<u>**Electromagnetism Project**</u>

Study of Cavities in Electromagnetism

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I. Introduction

Electromagnetic cavities are of major interest in physics, as they allow us to study the effect of trapping electromagnetic fields within finite spaces. The goal of this project is to study the interaction between a polarizable dielectric particle inside a Fabry-Perot cavity and a Gaussian laser beam, such that electromagnetic radiation gets scattered off the particle and fills the cavity. As this is advanced question to ask, I start with the basics of wave propagation, cavity electrodynamics and wave-particle interaction to build up the required Hamiltonian. The future goal of this work is to analyze the Hamiltonian derived to study cavity cooling of the trapped particle.

In this report, I will first derive plane wave solutions to the Maxwell's equations to show the propagation of electromagnetic radiation thought matter. Then using the plane wave solution, I will study the propagation of light through different matter-interfaces and derive the coefficients of reflection and transmission at such boundaries. Once I have said coefficients, I will use them to analyse a Fabry-Perot cavity and get an expression for light transmitted through the same. After the cavity analysis, I will look at an experimental setup by Delic et al[1] with leviated quantum dot inside a Fabry-Perot cavity and derive the components of the Hamiltonian of the system.

II. Plane Wave Propagation

A very important feature of Maxwell's equations of Electromagnetism is that they allow travelling wave solutions. This is of significance because it represents the transport of energy from one point in space to another. Without the propagation of waves, we cannot have any of the other features we want to study.
We will first start with Maxwell's equations in matter.

$$\vec{\nabla}.\vec{D} = \rho_f \tag{1}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{2}$$

$$\vec{\nabla}.\vec{B} = 0 \tag{3}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \tag{4}$$

where we have,

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \tag{5}$$

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} \tag{6}$$

We can define the terms as :

$$\begin{aligned}
\vec{D} &\Rightarrow \text{Electric Displacement} \\
\rho_f &\Rightarrow \text{Free charges} \\
\vec{H} &\Rightarrow \text{Auxiliary Field} \\
\vec{J_f} &\Rightarrow \text{Free current} \\
\vec{M} &\Rightarrow \text{Magnetisation}
\end{aligned}$$

If there are no sources and we have an infinite medium, we can set :

$$\rho_f = 0 \quad ; \qquad \vec{J_f} = 0 \tag{7}$$

Then the Maxwell's equations become :

$$\vec{\nabla}.\vec{B} = 0 \tag{8}$$

$$\vec{\nabla}.\vec{D} = 0 \tag{9}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{10}$$

$$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = 0 \tag{11}$$

We will then assume that the solutions have harmonic time dependence : $e^{-i\omega t}$. The electric and magnetic fields can then be written as :

$$\begin{aligned}
\vec{E}(x,t) &= \vec{E}(\omega, x)e^{-i\omega t} \\
\vec{B}(x,t) &= \vec{B}(\omega, x)e^{-i\omega t} \\
\vec{D}(x,t) &= \vec{D}(\omega, x)e^{-i\omega t} \\
\vec{H}(x,t) &= \vec{H}(\omega, x)e^{-i\omega t}
\end{aligned}$$

Plugging these equations into the Maxwell's equations for source-free infinite medium, we get the following equations :

$$\vec{\nabla}.\vec{B} = 0 \tag{12}$$

$$\vec{\nabla}.\vec{D} = 0 \tag{13}$$

$$\vec{\nabla} \times \vec{E} - i\omega\vec{B} = 0 \tag{14}$$

$$\vec{\nabla} \times \vec{H} + i\omega\vec{D} = 0 \tag{15}$$

Now, we will assume that we have a uniform isotropic medium. Thus, polarisation and magnetisation in the material is proportional to the magnetic field.

$$\vec{P} = \epsilon_0 \chi_0 \vec{E}\,; \quad \Rightarrow \quad \epsilon = \epsilon_0(1+\chi_0)\,; \quad \Rightarrow \quad \vec{D} = \epsilon \vec{E} \tag{16}$$

$$\vec{M} = \chi_m \vec{H}\,; \quad \Rightarrow \quad \mu = \mu_0(1+\chi_m)\,; \quad \Rightarrow \quad \vec{B} = \epsilon \vec{H} \tag{17}$$

Plugging these forms of $\vec{D}$ and $\vec{H}$ into the Maxwell's equations, we can get all four equations in terms of $\vec{B}$ and $\vec{E}$. Focusing on equation(11), we get :

$$\vec{\nabla} \times \vec{B} + i\omega\mu\epsilon\vec{E} = 0 \tag{18}$$

Now, from equation(14) writing $\vec{B}$ in terms of $\vec{E}$ and plugging this in equation(18).

$$\vec{\nabla} \times \left(\frac{\vec{\nabla} \times \vec{E}}{i\omega} \right) + i\omega\mu\epsilon\vec{E} = 0$$

$$\Longrightarrow \frac{\vec{\nabla}(\vec{\nabla}.\vec{E}) - \vec{\nabla}^2\vec{E}}{i\omega} + i\omega\mu\epsilon\vec{E} = 0$$

We finally arrive at equation :

$$\boxed{\therefore\ \vec{\nabla}^2\vec{E} + \omega^2\mu\epsilon\vec{E} = 0} \tag{19}$$

This is known as the Helmholtz wave equation.

