CS 2214 B

Assignment - 3

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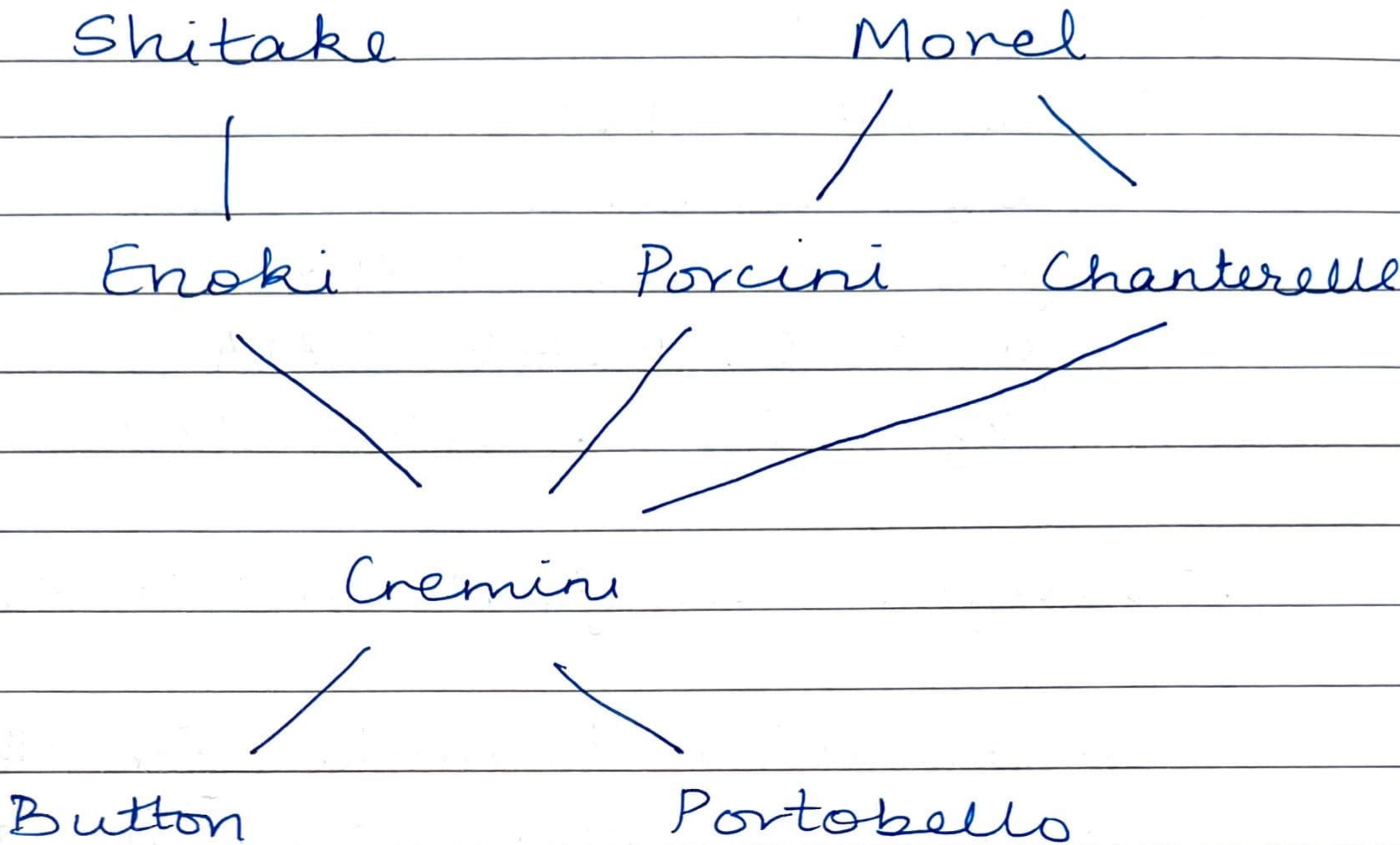
<u>Exercise 1</u> :

(a) Pairs of incomparable mushrooms :-

$M = \{$ Porcini, Chanterelle, Enoki, Shitake, Morel, Cremini, Portobello, Button $\}$.

Here, M defines a set of Mushrooms, along with a partial order $\succeq$ on M defined as $a \succeq b$ if a 'is better than' b or in other words if a 'covers' b. It is given that $(M, \succeq)$ is a poset and therefore all its relations are reflexive, antisymmetric and transitive.

Hasse Diagram



To obtain the set of incomparable mushrooms from poset $(M, \preceq)$, any two elements (x, y) should be $x \npreceq y$ and $y \preceq x$.

