f_s, f_s^\dagger \implies$ annihilation and creation operator for u (+ve $\boldsymbol{p}$ solutions) respectively

$\hat{f}_s, \hat{f}_s^\dagger \implies$ annihilation and creation operator for v (-ve $\boldsymbol{p}$ solutions) respectively

The Hamiltonian is given as :

$$H = \int d^3x \left(\Pi_A\Phi^A - \mathcal{L}\right) = \int d^3x \left[\frac{\delta\mathcal{L}}{\delta\dot{\psi}(x)}\dot{\psi}(x)\right] - L \tag{1.70}$$

From (1.54), we can find the values as :

$$\frac{\delta\mathcal{L}}{\delta\dot{\psi}} = \overline{\psi}(i\gamma^0)$$

and :

$$\frac{\delta\mathcal{L}}{\delta\dot{\psi}}\dot{\psi} = \overline{\psi}\left(i\gamma^0\dot{\psi}\right)$$

Using the above two values, we can calculate :

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta\dot{\psi}}\dot{\psi} - \mathcal{L} &= \overline{\psi}\left(i\gamma^0\dot{\psi}\right) - \overline{\psi}\left(i\gamma^0\dot{\psi}\right) - \overline{\psi}\left(i\gamma^i\nabla_i\psi\right) + \overline{\psi}m\psi\\ \therefore \frac{\delta\mathcal{L}}{\delta\dot{\psi}}\dot{\psi} - \mathcal{L} &= \overline{\psi}\left[-i\boldsymbol{\gamma}.\boldsymbol{\nabla} + m\right]\psi\end{aligned}$$

Thus, the Hamiltonian can be written as :

$$H = \int d^3x\,\overline{\psi}\left(-i\boldsymbol{\gamma}.\boldsymbol{\nabla} + m\right)\psi \tag{1.71}$$

Before we calculate the explicit value of the Hamiltonian, we need to find the action of $(-i\boldsymbol{\gamma}.\boldsymbol{\nabla})$ operator on ψ.

a.) Finding the action of $(-i\boldsymbol{\gamma}.\boldsymbol{\nabla})$ on $\boldsymbol{u_s(p)}\exp\left(-\boldsymbol{ip.x}\right)$ and $\boldsymbol{v_s(p)}\exp\left(\boldsymbol{ip.x}\right)$.
We can write $(-i\boldsymbol{\gamma}.\boldsymbol{\nabla})$ as $(\boldsymbol{\gamma}.\boldsymbol{p}+m)$ as $\boldsymbol{p}=-i\nabla$.
Using the definition of u-spinor from (1.43) :

$$
\begin{aligned}
(\not p - m)\, u_s(\boldsymbol{p}) &= 0 \\
\implies (\gamma^\mu p_\mu - m)\, u_s(\boldsymbol{p}) &= 0 \\
\implies \left(\gamma^0 p_0 - \gamma^i p_i - m\right) u_s(\boldsymbol{p}) &= 0 \\
\implies -\gamma^i p_i u_s(\boldsymbol{p}) &= \left(m - \gamma^0 p_0\right) u_s(\boldsymbol{p}) \\
\implies -\gamma^i p_i u_s(\boldsymbol{p}) &= \left(m - \gamma^0 E_p\right) u_s(\boldsymbol{p}) \\
\therefore \left(\gamma^i p_i + m\right) u_s(\boldsymbol{p}) \exp(-ip.x) &= \gamma^0 E_p u_s(\boldsymbol{p}) \exp(-ip.x)
\end{aligned}
$$

We can find similar expression for $v_s(\boldsymbol{p})$ following similar steps starting from (1.44). Thus, we get the final equations as :

$$
\boxed{
\begin{aligned}
(-\boldsymbol{\gamma}.\boldsymbol{\nabla} + m)\, u_s(\boldsymbol{p}) \exp(-ip.x) &= \gamma^0 E_p u_s(\boldsymbol{p}) \exp(-ip.x) \qquad (1.72)\\
(-\boldsymbol{\gamma}.\boldsymbol{\nabla} + m)\, v_s(\boldsymbol{p}) \exp(ip.x) &= -\gamma^0 E_p v_s(\boldsymbol{p}) \exp(ip.x) \qquad (1.73)
\end{aligned}
}
$$

We can now derive the Hamiltonian.

b.) Derivation of the explicit form of the Hamiltonian
We know the equations for $\psi(x)$ and $\overline{\psi}$ from (1.68) and (1.69). Thus, calculating :

$$
\begin{aligned}
\overline{\psi}\left[-i\boldsymbol{\gamma}.\boldsymbol{\nabla} + m\right]\psi &= \frac{\overline{\psi}}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2E_p}} \sum_{s=1,2} \Big[f_s(\boldsymbol{p})\, (-i\boldsymbol{\gamma}.\boldsymbol{\nabla} + m)\, u_s(\boldsymbol{p}) \exp(-ip.x) \\
&\qquad\qquad + \hat{f}_s^\dagger(\boldsymbol{p})\, (-i\boldsymbol{\gamma}.\boldsymbol{\nabla} + m)\, v_s(\boldsymbol{p}) \exp(ip.x) \Big] \\
&= \frac{\overline{\psi}}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2E_p}} \sum_{s=1,2} \left[f_s(\boldsymbol{p}) \gamma^0 E_p u_s(\boldsymbol{p}) \exp(-ip.x) - \hat{f}_s^\dagger(\boldsymbol{p}) \gamma^0 v_s(\boldsymbol{p}) \exp(ip.x) \right] \\
&= \frac{1}{(2\pi)^3} \int \frac{d^3p'}{\sqrt{2E_{p'}}} \\
&\qquad \times \int \frac{d^3p}{\sqrt{2E_p}} E_p \sum_{s,s'=1,2} \left[f_{s'}^\dagger(\boldsymbol{p'}) \overline{u}_{s'}(\boldsymbol{p'}) \exp(ip'.x) + \hat{f}_{s'}(\boldsymbol{p'}) \overline{v}_{s'}(\boldsymbol{p'}) \exp(-ip'.x) \right] \\
&\qquad\qquad \times \gamma^0 \left[f_s(\boldsymbol{p}) u_s(\boldsymbol{p}) \exp(-ip.x) - \hat{f}_s^\dagger(\boldsymbol{p}) v_s(\boldsymbol{p}) \exp(ip.x) \right]
\end{aligned}
$$

$$\therefore \overline{\psi}\left[-i\boldsymbol{\gamma}.\boldsymbol{\nabla}+m\right]\psi = \frac{1}{(2\pi)^3}\int\frac{d^3p'}{\sqrt{2E_{p'}}}$$
$$\times\int\frac{d^3p}{\sqrt{2E_p}}E_p\sum_{s,s'=1,2}\left[f^\dagger_{s'}(\boldsymbol{p'})u^\dagger_{s'}(\boldsymbol{p'})\exp(ip'.x)+\hat{f}_{s'}(\boldsymbol{p'})v^\dagger_{s'}(\boldsymbol{p'})\exp(-ip'.x)\right]$$
$$\times\left[f_s(\boldsymbol{p})u_s(\boldsymbol{p})\exp(-ip.x)-\hat{f}^\dagger_s(\boldsymbol{p})v_s(\boldsymbol{p})\exp(ip.x)\right]$$

Now, using the normalisation conditions given in (1.48) and (1.49), we can see that the cross terms will yield 0, and the $u^\dagger u$, $v^\dagger v$ terms will yield $2E_p\delta_{s's}$, if the momenta are the same. It kills the summation over s'. Putting this in the equation for Hamiltonian (1.71), we get :

$$H = \int d^3x\overline{\psi}\left(-i\boldsymbol{\gamma}.\boldsymbol{\nabla}+m\right)\psi$$
$$\implies H = \int\frac{d^3x}{(2\pi)^3}\int\frac{d^3p'}{\sqrt{2E_{p'}}}$$
$$\times\int\frac{d^3p}{\sqrt{2E_p}}E_p\sum_{s,s'=1,2}\Big[f^\dagger_{s'}(\boldsymbol{p'})f_s(\boldsymbol{p})u^\dagger_{s'}(\boldsymbol{p'})u_s(\boldsymbol{p})\exp\left(-i(p-p').x\right)$$
$$-\hat{f}_{s'}(\boldsymbol{p})\hat{f}^\dagger_s(\boldsymbol{p})v^\dagger_{s'}(\boldsymbol{p'})v_s(\boldsymbol{p})\exp\left(-i(p'-p).x\right)\Big]$$

Now, we can take the integration over space coordinates inside the summation, and use the definition of delta function as :

$$\delta^3(\boldsymbol{p}-\boldsymbol{p'}) = \frac{1}{(2\pi)^3}\int\exp\left(-i(\boldsymbol{p}-\boldsymbol{p'}).\boldsymbol{x}\right)d^3x \tag{1.74}$$

For $\exp(-i(p-p').x)$, we can write :

$$\frac{1}{(2\pi)^3}\int\exp\left(-i(p-p').x\right)d^3x = \frac{1}{(2\pi)^3}\int\exp(-i(p_0-p'_0)x_0)\times\exp\left(-i(\boldsymbol{p}-\boldsymbol{p'}).\boldsymbol{x}\right)d^3x$$

As the integration is not over x_0, we can take it out.

$$\implies\frac{1}{(2\pi)^3}\int\exp\left(-i(p-p').x\right)d^3x = \frac{1}{(2\pi)^3}\exp(-i(p_0-p'_0)x_0)\int\exp\left(-i(\boldsymbol{p}-\boldsymbol{p'}).\boldsymbol{x}\right)d^3x$$
$$\therefore\frac{1}{(2\pi)^3}\int\exp\left(-i(p-p').x\right)d^3x = \exp(-i(p_0-p'_0)x_0)\delta^3(\boldsymbol{p}-\boldsymbol{p'})$$

The integration has non-zero value for $\boldsymbol{p}=\boldsymbol{p'}$. Also, as we have defined our $p_0 = E_p = \sqrt{\boldsymbol{p'}^2+m^2}$, of $\boldsymbol{p}=\boldsymbol{p'}$, then $p_0 = p'_0$ and the remaining exponential becomes 1. Thus we can essentially write our integration as :

$$\therefore\delta^3(\boldsymbol{p}-\boldsymbol{p'}) = \frac{1}{(2\pi)^3}\int\exp\left(-i(p-p').x\right)d^3x$$

Putting this in the Hamiltonian equation :

$$H = \int \frac{d^3p'}{\sqrt{2E_{p'}}} \int \frac{d^3p}{\sqrt{2E_p}} E_p \\ \times \sum_{s',s=1,2} \left[f^\dagger_{s'}(\boldsymbol{p'}) f_s(\boldsymbol{p}) u^\dagger_{s'}(\boldsymbol{p'}) u_s(\boldsymbol{p}) \delta^3(\boldsymbol{p}-\boldsymbol{p'}) - \hat{f}_{s'}(\boldsymbol{p}) \hat{f}^\dagger_s(\boldsymbol{p}) v^\dagger_{s'}(\boldsymbol{p'}) v_s(\boldsymbol{p}) \delta^3(\boldsymbol{p'}-\boldsymbol{p}) \right]$$

So, the Hamiltonian is non-zero for $p = p'$. One of the momentum integrations get killed.

$$H = \int \frac{d^3p}{\sqrt{2E_p}} E_p \sum_{s',s=1,2} \left[f^\dagger_{s'}(\boldsymbol{p}) f_s(\boldsymbol{p}) u^\dagger_{s'}(\boldsymbol{p}) u_s(\boldsymbol{p}) - \hat{f}_{s'}(\boldsymbol{p}) \hat{f}^\dagger_s(\boldsymbol{p}) v^\dagger_{s'}(\boldsymbol{p}) v_s(\boldsymbol{p}) \right]$$

Now we are free to use (1.48) and (1.49) :

$$H = \int \frac{d^3p}{\sqrt{2E_p}} E_p(2E_p) \sum_{s',s=1,2} \delta_{s's} \left[f^\dagger_{s'}(\boldsymbol{p}) f_s(\boldsymbol{p}) - \hat{f}_{s'}(\boldsymbol{p}) \hat{f}^\dagger_s(\boldsymbol{p}) \right]$$

The summation over s' gets killed and we obtain our final expression for the Hamiltonian as :

$$H = \int d^3p\, E_p \sum_{s=1,2} \left[f^\dagger_s(\boldsymbol{p}) f_s(\boldsymbol{p}) - \hat{f}_s(\boldsymbol{p}) \hat{f}^\dagger_s(\boldsymbol{p}) \right] \tag{1.75}$$

But, with this form of the Hamiltonian, we can still obtain negative energy solutions. To avoid that, we have to assume that the creation and annihilation operator of both positive and negative momentum states follow anticommutation relations, instead of the commutation relation they followed in for scalar fields. This will be our normal ordering.

$$\boxed{\left[f_s(\boldsymbol{p}), f^\dagger_{s'}(\boldsymbol{p'}) \right]_+ = \left[\hat{f}_s(\boldsymbol{p}), \hat{f}^\dagger_{s'}(\boldsymbol{p'}) \right]_+ = \delta_{ss'} \delta^3(\boldsymbol{p}-\boldsymbol{p'})} \tag{1.76}$$

All other anticommutators are zero.
Thus, we obtain the normal ordered Hamiltonian as :

$$\boxed{:H: = \int d^3p\, E_p \sum_{s=1,2} \left[f^\dagger_s(\boldsymbol{p}) f_s(\boldsymbol{p}) + \hat{f}^\dagger_s(\boldsymbol{p}) \hat{f}_s(\boldsymbol{p}) \right]} \tag{1.77}$$

Noether's charge in normal ordered form can be written as :

$$:Q: = q \int d^3x :\psi^\dagger \psi := q \int d^3p \sum_{1,2} \left[f^\dagger_s(\boldsymbol{p}) f_s(\boldsymbol{p}) - \hat{f}^\dagger_s(\boldsymbol{p}) \hat{f}_s(\boldsymbol{p}) \right] \tag{1.78}$$

From the above equation (1.78), we can see that the Noether's charge for particles created by $f^\dagger$ is opposite to the particles created by $\hat{f}^\dagger$. So, the states created by $\hat{f}^\dagger$ are the anti-particles of the former.

We have also mentioned that all other anticommutators except the ones given in (1.76) are 0. This implies specifically :

$$f_s^\dagger(\boldsymbol{p}) f_s^\dagger(\boldsymbol{p}) = 0\,; \quad \hat{f}_s^\dagger(\boldsymbol{p}) \hat{f}_s^\dagger(\boldsymbol{p}) = 0$$

This means that two particles, or anti-particles of same spin and same momentum cannot be created. The anticommutators obey Pauli's exclusion principle.

