07/11/23 UNIT-IV: RECURRENCE RELATIONS

SEQUENCE

→ Arranging the numbers in a certain order (relation) is called a sequence.

→ A sequence is denoted by $\{a_n\}_{n=0}^{\infty}$ (or) $a_0, a_1, \ldots \infty$

SERIES

→ Sum of the numbers in a sequence is called a series. which is denoted by $\sum_{n=0}^{\infty} a_n$ (or) $a_0 + a_1 + a_2 + \ldots \infty$

Ex: $\{3^n\}_{n=0}^{\infty}$ is a sequence

$\sum_{n=0}^{\infty} 3^n$ is a series

GENERATING FUNCTION

→ Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Then its generating function (G.F) is defined by

$G(x) = \sum_{n=0}^{\infty} a_n x^n$ where x is an intermediate variable.

Note: We can obtain the generating function from the given sequence and vice versa.

Ex 1. Find the generating function of a sequence $\{\alpha_n\}_{n=0}^{\infty}$.

Sol: $G(x) = \alpha_0 + \alpha_1 x^1 + \alpha_2 x^2 + \ldots\ldots$

$= \sum_{n=0}^{\infty} \alpha_n x^n$

CLOSED FORM OF A GENERATING FUNCTION

→ The closed form of the generating function

$$G(x) = \sum_{n=0}^{\infty} \alpha x^n$$

$$= \alpha \sum_{n=0}^{\infty} x^n$$

$$= \alpha [1 + x + x^2 + \dots]$$

$$= \alpha (1-x)^{-1}$$

~~CLOSED~~ Ex: Closed form of $G(x) = \sum_{n=0}^{\infty} 3^n x^n$.

Sol: $G(x) = [1 + 3x + 9x^2 + 27x^3 + \dots]$

$\cancel{= 3[1 + x + x^2 + \dots]}$

$\cancel{= 3(1-x)^{-1}}$

$= \cancel{1 + 3x[1 + x + x^2}$

$= 1 + 3x + (3x)^2 + (3x)^3 + \dots$

$= (1-3x)^{-1}$

1. Find GF of the following sequences:

Sol: (i) $(1, 1, 1, 1)$

(ii) $\{(-1)^{n-1} n\}$

(iii) $\{a_r\}_{r=0}^{\infty}$ where $a_r = \begin{cases} 1 & 0 \le r \le 2 \\ 3 & 3 \le r \le 5 \\ 0 & r \ge 6 \end{cases}$

(iv) $\left\{\frac{1}{n!}\right\}_{n=0}^{\infty}$ e^x

(v) $C(n, k)$ where $k = 0, 1, 2, \dots$ $(1+x)^n$

(i) $(1,1,1,1)$

Sol: $G(x) = 1\cdot x^0 + 1\cdot x^1 + 1\cdot x^2 + 1\cdot x^3$

$$= (1+x+x^2+x^3)\frac{(1-x)}{(1-x)}$$

$$= \frac{(1-x^4)}{(1-x)}$$

$$= (1-x^4)(1-x)^{-1}$$

(ii) $\{(-1)^{n-1} n\}$

Sol: $G(x) = (-1)^{0-1}\cdot 0\cdot (x)^0 + (-1)^{1-1}\cdot 1\cdot (x)^1 + (-1)^{2-1}\cdot 2\cdot (x)^2 + \ldots\ldots$

$$= 0 + x - 2x^2 + 3x^3 - \ldots\ldots$$

$$= x[1 - 2x + 3x^2 - \ldots\ldots]$$

$$= x(1-x)^{-2}$$

(iii) $\{a_r\}_{r=0}^{\infty}$

Sol: $G(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \ldots$

$$= 1\cdot 1 + 1\cdot x + 1\cdot x^2 + 3\cdot x^3 + 3\cdot x^4 + 3\cdot x^5 + 0\cdot x^6 + 0\ldots$$

$$= 1 + x + x^2 + 3x^3 + 3x^4 + 3x^5$$

$$= 1 + x + x^2 + 3(x^3 + x^4 + x^5)$$

$$= 1 + x + x^2 + 3x^3(1 + x + x^2)$$

$$= (1+x+x^2)(1+3x^3)\frac{(1-x)}{(1-x)}$$

$$= \frac{(1-x^3)(1+3x^3)}{(1-x)}$$

$$= (1-x^3)(1+3x^3)(1-x)^{-1}$$

(iv) $\left\{\frac{1}{n!}\right\}_{n=0}^{\infty}$

Sol: $G(x) = \frac{1}{0!}\cdot x^0 + \frac{1}{1!}\cdot x^1 + \frac{1}{2!}\cdot x^2 + \frac{1}{3!}\cdot x^3 + \ldots$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$$

$$= e^x$$

(iv) $C(n,k)$ where $n = 0, 1, 2, \ldots\ldots$

Sol: Here, $C(n,k) = {}^nC_k$

So, $G(x) = {}^nC_0 \cdot x^0 + {}^nC_1 \cdot x^1 + {}^nC_2 \cdot x^2 + {}^nC_3 \cdot x^3 + \ldots$

$$= {}^nC_0 \cdot 1^n x^0 + {}^nC_1 \cdot x^1 \cdot 1^{n-1} + {}^nC_2 \cdot x^2 \cdot 1^{n-2} + {}^nC_3 \cdot x^3 \cdot 1^{n-3} + \ldots$$

$$= (1+x)^n$$

2. Find the GF of the n^{th} Fibonacci term, where the Fibonacci series is given by $f_n = f_{n-1} + f_{n-2}$ $\forall\ n \geq 2$.

Sol: Let the fibonacci sequence be $\{f_n\}_{n=0}^{\infty}$

$$\Rightarrow G(x) = \sum_{n=0}^{\infty} f_n x^n \quad \text{— (1)}$$

Given, $f_n = f_{n-1} + f_{n-2}$ $\forall\ n \geq 2$ — (2)

~~2 Find the GF of the n^{th} Fibonacci term, where the Fibonacci series is given by $f_n = f_{n-1} + f_{n-2}$ $\forall\ n \geq$~~

~~Sol~~: Multiply by x^n and apply $\sum_{n=2}^{\infty}$

$$\sum_{n=2}^{\infty} f_n \cdot x^n = \sum_{n=2}^{\infty} f_{n-1} \cdot x^n + \sum_{n=2}^{\infty} f_{n-2} \cdot x^n$$

$$\Rightarrow f_2 x^2 + f_3 x^3 + \ldots\ldots = f_1 x^2 + f_2 x^3 + \ldots\ldots + f_0 \cdot x^2 + f_1 \cdot x^3 + \ldots\ldots$$

$G(x) - f_0 x^0 - f_1 x = x(f_1 x + f_2 x^2 + \ldots) + x^2(f_0 + f_1 x + \ldots)$

[From ①]

$\Rightarrow G(x) - f_0 x^0 - f_1 x = x[G(x) - f_0] + x^2 \cdot G(x)$

$\Rightarrow$ ~~$f_0 x$~~ $G(x)[1 - x - x^2] = f_0 + f_1 x - f_0 x$

$\Rightarrow \quad G(x) = \cancel{\dfrac{f_0(1-x) + f_1 x}{1 - x - x^2}} \quad \dfrac{f_0 + (f_1 - f_0)x}{1 - x - x^2}$

3. Find the sequence for the generating function $(2+x)^3$.

Sol: $(2+x)^3 = 8 + 12x + 6x^2 + x^3$

$\therefore$ Sequence $= (8, 12, 6, 1)$

4. Find the sequence of $2x^2 \cdot (1-x)^{-1}$.

Sol: $G(x) = 2x^2(1-x)^{-1}$

$= 2x^2[1 + x + x^2 + x^3 + \ldots]$

$= 2x^2 + 2x^3 + 2x^4 + 2x^5 + \ldots$

$\therefore$ Sequence $= (0, 0, 2, 2, 2, 2, \ldots)$

5. Find the sequence of the following GF: $3x^3 + e^{2x}$

Sol: $G(x) = 3x^3 + e^{2x}$

$= 3x^3 + 1 + 2x + \dfrac{(2x)^2}{2!} + \dfrac{(2x)^3}{3!} + \ldots$

$= 3x^3 + 1 + 2x + \dfrac{2^2 x^2}{2!} + \dfrac{2^3 x^3}{3!} + \ldots$

$= 1 + 2x + 2x^2 + \left(3 + \dfrac{8}{6}\right)x^3 + \ldots$

$\therefore$ Sequence $= \left(1, 2, 2, \dfrac{13}{3}, \ldots\right)$

TABLE OF GENERATING FUNCTION

Sequence	GF
$C(n,r)$	$(1+x)^n$
$C(n-1+r,r)$	$(1+x)^{-n}$
$C(n-1+r,r).a^r$	$(1-ax)^{-n}$
$C(n-1+r,r)(-a)^r$	$(1+ax)^{-n}$
1^n	$(1-x)^{-1}$
$(-1)^n$	$(1+x)^{-1}$
a^n	$(1-ax)^{-1}$
$(-a)^n$	$(1+ax)^{-1}$
n	$x(1-x)^{-2}$
$n+1$	$(1-x)^{-2}$
$n(n+1)$	$2x(1-x)^{-3}$
n^2	$x(1+x)(1-x)^{-3}$
$n.a^n$	$ax(1-ax)^{-2}$
$n^2.a^n$	$ax(1+ax)(1-ax)^{-3}$
$(n+1)(n+2)$	$2(1-x)^{-3}$

Note:

Let the generating functions,

$A(x) = \sum_{r=0}^{\infty} a_r . x^r$ and $B(x) = \sum_{s=0}^{\infty} a_s . x^s$, then

(a) $A(x) \pm B(x) = \sum_{n=0}^{\infty} (a_r . x^r + a_s . x^s)$ $[n=r+s]$

(b) $A(x) . B(x) = \sum_{n=0}^{\infty} (a_r . a_s) x^{r+s}$ $[n=r+s]$

1. Calculating the coefficients of the generating function

(a) find the coefficients of :

(a) x^{32} in $(1+x^5+x^9)^{10}$.

