



Competitive  
Programming and  
Mathematics  
Society

# **Mathematics Workshop**

## Geometry Fundamentals

**CPMsoc Maths**

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- SCAN THE ATTENDANCE FORM

# Welcome



We would like to thank everyone for coming, even if its just for the pizza :D  
We are looking forward to expanding our activities from here onwards, if you have any ideas for what you think we can do to satisfy your interests, please let us know!!

# Attendance form



# Clarifications and Assumptions

- This will cover Geometry on the Euclidean Plane. We will not use the coordinate nor complex plane.
- We will mainly discuss angle relations. This tends to be easier for beginners.
- As the slides have said, this is a beginner workshop - do not expect to solve hard questions just from this. Skill come from practice.
- We assume you are familiar with the following:
  - Angles of a triangle add to  $180^\circ$ .
  - Equal angles that arise from parallel lines.
  - Vertically opposite angles.
  - Similar and congruent triangles.

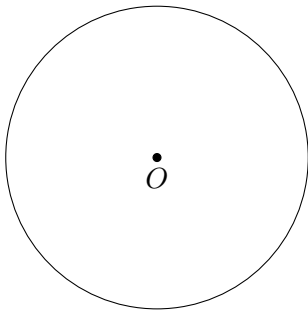
# Notation



- $AB$  refers to the line passing through points  $A, B$ . This is used interchangeably with  $\overline{AB}$  as the *length* of the line segment  $AB$ . What it will refer to is based on context, but  $\overline{AB}$  and  $|AB|$  may be used to denote the line and length respectively to resolve ambiguity.
- Unless otherwise mentioned,  $\angle BAC$  will refer to the non-reflex angle (the angle less than  $180^\circ$ ).
- $A - B - C$  means that points  $A, B, C$  are collinear.
- $(ABC)$  refers to the circle passing through points  $A, B, C$ .  $(AB)$  refers to the circle with diameter  $AB$ .
- $AB \cap CD = E$  means that  $E$  is the intersection of lines  $AB$  and  $CD$ .  
 $AB \cap (XYZ) = \{P, Q\}$  means that line  $AB$  and circle  $(XYZ)$  intersect at points  $P$  and  $Q$ .
- The foot of the perpendicular of  $A$  to  $BC$  is the point  $D$  on line  $BC$  such that  $AD \perp BC$ .

# What is a circle

A circle is the set of points that are a fixed distance from a certain point (called the "centre").



We usually denote the centre of circles as  $O$ .

# Inscribed Angle Theorem

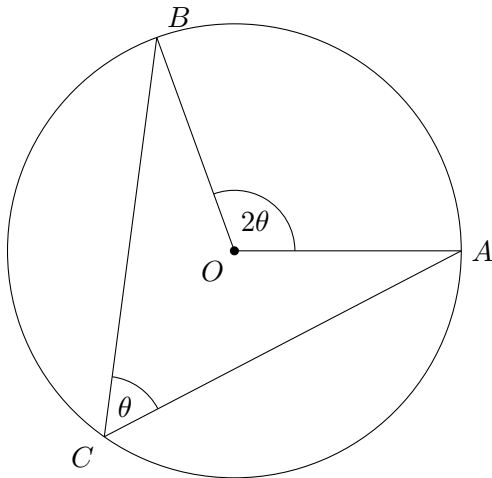


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## Theorem: (Inscribed Angle Theorem)

Let  $A, B, C$  be points on a circle with centre  $O$ . Then  $2\angle ACB = \angle AOB$





# Inscribed Angle Theorem

Let  $\angle ACO = \alpha$  and  $\angle BCO = \beta$ .

Then, we find that  $\angle CAO = \alpha$  and  $\angle BCO = \beta$  as  $\triangle AOC$  and  $\triangle BOC$  are isosceles triangles. Thus

$$\begin{aligned}\angle AOB &= 360^\circ - \angle AOC - \angle BOC \\ &= 360^\circ - (180^\circ - 2\alpha) - (180^\circ - 2\beta) \\ &= 360^\circ - 180^\circ + 2\alpha - 180^\circ + 2\beta \\ &= 2\alpha + 2\beta \\ &= 2\angle ACB.\end{aligned}$$

since  $\angle ABC = \angle ACO + \angle BCO = \alpha + \beta$ .

