Paradox of Low-Energy Excitations about AdS Black Holes

Submitted By

ASHMITA PANDA

School of Physical Sciences

National Institute of Science, Education and Research (NISER), Bhubaneswar

Under the Guidance of

Prof. Suvrat Raju

String Theory and Quantum Gravity Group

International Centre for Theoretical Sciences (ICTS), Bengaluru

Project Report

S.N. Bhatt Memorial Excellence Fellowship Program

ICTS, Bengaluru

July, 2022

Acknowledgement

I would like to express my deep and sincere gratitude to my guide Prof. Suvrat Raju. The project would not have been possible without his guidance and support. His expertise on the field allowed me to delve deeper into the subject matter. I also would like to thank Chandramouli Chowdhury, who was constantly availble to clarify my doubts and provide valuable feedback.

I would also like to express my gratitude towards the S.N.Bhatt Program 2022 and all its organisers for selecting me for the fellowship. It was a great learning oppurtunity.

Lastly, I would also like to thank all my fellow S.N. Bhatt scholars and the students of ICTS, who welcomed us with open arms, for making this stay an incredibly enjoyable experience. I would cherish the memories made and the bonds formed for the rest of my life.

Ashmita Panda
5th Year Integrated M.Sc.
School of Physical Sciences
NISER, Bhubaneswar

Abstract

A version of the Information Paradox asks the question : "Can holography describe the black-hole interior?". This can be resolved by using a state-dependent mapping between the interior bulk observables and the boundary observables. However, this in turn leads to the paradox of low energy excitations in AdS black-holes. An appropriately chosen unitary operator in the boundary can create a locally strong excitation near the black-hole horizon. This seemingly violates the statistical mechanics principle that an operator with energy parametrically smaller than kT cannot create a strong excitation in a thermal system. In this project, we review this paradox and show that the boundary operator producing the small excitation and the bulk observables showcasing the resultant strong excitation do not lie in the same causal patch. No observer can both produce the small boundary excitation and also observe its effects in the bulk.

Contents

Chapter 1

Introduction to Black Holes and QFT in Curved Spacetime

Black holes are objects in the Universe described by simple solutions to the Einstein equations, which can provide us with greater insights about causal structure, thermodynamics and quantum gravity.

We also primarily deal with free fields that will be coupled to the curvature of spacetime. One key difference between quantum fields in curved and flat spaces is that the notion of particles in a space is an observer dependent notion in quantum fields in curved spaces. This is because we do not have any set of preferred observers, like inertial observers in curved spacetime. We have accelerated observers and thus the notion of particles becomes observer dependent.

1.1 Schwarzschild Black Hole

The simplest black hole solution is the Schwarzschild solution with the metric :

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1.1}$$

We can prove that the metric described by (1.1) is a solution of the vacuum Einstein equation, given as $R_{\mu\nu} = 0$, where $R_{\mu\nu}$ is the Ricci tensor.

The Schwarzschild solution is also the unique spherically symmetric, asymptotically flat solution to the Einstein equations. This is known as the **Birkhoff's Theorem**.
Thus, not only does the Schwarzschild solution describe a blackhole, but it also describes the spacetime outside any non-rotating spherically symmetric object.

1.1.1 Derivation of Schwarzschild Solution

For deriving the black hole solution, we impose the following primary conditions :

(i) **Spherical Symmetry** - There should not be any explicit dependence on θ and ϕ in the metric as it breaks spherical symmetry. This also implies that the constant "r" slices are like spheres.

(ii) **Time Translational Symmetry (Stationarity)** - This is a simplifying assumption that demands that the spacetime does not evolve with time. All metric components are independent of the time "t" coordinate.

(iii) **Asymptotic Flatness** - The condition of asymptotic flatness is required by nature. If the space-time is non-trivial at any point, the effect must reduce to flat space at far enough distances.

After imposing these conditions we obtain the metric of the following form :

$$ds^2 = A^{(0)} dr_{(0)}^2 + B^{(0)} dt_{(0)}^2 + 2C^{(0)} dr_{(0)} dt_{(0)} + D^{(0)} \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1.2}$$

The coefficients $A^{(0)}$, $B^{(0)}$, $C^{(0)}$ and $D^{(0)}$ are functions of the coordinates $\left(r_{(0)}\right)$. We can make several coordinate transformations to reduce the number of unknown functions, keeping stationarity and spherical symmetry of the metric intact.

Step 1: We define a new radial coordinate, as :

$$r = \sqrt{D^{(0)}} \tag{1.3}$$

Differentiating the above equation (1.3), we obtain $dr_{(0)}$ in terms of dr :

$$\begin{aligned} dr &= \frac{1}{2\sqrt{D^{(0)}}} \left[\frac{dD^{(0)}}{r_{(0)}}\right] dr_{(0)} \\ \implies dr_{(0)} &= 2\sqrt{D^{(0)}} \left[\frac{dD^{(0)}}{dr_{(0)}}\right]^{-1} dr \\ \therefore\ dr_{(0)} &= 2r \left(\frac{dr_{(0)}}{dr}\right) dr \end{aligned}$$

Substituting the values in the line element (1.2) :

$$ds^2 = A^{(0)} \times 4r^2 \left(\frac{dr_{(0)}}{dr}\right) dr^2 + B^{(0)} dt_{(0)}^2 + 2C^{(0)} \times 2r \left(\frac{dr_{(0)}}{dr}\right) dr dt_{(0)} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

The coefficients can now be written as functions of the new coordinate r, instead of $r_{(0)}$. Thus, we finally obtain the line element as :

$$ds^2 = A^{(1)} dr^2 + B^{(1)} dt_{(0)}^2 + 2C^{(1)} dr dt_{(0)} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1.4}$$

The coefficients $A^{(1)}$, $B^{(1)}$ and $C^{(1)}$ are now functions of r and the functional dependence due to $D^{(0)}$ has been replaced by r^2.

Step 2: Now we can redefine the coordinate $t_{(0)}$. We can define it in a way to remove the cross term. We can begin by defining the new coordinate 't' as :

$$t = t_{(0)} + f(r) \tag{1.5}$$

f is any suitable function which depends only on r. We cannot have a functional dependence on t as it would ruin the linear time dependence of the metric. Differentiating (1.5) :

$$dt_{(0)} = dt - f'(r)dr$$

Substituting the value in the line element (1.4) :

$$\begin{aligned} ds^2 &= A^{(1)}dr^2 + B^{(1)}\left[dt - f'(r)dr\right]^2 + 2C^{(1)}dr\left[dt - f'(r)dr\right] \\ \therefore\ ds^2 &= \left\{A^{(1)} + B^{(1)}\left[f'(r)\right]^2 - 2C^{(1)}f'(r)\right\}dr^2 + B^{(1)}dt^2 + 2\left\{C^{(1)} - B^{(1)}f'(r)\right\}drdt \end{aligned}$$

We choose $f'(r)$ such that the cross term $drdt$ vanishes.

$$\begin{aligned} \Longrightarrow C^{(1)} - B^{(1)}\left(\frac{df}{dr}\right) &= 0 \\ \therefore f &= \int\left(\frac{C^{(1)}}{B^{(1)}}\right)dr \end{aligned}$$

After this, we have no further freedom of coordination transformation. We obtain the line element, thus by extension the metric as :

$$ds^2 = A(r)dr^2 + B(r)dt^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1.6}$$

$A(r)$ and $B(r)$ can be determined by solving the Einstein equations.
Now as we are considering time independent (stationary) solutions, we do not have any initial conditions. But we will have boundary conditions arising from the condition of asymptotic flatness.
In flat space, $A(r) \longrightarrow 1$ and $B(r) \longrightarrow -1$. So the boundary condition is given as :

$$\lim_{r\longrightarrow\infty} A(r) = 1\,; \qquad \lim_{r\longrightarrow\infty} B(r) = -1 \tag{1.7}$$

We can redefine the functions $A(r)$ and $B(r)$ as :

$$A(r) = \exp\left[a(r)\right]\,; \quad \lim_{r\longrightarrow\infty} a(r) = 0 \tag{1.8}$$

$$B(r) = -\exp\left[b(r)\right]\,; \quad \lim_{r\longrightarrow\infty} b(r) = 0 \tag{1.9}$$

With this we now obtain the metric as follows :

$$ds^2 = -\exp\left[b(r)\right] dt^2 + \exp\left[a(r)\right] dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1.10}$$

We now need to solve the Einstein equation for vacuum : $R_{\mu\nu} = 0$. We know that the Ricci tensor is given as :

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\alpha\theta}\Gamma^\theta_{\mu\nu} - \Gamma^\alpha_{\nu\theta}\Gamma^\theta_{\mu\alpha} \tag{1.11}$$

The Christoffel symbols are given as :

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\left[\partial_\alpha g_{\mu\beta} + \partial_\mu g_{\alpha\beta} - \partial_\beta g_{\mu\alpha}\right] \tag{1.12}$$

$g_{\mu\nu}$ is the spacetime metric, which we obtain by reading off the line element (1.10). On solving the vacuum Einstein equation for the obtained metric, we get unknown functions $A(r)$ and $B(r)$ as :

$$A(r) = \exp\left[a(r)\right] = \left(1 - \frac{2m}{r}\right)^{-1} \tag{1.13}$$

$$B(r) = \exp\left[b(r)\right] = 1 - \frac{2m}{r} \tag{1.14}$$

Here, m is just some integration constant.

With the functional form of $A(r)$ and $B(r)$ we obtain the final metric as :

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1.15}$$

The metric has obtained the form of the Schwarzschild metric as given in (1.1), but we need to attach a physical meaning to the integration constant term m. To do that,we will consider the Newtonian limit :

$$\begin{gathered} g_{tt} = -\left(1 - 2\Phi\right) ; \Phi = \frac{GM}{r} \\ \implies 1 - \frac{2GM}{r} = 1 - \frac{2m}{r} \\ \therefore m = GM \end{gathered}$$

Here, G is the gravitational constant and M is the mass of the source of the gravitational field. Thus, we obtain the Schwarzschild solution for the vacuum Einstein equation.

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

We also note that m is not a function of t. Mass is conserved and we cannot have a solution in which mass is varying but everything outside is a vacuum as the mass has nowhere to flow to.

1.1.2 Kruskal Spacetime

If we look at the Schwarzschild metric, we notice that we obtain singularities at the point $r = 2GM$ and $r = 0$. We now want to ask the question if the singularity at $r = 2GM$ is a real singularity or just a coordinate artefact.

One way to determine if a singularity is a real singularity or not is to determine whether some curvature diverges at these points or not. Curvature invariants are scalars constructed out of Reimann tensor and covariant derivative appropriately contracted.

We can calculate curvature constants like the **Kretschmann Scalar** at the point $r = 2GM$ and obtain a finite value.

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{48G^2M^2}{r^6} \tag{1.16}$$

The above equation (1.16) for the Kretschmann scalar takes a finite value of $\dfrac{3}{4G^4M^4}$ at $r = 2GM$. While this does suggest that the singularity at $r = 2GM$ might not a real singularity, we cannot conclude this immediately, as there are an infinite number of curvature invariants and all of them need to be finite at $r = 2GM$. We call the $r = 2GM$ as the horizon of the black hole.
We can conclusively resolve this issue if we find a coordinate system where the metric is regular, i.e., finite and invertible at $r = 2GM$.

We now define a new coordinate system, defined as :

$$dr_*^2 = \left(1 - \frac{2GM}{r}\right)^{-2} dr^2 \tag{1.17}$$

Solving this, we get the dependence of r and $r*$ as :

$$r_* = r + 2GM \log\left|\frac{r - 2GM}{2GM}\right| \tag{1.18}$$

This coordinate system is called the **tortoise coordinates** because when we approach the horizon, the change in r is increasingly slow.

Rewriting the line element (1.1) in the tortoise coordinates :

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)\left(dt^2 - dr_*^2\right) + r^2\left(\theta^2 + \sin^2\theta d\phi^2\right) \tag{1.19}$$

In this metric, light rays always travel in the radial direction.

$$ds^2 = 0 \implies \frac{dr}{dt} = \pm\left(1 - \frac{2GM}{r}\right) \implies \frac{dr_*}{dt} = \pm 1$$

So the null geodesics are given by :

$$t \pm r_* = \text{constant} \tag{1.20}$$

But in these coordinates too we have a singularity at $r = 2GM$ point.
We next introduce null coordinates, defined as :

$$u = t - r_* \; ; \quad v = t + r_* \tag{1.21}$$

Rewriting the metric (1.19) using the null coordinates :

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dudv + r^2\left(d\theta^2 + sin^2\theta d\phi^2\right) \tag{1.22}$$

As the metric is still singular at $r = 2GM$, we define the final coordinates, called the **Kruskal-Szekeres** coordinates.

$$U = -\exp\left(-\frac{u}{4GM}\right) \; ; \quad V = \exp\left(\frac{v}{4GM}\right) \tag{1.23}$$

Differentiating (1.23) to obtain dU and dV in terms of the null coordinates :

$$\begin{aligned} dU &= \frac{1}{4GM}\exp\left(-\frac{u}{4GM}\right) du \\ dV &= \frac{1}{4GM}\exp\left(\frac{v}{4GM}\right) dv \end{aligned}$$

Calculating the components of the metric :

$$\begin{aligned} dU &= \frac{1}{4GM}\exp\left(\frac{-u}{4GM}\right) du \\ dV &= \frac{1}{4GM}\exp\left(\frac{v}{4GM}\right) dv \end{aligned}$$

Inverting and multiplying to find $dudv$ in terms of $dUdV$:

$$dudv = 16G^2M^2\exp\left(\frac{u-v}{4GM}\right) dUdV = 16G^2M^2\exp\left(\frac{-2r_*}{4GM}\right) dUdV$$

Now replacing r_* with (1.18) :

$$\begin{aligned} & dudv = 16G^2M^2\exp\left(\frac{-2r}{4GM}\right)\exp\left(\frac{-4GM}{4GM}\log\left(\frac{r-2GM}{2GM}\right)\right) dUdV \\ \implies & dudv = 16G^2M^2\exp\left(\frac{-r}{2GM}\right)\left(\frac{2GM}{r-2GM}\right) dUdV \end{aligned}$$

Plugging in the value of $dudv$ in (1.22) :

$$ds^2 = -\left(\frac{r-2GM}{r}\right)16G^2M^2\exp\left(\frac{-r}{2GM}\right)\left(\frac{2GM}{r-2GM}\right)dUdV + r^2\left(\theta^2 + \sin^2\theta d\phi^2\right)$$

We finally obtain the metric in Kruskal coordinates as :

$$ds^2 = -\frac{32(GM)^3}{r}\exp\left(\frac{-r}{2GM}\right)dUdV + r^2\left(\theta^2 + \sin^2\theta d\phi^2\right) \tag{1.24}$$

We can also relate r and t to the Kruskal coordinates in the following way :

$$UV = \frac{r-2GM}{2GM}\exp\left(\frac{r}{2GM}\right) \tag{1.25}$$

$$\frac{U}{V} = -\exp\left(\frac{-t}{2GM}\right) \tag{1.26}$$

We note that the point $r = 2GM$ is no longer a singularity. Not only is the region $r > 2GM$ non-singular, but also extremely smooth. We also note that $r = 0$ point is a true singularity, which we have not been able to remove by just coordinate transformations.
Kruskal spacetime is the maximal extension of the Schwarzschild solution. By maximal extension, we mean that all geodesics end either at infinity or at a genuine singularity, which is $r = 0$.

