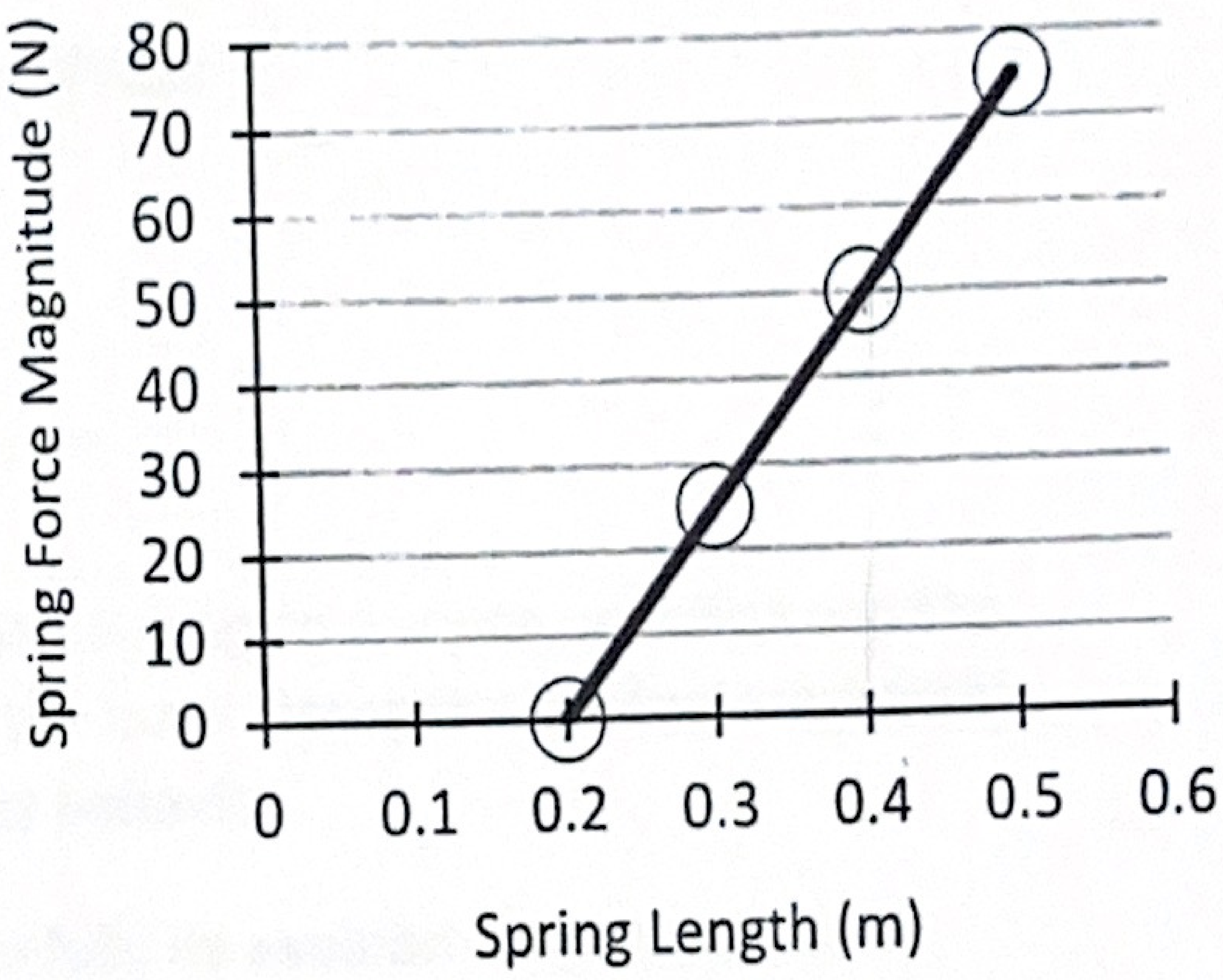
Ideal springs can stretch (and compress) forever, but real springs have elastic limits – deflection levels when they stop following Hooke's Law and can experience permanent damage or breakage. Springs are also usually treated as having no mass, though this assumption is not valid when the mass of the spring is comparable to the masses to which it is attached.