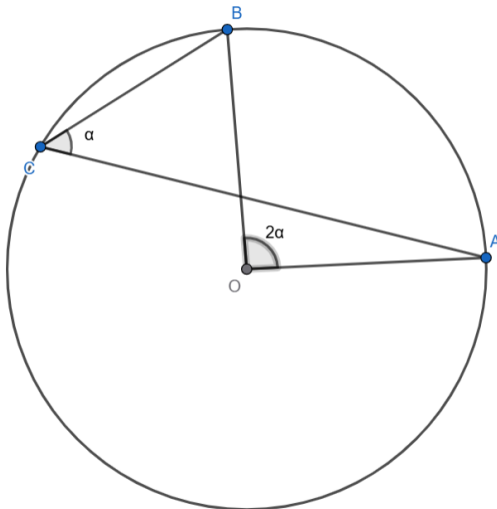
# Inscribed Angle Theorem



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What are we missing?

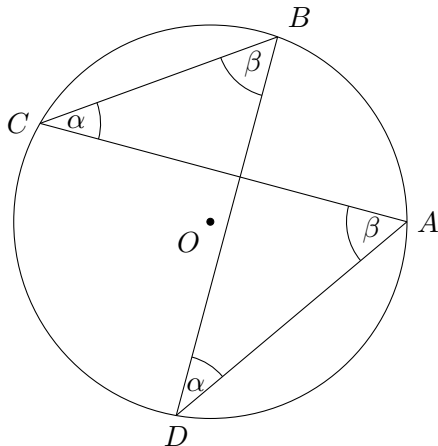


# Bowtie Theorem



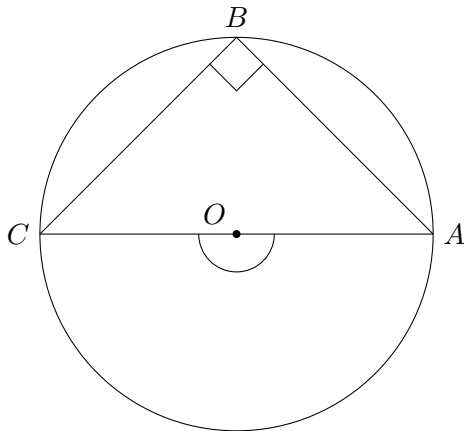
## Theorem: (Bowtie Theorem)

Let points  $A, B, C, D$  lie on a circle such that  $C$  and  $D$  are on the same side of line  $AB$ . Then  $\angle ACB = \angle ADB$ .



# Inscribing Right Angles

What happens if  $\angle ACB = 90^\circ$ ?

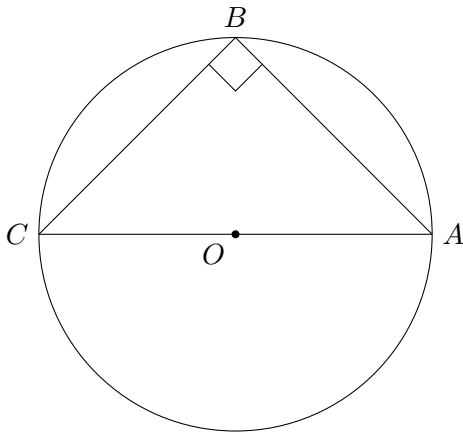


# Thales' Theorem



## Theorem: (Thales' Theorem)

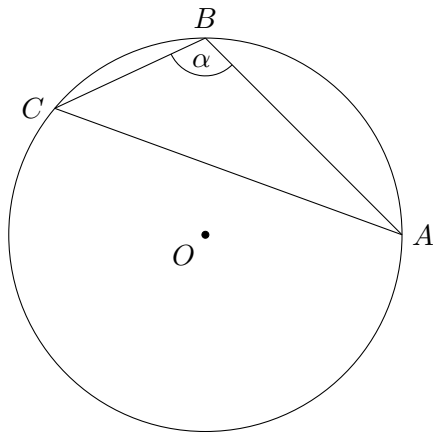
Let  $AB$  be a diameter of a circle  $\Gamma$ . Then for any point  $C$  that lies on the circle,  $\angle ACB = 90^\circ$ .