We can repeat the same procedure, but by writing $\vec{E}$ in term of $\vec{B}$. Repeating the same procedure, we arrive at the Helmholtz equation for $\vec{B}$:

$$\boxed{\therefore\ \vec{\nabla}^2\vec{B} + \omega^2\mu\epsilon\vec{B} = 0} \tag{20}$$

Let us assume plane electromagnetic wave solutions of frequency ω and wave vector $\vec{k} = k\vec{n}$.

$$\vec{E}(\vec{x},t) = \vec{E}_0 \exp(ik\vec{n}.\vec{x} - i\omega t) \tag{21}$$

$$\vec{B}(\vec{x},t) = \vec{B}_0 \exp(ik\vec{n}.\vec{x} - i\omega t) \tag{22}$$

Focusing on $\vec{E}$ in the Helmholtz wave equation(19) :

$$\begin{aligned}
&\vec{\nabla}^2\vec{E} + \omega^2\mu\epsilon\vec{E} = 0 \\
\Longrightarrow\ &\vec{\nabla}.\left[E_0(ik)\left\{n_x\hat{x} + n_y\hat{y} + n_z\hat{z}\right\}e^{ik\vec{n}.\vec{x}-i\omega t}\right] + \omega^2\mu\epsilon\vec{E} = 0 \\
\Longrightarrow\ &E_0(ik)\left[\frac{\partial}{\partial x}\left(n_x e^{ik\vec{n}.\vec{x}-i\omega t}\right) + \frac{\partial}{\partial y}\left(n_y e^{ik\vec{n}.\vec{x}-i\omega t}\right) + \frac{\partial}{\partial z}\left(n_z e^{ik\vec{n}.\vec{x}-i\omega t}\right)\right] + \omega^2\mu\epsilon\vec{E} = 0 \\
\Longrightarrow\ &E_0(ik)^2(n_x^2 + n_y^2 + n_z^2)e^{ik\vec{n}.\vec{x}-i\omega t} + \omega^2\mu\epsilon\vec{E} = 0 \\
\Longrightarrow\ &-k^2\vec{n}.\vec{n}\vec{E} + \omega^2\mu\epsilon\vec{E} = 0
\end{aligned}$$

Thus we finally obtain the equation :

$$k^2\vec{n}.\vec{n} = \omega^2\mu\epsilon \tag{23}$$

As the phase velocity of electromagnetic wave is given as :

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}; \quad \implies \quad k^2 = \omega^2\mu\epsilon \tag{24}$$

This tells us that for equation(23) to be satisfied by our plane wave solutions, we must have $\vec{n}.\vec{n} = 1$, i.e., $\vec{n}$ must be a unit vector.

Now, let us go back to the Maxwell's equations (8) and (9), focusing on them one at a time, starting with equation(9).

$$\begin{aligned}
&\vec{\nabla}.\vec{E} = 0 \\
\implies &\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \\
\implies &\left[(ik)n_xE_x + (ik)n_yE_y + (ik)n_zE_z\right]e^{ik\vec{n}.\vec{x}-i\omega t} = 0
\end{aligned}$$

This equation simplifies down to :

$$\vec{n}.\vec{E} = 0 \tag{25}$$

This tells us that $\vec{E}$ is perpendicular to the direction of propagation of the wave given by $\vec{n}$. We can repeat the similar analysis starting from equation(8), and we will obtain :

$$\vec{n}.\vec{B} = 0 \tag{26}$$

Thus, $\vec{B}$ is also perpendicular to the direction of propagation of the wave, $\vec{n}$.
As both $\vec{E}$ and $\vec{B}$ are both perpendicular to direction of propagation, so electromagnetic field is a plane transverse wave.
Finally, we can look at equation(14) to write $\vec{B}$ in terms of $\vec{E}$.

$$\vec{B} = \sqrt{\mu\epsilon}\hat{n} \times \vec{E} \tag{27}$$

So we see that magnetic field is perpendicular to the direction of propagation as well to the electric field. It is also much smaller in magnitude compared to the electric field. The term $\sqrt{\mu\epsilon}$ is inverse of the speed of light in that medium, which is smaller than the speed of light in vacuum.

III. Electromagnetic Waves at Interface between Media

When light (electromagnetic wave) approaches an interface between two different media, it undergoes reflection and refraction at the surface. In this section, we will explore that phenomenon and derive the coefficient of reflection and refraction at said surface.

Let us consider two media, with permeability and permittivity given as μ_1, ϵ_1 and μ_2, ϵ_2. They have refractive indices given as :

$$n_1 = \sqrt{\frac{\mu_1 \epsilon_1}{\mu_0 \epsilon_0}} \tag{28}$$

$$n_2 = \sqrt{\frac{\mu_2 \epsilon_2}{\mu_0 \epsilon_0}} \tag{29}$$

The interface lies at x=0. A wave vector with propagation direction given as $\vec{k}$ is incident on the interface, going towards medium 2 from medium 1.

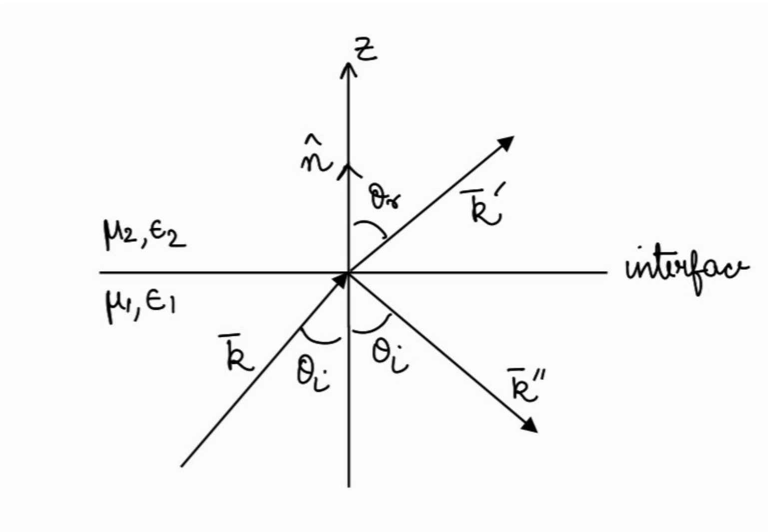

Fig 1. Wave vector incident on interface of two media

We can now write the equations for the incident, reflected and refracted rays.