1. The plot to the right shows hypothetical data for a spring being stretched. What is the rest length of this spring?

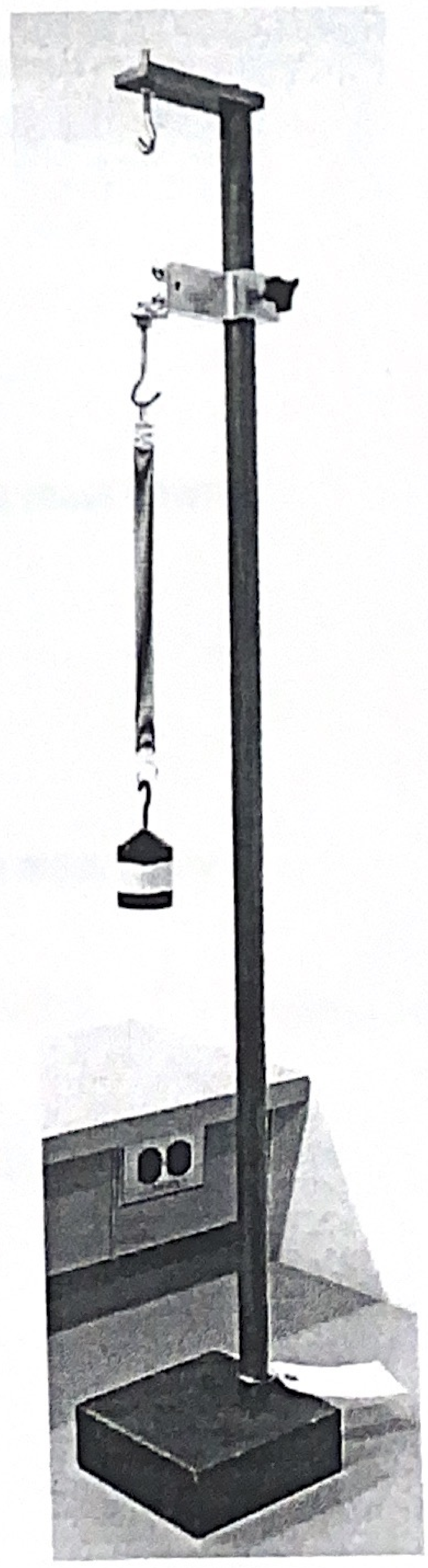
$\boxed{0.2 m}$

2. What is the stiffness of this hypothetical spring?

$$k = \frac{F}{x} = \frac{50-0\ [N]}{0.4-0.2\ [m]} = \boxed{250\ N/m}$$

Now that you've spent some time thinking and talking about how springs work, obtain the following items for your group:

- ❑ Laboratory bench with computer
- ❑ Stand with heavy base, vertical metal rod, and adjustable hooked cantilever
- ❑ Linear extension spring, labeled with its stiffness
- ❑ Hooked mass marked with masking tape (for high visibility during motion capture)



3. To keep today's lab short, we have given you the stiffness of your spring rather than having you measure it for yourself. What is the stiffness of your given spring? We call this *k*.

$k = 25.4\ N/m$

4. What is the mass of your given hooked mass? We call this *m*.

$m = 500g = 0.5kg$

To understand the simple harmonic oscillations of a mass hanging on a spring, you first need to think carefully about the force exerted by the spring on the mass. **Hang the hooked mass from your spring, stop it from oscillating, and contemplate its behavior.**

We call the vertical position where the mass hangs peacefully its **equilibrium position**, since nothing in the system is changing. This is a different position from the spring's rest length. Your analysis will involve two variables, defined as follows:

F_s	Vertical **force** exerted on the mass by the spring Zero is when the spring is at its rest length, and positive is **down**
p_y	Vertical **position** of the mass, measured from the mass's equilibrium position Zero is the position where the mass hangs still, and positive is **down**

5. What force does the spring exert on the mass when it is hanging stationary only under the influence of gravity? Use symbols rather than numbers, and make sure the sign is correct. In this class, we follow the convention that g=9.81 m/s^2 (a positive number).