# Inscribing Obtuse Angles



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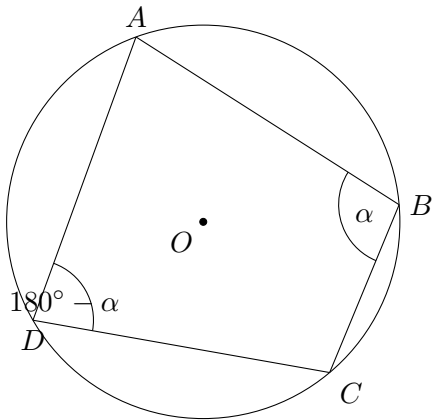


# Angles on Opposite Arcs



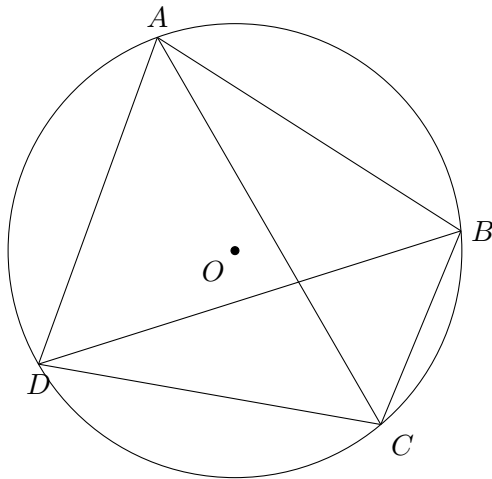
## Lemma:

Let  $A, B, C, D$  be points on a circle where  $C$  and  $D$  are on opposite sides of line  $AB$ .  
Then  $\angle ACB + \angle ADC = 180^\circ$ .



# Cyclic Quadrilaterals

A cyclic quadrilateral is a quadrilateral whose vertices all lie on a circle.





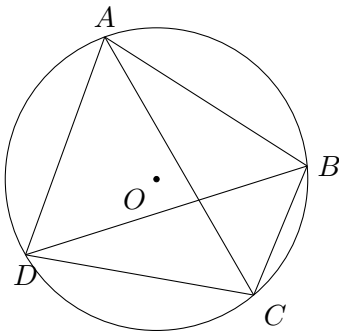
# Cyclic Quadrilaterals



## Lemma: (Cyclic Quads)

Let  $ABCD$  be a quadrilateral, then the following are equivalent:

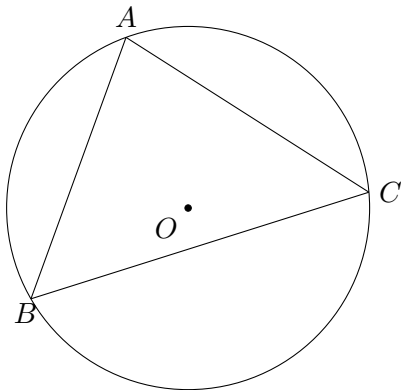
- $\angle ACB = \angle ADB$
- $\angle ABC + \angle ADC = 180^\circ$
- $ABCD$  is a cyclic quadrilateral.



# What about cyclic triangles?



Do the vertices of a triangle always lie on a circle?



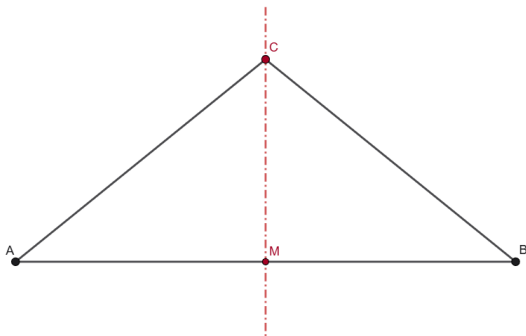
Yes they do!

# What about cyclic triangles?



We can look to find the centre of this circle for each triangle.

The perpendicular bisector of a line segment  $AB$  as the line  $\ell$  that passes through the midpoint of  $AB$ , and is perpendicular to  $AB$ .



We can see  $\triangle ACM \equiv \triangle BCM$  and so  $AC = BC$ .