In this metric, the $r = 2GM$ point corresponds to $U = 0$ or $V = 0$. If we have $U = 0$, then $t = \infty$ and if $V = 0$, then we have $t = -\infty$. The singularity at $r = 0$ corresponds to $UV = 1$. We can represent the spacetime in Kruskal coordinates with the Kruskal diagram.

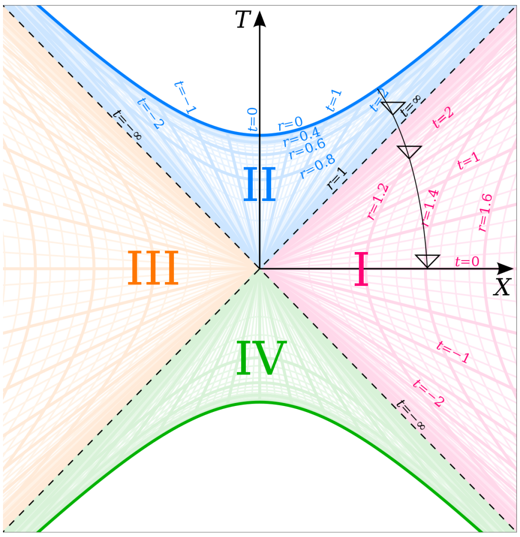

Fig. 1.1: Kruskal Diagram of Schwarzschild Metric
Source : https://en.wikipedia.org/wiki/Kruskal%E2%80%93Szekeres_coordinates

Here in the above diagram $2GM$ has been normalised to 1. So, $r = 1$ becomes the horizon and $r = 0$ remains the true singularity.

In the diagram we have lines of constant r and constant t. Region I is the physical space corresponding to $r > 1$ ($2GM = 1$). The $t = \infty$, i.e., $U = 0$ line corresponds to the future horizon and the $t = -\infty$, i.e., $V = 0$ line corresponds to the past horizon.

The horizon separates the space into two parts : $r > 2GM$ which is outside the black hole and $r < 2GM$ which is inside the black hole. Once an observer crosses the horizon, they have no choice but to fall into the singularity. They cannot escape back.

The region IV is sometimes called the white hole and the $V = 0, t = -\infty$ horizon is sometimes called the white hole horizon. It is not a physical region in spacetime, but a result of the maximal extension of the Schwarzschild metric.

1.1.3 Penrose Diagram

While we have maximally extended the Schwarzschild metric with the help of Kruskal coordinates, we still have infinities in the diagram. We can get rid of the infinities by performing a coordinate transformation as follows :

$$\chi = \tan^{-1} U \ ; \quad \xi = \tan^{-1} V \tag{1.27}$$

The $\tan^{-1} x$ function maps infinities to $\pm\frac{\pi}{2}$. χ and ξ directly give us access to the complete UV plane. But we also need to note that the entire plane is not physical. We have a singularity at $UV = 1$. Cutting off the spacetime region at the singularity, we obtain a diagram very similar to the Kruskal diagram.

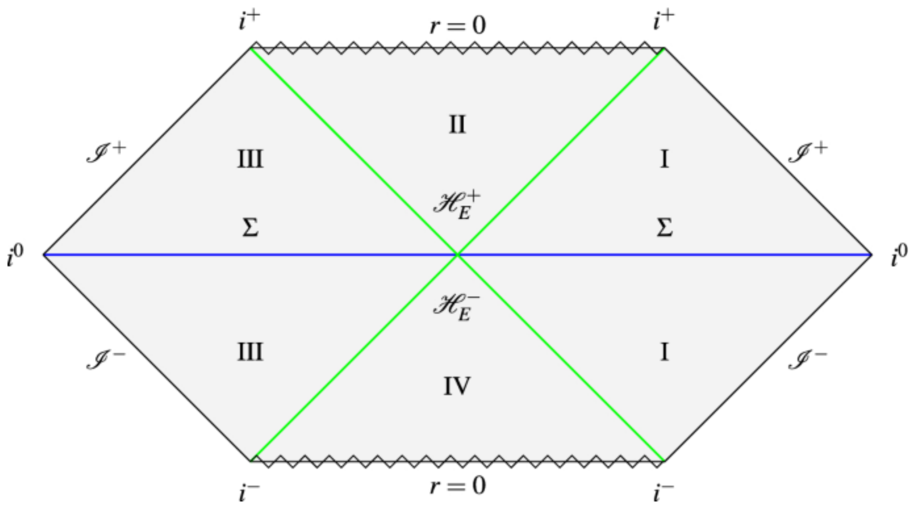

Fig. 1.2: Penrose Diagram

Source : https://link.springer.com/article/10.1007/s10701-021-00432-1

Here $\mathscr{I}^-$ represents past null infinity, $\mathscr{I}^+$ represents future null infinity, i^+ stands for future time like infinity, i^- stands for past time like infinity and i^0 stands for space like infinity.

$r = 0$ represent the singularities.
Region I is our Universe, region III represents a parallel Universe, region II is the black hole and region IV represents a white hole.

The Schwarzschild metric represents a black hole that has always been in existence, and will continue to be in existence. But in the physical world, we expect to see black holes which have been formed at some time in the history of the Universe, most probably due a collapse of a star or dust ball. For such black holes, the metric will be different from the Schwarzschild metric. The geometries of such black holes will also not remain the same throughout.We expect to see geometries that started out as stars and then collapsed to form black holes.

1.2 Scalar Field in Curved Spacetime

We will mainly discuss about a scalar field in curved spacetime.
Starting with the Lagrangian density of a minimally coupled scalar field, i.e., a scalar field just coupled to the background metric.

$$\mathscr{L} = \frac{1}{2}\sqrt{-g}\left\{g^{\mu\nu}\left(\partial_\mu\phi\right)\left(\partial_\nu\phi\right) - m^2\phi^2\right\} \tag{1.28}$$

We will consider the case where we have the background metrics fixed and the perturbative fields moving on some fixed metric.
The Euler-Lagrangian equations for fields are given as :

$$\partial_\mu\left[\frac{\delta\mathscr{L}}{\delta\left(\partial_\mu\phi\right)}\right] = \frac{\delta\mathscr{L}}{\delta\phi} \tag{1.29}$$

Using the same to obtain the equations of motion for the minimally coupled scalar field.
Calculating the RHS :

$$\frac{\delta\mathscr{L}}{\delta\phi} = \frac{1}{2}\sqrt{-g}\left(-m^2 2\phi\right) = -\sqrt{-g}\, m^2\phi$$

Calculating the LHS :

$$\begin{aligned}
\frac{\delta\mathscr{L}}{\delta\left(\partial_\mu\phi\right)} &= \frac{1}{2}\sqrt{-g}\left\{g^{\alpha\beta}\left(\partial_\alpha\phi\right)\delta^\mu_\beta + g^{\alpha\beta}\left(\partial_\beta\phi\right)\delta\right\} \\
\therefore\ \frac{\delta\mathscr{L}}{\delta\left(\partial_\mu\phi\right)} &= \sqrt{-g}\, g^{\mu\nu}\left(\partial_\nu\phi\right) \\
\implies \partial_\mu\left[\frac{\delta\mathscr{L}}{\delta\left(\partial_\mu\phi\right)}\right] &= \partial_\mu\left[\sqrt{-g}\, g^{\mu\nu}\left(\partial_\nu\phi\right)\right]
\end{aligned}$$

We can then write the final equation of motion as :

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\, g^{\mu\nu}\left(\partial_\nu\phi\right)\right] + m^2\phi = 0 \tag{1.30}$$

The set of classical solutions form a linear space.

To quantise the theory, we will adopt the canonical quantisation approach. For this approach, we need a notion of momentum, as it requires commutation relation between the canonical degrees of freedom and their canonically conjugate momenta. The momenta is defined as the derivative of the action with respect to the time derivative of the canonical degree of freedom.

Now, even before we define the canonical momenta, we need a notion of what time is. One of the main difference between flat space and curved space is that there is no canonical notion of time in curved spacetime. We can argue that even flat spacetime has no notion of canonical time as we can boost any frame. But in this case, all observers in different boosted frames agree on the presence/absence and number of particles. In short, the boosts take positive frequency solutions to positive frequency solutions amongst frames.
The curved spacetime background does not always respect the fact that positive frequency solutions in one frame go to positive frequency solutions in other frames.

Since we have to make an explicit choice of time, we choose x^0 to be the time coordinate, i.e., $t \equiv x^0$. With this choice of time, we can write down the canonical momenta as :

$$\Pi(t, \boldsymbol{x}) = \frac{\delta \mathscr{L}}{\delta\left(\partial_0 \phi\right)} = \sqrt{-g}\, g^{0\mu} \partial_\mu \phi\, (t, \boldsymbol{x}) \tag{1.31}$$

Different observers can choose different time surfaces, i.e., different foliations. All choices are equally valid.
The three canonical commutation relations are imposed on equal time are as follows :

$$[\phi(t, \boldsymbol{x}), \Pi(t, \boldsymbol{y})] = i\delta(\boldsymbol{x} - \boldsymbol{y}) \tag{1.32}$$

$$[\phi(t, \boldsymbol{x}), \phi(t, \boldsymbol{y})] = 0 \tag{1.33}$$

$$[\Pi(t, \boldsymbol{x}), \Pi(t, \boldsymbol{y})] = 0 \tag{1.34}$$

We now introduce creation and annihilation operators. The choice depends on the choice of basis.

$$\phi(t, \boldsymbol{x}) = \sum_i \left[a_i f_i(t, \boldsymbol{x}) + a_i^\dagger f_i^*(t, \boldsymbol{x})\right] \tag{1.35}$$

The summation runs over the complete basis of solutions.
Calculating the conjugate the momenta, from the definition of ϕ and (1.31).

$$\Pi(t, \boldsymbol{x}) = \sqrt{-g}\, g^{0\mu} \sum_i \left[a_i \partial_\mu f_i + a_i^\dagger \partial_\mu f_i^*\right] \tag{1.36}$$

We can impose the three commutation relations and find conditions that need to be satisfied by the creation and annihilation operators and the function f.

Starting with the commutation relation between field operators (1.33) :

$$\begin{aligned}&[\phi(t,\boldsymbol{x}),\phi(t,\boldsymbol{y})]=0\\ \implies &\phi(t,\boldsymbol{x})\phi(t,\boldsymbol{y})-\phi(t,\boldsymbol{y})\phi(t,\boldsymbol{x})=0\end{aligned}$$

Using the definition of ϕ from (1.35) :

$$\begin{aligned}\text{LHS}=\sum_{i,j}\left[a_i f_i(t,\boldsymbol{x})+a_i^\dagger f_i^*(t,\boldsymbol{x})\right]\left[a_j f_j(t,\boldsymbol{y})+a_j^\dagger f_j^*(t,\boldsymbol{y})\right]\\ -\sum_{i,j}\left[a_j f_j(t,\boldsymbol{y})+a_j^\dagger f_j^*(t,\boldsymbol{y})\right]\left[a_i f_i(t,\boldsymbol{x})+a_i^\dagger f_i^*(t,\boldsymbol{x})\right]\end{aligned}$$

Opening the brackets and evaluating :

$$\begin{aligned}\therefore\ \text{LHS}=\sum_{i,j}\Big\{[a_i,a_j]f_i(t,\boldsymbol{x})f_j(t,\boldsymbol{y})+[a_i,a_j^\dagger]f_i(t,\boldsymbol{x})f_j^*(t,\boldsymbol{y})+[a_i^\dagger,a_j]f_j^*(t,\boldsymbol{x})f_j(t,\boldsymbol{y})\\ +[a_i^\dagger,a_j^\dagger]f_i^*(t,\boldsymbol{x})f_j^*(t,\boldsymbol{y})\Big\}\end{aligned}$$

As the LHS$=0$, we see that the coefficients must be individually zero. Also, if we consider the case of $i=j$, we obtain :

$$\sum_i\Big\{[a_i,a_i]f_i(t,\boldsymbol{x})f_i(t,\boldsymbol{y})+[a_i,a_i^\dagger]\big[f_i(t,\boldsymbol{x})f_i^*(t,\boldsymbol{y})-f_i^*(t,\boldsymbol{x})f_i(t,\boldsymbol{y})\big]+[a_i^\dagger,a_i^\dagger]f_i^*(t,\boldsymbol{x})f_i^*(t,\boldsymbol{y})\Big\}=0$$