(b) x^9 & x^{25} in $(1+x^3+x^8)^{10}$

(c) x^{27} in $(x^4+x^5+x^6+\dots)^5$

(d) x^{20} in $(x+x^2+2x^3+x^4)(x^2+x^3+x^4+\dots)^5$.

(e) x^{12} in $\dfrac{x^2}{(1-2x)^{10}}$ and $\dfrac{x^2-3x}{(1-3x)^4}$

(a) Sol: $32 = 6(0)+1(5)+3(9)$ $\quad (6+1+3=10)$

$\therefore$ Coefficient of $x^{32} = \dfrac{10!}{6!\,1!\,3!}$

(b) For x^9

$9 = 7(0)+3(3)+0(8)$ $\quad (7+3+0=10)$

$\therefore$ Coefficient of $x^9 = \dfrac{10!}{7!\,3!\,0!}$

For x^{25}

$25 = 5(0)+3(3)+2(8)$

$\therefore$ Coefficient of $x^{25} = \dfrac{10!}{5!\,3!\,2!}$ $\quad (5+3+2=10)$

(c) ~~$27 = 0(4)+3(5)+2(6)$~~

Given, $G(x) = (x^4+x^5+x^6+\dots)^5$

$= [x^4(1+x+x^2+\dots)]^5$

P.T.O.

$= x^{20} (1+x+x^2+\ldots)^5$

$= x^{20} [(1-x)^{-1}]^5$

$= x^{20} (1-x)^{-5}$

$= x^{20} \sum C(5-1+r, r) \cdot x^r$

$[\because (1-x)^{-n} = C(n-1+r, r) \cdot x^r]$

$= x^{20} \sum C(4+r, r) \cdot x^r$

$= \sum C(4+r, r) \cdot x^{20+r}$

Coefficient of x^{27} is $x^{27} = x^{20+r}$

$\Rightarrow r = 7$

$\therefore$ Coefficient of x^{27} is $C(4+7, 7)$

$= C(11, 7) = {}^{11}C_7 = \dfrac{11!}{4!\,7!}$

(d) $(x+x^2+2x^3+x^4)(x^2+x^3+x^4+\ldots)^5$

$(x+x^2+2x^3+x^4)\, [x^2(1+x+x^2+\ldots)]^5$

$(x+x^2+2x^3+x^4) \cdot x^{10} [(1-x)^{-1}]^5$

$(x+x^2+2x^3+x^4) \cdot x^{10} (1-x)^{-5}$

$\Rightarrow (x+x^2+2x^3+x^4) \cdot x^{10} \sum C(5-1+r, r) \cdot x^r$

$[\because (1-x)^{-n} = C(n-1+r, r) \cdot x^r]$

$\Rightarrow (x+x^2+2x^3+x^4) \cdot \sum C(4+r, r)\, x^{10+r}$

$\therefore$ Coefficient of x^{20} is

$\sum C(4+r, r) \cdot x^{11+r} + \sum C(4+r, r) \cdot x^{12+r}$

$+ 2 \sum C(4+r, r)\, x^{13+r} + \sum C(4+r, r)\, x^{14+r}$

$\therefore$ Coefficient of x^{20} is.

$C(4+9,9)+C(4+8,8)+2C(4+7,7)+C(4+6,6)$

$\Rightarrow C(13,9)+C(12,8)+2C(11,7)+C(10,6)$

$\Rightarrow \frac{13!}{4!.9!}+\frac{12!}{4!8!}+2\cdot\frac{11!}{4!7!}+\frac{10!}{4!6!}$

(e) $\frac{x^2}{(1-2x)^{10}}$

$=x^2(1-2x)^{-10}$

$=x^2\sum_{r=0}^{\infty}C(10-1+r,r).(2x)^r$

$=\sum_{r=0}^{\infty}C(9+r,r).2^r.x^{r+2}$

$\therefore$ Coefficient of x^{r+2} is $x^{12}=x^{r+2}$

$\Rightarrow r=10$

$\therefore$ Coefficient of x^{12} is $C(9+10,10).2^{10}$

$\Rightarrow 2^{10}.C(19,10)$

$\frac{x^2-3x}{(1-3x)^4}$

$=(x^2-3x)(1-3x)^{-4}$

$=(x^2-3x)\sum_{r=0}^{\infty}C(4-1+r,r).(3x)^r$

$=(x^2-3x)\sum C(3+r,r).3^r.x^r$

$=\sum C(3+r,r).3^r.x^{r+2}-3.\sum C(3+r,r).x^{r+1}$

$\therefore$ Coefficient of x^{12} is

$C(3+10,10).3^{10}-3.C(3+11,11).3^{11}$

$= C(13,10) \cdot 3^{10} - C(14,11) \cdot 3^{12}$

$= 3^{10}\left[C(13,10) - 9C(14,11)\right]$

<u>08/11/23</u>

2 Find GF for that determines no. of non negative integer solution on $e_1+e_2+e_3+e_4+e_5=20$ under the constraints

$0 \le e_1 \le 3,\ 0 \le e_2 \le 4,\ 2 \le e_3 \le 6,\ 2 \le e_4 \le 5,\ e_5 \le 9,$

where e_5 is an odd.

<u>Sol</u>:

Let $f_1(x) = x^0 + x^1 + x^2 + x^3$ [Constraints on e_i]

& $f_2(x) = x^0 + x^1 + x^2 + x^3 + x^4$ $[0 \le e_2 \le 4]$

$f_3(x) = x^2 + x^3 + x^4 + x^5 + x^6$ $[2 \le e_3 \le 6]$

$f_4(x) = x^2 + x^3 + x^4 + x^5$ $[2 \le e_4 \le 5]$

$f_5(x) = x + x^3 + x^5 + x^7 + x^9$ [$e_5 \le 9$, where e_5 is odd

$\therefore G(x) = f_1(x)\, f_2(x)\, f_3(x)\, f_4(x)\, f_5(x)$

$= (1+x+x^2+x^3)(1+x+x^2+x^3+x^4)$
$(x^2+x^3+x^4+x^5+x^6)(x^2+x^3+x^4+x^5)$
$(x+x^3+x^5+x^7+x^9)$

$= \dfrac{(1-x^4)}{(1-x)} \cdot \dfrac{(1-x^5)}{(1-x)} \cdot x^2\left[1+x+x^2+x^3+x^4\right]$
$\cdot x^2\left[1+x+x^2+x^3\right] \cdot x\left[1+x^2+x^4+x^6+x^8\right]$

$= \dfrac{(1-x^4)(1-x^5)}{(1-x)^2} \cdot x^2 \dfrac{(1-x^5)}{(1-x)} \cdot x^2 \dfrac{(1-x^4)}{(1-x)} \cdot x\left[1+t+t^2+t^3+t^4\right]$

[let $t = x^2$]

$$= \frac{(1-x^4)^2(1-x^5)^2}{(1-x)^4} \cdot x^5 \cdot \frac{(1-t^5)}{1-t}$$

$$= \frac{(1-x^4)^2(1-x^5)^2}{(1-x)^4} (x^5) \frac{(1-x^{10})}{(1-x^2)} \qquad [\because t = x^2]$$

3. In how many ~~days~~ ways we can distribute 24 chocolates to 4 students so that each student gets atleast 3 but not more than 8 chocolates.

Sol: Consider, e_1, e_2, e_3, e_4 are 4 students. Distribute 24 chocolates to them, i.e;

$$e_1 + e_2 + e_3 + e_4 = 24 \quad — ①$$

and given constraints $3 \le e_i \le 8$ for $i = 1, 2, 3, 4$.

Now, we would like to find number of integral solution of ① [Find coefficient of x^{24} in the GF of ①

Now, $f_1(x) = x^3 + x^4 + x^5 + x^6 + x^7 + x^8 = f_2(x) = f_3(x) = f_4(x)$

$[\because 3 \le e_i \le 8 \;\; \forall\, i = 1, 2, 3, 4]$

$$G(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \cdot f_4(x)$$

$$= (x^3 + x^4 + x^5 + x^6 + x^7 + x^8)^4$$

$$= (x^3)^4 \left[1 + x + x^2 + x^3 + x^4 + x^5\right]^4$$

$$= x^{12} \cdot \left[\frac{(1-x^6)}{(1-x)}\right]^4$$

$$= x^{12} \cdot \frac{(1-x^6)^4}{(1-x)^4}$$

$$= x^{12} \cdot (1-x^6)^4 \cdot (1-x)^{-4}$$

$\rightarrow$

$$= x^{12} \sum_{r=0}^{\infty} C(4,r)(-x^{6})^r \sum_{s=0}^{\infty} C(4-1+s,s).x^s$$

$$= \cancel{x^{12}} \sum_{r=0}^{\infty} C(4,r)(-1)^r . x^{6r} . x^{12} \sum_{s=0}^{\infty} C(3+s,s).x^s$$

$$= \sum_{r=0}^{\infty} C(4,r)(-1)^r x^{12+6r} \sum_{s=0}^{\infty} C(3+s,s). x^s$$

