CS 2209 B

Assignment - 3

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Exercise 1 :-

① Everybody has a mother and a father.

Let $f(x)$ = x has a father

Let $M(x)$ = x has a mother

$\forall$ = For all.

WFF : $\forall x(M(x) \wedge f(x))$

② All users must enter passwords that include special characters.

User(x) = x is a user.

Let $p(x,y)$ = x must enter passwords 'y'

Let $C(y)$ = 'y' includes special characters.

Here, y : ~~characters~~ passwords, x : Users.

$\exists$ = For some, $\forall$ = For all.

WFF : $\forall x(User(x) \rightarrow \exists y(P(y,x) \wedge C(y)))$

~~$\exists y \forall x(p(x) \rightarrow S(y))$~~

③ Every prize was won by a Girl.

Let $P(x)$ = 'x' is a prize

Let $W(x,y)$ = 'x' is won by 'y'

, $G(y)$ = y is a girl.

$\forall$ = for all.

WFF : $\forall x(P(x) \rightarrow (\exists y(G(y) \wedge W(x,y))))$

④ No lunatics is fit to serve on a Jury

Let $L(x)$ = x is a lunatic.

$J(x)$ = x is fit to serve a Jury

Here, x = person ; $\forall x$ For all.

WFF : $\forall x\ (L(x) \rightarrow \sim J(x))$

⑤ Every Sane person can do logic

Let $S(x)$ = x is Sane

$L(x)$ = x can do Logic

Here, 'x' is a person ; $\forall$ = for all

WFF : $\forall x\ (S(x) \rightarrow L(x))$

⑥ No student has more than one Student number.

Let $N(x)$ = x ~~has~~ is a Student ~~number~~

$O(x)$ = x has more than one Student number

Here, x = Student ; $\forall$ = For all

WFF : $(\forall x(N(x) \wedge \neg O(x)))$

⑦ Whoever contributes to the Times is a writer.

Let $T(x)$ = x is a contributer to Times

$W(x)$ = x is a writer.

Here, x = is a person ; $\forall x$ = For all 'x'

WFF : $\forall x\ (T(x) \rightarrow W(x))$

Exercise 2 :-

- ForAll(ϕ) : The set of universally quantified variables

if ϕ is a variable, then ForAll(ϕ) $= \emptyset$

if ϕ is a predicate with variable $a_1, a_2, a_3, \ldots a_n$, then ForAll($\phi$) $= \{a_1, a_2 \ldots a_n\}$.

if ϕ is the negation of a WFF ψ, then ForAll(ϕ) = ForAll($\neg\psi$).

if ϕ is a conjunction of WFF ψ_1 and ψ_2, then ForAll(ϕ) = ForAll(ψ_1) $\wedge$ ForAll(ψ_2).

if ϕ is a disjunction of WFF ψ_1 and ψ_2, then ForAll(ϕ) = ForAll(ψ_1) $\vee$ ForAll(ψ_2).

if ϕ is an implication of WFF ψ_1 and ψ_2, then ForAll(ϕ) = ForAll(ψ_1) $\longrightarrow$ ForAll(ψ_2).

If ϕ is a universal quantifier of a variable 'x' over a WFF (ψ), then ForAll(ϕ) = ~~ϕ~~ $\forall x \psi$

if ϕ is a existential quantifier of a variable 'y' over a WFF(ψ), then ForAll(ϕ) = $\exists y \psi$.

Base Case : if ϕ is a predicate with no quantifiers, then ForAll(ϕ) is empty.

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- Predicates (ϕ) : The set of Predicates.

If ϕ is a variable or constant, then Predicates(ϕ) = $\emptyset$.

if ϕ is a predicate with variables $a_1, a_2, \dots a_n$, then Predicates (ϕ) = $\{\phi\}$.

if ϕ is a negation of WFF ψ, then Predicates (ϕ}. = Predicates ($\neg\psi$).

if ϕ is a conjunction of WFF ψ_1 and ψ_2, then Predicates (ϕ) = Predicates(ψ_1) $\wedge$ Predicates(ψ_2)

if ϕ is a disjunction of WFF ψ_1 and ψ_2, then Predicates (ϕ) = Predicates(ψ_1) $\vee$ Predicates(ψ_2)

if ϕ is an implication of WFF ψ_1 and ψ_2, then Predicates (ϕ) = Predicates(ψ_1) $\rightarrow$ Predicates(ψ_2)

if ϕ is universal $(\forall x)$ quantifier of a variable 'x' over a WFF (ψ), then Predicates (ϕ) = Predicates (ψ).