1.5.3 Fock space for fermions

We can define the vacuum as :

$$f_s(\boldsymbol{p})|0\rangle = 0\,; \quad \hat{f}_s|0\rangle(\boldsymbol{p}) = 0 \quad \forall \boldsymbol{p} \text{ and } s$$

States contain particles and antiparticles are constructed by the action of $f_s^\dagger$ and $\hat{f}_s^\dagger$ respectively on the vacuum.

Now we wish to find the commutation relations between H, $f_r^\dagger$ and $\hat{f}_s^\dagger$.

a.) Finding the commutation relations $\left[H, f_r^\dagger(\boldsymbol{k})\right]$ and $\left[H, \hat{f}_r^\dagger(\boldsymbol{k})\right]$
Using the equation for H given in (1.77) and the commutation relations in (1.76), we can write :

$$\begin{aligned}
\left[H, f_r^\dagger(\boldsymbol{k})\right] &= H f_r^\dagger(\boldsymbol{k}) - f_r^\dagger(\boldsymbol{k}) H \\
&= \int d^3p E_p \sum_{s=1,2} \Big[f_s^\dagger(\boldsymbol{p}) f_s(\boldsymbol{p}) f_r^\dagger(\boldsymbol{k}) + \hat{f}_s^\dagger(\boldsymbol{p}) \hat{f}_s(\boldsymbol{p}) f_r^\dagger(\boldsymbol{k}) \\
&\qquad\qquad - f_r^\dagger(\boldsymbol{p}) f_s^\dagger(\boldsymbol{p}) f_s(\boldsymbol{p}) - f_r^\dagger(\boldsymbol{k}) \hat{f}_s^\dagger(\boldsymbol{p}) \hat{f}_s(\boldsymbol{p}) \Big] \\
&= \int d^3p E_p \sum_{s=1,2} \Big\{ f_s^\dagger(\boldsymbol{p}) \left[f_s(\boldsymbol{p}), f_r^\dagger(\boldsymbol{k})\right]_+ - f_s^\dagger(\boldsymbol{p}) f_r^\dagger(\boldsymbol{k}) f_s(\boldsymbol{p}) \\
&\qquad\qquad - f_r^\dagger(\boldsymbol{k}) f_s^\dagger(\boldsymbol{p}) f_s(\boldsymbol{p}) + \hat{f}_s^\dagger(\boldsymbol{p}) \hat{f}_s(\boldsymbol{p}) f_r^\dagger(\boldsymbol{k}) - f_r^\dagger(\boldsymbol{k}) \hat{f}_s^\dagger(\boldsymbol{p}) \hat{f}_s(\boldsymbol{p}) \Big\} \\
&= \int d^3p E_p \sum_{s=1,2} \Bigg\{ f_s^\dagger(\boldsymbol{p}) \delta_{rs} \delta^3(\boldsymbol{p}-\boldsymbol{k}) - \cancelto{0}{\left[f_s^\dagger(\boldsymbol{p}), f_r^\dagger(\boldsymbol{k})\right]}_+ f_s(\boldsymbol{p}) \\
&\qquad\qquad + \hat{f}_s^\dagger(\boldsymbol{p}) \cancelto{0}{\left[\hat{f}_s(\boldsymbol{p}), f_r^\dagger(\boldsymbol{k})\right]}_+ - \hat{f}_s^\dagger(\boldsymbol{p}) f_r^\dagger(\boldsymbol{k}) \hat{f}_s(\boldsymbol{p}) - f_r^\dagger(\boldsymbol{k}) \hat{f}_s^\dagger(\boldsymbol{p}) f_s(\boldsymbol{p}) \Bigg\} \\
&= \int d^3p E_p \sum_{s=1,2} \Bigg\{ f_s^\dagger(\boldsymbol{p}) \delta_{rs} \delta^3(\boldsymbol{p}-\boldsymbol{k}) - \cancelto{0}{\left[f_s^\dagger(\boldsymbol{p}), f_r^\dagger(\boldsymbol{k})\right]}_+ \hat{f}_s(\boldsymbol{p}) \Bigg\}
\end{aligned}$$

Thus, we finally obtain :

$$\left[H, f_r^\dagger(\boldsymbol{k})\right] = E_k f^\dagger(\boldsymbol{k})$$

By similar calculation, we can find the other commutation relation too. We finally obtain :

$$\boxed{\begin{aligned} \left[H, f_r^\dagger(\boldsymbol{k})\right] &= E_k f^\dagger(\boldsymbol{k}) \qquad (1.79) \\ \left[H, \hat{f}_r^\dagger(\boldsymbol{k})\right] &= E_k \hat{f}^\dagger(\boldsymbol{k}) \qquad (1.80) \end{aligned}}$$

We can say from the above relations that $f_r^\dagger$ and $\hat{f}_r^\dagger$ both create positive energy states, but the former state is a particle and the latter is an antiparticle.

1.6 Propagator of Fields

To understand how particles propagate through spacetime, we need to define a propagator. We will first define a propagator for the scalar fields, and then use similar calculations to define the propagator for fermionic fields.

1.6.1 Propagator of Klein-Gordon field

Beginning with the Klein-Gordon equation coupled to a source term :

$$\left(\Box + m^2\right)\phi(x) = J(x) \tag{1.81}$$

We can define the Green's function as :

$$\left(\Box_x + m^2\right)G(x - x') = -\delta(x - x') \tag{1.82}$$

$\Box_x$ denotes that the derivatives are taken with respect to unprimed frame x^μ, and not x'^μ. The solution of the above equation can be written in the form :

$$\phi(x) = \phi_0(x) - \int d^4x' G(x - x')J(x') \tag{1.83}$$

$\phi_0(x)$ represents any solution of the free Klein-Gordon equation, i.e., with $J(x) = 0$.

Taking the Fourier transform of the above equation, we obtain the equation for $G(x - x')$ as :

$$G(x - x') = \int \frac{d^4p}{(2\pi)^4} \exp\left(-ip.(x - x')\right) G(p) \tag{1.84}$$

$G(p)$ are the fourier components. To find the form of $G(p)$ we need to put the equation of $G(x - x')$ in (1.84) in (1.82).

$$-\delta^4(x - x') = \left(\Box_x + m^2\right)G(x - x')$$

$$\implies -\delta^4(x-x') = \left(\Box_x + m^2\right)\int \frac{d^4p}{(2\pi)^4}\exp\left(-ip.\left(x-x'\right)\right)G(p)$$
$$\implies -\delta^4(x-x') = \int \frac{d^4p}{(2\pi)^4}\left[\left(\partial^\mu\partial_\mu + m^2\right)\exp(-ip_\mu x^\mu)\right]\exp(-ip.x')G(p)$$
$$\implies -\delta^4(x-x') = \int \frac{d^4p}{(2\pi)^4}\left[\left(g^{\mu\nu}\partial_\nu\partial_\mu + m^2\right)\exp(-ip_\mu x^\mu)\right]\exp(-ip.x')G(p)$$
$$\implies -\delta^4(x-x') = \int \frac{d^4p}{(2\pi)^4}\left[\left(g^{\mu\nu}(-ip_\mu)\partial_\nu\left\{\exp(-ip_\nu x^\nu)\right\} + m^2\exp(-ip_\mu x^\mu)\right]\exp(-ip.x')G(p)$$
$$\implies -\delta^4(x-x') = \int \frac{d^4p}{(2\pi)^4}\left[\left(g^{\mu\nu}(-ip_\mu)(-ip_\nu) + m^2\right)\exp(-ip_\mu x^\mu)\right]\exp(-ip.x')G(p)$$
$$\implies -\delta^4(x-x') = \int \frac{d^4p}{(2\pi)^4}\left[\left(-p_\mu p^\mu + m^2\right)\exp(-ip_\mu x^\mu)\right]\exp(-ip.x')G(p)$$
$$\implies -\delta^4(x-x') = \int \frac{d^4p}{(2\pi)^4}(-p^2 + m^2)\exp(-ip.(x-x')G(p)$$

Using the equation for $\delta^4(x-x')$:

$$\delta^4(x-x') = \int \frac{d^4p}{(2\pi)^4}\exp\left(-ip.(x-x')\right) \tag{1.85}$$

We get the equation for $G(p)$ as :

$$G(p) = \frac{1}{p^2 - m^2} = \frac{1}{(p^0)^2 - E_p^2} \tag{1.86}$$

The above equation has a singularity at $p^0 = \pm E_p$. To avoid this problem, we use a prescription by Feynman and define Feynman propagator for scalar field as :

$$\Delta_F(p) = \frac{1}{p^2 - m^2 + i\varepsilon'} = \frac{1}{(p^0)^2 - (E_p - i\varepsilon)^2} \tag{1.87}$$

a.) Finding the relation between ε and ε'.

ε and ε' are infinitesimally small positive quantities, so we will ignore terms of $\mathcal{O}(\varepsilon)$. We know :

$$E_p^2 = \boldsymbol{p}^2 + m^2 = p^2 - (p^0)^2 + m^2$$

We can see from (1.87) :

$$p^2 - m^2 + i\varepsilon' = (p^0)^2 - \left(E_p^2 - i\varepsilon\right)^2$$
$$(p^0)^2 - E_p^2 + i\varepsilon = (p^0)^2 - (E_p^2 - 2iE_p\varepsilon')$$
$$\therefore \varepsilon' = 2E_p\varepsilon$$

As we will set ε and ε' to zero in the end, we can avoid distinguishing between them.
Also, by introducing the ε parameter, we have made the propagator complex. It is no longer a classical propagator.

We can write (1.87) in the form :

$$\Delta_F(p) = \frac{1}{2E_p}\left[\frac{1}{p^0 - (E_p - i\varepsilon)} - \frac{1}{p^0 + (E_p - i\varepsilon)}\right] \tag{1.88}$$

b.) Derive the form of $\boldsymbol{\Delta_F(p)}$ given in (1.88)

We know :

$$\begin{aligned}
\Delta_F(p) &= \frac{1}{(p^0)^2 - (E_p - i\varepsilon)^2} \\
\implies \Delta_F(p) &= \frac{1}{[p^0 - (E_p - i\varepsilon)]\,[p^0 + (E_p - i\varepsilon)]} \\
\implies \Delta_F(p) &= \frac{1}{2(E_p - i\varepsilon)}\,\frac{[p^0 + (E_p - i\varepsilon)] - [p^0 - (E_p - i\varepsilon)]}{[p^0 - (E_p - i\varepsilon)]\,[p^0 + (E_p - i\varepsilon)]}
\end{aligned}$$

Taking $\varepsilon \longrightarrow 0$, we get :

$$\therefore \Delta_F(p) = \frac{1}{2E_p}\left[\frac{1}{p^0 - (E_p - i\varepsilon)} - \frac{1}{p^0 + (E_p - i\varepsilon)}\right]$$

Putting the expression (1.88) in (1.84) :

$$\begin{aligned}
\Delta_F(x - x') &= \int \frac{d^4p}{(2\pi)^4} \exp\left(-ip.(x - x')\right) \Delta_F(p) \\
\implies \Delta_F(x - x') &= \int \frac{d^4p}{(2\pi)^4} \exp\left(-ip.(x - x')\right) \frac{1}{2E_p}\left[\frac{1}{p^0 - (E_p - i\varepsilon)} - \frac{1}{p^0 + (E_p - i\varepsilon)}\right] \\
\implies \Delta_F(x - x') &= \int \frac{d^4p}{(2\pi)^4} \exp\left(-ip_\mu(x^\mu - x'^\mu)\right) \frac{1}{2E_p}\left[\frac{1}{p^0 - (E_p - i\varepsilon)} - \frac{1}{p^0 + (E_p - i\varepsilon)}\right]
\end{aligned}$$

Raising and lowering indices inside exponential term :

$$\begin{aligned}
\implies \Delta_F(x - x') &= \int \frac{d^3p}{(2\pi)^3} \frac{\exp\left(ip^j(x_j - x'_j)\right)}{2E_p} \int_{-\infty}^{+\infty} \frac{dp^0}{2\pi} \exp\left(-ip^0(t - t')\right) \\
&\qquad \times \left[\frac{1}{p^0 - (E_p - i\varepsilon)} - \frac{1}{p^0 + (E_p - i\varepsilon)}\right] \\
\implies \Delta_F(x - x') &= \int \frac{d^3p}{(2\pi)^3} \frac{\exp\left(i\boldsymbol{p}.(\boldsymbol{x} - \boldsymbol{x'})\right)}{2E_p} \int_{-\infty}^{+\infty} \frac{dp^0}{2\pi} \exp\left(-ip^0(t - t')\right) \\
&\qquad \times \left[\frac{1}{(p^0 - E_p) + i\varepsilon} - \frac{1}{(p^0 + E_p) - i\varepsilon}\right]
\end{aligned}$$

Using the result from complex analysis :

$$\lim_{\varepsilon \longrightarrow 0} \int_{-\infty}^{+\infty} d\xi \frac{\exp(-i\xi t)}{\xi + i\varepsilon} = -2\pi i \Theta(t) \tag{1.89}$$

We are using the same form of Θ as defined in (1.63).

We will use this result to calculate integration over p^0.

c.) Finding the integration over p^0.
For the first term in the integration, let

$$p^{0'} = p^0 - E_p$$

So, we can write :

$$\int_{-\infty}^{+\infty} \frac{dp^0}{2\pi} \frac{\exp\left(-ip^0(t-t')\right)}{(p^0 - E_p) + i\varepsilon} = \int_{-\infty}^{+\infty} \frac{dp^{0'}}{2\pi} \frac{\exp\left(-ip^{0'}(t-t')\right)}{p^{0'} + i\varepsilon} \exp\left(-iE_p(t-t')\right)$$
$$\therefore \int_{-\infty}^{+\infty} \frac{dp^0}{2\pi} \frac{\exp\left(-ip^0(t-t')\right)}{(p^0 - E_p) + i\varepsilon} = -i\Theta(t-t') \exp\left(-iE_p(t-t')\right)$$

We can do a similar substitution for the second term and obtain :

$$\int_{-\infty}^{+\infty} \frac{dp^0}{2\pi} \frac{\exp\left(-ip^0(t-t')\right)}{(p^0 + E_p) - i\varepsilon} = i\Theta(t'-t) \exp\left(-iE_p(t-t')\right)$$

Putting these values in the original integration, we obtain :

$$\Delta_F(x-x') = -i \int \frac{d^3p}{(2\pi)^3 2E_p} \left[\Theta(t-t') \exp\left(i\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{x'}) - iE_p(t-t')\right)\right.$$
$$\left. + \Theta(t'-t) \exp\left(i\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{x'}) + iE_p(t-t')\right)\right]$$
$$\therefore i\Delta_F(x-x') = \int \frac{d^3p}{(2\pi)^3 2E_p} \left[\Theta(t-t') \exp\left(-ip.(x-x')\right) + \Theta(t'-t) \exp\left(ip.(x-x')\right)\right] \quad (1.90)$$

We have used the relation $p^0 = E_p$ to reach to the final equation (1.90).