$F_s = -mg = -0.5 \times 9.81 = \boxed{-4.905 N}$

6. Compared to the hanging case, what happens to the spring force if you pull the mass down by a small amount and hold it there? Answer in words.

The spring force will increase in magnitude

7. Compared to the hanging case, what happens to the spring force if you lift the mass up by a small amount and hold it there? Answer in words.

The spring force will decrease in magnitude. (assuming the spring's equilibrium point is not being crossed)

8. Write a symbolic expression for the spring force F_s in terms of the mass's position p_y, involving any system parameters that you need. Do not substitute in numerical values yet.

$$F\,[N] = k\,[N/m]\; P_y\,[m]$$

$$N = \frac{N}{m} \times m \; ?$$

Test case: $k = 10\ N/m$, $P_y = 1m$

$$10\ N/m \times 1m = 10N$$

Now let $P_y = 2m$

$$10\ N/m \times 2m = 20N$$ ✓

It makes sense that pulling the spring further would create a bigger force.

✓ Writing **symbolic expressions** is very useful in engineering because it allows you to check your work. For the equation you just wrote, check that the **units** match for all terms in the equation, and check specific **test cases** to make sure the output matches your intuition. Fix anything you notice. *Then have someone on the teaching team check your equation.*

Investigation 2: Understanding Oscillations

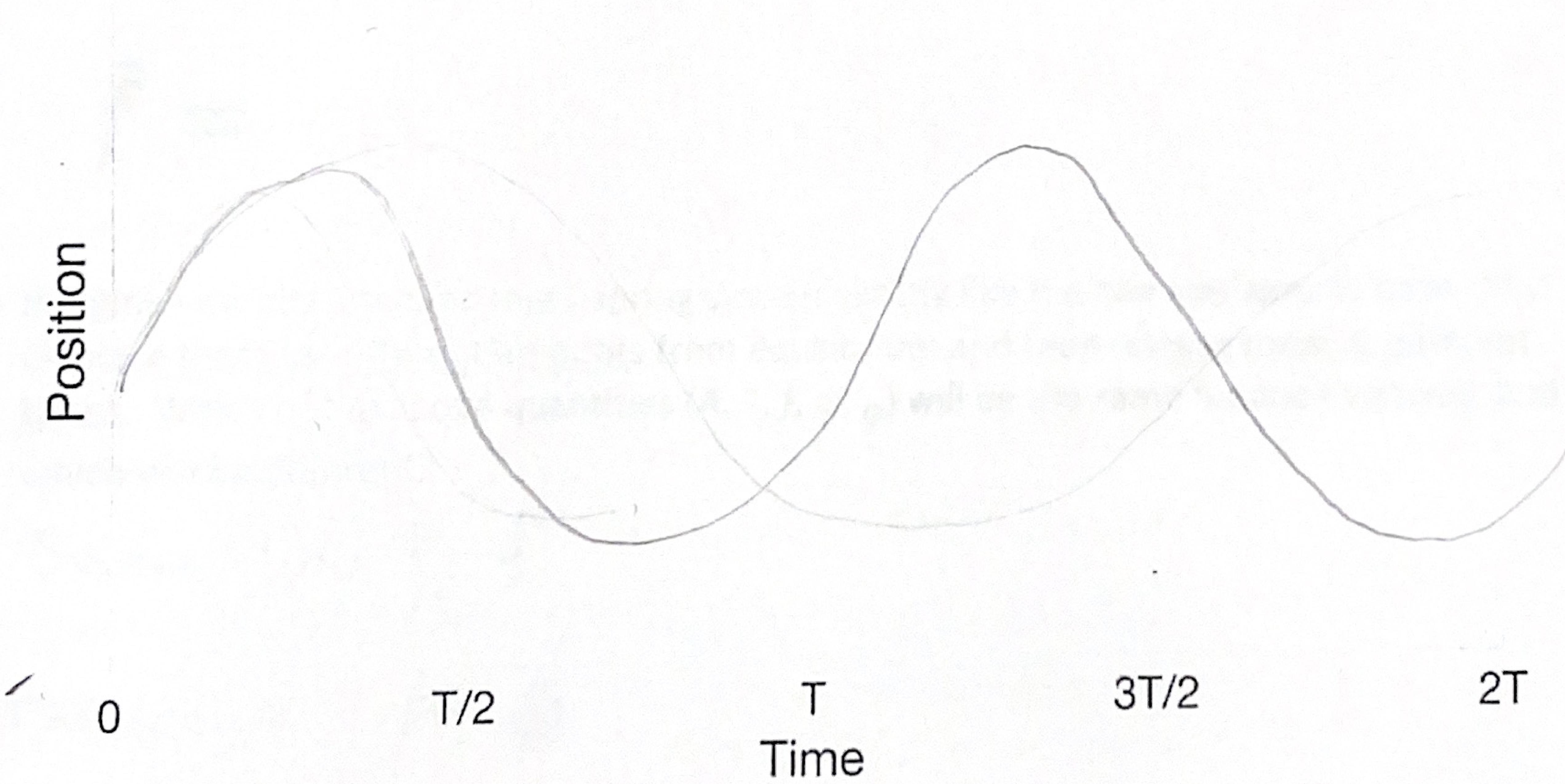
This investigation leads you through a study of the oscillations that occur when you connect a mass to the bottom of your spring and allow it to bounce.