Thus, we can conclude the commutation relations of the creation and annihilation operators as :

$$[a_i,a_j^\dagger]=\delta_{ij}\quad\text{and}\quad[a_i,a_j]=0=[a_i^\dagger,a_j^\dagger]\tag{1.37}$$

Imposing the commutation relation between ϕ and Π as given in (1.32) :

$$\begin{aligned}&[\phi(t,\boldsymbol{x}),\Pi(t,\boldsymbol{y})]=i\delta(\boldsymbol{x}-\boldsymbol{y})\\ \implies &\phi(t,\boldsymbol{x})\Pi(t,\boldsymbol{y})-\Pi(t,\boldsymbol{y})\phi(t,\boldsymbol{x})=i\delta(\boldsymbol{x}-\boldsymbol{y})\end{aligned}$$

Substituting the forms of ϕ and Π from (2.8) and (1.36) :

$$\begin{aligned}\text{LHS}=&\left\{\sum_i\left[a_i f_i(t,\boldsymbol{x})+a_i^\dagger f_i^*(t,\boldsymbol{x})\right]\right\}\left\{\sqrt{-g}\,g^{0\mu}\sum_j\left[a_j\partial_\mu f_j+a_j^\dagger\partial_\mu f_j^*\right]\right\}\\ &-\left\{\sqrt{-g}\,g^{0\mu}\sum_j\left[a_j\partial_\mu f_j+a_j^\dagger\partial_\mu f_j^*\right]\right\}\left\{\sum_i\left[a_i f_i(t,\boldsymbol{x})+a_i^\dagger f_i^*(t,\boldsymbol{x})\right]\right\}\\ \implies\ \text{LHS}=&\sqrt{-g}\sum_{i,j}\Big\{[a_i,a_j]f_i(t,\boldsymbol{x})g^{0\mu}\partial_\mu f_j(t,\boldsymbol{y})+[a_i,a_j^\dagger]f_i(t,\boldsymbol{x})g^{0\mu}\partial_\mu f_j^*(t,\boldsymbol{y})\\ &+[a_i^\dagger,a_j]f_i^*(t,\boldsymbol{x})g^{0\mu}\partial_\mu f_j(t,\boldsymbol{y})+[a_i^\dagger,a_j^\dagger]f_i^*(t,\boldsymbol{x})g^{0\mu}\partial_\mu f_j^*(t,\boldsymbol{y})\Big\}\end{aligned}$$

$$\therefore \text{ LHS} = \sqrt{-g}\sum_i \left\{f_i(t,\boldsymbol{x})g^{0\mu}\partial_\mu f_i^*(t,\boldsymbol{y}) + f_i^*(t,\boldsymbol{x})g^{0\mu}\partial_\mu f_i(t,\boldsymbol{y})\right\} = \text{RHS} = i\delta(\boldsymbol{x}-\boldsymbol{y})$$

Thus, we obtain the relation :

$$\sum_i \left\{f_i(t,\boldsymbol{x})g^{0\mu}\partial_\mu f_i^*(t,\boldsymbol{y}) + f_i^*(t,\boldsymbol{x})g^{0\mu}\partial_\mu f_i(t,\boldsymbol{y})\right\} = \frac{i\delta(\boldsymbol{x}-\boldsymbol{y})}{\sqrt{-g}} \tag{1.38}$$

In flat space, these mode functions are just plane waves. But here, we have a background metric different from flat space, and the mode functions do not necessarily have to be plane waves.

We now define the vacuum as the state that gets annihilated by all annihilation operators, i.e.,

$$a_i|\Omega\rangle = 0 \quad \forall\, i \tag{1.39}$$

We need to note that the annihilation operators depend on the choice of basis. Another observer may choose a different set of basis, resulting in a different set of annihilation operators which may not satisfy the previously chosen vacuum condition.

Let us say another observer has b_i, $b_i^\dagger$ as their annihilation operator in their basis g_i, giving the expansion of ϕ as :

$$\phi(t,\boldsymbol{x}) = \sum_i \left[b_i g_i(t,\boldsymbol{x}) + b_i^\dagger g_i^*(t,\boldsymbol{x})\right] \tag{1.40}$$

Now as we know they are both complete basis of solutions, there needs to be a linear transformation between them.
So, we must have α and β such that :

$$a_i = \sum_j \left[\alpha_{ij}b_j + \beta_{ji}^* b_j^\dagger\right] \tag{1.41}$$

$$a_i^\dagger = \sum_j \left[\alpha_{ji}^* b_j^\dagger + \beta_{ji}b_j\right] \tag{1.42}$$

These α and β are called the Bogoliubov coefficients between a and b.
So, starting from their vacuum, both ‘a’ and ‘b’ can construct their own Fock Space. But these vacuums may not be the same. We need to note that, at the end, the full Hilbert space remains the same. What changes is what one calls each state. The state ‘a’ calls vacuum may be different to what ‘b’ calls vacuum but they both belong to the same Hilbert space.

$$a_i|\Omega_a\rangle = 0$$

$$\implies \sum_j \left(\alpha_{ij}b_j + \beta_{ji}^* b_j^\dagger\right)|\Omega_a\rangle = 0$$

Taking an ansatz for the vacuum of 'a' in terms of vacuum of 'b' :

$$|\Omega_a\rangle = \exp\left(\frac{1}{2}b_j^\dagger c_{jk} b_k^\dagger\right)|\Omega_b\rangle \tag{1.43}$$

Using the ansatz in the previous equation :

$$\sum_j \left(\alpha_{ij} b_j + \beta_{ji}^* b_j^\dagger\right)\exp\left(\frac{1}{2}b_j^\dagger c_{jk} b_k^\dagger\right)|\Omega_b\rangle = 0 \tag{1.44}$$

Now evaluating the action of b_m on $|\Omega_a\rangle$, using the ansatz :

$$\begin{aligned}
& b_m|\Omega_a\rangle = b_m \exp\left(\frac{1}{2}b_j^\dagger c_{jk} b_k^\dagger\right)|\Omega_b\rangle \\
\implies\ & b_m|\Omega_a\rangle = b_m\left[1 + \frac{1}{2}b_j^\dagger c_{jk} b_k^\dagger \times \frac{1}{2} + \frac{1}{2}b_j^\dagger c_{jk} b_k^\dagger \times \frac{1}{2}b_p^\dagger c_{pq} b_q^\dagger \times \frac{1}{2^2} + \cdots\right]|\Omega_b\rangle \\
\implies\ & b_m|\Omega_a\rangle = \cancelto{0}{b_m|\Omega_b\rangle} + \frac{1}{4}b_m b^\dagger c_{jk} b_k^\dagger|\Omega_b\rangle + \frac{1}{16}b_m b_k^\dagger c_{jk} b_k^\dagger b_p^\dagger c_{pq} b_q^\dagger|\Omega_b\rangle + \cdots
\end{aligned}$$

Now using the commutation relations of creation and annihilation operators from (2.10) :

$$\begin{aligned}
\implies\ b_m|\Omega_a\rangle = {} & \frac{1}{4}\delta_{mj} c_{jk} b_k^\dagger|\Omega_b\rangle + \frac{1}{4}b_j^\dagger c_{jk}\delta_{mk}|\Omega_b\rangle + \frac{1}{4}b_j^\dagger c_{jk} b_k^\dagger \cancelto{0}{b_m|\Omega_b\rangle} + \frac{1}{16}\delta_{mj} c_{jk} b_k^\dagger b_p^\dagger c_{pq} b_q^\dagger|\Omega_b\rangle \\
& + \frac{1}{16}b_j^\dagger c_{jk}\delta_{mk} b_p^\dagger c_{pq} b_q^\dagger|\Omega_b\rangle + \frac{1}{16}b_j^\dagger c_{jk} b_k^\dagger \delta_{mp} c_{pq} b_q^\dagger|\Omega_b\rangle + \frac{1}{16}b_j^\dagger c_{jk} b_k^\dagger b_p^\dagger c_{pq}\delta_{mq}|\Omega_b\rangle \\
& + \frac{1}{16}b_j^\dagger c_{jk} b_k^\dagger b_p^\dagger c_{pq} b_q^\dagger \cancelto{0}{b_m|\Omega_b\rangle} + \cdots
\end{aligned}$$

Arranging all dummy variables appropriately, and evaluating the delta functions, we obtain :

$$\begin{aligned}
\implies\ & b_m|\Omega_a\rangle = \frac{1}{2}c_{mk} b_k^\dagger|\Omega_b\rangle + \frac{1}{2}c_{mk} b_k^\dagger\left(\frac{1}{2}b_p^\dagger c_{pq} b_q^\dagger\right)|\Omega_b\rangle + \cdots \\
\implies\ & b_m|\Omega_a\rangle = \frac{1}{2}c_{mk} b_k^\dagger\left(1 + \frac{1}{2}b_p^\dagger c_{pq} b_q^\dagger + \cdots\right)|\Omega_b\rangle \\
\implies\ & b_m|\Omega_a\rangle = \frac{1}{2}c_{mk} b_k^\dagger \exp\left(\frac{1}{2}b_p^\dagger c_{pq} b_q^\dagger\right)|\Omega_b\rangle
\end{aligned}$$

Thus, we finally obtain :

$$b_j|\Omega_a\rangle = \frac{1}{2}c_{jm} b_m^\dagger|\Omega_a\rangle \tag{1.45}$$

Now using the result (1.45) in (1.44) :

$$\begin{aligned}
\sum_j \left(\alpha_{ji} b_j|\Omega_a\rangle + \beta_{ji}^* b_j^\dagger|\Omega_a\rangle\right) = 0 \\
\sum_j \left(\alpha_{ji} c_{jm} b_m^\dagger + \beta_{ji}^* b_j^\dagger\right)|\Omega_a\rangle = 0
\end{aligned}$$

Renaming dummy variables :

$$\sum_j \left(\alpha_{mi} c_{mj} + \beta^*_{ji}\right) b^\dagger_j |\Omega_a\rangle = 0$$

Thus, we finally obtain the relation :

$$c = -\beta^* \alpha^{-1} \tag{1.46}$$

So, from the above calculation we can conclude that the ‘a’-vacuum is a specific excited state of the ‘b’-vacuum. It is the state with two particle pairs on top of the ‘b’-vacuum. This further enforces the statement that the number of particles is an observer-dependent quantity.

Thus, the notion of a vacuum is not an invariant quantity but an observer-dependent quantity. The invariant quantities are the correlation functions between the value of the fields at given points, i.e., $\langle \phi(x_1)\phi(x_2)\rangle$ and so on. Here we are considering two different spacetime points ($x_1 \neq x_2$). As long as the observers are measuring the same given set of distinct points, they will measure the correlation functions to be exactly the same.

Chapter 2

Low Energy Excitations of a Thermal System

In this chapter, we show that there exists a bound on the effect of low energy excitation of a thermal system. This is a general result of statistical mechanics, and not specific to AdS/CFT.

Let us consider a system with large number of degrees of freedom and the set of energy eigenstates around some energy E. The states have energy in the range $E \pm \Delta$, where $\Delta << E$. We denote the Hilbert space spanned by the states as $\mathcal{H}_E$. The dimension of this space is given as $\dim(\mathcal{H}_E) = e^S$, where S is the entropy, with $S \propto E$. Picking a state $|\Psi\rangle$ using the Haar measure and then by the eigenstate thermalisation hypothesis for any coarse-grained observable A_α, we can write :

$$\langle\Psi|A_\alpha|\Psi\rangle = \frac{1}{Z(\beta)}\operatorname{Tr}\left(e^{-\beta H}A_\alpha\right) + O\left(\frac{1}{\sqrt{S}}\right) \tag{2.1}$$

$Z(\beta)$ is the partition function, and $\beta = \dfrac{\partial S}{\partial E}$ is the inverse effective temperature of the state. For any coarse-grained probe, the pure state $|\Psi\rangle$ is effectively thermal.