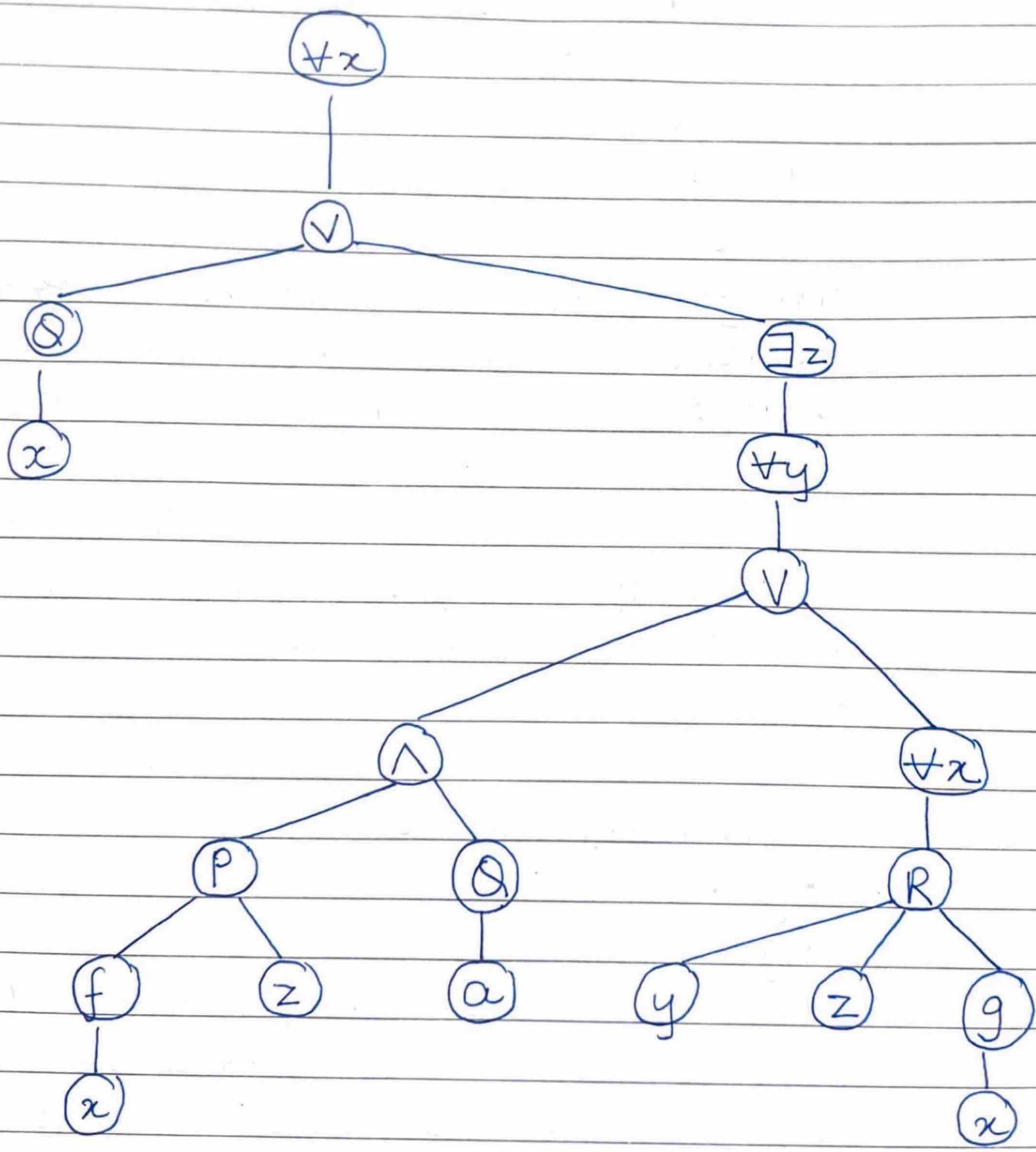
if ϕ is existential $(\exists x)$ quantifier of a variable 'x' over a WFF (ψ), then Predicates (ϕ) = Predicates (ψ).

Base Case : if ϕ is a predicate with no quantifier, then Predicates (ϕ) ~~is~~ contains only ϕ.

Exercise 3 :-

① Syntax Tree -

$\phi = \forall x (Q(x) \vee \exists z \forall y ((P(f(x), z) \wedge Q(a)) \vee \forall x\, R(y, z, g(x))))$



② Sub Formulas in ϕ :

- $Q(x)$
- $P(f(x), z) \wedge Q(a)$
- $\forall x R(y, z, g(x))$
- $\exists z \forall y (P(f(x), z) \wedge Q(a)) \vee \forall x R(y, z, g(x))$
- $\forall x (Q(x) \vee \exists z \forall y ((P(f(x), z) \wedge Q(a)) \vee \forall x R(y, z, g(x)$
- $Q(a)$
- $P(f(x), z)$

Terms in ϕ :

- x (Variable)
- y (Variable)
- z (Variable
- $f(x)$
- $g(x)$
- a (Constant)

Predicates in ϕ :

- P
- Q
- R

Functions in ϕ :

- $f(x)$
- $g(x)$

Free Variables in ϕ :

There are no free variables in ϕ since there is no leaf node in the parse tree of ϕ which doesn't have a path upward to $\forall$ or $\exists$.

