2209B
Assignment - 2

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Exercise 1 : Using Natural Deduction

1. $(p \vee q), \neg q \vdash p$

1	$(p \vee q)$	Premise
2	$\neg q$	Premise
3	p	Assumption [$\vee e_1$ 1]
4	q	Assumption [$\vee e_2$ 1]
5	$\perp$	$\perp i$ 2,4
6	p	$\perp e$ 3,5
7	p	$\vee e$ 1,3,4-6

2. $\vdash ((\neg p \vee q) \longrightarrow (p \longrightarrow q))$

1	$(\neg p \vee q)$	Assumption
2	p	Assumption
3	q	$\vee e_2$ 1
4	$(p \longrightarrow q)$	$\longrightarrow i$ 2 , 3
5	$(\neg p \vee q) \rightarrow (p \longrightarrow q)$	$\longrightarrow i$ 1 , 4

3. $\vdash (\neg p \rightarrow (p \rightarrow (p \rightarrow q)))$

1	$\neg p$	Assumption
2	$p \rightarrow (p \rightarrow q)$	Assumption $(\rightarrow_i 1)$
3	p	Assumption $(\rightarrow_{e_1} 2)$
4	$(p \rightarrow q)$	$\rightarrow_e$ 2, 3
5	q	Modus Ponens 3, 4
6	$\neg p \rightarrow (p \rightarrow (p \rightarrow q))$	$\rightarrow_i$ 2-5

4. $\vdash ((p \wedge q) \rightarrow p)$

1	$(p \wedge q)$	Assumption
2	p	$\wedge_{e_1}$ 1
3	$(p \wedge q) \rightarrow p$	$\rightarrow_i$ 1, 2

5. $(p \rightarrow q), (r \rightarrow s) \vdash ((p \wedge r) \rightarrow (q \wedge s))$

1	$(p \rightarrow q)$	Premise
2	$(r \rightarrow s)$	Premise
3	$(p \wedge r)$	Assumption
4	p	$\wedge_{e_1}$ 3
5	r	$\wedge_{e_2}$ 3
6	q	$\rightarrow_e$ 1, 4
7	s	$\rightarrow_e$ 2, 5
8	$(q \wedge s)$	$\wedge_i$ 6, 7
9	$(p \wedge r) \rightarrow (q \wedge s)$	$\rightarrow_i$ 3-8

Exercise 2 :

To prove : $((\phi_1 \land \phi_2) \land \phi_3) \dashv\vdash (\phi_1 \land (\phi_2 \land \phi_3))$

Associativity

Using the process of Natural Deduction to show —

$$((\phi_1 \land \phi_2) \land \phi_3) \vdash (\phi_1 \land (\phi_2 \land \phi_3))$$

1	$(\phi_1 \land \phi_2) \land \phi_3$	Premise
2	$(\phi_1 \land \phi_2)$	$\land e_1$ 1
3	ϕ_3	$\land e_2$ 1
4	ϕ_1	$\land e_1$ 2
5	ϕ_2	$\land e_2$ 2
6	$(\phi_2 \land \phi_3)$	$\land i$ 5 , 3
7	$\phi_1 \land (\phi_2 \land \phi_3)$	$\land i$ 4 , 6

Using the process o Natural Deduction to show —

$$(\phi_1 \land (\phi_2 \land \phi_3)) \vdash ((\phi_1 \land \phi_2) \land \phi_3)$$

1	$\phi_1 \land (\phi_2 \land \phi_3)$	Premise
2	$(\phi_2 \land \phi_3)$	$\land e_2$ 1
3	ϕ_1	$\land e_1$ 1
4	ϕ_2	$\land e_1$ 2
5	ϕ_3	$\land e_2$ 2
6	$(\phi_1 \land \phi_2)$	$\land i$ 3 , 4
7	$(\phi_1 \land \phi_2) \land \phi_3$	$\land i$ 5 , 6

Therefore, the two formulas are provably equivalent , denoted by $\dashv\vdash$, since we have shown that if $\phi = ((\phi_1 \land \phi_2) \land \phi_3)$ and $\psi = (\phi_1 \land (\phi_2 \land \phi_3))$, then $\phi \vdash \psi$ and $\psi \vdash \phi$.

