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CHAPTER-2
POLYNOMIALS

EXERCISE 2.1

1. The graphs of $y = p(x)$ are given in Fig. 2.10 below, for some polynomials $p(x)$. Find the number of zeroes of $p(x)$, in each case.

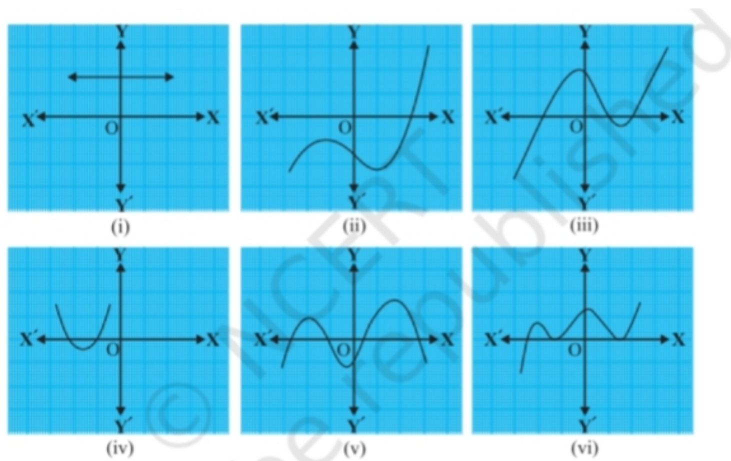

Fig. 2.10

2. **2.3 Relationship between Zeroes and Coefficients of a Polynomial**

   You have already seen that zero of a linear polynomial $ax + b$ is $-\frac{b}{a}$. We will now try to answer the question raised in Section 2.1 regarding the relationship between zeroes and coefficients of a quadratic polynomial.

   For this, let us take a quadratic polynomial, say $p(x) = 2x^2 - 8x + 6$. In Class IX, you have learnt how to factorise quadratic polynomials by splitting the middle term. So, here we need to split the middle term '$-8x$' as a sum of two terms, whose product is $6 \times 2x^2 = 12x^2$. So, we write:

$$2x^2-8x+6 = 2x^2-6x-2x+6 = 2x(x-3)-2(x-3) = (2x-2)(x-3) = 2(x-1)(x-3)$$

So, the value of $p(x) = 2x^2 - 8x + 6$ is zero when $x - 1 = 0$ or $x - 3 = 0$, that is, when $x = 1$ or $x = 3$. So, the zeroes of $2x^2 - 8x + 6$ are 1 and 3.

Observe that:

Sum of its zeroes $= 1 + 3 = 4 = \frac{-(-8)}{2} = \frac{-b}{a}$

Product of its zeroes $= 1 \times 3 = 3 = \frac{6}{2} = \frac{c}{a}$

Let us take one more quadratic polynomial, say, $p(x) = 3x^2 + 5x - 2$. By the method of splitting the middle term:

$$\begin{aligned} 3x^2 + 5x - 2 &= 3x^2 + 6x - x - 2 \\ &= 3x(x+2) - 1(x+2) \\ &= (3x-1)(x+2) \end{aligned}$$

Hence, the value of $3x^2 + 5x - 2$ is zero when either $3x - 1 = 0$ or $x + 2 = 0$, i.e., when $x = \dfrac{1}{3}$ or $x = -2$.

Observe that:

Sum of its zeroes $= \dfrac{1}{3} + (-2) = -\dfrac{5}{3} = \frac{-b}{a}$

Product of its zeroes $= \dfrac{1}{3} \times (-2) = -\dfrac{2}{3} = \frac{c}{a}$

In general, if α and β are the zeroes of the quadratic polynomial $p(x) = ax^2 + bx + c$, $a \neq 0$, then $x - \alpha$ and $x - \beta$ are the factors of $p(x)$. Therefore,

$$\begin{aligned} ax^2 + bx + c &= k(x-\alpha)(x-\beta) \\ &= k\left[x^2 - (\alpha+\beta)x + \alpha\beta\right] \\ &= kx^2 - k(\alpha+\beta)x + k\alpha\beta \end{aligned}$$

Comparing the coefficients of x^2, x, and constant terms on both sides, we get:

$$a = k, \quad b = -k(\alpha+\beta), \quad c = k\alpha\beta$$

This gives:

$\alpha + \beta = \frac{-b}{a}$

$\alpha\beta = \frac{c}{a}$

where α and β are Greek letters pronounced as 'alpha' and 'beta' respectively. We will use later one more letter 'γ', pronounced as 'gamma'.