Incident Ray :

$$\vec{E}(\vec{x}, t) = \vec{E}_0 e^{i\vec{k}.\vec{x} - i\omega t} \tag{30}$$

$$\vec{B}(\vec{x}, t) = \sqrt{\mu_1 \epsilon_1} \frac{\vec{k} \times \vec{E}_1}{k} \tag{31}$$

Refracted Ray :

$$\vec{E'}(\vec{x},t) = \vec{E'}_0 e^{i\vec{k'}.\vec{x}-i\omega t} \tag{32}$$

$$\vec{B'}(\vec{x},t) = \sqrt{\mu_2\epsilon_2}\frac{\vec{k'}\times\vec{E'}}{k'} \tag{33}$$

Reflected Ray :

$$\vec{E''}(\vec{x},t) = \vec{E''}_0 e^{i\vec{k''}.\vec{x}-i\omega t} \tag{34}$$

$$\vec{B''}(\vec{x},t) = \sqrt{\mu_1\epsilon_1}\frac{\vec{k''}\times\vec{E''}}{k} \tag{35}$$

$\vec{k}$, $\vec{k'}$ and $\vec{k''}$ are wave vectors, whose magnitude is given as :

$$|\vec{k}| = |\vec{k''}| = k = \omega\sqrt{\mu_1\epsilon_1} \tag{36}$$

$$|\vec{k'}| = k' = \omega\sqrt{\mu_2\epsilon_2} \tag{37}$$

We now take into account the boundary conditions at the interface at $x = 0$. The normal components of $\vec{B}$ and $\vec{D}$ are continuous across the boundary along with parallel components of $\vec{E}$ and $\vec{H}$. Let $\vec{n}$ be a unit vector perpendicular to the interface (not propagation vector). We can then write the boundary conditions mathematically as :

$$\vec{D}^{\perp}_{bottom} - \vec{D}^{\perp}_{top} = 0 \implies \left[\epsilon_1(\vec{E}_0 + \vec{E''}_0) - \epsilon_2\vec{E'}_0\right].\hat{n} = 0 \tag{38}$$

$$\vec{B}^{\perp}_{bottom} - \vec{B}^{\perp}_{top} = 0 \implies \left[\vec{k}\times\vec{E}_0 + \vec{k''}\times\vec{E''}_0 - \vec{k'}\times\vec{E'}_0\right].\hat{n} = 0 \tag{39}$$

$$\vec{E}^{\parallel}_{bottom} - \vec{E}^{\parallel}_{top} = 0 \implies \left[\vec{E}_0 + \vec{E''}_0 - \vec{E'}_0\right]\times\hat{n} = 0 \tag{40}$$

$$\vec{H}^{\parallel}_{bottom} - \vec{H}^{\parallel}_{top} = 0 \implies \left[\frac{1}{\mu_1}(\vec{k}\times\vec{E}_0 + \vec{k''}\times\vec{E''}_0) - \frac{1}{\mu_2}(\vec{k'}\times\vec{E'})\right]\times\hat{n} = 0 \tag{41}$$

These equations cannot be solved further without knowing the polarisation of the incoming electromagnetic wave. We can consider two possible cases :

i.) polarisation vector is perpendicular to the plane of incidence, i.e., electric field goes into the plane (or out of the plane) while the magnetic field lies on the plane.

ii.) polarisation vector is parallel to the plane of incident, i.e, electric field lies on the plane while magnetic field goes out of the plane (or into the plane).

All other cases of circular or elliptical polarisation can be solved as combination of the above two cases. Let us solve for both of these scenarios case by case.

3.1 Electric Field Perpendicular to Plane of Incidence

When the electric field is perpendicular to the plane of incidence, it has to be parallel to the surface of the interface.

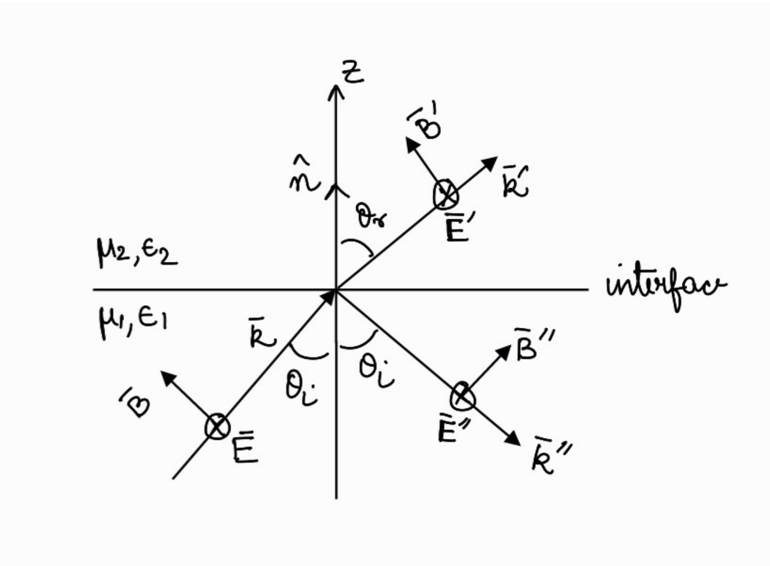

Fig 2. Electric Field perpendicular to plane of incidence

We can then solve for the boundary conditions derived above. As $\vec{E}$ is parallel to the surface, hence perpendicular to $\hat{n}$, the first boundary condition in equation(38) gives zero on the LHS. From equation(40) and equation(41), we have :

$$E_0 + E_0'' - E_0' = 0 \tag{42}$$

$$\sqrt{\frac{\epsilon_1}{\mu_1}}(E_0 - E_0'')\cos\theta_i - \sqrt{\frac{\epsilon_2}{\mu_2}}E_0'\cos\theta_r = 0 \tag{43}$$

Now we know from the Snell's law that :

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{n_2}{n_1} \tag{44}$$

Using the Snell's law in equation(44) the equations(42,43) for incident, reflected and refracted rays can be solved as follows :

$$\frac{E_0'}{E_0} = \frac{2n_1\cos\theta_i}{n\cos\theta_i + \frac{\mu_1}{\mu_2}\sqrt{n_2^2 - n_1^2\sin\theta_i{}^2}} \tag{45}$$

$$\frac{E_0''}{E_0} = \frac{n\cos\theta_i - \frac{\mu_1}{\mu_2}\sqrt{n_2^2 - n_1^2\sin\theta_i{}^2}}{n\cos\theta_i + \frac{\mu_1}{\mu_2}\sqrt{n_2^2 - n_1^2\sin\theta_i{}^2}} \tag{46}$$

So, equation(45) gives us the coefficient of transmission (or refraction) and equation(46) gives coefficient of reflection.