Activity 2.1: Describing Your System's Oscillations

While you can make a mass hang still at the end of a vertical spring, it clearly prefers to bounce. The technical term for this general type of repeating behavior is *oscillation.*

Put the mass on the bottom of your spring if it's not already there. Lift the mass up by a small amount and let it go.

9. Watch the movement of the mass, and then draw your prediction for the shape of its movement on the following graph, taking note of the axis units. One cycle (T) is the length of time it takes the mass to return to the same dynamic state (position and velocity).



Oscillations like this are typically described with the following five quantities:

A Amplitude, the maximum displacement from the equilibrium position, typically in meters

T Period, the time it takes for the motion to repeat itself, typically in seconds

f Frequency, the number of oscillation cycles per unit of time, typically in Hz, cycles/s

ω Angular frequency, the number of radians (a dimensionless unit) per second, rad/s, s^{-1}

φ Phase constant, a shift of the oscillation forward or backward in time, typically in radians

The second, third, and fourth quantities pertain to the cyclic timing of the oscillation. We can write several mathematical relationships between these quantities, as follows:

$$1 \text{ cycle} = 2\pi \text{ radians}$$

$$\boxed{f = \frac{1 \text{ cycle}}{T}} \qquad \boxed{\omega = \frac{2\pi \text{ radians}}{T}}$$

$$f = \omega \cdot \frac{1 \text{ cycle}}{2\pi \text{ radians}} \qquad \omega = f \cdot \frac{2\pi \text{ radians}}{1 \text{ cycle}}$$

10. Start your mass-spring system oscillating up and down again. Where is the center point of the oscillation? Does it depend on the amplitude?

Center point is where mass hangs when undisturbed/still.
Does not depend on amplitude.

11. Use your eyes and a stopwatch to obtain a good experimental estimate of the frequency of this oscillation. We will call this $\hat{f}$, where the angular hat above the variable name signifies an estimate. Show your work. Ignore uncertainty for now.

$\hat{f} = 0.15$

12. Imagine you had a second mass-spring system exactly like the one you already have. You displace them by different amounts from equilibrium and then release them at different times. Which of the above quantities (A, T, f, ω, φ) will be the same for both systems, and which will be different?

Same: ω, T, f

Different: A, ϕ

Recording the Movie: Now your system is ready to record a movie. As you did before, lift up the mass, let go, and let it oscillate. It is best if the mass does not swing back and forth. Large vertical motions are preferable to small ones, but the mass should not touch the edges of the frame. When your mass is bouncing in this way, follow the steps below to record a movie.

Recording and Analyzing a Movie

A. Mocap: Follow mocap procedure posted on canvas - use calibration section

B. Examine Graphs: Look at the graphs that MATLAB produces.

13. Briefly describe the shape of the mass-spring system's position-time graph. How does it compare to your prediction from above?