From the given set of rankings, we can elimanate the ones that ~~so~~ are comparable.
For example, Cremini is better than Button. So we can exclude (Cremini and Button). We also exclude pairs based on transitvity that is if $a \geq b$ and $b \geq c$, then $c \leq b \leq a$ and so $c \leq a$.
For example, it is given that Chanterelle is better than Cremini, Cremini is better than Button, Cremini is better than Portobello. So, we know that Chanterelle is better than Cremini, Button, Portobello. Hence, we can exclude, (Chanterelle, Cremini), (Chanterelle, Button) (Chanterelle, Portobello), (Cremini, Button), (Cremini, Portobello).
Applying these rules, we get the following incomparable pairs:

1. (Portobello, Button)
2. (Chanterelle, Porcini)
3. (Chanterelle, Enoki)
4. (Chanterelle, Shitake)
5. (Morel, Shitake)
6. (Morel, Enoki)
7. (Shitake, Porcini)
8. (Enoki, Porcini).

(b) An element 'm' of the set M with the partial order $\succeq$ is minimal if for any $a \in M$ such that $a \succeq m$, then $m = a$. It is maximal if for any $a \in M$ such that $m \succeq a$, then $m = a$.

The mushroom elements that are ranked the highest are: Morel, Shitake

This is because the ~~se~~ Poset M has the following rankings:

Cremini $\succeq$ Button, Portobello,
Chanterelle $\succeq$ Cremini, Button, Portobello
Porcini $\succeq$ Cremini, Button, Portobello
Enoki $\succeq$ Cremini, Portobello, Button
Shitake $\succeq$ Enoki, Cremini, Portobello, Button
Morel $\succeq$ Chanterelle, Cremini, Portobello, Button
Porcini

Note: Morel, Shitake, ~~Enoki~~ are although not comparable to all the elements of the Poset, but as per the given rankings, it is not the case that any element is better than them.

The mushroom elements that are ranked the lowest are: Button, Portobello.

This is ~~because~~ because these are 'not better than' any other element of poset.

(c) Two Relative rankings to be added are:

1. Morel is better than Shitake
2. Portobello is better than Button.

By adding these, the now unique maximal element is 'Morel' because for the poset $(M, \succeq)$, ~~let us~~ we know ~~assume~~ that 'Shitake' was another maximum element. Shitake is better than Enoki, Portobello, Button, Cremini. So, Shitake $\geq$ Enoki, Portobello, Button, Cremini. By introducting 'Morel is better than Shitake', we form (by transitivity) that Morel is better than Shitake, Enoki, Portobello, Button, Porcini, Chanterelle, Cremini. So, Morel $\geq$ Shitaki, Enoki, Portobello, Button, Porcini, Chanterelle, Cremini. Therefore, ~~Shitake ≥ Morel and~~ Morel ≤ Shitake. ~~By Antisymmetry of partial order, Shitake = Morel.~~ Therefore, Morel is unique ~~as~~ Maximul as 'it is better than' any other mushroom of the poset.

Similarily, by adding 'Portebello is better than Button', we have a new unique mimimal: 'Button'. For the poset $(M, \geq)$, ~~let us~~ we know ~~assume~~ that ~~Shit~~ Portebello was another minimum element. Portebello is only better than Button, i.e., Portebello $\geq$ Button. However, Button is the only mushroom ~~is~~ that 'is not better than' any other element of the Poset.

