CS 2214

Assignment 2 - due 9th February

Student name : Ashna Mittal
Student ID : 251206758

Q1(a) A : finite set
$R \subseteq A \times A$: relation

R is negatively transitive when :
$\forall x, y, z \in A, (x,y) \notin R \wedge (y,z) \notin R \rightarrow (x,z) \notin R$

R is asymmetric when :
$\forall x, y \in A, (x,y) \in R \leftrightarrow (y,x) \notin R$.

① R is transitive if and only if for every $x, y, z \in A$:
$(x,y) \in R$ and $(y,z) \in R \rightarrow (x,z) \in R$.

Asymmetric :
if $(x,y) \in R \longrightarrow (y,x) \notin R$
if $(y,z) \in R \longrightarrow (z,y) \notin R$

Negatively Transitive :
if $(x,y) \notin R \longrightarrow (x,y) \in R$
if $(y,z) \notin R \longrightarrow (z,y) \in R$.
if $(x,z) \notin R \longrightarrow (z,x) \in R$

From these, we can conclude that
if $(z,x) \in R \wedge (x,y) \in R \rightarrow (z,y) \in R$

Therefore, R is both asymmetric and negatively transitive, then R is transitive.

② No, Transitivity does not imply Asymmetricity and Negatively Transitivity.
Counterexample:
Let set $A = \{1, 2, 3\}$
relation $R = \{(1,2), (2,3), (3,1)\}$
for R to be transitive, we need:
$(1,2) \in R \land (2,3) \in R \rightarrow (1,3) \in R$
$(2,3) \in R \land (3,1) \in R \rightarrow (2,1) \in R$
Therefore, the relation R is transitive.
However, R is not Asymmetric because $(1,2) \in R$ by $(2,1) \notin R$ is not correct because $(2,1) \in R$ as per the rule of transitivity of R. Similarily, $(3,1) \in R$, if R were Asymmetric, then $(1,3) \notin R$, but it is not the case.

Also, R is not negatively transitive because $(1,2) \notin R \land (2,3) \notin R \rightarrow (1,3) \notin R$ is false.

Therefore, it is not always the case that if R is transitive, it should be Asymmetric and negatively transitive.

(b) R is anti - Transitive if :

$\forall a, b, c \in A, (a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \notin R$.

To prove: Anti Transitive relation is irreflexive.

Proof : Proof by Contradiction.

Irreflexive rule : $a \in A, (a,a) \notin R$

Let us assume that the relation R is Anti - Transitive but not Irreflexive. There exists an element a in A such that $(a,a) \in R$.

Condition for Anti Transitivity $\Rightarrow$

$$(a,a) \in R \wedge (a,a) \in R \rightarrow (a,a) \notin R$$

This condition contradicts our initial assumption that $(a,a) \in R$.

$\Rightarrow$ Anti - Transitive relation is Irreflexive.

Q2 Let ξ (universal set) = 150
Students (Oyster Mushroom) = 100
Students (maitake Mushroom) = 33
Students (Lion's mane mushroom) = 23
Students (Oyster And maitake) = 23
Students (maitake And Lion's Mane) = 12
Students (Oyster And Lion's Mane) = 10
Students (Oyster And Lion's Mane And Maitake) = 2

Let Oyster mushroom found by students be 'P'.
Let Maitake mushroom founded by students be 'Q'.
Let Lion's Mane mushroom found by students be 'R'.