To find Coefficient of x^{24}

$6r+s+12=24$

r	s
0	12
1	6
2	0

$\therefore$ Coefficient of x^{24} is

$$C(4,0).(-1)^0. C(3+12,12) + C(4,1).(-1)^1. C(3+6,6)$$
$$+ C(4,2)(-1)^2. C(3+0,0)$$

$$\Rightarrow - C(4,0).C(15,12) - C(4,1).C(9,6)$$
$$+ C(4,2).C(3,0)$$

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4. Using generating function ~~to~~ find the number of (i) non-negative & (ii) positive integer solutions of ~~$e_1+e_2+e_3+e_4 \le 8$~~ $e_1+e_2+e_3+e_4 = 25$, ~~$e_i \ge 0$~~ ~~$\forall i = 1,2,3$~~

Sol: (i) Non-negative integer solution

$e_1+e_2+e_3+e_4 = 25$, $e_i \ge 0 \;\forall i = 1,2,3,4$

Now,

Let $f_1(x) = x^0 + x^1 + x^2 + \dots\dots = f_2(x) = f_3(x) = f_4(x)$

$$G(x) = f_1(x).f_2(x).f_3(x).f_4(x)$$
$$= [f_1(x)]^4 = (1+x+x^2+\dots)^4$$
$$= [(1-x)^{-1}]^4$$
$$= (1-x)^{-4}$$
$$= \sum_{r=0}^{\infty} C(4-1+r, r).(x)^r$$
$$\cancel{= C(3+r, r)\, x^r}$$
$$= \sum_{r=0}^{\infty} C(3+r, r).x^r$$

For coefficient of x^{25} : $x^{25} = x^r$

$\Rightarrow r = 25$

$= \cancel{II}$

$= C(3+25, 25)$

$= C(28, 25)$

(ii) Positive Integer Solution

$e_1 + e_2 + e_3 + e_4 = 25$ $\quad e_i \geq 1 \quad \forall i = 1,2,3,4$

Now,

Let $f_1(x) = +x^1 + x^2 + \dots\dots = f_2(x) = f_3(x) = f_4(x)$

$$G(x) = f_1(x).f_2(x).f_3(x).f_4(x)$$
$$= [f_1(x)]^4$$
$$= (x + x^2 + x^3 + \dots)^4$$
$$= [x.(1+x+x^2+\dots)]^4$$
$$= [x(1-x)^{-1}]^4$$
$$= x^4.(1-x)^{-4}$$

$$= x^{4} \cdot \sum_{r=0}^{\infty} C(4-1+r, r) \cdot x^{r}$$

$$= \sum_{r=0}^{\infty} C(3+r, r) \cdot x^{r+4}$$

For coefficient of x^{25} ; $x^{r+4} = x^{25}$

$$\Rightarrow r = 21$$

$$\therefore = \cancel{\sum_{r=0}^{\infty}} C(3+21, 21)$$

$$= C(24, 21)$$

RECURRENCE RELATIONS

→ A recurrence relation for the sequence $\{a_n\}_{n=}^{\infty}$ is an equation that expresses a_n in terms of one or more previous terms namely a_{n-1}, $a_{n-2}, \ldots, a_1, a_0$.

→ A recurrence relation is also called ~~Reference~~ Difference equation.

Ex: 1. If a_n denotes n^{th} term of a G.P with a common ratio 'r', then difference equation is given by

$a_n = r \cdot a_{n-1} \quad \forall n \geq 1$

2. Ex: $a_n = n + a_{n-1} \quad \forall n \geq 1$

Note: If n, k are non-negative integers 0,1,2,...

A recurrence relation of the form

$C_0(n)a_n + C_1(n)a_{n-1} + \dots + C_k(n)a_{n-k} = f(n), \forall n \geq k$ — ①

where $C_0(n), C_1(n), \dots, C_k(n)$ and $f(n)$ are the functions of n. One is said to be a linear recurrence relation.

→ The difference between the greatest & least suffix in the given recurrence relation is called "Order of the recurrence relation."

→ Order of the recurrence relation RR ① is $(n-(n-k)) = k$.

<u>Note:</u>

1. If $C_0(n), C_1(n), \dots, C_k(n)$ are constants, then ① is called linear recurrence relation
2. If $f(n) = 0$ in ①, then the equation is called homogeneous recurrence relation.
3. If $f(n) \neq 0$ in ①, then the equation is called non-homogeneous recurrence relation.

FIBONACCI RECURRENCE RELATION

→ The recurrence relation $f_n = f_{n-1} + f_{n-2}$ $\forall n \geq 2$ with initial conditions $f_0 = 0, f_1 = 1$ is known as Fibonacci Recurrence Relation.

→ The numbers f_n generated by Fibonacci relation with initial conditions f_0 and f_1 are called Fibonacci numbers.

Q. Show that $F_0+F_1+F_2+\ldots\ldots+F_n = F_{n+2}-1$.

Sol: We know that, $F_n = F_{n-1}+F_{n-2}$

$$\Rightarrow F_{n+2} = F_{n+1}+F_n$$

$$\Rightarrow F_n = F_{n+2}-F_{n+1}$$

Now,
$$F_0 = F_2 - F_1$$
$$F_1 = F_3 - F_2$$
$$F_2 = F_4 - F_3$$
$$\vdots \quad \vdots \quad \vdots$$
$$F_n = F_{n+2} - F_{n+1}$$

$$F_0+F_1+F_2+\ldots+F_n = F_{n+2}-F_1$$
$$= F_{n+2}-1 \quad [\because F_1 = 1]$$

★★★

$$\boxed{\begin{array}{l} F_0+F_2+F_4+\ldots\ldots+F_{2n} = F_{2n+1} \\ F_0^2+F_1^2+\ldots\ldots+F_n^2 = F_n \cdot F_{n+1} \end{array}}$$

2. If the recurrence relation $a_n = a_{n-1}+3a_{n-2}$ $\forall$ $n \geq 2$ and $a_0=1, a_1=2$ find the values of a_3, a_4 & a_5.

3. If $a_n = 2^n + 5(3^n)$ $\forall$ $n \geq 0$

(i) Find a_0, a_1, a_2, a_3 and a_4.

(ii) Show that
$$a_2 = 5a_1 - 6a_0$$
$$a_3 = 5a_2 - 6a_1$$
$$a_4 = 5a_3 - 6a_2$$

(iii) Show that $a_n = 5a_{n-1} - 6a_{n-2}$, $\forall$ $n \geq 2$

2-Sol: Given recurrence relation is

$a_n = a_{n-1} + 3a_{n-2} \; \forall \; n \geq 2$ and $a_0 = 1, a_1 = 2$ — ①

Put $n = 2$ in ①

$\Rightarrow$ ~~$a_0 = a_1 +$~~ $a_2 = a_1 + 3a_0$

$\Rightarrow a_2 = (2) + 3(1)$

$a_2 = 5$

Put $n = 3$ in ①

$\Rightarrow a_3 = a_2 + 3a_1$

$= (5) + 3(2)$

$= 11$

Put $n = 4$ in ①

$\Rightarrow a_4 = a_3 + 3a_2$

$= (11) + 3(5)$

$= 26$

Put $n = 5$ in ①

$\Rightarrow a_5 = a_4 + 3a_3$

$= (26) + 3(11)$

$= 59$

~~Put n =~~

$\therefore a_3 = 11 ; a_4 = 26 ; a_5 = 59$

3-Sol: Given recurrence relation,

$a_n = 2^n + 5(3^n) \; \forall \; n \geq 0$ — ①

(i) Put $n = 0$ in ①

$a_0 = 2^0 + 5(3^0)$

$= 1 + 5(1)$

$= 6$

Put $n = 1$ in ①

$a_1 = 2^1 + 5(3^1)$

$= 2 + 15$

$= 17$

Put $n = 2$ in ①

$a_2 = 2^2 + 5(3^2)$

$= 4 + 45$

Put $n = 3$ in ①

$a_3 = 2^3 + 5(3^3)$

$= 8 + 135$

$= 143$

Put $n = 4$ in ①

$a_4 = 2^4 + 5(3^4)$

$= 16 + 405$

$= 421$

$\therefore a_0 = 6, \quad a_3 = 143$

$a_1 = 17, \quad a_4 = 421$

$a_2 = 49$

(ii) Consider,

$$5a_1 - 6a_0 = 5(17) - 6(6)$$
$$= 85 - 36$$
$$= 49 = a_2$$

$\therefore a_2 = 5a_1 - 6a_0$ — ②

(iii) $5a_2 - 6a_1 = 5(49) - 6(17)$
$$= 245 - 102$$
$$= 143 = a_3$$

$\therefore a_3 = 5a_2 - 6a_1$ — ③

$$5a_3 - 6a_2 = 5(143) - 6(49)$$
$$= 715 - 294$$
$$= 421 = a_4$$

$\therefore a_4 = 5a_3 - 6a_2$ — ④

Hence Proved

(iii) To prove $a_n = 5a_{n-1} - 6a_{n-2}, \forall n \geq 2$ — ⑤

Put $n=2$ in ⑤

$\Rightarrow a_2 = 5a_1 - 6a_0$ which is true [From ②]

Put $n=3$ in ⑤

$\Rightarrow a_3 = 5a_2 - 6a_1$ which is true [From ③]

Put $n=4$ in ⑤

$\Rightarrow a_4 = 5a_3 - 6a_2$ which is true [From ④]

$\therefore a_n = 5a_{n-1} - 6a_{n-2}, \forall n \geq 2$

Hence Proved

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SOLUTIONS OF RECURRENCE RELATIONS

→ There are three methods to solve recurrence relations:

1. Substitution Method (Implication Method)
2. Generating Function Method
3. Characteristic Roots Method

1. SUBSTITUTION METHOD

→ In this methods the recurrence relation for "a_n" is used repeatedly to solve for a general expression of a_n in terms of n.