Now considering a low-energy excitation of the system, i.e., a unitary operator such that :

$$\langle\Psi|U^\dagger HU|\Psi\rangle - \langle|H|\Psi\rangle = \delta E \tag{2.2}$$

We are considering the case in which the unitary operator increases the energy of the state by a small amount, i.e., $\delta E > 0$ but $\beta\delta E << 1$ and $\delta E << \Delta$. So we have $\left(\dfrac{\delta E}{E}\right) \propto \mathcal{O}\left(\dfrac{1}{S}\right)$ as $S \propto E$.
We can also write U as a map that takes $\mathcal{H}_E$ to $\mathcal{H}_{E+\delta E}$, i.e., $U : \mathcal{H}_E \longrightarrow \mathcal{H}_{E+\delta E}$. As the dimension of $\mathcal{H}_{E+\delta E}$ is larger than the dimension of $\mathcal{H}_E$, U is an injective map but not

surjective.

$$\dim\left(\mathcal{H}_{E+\delta E}\right) = e^{S+\frac{\partial S}{\partial E}\delta E} = e^{E+\beta\delta E} \tag{2.3}$$

The image of $\mathcal{H}_E$ under U forms a subspace of $\mathcal{H}_{E+\delta E}$. Let $|\Psi'\rangle$ be some typical state in $\mathcal{H}_{E+\delta E}$ picked by the Haar measure and let $\mathcal{P}_{U_E}$ be the projector onto the aforementioned subspace. Then we can write :

$$\langle\Psi'|\mathcal{P}_{U_E}|\Psi'\rangle = 1 - \beta\delta E + \mathcal{O}\left(\delta E^2\right) \tag{2.4}$$

$$\langle\Psi'|1 - \mathcal{P}_{U_E}|\Psi'\rangle = \beta\delta E + \mathcal{O}\left(\delta E^2\right) \tag{2.5}$$

Ignoring the higher order terms, we can write :

$$\implies \mathcal{P}_{U_E}|\Psi'\rangle = (1-\beta\delta E)\,|\Psi'\rangle$$
$$\implies (1-\mathcal{P}_{U_E})\,|\Psi'\rangle = \beta\delta E|\Psi'\rangle$$

Using this we can separate the states in $\mathcal{H}_{E+\delta E}$ into two groups : one in the subspace $U\left(\mathcal{H}_E\right)$ and the ones not in the subspace.

$$|\Psi_E\rangle = \frac{\mathcal{P}_{U_E}|\Psi'\rangle}{\sqrt{\langle\Psi'|\mathcal{P}_{U_E}|\Psi'\rangle}} \tag{2.6}$$

$$|\Psi_0\rangle = \frac{(1-\mathcal{P}_{U_E})\,|\Psi'\rangle}{\sqrt{\langle\Psi'|1-\mathcal{P}_{U_E}|\Psi'\rangle}} \tag{2.7}$$

State $|\Psi_E\rangle$ is in the subspace while $|\Psi_0\rangle$ is not in the subspace.
Using (2.4) and (2.5) in the definition of $|\Psi_E\rangle$ and $|\Psi_0\rangle$:

$$\implies \mathcal{P}_{U_E}|\Psi'\rangle = (1-\beta\delta E)^{1/2}\,|\Psi_E\rangle = \left(1-\frac{\beta\delta E}{2}\right)|\Psi_E\rangle + \text{higher order terms}$$
$$\implies (1-\mathcal{P}_{U_E})\,|\Psi'\rangle = (\beta\delta E)^{1/2}\,|\Psi_0\rangle + \text{higher order terms}$$

Adding the above two expressions :

$$\therefore\ |\Psi'\rangle = \left(1-\frac{\beta\delta E}{2}\right)|\Psi_E\rangle + (\beta\delta E)^{1/2}\,|\Psi_0\rangle + \text{higher order terms} \tag{2.8}$$

The difference in temperature between the two ensembles is given as :

$$\delta\beta = \frac{\partial\beta}{\partial E}\delta E = -\frac{\beta^2}{C_V}\delta E \tag{2.9}$$

We can write this as we know the specific heat at constant volume C_V is given as :

$$C_V = \left.\frac{\partial E}{\partial\beta}\right|_V \times \frac{\partial\beta}{\partial T} = \left.\frac{\partial E}{\partial\beta}\right|_V \left(-\beta^2\right)$$
$$\therefore \left.\frac{\partial\beta}{\partial E}\right|_V = -\frac{\beta^2}{C_V}$$

Thus, statistically the states $|\Psi'\rangle$ are very similar to a typical state picked from the Hilbert space $\mathcal{H}_E$. So, the expectation value of a coarse-grained observable remains almost the same across the two ensembles.

$$\langle\Psi'|A_\alpha|\Psi'\rangle = \langle\Psi|A_\alpha|\Psi\rangle + \mathcal{O}\left(\frac{1}{\sqrt{S}}\right) \tag{2.10}$$

We can calculate the change in the expectation value of an observable A_α due to the excitation, i.e., action of the unitary operator as :

$$\delta\langle A_\alpha\rangle = \left|\langle\Psi|U^\dagger A_\alpha U|\Psi\rangle - \langle\Psi|A_\alpha|\Psi\rangle\right| \tag{2.11}$$

Using (2.10) and the fact that $|\Psi_E\rangle = U|\Psi\rangle$, we can write :

$$\begin{aligned}
\implies \delta\langle A_\alpha\rangle &= \left|\langle\Psi'|A_\alpha|\Psi'\rangle - \langle\Psi_E|A_\alpha|\Psi_E\rangle\right| \\
\implies \delta\langle A_\alpha\rangle &= \Bigg|\left\{\left(1-\frac{\beta\delta E}{2}\right)\langle\Psi_E| + \sqrt{\beta\delta E}\langle\Psi_0|\right\}|A_\alpha|\left\{\left(1-\frac{\beta\delta E}{2}\right)|\Psi_E\rangle + \sqrt{\beta\delta E}|\Psi_0\rangle\right\} \\
&\qquad - \langle\Psi_E|A_\alpha|\Psi_E\rangle\Bigg| \\
\implies \delta\langle A_\alpha\rangle &= \Bigg|\left(1-\frac{\beta\delta E}{2}\right)^2\langle\Psi_E|A_\alpha|\Psi_E\rangle + \left(1-\frac{\beta\delta E}{2}\right)\sqrt{\beta\delta E}\langle\Psi_E|A_\alpha|\Psi_0\rangle \\
&\qquad + \sqrt{\beta\delta E}\left(1-\frac{\beta\delta E}{2}\right)\langle\Psi_0|A_\alpha|\Psi_E\rangle + \beta\delta E\langle\Psi_0|A_\alpha|\Psi_0\rangle - \langle\Psi_E|A_\alpha|\Psi_E\rangle\Bigg|
\end{aligned}$$

Thus, we obtain the expression as :

$$\delta\langle A_\alpha\rangle = \sqrt{\beta\delta E}\left|\langle\Psi_E|A_\alpha|\Psi_0\rangle + \langle\Psi_0|A_\alpha|\Psi_E\rangle\right| \tag{2.12}$$

Calculating term by term :

$$\begin{aligned}
\left|\langle\Psi_E|A_\alpha|\Psi_0\rangle\right|^2 &= \langle\Psi_E|A_\alpha|\Psi_0\rangle\langle\Psi_0|A_\alpha^\dagger|\Psi_E\rangle \\
\implies \left|\langle\Psi_E|A_\alpha|\Psi_0\rangle\right|^2 &= \langle\Psi_E|A_\alpha\left(|\Psi_0\rangle\langle\Psi_0| + |\Psi_E\rangle\langle\Psi_E|\right)A_\alpha^\dagger|\Psi_E\rangle - \left|\langle\Psi_E|A_\alpha|\Psi_E\rangle\right|^2
\end{aligned}$$

The operator $|\Psi_0\rangle\langle\Psi_0| + |\Psi_E\rangle\langle\Psi_E|$ is a projector, as $|\Psi_E\rangle$ and $|\Psi_0\rangle$ are orthogonal to each other. For any projector P, we have the property :

$$\langle\Psi_E|A_\alpha P A_\alpha^\dagger|\Psi_E\rangle = \langle\Psi_E|A_\alpha P P A_\alpha^\dagger|\Psi_E\rangle = \left|PA^\dagger|\Psi_E\rangle\right|^2 \leq \left|A_\alpha^\dagger|\Psi_E\rangle\right|^2 \tag{2.13}$$

Using the above property, we obtain at the result :

$$\left|\langle\Psi_E|A_\alpha|\Psi_0\rangle\right|^2 \leq \langle\Psi_E|A_\alpha A_\alpha^\dagger|\Psi_E\rangle - \left|\langle\Psi_E|A_\alpha|\Psi_E\rangle\right|^2 \tag{2.14}$$

Similarly, we can derive

$$\left|\langle\Psi_E|A_\alpha|\Psi_0\rangle\right|^2 \leq \langle\Psi_0|A_\alpha A_\alpha^\dagger|\Psi_0\rangle - \left|\langle\Psi_0|A_\alpha|\Psi_0\rangle\right|^2 \tag{2.15}$$

From the above analysis, we can deduce that the correlators in $|\Psi_E\rangle$ are thermal, upto the leading order in $\beta\delta E$ as we expect the correlators in $|\Psi_0\rangle$ to be thermal.

Combining all the results obtained till now, we can write the change in the expectation value of A_α as :

$$\delta\langle A_\alpha\rangle \leq 2\sqrt{\beta\delta E}\,\sigma_\alpha \tag{2.16}$$

where σ_α is a deviation of A_α and is defined as :

$$\sigma_\alpha^2 = \frac{1}{Z(\beta)}\min\left[\mathrm{Tr}\left(e^{-\beta H}A_\alpha^\dagger A_\alpha\right), \mathrm{Tr}\left(e^{-\beta H}A_\alpha A_\alpha^\dagger\right)\right] - \left|\frac{1}{Z(\beta)}\mathrm{Tr}\left(e^{-\beta H}A_\alpha\right)\right|^2 \tag{2.17}$$

In the limit of $\beta\delta E \ll 1$, the change in the expectation value of A_α is very small. Thus, in conclusion we can say that it is impossible to definitely excite a thermal system with energy less than kT.

The result we have derived above is a general statement and applies to all statistical systems. Marolf and Polchinski have suggested in their paper [1506.01337] that a state-dependent mapping of the blackhole interior violates this general principle of statistical mechanics. But we argue that such violations are unobservable, i.e, no observer or collection of observers can both produce the small excitation and also detect the violation of this principle. They do not lie in the same causal patch.

Chapter 3

Holographic Setup and Causal Patches

We wish to examine the statistical mechanics principle stated in the previous section in the context of AdS blackholes. In this chapter, we discuss the setup and define the causal patch we would be focussing on.

3.1 CFT Conventions

We are considering a CFT with a holographic dual and a large AdS Schwarzschild black hole in the dual geometry. The metric for the black hole is given as :

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \tag{3.1}$$

$$f(r) = 1 - \frac{c_d M}{r^{d-2}} + r^2 \tag{3.2}$$

$$c_d = 8(d-1)^{-1}\pi^{\frac{2-d}{2}}\Gamma(d/2) \tag{3.3}$$

where, M is the mass of the black hole and d is the dimension of the space. We are considering a AdS_{d+1} space.
To probe the space better, we go to the tortoise coordinates r_*, defined as :

$$dr_* = \frac{dr}{f(r)} \tag{3.4}$$

We can analyse the expression at different regions of space to obtain an expression for r_*.

<u>Case 1</u> : At the boundary $(r \longrightarrow \infty)$

At the boundary we can approximate $f(r) \simeq r^2$. This gives us :

$$\int_0^{r_*} dr_* = \int_r^\infty \frac{dr}{r^2}$$

$$\therefore\ r_* = \frac{1}{r} \tag{3.5}$$

So, at the boundary as $r \longrightarrow \infty$, we have $r_* \longrightarrow 0$.

Case 2 : Near the horizon ($r \longrightarrow r_0$)

Near the horizon, the expression for $f(r)$ is given as :

$$f(r) = \frac{4\pi}{\beta}\,(r - r_0) \tag{3.6}$$

β is the inverse temperature of the black hole. Thus, we obtain the expression for r_* as :

$$\begin{aligned} \int dr_* &= \frac{\beta}{4\pi}\int \frac{dr}{r - r_0} \\ \therefore\ r_* &= \frac{\beta}{4\pi}\ln{(r - r_0)} \end{aligned} \tag{3.7}$$

So, near the horizon as $r \longrightarrow r_0$ we have $r_* \longrightarrow -\infty$.

Thus, the line element in the tortoise coordinates is given as :

$$ds^2 = -f(r)\left[dt^2 - dr_*^2\right] + r^2(r_*)d\Omega^2 \tag{3.8}$$

We note that the null rays travel along $r_* - t$ and $r_* + t$ paths.

3.1.1 Kruskal Coordinates

To map the entire AdS space, we move to the Kruskal coordinates. We divide the space into two regions, one outside the horizon and one inside, and accordingly define two sets of the Kruskal coordinates.

Outside the horizon :

Outside the black hole horizon, the Kruskal coordinates are defined as :

$$U_K = -\exp\left[\frac{2\pi}{\beta}(r_* - t)\right]\ ;\quad V_K = \exp\left[\frac{2\pi}{\beta}(r_* + t)\right] \tag{3.9}$$

We can write the line element in terms of these coordinates by calculating dU_K and dV_K.

$$ds^2_{out} = -f(r)\frac{\beta^2}{4\pi^2}\exp\left(-\frac{4\pi}{\beta}r_*\right)dU_K dV_K + r^2(r_*)d\Omega^2 \tag{3.10}$$

The constant r_* and t lines are given as :

$$U_K V_K = -\exp\left(\frac{4\pi}{\beta} r_*\right) \quad \frac{U_K}{V_K} = -\exp\left(\frac{4\pi}{\beta} t\right) \tag{3.11}$$

Inside the horizon :

Inside the black hole, the Kruskal coordinates are defined as :

$$U_K = \exp\left[\frac{2\pi}{\beta}(r_* - t)\right] \ ; \quad V_K = \exp\left[\frac{2\pi}{\beta}(r_* + t)\right] \tag{3.12}$$

We can write the line element in terms of these coordinates by calculating dU_K and dV_K.

$$ds^2_{in} = -f(r)\frac{\beta^2}{4\pi^2}\exp\left(-\frac{4\pi}{\beta}r_*\right) dU_K dV_K + r^2(r_*) d\Omega^2 \tag{3.13}$$

The constant r_* and t lines are given as :

$$U_K V_K = \exp\left(\frac{4\pi}{\beta} r_*\right) \quad \frac{U_K}{V_K} = \exp\left(\frac{4\pi}{\beta} t\right) \tag{3.14}$$

We will be ignoring the region where $U_K \gg 1$ in the interior of the black hole, as it is inaccessible to any observers that we are interested in.

$$U_K \gg 1 \implies \exp\left[\frac{2\pi}{\beta}(r_* - t)\right] \gg 1 \implies r_* - t \gg 1 \implies r_* \gg t$$

Also, we are only interested in single sided black holes and observers who fall into the black hole from the horizon starting at time $t = 0$.

3.2 Causal Patches

Causal patches are of great importance in the analysis done by Prof. Raju. Here we define causal patches using a point on the singularity of the black hole.

Let $\mathcal{P}_C$ be a point on the singularity.The causal patch is then defined as the set of all points in the causal past of this point. We can picture this by considering the set of all past directed null rays emitted from this point in all possible directions. This past light cone intersects the boundary of the AdS space to form a sphere. Let the time at the intersection point be t_C. We extend this region to the past and cut it off at some value of $V_K = \exp\left(\frac{2\pi\vartheta}{\beta}\right)$, i.e., at $t = \vartheta$. Since we are not interested in the process of formation of the said black hole, we can consider it to be an eternal black hole and choose any general cutoff.