Now we will try to relate the Feynman propagator obtained in (1.90) to the quantised scalar fields.

The vacuum in Fock space of scalar fields is defined as :

$$a(p)|0\rangle = 0$$
$$\langle 0|a^\dagger(p) = 0$$

$a(p)$ and $a^\dagger(p)$ are annihilation and creation operators of scalar field, respectively.

Using the equation for the fourier decomposition of scalar field (1.67), we can find :

$$\phi(x)|0\rangle = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \exp(ip.x)|p\rangle$$
$$\langle 0|\phi(x') = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E'_p}} \exp(-ip'.x')\langle p'|$$

Thus, we get :

$$\langle 0|\phi(x')\phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3 2E_p} \exp\left(ip.(x-x')\right) \tag{1.91}$$

Putting this in (1.90), we see :

$$\begin{aligned} i\Delta_F(x-x') = \Theta(t-t')\langle 0|\phi(x)\phi(x')|0\rangle& \\ +\,\Theta(t'-t)\langle 0|\phi(x')\phi(x)|0\rangle& \end{aligned} \tag{1.92}$$

We can write (1.92) in a compact notation as :

$$i\Delta_F(x-x') = \langle 0|\mathscr{T}\left[\phi(x)\phi(x')\right]|0\rangle \tag{1.93}$$

where,

$$\mathscr{T}\left[A(x)B(x')\right] \equiv \begin{cases} A(x)B(x') & \text{if} \quad t>t' \\ B(x')A(x) & \text{if} \quad t'>t \end{cases} \tag{1.94}$$

is the time ordered product for scalar fields.

1.6.2 Propagator of Dirac field

Now to find how fermionic states propagate in space-time. So we will couple the Dirac equation to a source and write the equation as :

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi(x) = J(x) \tag{1.95}$$

Then, we can write the Dirac propagator $S(x-x')$ as :

$$\left(i\gamma^\mu\partial^x_\mu - m\right)S(x-x') = \delta^4(x-x') \tag{1.96}$$

The notation ∂^x_μ means that the derivative is taken with respect to unprimed frame $\{x\}$ and not with $\{x'\}$.
The solution is then given by the standard form :

$$\psi(x) = \psi_0(x) + \int d^4x' S(x-x')J(x')$$

Taking the fourier transform of the above equation we get :

$$S(x-x') = \int \frac{d^4p}{(2\pi)^4} \exp\left(-ip.(x-x')\right)S(p) \tag{1.97}$$

We will now attempt to find some conditions satisfied by $S(p)$.

a.) Finding the relation satisfied by $S(p)$.

In order to do so, we need to put the value of $S(x-x')$ given by (1.97) in (1.96).

$$\begin{aligned}
&\delta^4(x-x') = (i\not{\partial}-m)\,\delta(x-x')S(x-x')\\
\implies &\delta^4(x-x') = (i\not{\partial}-m)\int\frac{d^4p}{(2\pi)^4}\exp\left(-ip.(x-x')\right)S(p)\\
\implies &\delta^4(x-x') = \int\frac{d^4p}{(2\pi)^4}\left[(i\gamma^\mu\partial_\mu-m)\exp(-ip_\mu x^\mu)\right]\exp(ip.x')S(p)\\
\implies &\delta^4(x-x') = \int\frac{d^4p}{(2\pi)^4}\left[(-i^2\gamma^\mu p_\mu-m)\exp(-ip_\mu x^\mu)\right]\exp(ip.x')S(p)\\
\implies &\delta^4(x-x') = \int\frac{d^4p}{(2\pi)^4}\exp\left(-ip.(x-x')\right)(\gamma^\mu p_\mu-m)S(p)
\end{aligned}$$

As we know :

$$\delta^4(x-x') = \int\frac{d^4p}{(2\pi)^4}\exp\left(-ip.(x-x')\right)$$

We can write that :

$$(\not{p}-m)\,S(p) = 1 \tag{1.98}$$

Multiplying on both sides by $(\not{p}+m)$:

$$\begin{aligned}
&(\not{p}+m)\,(\not{p}-m)\,S(p) = (\not{p}+m)\\
\implies &\left(p^2-m^2\right)S(p) = (\not{p}+m)
\end{aligned}$$

As (p^2-m^2) is a scalar (a number), it can be taken in the denominator.

$$\therefore S(p) = \frac{\not{p}+m}{p^2-m^2} = \frac{\not{p}+m}{(p^0)^2-E_p^2} \tag{1.99}$$

The issue with this definition of $S(p)$ is that it cannot be defined at the poles, i.e., when $p^0=\pm E_p$. To avoid this problem we will define the Feynman propagator.

$$S_F(p) = \frac{\not{p}+m}{p^2-m^2+i\varepsilon} \tag{1.100}$$

Putting (1.100) in (1.97), we get :

$$S_F(x-x') = \int\frac{d^4p}{(2\pi)^4}\exp\left(-ip.(x-x')\right)\left[\frac{\not{p}+m}{p^2-m^2+i\varepsilon}\right] \tag{1.101}$$

Now,

$$\begin{aligned}
&(i\not{\partial}+m)\exp\left(-ip.(x-x')\right) = (i\not{\partial}+m)\exp(-ip_\mu x^\mu)\exp(ip.x')\\
\implies &(i\not{\partial}+m)\exp\left(-ip.(x-x')\right) = (i\gamma^\mu\partial_\mu+m)\exp(-ip_\mu x^\mu)\exp(ip.x')\\
\implies &(i\not{\partial}+m)\exp\left(-ip.(x-x')\right) = \left(-i^2\gamma^\mu p_\mu+m\right)\exp\left(-ip.(x-x')\right)
\end{aligned}$$

So, we can write (1.101) as :

$$S_F(x-x') = \int \frac{d^4p}{(2\pi)^4} \left[(i\not{\partial}+m)\exp\left(-ip.(x-x')\right)\right] \frac{1}{p^2-m^2+i\varepsilon}$$

Using (1.87) and (1.84), we can write $S_F(x-x')$ in terms of $\Delta_F(x-x')$:

$$S_F(x-x') = (i\not{\partial}+m)\,\Delta_F(x-x') \tag{1.102}$$

Putting in the value of $\Delta_F(x-x')$ from (1.90), we can write $S_F(x-x')$ as :

$$\begin{aligned} iS_F(x-x') &= \int \frac{d^3p}{(2\pi)^3 2E_p} \left[\Theta(t-t')\,(i\not{\partial}+m)\exp\left(-ip.(x-x')\right)\right. \\ &\qquad\qquad \left.+\Theta(t'-t)\,(i\not{\partial}+m)\exp\left(-ip.(x-x')\right)\right] \\ \therefore\, iS_F(x-x') &= \int \frac{d^3p}{(2\pi)^3 2E_p} \left[\Theta(t-t')\,(\not{p}+m)\exp\left(ip.(x-x')\right)\right. \\ &\qquad\qquad \left.-\Theta(t'-t)\,(\not{p}-m)\exp\left(ip.(x-x')\right)\right] \end{aligned} \tag{1.103}$$

Using similar methods that we have used to prove (1.93), we can also show that $iS_F(x-x')$ can be expressed as a time-ordered product of spinor fields.

$$\boxed{iS_{F_{ij}}(x-x') = \langle 0|\mathscr{T}\left[\psi_i(x)\overline{\psi}_j(x')|0\rangle\right]} \tag{1.104}$$

where,

$$\mathscr{T}\left[\psi_i(x)\overline{\psi}_j(x')\right] \equiv \begin{cases} \psi_i(x)\overline{\psi}_j(x') & \text{if} \quad t>t' \\ -\overline{\psi}_j(x')\psi_i(x) & \text{if} \quad t<t' \end{cases} \tag{1.105}$$

Chapter 2

S-Matrix Expansion and Wick's Theorem

In the previous chapter, we have studied about free fields. The Hamiltonian had terms quadratic in the same field operators. Physically, it means that at any point in space-time the Hamiltonian can create or annihilate a particle. So, if a particle gets annihilated, another can be created at the same point and essentially the same particle moves on, as they are indistinguishable.

If we have a Hamiltonian which is bilinear in two fields, then it can annihilate a particle of one type and create a particle of another type. Such terms in the total Hamiltonian can be treated as interaction terms.

2.1 Evolution Operator

We can write the total Hamiltonian of any system as :

$$H = H_0 + H_I \tag{2.1}$$

where,

$$\begin{aligned} H_0 &\implies \text{Free Hamiltonian with known eigenstates} \\ H_I &\implies \text{Interaction Hamiltonian} \end{aligned}$$

Let the state $\Psi(t)$ evolve under the total Hamiltonian given in (2.1). The equation is then given by :

$$i\frac{d}{dt}|\Psi(t)\rangle = (H_0 + H_I)\,|\Psi(t)\rangle \tag{2.2}$$

For the free Hamiltonian we have the state equation as :

$$i\frac{d}{dt}|\Psi_0(t)\rangle = H_0|\Psi_0(t)\rangle \tag{2.3}$$

Let us assume the initial state is $|i\rangle$. We can say $|i\rangle$ evolves in time under the complete Hamiltonian to become $\Psi(t)$. Had there been no interactions, the state $|i\rangle$ would have evolved under the free Hamiltonian H_0 to become $\Psi_0(t)$.

2.1.1 General form of evolution operators

We will define two different evolution operators, $U_0(t)$ and $U(t)$ to describe the evolution of states. They can be written as :

$$|\Psi_0(t)\rangle = U_0(t)|\Psi_0(-\infty)\rangle \equiv U_0(t)|i\rangle \tag{2.4}$$

$$|\Psi(t)\rangle = U_0(t)U(t)U_0^\dagger(t)|\Psi_0(t)\rangle \tag{2.5}$$

We will assume that all the states are normalised and the evolution operators are unitary for any time t. Normalisation of the states demands that $U_0(t)$ and $U(t)$ be unitary.

b.) Prove that $U_0(t)$ and $U(t)$ are unitary, if states are normalised.
For $U_0(t)$:

$$\begin{aligned} \langle\Psi_0(t)|\Psi_0(t)\rangle &= \langle i|U_0^\dagger(t)U_0(t)|i\rangle = \langle i|i\rangle \\ \implies U_0^\dagger(t)U_0(t) &= \mathbb{1} \end{aligned}$$

For $U(t)$:

$$\begin{aligned} \langle\Psi(t)|\Psi(t)\rangle &= \langle\Psi_0(t)|\left\{U_0(t)U(t)U_0^\dagger(t)\right\}^\dagger U_0(t)U(t)U_0^\dagger(t)|\Psi_0(t) \\ \implies \langle\Psi(t)|\Psi(t)\rangle &= \langle\Psi_0(t)|U_0(t)U^\dagger(t)\left\{U_0^\dagger(t)U_0(t)\right\}U(t)U_0^\dagger(t)|\Psi_0(t)\rangle \\ \implies \langle\Psi(t)|\Psi(t)\rangle &= \langle\Psi_0(t)|U_0(t)\left\{U^\dagger(t)U(t)\right\}U_0^\dagger(t)|\Psi_0(t)\rangle \end{aligned}$$

As $\Psi(t)$ and $\Psi_0(t)$ are normalised, we obtain

$$\therefore U^\dagger(t)U(t) = \mathbb{1}$$

Now we can write the state $\Psi(t)$ as :

$$|\Psi(t)\rangle = U_0(t)U(t)U_0^\dagger(t)|\Psi_0(t)\rangle$$

Using (2.4) :

$$\begin{aligned} \implies |\Psi(t)\rangle &= U_0(t)U(t)U_0^\dagger(t)U_0(t)|i\rangle \\ \therefore |\Psi(t)\rangle &= U_0(t)U(t)|i\rangle \end{aligned} \tag{2.6}$$

Rewriting (2.3) as :

$$i\frac{d}{dt}U_0(t) = H_0U_0(t)|i\rangle$$

If we consider $|i\rangle$ to be a complete set of states, we can reduce the above equation to operator equation :

$$i\frac{d}{dt}U_0(t) = H_0U_0(t) \tag{2.7}$$

Thus, we can write the general form of $U_0(t)$ as :

$$\boxed{U_0(t) = \exp\left(-iH_0(t)\right)} \tag{2.8}$$

Now considering the full Hamiltonian :

$$i\frac{d}{dt}|\Psi(t)\rangle = (H_0 + H_I)\,|\Psi(t)\rangle$$

Putting in the value of $\Psi(t)$ from (2.6), we get :

$$\begin{aligned} i\frac{d}{dt}\left[U_0(t)U(t)\right]|i\rangle &= (H_0+H_I)\left[U_0(t)U(t)\right]|i\rangle \\ \implies i\left[U_0(t)\frac{d}{dt}U(t) + U(t)\frac{d}{dt}U_0(t)\right]|i\rangle &= \left[H_0U_0(t)U(t) + H_IU_0(t)U(t)\right]|i\rangle \end{aligned}$$

As we have considered $|i\rangle$ to be a complete set of states, we can convert this to operator equation :

$$i\left[\frac{d\left\{U_0(t)\right\}}{dt}U(t) + U_0(t)\frac{dU(t)}{dt}\right] = H_0U_0(t)U(t) + H_IU_0(t)U(t)$$

From (2.7), we can write :

$$\begin{aligned} \implies \cancel{H_0U_0(t)Ut)} + iU_0(t)\frac{dU(t)}{dt} &= \cancel{H_0U_0(t)U(t)} + H_IU_0(t)U(t) \\ \implies iU_0(t)\frac{dU(t)}{dt} &= H_IU_0(t)U(t) \end{aligned}$$

Thus, we can write the final equation as :

$$\therefore i\frac{U(t)}{dt} = U_0^\dagger(t)H_IU_0(t)U(t) = H_I(t)U(t) \tag{2.9}$$

where,

$$H_I(t) = U_0^\dagger(t) H_I U_0(t) \tag{2.10}$$

The above equations (2.9) and (2.10) get acted from both sides by the set states denoted by $|i\rangle$. They represent the time-independent free states. Let us assume $|\beta\rangle$ is the set of basis vectors of $|i\rangle$.

Writing the matrix elements of $H_I(t)$ between $|\beta\rangle$:

$$\begin{aligned} \langle\beta|H_I(t)|\beta'\rangle &= \langle\beta|U_0^\dagger(t)H_I(t)U_0(t)|\beta'\rangle \\ \therefore \langle\beta|H_I(t)|\beta'\rangle &= \langle\beta,t|H_I(t)|t,\beta'\rangle \end{aligned} \tag{2.11}$$

$U_0(t)|\beta\rangle = |\beta,t\rangle$ represents the time-dependent free state into which $|\beta\rangle$ would have evolved into in time t, in the absence of interactions.