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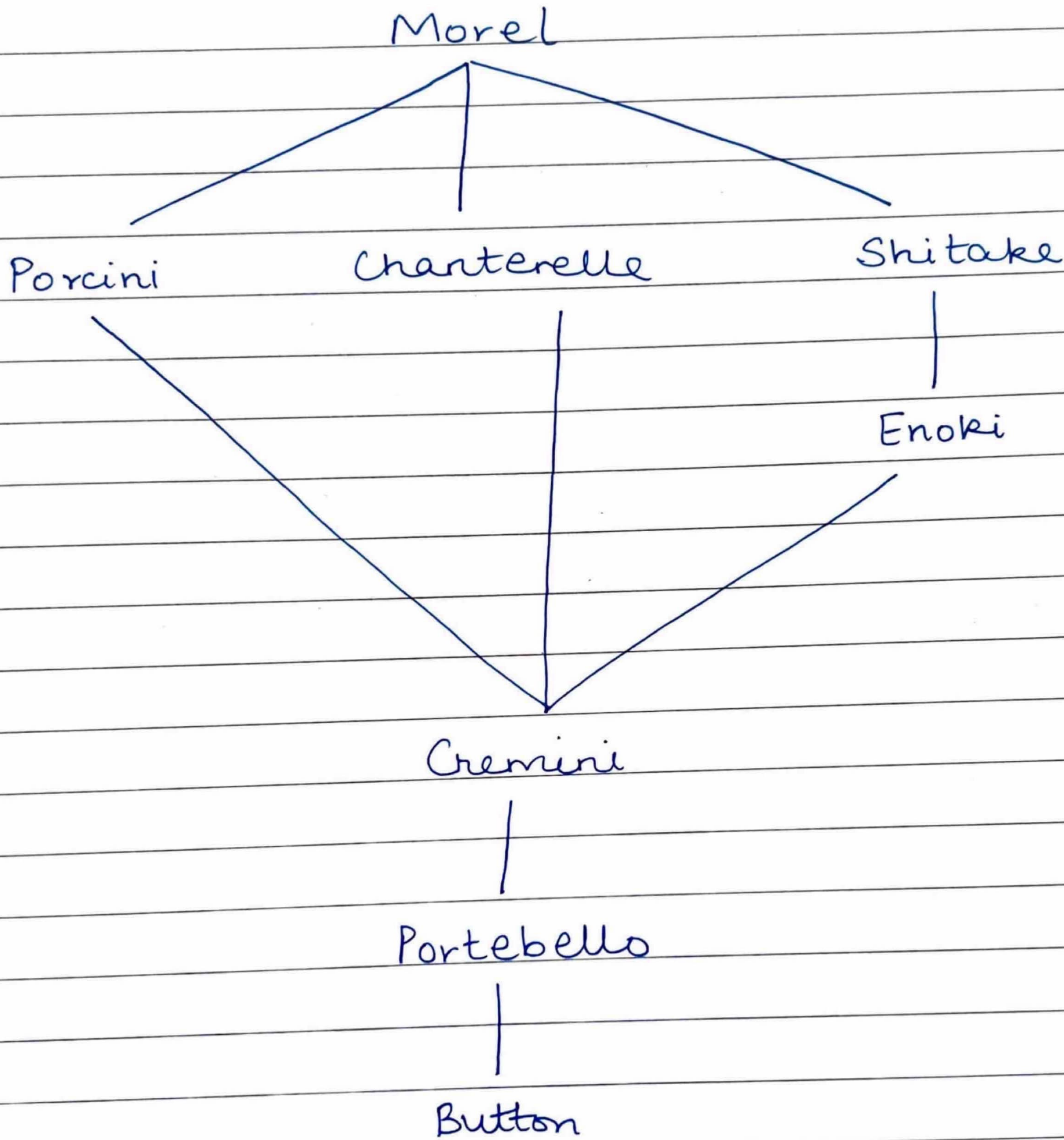
(d) The new set of ~~iso~~ incomparable pairs of mushrooms are :
1. Porcini , Chanterelle
2. Porcini , Enoki
3. Porcini , Shitake .
4. Chanterelle , Enoki
5. Chanterelle , Shitake .

We elimanated 'Button, Portebello' and 'Shitake, Morel' when we introduced the two new rankings in part (c). By transitivity, since 'Shitake is better than Enoki' and 'Morel is better than Shitake', we can eliminate the result, which is, 'Morel is better than Enoki' : {Morel, Enoki}.

(e) Hasse Diagram :



Excercise 2 :-

(a) $A = \mathbb{Z} \times \mathbb{Z}$

The set A is countably infinite since it can be indexed. It has a first, second element, and so on. There are surely infinite number of elements to count, but they are countable.

One possible bijection between A and $\mathbb{N}$: given a pair (m,h) in A, we associate it with a natural number $2^{(m-1)}(2n-1)$

The zigzag pattern ensures that we list all pairs of

~~This maps $(1,1)$ to 1, $(1,2)$ maps~~

integers exactly once. For a pair (m,h), $m+h=n$, starting

~~to using $n = (a+b)(a+b+1)/2 + b$~~

with $(0,n)$, $(1,n-1)$, $(2,n-2)$, ... Alternating between $(+)$ and $(-)$ I

~~This maps $(1,0)$ to 1, $(0,1)$ to 2,~~

from n to 8 h at each step.

~~etc.~~ Therefore, we can create a bijection between A and $\mathbb{N}$ by listing the element of A in the order: $(0,0)$, $(0,1)$, $(1,0)$, $(-1,0)$, $(0,-1)$, $(1,1)$, $(-1,-1)$, $(1,-1)$, $(-1,1)$, $(2,0)$, $(-2,0)$, $(0,2)$ $(0,-2)$, $(2,1)$, $(-2,-1)$, $(1,2)$, $(-1,-2)$, $(2,-1)$, $(-2,1)$, $(1,-2)$, $(-1,2)$ ....