Hence, $((\phi_1 \land \phi_2) \land \phi_3) \dashv\vdash (\phi_1 \land (\phi_2 \land \phi_3))$. { Associativity }

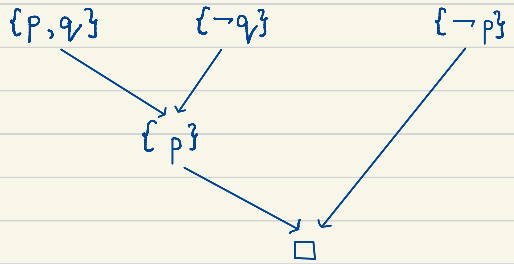
Exercise 3 :

1. ① $(p \vee q) , \neg q \vdash p$

To show $\phi_1 \wedge \phi_2 \wedge \dots \wedge \phi_n \wedge \neg \psi$ is a contradiction

Converting to CNF : $(p \vee q) \wedge \neg q \wedge \neg p$

$F = \{ \{p \vee q\}, \{\neg q\}, \{\neg p\} \}$



Since, the resolution rule generates a $\perp$ ($\square$: contradiction), the entailment holds.

② $\vdash ((\neg p \vee q) \longrightarrow (p \longrightarrow q))$

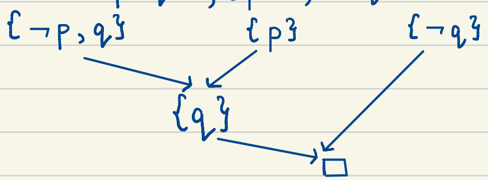
To show $\phi_1 \wedge \phi_2 \wedge \dots \wedge \phi_n \wedge \neg \psi$ is a contradiction

Converting to CNF : since $(a \rightarrow b) = (\neg a \vee b)$ and $\neg(a \vee b) = (\neg a \wedge \neg b)$

: $\neg(\neg(\neg p \vee q) \vee (\neg p \vee q)$

: $((\neg p \vee q) \wedge p \wedge \neg q)$

$F = \{ \{\neg p \vee q\}, \{p\}, \{\neg q\} \}$



Since, the resolution rule generates a $\perp$ ($\square$: contradiction), the entailment holds.

③ $\vdash (\neg p \rightarrow (p \rightarrow (p \rightarrow q)))$

To show $\phi_1 \wedge \phi_2 \wedge \dots \wedge \phi_n$ is a contradiction

Converting to CNF : Since $(a \rightarrow b) = (\neg a \vee b)$ and

$\neg(a \vee b) = (\neg a \wedge \neg b)$

$\therefore \neg(\neg(\neg p) \vee (\neg p \vee (\neg p \vee q)))$

$\therefore (\neg p \wedge (p \wedge (p \wedge \neg q)))$

$\therefore (\neg p \wedge p \wedge \neg q)$

$F = \{\{\neg p\}, \{p\}, \{\neg q\}\}$.

$$\begin{tikzcd}
\{\neg p\} \arrow[dr] & & \{p\} \arrow[dl] & \{\neg q\} \\
& \square & &
\end{tikzcd}$$

Some clauses like $\{\neg q\}$ do not participate in the derivation, even then, the resolution rule generates a $\perp$ ($\square$: contradiction), and ~~is~~ therefore holds the entailment.

4) $\vdash ((p \wedge q) \rightarrow p)$

To show $\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_n$ is a contradiction

Converting to CNF : $\neg(\neg(p \wedge q) \vee p)$

$: ((p \wedge q) \wedge \neg p)$

$: (p \wedge q \wedge \neg p)$

$F = \{ \{p\}, \{q\}, \{\neg p\} \}.$

$\{p\} \qquad \{q\} \qquad \{\neg p\}$

$\square$

Some clauses like $\{q\}$ do not participate in the derivation; even then, the resolution generates a $\perp$ ($\square$: contradiction), and therefore the entailment holds.

⑤ $(p \to q), (r \to s) \vdash ((p \wedge r) \to (q \wedge s))$

To show $\phi_1 \wedge \phi_2 \wedge \dots \wedge \phi_n \wedge \neg\psi$ is a contradiction

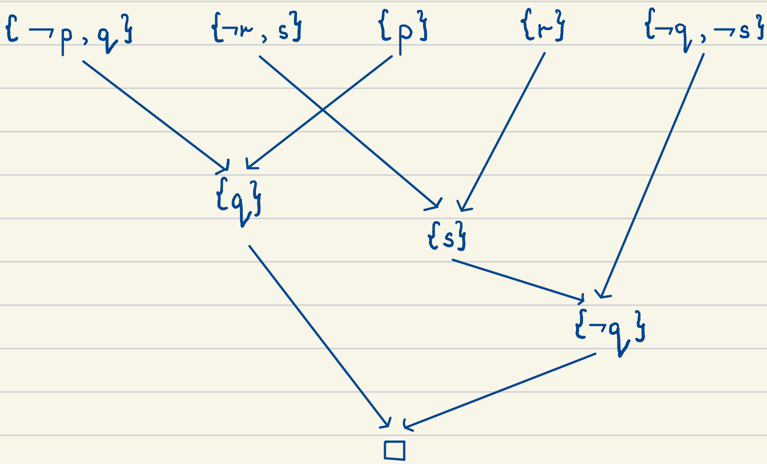
Converting to CNF $\Rightarrow$ since $(a \to b) = (\neg a \vee b)$ and $\neg(a \vee b) = (\neg a \wedge \neg b)$