We can see from (2.11) that the matrix elements of $H_I(t)$ in betweem time independent free states are equal to the matrix elements of H_I in between the time-dependent free states.

We will also assume that $|\beta\rangle$ states are a complete set of eigenstates of the total Hamiltonian at distant past ($t \longrightarrow -\infty$). It means, that in the distant past, all states were free, time-independent and there was no interaction, i.e., :

$$H_I \longrightarrow 0 \quad \text{and} \quad U(t) \longrightarrow 1 \quad \text{as} \quad t \longrightarrow -\infty \tag{2.12}$$

From (2.9), we can write the general solution for $U(t)$ as :

$$U(t) = U(t_0) - i\int_{t_0}^{t} dt_1 H_I(t_1)U(t_1)$$

From (2.12), the equation reduces to :

$$U(t) = 1 - i\int_{-\infty}^{t} dt_1 H_I(t_1)U(t_1)$$

Similarly, we can write $U(t_1)$ as $1 - i\int_{-\infty}^{t_1} dt_2 H_I(t_2)U(t_2)$:

$$\implies U(t) = 1 - i\int_{-\infty}^{t} dt_1 H_I(t_1) + (-i)^2\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 H_I(t_1)H_I(t_2)U(t_2)$$

We can repeat the substitution process indefinitely.

We obtain the general form of $U(t)$ as :

$$\boxed{\therefore U(t) = 1 + \sum_{n=1}^{\infty} (-i)^n \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n H_I(t_1) H_I(t_2) \cdots H_I(t_n)} \tag{2.13}$$

2.1.2 $U(t)$ as a time-ordered product

We will now attempt to write (2.13) as a time-ordered product of interaction Hamiltonians. Recalling the definition of time-ordered scalar products from (1.94) and now defining it for general fields :

$$\mathscr{T}\left[A(x)B(x')\right] \equiv \begin{cases} A(x)B(x') & \text{if} \quad t > t' \\ \pm B(x')A(x) & \text{if} \quad t' > t \end{cases} \tag{2.14}$$

Let us consider the integration :

$$\begin{aligned} \frac{1}{2!}\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 \mathscr{T}\left[H_I(t_1)H_I(t_2)\right] = \frac{1}{2!}\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 H_I(t_1)H_I(t_2) \\ + \frac{1}{2!}\int_{-\infty}^{t} dt_2 \int_{-\infty}^{t_2} dt_1 H_I(t_2)H_I(t_1) \end{aligned}$$

The first term corresponds to $t_1 > t_2$ and the second term corresponds to $t_2 > t_1$. As t_1 and t_2 are dummy indices, they can be interchanged. Doing so, we obtain :

$$\implies \frac{1}{2!}\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 \mathscr{T}\left[H_I(t_1)H_I(t_2)\right] = \frac{1}{2!}.2\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 H_I(t_1)H_I(t_2)$$

So, for $n = 2$ $U(t)$ from (2.13), and using $\mathscr{T}\left[H_I(t_1)\right] = H_I(t_1)$ we can write :

$$\begin{aligned} U(t) = 1 + (-i)\int_{-\infty}^{2} dt_1 \mathscr{T}\left[H_I(t_1)\right] \\ + \frac{(-i)^2}{2!}\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 \mathscr{T}\left[H_I(t_1)H_I(t_2)\right] \end{aligned}$$

Generalising for any n (integer value) :

$$\boxed{U(t) = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 \cdots \int_{-\infty}^{t} dt_n \mathscr{T}\left[H_I(t_1)H_I(t_2)\cdots H_I(t_n)\right]} \tag{2.15}$$

We can use a more compact notation :

$$\mathscr{T}\left[\exp\left(-i\int_{-\infty}^{t} dt' H_I(t')\right)\right] \equiv 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 \cdots \int_{-\infty}^{t} dt_n \mathscr{T}\left[H_I(t_1)H_I(t_2)\cdots H_I(t_n)\right] \tag{2.16}$$

In this notation, $U(t)$ is written simply as :

$$\boxed{U(t) = \mathscr{T}\left[\exp\left(-i\int_{-\infty}^{t} dt' H_I(t')\right)\right]} \tag{2.17}$$

The operators on the right are the ones that appear earlier than the ones on the left.

We cannot solve exactly for $U(t)$ because for arbitrary H_I, the series on the right does not converge.

2.2 S-Matrix

The S-matrix is defined as the $t \longrightarrow \infty$ limit of $U(t)$.

$$\boxed{S = \lim_{t\longrightarrow\infty} U(t) = \mathscr{T}\left[\exp\left(-i\int_{-\infty}^{t} dt' H_I(t')\right)\right]} \tag{2.18}$$

As we proved in subsection 2.1.1, $U(t)$ is a unitary matrix. Thus, by extension, we can say that S is also a unitary matrix.

Let us consider a physical system in which the particles are initially free in the distant past, i.e., for $t \longrightarrow -\infty$. The interaction Hamiltonian is zero at this point, i.e., $H_I(t \longrightarrow -\infty) = 0$. We can denote these free time-independent states by $|i\rangle$. Let $|\beta\rangle$ be the basis of the time-independent free states.

At some time t, the interaction slowly comes into play. The states are no longer free and now evolve under total Hamiltonian H (2.1). For any time t, the state can be written as :

$$|\Psi(t)\rangle = U_0(t)U(t)|i\rangle$$

After the interaction has acted for some time, it gets turned off slowly. Long after the interaction has ceased to exist, i.e., for $t \longrightarrow \infty$ the states have become free again. We can represent the final state as :

$$|\Psi_f(t)\rangle = U_0(t)|f\rangle$$

$|i\rangle$ and $|f\rangle$ are some state in $|\beta\rangle$.

We can calculate the amplitude of transition from $|i\rangle$ to $|f\rangle$ as :

$$\lim_{t\longrightarrow\infty}\langle\Psi_f(t)|\Psi(t)\rangle = \lim_{t\longrightarrow\infty}\langle f|U_0^\dagger(t)U_0(t)U(t)|i\rangle$$

$$\therefore \lim_{t\longrightarrow\infty} \langle \Psi_f(t)|\Psi(t)\rangle = \lim_{t\longrightarrow\infty} \langle f|U(t)|i\rangle = \langle f|S|i\rangle \tag{2.19}$$

Thus, we see that on reducing the interaction problem to a problem of finding the probability amplitude of an initial state $|i\rangle$ evolving into some final state $|f\rangle$, we obtain the solution as the matrix elements of S-Matrix operator in the space of free states.

2.3 Wick's Theorem

Recalling the definition of S-Matrix from (2.18) and the notation from (2.16),

$$S = \lim_{t\longrightarrow\infty} U(t) = \mathscr{T}\left[\exp\left(-i\int_{-\infty}^{t} dt' H_I(t')\right)\right]$$

$$\therefore S = \lim_{t\longrightarrow\infty}\left(1 + \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 \cdots \int_{-\infty}^{t} dt_n \mathscr{T}\left[H_I(t_1)H_I(t_2)\cdots H_I(t_n)\right]\right)$$

We see that S-Matrix is a product of time-ordered interaction hamiltonians. Each of the interacton hamiltonians in itself is normal-ordered.

This leads to a problem. In normal ordering all creation operators are to the left of annihilation operators, so even those creation operators which appear at an earlier time, get placed left to the annihilation operators which appear later in time. This goes against the prescription of time ordering.

The problem can be rectified by using **Wick's Theorem**.

We will begin with time-ordering and attempt to reach normal-ordering. For simplicity, we will consider the time-ordered product of a scalar field.

$$\mathscr{T}\left[\phi(x)\phi(x')\right] = \Theta(t-t')\phi(x)\phi(x') + \Theta(t'-t)\phi(x')\phi(x) \tag{2.20}$$

We can write the RHS as a normal-ordered product. To do that, we will first write $\phi(x)$ as a sum of its annihilation operator and creation operator parts.

$$\phi(x) = \phi_+(x) + \phi_-(x)$$

The vacuum of the Fock space of scalars is defined by :

$$\phi_+(x)|0\rangle = 0 \quad \langle 0|\phi_-(x) = 0$$

Now, multiplying $\phi(x)$ and $\phi(x')$:

$$\begin{aligned}
\phi(x)\phi(x') &= [\phi_+(x) + \phi_-(x)]\,[\phi_+(x') + \phi_-(x')] \\
\therefore\ \phi(x)\phi(x') &= \phi_+(x)\phi_+(x') + \phi_+(x)\phi_-(x') + \phi_-(x)\phi_+(x') + \phi_-(x)\phi_-(x')
\end{aligned}$$

Writing it as a normal ordered product :

$$\begin{aligned}
:\phi(x)\phi(x'): &= \phi_+(x)\phi_+(x') + \phi_-(x')\phi_+(x') + \phi_-(x)\phi_+(x') + \phi_-(x)\phi_+(x') \\
\implies :\phi(x)\phi(x'): &= \phi(x)\phi(x') - \phi_+(x)\phi_-(x') + \phi_-(x')\phi_+(x) \\
\therefore :\phi(x)\phi(x'): &= \phi(x)\phi(x') - [\phi_+(x), \phi_-(x')]
\end{aligned} \tag{2.21}$$

Finding the matrix elements of (2.21) between the vacuum of Fock state :

$$\begin{aligned}
\langle 0| :\phi(x)\phi(x'): |0\rangle &= 0 \\
\implies \langle 0|\phi(x)\phi(x')|0\rangle - \langle 0|\,[\phi_+(x), \phi_-(x')]\,|0\rangle &= 0
\end{aligned}$$

Thus, we get :

$$\langle 0|\phi(x)\phi(x')|0\rangle = \langle 0|\,[\phi_+(x), \phi_-(x')]\,|0\rangle$$

Since the commutator is a number :

$$\begin{aligned}
\implies \langle 0|\phi(x)\phi(x')|0\rangle &= [\phi_+(x), \phi_-(x')]\,\langle 0|0\rangle \\
\therefore\ \langle 0|\phi(x)\phi(x')|0\rangle &= [\phi_+(x), \phi_-(x')]
\end{aligned} \tag{2.22}$$

Then, using (2.22) in (2.21), we get :

$$\phi(x)\phi(x') = :\phi(x)\phi(x'): + \langle 0|\phi(x)\phi(x')|0\rangle \tag{2.23}$$

Putting (2.23) in (2.20), we get :

$$\begin{aligned}
\mathscr{T}\,[\phi(x)\phi(x')] &= \Theta(t-t')\phi(x)\phi(x') + \Theta(t-t')\phi(x')\phi(x) \\
&= \Theta(t-t') :\phi(x)\phi(x'): + \Theta(t-t')\langle 0|\phi(x)\phi(x')|0\rangle + \Theta(t'-t) :\phi(x')\phi(x): \\
&\qquad + \Theta(t'-t)\langle 0|\phi(x')\phi(x)|0\rangle
\end{aligned}$$

For scalar fields, we can write $:\phi(x)\phi(x') :=: \phi(x')\phi(x):$

$$\begin{aligned}
\implies \mathscr{T}\,[\phi(x)\phi(x')] = &\ [\Theta(t-t') + \Theta(t'-t)] :\phi(x')\phi(x): \\
&+ \langle 0|\,[\Theta(t-t')\phi(x)\phi(x') + \Theta(t'-t)\phi(x')\phi(x)]\,|0\rangle
\end{aligned}$$

Using (2.20) and definition of $\Theta(z)$ from (1.63), we can write :

$$\boxed{\mathscr{T}\left[\phi(x)\phi(x')\right] =: \phi(x)\phi(x') : + \langle 0|\mathscr{T}\left[\phi(x)\phi(x')\right]|0\rangle} \tag{2.24}$$

This is the **Wick's Theorem**.

We can define a more compact notation for the second term as :

$$\boxed{\langle 0|\mathscr{T}\left[\phi(x)\phi(x')\right]|0\rangle \equiv \underbrace{\phi(x)\phi}(x')} \tag{2.25}$$

This is called **Wick's Contraction**.

Rewriting (2.24) using (2.25) for any general fields :

$$\boxed{\mathscr{T}\left[\Phi(x)\Phi'(x')\right] =: \Phi(x)\Phi'(x') : + \underbrace{\Phi(x)\Phi'}(x')} \tag{2.26}$$

Using (2.26) the Feyman operators for scalar and Dirac fields can be written as :

$$i\Delta_F(x_1 - x_2) = \underbrace{\phi(x_1)\phi^\dagger}(x_2) = \underbrace{\phi^\dagger(x_2)\phi}(x_1) \tag{2.27}$$

$$iS_{F_{ij}}(x_1 - x_2) = \underbrace{\psi_i(x_1)\overline{\psi}_j}(x_2) = -\underbrace{\overline{\psi}_j(x_2)}\,\psi_i(x_1) \tag{2.28}$$

Chapter 3

Quantisation of Electromagnetic Field

A photon is a quantum of the electromagnetic field, and a spin-1 particle. In this chapter we will attempt to quantise the electromagnetic field and see how the photon arises from that excercise.

3.1 Classical Theory of Electromagnetic Fields

Classically, the electromagnetic fields are described by the four Maxwell equations :

$$\boldsymbol{\nabla} . \boldsymbol{E} = \rho \tag{3.1}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{j} \tag{3.2}$$

The above two equations are constrained by the last two equations :

$$\boldsymbol{\nabla} . \boldsymbol{B} = 0 \tag{3.3}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \tag{3.4}$$

3.1.1 The Field Strength Tensor

$\boldsymbol{E}$ and $\boldsymbol{B}$ can be expressed in terms of a vector field $\boldsymbol{A}$ and a scalar field φ.

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \tag{3.5}$$

$$\boldsymbol{E} = -\boldsymbol{\nabla}\varphi - \frac{\partial \boldsymbol{A}}{\partial t} \tag{3.6}$$

φ and $\boldsymbol{A}$ can be expressed as a 4-vector A^μ, given by :

$$A^\mu \equiv \left(A^0, \boldsymbol{A}\right) = (\varphi, \boldsymbol{A}) \tag{3.7}$$

The previous description of the electric and magnetic fields in terms of the four Maxwell's equations is not manifestly covariant.
Now, after defining the 4-vector A^μ, it can be written in a manifestly covariant manner as :

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.8}$$

$F_{\mu\nu}$ is termed as the Field Strength Tensor.

a.) Determining the components of $F_{\mu\nu}$
From (3.8):

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

μ and ν can take values from 0 to 3.