3.2 Electric Field Parallel to Plane of Incidence

When electric field is parallel to the plane of incidence, it has to be perpendicular to the surface of the interface.

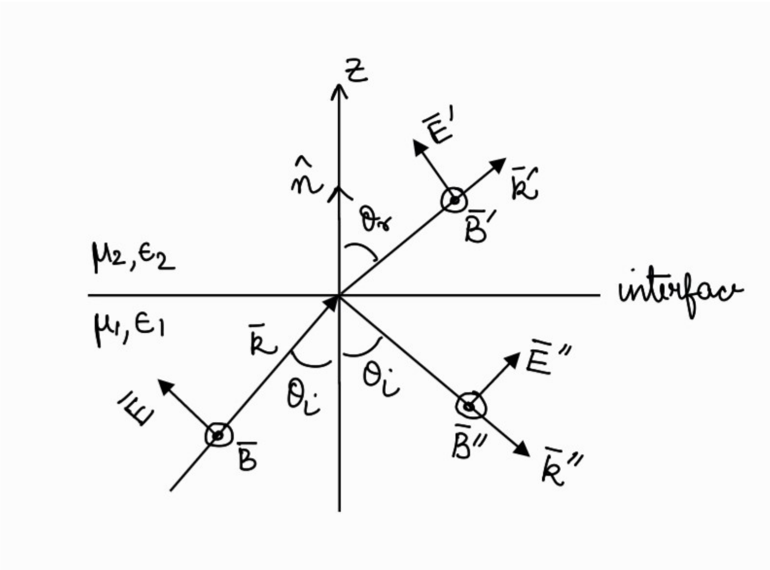

Fig 3. Electric Field parallel to plane of incidence

We can focus on equations(40, 41) to obtain equations for the electric field amplitude.

$$\cos\theta_i(E_0 - E_0'') - \cos\theta_r E_0' = 0 \tag{47}$$

$$\sqrt{\frac{\epsilon_1}{\mu_1}}(E_0 + E_0'') - \sqrt{\frac{\epsilon_2}{\mu_2}}E_0' = 0 \tag{48}$$

Now using Snell's law from equation(44), we can simplify the above equations as :

$$\frac{E_0'}{E_0} = \frac{2n_1 n_2 \cos\theta_i}{\frac{\mu_1}{\mu_2}n_2^2\cos\theta_i + n_1\sqrt{n_2^2 - n_1^2\sin\theta_i}^2} \tag{49}$$

$$\frac{E_0''}{E_0} = \frac{\frac{\mu_1}{\mu_2}n_2^2\cos\theta_i - n_1\sqrt{n_2^2 - n_1^2\sin\theta_i}^2}{\frac{\mu_1}{\mu_2}n_2^2\cos\theta_i + n_1\sqrt{n_2^2 - n_1^2\sin\theta_i}^2} \tag{50}$$

Thus, for parallel incidence equation(49) gives us the coefficient of refraction (or transmission) and equation(50) gives us the coefficient of reflection.

3.3 Reflection and Transmission for Normal Incidence

While all the equations derived are valid for any angle of incidence, we are mainly interested in the case of normal incidence, i.e., $\theta_i = \theta_r = 90°$. When we look at experimental setup used by Delic et al[1], they have normal incidence of light in their cavity, thus we wish to look at that specific case.

Implementing $\theta_i = \theta_r = 90°$ in equations (45, 46, 49, 50) we observe that the four of them boil down to only two equations :

$$\boxed{\begin{aligned} \frac{E_0'}{E_0} &= \frac{2n_1}{n_1+n_2} \implies \text{coefficient of transmission} \\ \frac{E_0''}{E_0} &= \frac{n_2-n_1}{n_1+n_2} \implies \text{coefficient of reflection} \end{aligned}} \quad \begin{aligned} (51) \\ (52) \end{aligned}$$

Thus, for normal incidence of light, it does not matter if the electromagnetic wave is polarised parallel or perpendicular to the plane of incidence.
Now having derived the coefficients of reflection and transmission, we can start looking at specific cavities and study their properties.

IV. Fabry-Perot Cavity

Fabry-Perot cavity is an open optical resonator consisting of two parallel mirrors, with either same or different reflectivities. The medium inside the cavity can be either empty ($n = 1$) or be filled some dielectric ($n \neq 1$).
I will be exploring the two distinct cases here, both for empty cavities :

i.) both mirrors have the same coefficients of reflection and transmission

ii.) both mirrors have different coefficients of reflection and transmission

4.1 Identical Mirrors

We will first consider the case when both mirrors are identical.

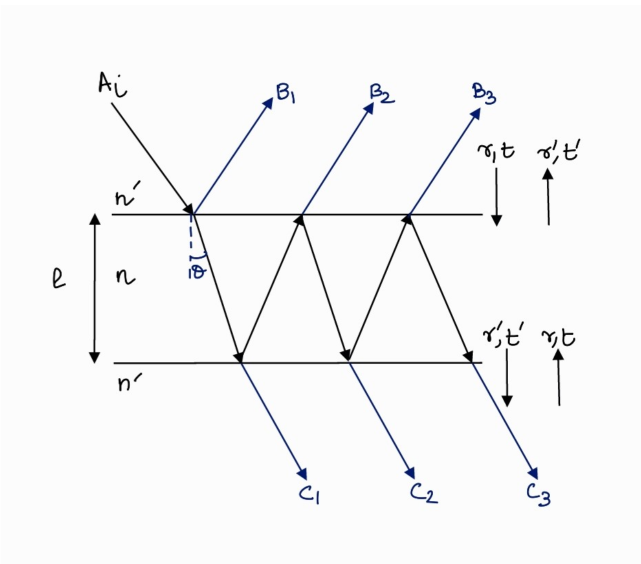

Fig 4. Identical Mirrors Fabry-Perot Elaton

Let r and t be coefficients of reflection and transmission respectively. The length of the cavity is given as l. After two successive reflections, the path difference is given as :

$$\begin{aligned} \delta L &= AB + BC = BC.\cos 2\theta + BC \\ \implies \delta L &= l\frac{\cos 2\theta}{\cos\theta} + \frac{l}{\cos\theta} \\ \implies \delta L &= \frac{l}{\cos\theta}(\cos 2\theta + 1) = \frac{l}{\cos\theta} 2\cos^2\theta \\ \therefore \delta L &= 2l\cos\theta \end{aligned}$$