The position time graph looks like a sine wave, matching my prediction.

14. Use the data cursor tool in the MATLAB figure window to estimate the frequency of the mass-spring system's oscillations. Use standard units. In newer MATLAB versions, you can simply click on the plotted lines/curves/etc. to get the data. $f = \frac{1}{T}$ $T = 0.75$ s

$$\hat{f} = 2.54 \quad 1.33\ \text{Hz}$$

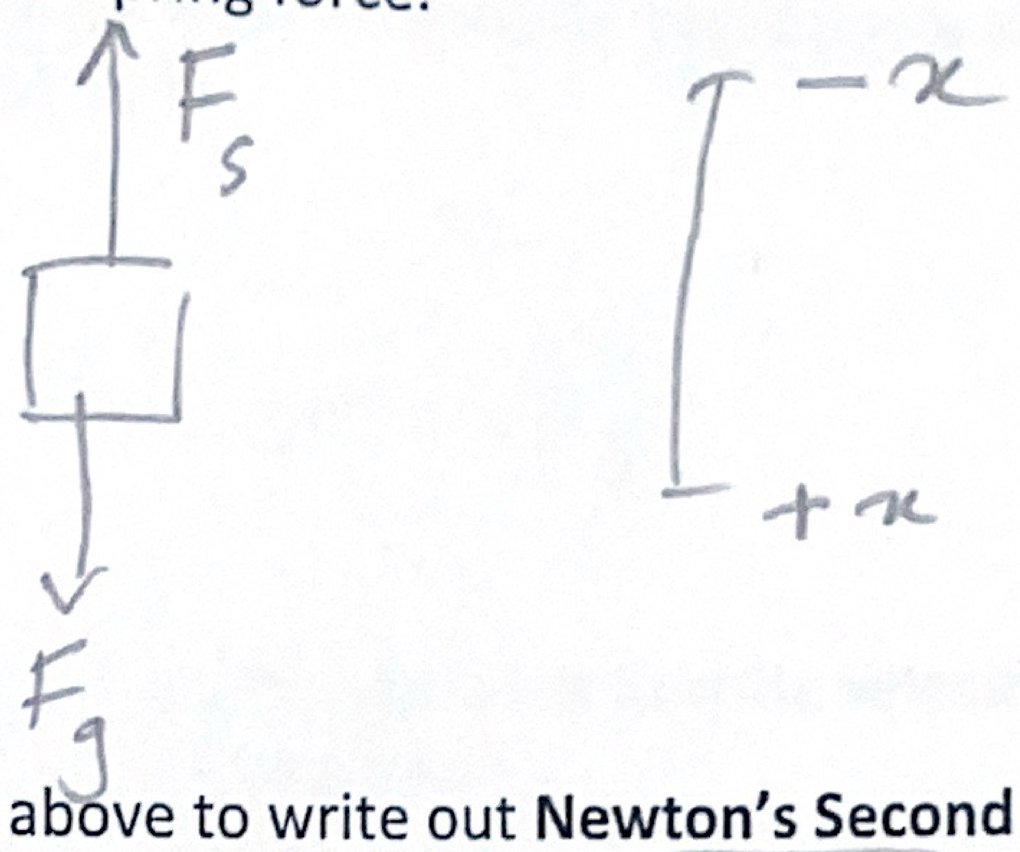
If this frequency estimate from your camera data does not nominally agree with your stop-watch estimate, figure out where you have made an error, and correct it.

15. Record another movie of the mass bouncing. Make this oscillation larger or smaller than the previous one. Is the frequency the same as that of the first motion?

The frequency is about the same

Save and submit the generated position, velocity, and acceleration graphs.

16. Follow the steps above to draw a general **free-body diagram** for the mass as it oscillates. Use the symbol F_s for the spring force.