For $\mu = \nu$:

$$F_{\mu\mu} = \partial_\mu A_\mu - \partial_\mu A_\mu = 0$$

Thus, all diagonal terms are 0.
For $\mu = 0$ and $\nu = i$:

$$F_{0i} = \partial_0 A_i - \partial_i A_0$$
$$\therefore F_{0i} = -\partial_0 A^i - \boldsymbol{\nabla}\varphi = E^i$$

For $\nu = 0$ and $\mu = i$:

$$F_{i0} = \partial_i A_0 - \partial_0 A_i$$
$$\therefore F_{i0} = \boldsymbol{\nabla}\varphi + \partial_0 A^i = -E^i$$

Now, for $\mu = i$ and $\nu = j$:

$$F_{ij} = \partial_i A_j - \partial_j A_i = \partial_j A^i - \partial_i A^j$$

So, we get :

$$F_{12} = \partial_2 A^1 - \partial_1 A^2 = -B^3$$
$$F_{13} = \partial_3 A^1 - \partial_1 A^3 = B^2$$
$$F_{21} = \partial_1 A^2 - \partial_2 A^1 = B^3$$
$$F_{23} = \partial_3 A^2 - \partial_2 A^3 = -B^1$$
$$F_{31} = \partial_1 A^3 - \partial_3 A^1 = -B^2$$
$$F_{32} = \partial_2 A^3 - \partial_3 A^2 = B^2$$

Thus, using all the values calculated above, we can finally write the form of the $F_{\mu\nu}$ matrix as :

$$F_{\mu\nu} = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & -B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{bmatrix} \tag{3.9}$$

Equations (3.1) and (3.2) can be re-written in terms of $F_{\mu\nu}$ as :

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{3.10}$$

where j^ν is the current density 4-vector which incorporates the sources.

$$j^\nu \equiv \left(j^0, \boldsymbol{j}\right) = (\rho, \boldsymbol{j}) \tag{3.11}$$

Equations (3.3) and (3.4) can be re-written as :

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \tag{3.12}$$

3.1.2 Lagrangian of EM field

The Lagrangian of EM fields can be written as :

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu \tag{3.13}$$

Current density j^μ depends on the field operators of the source fields.

If we consider free fields, then we can write the Lagrangian as :

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Let us consider the transformation :

$$A_\mu \longrightarrow A'_\mu = A_\mu + \partial_\mu \theta \tag{3.14}$$

θ is a parameter which depends on spacetime at every point. Thus, the transformation is local. The value of A_μ is affected differently at every point.

a.) Computing the change in Lagrangian (3.13) of the field due to the transformation given in (3.14)

Considering the transformation in (3.14) :

$$A_\mu \longrightarrow A'_\mu = A_\mu + \partial_\mu \theta$$

Then the change in $F_{\mu\nu}$ can be calculated as :

$$\begin{aligned} F'_{\mu\nu} &= \partial_\mu A'_\nu - \partial_\nu A'_\mu \\ &= \partial_\mu \left(A_\nu + \partial_\nu \theta\right) - \partial_\nu \left(A_\mu + \partial_\mu \theta\right) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu \partial_\nu \theta - \partial_\nu \partial_\mu \theta \end{aligned}$$

As ∂_μ and ∂_ν commute, we get the change as :

$$\therefore F'_{\mu\nu} = F_{\mu\nu}$$

The change in j^ν is then found as :

$$j'^{\nu} = \partial_\mu F'^{\mu\nu} = \partial_\mu F^{\mu\nu} = j^\nu$$

Now we can find the change in the lagrangian (3.13) as :

$$\begin{aligned} \mathscr{L}' &= -\frac{1}{4} F'_{\mu\nu} F'^{\mu\nu} - j'^{\mu} A'_\mu \\ &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu \left(A_\mu + \partial_\mu \theta\right) = \mathscr{L} - j^\mu \partial_\mu \theta \\ \therefore \delta \mathscr{L} &= -j^\mu \partial_\mu \theta = -\partial_\mu \left(j^\mu \theta\right) + \theta \left(\partial_\mu j^\mu\right) \end{aligned}$$

j^μ is a conserved current so we can write $\partial_\mu j^\mu = 0$. Thus, we get the change in the lagrangian as :

$$\therefore \delta \mathscr{L} = -\partial_\mu \left(j^\mu \theta\right)$$

The Lagrangian changes by a total divergence, so $\mathscr{L}$ and $\mathscr{L}'$ are equivalent. Thus, we can say that the Lagrangian for the EM field remains invariant under the local transformation of A^μ field given in (3.8).

3.2 Problems with Quantisation of EM Field

If we attempt to quantise the classical theory of Electromagnetic Fields directly, we find that the Green's function does not exist.

3.2.1 Non-existence of propagator

We can show that the propagator for EM fields does not exist for the current Lagrangian being used. To show this we first need to write the equation for the EM fields.

a.) Writing the equation for EM fields.
We know from (3.10) that the equation for EM fields can be written as :

$$j^\nu = \partial_\mu F^{\mu\nu}$$

From (3.8) :

$$\begin{aligned}
&\implies j^\nu = \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \\
&\implies j^\nu = \partial_\mu \left(\partial^\mu A^\nu\right) - \partial_\mu \left(\partial^\nu A^\mu\right) \\
&\implies j^\nu = \Box A^\nu - \partial_\mu \partial^\nu A^\mu
\end{aligned}$$

Raising and lowering some operators on RHS :

$$\begin{aligned}
&\implies j^\nu = g^{\nu\alpha}\Box A_\alpha - g^{\mu\alpha} g_{\mu\beta}\partial^\beta \partial^\nu A_\alpha \\
&\implies j^\nu = g^{\nu\alpha}\Box A_\alpha - \delta^\alpha_\beta \partial^\beta \partial^\nu A_\alpha \\
&\implies j^\nu = g^{\nu\alpha}\Box A_\alpha - \partial^\alpha \partial^\nu A_\alpha
\end{aligned}$$

Changing a few dummy indices we get the final equation of motion as :

$$\left(g^{\nu\mu}\Box - \partial^\mu\partial^\nu\right) A_\mu = j^\nu \tag{3.15}$$

b.) Introducing the Green's function and finding its Fourier transform.
Replacing ν with λ in (3.15) we can introduce the propagator $D_{\mu\nu}(x - x')$ as :

$$\left(g^{\lambda\mu}\Box - \partial^\lambda\partial^\mu\right) D_{\mu\nu}(x - x') = g^\lambda_\nu \delta^4(x - x') \tag{3.16}$$

All the derivatives are taken with respect to x and not x'.

Defining the Fourier transform of the propagator:

$$D_{\mu\nu}(x - x') = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \exp\left(-ik.(x - x')\right) D_{\mu\nu}(k) \tag{3.17}$$

Putting this in the (3.16) :

$$\begin{aligned}
\text{LHS} = g^{\lambda\mu} &\int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} D_{\mu\nu}(k)\partial^\alpha\partial_\alpha \exp\left(-ik.(x - x')\right) \\
&- \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} D_{\mu\nu}(k)\partial^\lambda\partial^\mu \exp\left(-ik.(x - x')\right)
\end{aligned}$$

$$= g^{\lambda\mu} \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} (-ik_\alpha)(-ik^\alpha) D_{\mu\nu}(k) \exp\left(-ik.(x-x')\right)$$
$$- \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} (-ik^\mu)(-ik^\lambda) D_{\mu\nu}(k) \exp\left(-ik.(x-x')\right)$$

Using (1.85) :

$$\text{LHS} = -\left[g^{\lambda\mu}k^2 - k^\lambda k^\mu\right] D_{\mu\nu}(k)\delta^4(x-x') \tag{3.18}$$

Thus, from (3.18) and (3.16) we can write :

$$-\left[g^{\lambda\mu}k^2 - k^\lambda k^\mu\right] D_{\mu\nu}(k) = g^\lambda_\nu \tag{3.19}$$

As $D_{\mu\nu}(k)$ is a rank two tensor depending only on k, we can guess the general form as :

$$D_{\mu\nu}(k) = a g_{\mu\nu} + b k_\mu k_\nu \tag{3.20}$$

a and b are Lorentz invariant quantities.

Using (3.20) in (3.19) :

$$\begin{aligned}
g^\lambda_\nu &= -\left[g^{\lambda\mu}k^2 - k^\lambda k^\mu\right]\left[a g_{\mu\nu} + b k_\mu k_\nu\right] \\
\implies g^\lambda_\nu &= -\left[a g^\lambda_\nu k^2 + b g^{\lambda\mu} k^2 k_\mu k_\nu - a k^\lambda k^\mu g_{\mu\nu} - b k^\lambda k^\mu k_\mu k_\nu\right] \\
\implies g^\lambda_\nu &= -\left[a k^2 g^\lambda_\nu + \cancel{b^2 k^\lambda k_\nu} - a k^\lambda k_\nu - \cancel{b k^2 k^\lambda k_\nu}\right] \\
\therefore g^\lambda_\nu &= -a k^2 g^\lambda_\nu + a k^\lambda k_\nu
\end{aligned} \tag{3.21}$$

All terms involving b get cancelled out in (3.21). So, from (3.21) we can write :

$$a = -\frac{1}{k^2}\,; \quad k^\lambda k_\nu = 0$$

So the propagator does not exist.

3.2.2 Non-existence of Hamiltonian formalism

We can also show that for the Lagrangian we have written the Hamiltonian formalism does not exist.

We can find the conjugate momenta of A^μ as :

$$\Pi^\mu = \frac{\delta L}{\delta \dot{A}^\mu} = F^{\mu 0}$$

From above we can immediately see :

$$\Pi^0 = F^{00} = 0$$

So, one of the conjugate momenta does not exist for A^μ. Then, we cannot find $\dot{A}^\mu$ as the equation of conjugate momenta cannot be inverted.
Thus, we can conclude that the Hamiltonian formalism does not exist.

These problems arise because we are trying to represent the $\boldsymbol{E}$ and $\boldsymbol{B}$ with a 4-vector with four independent components while they themselves have only two independent components. The first two Maxwell equations (3.1) and (3.2) are written for 6 independent quantities, but the last two equations (3.3) and (3.4) impose four constraints on them. So there only two degrees of freedom.
Thus we cannot have four independent components of A^μ.

3.3 Modifying the Classical Lagrangian

As we saw in subsection 3.1.2, our choice of A_μ is not unique. Any A'_μ related to a given A_μ as per (3.14) is also a valid 4-vector to describe the EM fields. This gives us too much freedom in the choice of A_μ. We will choose one representation from each family of A'_μ defined by θ.

We can make this choice by enforcing some conditions on A_μ. This procedure is called **gauge fixing**.
We will choose the Lorentz gauge to work with, which is given by :

$$\partial_\mu A^\mu = 0 \tag{3.22}$$

Adding the gauge fixing term in the Lagrangian, we will re-write it as :

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu - \frac{1}{2\xi} \left(\partial_\mu A^\mu\right)^2 \tag{3.23}$$

This does not disturb the Lagrangian as long as we use only those A_μ which satisfy (3.22).
We can always transform A_μ as per (3.14) such that (3.22) is valid.

a.) Checking if Π^0 still vanishes indentically.
Now after this transformation, we will attempt to see if we can write the Hamiltonian formalism for the EM fields. The conjugate momenta is defined as :

$$\Pi^\mu = \frac{\delta L}{\delta \left(\partial_0 A_\mu\right)} \tag{3.24}$$

Starting with the new Lagrangian (3.23) :

$$
\begin{aligned}
\mathscr{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 - j^\mu A_\mu \\
&= -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 - j^\mu A_\mu \\
&= -\frac{1}{2}\left[g^{\mu\alpha}g^{\nu\beta}\left(\partial_\mu A_\nu\right)\left(\partial_\alpha A_\beta\right) - g^{\mu\alpha}g^{\nu\beta}\left(\partial_\mu A_\nu\right)\left(\partial_\beta A_\alpha\right)\right] \\
&\qquad\qquad - \frac{1}{2}\left[\frac{1}{\xi}g^{\mu\alpha}g^{\nu\beta}\left(\partial_\beta A_\nu\right)\left(\partial_\mu A_\alpha\right)\right] - j^\mu A_\mu
\end{aligned}
$$

Now from (3.24 :)

$$
\begin{aligned}
\frac{\delta\mathscr{L}}{\delta\left(\partial_0 A_\mu\right)} &= -\frac{1}{2}\left[g^{\mu\alpha}g^{\nu\beta}\frac{\delta\left(\partial_\mu A_\nu\right)}{\delta\left(\partial_0 A_\mu\right)}\left(\partial_\alpha A_\beta\right) + g^{\mu\alpha}g^{\nu\beta}\left(\partial_\mu A_\nu\right)\frac{\delta\left(\partial_\alpha A_\beta\right)}{\delta\left(\partial_0 A_\mu\right)}\right. \\
&\qquad -g^{\mu\alpha}g^{\nu\beta}\frac{\delta\left(\partial_\mu A_\nu\right)}{\delta\left(\partial_0 A_\mu\right)}\left(\partial_\beta A_\alpha\right) - g^{\mu\alpha}g^{\nu\beta}\left(\partial_\mu A_\nu\right)\frac{\delta\left(\partial_\beta A_\alpha\right)}{\delta\left(\partial_0 A_\mu\right)} \\
&\qquad \left.+\frac{1}{\xi}g^{\mu\alpha}g^{\nu\beta}\frac{\delta\left(\partial_\beta A_\nu\right)}{\delta\left(\partial_0 A_\mu\right)}\partial_\mu A_\alpha + \frac{1}{\xi}g^{\mu\alpha}g^{\nu\beta}\left(\partial_\beta A_\nu\right)\frac{\delta\left(\partial_\mu A_\alpha\right)}{\delta\left(\partial_0 A_\mu\right)}\right] \\
\implies \frac{\delta\mathscr{L}}{\delta\left(\partial_0 A_\mu\right)} &= -\frac{1}{2}\left[g^{\mu\alpha}g^{\nu\beta}\delta^0_\mu\delta^\mu_\nu\left(\partial_\alpha A_\beta\right) + g^{\mu\alpha}g^{\nu\beta}\left(\partial_\mu A_\nu\right)\delta^0_\alpha\delta^\mu_\beta\right. \\
&\qquad \left.-g^{\mu\alpha}g^{\nu\beta}\left(\delta^0_\mu\delta^\mu_\nu\right)\left(\partial_\beta A_\alpha\right) - g^{\mu\alpha}g^{\nu\beta}\left(\partial_\mu A_\nu\right)\delta^0_\beta\delta^\mu_\alpha\right] \\
&\qquad -\frac{1}{2}\left[\frac{1}{\xi}g^{\mu\alpha}g^{\nu\beta}\delta^0_\beta\delta^\mu_\nu\left(\partial_\mu A_\alpha\right) + \frac{1}{\xi}g^{\mu\alpha}g^{\nu\beta}\left(\partial_\beta A_\nu\right)\delta^0_\mu\delta^\mu_\alpha\right] \\
\implies \frac{\delta\mathscr{L}}{\delta\left(\partial_0 A_\mu\right)} &= -\frac{1}{2}\left[\delta^0_\mu\delta^\mu_\nu\left(\partial^\mu A^\nu\right) + \left(\partial 6\alpha A^\beta\right)\delta^0_\alpha\delta^\mu_\beta\right. \\
&\qquad \left.-\delta^0_\mu\delta^\mu_\nu\left(\partial^\nu A^\mu\right) - \left(\partial^\alpha A^\beta\right)\delta^0_\beta\delta^\mu_\alpha\right] \\
&\qquad -\frac{1}{2}\left[\frac{1}{\xi}\delta^0_\beta\delta^\mu_\nu g^{\nu\beta}\left(\partial_\mu A^\mu\right) + \frac{1}{\xi}g^{\mu\alpha}\left(\partial_\beta A^\beta\right)\delta^0_\mu\delta^\mu_\alpha\right] \\
\implies \frac{\delta\mathscr{L}}{\delta\left(\partial_0 A_\mu\right)} &= -\frac{1}{2}\left[\partial^0 A^\mu + \partial^0 A^\mu - \partial^\mu A^0 - \partial^\mu A^0 + \frac{1}{\xi}g^{\mu 0}\left(\partial_\nu A^\nu\right) + \frac{1}{\xi}g^{0\mu}\left(\partial_\beta A^\beta\right)\right]
\end{aligned}
$$