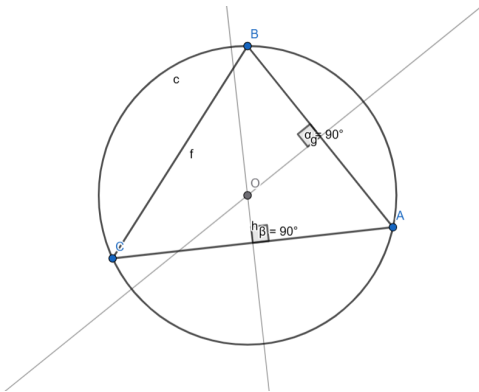
# A Centre of a Triangle



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Let the perpendicular bisectors of  $AB$  and  $AC$  meet at some point  $P$ .



Then we can see  $AP = BP$ , and  $AP = CP$  and so the circle with centre  $P$  with radius  $r = AP$  would pass through our 3 vertices.

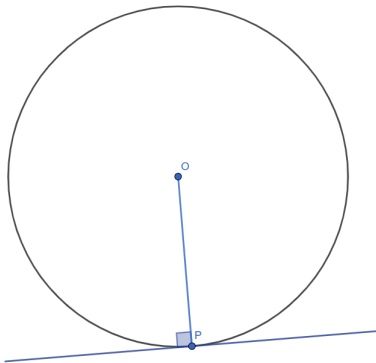
# Tangents



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A tangent  $\ell$  to a circle  $\Gamma$  is a line that intersects it at only one point. We can alternatively say that  $\ell$  is tangent to  $\Gamma$ .

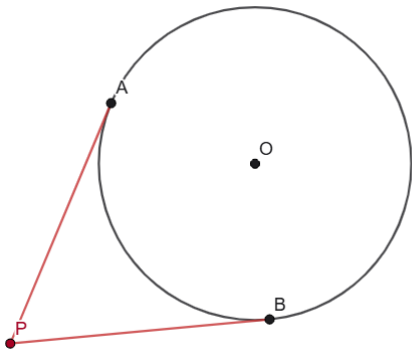


If they intersect at a point  $P$ , then  $OP$  is perpendicular to line  $\ell$ .

# Ice Cream Cone Lemma

## Lemma: (Ice Cream Cone)

Let  $A$  and  $B$  be points on a circle. Then if  $PA$  and  $PB$  are tangents, we get that  $|PA| = |PB|$ .



# Ice Cream Cone Lemma

Let  $O$  be the centre of the circle. We can see that for triangles  $\triangle AOP$  and  $\triangle BOP$  :

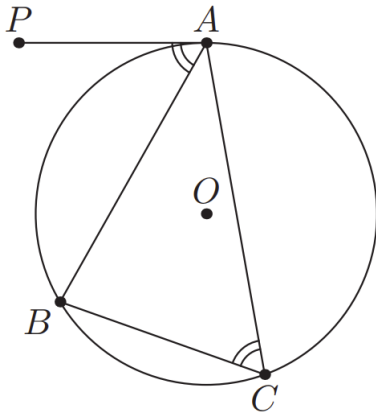
- $OP$  is shared.
- $AO = BO$  as they are both radii
- $\angle OAP = \angle OBP = 90^\circ$

and so they are congruent from *RHS*. Thus, we find that  $AP = BP$  as they are corresponding sides.

# Alternate Segment Theorem

## Theorem: (Alternate Segment Theorem)

Let  $A, B, C$  be points on a circle  $\Gamma$ , and  $AP$  be a tangent. Then  $\angle PAB = \angle ACB$ .





# Alternate Segment Theorem



We find that  $\angle BAO = 90^\circ - \angle PAB$ . Thus, we find that

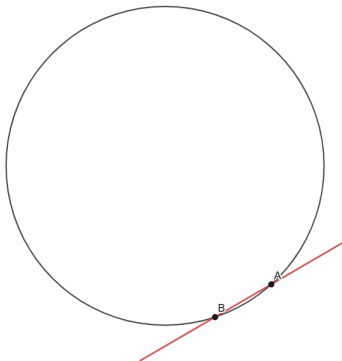
$$\begin{aligned}\angle ACB &= \frac{1}{2}\angle AOB \\ &= \frac{1}{2}(180^\circ - 2\angle BAO) \\ &= \frac{1}{2}(180^\circ - 2(90^\circ - \angle PAB)) \\ &= \frac{1}{2}(180^\circ - 180^\circ + 2\angle PAB) \\ &= \angle PAB.\end{aligned}$$

as desired.