So, of each positive integer 'n', we have exactly $4n$ elements that have a sum of absolute values of coordinates equal to n.

So $|A| = |\mathbb{N}|$.

(b) Knights : Always tell the truth
Knaves : Always lie
B : 'I am not a knight'

when randomly picked to answer the question : 'Are you a knight', a knight would always tell the truth and therefore answer : 'Yes, I am a knight'. Therefore, knights cannot be part of our set B. However, a knave would always lie and say 'Yes, I am a knight', even though they are not. Therefore, they would not own up and would so not say that 'I am not a knight'. Therefore, they too cannot be part of our set B. Since both knights and knaves are not part of set B, it is a ϕ set.

$$B = \{ \phi \}.$$

The only element in set B would be the element 'null' or '{ }' or 'ϕ'. Thus, the set is FINITE

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(c) Set C : all real numbers in $[1, 2)$

Let us assume that there are 'n' number of elements in C and so it is countable.
Using Proof by Contradiction, we can show that it is uncountable.
Using Cantor's Diagonal Arguement :
Let x, be an element of $[1,2)$. We can show that x is not listed among r_1, r_2, r_3, etc. Assign the digit d_i of x to be different from d_i of r_i for all $i = 0, 1, 2 \ldots$

set C :

r_1	=	1 · 0 1 2 3 4 5	(0th element)
r_2	=	1 · 1 2 3 4 5 6	(1st element)
r_3	=	1 · 2 3 4 5 6 7	(2nd element)
r_4	=	1 · 3 4 5 6 7 8	(3rd element)
r_5	=	1 · 4 5 6 7 8 9 . . .	(4th element)
r_6	=	1 · 5 6 7 8 9 0	(5th element)
⋮			
		1 · 1 3 5 9 7 1 . . .	(ith element)

Surely, $x \in [1, 2)$ since its whole number digit is 1 and every digit after decimal place is among $\{0, 1, 2 \ldots 9\}$. x differs from r_1 by digit d_1, etc. Hence x is not in the list $r_1, r_2 \ldots$ and our assumption was incorrect.
Since $[1, 2) \subset R$, it follows that set C is <u>UNCOUNTABLE</u>.

(d) set D of univariate polynomials in x of degree ~~[illegible]~~ ≤ 100 whose coefficients are Integers.

The set D is countably Infinite. The set D can be indexed even though there are infinite number of elements to count.
One possible bijection between D and N is :-
Given a polynomial $p(x)$ in set D, we can associate it with a natural number $n = a_m 10^m + a_{(m-1)} 10^{m-1} + \dots a_1 10 + a_0$, where $a_m, a_{(m-1)} \dots a_0$ are coefficients of $p(x)$. This bijection maps the polynomial $a_n x^n + a_{(n-1)} x^{n-1} + \dots a_1 x + a_0$ to the natural numbers $a_n 10^n + a_{(n-1)} 10^{n-1} + \dots a_1 10 + a_0$. It covers every natural number, which makes D an infinitely countable set. Conversely, given an infinite sequence of Integers, we can construct a polynomial by using the sequence as its coefficients.

So $|D| = |N|$.

Exercise 3 :-

(a) $f : X \longrightarrow Y$

$A, B \subseteq X$

$(A \cap B) = f(A) \cap f(B)$, then f is injective

Using Contrapositivity of injectivity, we assume that f is not injection and so $(A \cap B) \neq f(A) \cap f(B)$

Since f is not injective there exists $(x, y) \in X$ such that $x \neq y$ and $f(x) \neq f(y)$.

Assuming, Let $A = \{x\}$ and let $B = \{y\}$. Since $x \neq y$, $A \cap B = \emptyset$ and also since $f(x) \neq f(y)$, $f(A) \neq f(B)$ and so $f(A) \cap f(B) = \emptyset$

To conclude, since $f(A \cap B) = f(A) \cap f(B) = \emptyset$, then f in injective.

(b) $f: X \longrightarrow Y$

if f is injective, then $f(A) \cap f(B) = f(A \cap B)$

To show: $f(A \cap B) \subseteq f(A) \cap f(B)$

and

$$f(A) \cap f(B) \subseteq f(A \cap B)$$

Assuming f is injective, $f(x) = f(y)$, where $x = y$.

Let $A = \{x\}$ and $B = \{y\}$

and let $f(x) = f(y)$, since $x = y$ and $A \cap B = \{x\}$.

$f(A \cap B) = f(x)$

Since we concluded before that $x = y$, we know $f(x) = f(y)$.