The intersection of the causal patch with the boundary of the AdS space is denoted by $\mathcal{B}_C$ and the causal patch itself is denoted by C. We have a one-parameter family of causal patches that can all be labelled using t_C.

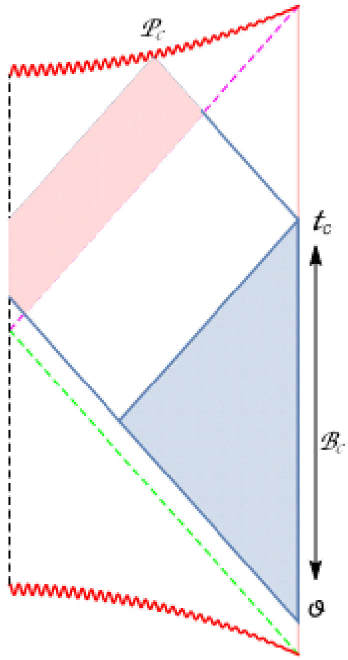

Fig. 3.1: Causal Patches
Source : http://arxiv.org/abs/1604.03095

Physically if we think about a set of observers who jump into the black hole from the boundary and end their life at the point $\mathcal{P}_C$ in the singularity, then the causal patch we have defined becomes the set of all points in the spacetime which can influence or effect the observers before their trajectory terminates at the singularity.

We can also define two additional geometrical regions which might be useful. The causal wedge of $\mathcal{B}_C$ is defined as the set of all points in the bulk such that some parts of their causal past and causal future lie in $\mathcal{B}_C$. It is denoted by $\wedge_C$ and marked in blue in the figure (3.1). The intersection of the causal patch with the interior of the black hole is denoted by $\vee_C$. It is marked in pink in the figure (3.1).

3.3 Source Deformed Correlators

In this section we aim to describe how to write the correlators of bulk operators in between actively produced non-equilibrium states of the CFT hamiltonian.

Let H be the CFT hamiltonian in the absence of all sources. We allow the CFT to thermalise so that it goes into an equilibrium state $|\Psi\rangle$. We then deform thr hamiltonian using a source term.

$$H^J(t) = H + J(t)A^J_\alpha(t) \tag{3.15}$$

$J(t)$ is the source term. $A_\alpha(t)$ is some element of algebra made out of local operators at time t. $A^J_\alpha(t)$ is the new Heisenberg operator defined in terms of the old operator A_α as :

$$A^J_\alpha(t) = \overline{\mathcal{T}}\left[\exp\left\{i\int_\vartheta^t J(x)A_\gamma(x)dx\right\}\right] A_\alpha(t)\mathcal{T}\left[\exp\left\{-i\int_\vartheta^t J(x)A_\gamma(x)dx\right\}\right] \tag{3.16}$$

$\mathcal{T}$ and $\overline{\mathcal{T}}$ are the time ordering and anti-time ordering symbols respectively.

The expectation value of a single source deformed observable can be written as :

$$\langle\Psi|A^J_\alpha(t)|\Psi\rangle = \langle\Psi|\overline{\mathcal{T}}\left[\exp\left\{i\int_\vartheta^t J(x)A_\gamma(x)dx\right\}\right] A_\alpha(t)\mathcal{T}\left[\exp\left\{-i\int_\vartheta^t J(x)A_\gamma(x)dx\right\}\right]|\Psi\rangle \tag{3.17}$$

Thus, modifying the Hamiltonian with the source looks equivalent to evaluating the correlators in the non-equilibrium states $|\Psi^{ne}\rangle$ after time t, where :

$$|\Psi^{ne}\rangle = \mathcal{T}\left[\exp\left\{-i\int_\vartheta^t J(x)A_\gamma(x)dx\right\}\right]|\Psi\rangle \tag{3.18}$$

We can consider a special case where the observables are just some product of localised operators in the bulk.

$$\phi^J(t, r_*, \Omega) = \overline{\mathcal{T}}\left[\exp\left\{i\int_\vartheta^t J(x)A_\gamma(x)dx\right\}\right] \phi(t, r_*, \Omega)\mathcal{T}\left[\exp\left\{-i\int_\vartheta^t J(x)A_\gamma(x)dx\right\}\right] \tag{3.19}$$

In the time ordered unity, when the integral runs from $[t + r_*, t]$, then bulk field commutes with the operators as they become causally separated. Recalling that $A_\alpha(t)$ are localised on the boundary, we can say by bulk locality that whenever $t + r_* < x$, we have $[\phi(t, r_*\Omega), A_\alpha(x)] = 0$. Thus, we can write :

$$\phi^J(t, r_*, \Omega) = W^\dagger(t + r_*)\phi(t, r_*, \Omega)W(t + r_*) \tag{3.20}$$

where,

$$W(t + r_*) \equiv \mathcal{T}\left[\exp\left\{-i\int_\vartheta^{t+r_*} J(x)A_\gamma(x)dx\right\}\right] \tag{3.21}$$

We can now write the correlator of the deformed bulk field as :

$$\begin{aligned}\langle\Psi|\phi^J(t_1, r_{*1}, \Omega_1)\cdots\phi^J(t_n, r_{*n}, \Omega_n)|\Psi\rangle = \langle\Psi|W^\dagger(t_1 + r_{*1})\phi(t_1, r_{*1}, \Omega_1)W(t_1 + r_{*1})\cdots \\ \cdots W^\dagger(t_n + r_{*n})\phi(t_n, r_{*n}, \Omega_n)W(t_n + r_{*n})|\Psi\rangle\end{aligned} \tag{3.22}$$

Now if we define our unitary operator appropriately, we can write the correlators of bulk fields as the expectation value of some operator between non-equilibrium states.

$$\langle\Psi|\phi^J(t_1, r_{*1}, \Omega_1)\cdots\phi^J(t_n, r_{*n}, \Omega_n)|\Psi\rangle = \langle\Psi|U^\dagger A_\alpha U|\Psi\rangle \tag{3.23}$$

where,

$$U = \exp\left[-i\int_{\vartheta}^{t_C} A_\alpha(t)dt\right] \tag{3.24}$$

$$\begin{aligned}A_\alpha = U\, W^\dagger(t_1 + r_{*1})\phi(t_1, r_{*1}, \Omega_1)W(t_1 + r_{*1})\cdots \\ \times W^\dagger(t_n + r_{*n})\phi(t_n, r_{*n}, \Omega_n)W(t_n + r_{*n})\, U^\dagger\end{aligned} \tag{3.25}$$

We note that $\forall i$ the coordinates (t_i, r_{*i}, Ω_i) lies in the causal patch. So, $(t_i + r_{*i}, 0, \Omega')$ also lies in the causal patch $\forall\Omega'$. Thus, the operator A_α lies completely in the causal patch and the unitary operator U lies completely in the boundary of the causal patch.

In conclusion, a correlator with source-deformed fields can be reduced to a correlator of ordinary fields with insertions only in the causal patch and an excitation on the boundary of the causal patch.

Operationally, an observer measure the expectation value $\langle\Psi|A_\alpha|\Psi\rangle$, actively deform the Hamiltonian by turning on a source and subsequently jump into the black hole to measure the LHS of the equation (3.23) after crossing the horizon to obtain the change in the expectation value of the observable A_α due to the action of the unitary operator U defined in (3.24).

An important point to note is that the source being considered is dual to the local operators on the boundary. We do not consider cases with bulk sources as they would violate local energy conservation. Thus, we consider only boundary sources.

3.4 Transfer Functions

In the process of reconstructing the bulk from the boundary, a crucial role is played by the transfer functions which map the bulk operators to the boundary operators.

In the causal patch outside the horizon, characterised by : $\exp\left(\frac{2\pi\vartheta}{\beta}\right) < V < \exp\left(\frac{2\pi t_C}{\beta}\right)$ and $U < 0$, the bulk field can be mapped to the boundary operators via the mapping :

$$\phi(t, r_*, \Omega) = \sum_{l,\omega} D^0(\omega, l)\mathcal{O}_{\omega,l}\zeta_{\omega,l}(r_*)e^{-i\omega t}Y_l(\Omega) + \text{h.c.} \tag{3.26}$$

In the causal patch inside the horizon, characterised by : $\exp\left(\frac{2\pi\vartheta}{\beta}\right) < V < \exp\left(\frac{2\pi t_C}{\beta}\right)$ and

$0 < U < \exp\left[\frac{2\pi}{\beta}\left(2r_{*s} - t_C\right)\right]$, the bulk field can be mapped to the boundary operators via the mapping :

$$\phi(t, r_*, \Omega) = \sum_{l,\omega} Y_l(\Omega) D^0(\omega, l) \tilde{\zeta}^-_{\omega,l}(r_*) \left(\mathcal{O}_{\omega,l} e^{2i\delta_{\omega,l}} e^{-i\omega t} + \tilde{\mathcal{O}}_{\omega,l} e^{i\omega t}\right) + \text{h.c.} \tag{3.27}$$

$\zeta_{\omega,l}(r_*)$ and $\tilde{\zeta}^-_{\omega,l}$ are radial functions that ensure that the full mode satisfies the wave equation both inside and outside the horizon. They are also chosen appropriately to satisfy the boundary conditions.
Assuming that the bulk field has mass m, we can write for modes outside the horizon :

$$\left(\Box - m^2\right) \zeta_{\omega,l}(r_*) e^{-i\omega t} Y_l(\Omega) = 0 \tag{3.28}$$

$$\zeta_{\omega,l}(r_*) \xrightarrow[r_* \to -\infty]{} e^{i\omega r_*} + e^{2i\delta_{\omega,l}} e^{-i\omega r_*} \tag{3.29}$$

The mode inside the black hole satisfies the condition :

$$\tilde{\zeta}^-_{\omega,l}(r_*) \xrightarrow[r_* \to -\infty]{} e^{-i\omega r_*} \tag{3.30}$$

The phase $\delta_{\omega,l}$ and the function $D^0(\omega, l)$ are determined by enforcing the condition that $\zeta_{\omega,l}$ is normalizable at the boundary such that the normalizable part has the coefficient :

$$\zeta_{\omega,l}(r_*) \xrightarrow[r_* \to 0]{} \frac{1}{D^0(\omega, l)} (r_*)^\Delta \tag{3.31}$$

Δ is the dimension of the operator $\mathcal{O}$.

$\mathcal{O}_{\omega,l}$ are the modes of the boundary operators and $\tilde{\mathcal{O}}_{\omega,l}$ are the mirror operators. Given an equilibrium state, the mirror operators are defined through a set of linear equations given as :

$$\tilde{\mathcal{O}}_{\omega,l} A_\alpha |\Psi\rangle = e^{-\frac{\beta\omega}{2}} A_\alpha \mathcal{O}^\dagger_{\omega,l} |\Psi\rangle \tag{3.32}$$

Thus, this completely specifies the action of operators on the subspace formed by low energy excitations of the equilibrium states.

Chapter 4

Stability of AdS Correlators in a Causal Patch

In this section we prove that AdS correlators obey the statistical mechanics inequality (2.16) whenever the excitation U and the correlator to be measured lie in the same causal patch.

It is also important to ask why we even expect the inequality to be violated. The bulk operators $\mathcal{O}_{\omega,l}$ are simple rewritings of ordinary operators, as by definition they are the modes of the boundary operators. So, they obey the inequality given in (2.16). On the contrary, the mirror operators $\tilde{\mathcal{O}}_{\omega,l}$, defined via the the set of linear equations (3.32), are state-dependent. It means it is not possible to find a globally defined operator $\tilde{\mathcal{O}}_{\omega,l}$ such that it satisfies (3.32) for all equilibrium states $|\Psi\rangle$. This state-dependence may cause the operators to behave very differently, and we cannot claim for sure that they will satisfy the inequality (2.16) at all times. We show here that the inequality holds whenever $\tilde{\mathcal{O}}_{\omega,l}$ are localised in the same causal patch as the excitation U.

4.1 Boundary Side Analysis

To analyse the AdS wave functions, we attempt to solve the Klein-Gordon equation for scalar fields coupled to an AdS_{d+1} space metric (3.1).