Once we have the path difference, we can write the phase difference as :

$$\boxed{\delta = \frac{2\pi(\delta L)n}{\lambda} = \frac{4n\pi l\cos\theta}{\lambda}} \tag{53}$$

λ is the wavelength of the incident ray and θ is the angle of refraction.
The coefficient of reflection is directional, while coefficient of transmission is not. They can be related as :

$$r' = -r\,; \quad t' = t\,; \quad r^2 + tt' = 1 \tag{54}$$

We can now look at the amplitudes of the reflected and transmitted waves. A_i is the amplitude of the incident ray.
The amplitude of partial reflections can be written as :

$$B_1 = rA_i\,; \quad B_2 = tt'r'A_ie^{i\delta}\,; \quad B_3 = tt'(r')^3A_ie^{2i\delta} \quad \text{and so on}\ldots \tag{55}$$

If we add all the partial reflections we can get the total reflected amplitude.

$$\begin{aligned} A_r &= B_1 + B_2 + B_3 + \cdots \\ \implies A_r &= A_i\left\{r + tt'r'e^{i\delta}(1 + r'^2e^{i\delta} + r'^4e^{2i\delta} + \cdots)\right\} \end{aligned}$$

This forms a geometric series, which can sum up to obtain the final expression.

$$\boxed{A_r = \left\{\frac{(1-e^{i\delta})\,r}{1-r^2e^{i\delta}}\right\}A_i} \tag{56}$$

Now focusing on transmitted waves, we can follow a similar pattern to find the partial transmissions.

$$C_1 = tt'A_i\,; \quad C_2 = tt'(r')^2e^{i\delta}A_i\,; \quad C_3 = tt'(r')^4e^{2i\delta}A_i \quad \text{and so on}\ldots \tag{57}$$

Adding up all the partial transmissions, we can find the total transmitted amplitude.

$$A_t = C_1 + C_2 + C_3 + \cdots$$

$$\implies A_t = A_i tt'(1 + r'^2 e^{i\delta} + r'^4 e^{2i\delta} + \cdots)$$

This is a geometric series as well, which can be summed up as follows :

$$\boxed{A_t = \left\{ \frac{tt'}{1 - r^2 e^{i\delta}} \right\} A_i} \tag{58}$$

The transmitted and reflected intensities are given as :

$$I_t = A_t A_t^* \; ; \quad I_r = A_r A_r^* \tag{59}$$

We can use MATHEMATICA to plot the ratio of transmitted intensity to incident intensity as a function of the length of the cavity, once we choose constants appropriately. I have chosen $\theta = 0\circ$, $\lambda = 1$ and $n = 1$ (vacuum inside cavity). Then for different values of reflectivity of mirror chosen, the ratio of transmitted light gets modified.

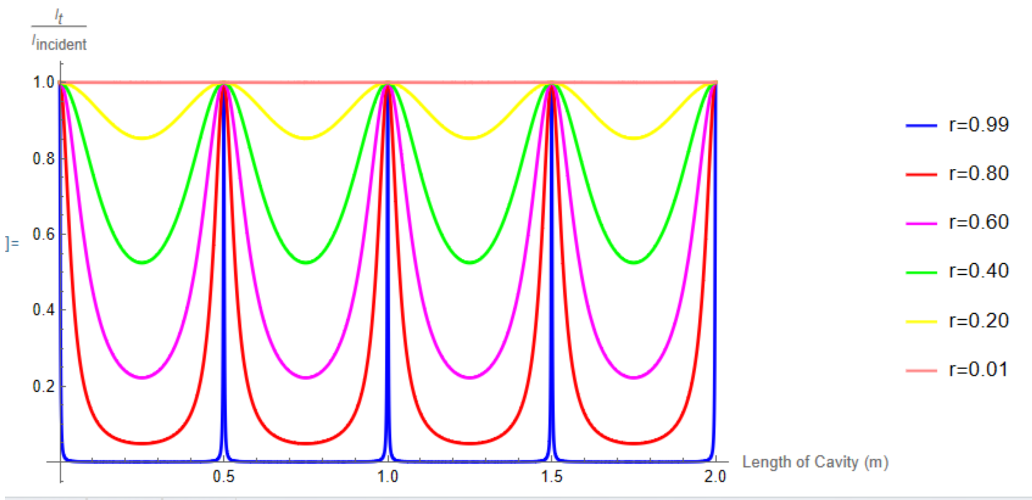

Fig 5. Ratio of transmitted light for identical mirrors

4.2 Non-Identical Mirrors

We will now consider the case where we have non-identical mirrors.

Let r_1, t_1 and r_2, t_2 be coefficients of reflection and transmission of the mirrors. The length of the cavity is still given by l and the value of phase difference after two successive reflections remains the same as in equation(53).
(The image is included in the next page.)

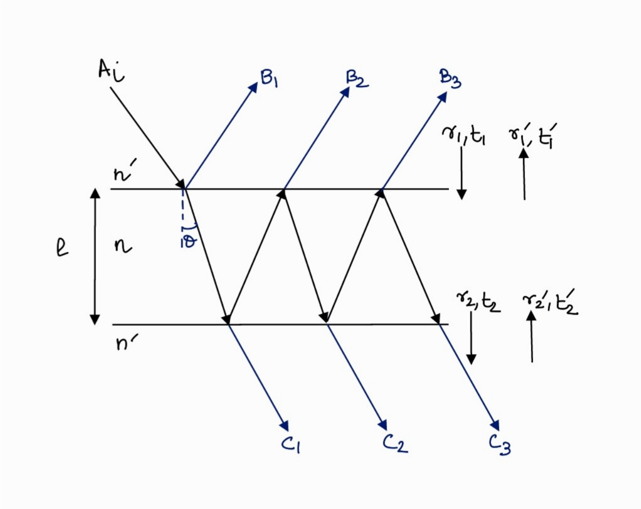

Fig 6. Non-identical Mirrors Fabry-Perot Elaton

The incident wavelength is λ and amplitude is A_i. The coefficient of reflections and transmissions are related as :

$$r_1' = -r_1\,; \quad r_2' = -r_2\,; \quad t_1' = t - 1\,; \quad t_2' = t_2\,; \quad r_1^2 + t_1 t_1' = 1\,; \quad r_2^2 + t_2 t_2' = 1 \tag{60}$$