$\Rightarrow f(x) \cap f(y) = f(x) = f(A \cap B)$.

Therefore, $f(A \cap B) \subseteq f(A) \cap f(B)$.

$f(A) \cap f(B) = f(x) \cap f(y) = f(x)$

since, $x = y$ and because $(A \cap B) = \{x\}$, then $f(A \cap B) = f(x) = f(x) \cap f(y)$.

Therefore, $f(A) \cap f(B) = f(A \cap B)$.

~~Therefo~~

Hence, f is injective and $f(A \cap B) = f(A) \cap f(B)$.

Exercise 4 :

(i) $a_n = \frac{1}{3} a_{n-1} \quad ; \quad n > 1 \quad , a_1 = 7$

for $n = 2$, $a_2 = \frac{a_1}{3} = \frac{1}{3}(7) = \frac{7}{3}$.

for $n = 3$, $a_3 = \frac{a_2}{3} = \frac{1}{3}\left(\frac{7}{3}\right) = \frac{7}{(3)^2}$

for $n = 4$, $a_4 = \frac{a_3}{3} = \frac{1}{3}\left(\frac{7}{9}\right) = \frac{7}{(3)^3}$

for $n = 5$, $a_5 = \frac{a_4}{3} = \frac{1}{3}\left(\frac{7}{3^3}\right) = \frac{7}{(3)^4}$

Therefore, final result $= \left\{7, \frac{7}{3}, \frac{7}{9}, \frac{7}{27}, \ldots\right\}$

The Required Formula for the above Geometric Progression :

$$f(n) = 7 \times \left(\frac{1}{3}\right)^{n-1}$$

General Term $\Rightarrow$ $a_n = \frac{7}{(3)^{n-1}}$

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(ii) $a_n = 4a_{n-1} + 6 \; ; \; n > 0, \; a_0 = 1$

for $n=1$, $a_1 = 4a_0 + 6 \Rightarrow 4.1 + 6 = 10$
for $n=2$, $a_2 = 4a_1 + 6 \Rightarrow 4(10) + 6 = 46$
for $n=3$, $a_3 = 4a_2 + 6 \Rightarrow 4(46) + 6 = 190$
for $n=4$, $a_4 = 4a_3 + 6 \Rightarrow 4(190) + 6 = 766$

Therefore, final result $= \{1, 10, 46, 190 \ldots\}$

$$a_n = 4a_{n-1} + 6$$
$$= 4(4a_{n-2} + 6) + 6$$
$$= (4^2 a_{n-2}) + (6) + (4 * 6)$$
$$= 4^2 (4a_{n-3} + 6) + (4 \times 6) + 6$$
$$= 4^3 (4a_{n-3}) + (4^2 \times 6) + (4 \times 6) + 6$$
$$\vdots$$
$$= (4^n) a_0 + 6 (4^{n-1} + 4^{n-2} + \ldots 4^1 + 4^0)$$

since $a_0 = 1$, $a_n = 4^n + 6(4^{n-1} + 4^{n-2} + \ldots 4^1 + 4^0)$

let $S = 4^0 + 4^1 \ldots 4^{n-2} + 4^{n-1}$

$a = 4^0 = 1 \; ; \; r = 4$, terms $= n$.

we can say $a_n = 4^n + 6 \sum_{i=0}^{n} a \cdot r^i$

$$\sum_{i=0}^{n} a \cdot r^i = \frac{ar^{n-1} - a}{r-1}, \quad r \neq 1$$

sum of n term of a finite GP:

$$S = \frac{a(r^n - 1)}{r-1}$$

$$\sum_{i=0}^{n} a \cdot r^i = S = \frac{1(4^n - 1)}{4-1} = \frac{4^n - 1}{3}$$

original $a_n = 4^n + 6[(4^n - 1)/3] = 3(4)^n - 2$

Therefore, $a_n = \boxed{3(4)^n - 2}$ is the solved recurrence relation.

(iii) $a_{n+1} = 2.n.a_n$; $n > 1$, $a_1 = 1$

$a_1 = 1$

$a_{1+1} = 2.1.1 = 2$

$a_3 = 2.2.2 = 8$

$a_4 = 2.3.8 = 48$

$a_5 = 2.4.48 = 384$

$a_6 = 2.5.384 = 3840$

Therefore, the final result $= \{1, 2, 8, 48, 384..\}$

The Required Formula for the above geometric Progression (in terms of Product Notation):

$$f(n) = (n-1)! \times 2^{(n-1)}$$

$$\text{General Term} = \left[\prod_{i=1}^{n} (n-1)\right] \times 2^{(n-1)}.$$