$: (\neg p \vee q) \wedge (\neg r \vee s) \wedge \neg(\neg(p \wedge r) \vee (q \wedge s))$

$: (\neg p \vee q) \wedge (\neg r \vee s) \wedge ((p \wedge r) \wedge \neg(q \wedge s))$

$: (\neg p \vee q) \wedge (\neg r \vee s) \wedge (p \wedge r \wedge (\neg q \vee \neg s))$

$F = \{\{\neg p \vee q\}, \{\neg r \vee s\}, \{p\}, \{r\}, \{\neg q \vee \neg s\}\}$



Since, the resolution rule generates a $\perp$ ($\square$: contradiction), the entailment holds.

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⑥ $((\phi_1 \wedge \phi_2) \wedge \phi_3) \vdash (\phi_1 \wedge (\phi_2 \wedge \phi_3))$

To show $\phi_1 \wedge \phi_2 \wedge \dots \wedge \phi_n$ is a contradiction

Converting to CNF : $\cancel{(\phi_1 \wedge \phi_2) \wedge \phi_3}$

$: ((\phi_1 \wedge \phi_2) \wedge \phi_3) \wedge \neg(\phi_1 \wedge (\phi_2 \wedge \phi_3))$

$: ((\phi_1 \wedge \phi_2) \wedge \phi_3) \wedge (\neg\phi_1 \vee (\neg\phi_2 \vee \neg\phi_3))$

$F = \{\{\phi_1\}, \{\phi_2\}, \{\phi_3\}, \{\neg\phi_1, \neg\phi_2, \neg\phi_3\}\}$

$\{\phi_1\}$ $\{\phi_2\}$ $\{\phi_3\}$ $\{\neg\phi_1, \neg\phi_2, \neg\phi_3\}$

$\{\neg\phi_2, \neg\phi_3\}$

$\{\neg\phi_2\}$

$\square$

Since, the resolution rule generates a $\perp$ ($\square$: contradiction), the entailment holds.

⑦ $(\phi_1 \wedge (\phi_2 \wedge \phi_3)) \vdash ((\phi_1 \wedge \phi_2) \wedge \phi_3)$

To show $\phi_1 \wedge \phi_2 \wedge \dots \wedge \phi_n$ is a contradiction

Converting to CNF : $(\phi_1 \wedge (\phi_2 \wedge \phi_3)) \wedge \neg((\phi_1 \wedge \phi_2) \wedge \phi_3)$

$\therefore\ \phi_1 \wedge (\phi_2 \wedge \phi_3) \wedge ((\neg\phi_1 \vee \neg\phi_2) \vee \neg\phi_3)$

$F = \{\{\phi_1\}, \{\phi_2\}, \{\phi_3\}, \{\neg\phi_1, \neg\phi_2, \neg\phi_3\}\}$

$\{\phi_1\}$ $\{\phi_2\}$ $\{\phi_3\}$ $\{\neg\phi_1, \neg\phi_2, \neg\phi_3\}$

$\{\neg\phi_1, \neg\phi_2\}$

$\{\neg\phi_1\}$

$\square$

Since, the resolution rule generates a $\perp$ ($\square$: contradiction), the entailment holds.

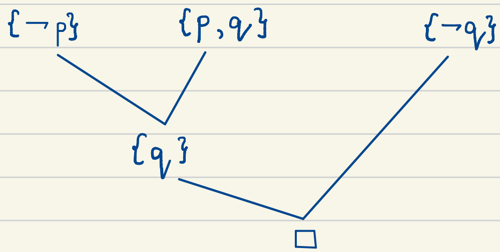
Exercise 4 :

1. $\neg p , (p \vee q) \vdash q$

To show $\phi_1 \wedge \phi_2 \wedge \dots \wedge \phi_n \wedge \neg \psi$ is a contradiction

Converting to CNF : $\neg p \wedge (p \vee q) \wedge \neg q$

$F = \{ \{\neg p\} , \{p \vee q\} , \{\neg q\} \}$



Since, the resolution rule generates a $\perp$ ($\square$: contradiction), the entailment holds and the statement is VALID.