So, the zeroes are $x = \frac{1}{3}$ and $x = -2$.

Observe that:

Sum of zeroes $= \frac{-5}{3}$

Product of zeroes $= \frac{-2}{3}$

In general, for $p(x) = ax^2 + bx + c$, zeroes α and β:

$\alpha + \beta = \frac{-b}{a}$

$\alpha\beta = \frac{c}{a}$

Example 2

Find the zeroes of $x^2 + 7x + 10$, and verify the relationship between zeroes and coefficients.

Solution: Factorizing:

$$x^2 + 7x + 10 = (x + 2)(x + 5)$$

Zeroes are $x = -2$ and $x = -5$.

Observe that:

Sum of zeroes $= -2 + (-5) = -7 = \frac{-7}{1}$

Product of zeroes $= (-2) \times (-5) = 10 = \frac{10}{1}$

Example 3

Find the zeroes of $x^2 - 3$ and verify the relationship.

Solution: Using identity $a^2 - b^2 = (a - b)(a + b)$:

$$x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$$

Zeroes are $x = \sqrt{3}$ and $x = -\sqrt{3}$.

Observe that:

Sum of zeroes $= \sqrt{3} + (-\sqrt{3}) = 0$

Product of zeroes $= (\sqrt{3})(-\sqrt{3}) = -3$

Example 4

Find a quadratic polynomial, the sum and product of whose zeroes are -3 and 2, respectively.

Solution: Let the quadratic polynomial be $ax^2 + bx + c$, and its zeroes be α and β.

We have:

$\alpha + \beta = -3 = \frac{-b}{a}$

$\alpha\beta = 2 = \frac{c}{a}$

If $a = 1$, then $b = 3$, $c = 2$. So, one quadratic polynomial which fits the given conditions is:

$$x^2 + 3x + 2$$

Any other quadratic polynomial that fits these conditions will be of the form $k(x^2 + 3x + 2)$, where $k \in \mathbb{R}$.

Let us now look at cubic polynomials. Do you think a similar relation holds between the zeroes of a cubic polynomial and its coefficients?

Consider:

$$p(x) = 2x^3 - 5x^2 - 14x + 8$$

You can check that $p(x) = 0$ for:

$$x = 4, \quad x = -2, \quad x = \frac{1}{2}$$

Since a cubic polynomial can have at most three zeroes, these are the zeroes of $p(x)$.

Observe:

Sum of the zeroes: $4 + (-2) + \dfrac{1}{2} = \dfrac{5}{2} = \frac{-(-5)}{2} = \frac{-b}{a}$

Product of the zeroes: $4 \times (-2) \times \dfrac{1}{2} = -4 = \frac{-8}{2} = \frac{-d}{a}$

Sum of the products taken two at a time:

$$(4 \times -2) + (-2 \times \frac{1}{2}) + \left(\frac{1}{2} \times 4\right) = -8 - 1 + 2 = -7 = \frac{c}{a}$$

In general, if α, β, γ are the zeroes of the cubic polynomial $ax^3 + bx^2 + cx + d$, then:

$\alpha + \beta + \gamma = \frac{-b}{a}$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

$\alpha\beta\gamma = \frac{-d}{a}$

Let us consider an example.

Example 5 : Verify that 3, -1, $-\frac{1}{3}$ are the zeroes of the cubic polynomial $p(x) = 3x^3 - 5x^2 - 11x - 3$, and then verify the relationship between the zeroes and the coefficients.

