4.4

- Proof by cases
- Division theorem

Proof by case

$p \to r$
$q \to r$
$p \vee q$
$\therefore\ r$

Proposition

If $x \in \mathbb{R}$, then $x^2 \geq 0$

Proof:

suppose $x \in \mathbb{R}$

Either $x \geq 0$ or $x \leq 0$

Case 1: suppose $x \geq 0$

Then $x \cdot x \geq 0$

so $x^2 \geq 0$

Case 2: supp. $x \leq 0$

then: $x \cdot x \geq x \cdot 0$

$x^2 \geq 0$ $\square$

- If $a \leq b$ and $c \geq 0$ then $ac \leq bc$
- If $a \leq b$ and $c \leq$

• Definition

For any real number x, the **absolute value of x**, denoted $|x|$, is defined as follows:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

$|7| = 7$ $\quad 7 \geq 0$

$|-3| = -(-3) = 3$ $\quad -3 < 0$

$|0| = 0$ $\quad 0 \geq 0$

Prop: For any real number r, $|r| \geq 0$

Proof: suppose $r \in \mathbb{R}$, then either $r \geq 0$ or $r < 0$.

Case 1: If $r \geq 0$ then $|r| = r \geq 0$

Case 2: If $r < 0$ then $|r| = -r$

since $r < 0$

$-r \quad -r$

$0 < -r$

So $|r| = -r > 0$

so $|r| \geq 0$

Lemma 4.4.4

For all real numbers r, $-|r| \leq r \leq |r|$.

Proof:

Suppose r is any real number. We divide into cases according to whether $r \geq 0$ or $r < 0$.

Case 1 ($r \geq 0$): In this case, by definition of absolute value, $|r| = r$. Also, since r is positive and $-|r|$ is negative, $-|r| < r$. Thus it is true that

$$-|r| \leq r \leq |r|.$$

Proof: supp. $r \in \mathbb{R}$

then $r \geq 0$ or $r < 0$

case 1: If $r \geq 0$ then $|r| = r$ so $r \leq |r|$

$r = |r| \geq 0 \geq -|r|$

$-|r| \leq r \leq |r|$

case 2: If $r < 0$, then $|r| = -r$ so $-|r| = r$

$-|r| \leq r$

since $r < 0$ and $|r| \geq 0$, $-|r| \leq r \leq |r|$

Theorem 4.4.1 The Quotient-Remainder Theorem

Given any integer n and positive integer d, there exist unique integers q and r such that

$$n = dq + r \quad \text{and} \quad 0 \le r < d.$$

We call q the "quotient" and r remainder

$n = 57$ $d = 13$

$n = 13 \cdot \underset{q}{4} + \underset{0 \le r < 13}{5}$

$$\begin{array}{r} 4 \\ 13 \overline{) 57} \\ -52 \\ \hline 5 \end{array}$$

Theorem 4.4.1 The Quotient-Remainder Theorem

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There exists integers q and r such that

$n = dq + r$ $0 \le r < d$

and these numbers are unique. There is no other way to write in this form.

$\overset{n}{57} = \overset{d}{13} \cdot 4 + 5 \leftarrow 0 \le 5 < 13$

$57 = 13 \cdot 3 + 18$ $0 \le 5 < 13$

Given an integer n, and a positive integer d, we define:

$n \text{ div } d = q$

$n \text{ modulo } d = r$

where q and r are the unique integer satisfying.

$n = q = qd + r, \; 0 \le r < d$

$\frac{n}{57} = 13 \cdot 4 + 5$ $57 \text{ div } 13 = 4$

$57 \text{ mod } 13 = 5$

Example 4.4.4 Solving a Problem about *mod*

Suppose m is an integer. If m *mod* $11 = 6$, what is $4m$ *mod* 11?

Since $m \bmod 11 = 6$ then

$m = 11 \cdot q + 6$

where $q \in \mathbb{Z}$

$4m = 4(11q + 6)$

$= 44q + 24$

$= 11(4q) + 22 + 2$

$4m = 11(4q + 2) + 2$

$4m \bmod 11 = 2$

Theorem 4.4.3

The square of any odd integer has the form $8m + 1$ for some integer m.

Hint: Prove this by using the quotient remainder theorem with d = 4

Proof:

Suppose n is an odd integer.

By the quotient remainder theorem,

with d = 4, either $n = 4k$ or $n = 4k+1$ or $n = 4k+2$ or $n = 4k+3$ for some $k \in \mathbb{Z}$

Since n is odd, $n \neq 4k$ and $n \neq 4k+2 = 2(2k+1)$

So, either $n = 4k+1$ or $4k+3$

Case 1: If $n = 4k+1$ then $n^2 = (4k+1)$

$= 16k^2 + 8k + 1 = 8(2k^2 + k)$ where $2k^2 + k \in \mathbb{Z}$

Case 2: If $n = 4k+3$ then

$n^2 = (4k+3)^2 = 16k^2 + 24k + 9$

$= 16k^2 + 24k + 8 + 1 = 8(2k^2 + 3k + 1) + 1$

where $2k^2 + 3k + 1 \in \mathbb{Z}$