17. Use the FBD you drew above to write out **Newton's Second Law** for the oscillating mass-spring system. Do not make any substitutions yet. (Being methodical and writing out all the steps of a calculation is the best way to make sure you do it correctly.)

$$\sum F = ma = F_g + (-F_s) = F_g - F_s$$

18. Substitute your symbolic spring force expression from Investigation 1 into your equation from the previous equation. Don't substitute in any numbers yet. Then solve for a_y in terms of p_y and any system parameters. Simplify the equation, and box your answer.

$$\sum F = ma = mg - kp_y$$

$$kp_y + ma = mg$$

$$\boxed{a = \frac{mg - kp_y}{m}}$$

$$p_y = \frac{mg - ma}{k}$$

$$a = g - \frac{kp_y}{m} = g - \frac{k}{m} p_y$$

This analytical result gives you a way to calculate the mass's acceleration a_y given just its vertical spring deflection p_y, the spring stiffness k, and the magnitude of the mass m. Think about what this equation is telling you. Because acceleration is the second time-derivative of position, these two variables are related in a special way, and this equation constitutes a *second-order ordinary differential equation (ODE).*

You will learn detailed methods for how to solve ODEs in future math and engineering classes. For now, we will simply find formulas for the mass's position, velocity, and acceleration (p_y, v_y , a_y) that satisfy this relationship.

19. Look at your saved graph of the mass's movement and think about its shape over time. Then use the symbols A, ω, and φ to write an appropriate expression for the mass's vertical position as a function of time.

$p_y(t) = A\cos(\omega t + \phi)$

20. Differentiate your position expression twice to write velocity and acceleration also as functions of time. Don't forget the chain rule.

$v_y(t) = -A\omega \sin(\omega t + \phi)$

$a_y(t) = -A\omega^2 \cos(\omega t + \phi)$

21. Substitute your expressions for $p_y(t)$ and $a_y(t)$ into the second-order ODE you obtained from Newton's Second Law. This solution should not create a mathematical contradiction. Determine the **symbolic equation for the angular frequency** at which the mass-spring system will oscillate. This frequency is known as the natural frequency ω_n, and it is a function of the system's fundamental parameters. Show your work and box your answer.

$$-A\omega^2\cos(\omega t+\phi) = g - \frac{k}{m}[A\cos(\omega t+\phi)]$$

$$A\cos(\omega t+\phi)\left[-\omega^2+\frac{k}{m}\right] = g \rightarrow -\omega^2 - \frac{k}{m} = \frac{g}{A\cos(\omega t+\phi)}$$

$$\therefore \omega^2 = \frac{g}{p_y} + \frac{k}{m} \rightarrow \boxed{\omega = \sqrt{\frac{g}{p_y} + \frac{k}{m}}}$$

22. Given the stiffness and mass values for your components, as well as the equation you found in the previous question, at what angular frequency should your mass-spring system oscillate? Ignore uncertainty for now.

$k = 25.4\,N/m \quad m = 0.5\,kg \quad g = 9.81\,m/s^2 \quad P_y = 0.045\,m$

$$\omega_n = \sqrt{\frac{g}{P_y} + \frac{k}{m}} = \sqrt{\frac{9.81}{0.045} + \frac{25.4}{0.5}} = 16.395$$

23. What is this predicted natural frequency in cycles per second? This unit of measurement is usually written as hertz, abbreviated Hz. Keep in mind that one cycle equals 2π radians.

$\omega = 2\pi f \rightarrow f = \omega / 2\pi =$

$$f_n = 16.395 / 2\pi = 2.61\ Hz$$

24. What is the percent error between your camera-based frequency measurement and your system's predicted natural frequency? If your error is more than 5%, you have probably made a mistake. Find the source of the discrepancy and fix the error in your work above.