As β is a dummy index, we can change it to ν :

$$
\begin{aligned}
\implies \frac{\delta\mathscr{L}}{\delta\left(\partial_0 A_\mu\right)} &= -\dot{A}^\mu + \partial^\mu A^0 - \frac{1}{\xi}g^{\mu 0}\left(\partial_\nu A^\nu\right) \\
&= -\dot{A}^\mu + \delta^0_\mu\delta^\mu_\nu\partial^\nu A^\mu - \frac{1}{\xi}g^{\mu 0}\left(\partial_\nu A^\nu\right) \\
&= -\dot{A}^\mu + \delta^0_\mu\partial^\mu A^\mu - \frac{1}{\xi}g^{\mu 0}\left(\partial_\nu A^\nu\right) \\
&= -\dot{A}^\mu + \delta^0_\mu g^{\mu\nu}\left(\partial_\nu A^\mu\right) - \frac{1}{\xi}g^{\mu 0}\left(\partial_\nu A^\nu\right)
\end{aligned}
$$

$g^{\mu\nu}$ is non-zero for only $\mu = \nu$, so :

$$
= -\dot{A}^\mu + \delta^0_\nu g^{\mu\nu}\left(\partial_\nu A^\nu\right)\frac{1}{\xi}\left(\partial_\nu A^\nu\right)
$$

Thus we get the final equation as :

$$\Pi^{\mu} = \frac{\delta \mathscr{L}}{\delta\left(\partial_0 A_{\mu}\right)} = -\dot{A}^{\mu} + \left(1 - \frac{1}{\xi}\right) g^{\mu 0}\left(\partial_{\nu} A^{\nu}\right) \tag{3.25}$$

So, we get Π^0 as :

$$\Pi^{0} = -\dot{A}^{0} + \left(1 - \frac{1}{\xi}\right)\left(\partial_{\nu} A^{\nu}\right) \tag{3.26}$$

It no longer vanished identically.

So, we can now attempt to write the propagator of the EM field.

3.4 Propagator of Electromagnetic Field

From (3.23) we can see that the new Lagrangian after introducing the Gauge fixing term can be written as :

$$\begin{aligned}
\mathscr{L} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^{\mu} A_{\mu} - \frac{1}{2\xi}\left(\partial_{\mu} A^{\mu}\right)^2 \\
\therefore \mathscr{L} &= -\frac{1}{2} g^{\mu\alpha} g^{\nu\beta}\left[\left(\partial_{\mu} A_{\nu}\right)\left(\partial_{\alpha} A_{\beta}\right) - \left(\partial_{\mu} A_{\nu}\right)\left(\partial_{\beta} A_{\alpha}\right) + \frac{1}{\xi}\left(\partial_{\beta} A_{\nu}\right)\left(\partial_{\mu} A_{\alpha}\right)\right] - j^{\mu} A_{\mu}
\end{aligned}$$

3.4.1 Equations of Motion for the modified Lagrangian

We will attempt to find the equation of motion for the modified Lagrangian.

Euler-Lagrange equations :

$$\partial_{\mu}\left(\frac{\partial \mathscr{L}}{\partial\left(\partial_{\mu} A_{\nu}\right)}\right) = \frac{\partial \mathscr{L}}{\partial A_{\nu}}$$

For LHS :

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial\left(\partial_{\mu} A_{\nu}\right)} &= -\frac{1}{2} g^{\mu\alpha} g^{\nu\beta}\left[\left(\partial_{\alpha} A_{\beta}\right) + \left(\partial_{\mu} A_{\nu}\right)\delta_{\alpha}^{\nu}\delta_{\beta}^{\mu} - \left(\partial_{\beta} A_{\alpha}\right) - \left(\partial_{\mu} A_{\nu}\right)\delta_{\alpha}^{\nu}\delta_{\beta}^{\mu} + \frac{1}{\xi}\delta_{\beta}^{\mu}\left(\partial_{\mu} A_{\alpha}\right) + \frac{1}{\xi}\left(\partial_{\beta} A_{\nu}\right)\delta_{\alpha}^{\nu}\right] \\
&= -\frac{1}{2}\left[\partial^{\mu} A^{\nu} + \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} - \partial^{\nu} A^{\mu} + \frac{1}{\xi}\left(\partial^{\nu}\partial^{\mu} + \partial^{\nu}\partial^{\mu}\right)\right] \\
&= -\partial^{\mu} A^{\nu} + \left(1 - \frac{1}{\xi}\right)\partial^{\nu} A^{\mu} \\
\therefore \frac{\partial \mathscr{L}}{\partial\left(\partial_{\mu} A_{\nu}\right)} &= -\partial^{\mu} A^{\nu} + \left(1 - \frac{1}{\xi}\right) g^{\mu\nu}\partial_{\mu} A^{\mu}
\end{aligned}$$

Now we can find the complete RHS as :

$$\begin{aligned}
\partial_{\mu}\left(\frac{\partial \mathscr{L}}{\partial\left(\partial_{\mu} A_{\nu}\right)}\right) &= -\partial_{\mu}\partial^{\mu} A^{\nu} + \left(1 - \frac{1}{\xi} g^{\mu\nu}\right)\partial_{\mu}\left(\partial_{\mu} A^{\mu}\right) \\
&= -\Box A^{\nu} + \left(1 - \frac{1}{\xi}\right)\partial^{\nu}\left(\partial_{\mu} A^{\mu}\right)
\end{aligned}$$

Thus, we finally get the equation of motion as :

$$\Box A^{\nu} - \left(1 - \frac{1}{\xi}\right) \partial^{\nu} \left(\partial_{\mu} A^{\mu}\right) = j^{\nu} \tag{3.27}$$

Now, raising and lowering a few indices :

$$g^{\mu\nu} \Box A_{\mu} - \left(1 - \frac{1}{\xi}\right) \partial^{\nu} \left(\partial^{\mu} A^{\mu}\right) = j^{\nu}$$

Thus, the final equations of motion can be written as :

$$\boxed{\left\{ g^{\mu\nu} \Box - \left(1 - \frac{1}{\xi}\right) \partial^{\nu} \partial^{\mu} \right\} A_{\mu} = j^{\nu}} \tag{3.28}$$

3.4.2 Propagator for the modified Lagrangian

We can define the propagator as we did in (3.16) :

$$\left\{ g^{\lambda\mu} \Box - \left(1 - \frac{1}{\xi} \partial^{\lambda} \partial^{\mu}\right) \right\} D_{\mu\nu}(x - x') = g^{\lambda}_{\nu} \delta^{4}(x - x') \tag{3.29}$$

Taking the fourier transform, we get :

$$- \left\{ g^{\lambda\mu} k^{2} - \left(1 - \frac{1}{\xi}\right) k^{\lambda} k^{\mu} \right\} D_{\mu\nu}(k) = g^{\lambda}_{\nu} \tag{3.30}$$

Using the general form of $D_{\mu\nu}(k)$ as defined in (3.20) :

$$D_{\mu\nu}(k) = a g_{\mu\nu} + b k_{\mu} k_{\nu}$$

Putting this value in the equation for the Fourier transform (3.30), we get :

$$\begin{aligned} g^{\lambda}_{\nu} &= - \left[g^{\lambda\mu} k^{2} - \left(1 - \frac{1}{\xi}\right) k^{\lambda} k^{\mu} \right] \left(a g_{\mu\nu} + b k_{\mu} k_{\nu}\right) \\ &= - \left[a g^{\lambda}_{\nu} k^{2} + b k^{2} k^{\lambda} k_{\nu} - a \left(1 - \frac{1}{\xi}\right) k^{\lambda} k^{\nu} - b \left(1 - \frac{1}{\xi}\right) k^{2} k^{\lambda} k_{\nu} \right] \\ \therefore g^{\lambda}_{\nu} &= - a k^{2} g^{\lambda}_{\nu} \left[a \left(1 - \frac{1}{\xi}\right) - b \frac{1}{\xi} k^{2} \right] k^{\lambda} k_{\nu} \end{aligned}$$

We can see immediately that this time the b term has not been eliminated completely. We can solve for $D_{\mu\nu}(k)$.
For $\lambda = \nu$:

$$\begin{aligned} 1 &= - a k^{2} + \left[a \left(1 - \frac{1}{\xi}\right) - \frac{b}{\xi} k^{2} \right] k^{2} \\ \implies 1 &= - \cancel{a k^{2}} + \cancel{a k^{2}} - \frac{a k^{2}}{\xi} - \frac{b k^{4}}{\xi} \end{aligned}$$

$$\implies 1 = -\frac{1}{\xi}\left(ak^2 + bk^4\right)$$

Let:

$$a = -\frac{1}{k^2}$$

Then we obtain b as :

$$b = \frac{1-\xi}{k^4}$$

Thus,

$$a = -\frac{1}{k^2}\,; \quad b = \frac{1-\xi}{k^4} \tag{3.31}$$

Then we obtain the propagator as :

$$D_{\mu\nu}(k) = -\frac{1}{k^2}\left[g_{\mu\nu} - (1-\xi)\,\frac{k_\mu k_\nu}{k^2}\right] \tag{3.32}$$

The Feynman propagator can be written as :

$$\boxed{D_{\mu\nu}(k) = -\frac{1}{k^2 + i\varepsilon}\left[g_{\mu\nu} - (1-\xi)\,\frac{k_\mu k_\nu}{k^2}\right]} \tag{3.33}$$

The parameter ξ labels a family of theories, classical in nature, which are equivalent to Maxwell's theory in our chosen gauge : $\partial_\mu A^\mu = 0$.

We will choose the value of $\xi = 1$, called the **Feynman-'t Hooft gauge**. The propagator in this Gauge becomes :

$$\boxed{iD_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2 + i\varepsilon}} \tag{3.34}$$

In this gauge, the equation of motion (3.28) becomes :

$$\Box A^\mu = j^\mu \tag{3.35}$$

It looks analogous to the equations for four massless scalar fields.

3.5 Fourier Decomposition of Photon Field

As for the scalar and Dirac fields, we can write the Fourier decomposition of the photon field as :

$$A^\mu(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \sum_{r=0}^{3} \left[\epsilon_r^\mu(k) a_r(k) \exp(-ik.x) + \epsilon_r^{*\mu}(k) a_r^\dagger(k) \exp(ik.x)\right] \tag{3.36}$$

where, $\omega_k = |\boldsymbol{k}|$.

ϵ_r^μ are a set of four linearly independent 4-vectors, called polarisation vectors. They carry the transformation property of A^μ field, A^μ transforms as a 4-vector and apart from the polarisation vectors, all other components of (3.36) transform as Lorentz scalars.

We can write the orthonormality condition of polarisation vectors as :

$$\boxed{\epsilon_r^\mu \epsilon_{s\mu}^* = \zeta_r \delta_{rs}} \tag{3.37}$$

where,

$$\zeta_0 = -1\,; \quad \zeta_1 = \zeta_2 = \zeta_3 = 1 \tag{3.38}$$

The completeness relation is given as :

$$\boxed{\sum_{r=0}^{3} \zeta_r \epsilon_r^\mu \epsilon_r^{*\nu} = -g^{\mu\nu}} \tag{3.39}$$

3.5.1 Specific choice of Polarisation Vectors

We can choose some specific values of the polarisation vector to get some more insight.

Consider a time-like vector n^μ, which satisfies :

$$n^\mu n_\mu = 1 \quad n^0 > 1$$

We can set ϵ_0^μ to n^μ. We call this vector the scalar polarisation vector.
Then, we can choose ϵ_3^μ to be the longitudinal polarisation vector in the n-k plane. We will mostly use this choice. The other two polarisation vectors are defined orthogonal to the n-k plane. They are called transverse polarisation vectors.

$$\epsilon_0^\mu = n^\mu \tag{3.40}$$

$$\epsilon_3^\mu = \frac{k^\mu - (k.n)\, n^\mu}{\sqrt{(k.n)^2 - k^2}} \tag{3.41}$$

$$\epsilon_r^\mu(k)\epsilon_{s\mu}^*(k) = -\delta_{rs} \quad r = 1, 2 \tag{3.42}$$

3.5.2 Commutation relations

We can guess the commutation relations from scalar field theory, and attempt to write them as :

$$[A_\mu(t, \boldsymbol{x}), \Pi^\nu(t, \boldsymbol{y})] = i\delta_\mu^\nu \delta^3(\boldsymbol{x} - \boldsymbol{y}) \tag{3.43}$$

The other commutation relations implied are :

$$[A_\mu(t,\boldsymbol{x}), A_\nu(t,\boldsymbol{y})] = 0 \tag{3.44}$$

$$[\Pi_\mu(t,\boldsymbol{x}), \Pi_\nu(t,\boldsymbol{y})] = 0 \tag{3.45}$$

The creation and annihilation operators obey the relations :

$$\left[a_r(k), a_s^\dagger(k')\right] = \zeta_r \delta_{rs} \delta^3(\boldsymbol{k} - \boldsymbol{k'}) \tag{3.46}$$

But this does not give us the correct commutation relation for a_0 and $a_0^\dagger$. This problem arises from the fact that we have four independent polarisation vectors, when we only need two to completely define the electromagnetic fields. We will address this issue in the next section by considering the physical states of EM field.