# What does $AA$ mean?

Sometimes you will see  $AA$  being used in a question or solution, and it will refer to the tangent at  $A$ . Why is this?

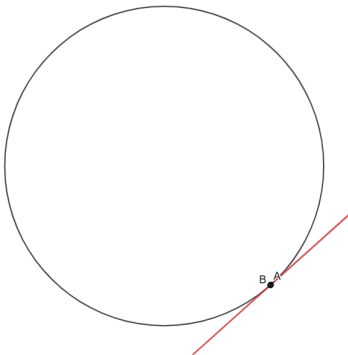
Essentially, this is because the tangent is the line  $AB$  for points  $A, B$  on a circle, where  $B$  limits to point  $A$ .



# What does $AA$ mean?

Sometimes you will see  $AA$  being used in a question or solution, and it will refer to the tangent at  $A$ . Why is this?

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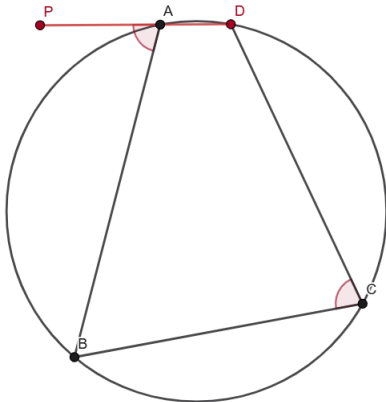
# Alternate Segment Theorem



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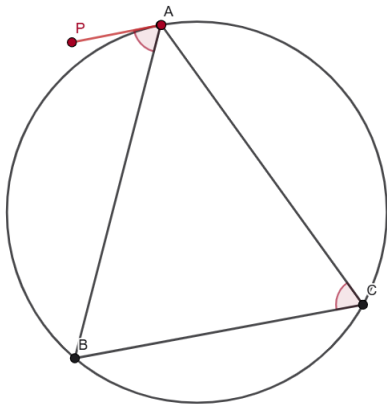


Another way to think about the alternate segment theorem is through a degenerate "cyclic quad", with two of the vertices being the same.



# Alternate Segment Theorem

Another way to think about the alternate segment theorem is through a degenerate "cyclic quad", with two of the vertices being the same.

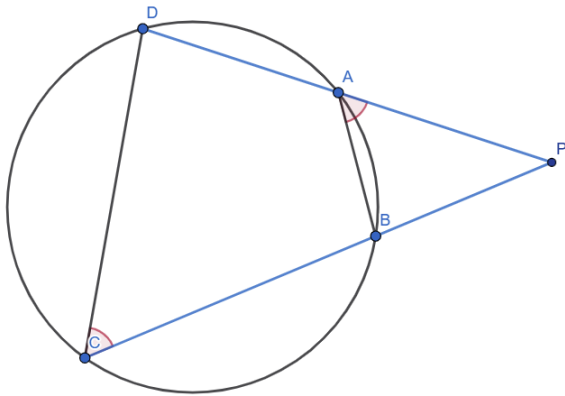


# Power of a Point Pt. 1



## Lemma: (Power of A Point)

Let  $ABCD$  be a cyclic quadrilateral, and lines  $AD \cap BC = P$ . Then  $PA \times PD = PB \times PC$ .



# Power of a Point Pt. 1



As  $ABCD$  is cyclic, we have:

$$\blacksquare \angle PAB = 180^\circ - \angle DAB = \angle PCD$$

$$\blacksquare \angle PBA = 180^\circ - \angle ABC = \angle PDC$$

and so we can see  $\triangle PAB \simeq \triangle PCD$  from AA similarity. Thus, we get that

$$\frac{PA}{PB} = \frac{PC}{PD}$$
$$PA \times PD = PB \times PC$$

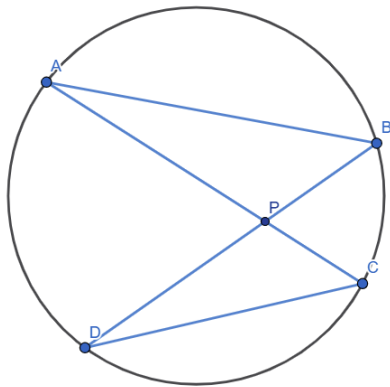
as desired.