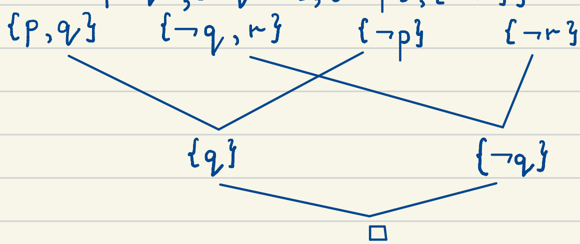
2. $(p \vee q), (\neg q \vee r) \vdash (p \vee r)$

To show $\phi_1 \wedge \phi_2 \wedge \dots \wedge \phi_n \wedge \neg \psi$ is a contradiction

Converting to CNF : $(p \vee q) \wedge (\neg q \vee r) \wedge \neg (p \vee r)$

: $(p \vee q) \wedge (\neg q \vee r) \wedge \neg p \wedge \neg r$

$F = \{ \{p \vee q\}, \{\neg q \vee r\}, \{\neg p\}, \{\neg r\} \}$



Since, the resolution rule generates a $\perp$ ($\square$: contradiction), the entailment holds and the statement is VALID.

3. $(p \to q), (s \to t) \vdash ((p \vee s) \to (q \wedge t))$

To show $\emptyset_1 \wedge \emptyset_2 \wedge \ldots \wedge \emptyset_n \wedge \neg\psi$ is a contradiction

Converting to CNF : since $(a \to b) = (\neg a \vee b)$ and $\neg(a \vee b) = (\neg a \wedge \neg b)$

: $(\neg p \vee q) \wedge (\neg s \vee t) \wedge \neg(\neg(p \vee s) \vee (q \wedge t))$

: $(\neg p \vee q) \wedge (\neg s \vee t) \wedge (p \vee s) \wedge (\neg q \vee \neg t)$

$F = \{\{\neg p \vee q\}, \{\neg s \vee t\}, \{p \vee s\}, \{\neg q \vee \neg t\}\}$

$\{\neg p, q\}$ $\quad \{\neg s, t\}$ $\quad \{p, s\}$ $\quad \{\neg q, \neg t\}$

$\{s, q\}$ $\quad \{\neg s, \neg q\}$

$\square$

Since, the resolution rule generates a $\perp$ ($\square$: contradiction) the entailment holds and the statement is VALID.

Exercise 5 :

To prove : $((p \land q) \to r), (p \to s), ((s \land u) \to t), ((t \land r) \to \bot), p \vdash (\neg q \lor \neg u)$

Since, there are both FACTS (A horn clause with only 1 positive literal) and CONSTRAINTS (A horn clause without a positive literal), Forward chaining is possible.

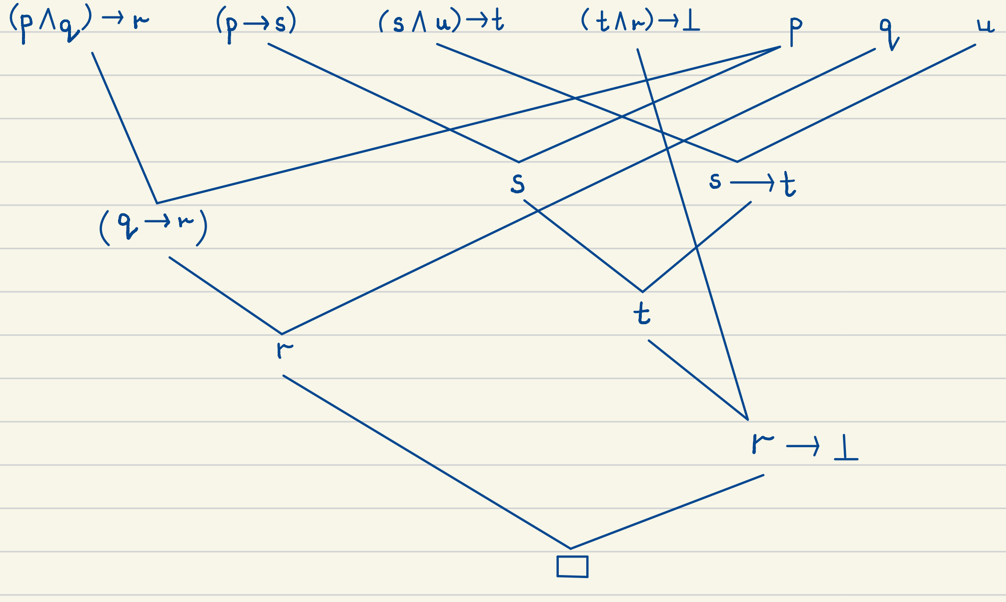
To show $\phi_1 \land \phi_2 \land \dots \land \phi_n \land \neg \psi$ is a contradiction

$$\vdash \neg(\neg q \lor \neg u) = \vdash (q \land u) = \vdash q \land u$$

FACTS : $\{p\}, \{q\}, \{u\}$

CONSTRAINTS : $\{\neg t, \neg r\}$

RULES : $\{\neg p, \neg q\}, \{\neg p\}, \{\neg s, \neg u\}$



Since, the resolution for horn formulas (forward chaining) generates a $\bot$ ($\Box$: contradiction), the entailment holds.