1. Solve the recurrence relation $a_n = a_{n-1} + f(n)$ $\forall\ n \geq 1$ by substitution method.

Sol: $a_1 = a_0 + f(1)$

$a_2 = a_1 + f(2)$

$= a_0 + f(1) + f(2)$

$a_3 = a_2 + f(3)$

$= a_0 + f(1) + f(2) + f(3)$

$a_n = a_0 + f(1) + f(2) + \dots\dots + f(n-1) + f(n)$

$= a_0 + \sum_{k=1}^{n} f(k)$

$= a_0 + \sum_{n=1}^{k} f(n)$

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2. Solve the recurrence relation $a_n = a_{n-1} + n \cdot 3^n \; \forall \; n \geq 1$.

Sol: $a_1 = a_0 + 1 \cdot 3^1$

$= a_0 + 1 \cdot 3$

$a_2 = a_1 + 2 \cdot 3^2$

$= a_0 + \cancel{3+18} \; 1 \cdot 3 + 2 \cdot 3^2$

$a_3 = a_2 + 3 \cdot 3^3$

$= a_0 + \cancel{3+18+81} \; 1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3$

$a_4 = a_3 + 4 \cdot 3^4$

$= a_0 + 1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + 4 \cdot 3^4$

$\Rightarrow a_n = a_{n-1} + n \cdot 3^n$

$= a_0 + 1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + 4 \cdot 3^4 + \ldots + n \cdot 3^n$

$= a_0 + \sum_{K=1}^{n} K \cdot 3^K$

3. Solve the recurrence relation $a_n = a_{n-1} + n^2$, where $a_0 = 7 \; \forall \; n \geq 1$.

Sol: $a_1 = a_0 + 1^2$

$= a_0 + 1^2$

$a_2 = a_1 + 2^2$

$= a_0 + 1^2 + 2^2$

$a_3 = a_2 + 3^2$

$= a_0 + 1^2 + 2^2 + 3^2$

$a_4 = a_3 + 4^2$

$= a_0 + 1^2 + 2^2 + 3^2 + 4^2$

$\Rightarrow a_n = a_{n-1} + n^2$

$= a_0 + 1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = a_0 + \sum^{n}$

$= a_0 + \sum_{k=1}^{n} k^2$

$= \cancel{7_0}\ 7 + \frac{k(k+1)(2k+1)}{6}$

$= 7 + \frac{n(n+1)(2n+1)}{6}$

. Solve the recurrence relation $a_n = 2a_{n-1} + 1$, $n \geq 1$, $a_1 = 7$

Sol: $a_1 = 2a_0 + 1$

$\Rightarrow 7 = 2a_0 + 1$

$\Rightarrow a_0 = \frac{7-1}{2} = 3$

$a_2 = 2a_1 + 1$

$= 2(7) + 1 = 15$

$a_3 = 2a_2 + 1$

$= 2(15) + 1 = 31$

$a_4 = 2a_3 + 1$

$= 2(31) + 1 = 63$

$a_1 = 2a_0 + 1$

$a_2 = 2a_1 + 1$

$= 2(2a_0 + 1) + 1$

$= 4a_0 + 2 + 1$

$\cancel{= 4a_0 + 15}$

$a_3 = 2a_2 + 1$

$= 2(4a_0 + 3) + 1$

$= 8a_0 + 31$

$a_1 = 2a_0 + 1$

$a_2 = 2a_1 + 1$

$= 2(2a_0 + 1) + 1$

$= 4a_0 + 2 + 1$

$a_3 = 2a_2 + 1$

$\cancel{= 2(4a_0 + 2 + 1) + 1}$ $= 2(2a_1 + 1) + 1$

$\cancel{= 8a_0 + 2^2 + 2 + 1}$ $= 4a_1 + 2 + 1$

$\rightarrow$

$a_4 = 2a_3 + 1$

$= \cancel{2(8a_0 + 2^2 + 2 + 1) + 1} = 2(4a_1 + 2 + 1) + 1$

$= \cancel{16a_0 + 2^3 + 2^2 + 2 + 1} = 8a_1 + 2^2 + 2 + 1$

$= \cancel{2^4 a_0 + 2^3 + 2^2 + 2 + 1} = 2^3 a_1 + 2^2 + 2 + 1$

$\Rightarrow a_n = 2a_{n-1} + 1$

$= 2^{n-1} a_1 + 2^{n-2} + 2^{n-3} + \ldots + 2^2 + 2 + 1$

$= 2^{n-1} a_1 + (1 + 2 + 2^2 + \ldots + 2^{n-2})$

$= 2^{n-1} a_1 + 1 \cdot \frac{(2^{n-2} - 1)}{(2-1)}$

$= 2^{n-1} a_1 + (2^{n-2} - 1)$

$= 2^{n-1} \cdot 7 + (2^{n-2} - 1)$

5. Solve the recurrence relation $a_n = a_{n-1} + 2$, $a_0 =$

6. Solve the recurrence relation $a_n = a_{n-1} + n$, $a_0 = 1$.

7. Solve the recurrence relation $a_n = a_{n-1} + 2n + 3$, $a_0 = 4$.

8. Solve the recurrence relation $a_n = 3a_{n-1} + 1$, $a_0 =$

5 Sol: $a_1 = a_0 + 2$

$a_2 = a_1 + 2$

$= a_0 + 2 + 2$

$a_3 = a_2 + 2$

$= a_0 + 2 + 2 + 2$

$\Rightarrow a_4 = a_3 + 2$

$= a_0 + 2 + 2 + 2 + 2$

$\Rightarrow a_n = a_{n-1} + 2$

$= a_0 + \underbrace{2 + 2 + 2 + \cdots\cdots + 2}_{n\text{-times}}$

$= \cancel{a_0 + \sum 2n} \quad a_0 + \sum_{k=1}^{n} 2k$

$= a_0 + \sum_{n=1}^{k} 2n$

6. Sol: $a_1 = a_0 + 1$

$a_2 = a_1 + 2$

$= a_0 + 1 + 2$

$a_3 = a_2 + 3$

$= a_0 + 1 + 2 + 3$

$a_4 = a_3 + 4$

$= a_0 + 1 + 2 + 3 + 4$

$\Rightarrow a_n = a_{n-1} + n$

$= a_0 + 1 + 2 + 3 + 4 + \cdots\cdots + n$

$= a_0 + \sum_{k=1}^{n} k$

$= a_0 + \sum_{n=1}^{k} n$

7. Sol: $a_1 = a_0 + 2\cdot 1 + 3$

$= a_0 + 2 + 3$

$a_2 = a_1 + 2\cdot 2 + 3$

$= a_1 + 4 + 3$

$\cancel{a_3 = a_2 +}$

$= a_0 + 2 + 3 + 4 + 3 \quad \cancel{= a_0 + 2 + 2\cdot 2}$

$= a_0 + 2 + 4 + 3(2)$

$= a_0 + 2.1 + 2.2 + 3(2)$

$a_3 = a_2 + 2.3 + 3$

$= a_0 + 2 + 3 + 4 + 3 + 6 + 3$

$= a_0 + 2 + 4 + 6 + 3(3)$

$= a_0 + 2.1 + 2.2 + 2.3 + 3(3)$

$\Rightarrow a_n = a_{n-1} + 2n + 3$

$= a_0 + 2.1 + 2.2 + 2.3 + \ldots\ldots + 2.n + 3(n)$

$= a_0 + \sum_{k=1}^{n} 2k + 3n$

$\Rightarrow \boxed{a_n = a_0 + 3n + \sum_{n=1}^{k} 2n}$

8. <u>Sol:</u> ~~$a_1 = 3a_0$~~ Given $a_0 = 4$

$\Rightarrow \boxed{a_n = 4 + 3n + \sum_{n=1}^{k} 2n}$

8. <u>Sol:</u> $a_1 = 3a_0 + 1$

$a_2 = 3a_1 + 1$

$= 3(3a_0 + 1) + 1$

$= 3^2 a_0 + 3 + 1$

$a_3 = 3a_2 + 1$

$= 3(3^2 a_0 + 3 + 1) + 1$

$= 3^3 a_0 + 3^2 + 3 + 1$

$a_4 = 3a_3 + 1$

$= 3(3^3 a_0 + 3^2 + 3 + 1) + 1$

$= 3^4 a_0 + 3^3 + 3^2 + 3 + 1$

$$\Rightarrow a_n = 3a_{n-1} + 1$$

$$= 3^n a_0 + 3^{n-1} + 3^{n-2} + \cdots\cdots + 3^2 + 3 + 1$$

$$= 3^n a_0 + \sum_{K=1}^{n} 3^{K-1}$$

$$\Rightarrow a_n = 3^n a_0 + \sum_{K=1}^{n} 3^{K-1}$$

Given, $a_0 = 1$

$$\Rightarrow a_n = 3^n \cdot 1 + \sum_{n=1}^{K} 3^{n-1}$$

$$\Rightarrow \boxed{a_n = 3^n + \sum_{n=1}^{K} 3^{n-1}}$$

II GENERATING FUNCTION METHOD

→ Recurrence relations can also be solved using generating function.