① Students (only Oyster mushroom)

$= P - (P \cap Q) - (P \cap R) + (P \cap Q \cap R)$

$= 100 - 23 - 10 + 2$

$n(\text{only } P) = 69$

② Students (only Maitake mushroom)

$= Q - (P \cap Q) - (R \cap Q) + (P \cap Q \cap R)$

$= 33 - 23 - 12 + 2$

$n(\text{only } Q) = 0$

③ Students (only Lion's mane Mushroom)

$= R - (P \cap R) - (R \cap Q) + (P \cap Q \cap R)$

$= 23 - 10 - 12 + 2$

$n(\text{only } R) = 3$

(4) Students (no mushroom)

$= \mathcal{E}(\text{universal}) - \cancel{\text{students}}\ n(P) - n(Q)$

$- n(R) + n(P \cap Q) + n(P \cap R) +$

$n(P \cap Q) - n(P \cap Q \cap R)$

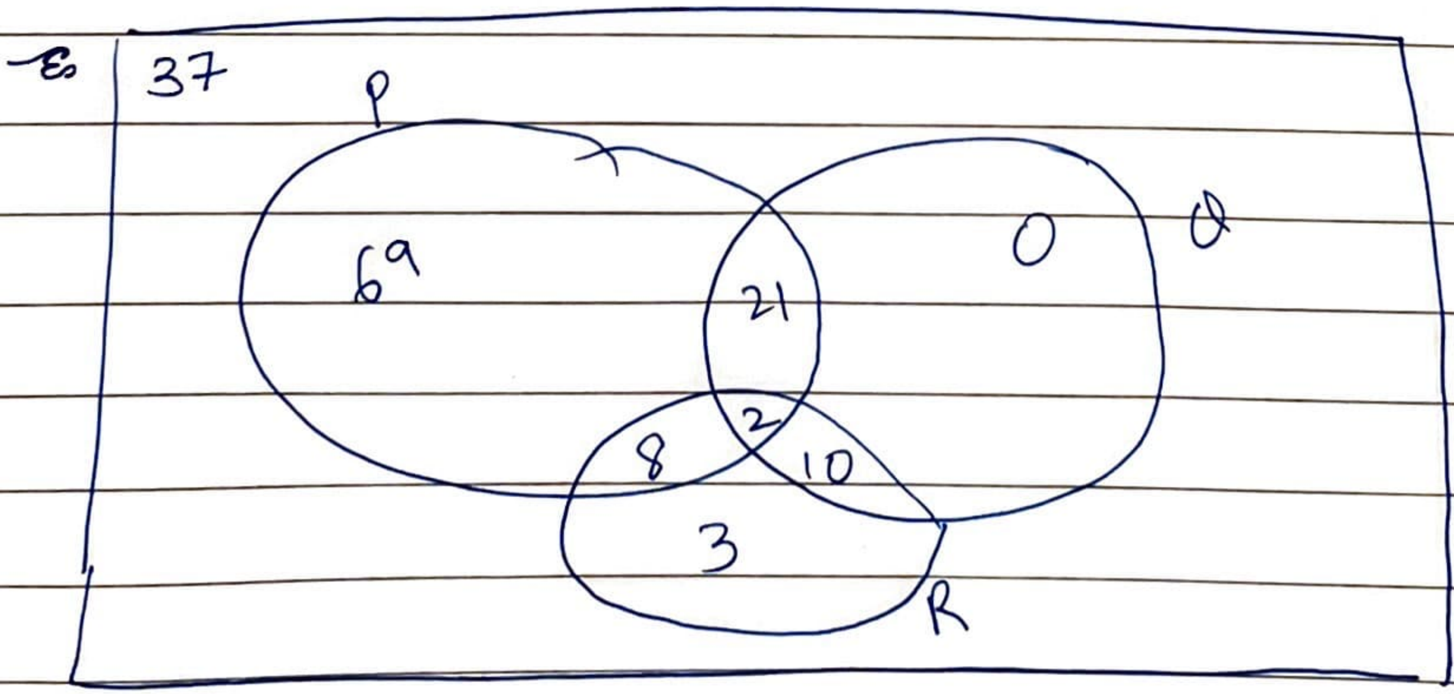
$= 150 - 100 - 33 - 23 + 23 +$

$12 + 10 - 2$

$= 150 - 113$

$n(\text{none}) = 37$

Venn Diagram



Q3 $\mathbb{Z}_{100} = \{x \in \mathbb{Z} \mid |x| \leq 100\}$

(a) An equivalence relation is reflexive, Symmetric and Transitive.
To show: a binary relation R is reflexive, Symmetric, transitive and therefore equivalent.