$$\frac{2.54 - 2.61}{2.61} = 2.7\%$$

$$\frac{actual - pred}{act}$$

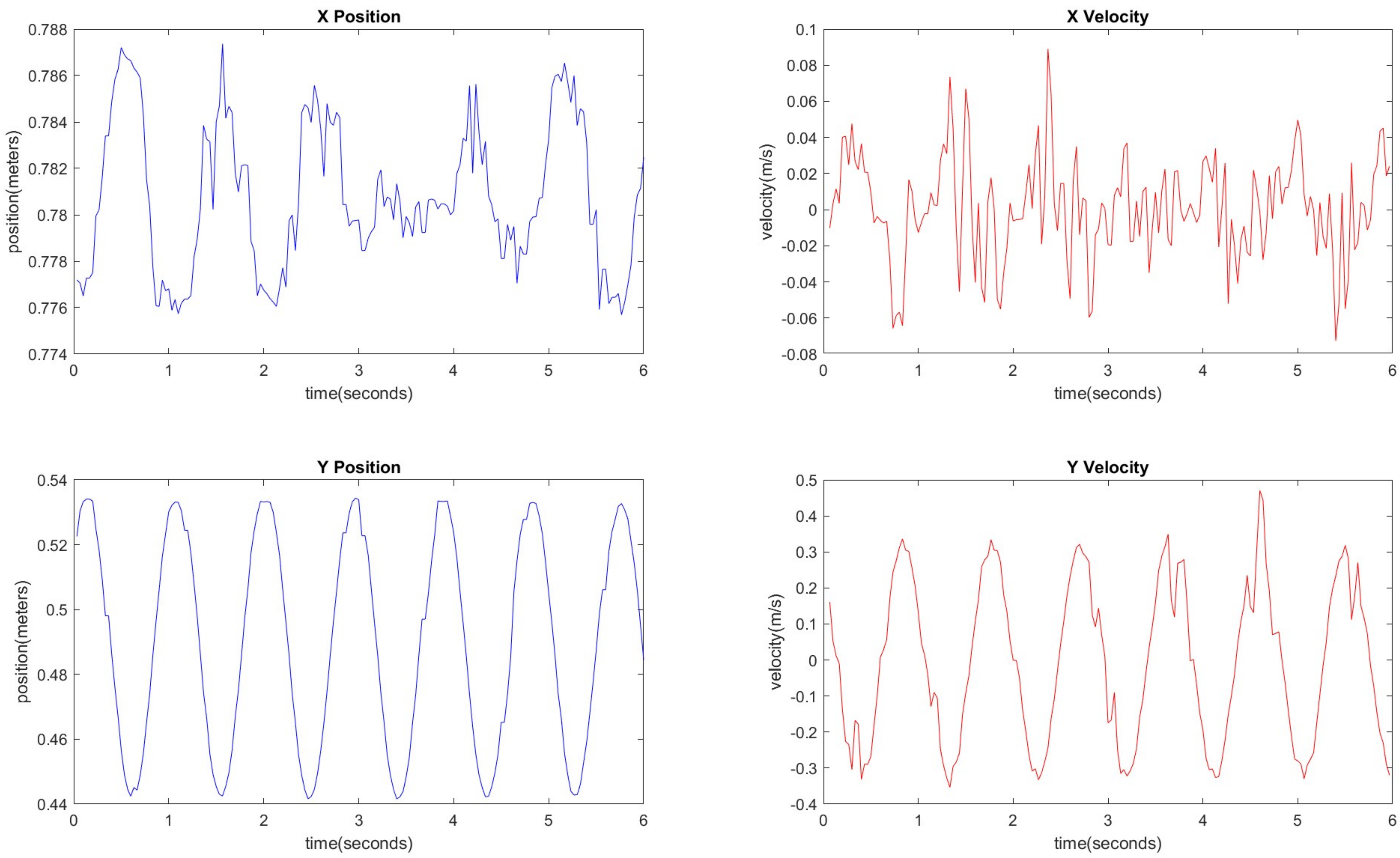
25. If you wanted to decrease the natural frequency of your mass-spring system (make it oscillate more slowly), should you increase or decrease the mass? Explain.

Increase. $\omega = \sqrt{\frac{k}{m}} \rightarrow \omega \propto \frac{1}{\sqrt{m}}$

$\Rightarrow$ If k was to be decreased, m must increase

26. Test your prediction by placing an appropriately sized mass on your spring and starting it bouncing. Watch the new oscillations with your eyes. Are they faster or slower?

Slower


X Position
position(meters)
time(seconds)
X Velocity
velocity(m/s)
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