→ Some equivalent expressions are used to solve the recurrence relations which are given by:

If $G(x) = \sum_{n=0}^{\infty} a_n . x^n$, then

(i) $\sum_{n=k}^{\infty} a_n x^n = G(x) - \sum_{n=0}^{k-1} a_n x^n$

(ii) $\sum_{n=k}^{\infty} a_{n-1} x^n = x\left[G(x) - \sum_{n=0}^{k-2} a_n x^n\right]$

$= x\left[G(x) - (a_0 + a_1 x + a_2 x^2 + \dots + a_{k-2}x^{k-2})\right]$

(iii) $\sum_{n=k}^{\infty} a_{n-2} x^n = x^2\left[G(x) - \sum_{n=0}^{k-3} a_n x^n\right]$

$= x^2\left[G(x) - (a_0 + a_1 x + a_2 x^2 + \dots + a_{k-3}x^{k-3})\right]$

(iv) $\sum_{n=k}^{\infty} a_{n-k} x^n = x^k\left[G(x)\right]$

1. Solve the following ~~generating~~ recurrence relations by generating function method.

(a) $a_n - 7a_{n-1} + 10a_{n-2} = 0 \quad \forall\ n \geq 2$

(b) $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0 \quad \forall\ n \geq 3$

(c) $a_n - 9a_{n-1} + 20a_{n-2} = 0 \quad \forall\ n \geq 2$

(d) $a_n - 3a_{n-1} - 2 = 0$ with $a_0 = 1 \quad \forall\ n \geq 1$.

Sol:

~~(a) $a_n - 7a_{n-1} + 10a_{n-2} = 0 \quad \forall\ n \geq 2$~~

(a) Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ — ①

Sol:

Given $a_n - 7a_{n-1} + 10a_{n-2} = 0$ — ②

Multiply by x^n and take $\sum_{n=2}^{\infty}$ on both sides of ②

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - 7\sum_{n=2}^{\infty} a_{n-1} x^n + 10 \sum_{n=2}^{\infty} a_{n-2} \cdot x^n = \sum_{n=2}^{\infty} 0 \cdot x^n$$

$$\Rightarrow [G(x) - a_0 - a_1 x] - 7[x(G(x) - a_0)] + 10[x^2(G(x))] = 0$$

$$\Rightarrow G(x)[1 - 7x + 10x^2] - a_0(1-7x) - a_1 x = 0$$

$$\Rightarrow G(x) = \frac{a_0(1-7x) - a_1 x}{(1 - 7x + 10x^2)}$$

$$= \frac{a_0 + x(a_1 - 7a_0)}{(1-7x+10x^2)} = \frac{a_0 + x(a_1 - 7a_0)}{(1-2x)(1-5x)}$$

$$= \frac{c_1}{1-2x} + \frac{c_2}{1-5x}$$

$$= c_1(1-2x)^{-1} + c_2(1-5x)^{-1}$$

$$= c_1 \sum_{n=0}^{\infty} (2x)^n + c_2 \sum_{n=0}^{\infty} (5x)^n$$

$$= c_1 \sum_{n=0}^{\infty} 2^n x^n + c_2 \sum_{n=0}^{\infty} 5^n x^n$$

$$G(x) = \sum_{n=0}^{\infty} [c_1 2^n + c_2 5^n] x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} [c_1 \cdot 2^n + c_2 5^n] x^n$$

$\therefore a_n = c_1 \cdot 2^n + c_2 \cdot 5^n$ [$\because$ By comparing coefficient of x^n]

→

(b) $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0 \quad \forall\ n \geq 3$

Sol: Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$. — (1)

Given, $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$ — (2)

Multiply by x^n and take $\sum_{n=3}^{\infty}$ on both sides of (2)

$$\sum_{n=3}^{\infty} a_n x^n - 8\sum_{n=3}^{\infty} a_{n-1} \cdot x^n + 21\sum_{n=3}^{\infty} a_{n-2} \cdot x^n - 18\sum_{n=3}^{\infty} a_{n-3} x^n = \sum_{n=3}^{\infty} 0 \cdot x^n$$

$$\Rightarrow [G(x) - a_0 - a_1 x - a_2 x^2] - 8[x(G(x) - a_0 - a_1 x)]$$
$$+ 21[x^2(G(x) - a_0)] - 18[x^3 G(x)] = 0$$

$$\Rightarrow G(x)[1 - 8x + 21x^2 - 18x^3] - a_0(1 - 8x - 21x^2) \cancel{- a_0(1 - 8x - 21x^2)}$$
$$- a_1(x - 8x^2) - a_2 x^2 = 0$$

$$\cancel{\Rightarrow G(x) = \frac{28a_0 - a_1(7x) - a_2 x^2}{-18x^3 + 21x^2 - 8x + 1}}$$

$$\cancel{\Rightarrow G(x) = \frac{28a_0 - x(7a_1) - a_2 x^2}{-18x^3 + 21x^2 - 8x + 1}}$$

$$\Rightarrow G(x)[1 - 8x + 21x^2 - 18x^3] = a_0(1 - 8x - 21x^2) + a_1(x - 8x^2) + a_2 x^2$$

$$= \cancel{x^2(-21a_0 + a_2) + x(-8a_0}$$

$$= x^2(-21a_0 - 8a_1 + a_2) + x(-8a_0 + a_1) + a_0$$

$$\Rightarrow G(x) = \frac{x^2(-21a_0 - 8a_1 + a_2) + x(-8a_0 + a_1) + a_0}{1 - 8x + 21x^2 - 18x^3}$$

$$\Rightarrow G(x) = \frac{a_0 + x(a_1 - 8a_0) + x^2(a_2 - 8a_1 + 21a_0)}{(1-2x)(1-3x)^2}$$

$$= \frac{c_1}{1-2x} + \frac{c_2}{1-3x} + \frac{c_3}{(1-3x)^2}$$

$$= c_1(1-2x)^{-1} + c_2(1-3x)^{-1} + c_3(1-3x)^{-2}$$

$$= c_1 \sum_{n=0}^{\infty} (2x)^n + c_2 \sum_{n=0}^{\infty} (3x)^n + c_3 \sum C(2-1+r, r) \cdot (3x)^r$$

$$= c_1 \sum_{n=0}^{\infty} 2^n x^n + c_2 \sum_{n=0}^{\infty} 3^n x^n + c_3 \sum C(1+r, r) \cdot (3x)^r$$

$$\Rightarrow G(x) = c_1 \sum_{n=0}^{\infty} 2^n x^n + c_2 \sum_{n=0}^{\infty} 3^n x^n + c_3 \sum C(1+n, n) 3^n x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left[c_1 2^n + c_2 3^n + c_3 (n+1, n) 3^n\right] x^n$$

$$\therefore a_n = c_1 2^n + c_2 3^n + c_3 (n+1, n) \cdot 3^n$$

(c) $a_n - 9a_{n-1} + 20a_{n-2} = 0 \quad \forall\, n \geq 2$ — (2)

Sol:

Multiply with x^n and take $\sum_{n=2}^{\infty}$ on both sides of (2)

$$\sum_{n=2}^{\infty} a_n x^n - 9 \sum_{n=2}^{\infty} a_{n-1} x^n + 20 \sum_{n=2}^{\infty} a_{n-2} x^n = \sum_{n=2}^{\infty} 0 \cdot x^n$$

$$\Rightarrow [G(x) - a_0 - a_1 x] - 9[x(G(x) - a_0)] + 20[x^2(G(x))] = 0$$

$$\Rightarrow G(x)[1 - 9x + 20x^2] - a_0 - a_1 x + 9x \cdot a_0 = 0$$

$$\Rightarrow G(x) = \frac{a_0 + a_1 x - 9x a_0}{20x^2 - 9x + 1}$$

$$\Rightarrow G(x) = \frac{a_0 + x(a_1 - 9a_0)}{(1-4x)(1-5x)}$$

$$= \frac{C_1}{1-4x} + \frac{C_2}{1-5x}$$

$$= C_1(1-4x)^{-1} + C_2(1-5x)^{-1}$$

$$= C_1 \sum_{n=0}^{\infty} (4x)^n + C_2 \sum_{n=0}^{\infty} (5x)^n$$

$$\Rightarrow G(x) = C_1 \sum_{n=0}^{\infty} 4^n . x^n + C_2 \sum_{n=0}^{\infty} 5^n . x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} [C_1 . 4^n + C_2 . 5^n] x^n$$

$$\therefore \boxed{a_n = C_1 . 4^n + C_2 . 5^n}$$

(d) $a_n - 3a_{n-1} - 2 = 0$ with $a_0 = 1$ $\forall$ $n \geq 1$

Sol: Let $\sum_{n=0}^{\infty} a_n x^n = G(x)$ — ①

Given, $a_n - 3a_{n-1} - 2 = 0$ with $a_0 = 1$ $\forall$ $n \geq 1$
└ ②

Multiply with x^n and taking $\sum_{n=1}^{\infty}$ on both sides of ②

$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n - 3\sum_{n=1}^{\infty} a_{n-1} x^n - 2\sum_{n=1}^{\infty} x^n = 0$$

$$\Rightarrow [G(x) - a_0] - 3[x(G(x))] = 2\cancel{\sum_{n=1}^{\infty}}(x + x^2 + x^3 + \dots)$$

$$\Rightarrow G(x)(1-3x) - a_0 = 2x(1 + x + x^2 + \dots)$$

$$\Rightarrow G(x)(1-3x) - a_0 = 2x(1-x)^{-1}$$

$$\Rightarrow G(x) = \frac{a_0 + 2x(1-x)^{-1}}{1-3x}$$

$$= \frac{a_0}{(1-3x)} + \frac{2x(1-x)^{-1}}{\cancel{(1-3x)}(1-3x)}$$

$$= \frac{a_0}{(1-3x)} + \frac{2x}{(1-x)(1-3x)}$$

$$= \frac{1}{1-3x} + \frac{c_1}{1-x} + \frac{c_2}{1-3x}$$

$$= (1-3x)^{-1} + c_1(1-x)^{-1} + c_2(1-3x)^{-1}$$

$$= c_1(1-x)^{-1} + (1+c_2)(1-3x)^{-1}$$

$$\Rightarrow G(x) = c_1 \sum_{n=0}^{\infty} x^n + (1+c_2)\sum_{n=0}^{\infty} 3^n x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} [c_1 \cdot 1^n + (1+c_2)3^n] x^n$$

$$\therefore \boxed{a_n = c_1 \cdot 1^n + (1+c_2)3^n}$$

3. CHARACTERISTIC ROOTS METHOD

→ Let $c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \dots + c_k a_{n-k} = 0$ — (1)

$\forall\ n \geq k$, $c_k \neq 0$ is a linear recurrence relation (difference relation) with an order 'k', then the characteristic equation is given by:

$$t^k + c_1 t^{k-1} + c_2 t^{k-2} + c_3 t^{k-3} + \dots + c_k t^0 = 0 \quad \text{— (2)}$$

Note:

The order of the recurrence relation is same as the degree of the characteristic equation and also equal to the number of arbitrary constants.