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(3) Model for ϕ :

Considering a structure S and the given formula ϕ have the same signature (F, P).

Let the universal set $U_S = \{1, 2, 3\}$

Let the Predicate interpretations be:

- $Q(x)$ is true for $x = 1$ and $x = 3$
- $P(x, y)$ is true for $(x,y) = (1,2), (2,1), (3,3)$
- $R(x, y, z)$ is true for $(x,y,z) = (1,2,3), (2,3,1)$

Let the Function interpretations be:

$f(x) = x - 1$

$g(x) = x + 1$.

~~With~~ the above interpretations, we can check that the sentence ϕ is true in this method:

- For every x in U_S, $Q(x)$ is true for $x = 1, 3$.
- For every x in U_S, there exists a 'z' in U_S such that $P((f(x), z) \wedge Q(a))$ is true for $(x, z) = (1, 2), (2, 2), (3, 3)$.
- for every x in U_S, and y in U_S, $R(y, z, g(x))$ is true for $(x, y, z) = (1, 2, 3)$ and $(2, 3, 1)$.

Therefore, this model satisfies the ϕ.

Exercise 4 :-

① $\forall x \forall y\ (x+y = x \to y = 0)$

N : ~~Yes~~ No, the sentence doesn't holds. Suppose x and y are natural numbers such that $x+y = x$, then by natural number definition, $y=0$ and $0 \notin N$.

Z : ~~No~~ Yes, the sentence ~~doesn't~~ holds. Suppose x and y are two integers such that $x+y = x$, then by ~~natural~~ Integer Number definition, $y = 0$.

R : Yes, the sentence holds; Suppose x and y are real numbers such that $x+y=x$, then by real number definition, $y=0$.

② $\forall x \forall y\ (x \times y = x \to y = 1)$

N : Yes, the sentence holds. Suppose x and y are two natural numbers such that $x \times y = x$, then by natural number definition, $y = 1$.

Z : ~~Yes~~ No, the sentence doesn't holds. Suppose x and y are two integers $(0,5)$ such that $x \times y = x$ $(0 \times 5 = 0)$, then ~~by Integer number definition~~, $y = 5 \neq 1$

R : ~~Yes~~ No, the sentence doesn't holds; Suppose x and y are two real numbers $(0,-2)$, such that $x \times y = x$ $(0 \times -2) = 0$, Then ~~by real number definition~~, $y = -2 \neq 1$

③ $\exists x \forall y\ (x \times y = x + y \rightarrow y = x)$

N : Yes, the sentence holds. Suppose $x = 2$, $y = 2$, then $x \times y = x + y = 4$ and $x = y = 2$. Hence, it holds in Natural numbers.

Z : Yes, the sentence holds. Suppose $x = y = 2$, then $x \times y = x + y = 4$ and $x = y$. Hence, it holds in Integer numbers.

R : Yes, the sentence holds. Suppose $x = 2$, $y = 2$, then $x \times y = x + y = 4$ and $x = y = 2$. Hence, it holds in Real numbers.

④ $\forall x \exists y\ (x \times y = x + y \longrightarrow y = x)$

N : No, the sentence doesn't hold. Suppose, we assume $x = 3$, $y = 3$
$3 \times 3 \neq 3 + 3$ or if $x = 3$, $y = x / x - 1 = \frac{3}{2}$
then $3/2 \notin N$ and $x \neq y$.

Z : No, the sentence doesn't hold. Suppose, $x = 3$, $y = x / x - 1 = 3/2$.
$3/2 \notin Z$ and $x \neq y$.

R : No, the sentence doesn't hold. Suppose, $x = 3$, $y = 3/2$. $x \neq y$. even though $x \times y = x + y$.

Exercise 5 :-

① Group Theory is (finitely) axiomatizable

Yes; A Theory is finitely axiomatizable if Theory = Cons(F) for some finite sets of axioms F. For Group Theory, we have three possible set of axioms

a. Associativity : $\forall x \forall y \forall z (((x*y)*z) = (x*(y*z)))$

b. Identidy element : $\exists y \forall x (x*y = y*x = x)$

c. Inverse element : $\exists z \forall x \exists y (x*y = y*x = z)$

Therefore, It is true that Group Theory is finitely axiomatizable.

② Group Theory is Complete.