3.6 Physical States

3.6.1 Need for defining Physical States

We will start by first defining the normalised vacuum state as :

$$a_r(k)|0\rangle = 0\,; \quad \forall k\,, \forall r \tag{3.47}$$

Then the state containing a single photon with momentum k and polarisation vector ϵ_r is given by :

$$|k, r\rangle \equiv a_r^\dagger(k)|0\rangle \tag{3.48}$$

Now, attempting to take the norm of this field for $r = 0$,i.e., a state with polarisation vector ϵ_0 :

$$\langle k, 0|k', 0\rangle = \langle 0|a_0(k) a^\dagger(k')|0\rangle$$

Using the commutation relations described in (3.46) :

$$\begin{aligned}
-\delta^3(\boldsymbol{k} - \boldsymbol{k'}) &= a_0(k)a_0^\dagger(k') - a_0^\dagger(k')a_0(k) \\
\implies -\langle 0|\delta^3(\boldsymbol{k} - \boldsymbol{k'})|0\rangle &= \langle 0|a_0(k)a_0^\dagger(k') - a_0^\dagger(k')a_0(k)|0\rangle \\
\implies -\delta^3(\boldsymbol{k} - \boldsymbol{k'})\langle 0|0\rangle &= \langle 0| \left[a_0(k), a_0^\dagger(k')\right] |0\rangle = \langle 0|a_0(k)a_0^\dagger(k')|0\rangle \\
\implies -\delta^3(\boldsymbol{k} - \boldsymbol{k'}) &= \langle 0| \left[a_0(k), a_0^\dagger(k')\right] |0\rangle = \langle k, 0|k', 0\rangle \\
\therefore \langle k, 0|k', 0\rangle &= -\delta^3(\boldsymbol{k} - \boldsymbol{k'})
\end{aligned}$$

We observe that the norm is negative, which should not be. This issue is again an offshoot of the problem of degrees of freedom.

Electromagmetic fields have two degrees of freedom but we are using four independent polarisation vectors to describe them.

3.6.2 Defining Physical States

We have to take into account that we are attempting to quantise the Maxwell's equations under the Lorentz gauge (3.22), not for all A_μ. But, in operator formalism we cannot directly set $\partial_\mu A^\mu$ to zero.

The problem was solved by Gupta and Bleuler. Setting $\partial_\mu A^\mu = 0$ is not necessary, we just need to have two states $\Psi\rangle$ and $\Psi'\rangle$ such that the matrix element of $\partial_\mu A^\mu$ between the states gives 0, i.e., :

$$\langle\Psi'|\partial_\mu A^\mu|\Psi\rangle = 0 \tag{3.49}$$

We can now use this to define a **physical** state $|\Psi\rangle$ of the field as :

$$\partial_\mu A^\mu_+|\Psi\rangle = 0\,; \quad \langle\Psi|\partial_\mu A^\mu_- = 0 \tag{3.50}$$

where,

$$\begin{aligned}\partial_\mu A^\mu_+ &= \text{Annihilation operator part of } A^\mu \\ \partial_\mu A^\mu_- &= \text{Creation operator part of } A^\mu\end{aligned}$$

So, the total operator $\partial_\mu A^\mu$ can be written as :

$$\partial_\mu A^\mu = \partial_\mu A^\mu_- + \partial_\mu A^\mu_+ \tag{3.51}$$

Now, with these definitions, (3.49) is directly satisfied. So $\partial_\mu A^\mu_+$ operator can now be written as :

$$\partial_\mu A^\mu_+(x) = -i\int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \sum_{r=0}^{3} \epsilon^\mu_r(k) a_r(k) \exp(-ik.x) \tag{3.52}$$

a.) Lorentx invariance of $\partial_\mu A^\mu_+$.

Let us consider a coordinate transformation :

$$\{x^\mu\} \longrightarrow \{y^\mu\} = \Lambda^\mu_\nu x^\nu$$

So writing $\partial_\mu A^\mu_+$ in the new coordinate system :

$$\partial_\mu A^\mu_+(y) = -i\int \frac{d^3k'}{\sqrt{(2\pi)^3 2\omega_{k'}}} \sum_{r=0}^{3} k'_\mu \epsilon'^\mu_r(k') a_r(k') \exp(-ik'.y)$$

We can write the transformation of k and ϵ_r as :

$$k'_\mu = \Lambda^\nu_\mu k_\nu \,;\, \epsilon'^\mu_r = \Lambda^\mu_\nu \epsilon^\nu_r$$

Thus,

$$\partial_\mu A^\mu_+(y) = -i \int \frac{d^3k'}{\sqrt{(2\pi)^3 2\omega_{k'}}} \sum_{r=0}^{3} \Lambda^\nu_\mu \Lambda^\mu_\nu k_\nu \epsilon'^\mu_r(k') a_r(k') \exp(-ik.x)$$

As $\Lambda^\nu_\mu \Lambda^\mu_\nu = 1$:

$$\partial_\mu A^\mu_+(y) = \partial_\mu A^\mu_+(x)$$

It is Lorentz invariant. So, we can work with it in any frame.

3.6.3 Finding the Hamiltonian

We will try to find the Hamiltonian for EM fields. We can work on any chosen frame as $\partial_\mu A^\mu_+$ operator is Lorentz invariant.

Let us consider a frame where :

$$k^\mu = (\omega_k, 0, 0, \omega_k)\,; \quad k_\mu = (\omega_k, 0, 0, -\omega_k)$$

Let the polarisation vectors be :

$$\begin{aligned} \epsilon^\mu_0 &= (1, 0, 0, 0) \\ \epsilon^\mu_1 &= (0, 1, 0, 0) \\ \epsilon^\mu_2 &= (0, 0, 1, 0) \\ \epsilon^\mu_3 &= (0, 0, 0, 1) \end{aligned}$$

Then, we can find the product of k_μ with the polarisation vectors as :

$$k_\mu \epsilon^\mu_0 = \omega_k\,; \quad k_\mu \epsilon^\mu_1 = 0\,; \quad k_\mu \epsilon^\mu_2 = 0\,; \quad k_\mu \epsilon^\mu_3 = -\omega_k$$

Putting these values in the (3.52) :

$$\partial_\mu A^\mu_+(x) = -i \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \omega_k \left[a_0(k) - a_3(k)\right] \exp(-ik.x)$$

We also need to satisfy (3.50) :

$$\begin{aligned} &\implies \partial_\mu A^\mu_+(x)|\Psi\rangle = -i \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \omega_k \exp(-ik.x) \left[a_0(k) - a_3(k)\right] |\Psi\rangle = 0 \\ &\implies \langle\Psi|\partial_\mu A^\mu_-(x) = i \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \omega_k \exp(-ik.x) \langle\Psi| \left[a^\dagger_0(k) - a^\dagger_3(k)\right] = 0 \end{aligned}$$

From the above two equations, we see that the annihilation and creation operators satisfy the relations :

$$a_0(k)|\Psi\rangle = a_3(k)|\Psi\rangle \tag{3.53}$$
$$\langle\Psi|a_0^\dagger(k) = \langle\Psi|a_3^\dagger(k) \tag{3.54}$$

In the Feynman-t Hooft gauge, $\xi = 1$, we get the conjugate momenta (3.26) as :

$$\Pi^\mu = -\dot{A}^\mu$$
$$\therefore \Pi^\mu = i\int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} k_0 \sum_{r=0}^{3}\left[\epsilon_r^\mu(k)a_r(k)\exp(-ik.x) - \epsilon_r^{*\mu}a_r^\dagger(k)\exp(ik.x)\right] \tag{3.55}$$

The Hamiltonian is given as :

$$H = \int d^3x\left(\dot{A}^\mu\Pi_\mu\right) \tag{3.56}$$

Evaluation using (3.55) :

$$H = (-i)(i)\int d^3x \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\omega_k \int \frac{d^3k'}{\sqrt{(2\pi)^3 2\omega_k'}}\omega_k'$$
$$\sum_{r,s=0}^{3}\left[\epsilon_r^\mu(k)a_r(k)\exp(-ik.x) - \epsilon_r^{*\mu}(k)a_r^\dagger(k)\exp(ik.x)\right]$$
$$\times\left[\epsilon_{s\mu}(k')a_s(k')\exp(-ik'.x) - \epsilon_{s\mu}^*(k')a_s^\dagger(k')\exp(ik'.x)\right]$$

$$= \frac{1}{(2\pi)^3}\int d^3x \int \frac{d^3k\,\omega_k}{\sqrt{2\omega_k}}\int \frac{d^3k'\,\omega_k'}{\sqrt{2\omega_k'}}\sum_{r,s=0}^{3}\left[\epsilon_r^\mu(k)\epsilon_{s\mu}(k')a_r(k)\exp(-ik.x)\exp(-ik'.x)\right.$$
$$-\epsilon_r^\mu(k)\epsilon_{s\mu}^*(k')a_r(k)a_s^\dagger(k')\exp(-ik.x)\exp(ik'.x) \; -\epsilon_r^{*\mu}(k)\epsilon_{s\mu}(k')a_r^\dagger(k)a_s(k')\exp(ik.x)\exp(-ik'.x)$$
$$\left.+\epsilon_r^{*\mu}(k)\epsilon_{s\mu}^*(k')a_r^\dagger(k)a_s^\dagger(k')\exp(ik.x)\exp(ik'.x)\right]$$

Pushing the integration over x inside and using the definition of Dirac delta function (1.74) :

$$= \int \frac{d^3k\,\omega_k}{\sqrt{2\omega_k}}\int \frac{d^3k'\,\omega_k'}{\sqrt{2\omega_k'}}\sum_{r,s=0}^{3}\left[\epsilon_r^\mu(k)\epsilon_{s\mu}(k')a_r(k)\delta^3(\boldsymbol{k}-(-\boldsymbol{k'})) \; -\epsilon_r^\mu(k)\epsilon_{s\mu}^*(k')a_r(k)a_s^\dagger(k')\delta^3(\boldsymbol{k}-\boldsymbol{k'})\right.$$
$$\left.-\epsilon_r^{*\mu}(k)\epsilon_{s\mu}(k')a_r^\dagger(k)a_s(k')\delta^3(\boldsymbol{k'}-\boldsymbol{k}) \; +\epsilon_r^{*\mu}(k)\epsilon_{s\mu}^*(k')a_r^\dagger(k)a_s^\dagger(k')\delta^3(-\boldsymbol{k}-\boldsymbol{k'})\right]$$

Using the orthonormality relation of polarisation vectors from (3.37) :

$$= \int \frac{d^3k\,\omega_k}{\sqrt{2\omega_k}}\int \frac{d^3k'\,\omega_k'}{\sqrt{2\omega_k'}}\sum_{r,s=0}^{3}\left[0 + \zeta_r\delta_{rs}\delta^3(\boldsymbol{k}-\boldsymbol{k'})a_r(k)a_s^\dagger(k') \; +\zeta_r\delta_{rs}\delta^3(\boldsymbol{k}-\boldsymbol{k'})a_r^\dagger(k)a_s(k') + 0\right]$$

One integration over k gets killed and the summation over one variable, say s gets killed. So we obtain :

$$= \int \frac{d^3k\, \omega_k.\omega_k}{2\omega_k} \sum_{r=0}^{3} \left[\zeta_r a_r(k) a_r^\dagger(k) \; + \zeta_r a_r^\dagger(k) a_r(k) \exp(ik.x)\right]$$

Now, using the commutation relation of annihilation and creation operators given in (3.46) :

$$= \int \frac{d^3k\, \omega_k}{2} \sum_{r=0}^{3} \left[\zeta_r a_r^\dagger(k) a_r(k) \; + \zeta_r a_r^\dagger(k) a_r(k)\right] = \int d^3k\, \omega_k \sum_{r=0}^{3} \left[\zeta_r a_r^\dagger(k) a_r(k)\right]$$

Thus, we obtain the normal-ordered Hamiltonian as :

$$\boxed{: H : = \int d^3k \sum_{r=1}^{3} \left[a_r^\dagger(k) a_r(k) - a_0^\dagger(k) a_r(k)\right]} \tag{3.57}$$

Now we can take the matrix element of the Hamiltonian between any two physical states :

$$\begin{aligned} \langle \Psi' | : H : | \Psi \rangle = \int d^3k \Big\{ &\langle \Psi' | a_3^\dagger(k) a_3(k) - a_0^\dagger(k) a_0(k) | \Psi \rangle \Big\} \\ &+ \sum_{r=1}^{2} \Big[\langle \Psi' | a_r^\dagger(k) a_r(k) - a_0^\dagger(k) a_0(k) | \Psi \rangle \Big] \end{aligned}$$

The first term vanishes due to (3.53) and (3.54)

$$\therefore \langle \Psi' | : H : | \Psi \rangle = \int d^3k \sum_{r=1}^{2} \Big[\langle \Psi' | a_r^\dagger(k) a_r(k) - a_0^\dagger(k) a_0(k) | \Psi \rangle \Big] \tag{3.58}$$

So we can see from (3.58), that only two polarisation states, i.e., the transverse polarisation states contribute to the Hamiltonian of the Electromagnetic Field.

Physical quantities do not depend on the scalar polarisation vector, so we can keep using the commutation relation in (3.46) as the problem of negative norm never comes up.

Chapter 4

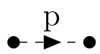
Vacuum Polarization Tensor

Vacuum Polarization is a process which describes the phenomenon of production of virtual electron-positron pairs as a photon propagates. This is also referred to as the Self-Energy of the photon.
The amplitude of vacuum polarization is denoted by $\pi_{\mu\nu}(k)$, where k is the momentum of the external photon.

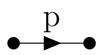
We will first try to understand how the situation arises and then find the amplitude directly from the Feynman diagrams.

4.1 Feynman Rules and Feynman Diagrams

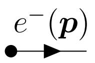
The Feynman diagrams for fermions and photons :

Scalar Propagator (Internal Line) : 

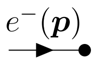
$= i\Delta_F(p)$

Fermion propagator (Internal Line) : 

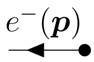
$= iS_F(p)$

External Fermion Line : 

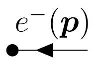
$= \overline{u}_s(\boldsymbol{p})$

External Fermion Line : 

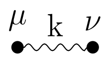
$= u_s(\boldsymbol{p})$

External Anti-Fermion Line : 

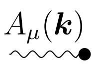
$= \overline{v}_s(\boldsymbol{p})$

External Anti-Fermion Line : 

$= v_s(\boldsymbol{p})$

Photon propgator (Internal Line) : 

$= iD^{\mu\nu}(k)$

External Photon Line : 

$= \epsilon_\mu(k)$

$$\text{External Photon Line}: \quad \overset{A_\mu(\boldsymbol{k})}{\bullet\!\!\sim\!\sim\!\sim} = \epsilon^*_\mu(k)$$

Feynman Rules for fermions and photons :

i. For each vertex we need to put : $(-i\lambda) \int d^4x$

ii. For each external line of momentum p going to x : $\exp(-ip.x)$

iii. For each loop we need to put : $\int \dfrac{d^4p}{(2\pi)^4}$

iv. Summing over all Dirac indices of a loop implies taking a trace over the factors obtained around the loop.

v. For each fermion loop, we need to put in a factor of -1.

vi. Momentum must be conserved at each vertex.