Solution : Comparing the given polynomial with $ax^3 + bx^2 + cx + d$, we get:

$$a = 3, \quad b = -5, \quad c = -11, \quad d = -3$$

Check each root:

$$\begin{aligned}
p(3) &= 3 \times 3^3 - 5 \times 3^2 - 11 \times 3 - 3 = 81 - 45 - 33 - 3 = 0, \\
p(-1) &= 3 \times (-1)^3 - 5 \times (-1)^2 - 11 \times (-1) - 3 = -3 - 5 + 11 - 3 = 0, \\
p\left(-\frac{1}{3}\right) &= 3 \times \left(-\frac{1}{3}\right)^3 - 5 \times \left(-\frac{1}{3}\right)^2 - 11 \times \left(-\frac{1}{3}\right) - 3 \\
&= -\frac{1}{9} - \frac{5}{9} + \frac{11}{3} - 3 \\
&= -\frac{6}{9} + \frac{11}{3} - 3 = -\frac{2}{3} + \frac{11}{3} - 3 \\
&= 3 - 3 = 0
\end{aligned}$$

Therefore, $3, -1$ and $-\frac{1}{3}$ are the zeroes of $3x^3 - 5x^2 - 11x - 3$. So, we take $\alpha = 3$, $\beta = -1$, and $\gamma = -\frac{1}{3}$.

Now, verify the relationships:

$$\alpha + \beta + \gamma = 3 + (-1) + \left(-\frac{1}{3}\right) = \frac{5}{3} = \frac{-b}{a} = \frac{-(-5)}{3}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = 3 \times (-1) + (-1) \times \left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right) \times 3 = -3 + \frac{1}{3} - 1 = -\frac{11}{3} = \frac{c}{a}$$

$$\alpha\beta\gamma = 3 \times (-1) \times \left(-\frac{1}{3}\right) = 1 = \frac{-d}{a}$$

Not from the examination point of view.

EXERCISE 2.2

1. Find the zeroes of the following quadratic polynomials and verify the relationship between the zeroes and the coefficients.
   (i) $x^2 - 2x - 8$
   (ii) $4s^2 - 4s + 1$
   (iii) $6x^2 - 3 - 7x$
   (iv) $4u^2 + 8u$
   (v) $t^2 - 15$
   (vi) $3x^2 - x - 4$
2. Find a quadratic polynomial each with the given numbers as the sum and product of its zeroes respectively.
   (i) $\frac{1}{4}, -1$
   (ii) $\frac{1}{2}, \frac{3}{2}$
   (iii) 0, 5
   (iv) 1, 1
   (v) $\frac{1}{4}, -\frac{1}{4}$
   (vi) 4, 1

2.4 Division Algorithm for Polynomials

You know that a cubic polynomial has at most three zeroes. However, if you are given only one zero, can you find the other two?

For this, let us consider the cubic polynomial $x^3 - 3x^2 - x + 3$.

If we tell you that one of its zeroes is 1, then you know that $x-1$ is a factor of x^3-3x^2-x+3.

So, you can divide $x^3 - 3x^2 - x + 3$ by $x - 1$, as you have learnt in Class IX, to get the quotient $x^2 - 2x - 3$.

Next, you could get the factors of $x^2 - 2x - 3$ by splitting the middle term, as $(x+1)(x-3)$. This would give you:

$$x^3 - 3x^2 - x + 3 = (x - 1)(x^2 - 2x - 3) = (x - 1)(x + 1)(x - 3)$$

So, all the three zeroes of the cubic polynomial are now known to you as 1, -1, and 3.

Let us discuss the method of dividing one polynomial by another in some detail.

Before noting the steps formally, consider an example.

Example 6: Divide $2x^2 + 3x + 1$ by $x + 2$.

Solution: Note that we stop the division process when either the remainder is zero or its degree is less than the degree of the divisor. So, here the quotient is $2x - 1$ and the remainder is 3.

Also,

$$(2x - 1)(x + 2) + 3 = 2x^2 + 3x - 2 + 3 = 2x^2 + 3x + 1$$

i.e.,

$$2x^2 + 3x + 1 = (x + 2)(2x - 1) + 3$$

Therefore,

$$\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}$$

Example 7: Divide $3x^3 + x^2 + 2x + 5$ by $1 + 2x + x^2$.