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[g^{\mu\nu}\sqrt{-g}\partial_\nu\phi\right] - m^2\phi = 0 \tag{4.1}$$

$-g$ gives us the determinant of the metric. Using the metric definition in terms of tortoise coordinates (3.8), we can find the square root of metric determinant as :

$$\sqrt{-g} = f(r)r^{d-1} \tag{4.2}$$

Plugging this value in (4.1), we obtain :

$$\implies \frac{1}{f(r)r^{d-1}}\partial_\mu\left\{g^{\mu\nu}f(r)r^{d-1}\partial_\nu\phi\right\}-m^2\phi=0$$

Recalling that the non-diagonal elements of the metric are zero, we can write :

$$\begin{aligned}\implies \frac{1}{f(r)r^{d-1}}\partial_t\left\{-\frac{1}{f(r)}f(r)r^{d-1}\partial_t\phi\right\}+\frac{1}{f(r)r^{d-1}}\partial_{r_*}\left\{\frac{1}{f(r)}f(r)r^{d-1}\partial_{r_*}\phi\right\}\\ +\frac{1}{f(r)r^{d-1}}\partial_\Omega\left\{\frac{1}{r^2}f(r)r^{d-1}\partial_\Omega\phi\right\}-m^2\phi=0\end{aligned}$$

We obtain the final equation as :

$$-\frac{1}{f(r)}\partial_t^2\phi+\frac{1}{f(r)r^{d-1}}\partial_{r_*}r^{d-1}\partial_{r_*}\phi+\frac{1}{r^2}\Box_\Omega\phi-m^2\phi=0 \tag{4.3}$$

Taking an ansatz for modes of the bulk field :

$$\phi_{\omega,l}(t,r_*,\Omega)=r^{\frac{1-d}{2}}\chi_{\omega,l}(r_*)e^{-i\omega t}Y_l(\Omega) \tag{4.4}$$

If we insert the expression for modes of bulk field (4.4) in the differential equation (4.3) :

$$\begin{aligned}\implies -\frac{1}{f(r)}\partial_t^2\left\{r^{\frac{1-d}{2}}\chi_{\omega,l}(r_*)e^{-i\omega t}Y_l(\Omega)\right\}+\frac{1}{f(r)r^{d-1}}\partial_{r_*}\left[r^{d-1}\partial_{r_*}\left\{r^{\frac{1-d}{2}}\chi_{\omega,l}(r_*)e^{-i\omega t}Y_l(\Omega)\right\}\right]\\ +\frac{1}{r^2}r^{\frac{1-d}{2}}\chi_{\omega,l}(r_*)e^{-i\omega t}\Box_\Omega Y_l(\Omega)-m^2\left\{r^{\frac{1-d}{2}}\chi_{\omega,l}(r_*)e^{-i\omega t}Y_l(\Omega)\right\}=0\end{aligned} \tag{4.5}$$

Focussing on the second term :

$$\begin{aligned}
\partial_{r_*}\left\{r^{d-1}\partial_{r_*}r^{\frac{1-d}{2}}\chi_{\omega,l}(r_*)\right\}&=\left\{\partial_{r_*}r^{d-1}\right\}\left\{\partial_{r_*}r^{\frac{1-d}{2}}\chi_{\omega,l}(r_*)\right\}+r^{d-1}\partial_{r_*}^2\left\{r^{\frac{1-d}{2}}\chi_{\omega,l}(r_*)\right\}\\
\implies \partial_{r_*}\left\{r^{d-1}\partial_{r_*}r^{\frac{1-d}{2}}\chi_{\omega,l}(r_*)\right\}&=\left\{(d-1)r^{d-2}\frac{dr}{dr_*}\right\}\left\{\left(\frac{1-d}{2}\right)r^{\frac{1-d}{2}-1}\frac{dr}{dr_*}\chi_{\omega,l}(r_*)+r^{\frac{1-d}{2}}\partial_{r_*}\chi_{\omega,l}(r_*)\right\}\\
&\quad+r^{d-1}\partial_{r_*}\left\{\left(\frac{1-d}{2}\right)r^{\frac{1-d}{2}-1}\frac{dr}{dr_*}\chi_{\omega,l}(r_*)+r^{\frac{1-d}{2}}\partial_{r_*}\chi_{\omega,l}(r_*)\right\}\\
\implies \partial_{r_*}\left\{r^{d-1}\partial_{r_*}r^{\frac{1-d}{2}}\chi_{\omega,l}(r_*)\right\}&=(d-1)r^{d-2}f(r)\left\{\left(\frac{1-d}{2}\right)r^{\frac{-d-1}{2}}f(r)\chi_{\omega,l}(r_*)+r^{\frac{1-d}{2}}\partial_{r_*}\chi_{\omega,l}(r_*)\right\}\\
&\quad+r^{d-1}\left(\frac{1-d}{2}\right)\left\{\left(\frac{-d-1}{2}\right)r^{\frac{-d-1}{2}-1}f^2(r)\chi_{\omega,l}(r_*)\right\}\\
&\quad+r^{d-1}\left(\frac{1-d}{2}\right)\left\{r^{\frac{-d-1}{2}}f'(r)f(r)\chi_{\omega,l}(r_*)+r^{\frac{-d-1}{2}}f(r)\partial_{r_*}\chi_{\omega,l}(r_*)\right\}\\
&\quad+r^{d-1}\left\{\left(\frac{1-d}{2}\right)r^{\frac{1-d}{2}-1}f(r)\partial_{r_*}\chi_{\omega,l}(r_*)+r^{\frac{1-d}{2}}\partial_{r_*}^2\chi_{\omega,l}(r_*)\right\}\\
\implies \partial_{r_*}\left\{r^{d-1}\partial_{r_*}r^{\frac{1-d}{2}}\chi_{\omega,l}(r_*)\right\}&=r^{\frac{d-1}{2}}\partial_{r_*}^2\chi_{\omega,l}(r_*)+r^{\frac{d-5}{2}}\frac{2r(1-d)}{4}f'(r)f(r)\chi_{\omega,l}(r_*)\\
&\quad+r^{\frac{d-5}{2}}\frac{(1-d)(d-3)}{4}f^2(r)\chi_{\omega,l}(r_*)
\end{aligned}$$

Thus, we finally obtain :

$$\partial_{r_*}\left\{r^{d-1}\partial_{r_*}r^{\frac{1-d}{2}}\chi_{\omega,l}(r_*)\right\} = r^{\frac{d-1}{2}}\partial^2_{r_*}\chi_{\omega,l}(r_*) + \frac{1}{4}\left\{(d-3)f(r)+2rf'(r)\right\}r^{\frac{d-5}{2}}(1-d)f(r)\chi_{\omega,l}(r_*) \tag{4.6}$$

Plugging this back into the differential equation (4.5) and noting that $\Box_\Omega Y_l(\Omega) = -l(l+d-2)Y_l(\Omega)$, we obtain :

$$\begin{aligned}\implies -\frac{1}{f(r)}r^{\frac{1-d}{2}}\chi_{\omega,l}(r_*)Y_l(\Omega)(-i\omega)^2e^{-i\omega t}\\ +\frac{1}{f(r)r^{d-1}}e^{-i\omega t}Y_l(\Omega)\left[r^{\frac{d-1}{2}}\partial^2_{r_*}\chi_{\omega,l}(r_*) + \frac{1}{4}\left\{(d-3)f(r)+2rf'(r)\right\}r^{\frac{d-5}{2}}(1-d)f(r)\chi_{\omega,l}(r_*)\right]\\ +\frac{1}{r^2}r^{\frac{1-d}{2}}\chi_{\omega,l}(r_*)e^{-i\omega t}\left[-l(l+d-2)Y_l(\Omega)\right] - m^2r^{\frac{1-d}{2}}\chi_{\omega,l}(r_*)e^{-i\omega t}Y_l(\Omega) = 0\end{aligned}$$

Removing the common terms and simplifying :

$$\begin{aligned}\implies \frac{r^{\frac{d-1}{2}}}{f(r)r^{d-1}}\partial^2_{r_*}\chi_{\omega,l}(r_*) = \frac{\omega^2 r^{\frac{1-d}{2}}}{f(r)}\chi_{\omega,l}(r_*) - \frac{r^{\frac{d-5}{2}}}{4r^{d-1}}\left\{(d-3)f(r)+2rf'(r)f(r)\right\}(1-d)\chi_{\omega,l}(r_*)\\ +\frac{r^{\frac{1-d}{2}}}{r^2}l(l+d-2)\chi_{\omega,l}(r_*) + m^2r^{\frac{1-d}{2}}\chi_{\omega,l}(r_*)\end{aligned}$$

We obatin the final differential equation with respect to $\chi_{\omega,l}$ as :

$$\partial^2_{r_*}\chi_{\omega,l}(r_*) = -\left[\omega^2 + \frac{(d-3)(1-d)}{4}\frac{f^2(r)}{r^2} - l(l+d-2)\frac{f(r)}{r^2} + \frac{(1-d)}{2}\frac{f'(r)f(r)}{r}\right]\chi_{\omega,l}(r_*) \tag{4.7}$$

We can also re-write the equation as :

$$\partial^2_{r_*}\chi_{\omega,l}(r_*) + \gamma^2\chi_{\omega,l} - \frac{\left(m^2+\frac{d^2-1}{4}\right)}{r_*^2}\chi_{\omega,l}(r_*) = V_{bd}\chi_{\omega,l}(r_*) \tag{4.8}$$

where,

$$\gamma^2 = \omega^2 - l(l+d-2) - \frac{m^2+(d-2)(d-1)}{3} \tag{4.9}$$

$$\begin{aligned}V_{bd}(r_*) = l(l+d-2)\left[\frac{f(r)}{r^2}-1\right] + \frac{(d-3)(d-1)}{4}\left[\frac{f^2(r)}{r^2}-\frac{1}{r_*^2}-\frac{4}{3}\right]\\ + m^2\left[f(r)-\frac{1}{r_*^2}-\frac{1}{3}\right] + \frac{d-1}{2}\left[\frac{f'(r)f(r)}{r}-\frac{2}{r_*^2}-\frac{2}{3}\right]\end{aligned} \tag{4.10}$$

$V_{bd}(r_*)$ is finite at all values of $r \in (-\infty, 0]$ and near the boundary $V_{bd}(r_*) = O(r_*^2)$.

We can solve the equation (4.8) with Green's functions both near the boundary and the horizon. We start with a "free" equation near the boundary, in which we neglect V_{bd}.

$$\partial_{r_*}^2 \chi_{\omega,l}^0(r_*) + \gamma^2 \chi_{\omega,l}^0 - \frac{\left(m^2 + \frac{d^2-1}{4}\right)}{r_*^2} \chi_{\omega,l}^0(r_*) = 0 \tag{4.11}$$

Now solving the above equation, we obtain solutions of the form :

$$\chi_{\omega,l}^0(r_*) = c_1 \sqrt{r_*} \frac{J_\nu}{\gamma^\nu}(\gamma r_*) + c_2 \sqrt{r_*} \gamma^\nu Y_\nu(\gamma r_*) \tag{4.12}$$

where,

$$\frac{1}{\gamma^\nu} J_\nu(\gamma r_*) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s+1)\Gamma(\nu+s+1)} \gamma^{2s} \left(\frac{r_*}{2}\right)^{\nu+2s} \tag{4.13}$$

$$\gamma^\nu Y_\nu(\gamma r_*) = \frac{J_\nu(\gamma r_*)\cos(\pi\nu) - J_{-\nu}(\gamma r_*)}{\sin(\nu\pi)} \tag{4.14}$$

$$\nu = \sqrt{m^2 + \frac{d^2-1}{4}} \tag{4.15}$$

In the $r_* \longrightarrow 0$ limit near the boundary, there could be divergent terms in $Y_\nu(\gamma r_*)$, so we set $c_2 = 0$. Thus, for normalisable solutions, we have :

$$\chi_{\omega,l}^0(r_*) = \frac{\sqrt{r_*}}{\gamma^\nu} J_\nu(\gamma r_*) \tag{4.16}$$

Also, near the boundary $\chi_{\omega,l}^0 \longrightarrow \chi_{\omega,l}$. Now evaluating the terms in near the boundary limit :

$$r^{\frac{1-d}{2}} \chi_{\omega,l}(r_*) \overset{r_*\to 0}{\simeq} r_*^{\frac{d-1}{2}} \sqrt{r_*} \frac{(-1)^0}{\Gamma(1)\Gamma(\nu+1)} \left(\frac{1}{2}\right)^\nu r^\nu + \text{higher order terms } (s>0) \tag{4.17}$$

Thus, we can write the mode of the bulk fields as :

$$\phi_{\omega,l}(t, r_*, \Omega) = e^{-i\omega t} Y_l(\Omega) r^{\frac{1-d}{2}} \chi_{\omega,l}(r_*) \xrightarrow[r_*\to 0]{} e^{-i\omega t} Y_l(\Omega) r_*^{\frac{d}{2}+\nu} \tag{4.18}$$

As in the expression of $\chi_{\omega,l}^0$ only even powers of γ appear, $\chi_{\omega,l}^0$ has no poles or branch cuts in ω at any finite r_* at any finite value of ω.

For the entire solution, the Green's function can be written as :

$$G(r'_*, r_*) = \frac{\pi}{2} \sqrt{r'_* r_*} \left\{ J_\nu(\gamma r_*) Y_\nu(\gamma r'_*) - Y_\nu(\gamma r_*) J_\nu(\gamma r'_*) \right\} \theta(r'_* - r_*) \tag{4.19}$$

Substituting the expressions of J_ν and Y_ν :

$$G(r'_*, r_*) = \frac{\pi \sqrt{r_* r'_*}}{2\sin(\nu\pi)} \sum_{s,t=0}^{\infty} \frac{(-1)^{s+t} \gamma^{2(s+t)} \left(\frac{r'_*}{2}\right)^{2s+\nu} \left(\frac{r_*}{2}\right)^{2t-\nu}}{\Gamma(s+1)\Gamma(\nu+s+1)\Gamma(t+1)\Gamma(t-\nu+1)} - (r'_* \leftrightarrow r_*) \tag{4.20}$$

Using the Green's function and the solution of $\chi^0_{\omega,l}$, we can construct the full solution as :

$$\chi^{(n)}_{\omega,l}(r_*) = \int_0^{r_*} \chi^{(n-1)}_{\omega,l}(r'_*)G(r'_*, r_*)V_{bd}(r'_*)dr'_* \tag{4.21}$$

$$\chi_{\omega,l}(r_*) = \sum_{n=0}^{\infty} \chi^{(n)}_{\omega,l}(r_*) \tag{4.22}$$

The full solution has no branch cuts or roots at finite ω and finite r_*.
Thus, we obtain the modes of the bulk fields $\phi_{\omega,l}(r_*)$.