Q. Find the characteristic equations of the following recurrence relations:

(a) $a_n + 3a_{n-1} + 4a_{n-2} + 5a_{n-3} = 0$. $[t^3 + 3t^2 + 4t + 5 = 0]$

(b) $a_n + 7a_{n-2} + a_{n-3} = 0$. $[t^3 + 7t + 1 = 0]$

(c) $a_n + 6a_{n-3} = 0$. $[t^3 + 6 = 0]$

Procedure to find the solution of a recurrence relation by characteristic root method

Step I: Find the characteristic equation of given recurrence relation ①, i.e,

$$c_0 t^k + c_1 t^{k-1} + c_2 t^{k-2} + \dots + c_k = 0 \quad -②$$

Step II: Find the roots of characteristic equation ②

$\rightarrow$ Let $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_k$ are roots then the solution is:

Case (i): If roots are distinct

$$a_n = D_1\alpha_1^n + D_2\alpha_2^n + D_3\alpha_3^n + \dots + D_k\alpha_k^n \quad -③$$

Case (ii): If $\alpha_1, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_3, \dots, \alpha_k$ are roots and α_1, α_2 are repeated

Homogeneous solution is

$$a_n = (D_1 + nD_2 + n^2 D_3)\alpha_1^n + (D_4 + nD_5)\alpha_2^n + D_6\alpha_3^n + \dots + D_k\alpha_k^n$$

2. Find the solution for the following recurrence relations by using Characteristic roots method

(a) $a_n - 7a_{n-1} + 10a_{n-2} = 0 \quad \forall\ n \geq 2$

(b) $a_n - 7a_{n-1} + 12a_{n-2} = 0 \quad \forall\ n \geq 2$

(c) $a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = 0 \quad \forall\ n \geq 3$

(d) $a_n - 3a_{n-1} + 2a_{n-2} = 0 \quad \forall\ n \geq 2$

(e) $a_n = 2(a_{n-1} - a_{n-2}) \quad \forall\ n \geq 2,\ a_0 = 1,\ a_1 = 2$

(a) Sol:

Characteristic equation is

$$t^2 - 7t + 10 = 0$$

$$\Rightarrow (t-2)(t-5) = 0$$

$$\Rightarrow t = 2, 5$$

$\therefore$ Solution $= a_n = c_1 2^n + c_2 5^n$

(b) Sol:

Characteristic equation is

$$t^2 - 7t + 12 = 0$$

$$\Rightarrow (t-3)(t-4) = 0$$

$$\Rightarrow t = 3, 4$$

$\therefore$ Solution $= c_1 3^n + c_2 4^n$

(c) Sol:

Characteristic equation is

$$t^3 - 7t^2 + 16t - 12 = 0.$$

$$\Rightarrow (t-2)(t-2)(t-3) = 0$$

$$\Rightarrow t = 2, 2, 3$$

$\therefore$ Solution $= (c_1 + c_2 n)2^n + c_3 3^n$

(d) Sol:

$$\cancel{a_n = 2(a_{n-1} - a_{n-2})}$$

$$\cancel{\Rightarrow a_n - 2a_{n-1} + 2a_{n-2} = 0}$$

~~Characteristic equation is~~

$$\cancel{t^2 - 2t + 2 = 0}$$

(d) Sol:

$\cancel{a_n = 3}$

Characteristic equation is

$$t^2 - 3t + 2 = 0$$

$$\Rightarrow (t-1)(t-2) = 0$$

$$\Rightarrow t = 1, 2$$

$\therefore$ Solution: $a_n = c_1 \cdot 1^n + c_2 \cdot 2^n$

(e) Sol:

$$a_n = 2(a_{n-1} - a_{n-2})$$

$$\Rightarrow a_n - 2a_{n-1} + 2a_{n-2} = 0$$

Characteristic equation is

$$t^2 - 2t + 2 = 0$$

$$\Rightarrow t = \frac{2 \pm \sqrt{4-8}}{2}$$

$$\Rightarrow t = 1 \pm i$$

Solution is $a_n = c_1(1+i)^n + c_2(1-i)^n$ — (1)

Given $a_0 = 1$

Put $n = 0$ in (1)

$$a_0 = c_1(1+i)^0 + c_2(1-i)^0$$

$$\Rightarrow 1 = c_1 + c_2 \quad \text{— ②}$$

Put $n=1$ in ①

$$a_1 = c_1(1+i) + c_2(1-i)$$

$$\Rightarrow 2 = c_1(1+i) + c_2(1-i) \quad \text{— ③}$$

Solving eqs ② and ③

$$(1-i) \quad c_1 + c_2 = 1$$

$$c_1(1+i) + c_2(1-i) = 2$$

$$(1-i)c_1 + \cancel{(1-i)c_2} = 1-i$$

$$(1+i)c_1 + \cancel{(1-i)c_2} = 2$$

$$(-) \qquad (-) \qquad (-)$$

$$(\cancel{1}-i-\cancel{1}-i)c_1 = 1-i-2$$

$$\Rightarrow -2ic_1 = -i-1$$

$$\Rightarrow 2ic_1 = i+1$$

$$\Rightarrow c_1 = \frac{1}{2}\left[\frac{i+1}{i}\right]$$

$$= \frac{1}{2}\left[\frac{-1+i}{-1}\right]$$

$$= \frac{-1}{2}[i-1]$$

$$= \frac{1-i}{2}$$

Put $c_1 = \frac{1-i}{2}$ in eq ②

$$\Rightarrow \frac{1-i}{2} + c_2 = 1$$

$$\Rightarrow c_2 = 1 - \left(\frac{1-i}{2}\right)$$

$$= \frac{2-1+i}{2}$$

$$= \frac{1+i}{2}$$

$\therefore c_1 = \frac{1-i}{2}$ and $c_2 = \frac{1+i}{2}$

$\therefore$ Solution is $a_n = \left(\frac{1-i}{2}\right)(1+i)^n + \left(\frac{1+i}{2}\right)\cdot(1-i)^n$

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SOLUTIONS OF NON-HOMOGENEOUS LINEAR RECURRENCE RELATION

→ To solve non-homogeneous recurrence relation, we have two methods. They are:

1. Characteristic Roots Method
2. Generating Function Method.

1. CHARACTERISTIC ROOTS METHOD

→ Let $c_0 a_n + c_1 a_{n-1} + \dots + c_k a_{n-k} = f(n)$ — ①, $f(n) \neq 0$, $c_k \neq 0$

→ In order to solve non-homogeneous linear recurrence relation, we have to find two parts of the solution. They are: ~~(a) Homog~~

(a) Homogeneous Solution $a_n^{(h)}$

(b) Particular Solution $a_n^{(p)}$

→ The complete solution is given by: $\boxed{a_n = a_n^{(h)} + a}$

(a) Homogeneous Solution $[a_n^{(h)}]$

→ The homogeneous solution of ① which satisfies LHS of the recurrence relation, i.e., $a_n^{(h)}$ satisfies

$c_0 a_n + c_1 a_{n-1} + \ldots\ldots + c_k a_{n-k} = 0$.

(b) Particular Solution $[a_n^{(p)}]$

→ A solution which satisfies the given non-homogeneous recurrence relation is called a particular solution.

PROCEDURE TO ~~SOLVE~~ FIND HOMOGENEOUS SOLUTION

→ Let the given homogeneous reccurrence relation be ①.

Step I: Find the characteristic equation of the recurrence relation

Step II: Find the roots of the characteristic equation

Step III: Find the homogeneous solution

$a_n^{(h)} = D_1\alpha_1^n + D_2\alpha_2^n + \ldots\ldots + D_k\alpha_k^n$ (if $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$ are ~~roots~~ distinct roots)

PROCEDURE TO FIND PARTICULAR SOLUTION

→ The particular solution of the non-homogeneous linear recurrence relation depends on the nature of the RHS of the recurrence relation

Rule I: If $f(n) =$ constant, then particular solution is

$$a_n^{(p)} = \text{constant} = \alpha$$

Rule II: If $f(n) = a^n$ (a is constant), a is not a characteristic root of ①, then particular solution is

$$a_n^{(p)} = \text{constant} \times a^n = \alpha \cdot a^n$$

Rule III: If $f(n)$ is a polynomial ~~equation~~ in n with degree m, i.e., $f(n) = b_0 + b_1 n + b_2 n^2 + \dots + b_m n^m$, provided that ① is not a characteristic root, then the particular solution is given by

$$a_n^{(p)} = \alpha_0 + \alpha_1 n + \alpha_2 n^2 + \dots + \alpha_m n^m$$

Rule IV: If $f(n)$ is polynomial in n and multiplied with a^n (a is constant), a is not a characteristic root. Then the particular solution is given by:

$$a_n^{(p)} = (\alpha_0 + \alpha_1 n + \alpha_2 n^2 + \dots + \alpha_m n^m) \cdot a^n$$

[$f(n) = (b_0 + b_1 n + b_2 n^2 + \dots + b_m n^m) \cdot a^n$

(Special Rule)

Rule 5: If $f(n) = a^n$, a is a characteristic root and repeated 'r' times, then

$$a_n^{(p)} = \alpha \cdot a^n \cdot n^r$$

(special rules)

Rule 6: If $f(n) = (b_0 + b_1 n + b_2 n^2 + \dots + b_m n^m) \cdot a^n$; a is characteristic root and repeated 'r' times, then the particular solution is:

$$a_n^{(p)} = (Q_0 + Q_1 n + Q_2 n^2 + \dots + Q_m n^m) \cdot a^n \cdot n^r$$

1. Solve the following recurrence relation by characteristic roots method:

(a) $a_n - 9a_{n-1} + 20a_{n-2} = \phi$.