Starting with looking at the partially reflected waves.

$$B_1 = r_1 A_i\,; \quad B_2 = (t_1 A_i) r_2 t_1' e^{i\delta}\,; \quad B_3 = (t_1 r_2 A_i) r_1' r_2 t_1' e^{2i\delta}\,; \quad B_4 = (t_1 r_2 A_i) r_1' r_2 r_1' r_2 t_1' e^{3i\delta} \quad \text{and so on} \ldots \tag{61}$$

The total reflected beam is then given as :

$$A_r = B_1 + B_2 + B_3 + B_4 + \cdots$$
$$\implies A_r = A_i \left\{ r_1 + t_1 t_1' r_2' e^{i\delta} (1 + r_2 r_1 e^{i\delta} + r_2^2 r_1'^2 e^{2i\delta} + \cdots) \right\}$$

This is a geometric series and can be summed up neatly.

$$\boxed{A_r = \left\{ r_1 + \frac{t_1 t_1' r_2 e^{i\delta}}{1 + r_2 r_1 e^{i\delta}} \right\} A_i} \tag{62}$$

Now solving for the partially transmitted waves :

$$C_1 = (t_1 A_i) t_2\,; \quad C_2 = (t_1 A_i r_2) r_1' t_2 e^{i\delta}\,; \quad C_3 = (t_1 r_2 A_i) r_1' r_2 r_1' t_2 e^{2i\delta}\,; \quad C_4 = (t_1 r_2 A_i) r_1' r_2 r_1' r_2 r_1' t_2 e^{3i\delta} \quad \ldots \tag{63}$$

Adding all the partially transmitted waves gives us the total transmitted wave.

$$A_t = C_1 + C_2 + C_3 + C_4 + \cdots$$
$$\implies A_t = A_i t_1 t_2 \left\{ 1 + r_2 r_1' e^{i\delta} + (r_2 r_1')^2 e^{2i\delta} + \cdots \right\}$$

This is also a geometric series which can be added up to give the total transmitted amplitude.

$$\boxed{A_t = \left\{ \frac{t_1 t_2}{1 + r_1 r_2 e^{i\delta}} \right\} A_i} \tag{64}$$

We can plot the ratio of transmitted intensity ($I_t = A_t A_t^*$) to the ratio of incident intensity as a function of length of cavity for different chosen values of reflectivity of the mirrors. We choose $\theta = 0°$ and $\lambda = 1$ for plotting purposes.

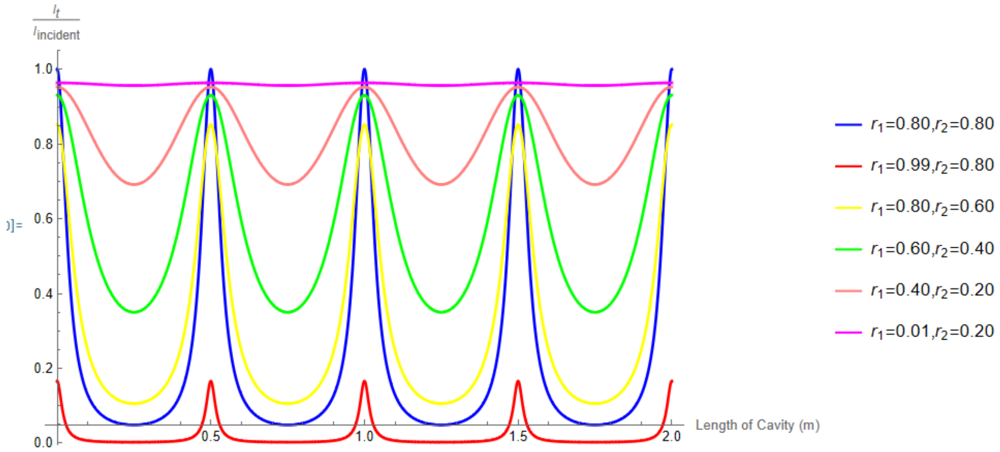

Fig 7. Ratio of transmitted light for non-identical mirrors

4.3 Conclusion

We see that for a 1-dimensional Fabry-Perot cavity, the light transmitted through the cavity is dependent on the length of the cavity, once we make a choice on the wavelength of light. The converse of this will be if we fix the length of the cavity, only intensities of certain wavelengths (or frequencies) will be successfully transmitted from the cavity. This implies that the cavity only supports certain modes, i.e., not light of all wavelengths can fill up the cavity. The length of the cavity has to be an integer or half-integer multiple of the wavelength of light filling the cavity.

As next steps, I am exploring the effect of inserting a thin glass slab inside the cavity and studying the effects on the transmitted light. Solving this system will the ray diagram approach has proven to be extremely challenging and I have starting looking at ways to solve the problem using Reflection and Transmission matrices approach.

V. Optically Levitated Quantum Dot in a Cavity

In this section I will look at the experimental setup used by Delic et al [1] to cool a levitated quantum dot in a cavity. They have used a Gaussian laser beam, focused using a microscope objective to levitate a quantum dot inside a cavity. The light scattering off the quantum dot fills up the cavity. In this process, the quantum dot loses momentum and steadily cools down. I will look at the setup component by component and try to derive the equations for them.