Transfer functions using boundary-side modes

We can now combine the modes of the bulk fields to the modes of the boundary operators to write down the solution for the bulk field.

$$\phi(t, r_*, \Omega) = \sum_{l,\omega} \mathcal{O}_{\omega,l} r^{\frac{1-d}{2}} \chi_{\omega,l}(r_*) e^{-i\omega t} Y_l(\Omega) + \text{h.c.} \tag{4.23}$$

Considering the expansion of the radial mode near the horizon,

$$r^{\frac{1-d}{2}} \chi_{\omega,l}(r_*) \xrightarrow[r_* \to -\infty]{} D^0(\omega, l) \left(e^{i\omega r_*} + e^{2i\delta_{\omega,l}} e^{-i\omega r_*}\right) \tag{4.24}$$

with appropriately defined $D^0(\omega, l)$ and $\delta_{\omega,l}$, we can define a function $\zeta_{\omega,l}(r_*)$ as :

$$\zeta_{\omega,l}(r_*) = \frac{1}{D^0(\omega, l)} r^{\frac{1-d}{2}} \chi_{\omega,l}(r_*) \tag{4.25}$$

$\zeta_{\omega,l}(r_*)$ is a sum of plane waves near the horizon. Using the new definition, the bulk field can be written as :

$$\phi(t, r_*, \Omega) = \sum_{l,\omega} \frac{a_{\omega,l}}{\sqrt{\omega}} \zeta_{\omega,l}(r_*) e^{-i\omega t} Y_l(\Omega) + \text{h.c.} \tag{4.26}$$

$a_{\omega,l}$ is the annihilation operator, defined as :

$$a_{\omega,l} = D^0(\omega, l) \sqrt{\omega} \mathcal{O}_{\omega,l} \tag{4.27}$$

$a^\dagger_{\omega,l}$ is the corresponding creation operator. We can show that the creation and annihilation operators satisfy the following commutation relation :

$$\left[a_{\omega,l}, a^\dagger_{\omega,l}\right] = \delta_{\omega,\omega'} \delta_{l,l'} \tag{4.28}$$

Looking into the analyticity properties of $\zeta_{\omega,l}$, we conclude that the singularities of the function only come from $D^0(\omega, l)$ as $\chi_{\omega,l}$ has no singularities. Now as $D^0(\omega, l)$ is defined as the coefficient of the outgoing waves : $e^{i\omega r_*}$, when $D^0(\omega, l)$ vanishes, the mode becomes purely ingoing at the horizon. These are called the quasinormal modes. They have the property that they all have $\Im(\omega) < 0$, i.e., $D^0(\omega, l)$ vanishes only when ω is in the lower half plane. Thus, $\zeta_{\omega,l}$ has poles only when ω is in the lower half plane and is regular in the upper half.

4.2 Analysis near the Horizon

Now considering the bulk modes starting from the horizon and solving outwards allows us to define "right-moving" modes.
We can write the differential equation for radial mode near the horizon as :

$$\partial_{r_*}^2 \zeta_{\omega,l}^{\pm}(r_*) + \omega^2 \zeta_{\omega,l}^{\pm}(r_*) = V_{hor}(r_*)\zeta_{\omega,l}^{\pm}(r_*) \tag{4.29}$$

where $\pm$ distinguishes the two independent solutions and V_{hor} is defined as :

$$V_{hor}(r_*) = -\left[\frac{(d-3)(1-d)}{4}\frac{f^2(r)}{r^2} - m^2 f(r) - l(l+d-2)\frac{f(r)}{r^2} + \frac{(1-d)}{2}\frac{f'(r)f(r)}{r}\right] \tag{4.30}$$

$\zeta_{\omega,l}^{+}$ corresponds to the right moving mode, which can be constructed using similar Green's function method. In the absence of potential, the right moving mode is given as :

$$\zeta_{\omega,l}^{+}(r_*) = e^{i\omega r_*} \tag{4.31}$$

The Green's function for differential equation (4.29) can be written as :

$$G_{hor}(r'_*, r_*) = \frac{1}{\omega}\sin\left(\omega(r_* - r'_*)\right)\theta(r_* - r'_*) \tag{4.32}$$

Thus the full solution of $\zeta_{\omega,l}^{+}$ can be defined as :

$$\zeta_{\omega,l}^{+(n)}(r_*) = \int_{-\infty}^{r_*} \zeta_{\omega,l}^{+(n-1)}(r'_*)V_{hor}(r')_* G_{hor}(r'_*, r_*)dr'_* \tag{4.33}$$

$$\zeta_{\omega,l}^{+}(r_*) = \sum_{n=0}^{\infty} \zeta_{\omega,l}^{+(n)}(r_*) \tag{4.34}$$

Near the horizon, the potential can be expanded as :

$$V_{hor}(r_*) = \sum_{m>0} V_m \exp\left(\frac{4\pi m r_*}{\beta}\right) \tag{4.35}$$

The closest poles occur at the singularity.
Using this expansion of the potential, we can write the radial wave function for $n = 1$ as :

$$\zeta_{\omega,l}^{+(1)}(r_*) = \sum_m \frac{\beta V_m \exp\left[\left(i\omega + \frac{4\pi m}{\beta}\right)r_*\right]}{4\pi m\left(2i\omega + \frac{4\pi m}{\beta}\right)} \tag{4.36}$$

The poles of the function occur at $\omega = \dfrac{2\pi i m}{\beta}$. So, for any general n, we have :

$$\zeta^{(n)}_{\omega,l}(r_*) = \sum_{m>0} D^{+(n)}_m(\omega, l) \exp\left[\left(i\omega + \frac{4\pi m}{\beta}\right) r_*\right] \tag{4.37}$$

We finally obtain the full solution as :

$$\zeta^{+}_{\omega,l}(r_*) = e^{i\omega r_*} D^{+}_{\omega,l}(r_*) \tag{4.38}$$

The function $D^{+}_{\omega,l}$ has poles only when $\dfrac{-i\omega\beta}{2\pi}$ is a positive integer.

We obtain similar solution for $\zeta^{-}_{\omega,l}$ which is related to $\zeta^{+}_{\omega,l}$ as :

$$\zeta^{-}_{\omega,l}(r_*) = \left(\zeta^{+}_{\omega,l}(r_*)\right)^* \tag{4.39}$$

The mode $\zeta^{-}_{\omega,l}$ has modes when $\dfrac{i\omega\beta}{2\pi}$ is a positive integer.

Transfer functions using horizon modes

Near the horizon, we can write the bulk field as a combination of left and right moving modes :

$$\phi(t, r_*, \Omega) = \phi_r(t, r_*, \Omega) + \phi_l(t, r_*, \Omega) \tag{4.40}$$

As we are solving for fields outside the black hole, the left and right moving modes are defined as follows :

$$\phi_r(t, r_*, \Omega) = \sum_{l,\omega} \frac{a_{\omega,l}}{\sqrt{\omega}} \zeta^{+}_{\omega,l}(r_*) e^{-i\omega t} Y_l(\Omega) + \text{h.c.} \tag{4.41}$$

$$\phi_l(t, r_*, \Omega) = \sum_{l,\omega} \frac{a_{\omega,l}}{\sqrt{\omega}} e^{2i\delta_{\omega,l}} \zeta^{-}_{\omega,l}(r_*) e^{-i\omega t} Y_l(\Omega) + \text{h.c.} \tag{4.42}$$

The analytic properties of the entire function is controlled by the analytic properties of $\zeta_{\omega,l}(r_*) = \zeta^{+}_{\omega,l}(r_*) + e^{2i\delta_{\omega,l}} \zeta^{-}_{\omega,l}(r_*)$, not by the individual components.

4.3 Crossing the Horizon

Behind the horizon the bulk fields are defined as :

$$\phi(t, r_*, \Omega) = \tilde{\phi}_l(t, r_*, \Omega) + \tilde{\phi}_r(t, r_*, \Omega) \tag{4.43}$$

where,

$$\tilde{\phi}_l(t, r_*, \Omega) = \sum_{l,\omega} \frac{a_{\omega,l}}{\sqrt{\omega}} e^{2i\delta_{\omega,l}} \tilde{\zeta}^-_{\omega,l}(r_*) e^{-i\omega t} Y_l(\Omega) + \text{h.c.} \tag{4.44}$$

$$\tilde{\phi}_r(t, r_*, \Omega) = \sum_{l,\omega} \frac{\tilde{a}_{\omega,l}}{\sqrt{\omega}} e^{i\omega t} Y_l^*(\Omega) \tilde{\zeta}^-_{\omega,l}(r_*) + \text{h.c.} \tag{4.45}$$

The operator $a_{\omega,l}$ appearing in $\tilde{\phi}_l(t, r_*, \Omega)$ is the same as those that appear for modes outside the horizon as the left moving modes are continuous across the horizon.
The operator $\tilde{a}_{\omega,l}$ are defined as :

$$\tilde{a}_{\omega,l} = \left(D^0(\omega, l)\right)^* \sqrt{\omega} \tilde{\mathcal{O}}_{\omega,l} \tag{4.46}$$

Now since the potential V_{hor} (4.35) is analytic across the horizon, we can continue the series expansion analytically inside the black hole as :

$$V_{hor}(r_*) = \sum_{m>0} (-1)^m V_m \exp\left(\frac{4\pi m r_*}{\beta}\right) \tag{4.47}$$

There are no differences in the analytic properties of $\zeta^\pm_{\omega,l}$ from the previous section.

4.4 Stability of Correlators

We finally want to prove the stability of AdS correlators under perturbations of the causal patch.

For that we start by considering a correlator involving an insertion of a right moving part of the field behind the horizon, in the absence of perturbations : $\langle \Psi | \tilde{\phi}_r(t, r_*, \Omega) A_\alpha | \Psi \rangle$. Using the definition of $\tilde{\phi}_r$ (4.45) and the property of mirror modes (3.32), we can write :

$$\langle \Psi | \tilde{\phi}_r(t, r_*, \Omega) A_\alpha | \Psi \rangle = \langle \Psi | A_\alpha \tilde{\phi}_r(t, r_*, \Omega) | \Psi \rangle = \langle \Psi | A_\alpha e^{-\frac{\beta H}{2}} \widehat{\phi}(t, r_*, \Omega) e^{\frac{\beta H}{2}} | \Psi \rangle \tag{4.48}$$

where,

$$\widehat{\phi}(t, r_*, \Omega) = \sum_{l,\omega} \frac{a_{\omega,l}}{\sqrt{\omega}} \tilde{\zeta}^+_{\omega,l}(r_*) e^{-i\omega t Y_l(\Omega)} + \text{h.c.} \tag{4.49}$$

The important point to note is that $\widehat{\phi}$ is entirely made up of ordinary operators. The coordinates are localised in $\vee_C$.
Thus, the expectation value of an operator containing mirror modes can be converted to an expectation value of ordinary operators for equilibrium states.

The operator $\widehat{\phi}$ has the property that it commutes with the modes of the boundary operators, i.e., for $(t',\Omega') \in \mathcal{B}_C$:

$$\left[\widehat{\phi}(t,r_*,\Omega), \mathcal{O}_{\omega,l}(t',\Omega')\right] = 0 \tag{4.50}$$

Now if we consider a unitary operator U_C made out of simple operators localised in $\mathcal{B}_C$ and $(t,r_*,\Omega) \in \vee_C$, then :

$$U_C^\dagger \widehat{\phi}(t,r_*,\Omega) U_C = \widehat{\phi}(t,r_*,\Omega) \tag{4.51}$$

Perturbations of the Causal Patch

Keeping these properties in mind, we can now introduce perturbations in our causal patch.

Recalling the result from the previous chapter that a correlator with source-deformed fields can be reduced to a correlator of ordinary fields with boundary excitations and insertions in the causal patch; we can characterise the perturbation by considering correlators involving mirror operators in a state excited by unitary operator U_C and another insertion of some local operators A_α : $\langle\Psi|U_C^\dagger \tilde{\phi}_r(t,r_*,\Omega) A_\alpha U_C|\Psi\rangle$.
By definition of the mirror operators (3.32) we have :

$$\langle\Psi|U_C^\dagger \tilde{\phi}_r(t,r_*,\Omega) A_\alpha U_C|\Psi\rangle = \langle\Psi|U_C^\dagger A_\alpha \tilde{\phi}_r(t,r_*,\Omega) U_C|\Psi\rangle = \langle\Psi|U_C^\dagger A_\alpha U_C e^{-\frac{\beta H}{2}} \widehat{\phi}(t,r_*,\Omega) e^{\frac{\beta H}{2}}|\Psi\rangle \tag{4.52}$$

Now we note that :

$$e^{-\frac{\beta H}{2}} U e^{\frac{\beta H}{2}}|\Psi\rangle = U|\Psi\rangle + O(\beta\delta E) \tag{4.53}$$

Recalling from chapter 2, for any coarse-grained operator A_α we have (2.1) :

$$\begin{aligned} &\operatorname{Tr}\left(e^{-\beta H} A_\alpha U\right) = Z(\beta)\langle\Psi|A_\alpha U|\Psi\rangle \\ \implies &\operatorname{Tr}\left(e^{-\beta H} A_\alpha U\right) = Z(\beta)\langle\Psi|A_\alpha e^{-\frac{\beta H}{2}} U e^{\frac{\beta H}{2}}|\Psi\rangle + O(\beta\delta E) \end{aligned}$$

Using the cyclicity property of trace :

$$\operatorname{Tr}\left(e^{-\beta H} A_\alpha U\right) = \operatorname{Tr}\left(e^{-\frac{\beta H}{2}} A_\alpha e^{\frac{-\beta H}{2}} U\right) + O(\beta\delta E) \tag{4.54}$$

Now evaluating the commutator in (4.52) using the property derived above and the cyclicity property of trace:

$$\begin{aligned} &Z(\beta)\langle\Psi|U_C A_\alpha U_C^\dagger \widehat{\phi}(t,r_*,\Omega) e^{\frac{\beta H}{2}}|\Psi\rangle = Z(\beta)\langle\Psi|U_C A_\alpha U_C^\dagger \widehat{\phi}(t,r_*,\Omega)|\Psi\rangle \\ \implies &Z(\beta)\langle\Psi|U_C A_\alpha U_C^\dagger \widehat{\phi}(t,r_*,\Omega) e^{\frac{\beta H}{2}}|\Psi\rangle = \operatorname{Tr}\left[e^{-\frac{\beta H}{2}} U_C A_\alpha U_C^\dagger e^{-\frac{\beta H}{2}} \widehat{\phi}(t,r_*,\Omega)\right] \\ \implies &Z(\beta)\langle\Psi|U_C A_\alpha U_C^\dagger \widehat{\phi}(t,r_*,\Omega) e^{\frac{\beta H}{2}}|\Psi\rangle = \operatorname{Tr}\left[e^{-\beta H} \widehat{\phi}(t,r_*,\Omega) e^{-\frac{\beta H}{2}} U_C A_\alpha U_C^\dagger e^{\frac{\beta H}{2}}\right] \\ \implies &Z(\beta)\langle\Psi|U_C A_\alpha U_C^\dagger \widehat{\phi}(t,r_*,\Omega) e^{\frac{\beta H}{2}}|\Psi\rangle = \operatorname{Tr}\left[e^{-\beta H} \widehat{\phi}(t,r_*,\Omega) e^{-\frac{\beta H}{2}} U_C A_\alpha e^{-\frac{\beta H}{2}} U_C^\dagger e^{\beta H}\right] \\ \implies &Z(\beta)\langle\Psi|U_C A_\alpha U_C^\dagger \widehat{\phi}(t,r_*,\Omega) e^{\frac{\beta H}{2}}|\Psi\rangle = \operatorname{Tr}\left[e^{-\frac{\beta H}{2}} U_C A_\alpha e^{-\frac{\beta H}{2}} U_C^\dagger \widehat{\phi}(t,r_*,\Omega)\right] \end{aligned}$$