(b) $a_n - a_{n-1} - 6a_{n-2} = -30$

(c) $a_n - 7a_{n-1} + 10a_{n-2} = 4^n$

(d) $a_n + a_{n-1} = 3n \cdot 2^n$

(e) $a_n - 3a_{n-1} - 4a_{n-2} = 4^n$.

(f) $a_{n+2} - 6a_{n+1} + 9a_n = 3 \cdot 2^n + 7 \cdot 3^n$

~~Sol~~ (a).

Sol: ~~Given~~ Let $a_n - 9a_{n-1} + 20a_{n-2} = 1$ — ①

The complete solution for ① is given by

$\boxed{a_n = a_n^{(h)} + a_n^{(p)}}$ — ②

The characteristic equation for ① is

$$t^2 - 9t^1 + 20 \cdot t^0 = 0$$

$$\Rightarrow t^2 - 9t + 20 = 0$$

$$\Rightarrow (t-4)(t-5) = 0$$

$$\Rightarrow t = 4, 5$$

$$\therefore a_n^{(h)} = c_1 \cdot 4^n + c_2 \cdot 5^n \quad — ③$$

Particular Solution

Since RHS of ① is constant, i.e., $f(n) = $ constant

Particular solution, $a_n^{(p)} = a$ — ④

~~From~~ (Since, it satisfies ①)

i.e., put $a_n^{(p)} = a$ in ①

$\Rightarrow a - 9a + 20a = 1$

$\Rightarrow 12a = 1$

$\Rightarrow \boxed{a = \frac{1}{12}}$ — ⑤

$\therefore a_n^{(p)} = \frac{1}{12}$ — ⑥

From ②, ③, ④

$\Rightarrow \boxed{a_n = c_1 \cdot 4^n + c_2 \cdot 5^n + \frac{1}{12}}$

(b)

Sol: Let $a_n - a_{n-1} - 6a_{n-2} = -30$ — ①

The complete solution for ① is given by

$\boxed{a_n = a_n^{(h)} + a_n^{(p)}}$ — ②

The characteristic equation for ① is

$t^2 - t^1 - 6 \cdot t^0 = 0$

$\Rightarrow t^2 - t - 6 = 0$

$\Rightarrow t^2 + 2t - 3t - 6 = 0$

$\Rightarrow t(t+2) - 3(t+2) = 0$

$\Rightarrow (t+2)(t-3) = 0$

$\Rightarrow t = -2, 3$

$\therefore a_n^{(h)} = c_1 \cdot (-2)^n + c_2 \cdot 3^n$

Particular Solution

Since RHS of ① is constant, i.e., $f(n)$ = constant

Particular solution, $a_n^{(p)} = \alpha$ — ④

Since, it satisfies ①

i.e., put $a_n^{(p)} = \alpha$ in ①

$\Rightarrow \cancel{\alpha} - \cancel{\alpha} - 6\alpha = -30$

$\Rightarrow \alpha = 5$ — ⑤

$\therefore a_n^{(p)} = 5$ — ⑥

From ②, ③, ④

$$\Rightarrow \boxed{a_n = c_1 \cdot (-1)^n \cdot 2^n + c_2 \cdot 3^n + 5}$$

(c)

Sol: Let $a_n - 7a_{n-1} + 10a_{n-2} = 4^n$ — ①

The complete solution for ① is given by

$$\boxed{a_n = a_n^{(h)} + a_n^{(p)}} \text{ — ②}$$

The characteristic equation for ① is

$t^2 - 7t + 10 = 0$

$\Rightarrow (t-2)(t-5) = 0$

$\Rightarrow t = 2, 5$

$\therefore a_n^{(h)} = c_1 \cdot 2^n + c_2 \cdot 5^n$ — ③

Particular Solution

Since RHS of ① is of the form a^n where a is not a characteristic root of ①, then

Particular Solution, $a_n^{(p)} = a \times 4^n$ — ~~3~~ ④

Since, it satisfies ①

i.e., $a \times 4^n - 7(a \times 4^{n-1}) + 10(a \times 4^{n-2}) = 4^n$

$\Rightarrow \cancel{(a - 7a + 10a) 4^n = 4^n}$

$\Rightarrow \cancel{16 - 4a = 1}$

$\cancel{a = \frac{1}{4}}$

$\Rightarrow a \cdot 4^{n-2}[4^2 - 7 \cdot 4 + 10] = 4^n$

$\Rightarrow \frac{a}{16}[16 - 28 + 10] = 1$

$\Rightarrow a = \frac{16}{-2}$

$\Rightarrow a = -8 \rightarrow$ ~~⑤~~

~~From~~ $\therefore a_n^{(p)} = -8 \cdot 4^n$ ⑤

From ②, ③, ⑤

$\therefore \boxed{a_n = c_1 \cdot 2^n + c_2 \cdot 5^n - 8 \cdot 4^n}$

(d) Sol:

Let $a_n + a_{n-1} = 3n \cdot 2^n$ — ①

The complete solution for ① is given by

$\boxed{a_n = a_n^{(h)} + a_n^{(p)}}$ — ②

The characteristic equation for ① is

$t^2 + t = 0$

$\Rightarrow \cancel{t(t+1) = 0}$ $\Rightarrow t = -1$

$\Rightarrow \cancel{t = 0, -1}$

$\therefore a_n^{(h)} = \cancel{\quad} C_2 \cdot (-1)^n$

$= \cancel{\quad} C_2 \cdot (-1)^n$

Particular Solution

Since, RHS of ① is in n and multiplied with a^n (a is constant and not a characteristic root), ~~then~~ i.e.;

$\cancel{a_n^{(p)}} \quad f(n) = (b_0 + b_1 n) \cdot a^n$

Particular solution, $a_n^{(p)} = (\alpha_0 + \alpha_1 n) \cdot 2^n$, 2 is not a characteristic root.

Since it satisfies ①

i.e., put $a_n^{(p)} = (\alpha_0 + \alpha_1 n) \cdot 2^n$ in ①

$\Rightarrow (\alpha_0 + \alpha_1 n) \cdot 2^n + (\alpha_0 + \alpha_1 (n-1)) \cdot 2^{n-1} = 3n \cdot 2^n$

$\Rightarrow \alpha_0 + \alpha_1 n + (\alpha_0 + \alpha_1 (n-1)) \cdot 2^{-1} = 3n$

$\Rightarrow \cancel{2\alpha_0 + 2\alpha_1 n} \; \alpha_0 + \alpha_1 n + \frac{\alpha_0}{2} + \frac{\alpha_1}{2}(n-1) = 3n$

$\Rightarrow \frac{3\alpha_0}{2} + \frac{\alpha_1}{2}(2n + n - 1) = 3n$

$\Rightarrow \frac{3\alpha_0}{2} + \frac{\alpha_1}{2}\overset{(3n-1)}{\cancel{(1+n)}} = 3n$ $\qquad \alpha_1 = 2$, $\alpha_0 = \frac{2}{3}$

$\Rightarrow \cancel{\frac{3\alpha_0}{2} + \frac{\alpha_1}{2} \cdot \frac{\alpha_1}{2} \cdot n + \left(\frac{3\alpha_0}{2} + \frac{\alpha_1}{2}\right) = 3n}$ $\qquad \Rightarrow \left(\frac{3\alpha_0}{2} - \frac{\alpha_1}{2}\right) + \frac{3\alpha_1 n}{2} = 3n$

Comparing like coefficients on b.s., we get

$\cancel{\frac{\alpha_1}{2} = 3}$ $\quad \frac{3\alpha_0}{2} - \frac{\alpha_1}{2} = 0 \qquad \frac{3\alpha_1}{2} = 3$

$\cancel{\Rightarrow \alpha_1 = 6}$ $\quad \Rightarrow \frac{3\alpha_0}{2} - \frac{2}{2} = 0 \qquad \Rightarrow \alpha_1 = 2$

$\Rightarrow \frac{3\alpha_0}{2} = 1$

$\Rightarrow \alpha_0 = \frac{2}{3}$

$\therefore a_n^{(p)} = \left(\frac{2}{3} + 2n\right) \cdot 2^n$ — ⑤

From ②, ③ and ⑤

$$\Rightarrow \boxed{a_n = c_1 \cdot 2^n + \left(\frac{2}{3} + 2n\right) 2^n}$$

(e) Sol:

Let

$a_n - 3a_{n-1} - 4a_{n-2} = 4^n$ — ①

The complete solution for ① is given ~~as~~ by

$$\boxed{a_n = a_n^{(h)} + a_n^{(p)}} \quad — ②$$

The characteristic equation for ① is given by

$$t^2 - 3t - 4 = 0$$

$$\Rightarrow (t+1)(t-4) = 0$$

$$\Rightarrow t = -1, 4$$

$$\therefore a_n^{(h)} = c_1 \cdot (-1)^n + c_2 (4)^n$$

$$= c_1 \cdot (-1)^n + c_2 \cdot 4^n \quad — ③$$

Particular Solution

Since RHS is of the form a^n where a is ~~not~~ a characteristic root and repeated r times (Here, $r = 1$), i.e.;

$$a_n^{(p)} = \alpha \cdot a^n \cdot n^r$$

~~$f(n) = (b_0 + b_1 n) \cdot a^n$~~

Particular solution; $a_n^{(p)} = \alpha \cdot 4^n \cdot n^1$

$$= \alpha \cdot 4^n \cdot n \quad — ④$$

Since it satisfies ①

i.e., $a \cdot 4^n \cdot n - 3[a \cdot 4^{n-1}(n-1)] - 4[a \cdot 4^{n-2} \cdot (n-2)]$
$= 4^n$

$\Rightarrow a \cdot n - 3[a \cdot 4^{-1}(n-1)] - 4[a \cdot 4^{-2} \cdot (n-2)] = 1$

$\Rightarrow a \cdot n - \frac{3}{4}a(n-1) - \frac{1}{4}a(n-2) = 1$

~~$\Rightarrow a \cdot n\left(1 - \frac{3}{4} - \frac{1}{4}\right) + \frac{3a}{4}$~~

$\Rightarrow a \cdot n - \frac{3a}{4} \cdot n + \frac{3a}{4} - \frac{an}{4} + \frac{2}{4} = 1$

$\Rightarrow a \cdot n\left(1 - \frac{3}{4} - \frac{1}{4}\right) + \frac{3a}{4} + \frac{1}{2} = 1$

$\Rightarrow$

$\frac{3a}{\cancel{4}\,2} = \frac{1}{\cancel{2}}$

$\Rightarrow a = \frac{2}{3}$

$\therefore a_n^{(p)} = \frac{2}{3} \cdot 4^n \cdot n$ — ⑤

From ②, ③ and ⑤

$\Rightarrow \boxed{a_n = C_1 \cdot (-1)^n + C_2 \cdot 4^n + \frac{2n}{3} \cdot 4^n}$

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Sol:

$a_{n+2} - 6a_{n+1} + 9a_n = 3 \cdot 2^n + 7 \cdot 3^n$ — ① $a_0 = 1$; $a_1 = 4$

e complete solution for ① is given by

$\boxed{a_n = a_n^{(h)} + a_n^{(p)}}$ — ②

e characteristic equation for ① is ~~given by~~

$t^2 - 6t + 9 = 0$

$\Rightarrow (t-3)^2 = 0$

$\Rightarrow t = 3, 3$

$\therefore a_n^{(h)} = (C_1 + nC_2) \cdot 3^n$ — ③

Particular Solution

Since RHS is of the form a^n [$f(n) = a^n$], a is a characteristic root and repeated r-times (Here, $r = 2$), then

$a_n^{(p)} = Q_0 2^n + Q_1 \cdot 3^n \cdot n^2$ — ④

Since, it satisfies ①

$\Rightarrow Q_0 \cdot 2^{n+2} + Q_1 \cdot 3^{n+2} \cdot (n+2)^2 - 6[Q_0 \cdot 2^{n+1} + Q_1 \cdot 3^{n+1} \cdot (n+1)^2$
$+ 9[Q_0 \cdot 2^n + Q_1 \cdot 3^n \cdot n^2] = 3 \cdot 2^n + 7 \cdot 3^n$

$\Rightarrow 2^n[Q_0 \cdot 2^2 - 6Q_0 \cdot 2 + 9Q_0] + 3^n[Q_1 \cdot 3^2 \cdot (n+2)^2 - 6Q_1 \cdot 3(n+1)^2$
$+ 9Q_1 \cdot n^2] = 3 \cdot 2^n + 7 \cdot 3^n$

$\Rightarrow 2^n[4Q_0 - 12Q_0 + 9Q_0] + 3^n[9Q_1(n+2)^2 - 18Q_1(n+1)^2$
$+ 9Q_1 n^2] = 3 \cdot 2^n + 7 \cdot 3^n$

~~$\Rightarrow 2^n[Q_0] + 3^n \, 9Q_1[9n^2 + 36 +$~~

$\Rightarrow 2^n[Q_0] + 3^n \cdot Q_1[\cancel{9n^2} + 36 + \cancel{36n} - \cancel{18n^2} - 18 - \cancel{36n} + \cancel{9n^2}]$
$= 3 \cdot 2^n + 7 \cdot 3^n$

$\Rightarrow 2^n[Q_0] + 3^n \cdot Q_1[18] = 3 \cdot 2^n + 7 \cdot 3^n$

$\Rightarrow 2^n[Q_0] + 3^n[18Q_1] = 3 \cdot 2^n + 7 \cdot 3^n$

Comparing like terms on b.s, we get:

$\Rightarrow Q_0 = 3$ $\quad 18Q_1 = 7$

$Q_1 = \frac{7}{18}$

$\therefore$ Particular Solution, $a_n^{(p)} = 3.2^n + \frac{7}{18}.3^n.n^2$ — (5)

From (2), (3) and (5)

$\therefore a_n = (c_1 + nc_2).3^n + 3.2^n + \frac{7}{18}.3^n.n^2$ — (6)

Given, $a_0 = 1$; $a_1 = 4$

Put $n=0$ in (6) $\Rightarrow a_0 = c_1 + 3$

$\Rightarrow 1 = c_1 + 3 \Rightarrow \boxed{c_1 = -2}$

Put $n=1$ in (6) $\Rightarrow a_1 = (c_1 + c_2).3 + 6 + \frac{7}{18}.3$

$\Rightarrow 4 = (-2 + c_2).3 + 6 + \frac{7}{6}$

$\Rightarrow -2 - \frac{7}{6} = 3(-2 + c_2)$

$\Rightarrow -2 + c_2 = \frac{-19}{18}$

$\Rightarrow c_2 = \frac{-19}{18} + 2$

$\boxed{c_2 = \frac{17}{18}}$

$\therefore \boxed{a_n = \left(-2 + \frac{17}{18}n\right).3^n + 3.2^n + \frac{7}{18}.3^n.n^2}$

$\Rightarrow \boxed{a_n = 3.2^n + 3^n\left[\frac{7}{18}n^2 + \frac{17}{18}n - 2\right]}$

2. GENERATING FUNCTION METHOD

P.T.O. $\longrightarrow$

GENERATING FUNCTION METHOD

Step 1: Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ — ①

Step 2: Multiply by x^n and take $\sum_{n=k}^{\infty}$ to the given non-homogeneous linear recurrence relation where k is the order of Recurrence relation

1. Solve the recurrence relation:

$a_n - a_{n-1} = 3(n-1)$; $\forall$ $n \geq 1$ & $a_0 = 2$

Sol: Given, $a_n - a_{n-1} = 3(n-1)$ $\forall$ $n \geq 1$ and $a_0 =$ — ①

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ — ②

Multiply ① by x^n and take $\sum_{n=1}^{\infty}$ on both sides of ①

$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 3 \sum_{n=1}^{\infty} (n-1) x^n$$

$$\Rightarrow [G(x) - a_0] - x[G(x)] = 3 \sum_{n=1}^{\infty} (n-1) x^n$$

$$\Rightarrow G(x)[1-x] - a_0 = 3 \sum_{n=1}^{\infty} (n-1) x^n$$

$$\Rightarrow G(x) = \frac{a_0 + 3 \sum_{n=1}^{\infty} (n-1) x^n}{(1-x)}$$

$$= \frac{2 + 3(x^2 + 2x^3 + 3x^4 + \ldots\ldots)}{(1-x)}$$

$$= \frac{2}{1-x} + \frac{3x^2(1 + 2x + 3x^2 + \ldots\ldots)}{1-x}$$

$$= \frac{2}{1-x} + \frac{3x^2(1-x)^{-2}}{1-x}$$

$$= \frac{2}{1-x} + \frac{3x^2}{(1-x)^3}$$

$$= \frac{2}{1-x} + 3x^2 \sum_{n=0}^{\infty} (1-x)^{-3}$$

$$= 2 \sum (1-x)^{-1} + 3x^2 \sum (1-x)^{-3}$$

$$= 2 \sum_{n=0}^{\infty} x^n + 3x^2 \sum_{r=0}^{\infty} C(2+r, r) . x^r$$

$$= 2 \sum_{n=0}^{\infty} (1)^n x^n + 3 \sum_{r=0}^{\infty} C(2+r, r) . x^{r+2}$$

$$= 2 \sum_{n=0}^{\infty} (1)^n x^n + 3 \sum_{n=0}^{\infty} C(n+2, n) . x^{n+2}$$

$$= 2 \sum_{n=0}^{\infty} (1)^n x^n + 3 \sum_{n=0}^{\infty} C(n-2+2, n-2) . x^n$$

$$= 2 \sum_{n=0}^{\infty} (1)^n x^n + 3 \sum_{n=0}^{\infty} C(n, n-2) . x^n$$

$$= 2 \sum_{n=0}^{\infty} (1)^n x^n + 3 \sum \frac{n!}{(n-2)!2!} . x^n$$

$$= 2 \sum_{n=0}^{\infty} (1)^n x^n + 3 \sum \frac{n(n-1)}{2} . x^n$$

$$\sum a_n x^n = \sum \left[2.(1)^n + \frac{3}{2} (n)(n-1) \right] . x^n$$

$$\Rightarrow \boxed{a_n = 2 + \frac{3}{2} n(n-1)}$$