Reflexive: For every $(x, y) \in \mathbb{Z}_{100} \times \mathbb{Z}_{100}$, we
~~have $x = y \Longrightarrow x^2 + y^2$~~
~~Since, $x = y$, $x^2 + y^2$ (substitution)~~
have $(x, y) R (a, b) \Rightarrow x^2 + y^2 = a^2 + b^2$
where $a = x$ and $b = y$.
$(x, y) R (x, y) \Rightarrow x^2 + y^2 = x^2 + y^2$

$\Rightarrow$ It is Reflexive.

Symmetric: Suppose $(x, y) R (a, b)$
$\Rightarrow x^2 + y^2 = a^2 + b^2$
To show: if $(x, y) R (a, b) = (a, b) R (x, y)$
$x^2 + y^2 = a^2 + b^2 \longrightarrow a^2 + b^2 = x^2 + y^2$
which is following $(a, b) R (x, y)$.

Transitive: Suppose $(x, y) R (a, b)$ and
$(a, b) R (c, d)$
$\Rightarrow$ $x^2 + y^2 = a^2 + b^2$ and $a^2 + b^2 = c^2 + d^2$
$\Rightarrow$ $x^2 + y^2 = c^2 + d^2$
$\Rightarrow$ $(x, y) R (c, d)$

Therefore, R is an equivalence relation on $\mathbb{Z}_{100} \times \mathbb{Z}_{100}$

(b) Z_{100} has 101 elements.

Number of elements in $(Z_{100} \times Z_{100}) \times (Z_{100} \times Z_{100})$ $= (101 \times 101) \times (101 \times 101)$

$$= 101^2 \times 101^2$$
$$= 10201 \times 10201$$
$$= (10201)^2$$
$$= 104060401$$

(b) Equivalence of $(5,10)$ can be found by finding all pairs $(x,y) \in Z_{100} \times Z_{100}$ such that $x^2 + y^2 = 5^2 + 10^2$

This is equivalent to finding all pairs $(x,y) \in Z_{100} \times Z_{100}$ such that $x^2 + y^2 = 25 + 100 = 125$.

Some elements in the equivalence class are $(5,10)$, $(10,5)$, ~~(10,45), (45,10)~~, $(-5,10)$, $(-10,5)$, $(-5,-10)$, $(-10,-5)$, $(5,-10)$, $(10,-5)$.

Q 4(a) To prove: $A \cup B = B \leftrightarrow A \subseteq B$

Let an element 'x' $\in A$.

if x is in B, then $x \in A \cup B$

$\Rightarrow A \cup B = B$.

if x is not in B, then $x \in A$ but $x \notin B$

$\Rightarrow x \in A \cup B$.

$\Rightarrow A \subseteq A \cup B$

Since $A \subseteq A \cup B$, if $A \cup B = B$, then $A \subseteq B$.

Therefore, $A \cup B = B$ if and only if $A \subseteq B$.

$A \cup B = B \longleftrightarrow A \subseteq B$.

(b) To prove: $A \subseteq B \cup C \leftrightarrow (A \setminus C \subseteq B) \wedge (A \setminus B \subseteq C)$

: $A \subseteq B \cup C \longleftrightarrow ((A \cap C) \subseteq B) \wedge ((A \cap B) \subseteq C)$

~~Bicondition rule: $(p \leftrightarrow q) \equiv ((p \rightarrow q) \wedge (q \rightarrow p))$~~

~~To prove: $(A \subseteq B \cup C) \rightarrow ((A \setminus C \subseteq B) \wedge (A \setminus B \subseteq C))$~~

Suppose, $x \in A$. if $x \in B \cup C$, then either $x \in B$ or $x \in C$. if $x \in B$, then $x \notin A$. So, $x \in (A - C)$.

if $x \in C$, then $x \notin A$. So, $x \in (A - B)$.