4.2 Local Gauge Invariance

For Scalar and Dirac fields, the Lagrangian is invariant under global transformations :

$$\begin{aligned} \phi &\longrightarrow \phi' = \exp(-ieQ\theta)\phi \\ \psi &\longrightarrow \psi' = \exp(-ieQ\theta)\psi \end{aligned}$$

where, θ is the global parameter.

Now, if we consider θ to be a local parameter instead of global, then we can see the that the Lagrangian of Dirac field (1.54) no longer remains invariant.

a.) Checking if the Lagrangian of the Dirac field is invariant under local transformation.

From (1.54) :

$$\begin{aligned} \mathscr{L}' &= \overline{\psi}'(i\partial\!\!\!/ - m)\psi' \\ &= \exp(ieQ\theta)\overline{\psi}\,(i\gamma^\mu\partial_\mu - m)\,[\exp(-ieQ\theta)\psi] \\ &= \exp(ieQ\theta)\overline{\psi}\,[i\gamma^\mu(-ieQ)(\partial_\mu\theta) - m]\exp(-ieQ\theta)\psi \\ &\qquad\qquad + \exp(ieQ\theta)\overline{\psi}\,[(i\partial\!\!\!/ - m)\psi]\exp(-ieQ\theta) \\ &= \overline{\psi}(i\partial\!\!\!/ - m)\psi + \overline{\psi}(eQ\gamma^\mu\partial_\mu\theta)\psi \end{aligned}$$

Thus, we obtain the transformation of the Lagrangian as :

$$\mathscr{L}' = \mathscr{L} + eQ\left\{\overline{\psi}\gamma^\mu\partial_\mu\theta\psi\right\} \tag{4.1}$$

4.2.1 Enforcing local symmetry on Fermion field

If we wish to make the Lagrangian invariant under local transformation we have to introduce new fields in the free Lagrangian expression (1.54). These terms have to transform like the extra term in (4.1), i.e., they have to transform like a 4-vector as $\partial_\mu \theta$ transforms as a 4-vector. Let us define the new Lagrangian as :

$$\mathscr{L} = \overline{\psi}(i\not{\partial} - m)\psi - eQ\overline{\psi}\gamma^\mu \psi A_\mu \tag{4.2}$$

A_μ is a 4-vector.
We demand that this lagrangian remains invariant under local transformations. So, we will need to find how A_μ transforms to satisfy that.

a.) Finding transformation law for A_μ.
Starting from the new Lagrangian and transforming ψ and A_μ as :

$$\begin{aligned} \psi &\longrightarrow \psi' = \exp(-ieQ\theta)\psi \\ A_\mu &\longrightarrow A'_\mu \end{aligned}$$

Thus,

$$\begin{aligned} \mathscr{L}' &= \overline{\psi}'(i\not{\partial} - m)\psi' - eQ\overline{\psi}'\gamma^\mu \psi' A'_\mu \\ &= \overline{\psi}(i\not{\partial} - m)\psi + eQ\overline{\psi}\gamma^\mu(\partial_\mu \theta)\psi - eQ\overline{\psi}\left[\exp(ieQ\theta)\right]\gamma^\mu \psi \left[\exp(-ieQ\theta)\right] A'_\mu \\ &= \overline{\psi}(i\not{\partial} - m)\psi - eQ\overline{\psi}\gamma^\mu \psi \left[A'_\mu - \partial_\mu \theta\right] \end{aligned}$$

For $\mathscr{L}' = \mathscr{L}$:

$$\implies \overline{\psi}(i\not{\partial} - m)\psi - eQ\overline{\psi}\gamma^\mu \psi \left[A'\mu - \partial_\mu \theta\right] = \overline{\psi}(i\not{\partial} - m)\psi - eQ\overline{\psi}\gamma^\mu \psi A_\mu$$

Thus, we obtain the transformation as :

$$A'_\mu = A_\mu + \partial_\mu \theta \tag{4.3}$$

We can see from (4.3) that the transformation of A_μ is the same as the local transformation that keeps the Lagrangian of the Electromagnetic Field invariant.

Thus, we can identify A_μ with the photon field. So, to define the complete Lagrangian we need to add the Lagrangian for the free photon field.

$$\boxed{\mathscr{L} = \overline{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eQ\overline{\psi}\gamma^\mu \psi A_\mu} \tag{4.4}$$

Here we originally had global symmetry for the free fermion field, and we introduced the photon field into it to enforce local symmetry.

This process is called gauging the global symmetry, and the photon field is called the gauge boson corresponding to the local symmetry that we have enforced. The coefficient of A_μ is called the coupling constant.

4.2.2 Minimal substitution

We can re-write the Lagrangian given in (4.4) as :

$$\boxed{\mathscr{L} = \overline{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}} \tag{4.5}$$

D_μ is called the gauge covariant derivative. It is given as :

$$\boxed{D_\mu = \partial_\mu + ieQA_\mu} \tag{4.6}$$

a.) Finding how $D_{\mu\nu}$ transforms for local symmetry.
Let :

$$\begin{aligned} \phi &\longrightarrow \phi' = \exp(-ieQ\theta)\phi \\ A_\mu &\longrightarrow A'_\mu = A_\mu + \partial_\mu\theta \end{aligned}$$

Now, evaluating $D'_\mu\phi'$:

$$\begin{aligned} D'_\mu\phi' &= \left\{\partial'_\mu + ieQA'_\mu\right\}\phi' \\ &= \{\partial_\mu + ieQ\,(A_\mu + \partial_\mu\theta)\}\exp(-ieQ\theta)\phi \\ &= -\cancel{ieQ(\partial_\mu\theta)\phi\exp(-ieQ\theta)} + \exp(-ieQ\theta)\phi + \exp(-ieQ\theta)ieQA_\mu + \cancel{ieQ(\partial_\mu\theta)\phi\exp(-ieQ\theta)} \end{aligned}$$

So we see that :

$$D'_\mu\phi' = \exp(-ieQ\theta)[D_\mu\theta] \tag{4.7}$$

Thus all terms which were invariant under global symmetry will remain invariant under local symmetry if we use gauge covariant derivative instead of the usual partial derivative.

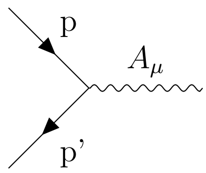
This substitution : $\partial_\mu \longrightarrow D_\mu$ is called **minimal substitution**.

4.3 Interacting Hamiltonian

For the theory of photons and fermions described by the Lagrangian (4.4), the interaction between the two fields is given by :

$$\mathscr{L}_{\text{int}} = -eQ\overline{\psi}\gamma^\mu\psi A_\mu \tag{4.8}$$

The Feynman rule for the vertex can be written as :



$$: \quad = -ieQ\gamma^\mu$$

The normal-ordered interaction Hamiltonian can be written as :

$$\begin{aligned}
&: \mathscr{H}_I : = -e : (\overline{\psi}_+ + \overline{\psi}_-)\gamma^\mu(\psi_+ + \psi_-)(A^\mu_+ + A^\mu_-) : \\
\implies &: \mathscr{H}_I : = -e : (\overline{\psi}_+\gamma^\mu\psi_+ + \overline{\psi}_+\gamma^\mu\psi_- + \overline{\psi}_-\gamma^\mu\psi_+ + \overline{\psi}_-\gamma^\mu\psi_-)(A^\mu_+ + A^\mu_-) : \\
\therefore \quad &: \mathscr{H}_I : = -e : \big(\overline{\psi}_+\gamma^\mu\psi_+A^\mu_+ + \overline{\psi}_+\gamma^\mu\psi_+A^\mu_- + \overline{\psi}_+\gamma^\mu\psi_-A^\mu_+ \\
&\qquad + \overline{\psi}_+\gamma^\mu\psi_-A^\mu_- + \overline{\psi}_-\gamma^\mu\psi_+A^\mu_+ + \overline{\psi}_-\gamma^\mu\psi_+A^\mu_- \\
&\qquad\qquad + \overline{\psi}_-\gamma^\mu\psi_-A^\mu_+ + \overline{\psi}_-\gamma^\mu\psi_-A^\mu_-\big)
\end{aligned}$$

Let us consider the 4-momenta of the two fermion lines to be p and p'. Let the 4-momenta of the photon be k. By momentum conservation at the vertex :

$$p' = \pm p \pm k$$

Squaring :

$$p'^2 = p^2 + k^2 \pm 2p.k$$

We will assume that the particles are on-shell, i.e., they obey $p^2 = m^2$. So :

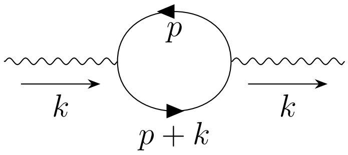
$$p'^2 = p^2 = m^2 \qquad \implies k^2 = 0 \qquad \implies p.k = 0$$

As the equations are Lorentz invariant, these conditions hold in every frame, especially in a frame where the electron is at rest, i.e., $p = (m, 0, 0, 0)$. In this frame, if $p.k = 0$ has to hold, then k has to be a space-like vector, i.e., $k = (0, \boldsymbol{k})$. Since $k^2 = 0$, $\boldsymbol{k}^2 = 0$. So, the photon does not exist for such a process.

Thus, none of the eight processes described by the normal-ordered interaction Hamiltonian are physical processes. We cannot have a physical process with only one QED vertex. We need atleast two.

4.4 Self-Energy Diagram of the Photon

One of the many possible processes involving two QED vertices is the Vacuum Polarisation. The feynman diagram for the process is given as :



By the Feynman rules, we can find the terms and factors showing up in the amplitude.

i. There are two fermions propagating between the two vertices, so there will be two fermion propagators of momentum p and $p+k$: $iS_F(p)$ and $iS_F(p+k)$

ii. There will be a vertex function : $ie\Gamma_\nu(k+p,p)$

iii. We have a fermion loop, so a factor of -1 will show up.

iv. Due to the loop, the amplitude will have $\int \dfrac{d^4p}{(2\pi)^4}$

v. We will take a trace over the factors around the loop, i.e., the fermion propagators and the vetex function.

vi. Numerical factor for the vertex, as shown in previous section : $ieQ\gamma_\mu$

The amplitude is denoted by $\pi_{\mu\nu}$ and can be written as :

$$i\pi_{\mu\nu}(k) = -\int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[ieQ\gamma_\mu iS_F(p) ie\Gamma_\nu(k+p,p) iS_F(k+p)\right] \tag{4.9}$$

To resolve this, we need to use the Ward-Takahashi Identity :

$$\boxed{q^\mu \Gamma_\mu(p,p-q) = Q\left[S_F^{-1}(p) - S_F^{-1}(p-q)\right]} \tag{4.10}$$

We will multiply (4.9) k^ν from the left on both sides :

$$k^\nu \pi_{\mu\nu} = ie^2 Q \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\gamma_\mu S_F(p) k^\nu \Gamma_\nu(k+p,p) S_F(k+p)\right]$$

Using the Ward-Takahashi Identity (4.10) :

$$\implies k^\nu \pi_{\mu\nu} = ie^2 Q^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\gamma_\mu S_F(p) \left(S_F^{-1}(p+k) - S_F^{-1}(p)\right) S_F(p+k)\right]$$

So, we obtain the expression as :

$$k^{\nu}\pi_{\mu\nu} = ie^2Q^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\gamma_\mu S_F(p) - S_F(k+p)\right] \tag{4.11}$$

Now, if we change the loop integration variable from p to $p' = p + k$, then the terms cancel. So we get :

$$k^{\nu}\pi_{\mu\nu}(k) = 0 \tag{4.12}$$

Now, if we multiply with k^{μ} instead of k^{ν} and perform the same calculations, we will obtain :

$$k^{\mu}\pi_{\mu\nu}(k) = 0 \tag{4.13}$$

So, $\pi_{\mu\nu}$ must be a rank-2 tensor depending on k only and satisfying both (4.12) and (4.13).

Just as for $D_{\mu\nu}(k)$ (3.20), we can write the most general guess for $\pi_{\mu\nu}(k)$ as :

$$\pi_{\mu\nu}(k) = ag_{\mu\nu} + bk_\mu k_\nu \tag{4.14}$$

Putting (4.14) in (4.12) :

$$\begin{aligned} k^{\nu}\pi_{\mu\nu}(k) &= k^{\nu}\left(ag_{\mu\nu} + bk_\mu k_\nu\right) \\ \implies 0 &= ak_\mu + bk^2 k_\mu \end{aligned}$$

Putting (4.14) in (4.13) :

$$\begin{aligned} k^{\mu}\pi_{\mu\nu}(k) &= k^{\mu}\left(ag_{\mu\nu} + bk_\mu k_\nu\right) \\ \implies 0 &= ak_\nu + bk^2 k_\nu \end{aligned}$$

Thus, we can deduce that for both the equations to be satisfied :

$$a = k^2\,; \quad b = -1$$

So, we can write the most general form of the Vacuum Polarization Tensor is given as :

$$\pi_{\mu\nu}(k) = \left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\Pi(k^2) \tag{4.15}$$

$\Pi(k^2)$ is a Lorentz scalar and is a function of k^2 only.

The Vacuum Polarization Tensor is proportional to two powers of external momentum and that makes sense as there are two external photon lines. The amplitude must be proportional to at least two powers of external momentum.

Chapter 5

Conclusion

Quantum field theory gives us an accurate description of the particles or energy packets (like photons), by describing them as quantisations of fields which can be created or destroyed at any space-time point. This immediately explains the indistinguishability of particles and gives a description which is relativistically accurate.
Dirac's Hamiltonian also shows that fermions have an intrinsic angular momentum, called spin, as a result of relativistic covariance of the Hamiltonian.

From the discussion of Local Gauge invariance, we concluded that if we want to impose local symmetry on the Fermion field, it must couple or interact with the photon field. Also, the interaction must have two vertices for it to be a physical process.

Many different physical processes can be described by solving the S-Matrix expansion to second order, but we focussed on the process of Vacuum Polarization, which is also called the Self-Energy Diagram of the photon.
We saw that as a photon propagates, it produces virtual electron-positron pairs, which are vert short lived. The amplitude of the process depends on the momentum of the incoming and outgoing photon, and the space-time metric. This is a purely quantum phenomenon.
The effects of Vacuum Polarization were more recently observed by TRISTAN Particle accelerator in Japan in 1997.

Thus, in this project I have studied the Fermion and Electromagnetic fields and then derived their Hamiltonian, the corresponding Feynman Propagators, S-Matrix Expansion and the Wick's Theorem. To conclude, I have used the above results and Ward-Takahashi Identity to derive the Vacuum Polarization Tensor.

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