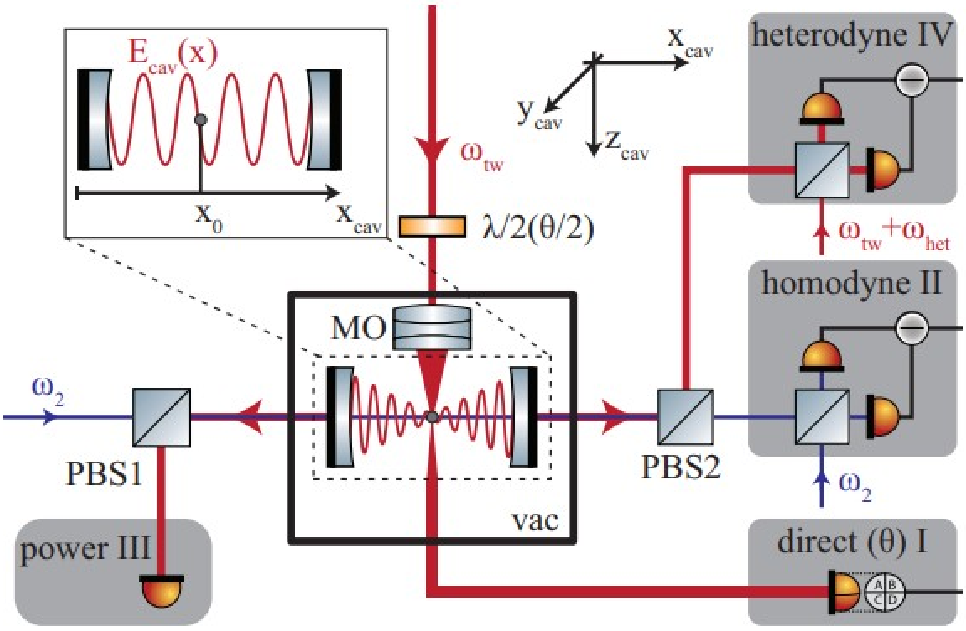

Fig 8. Experimental Set-up for Cooling of Quantum Dot[1]

5.1 Paraxial Helmholtz Equation

Probably the most important component in the experimental set-up above is the laser light which traps the nanoparticle. The dynamics of that laser beam is controlled by the wave equation given as :

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\vec{r},t) = 0 \tag{65}$$

This is the Helmholtz equation(19) derived in the first section. We have assumed zero charge density in space for vacuum (source free space).

To solve this equation(65), we will start with the separation of variables approach.

$$\phi(\vec{r},t) = \psi(\vec{r})T(t) \tag{66}$$

The Helmholtz equation is an eigenvalue problem of the Laplace operator, which allows us to write :

$$\nabla^2\phi(\vec{r},t) + k^2\phi(\vec{r},t) = 0 \tag{67}$$

We can now only focus on the r components as we use the separation of variables.

$$\nabla^2\psi(\vec{r}) + k^2\psi(\vec{r}) = 0 \tag{68}$$

We will then use the paraxial approximation, i.e., we will assume that the angle between the rays is so small that they are almost parallel to each other.

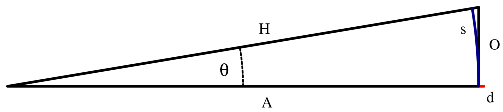

Fig 9. Paraxial Approximation

So, we have plane waves propagating along the z-axis. We can approximate it as :

$$\psi(\vec{r}) = u(\vec{r})e^{-ikz} \tag{69}$$

Physically this means that the wave is spreading out very slowly in all directions compared to its wavelength and it is a function of z. This also means that the amplitude of the wave is varying slowly. We can characterise this as :

$$\begin{aligned}
\delta u = \frac{\partial u}{\partial z}\delta z \ll u\,;dz \sim \lambda \\
\implies \frac{\partial u}{\partial z} \ll \frac{u}{\lambda} \sim ku \\
\implies \frac{\partial^2 u}{\partial z^2} \ll k\frac{\partial u}{\partial z} \ll k^2 u
\end{aligned}$$

This gives us the Slowly Varying Envelope Approximation (SVEA) :

$$\therefore \left|\frac{\partial^2 u}{\partial z^2}\right| \ll \left|2ik\frac{\partial u}{\partial z}\right| \tag{70}$$

We can now go back to equation(68) and look at the Laplace term in it.

$$\begin{aligned}
\nabla^2 = \nabla_\perp^2 + \partial_z^2 \\
\therefore \nabla^2\psi = \nabla_\perp^2\psi + \partial_z^2\psi = -k^2\psi
\end{aligned} \tag{71}$$

First solving the z-equation :

$$\begin{aligned}
\partial_z^2\psi = \partial_z^2\left[u(\vec{r})e^{-ikz}\right] \\
\implies \partial_z^2\psi = \frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z}e^{-ikz} - iku(r)e^{-ikz}\right]
\end{aligned}$$

$$\implies \partial_z^2\psi = \left[\frac{\partial^2 u}{\partial z^2} - 2ik\frac{\partial u}{\partial z} - iku(r).(-ik)\right] e^{-ikz}$$

We finally obtain the equation as :

$$\therefore \partial_z^2\psi = \left\{\frac{\partial^2 u}{\partial z^2} - 2ik\frac{\partial u}{\partial z} - k^2 u\right\} e^{-ikz} \tag{72}$$

Now plugging equation(72) in equation(71) :

$$\nabla_\perp^2 + \left\{\frac{\partial^2 u}{\partial z^2} - 2ik\frac{\partial u}{\partial z} - k^2 u\right\} e^{-ikz} + k^2\psi = 0$$
$$\implies \nabla_\perp^2\left(ue^{-ikz}\right) + \left\{\frac{\partial^2 u}{\partial z^2} - 2ik\frac{\partial u}{\partial z} - k^2 u\right\} e^{-ikz} + k^2\left(ue^{-ikz}\right) = 0$$

Cancelling off appropriate terms and using the SVEA(70), we finally arrive at the equation :

$$\boxed{\nabla_\perp^2 u - 2ik\frac{\partial u}{\partial z} = 0} \tag{73}$$

This is known as the Paraxial Helmholtz equation.

5.2 Gaussian Wave Profile

Now we we will consider a Gaussian wave profile, which is given as :

$$u(x,y,z) = \frac{1}{q(z)}\exp\left[-ik\frac{x^2+y^2}{2}.\frac{1}{q(z)}\right] \tag{74}$$

q is known as the complex beam parameter. It is given by the beam waist,$W(z)$ and radius of curvature, R of the beam.