Solution:

- First, write both polynomials in standard form:
  - Dividend: $3x^3 + x^2 + 2x + 5$ (already in standard form)
  - Divisor: $1 + 2x + x^2 = x^2 + 2x + 1$

Now, we divide using the standard polynomial division method:

1. Divide $3x^3$ by x^2 to get $3x$. Multiply and subtract:

$$3x(x^2 + 2x + 1) = 3x^3 + 6x^2 + 3x$$

   Subtracting from dividend:

$$(3x^3 + x^2 + 2x + 5) - (3x^3 + 6x^2 + 3x) = -5x^2 - x + 5$$

2. Divide $-5x^2$ by x^2 to get -5. Multiply and subtract:

$$-5(x^2 + 2x + 1) = -5x^2 - 10x - 5$$

Subtracting:

$$(-5x^2 - x + 5) - (-5x^2 - 10x - 5) = 9x + 10$$

3. **Step 3:** Since the degree of $9x + 10$ is less than that of $x^2 + 2x + 1$, the division stops.

Thus, the quotient is $3x - 5$ and the remainder is $9x + 10$.

Verification:

$$\begin{aligned}(x^2 + 2x + 1)(3x - 5) + (9x + 10) &= 3x^3 + 6x^2 + 3x - 5x^2 - 10x - 5 + 9x + 10 \\ &= 3x^3 + (6x^2 - 5x^2) + (3x - 10x + 9x) + (-5 + 10) \\ &= 3x^3 + x^2 + 2x + 5\end{aligned}$$

Hence, **Dividend = Divisor × Quotient + Remainder**.

This is similar to Euclid's division algorithm you studied in Chapter 1.

For polynomials, if $p(x)$ and $g(x)$ are polynomials with $g(x) \neq 0$, then:

$$p(x) = g(x) \cdot q(x) + r(x), \quad \text{where} \quad r(x) = 0 \quad \text{or} \quad \deg r(x) < \deg g(x)$$

This is known as the **Division Algorithm for Polynomials**.

Let us now take another example.

Example 8: Divide $3x^2 - x^3 - 3x + 5$ by $x - 1 - x^2$, and verify the division algorithm.

Solution:

- The given polynomials are not in standard form.
- To carry out division, write both the dividend and divisor in decreasing order of their degrees.

$$\text{Dividend} = -x^3 + 3x^2 - 3x + 5, \quad \text{Divisor} = -x^2 + x - 1$$

The division process is performed (as shown visually in the textbook).

We stop here since $\deg(3) = 0 < 2 = \deg(-x^2 + x - 1)$.

- **Quotient:** $x - 2$
- **Remainder:** 3

$$\text{Divisor} \times \text{Quotient} + \text{Remainder} = (-x^2 + x - 1)(x - 2) + 3$$

Compute:

$$\begin{aligned}(-x^2 + x - 1)(x - 2) &= -x^3 + x^2 - x + 2x^2 - 2x + 2 \\ &= -x^3 + 3x^2 - 3x + 2\end{aligned}$$

Adding remainder:

$$-x^3 + 3x^2 - 3x + 2 + 3 = -x^3 + 3x^2 - 3x + 5$$

Thus, **Dividend = Divisor $\times$ Quotient + Remainder** is verified.

Example 9: Find all the zeroes of $2x^4 - 3x^3 - 3x^2 + 6x - 2$, if you know that two of its zeroes are $\sqrt{2}$ and $-\sqrt{2}$.

Solution:

- Given zeroes: $\sqrt{2}$ and $-\sqrt{2}$
- So, $(x - \sqrt{2})(x + \sqrt{2}) = x^2 - 2$ is a factor of the polynomial.
- We divide the polynomial by $x^2 - 2$

1. First term of the quotient:
$$\frac{2x^4}{x^2} = 2x^2$$
2. Second term of the quotient:
$$\frac{-3x^3}{x^2} = -3x$$

3. Third term of the quotient:

$$\frac{x^2}{x^2} = 1$$

So, we get:

$$2x^4 - 3x^3 - 3x^2 + 6x - 2 = (x^2 - 2)(2x^2 - 3x + 1)$$

Now, we factorise the quadratic $2x^2 - 3x + 1$ by splitting the middle term:

- Split $-3x$ as $-2x - x$
- Factor: $2x^2 - 2x - x + 1 = 2x(x-1) - 1(x-1) = (2x-1)(x-1)$

So, the complete factorisation is:

$$2x^4 - 3x^3 - 3x^2 + 6x - 2 = (x^2 - 2)(2x - 1)(x - 1)$$

Therefore, the zeroes of the given polynomial are:

1. $x = \sqrt{2}$
2. $x = -\sqrt{2}$
3. $x = \frac{1}{2}$
4. $x = 1$