$\Rightarrow (A \subseteq (B \cup C))$ if and only if $(A - C) \subseteq B$ and $(A - B) \subseteq C$.

$\Rightarrow (A \subseteq B \cup C) \longleftrightarrow ((A - C) \subseteq B) \wedge ((A \cap B) \subseteq C)$

$\Rightarrow (A \subseteq B \cup C) \leftrightarrow (A \setminus C \subseteq B) \wedge (A \setminus B \subseteq C)$

Alternatively,

We must prove: ① $(A \subseteq B \cup C) \rightarrow (A \setminus C \subseteq B) \wedge (A \setminus B \subseteq C)$

② $(A \setminus C \subseteq B) \wedge (A \setminus B \subseteq C) \rightarrow (A \subseteq B \cup C)$

① Suppose, $(A \subseteq B \cup C)$ then $\forall x$, in A, if $x \notin C$, it must be in B and $\forall x$ in A, if $x \notin B$, it must be in C, because A is a subset of B union C.

Hence, $(A \subseteq B \cup C) \rightarrow (A \setminus C \subseteq B) \wedge (A \setminus B \subseteq C)$

② $\forall x$ in A which is not in C, it is in B and for every element x of A which is not in B, it is in C. So, $\forall x$ of A are either in B or in C. Therefore, A is a subset of B and C.

Hence, $(A \setminus C \subseteq B) \wedge (A \setminus B \subseteq C) \rightarrow (A \subseteq B \cup C)$

Hence,

$(A \subseteq B \cup C) \longleftrightarrow (A \setminus C \subseteq B) \wedge (A \setminus B \subseteq C)$

(c) To prove: $(A \Delta B) \Delta C = A \Delta (B \Delta C)$

let an element 'x' $\in A \Delta B$.

if x is in A, but not in B, then ~~x~~

$x \in (A \Delta B)$ and $x \notin C$.

So, $x \in (A \Delta C)$.

if x is in B, but not in A, then

$x \in (B \Delta A)$ and $x \notin C$.

So, $x \in (B \Delta C)$.

Therefore, $(A \Delta B) \Delta C = A \Delta (B \Delta C)$

Alternatively,

since $A \Delta B = (A \setminus B) \cup (B \setminus A)$

① Let an element 'x' $\in (A \Delta B) \Delta C$

$= (x \in (A \Delta B) \wedge x \notin C) \vee (x \notin (A \Delta B) \wedge x \in C)$

$[((x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B) \wedge x \notin C)] \vee$

$[((x \in A \wedge x \in B) \vee (x \notin A \wedge x \notin B) \wedge x \in C)]$

sol ① $= [(x \in A \wedge x \notin B \wedge x \notin C) \vee (x \notin A \wedge x \in B \wedge x \notin C)]$

$\vee [(x \in A \wedge x \in B \wedge x \in C) \vee (x \notin A \wedge x \notin B \wedge x \in C)]$

② Let an element 'x' $\in A \Delta (B \Delta C)$

$= (x \in A \wedge x \notin (B \Delta C)) \vee (x \notin A \wedge x \in (B \Delta C))$

$= [x \in A \wedge (\neg(x \in B \wedge x \in C) \wedge \neg(x \notin B \wedge x \in C)] \vee$

$[x \notin A \wedge ((x \in B \wedge x \notin C) \vee (x \notin B \wedge x \in C))]$

sol ② $= [x \in A \wedge ((x \notin B \wedge x \notin C) \vee (x \in B \wedge x \notin C)] \vee$

$[x \notin A \wedge ((x \in B \wedge x \notin C) \vee (x \notin B \wedge x \in C)]$

Since sol ① = sol ② $\Rightarrow (A \Delta B) \Delta C = A \Delta (B \Delta C)$