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi W^2(z)} \tag{75}$$

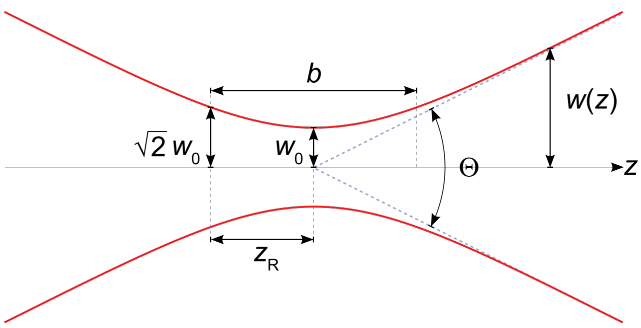

Fig 10. Gaussian Beam Profile

$W(z)$ is the beam waist which depending on z tells us that we are looking at a focused beam if $W(z)$ decreases then increases with z. We can plug the expression for $1/q(z)$ (74) in the expression for $u(x, y, z)$ (73) and use $r^2 = x^2 + y^2$ as well as $\lambda k = 2\pi$ to obtain the final expression for $u(r, z)$:

$$u(r,z) = \frac{1}{q(z)} \exp\left[-\frac{ikr^2}{2R(z)} - \frac{r^2}{W^2(z)}\right] \tag{76}$$

The beam waist and radius of curvature can be written as a function of z.

$$W(z) = W_0\sqrt{1 + \left(\frac{z}{z_R}\right)^2} = \frac{\lambda z_R}{\pi}\sqrt{1 + \left(\frac{z}{z_R}\right)^2} \tag{77}$$

$$R(z) = z\left[1 + \left(\frac{z_R}{z}\right)^2\right] \tag{78}$$

W_0 is the waist length at the centre of the beam (where it is focused) and z_R is the Rayleigh length, which is the length it takes the beam waist to expand by $\sqrt{2}$ times its value at the centre.

The Gaussian wave profile is a valid solution to the Paraxial Helmholtz solution. We can plug equation(76) into equation(69) and obtain the expression for the complete wave solution.

$$\psi(\vec{r}) = \frac{u_0}{q(z)} \exp\left[-\frac{ikr^2}{2R(z)} - \frac{r^2}{W^2(z)}\right] e^{-ikz}$$

$$\implies \psi(\vec{r}) = u_0\left[\frac{1}{R(z)} - i\frac{\lambda}{\pi W^2(z)}\right] \exp\left[-\frac{ikr^2}{2R(z)} - \frac{r^2}{W^2(z)}\right] e^{-ikz}$$

Using the expressions for radius of curvature (78) and beam waist (77), we can simplify further :

$$\implies \psi(\vec{r}) = \frac{u_0}{iz_R}\frac{(iz + z_R)\, e^{-ikz}}{\sqrt{1 + \left(\frac{z}{z_R}\right)^2}\sqrt{z^2 + z_R^2}} \exp\left[-\frac{ikr^2}{2R(z)} - \frac{r^2}{W^2(z)}\right]$$

By redefining some parameters for ease, we can come to the final equation for the Gaussian beam profile as :

$$\psi(\vec{r}) = u_1\frac{W_0}{W(z)} \exp\left(-\frac{r^2}{W^2(z)}\right) \exp\left[-i\left(kz + \frac{kr^2}{2R(z)} - \theta(z)\right)\right] \tag{79}$$

$$\theta(z) = \tan^{-1}\left(\frac{z}{z_R}\right) \Rightarrow \text{Guoy Phase Shift} \tag{80}$$

$$u_1 = \frac{u_0}{iz_R} \tag{81}$$

Thus, we get a solution of the Helmholtz paraxial equation in the form of a Gaussian electromagnetic field.

5.3 Hamiltonian of the Quantum Dot in Cavity

In the paper by Delic et al[1], they have used a Gaussian beam as an optical tweezer to levitate the polarizable particle. The particle then coherently scatters off the light from the optical tweezer field into the cavity. This process of scattering the photons into the initially empty produces a cooling mechanism on the particle itself.

The Hamiltonian of the system can be written as :

$$H = -\frac{\alpha}{2}|\vec{E}_{tw}|^2 - \frac{\alpha}{2}|\vec{E}_{cav}|^2 - \alpha\mathcal{R}\left(\vec{E}_{tw}.\vec{E}^*_{cav}\right) \tag{82}$$

Here, $\vec{E}_{tw}$ is the electric field of the optical tweezer which is given by the Gaussian beam profile we derived in the previous section(79). $\vec{E}_{cav}$ is the electric field of the radiation which gets scattered into the cavity by the quantum dot. To appropriately write down this term, we need to start from the quantum field theory description of photon and write the equation for the photon field. The final term in the Hamiltonian tells us about the interaction of the original tweezer field and the electromagnetic field filling up the cavity. It will be this term which contributes to the cooling mechanism of the particle. α is the polarizability of the quantum dot. If $\alpha = 0$, we will not have any interaction between the fields and the particle.

VI. Conclusion and Future Direction

Study of electromagnetic cavities is of utmost importance in Physics, especially in the field of optics. Many novel physical phenomenon, like optical levitation, cavity cooling, lasers, single-photon emitters can be accomplished by the study of cavities.

As a future direction to this project, there are two main aspects which I aim to study :

i.) Study the effect of introducing a glass slab into a 1-dimensional Fabry-Perot cavity

ii.) Derive an expression for the electric field filling the cavity starting from the equation for the photon field in quantum field theory.

As a long-term goal, I wish to show that cavity cooling of the particle can be achieved in this particular cavity and then try to explore similar phenomenon in different kinds of cavities.

References

[1] Uro š Delić, Manuel Reisenbauer, David Grass, Nikolai Kiesel, Vladan Vuletić, and Markus Aspelmeyer. Cavity cooling of a levitated nanosphere by coherent scattering. *Phys. Rev. Lett.*, 122:123602, Mar 2019.