No ; A Theory T is complete if for every sentence ϕ of the same signature either $\phi \in T$ or $\neg\phi \in T$. However, group theory has sentences that are neither provable nor refutable from the axioms. For example, the sentence 'there are infinitely many even numbers' ~~or~~ '~~there~~ exists ~~a~~ every group ~~order~~ of even ~~odd~~ ~~order~~ ~~than~~ that have the ~~et~~ element ~~of order 2~~ ' is nor provable nor refutable. Therefore, it is false that group theory is complete.

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③ {} is a theory
No ; A Theory is a set of sentences that are closed under logical entailment. Therefore, an empty set is not a theory because it doesn't contain even a ~~sig~~ single sentence. Hence, {} is a theory is false.

④. $\forall x P(x)$ is in Cons({}).
Yes, since {} is an empty set of axioms, there are none to contradict $\forall x P(x)$ and so it is true that $\forall x P(x)$ is in Cons({}).

⑤ $\forall x (P(x) \vee \neg P(x))$ is in Con({}).
Yes, since {} is an empty set of axioms, there are none to contradict $\forall x (P(x) \vee \neg P(x))$ and so it is true that $\forall x (P(x) \vee \neg P(x))$ is in Cons({}).

⑥ $\forall x (P(x))$ is in Cons$(\exists x (P(x) \wedge \neg P(x)))$
No, the given sentence '$\forall x (P(x))$' is inconsistent with the given Cons sentence '$\exists x (P(x) \wedge \neg P(x))$. The Cons sentence suggests that for an element 'x', $P(x)$ is true and $\neg P(x)$ is also true. This is a contradiction. Hence, it is false that $\forall x (P(x))$ is in Cons $(\exists x (P(x) \wedge \neg P(x)))$

Exercise 6 :-

① $\forall x \forall y\, (R(x,y) \vee R(y,x)) \vdash \forall x (R(x,x))$

Natural Deduction -

1	$\forall x \forall y (R(x,y) \vee R(y,x))$	Premise
2	x_0	
3	$\forall y (R(x_0,y) \vee R(y,x_0))$	$\forall x$ e 1
4	$(R(x_0,x_0) \vee R(x_0,x_0))$	$\forall y$ e 3
5	$R(x_0,x_0)$ $\cancel{\vee R(y_0,x_0)}$	idempotence $\cancel{\forall x \text{ e } 4}$
6	$\forall x (R(x,x))$	$\forall x$ i 2-5

Resolution -

$\forall x \forall y (R(x,y) \vee R(y,x)) \vdash \forall x\, R(x,x)$

$\forall x \forall y (R(x,y) \vee R(y,x)) \wedge \neg(\forall x (R(x,x)))$

$\forall x \forall y (R(x,y) \vee R(y,x)) \wedge \exists x \neg(R(x,x))$

$\cancel{\forall x \forall y (R(x,y) \vee R(y,x)) \wedge \exists}$

$F = \{\{R(x,y), R(y,x)\}\}, \{\neg R(c,c)\}$

$\{R(x,y), R(y,x)\}$ $\qquad\qquad$ $\{\neg R(c,c)\}$

$[c|x, c|y]$

$\square$

Since, the resolution results in a contradiction, the entailment holds.

② $\forall x(P(x) \vee Q(x)), \exists x(\neg P(x)) \vdash \exists x\, Q(x)$

Natural Deduction -

1	$\forall x(P(x) \vee Q(x))$	Premise
2	$\exists x(\neg P(x))$	Premise
3	$x_0 \quad \neg P(x_0)$	Assumption
4	$P(x_0) \vee Q(x_0)$	$\forall x$ e 1
5	$P(x_0)$	Assumption
6	$\perp$	$\neg$e 3, 5
7	$Q(x_0)$	$\perp$e 6
8	$\exists Q(x_0)$	$\exists x$ i 7
9	$Q(x)$	Assumption
10	$\exists x\, Q(x_0)$	$\exists x$ i 9
11	$\exists x\, Q(x_0)$	$\vee$e 4, 5-8, 9-10
12	$\exists x\, Q(x)$	$\exists x$ e 2, 3-11

Resolution -

$\equiv \forall x(P(x) \vee Q(x)) \wedge \exists x(\neg P(x)) \wedge \neg \exists x\, Q(x)$

$\equiv \forall x(P(x) \vee Q(x)) \wedge \exists x(\neg P(x)) \wedge \forall x \neg Q(x)$

$\cancel{\equiv \forall x(P(x) \vee Q(x)) \wedge}$ $F = \{P(x), Q(x)\}, \{\neg P(y)\}, \{\neg Q(c)\}$

$\{P(x), Q(x)\} \quad \{\neg P(y)\} \quad \{\neg Q(c)\}$

$[x/y]$

$\{Q(x)\}$

$[x/c]$

$\square$

Since, the resolution results in a contradiction, the entailment holds.