# Power of a Point Pt. 2



## Lemma: (Power of A Point)

Let  $ABCD$  be a cyclic quadrilateral, and lines  $AC \cap BD = P$ . Then  $PA \times PC = PB \times PD$ .



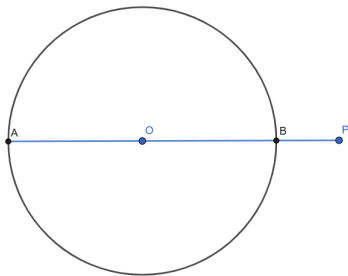


# Power of a Point

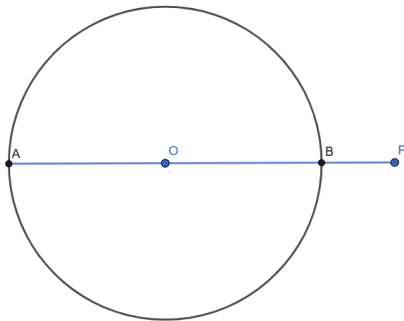
We can see that no matter which line we pick through our point  $P$ , the product will be fixed. We define the power of a point with respect to a circle to be this value if  $P$  is outside the circle, and the negative of this value if  $P$  is inside.

What is this value though?

We can pick the line through  $P$  and the centre  $O$  of the circle and let it intersect at points  $A$  and  $B$ .



# Power of a Point

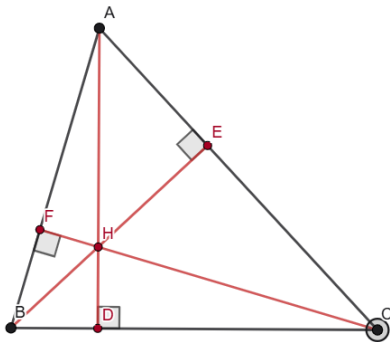


Now we see that if  $r$  is the radius of the circle,

$$\begin{aligned}PA \times PB &= (PO + OA) \times (PO - OB) \\&= (PO + r) \times (PO - r) \\&= PO^2 - r^2\end{aligned}$$

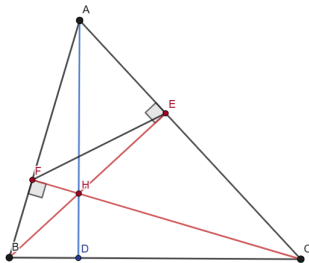
# The Orthocentre

We define the orthocentre as the intersection of the altitudes of the triangle. How do we know it exists?



# The Orthocentre

Let altitudes  $CF$  and  $BE$  intersect at  $H$ , and  $AH$  meet  $BC$  at  $D$ . We want to show that  $AD$  is an altitude. We can see that since  $\angle BFC = \angle BEC = 90^\circ$  and  $\angle AFH + \angle AEH = 180^\circ$ ,  $AFHE$  and  $BFEC$  are cyclic quadrilaterals.



Thus from bowtie,

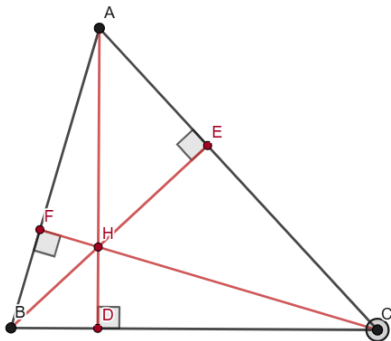
$$\angle HAE = \angle HFE = \angle CFE = \angle CBE$$

and so  $AEDB$  is also a cyclic quad. We thus get  $\angle ADB = \angle AEH = 90^\circ$  as desired.



# Lots of Cyclic Quads

We can see that there are a lot of cyclic quadrilaterals in our diagram. For example,  $AEHF$  and  $BFEC$  are (from before), but also  $BFHD$ ,  $CDHE$ ,  $AFDC$ , and  $AEDB$  are.