Now as $\widehat{\phi}(t, r_*, \Omega)$ and $U_C^\dagger$ commute :

$$\implies Z(\beta)\langle\Psi|U_C A_\alpha U_C^\dagger \widehat{\phi}(t, r_*, \Omega) e^{\frac{\beta H}{2}}|\Psi\rangle = \text{Tr}\left[e^{-\frac{\beta H}{2}} U_C A_\alpha e^{-\frac{\beta H}{2}} \widehat{\phi}(t, r_*, \Omega) U_C^\dagger\right]$$
$$\implies Z(\beta)\langle\Psi|U_C A_\alpha U_C^\dagger \widehat{\phi}(t, r_*, \Omega) e^{\frac{\beta H}{2}}|\Psi\rangle = \text{Tr}\left[e^{-\beta H} U_C A_\alpha e^{-\frac{\beta H}{2}} \widehat{\phi}(t, r_*, \Omega) U_C^\dagger e^{\frac{\beta H}{2}}\right]$$

Thus we finally obtain :

$$Z(\beta)\langle\Psi|U_C A_\alpha U_C^\dagger \widehat{\phi}(t, r_*, \Omega) e^{\frac{\beta H}{2}}|\Psi\rangle = \text{Tr}\left[e^{-\beta H} U_C A_\alpha e^{-\frac{\beta H}{2}} \widehat{\phi}(t, r_*, \Omega) e^{\frac{\beta H}{2}} U_C^\dagger\right] \tag{4.55}$$

The above result implies :

$$\langle\Psi|U_C A_\alpha \tilde{\phi}_r(t, r_*, \Omega) U_C^\dagger|\Psi\rangle = \langle\Psi|U_C A_\alpha e^{-\frac{\beta H}{2}} \widehat{\phi}(t, r_*, \Omega) e^{\frac{\beta H}{2}} U_C^\dagger|\Psi\rangle + O(\beta\delta E) \tag{4.56}$$

So we can conclude that even for non-equilibrium states, we can convert correlators with mirror operators into correlators with ordinary operators.

Now, we can write the change in the correlator under the excitation $U_C^\dagger$ as :

$$\delta\langle A_\alpha \tilde{\phi}_r(t, r_*, \Omega)\rangle = \langle\Psi|U_\mathcal{C} A_\alpha e^{-\frac{\beta H}{2}} \hat{\phi}(t, r_*, \Omega) e^{\frac{\beta H}{2}} U_\mathcal{C}^\dagger|\Psi\rangle - \langle\Psi|A_\alpha e^{-\frac{\beta H}{2}} \hat{\phi}(t, r_*, \Omega) e^{\frac{\beta H}{2}}|\Psi\rangle + O(\beta\delta E) \tag{4.57}$$

Thus we have successfully represented the change in the correlator involving mirror operators that were state dependent in terms of ordinary operators, i.e., state independent operators. The said ordinary operators obey the statistical mechanics inequality (2.16), and thus by extension the correlator of mirror operators also obey the inequality.

We can derive the change explicitly as :

$$\delta\langle A_\alpha \tilde{\phi}_r(t, r_*, \Omega)\rangle \leq 2\min(\tilde{\sigma_1}, \tilde{\sigma_2})\sqrt{\beta\delta E} \tag{4.58}$$

where the minimum of $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ gives the standard deviation of the operator $A_\alpha \tilde{\phi}_r(t, r_*, \Omega)$.

$$(\tilde{\sigma}_1)^2 = \langle\Psi|\tilde{\phi}_r^\dagger A_\alpha^\dagger A_\alpha \tilde{\phi}_r|\Psi\rangle - \left|\langle\Psi|A_\alpha \tilde{\phi}_r|\Psi\rangle\right|^2 \tag{4.59}$$
$$(\tilde{\sigma}_2)^2 = \langle\Psi|A_\alpha \tilde{\phi}_r \tilde{\phi}_r^\dagger A_\alpha^\dagger|\Psi\rangle - \left|\langle\Psi|A_\alpha \tilde{\phi}_r|\Psi\rangle\right|^2 \tag{4.60}$$

Thus, we proved that the statistical mechanics inequality is not violated if we choose the causal patches carefully.

Chapter 5

Conclusion

In this project, I reproduced some of the calculations of Prof. Raju's paper on Smooth Causal Patches for AdS black holes in which resolves the paradox of low-energy excitations about AdS black holes.
Statistical mechanics states that low energy excitations in a thermal system only have small effects on physical observables. Black hole geometry seems to violate this general principle as an analysis without the careful construction of causal patches may suggest that low energy excitations in the boundary may cause large effects in the bulk fields.

To summarise the important steps involved in the resolution of the paradox :

1. A causal patch was constructed using a point in the singularity of the AdS-Schwarzschild black hole being studied. Next, the CFT Hamiltonian was actively deformed using a source term dual to the local operators on the boundary.

2. The correlator with source deformed fields was shown to be equal to the correlator of ordinary fields with excitations localised on the boundary and insertions in the causal patch.

4. The Klein-Gordon equation for scalar fields was solved for regions both inside and outside the black hole. The bulk fields inside the black hole contained mirror operators which are state-dependent mappings that could violate the statistical mechanics principle.

5. It was shown next that correlators which involved mirror operators could be converted to correlators with ordinary state-independent operators, represented by $\widehat{\phi}(t, r_*, \Omega)$ for equilibrium states of CFT.

6. The operator $\widehat{\phi}(t, r_*, \Omega)$ has the property that it commutes with boundary operators when (t, r_*, Ω) lies in the causal patch inside the black hole.

7. As a consequence of this vanishing commutator, even for non-equilibrium states of CFT, the correlator involving mirror operators can be converted to correlators with ordinary state-independent operators.

8. Thus, as correlators involving ordinary operators obey the statistical mechanical constraints, correlators involving mirror operators also obey the constraints as long as they lie in the causal patch.

Bibliography

[1] Ahmed Almheiri, Donald Marolf, Joseph Polchinski, and James Sully. Black holes: complementarity or firewalls? *Journal of High Energy Physics*, 2013(2), feb 2013. URL: https://doi.org/10.1007%2Fjhep02%282013%29062, doi:10.1007/jhep02(2013)062.

[2] Souvik Banerjee, Jan-Willem Bryan, Kyriakos Papadodimas, and Suvrat Raju. A toy model of black hole complementarity. *Journal of High Energy Physics*, 2016(5), may 2016. URL: https://doi.org/10.1007%2Fjhep05%282016%29004, doi:10.1007/jhep05(2016)004.

[3] Raphael Bousso. Holographic probabilities in eternal inflation. *Physical Review Letters*, 97(19), nov 2006. URL: https://doi.org/10.1103%2Fphysrevlett.97.191302, doi:10.1103/physrevlett.97.191302.

[4] Raphael Bousso. Complementarity is not enough. *Physical Review D*, 87(12), jun 2013. URL: https://doi.org/10.1103%2Fphysrevd.87.124023, doi:10.1103/physrevd.87.124023.

[5] J. M. Deutsch. Quantum statistical mechanics in a closed system. *Phys. Rev. A*, 43:2046–2049, Feb 1991. URL: https://link.aps.org/doi/10.1103/PhysRevA.43.2046, doi:10.1103/PhysRevA.43.2046.

[6] Lukasz Fidkowski, Veronika Hubeny, Matthew Kleban, and Stephen Shenker. The black hole singularity in AdS/CFT. *Journal of High Energy Physics*, 2004(02):014–014, feb 2004. URL: https://doi.org/10.1088%2F1126-6708%2F2004%2F02%2F014, doi:10.1088/1126-6708/2004/02/014.

[7] Ben Freivogel, Robert A. Jefferson, Laurens Kabir, and I-Sheng Yang. Geometry of the infalling causal patch. *Physical Review D*, 91(4), feb 2015. URL: https://doi.org/10.1103%2Fphysrevd.91.044036, doi:10.1103/physrevd.91.044036.

[8] S.S. Gubser, I.R. Klebanov, and A.M. Polyakov. Gauge theory correlators from non-critical string theory. *Physics Letters B*, 428(1-2):105–114, may 1998. URL: https://doi.org/10.1016%2Fs0370-2693%2898%2900377-3, doi:10.1016/s0370-2693(98)00377-3.

[9] Alex Hamilton, Daniel Kabat, Gilad Lifschytz, and David A. Lowe. Local bulk operators in AdS/CFT correspondence: A boundary view of horizons and locality. *Physical Review D*, 73(8), apr 2006. URL: https://doi.org/10.1103%2Fphysrevd.73.086003, doi:10.1103/physrevd.73.086003.

[10] Daniel Harlow. Aspects of the papadodimas-raju proposal for the black hole interior. *Journal of High Energy Physics*, 2014(11), nov 2014. URL: https://doi.org/10.1007%2Fjhep11%282014%29055, doi:10.1007/jhep11(2014)055.

[11] Gary T. Horowitz and Veronika E. Hubeny. Quasinormal modes of AdS black holes and the approach to thermal equilibrium. *Physical Review D*, 62(2), jun 2000. URL: https://doi.org/10.1103%2Fphysrevd.62.024027, doi:10.1103/physrevd.62.024027.

[12] Irfan Ilgin and I-Sheng Yang. Causal patch complementarity: The inside story for old black holes. *Physical Review D*, 89(4), feb 2014. URL: https://doi.org/10.1103%2Fphysrevd.89.044007, doi:10.1103/physrevd.89.044007.

[13] Juan Maldacena. *International Journal of Theoretical Physics*, 38(4):1113–1133, 1999. URL: https://doi.org/10.1023%2Fa%3A1026654312961, doi:10.1023/a:1026654312961.

[14] Donald Marolf and Joseph Polchinski. Gauge-gravity duality and the black hole interior. *Physical Review Letters*, 111(17), oct 2013. URL: https://doi.org/10.1103%2Fphysrevlett.111.171301, doi:10.1103/physrevlett.111.171301.

[15] Donald Marolf and Joseph Polchinski. Violations of the born rule in cool state-dependent horizons. *Journal of High Energy Physics*, 2016(1), jan 2016. URL: https://doi.org/10.1007%2Fjhep01%282016%29008, doi:10.1007/jhep01(2016)008.

[16] Samir D Mathur. The information paradox: a pedagogical introduction. *Classical and Quantum Gravity*, 26(22):224001, oct 2009. URL: https://doi.org/10.1088%2F0264-9381%2F26%2F22%2F224001, doi:10.1088/0264-9381/26/22/224001.

[17] Kyriakos Papadodimas and Suvrat Raju. An infalling observer in AdS/CFT. *Journal of High Energy Physics*, 2013(10), oct 2013. URL: https://doi.org/10.1007%2Fjhep10%282013%29212, doi:10.1007/jhep10(2013)212.

[18] Kyriakos Papadodimas and Suvrat Raju. Black hole interior in the holographic correspondence and the information paradox. *Physical Review Letters*, 112(5), feb 2014. URL: https://doi.org/10.1103%2Fphysrevlett.112.051301, doi:10.1103/physrevlett.112.051301.

[19] Kyriakos Papadodimas and Suvrat Raju. State-dependent bulk-boundary maps and black hole complementarity. *Physical Review D*, 89(8), apr 2014. URL: https://doi.org/10.1103%2Fphysrevd.89.086010, doi:10.1103/physrevd.89.086010.

[20] Kyriakos Papadodimas and Suvrat Raju. Local operators in the eternal black hole. *Physical Review Letters*, 115(21), nov 2015. URL: https://doi.org/10.1103%2Fphysrevlett.115.211601, doi:10.1103/physrevlett.115.211601.

[21] Kyriakos Papadodimas and Suvrat Raju. Remarks on the necessity and implications of state-dependence in the black hole interior. *Physical Review D*, 93(8), apr 2016. URL: https://doi.org/10.1103%2Fphysrevd.93.084049, doi:10.1103/physrevd.93.084049.

[22] Suvrat Raju. The sensitivity of black holes to low energy excitations, 2017. URL: https://arxiv.org/abs/1703.10159, doi:10.48550/ARXIV.1703.10159.

[23] Suvrat Raju. Smooth causal patches for AdS black holes. *Physical Review D*, 95(12), jun 2017. URL: https://doi.org/10.1103%2Fphysrevd.95.126002, doi:10.1103/physrevd.95.126002.

[24] Leonard Susskind, Lá rus Thorlacius, and John Uglum. The stretched horizon and black hole complementarity. *Physical Review D*, 48(8):3743–3761, oct 1993. URL: https://doi.org/10.1103%2Fphysrevd.48.3743, doi:10.1103/physrevd.48.3743.

[25] Steven Weinberg. Quantum contributions to cosmological correlations. *Physical Review D*, 72(4), aug 2005. URL: https://doi.org/10.1103%2Fphysrevd.72.043514, doi:10.1103/physrevd.72.043514.

[26] Edward Witten. Anti de sitter space and holography. 1998. URL: https://arxiv.org/abs/hep-th/9802150, doi:10.48550/ARXIV.HEP-TH/9802150.