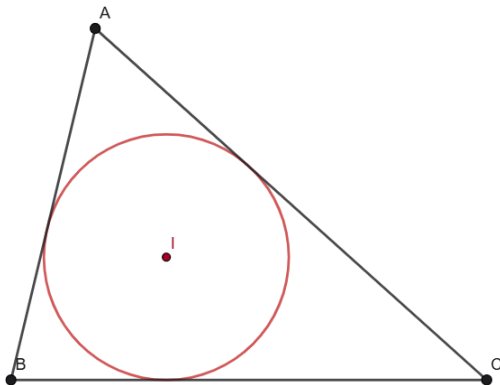
# The Incentre



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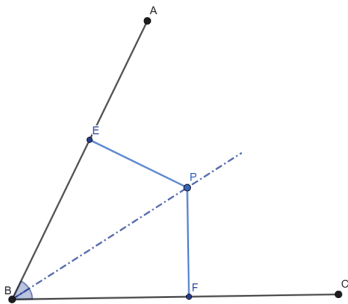


We know that there's a circle that passes through the vertices of any triangle.  
Does there exist a circle which is tangent to all of the sides of a triangle?



# The Incentre

The *angle bisector* of an angle  $\angle ABC$  is a line that splits that angle into two equal angles. Let  $P$  be a point on this line, and  $E, F$  be the feet of the perpendicular from  $P$  to  $AB, CB$

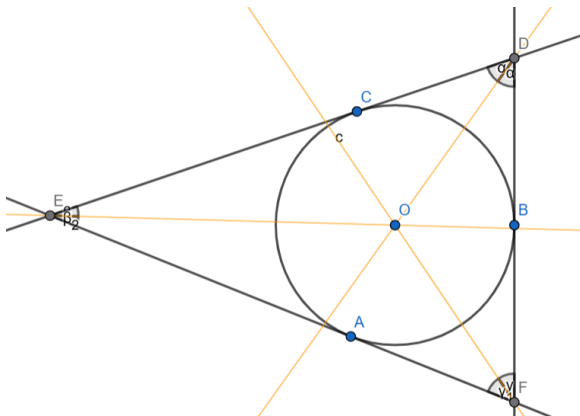


Then we can see that  $\triangle BEP \equiv \triangle BFP$  from *AAS* congruency.



# The Incentre

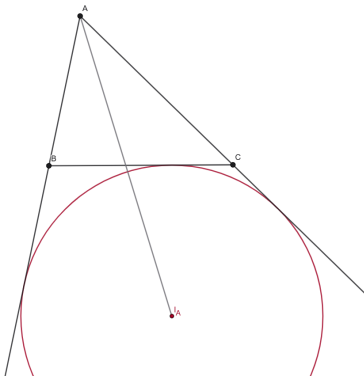
Let the angle bisectors of  $\angle BAC$  and  $\angle ABC$  of triangle  $ABC$  meet at a point  $I$ . Then consider the feet of the perpendiculars  $D, E, F$  from  $I$  to  $BC, AC$ , and  $AB$  respectively. We then see that  $ID = IE$  and  $ID = IF$ , and so the circle with centre  $I$  and radius  $r = ID$  is going to be tangent to all 3 sides.



# The Excentre



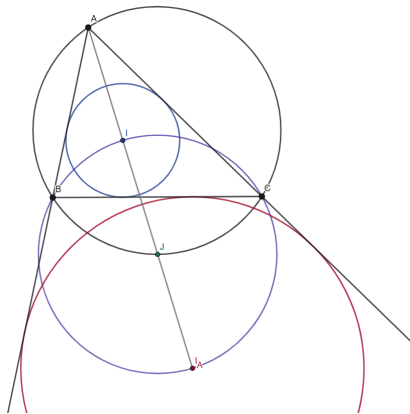
The  $A$ -excentre of a triangle  $ABC$  is the circle that is tangent to side  $BC$ , ray  $AB$  at a point beyond  $B$  and  $AC$  at a point beyond  $C$ . We can see it is the intersection of the external angle bisectors of  $\angle ABC$  and  $\angle ACB$  along with the internal angle bisector of  $\angle BAC$ .



# Incentre/Excentre Lemma

## Lemma: (Incentre/Excentre Lemma)

Let  $I$  and  $I_A$  are the incentre and  $A$ -excentre of triangle  $ABC$ . Then  $I, I_A, B, C$  lie on a circle with diameter  $II_A$ , with its centre on  $(ABC)$ .



# Attendance form :D



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# Further events



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Please join us for:

- Rookie Code Rumble (Wed 11th June)
- Chicken Contest Debrief (Thur June 19th)