Note: Let A and B be two non-empty, finite disjoint sets.

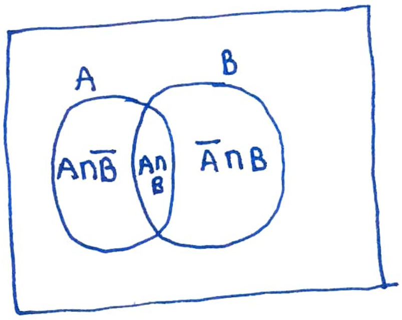
Then $|A \cup B| = |A| + |B|$.

Principle of inclusion-exclusion:

Let A and B be two non-empty, finite subsets of an universal set U.

Then $|A \cup B| = |A| + |B| - |A \cap B|$.

Proof: Let U be an universal set and A and B be two non-empty finite subsets of U.



From the venn diagram, we have

$$A = (A \cap \bar{B}) \cup (A \cap B)$$

and

$$B = (\bar{A} \cap B) \cup (A \cap B).$$

Clearly, $A \cap \bar{B}$, $A \cap B$ and $\bar{A} \cap B$ are disjoint, and

$$A \cup B = (A \cap \bar{B}) \cup (A \cap B) \cup (\bar{A} \cap B).$$

$$\Rightarrow |A \cup B| = |A \cap \bar{B}| + |A \cap B| + |\bar{A} \cap B|, \quad \text{—— (I)}$$

$$|A| = |A \cap \bar{B}| + |A \cap B| \quad \text{and} \quad \text{—— (II)}$$

$$|B| = |\bar{A} \cap B| + |A \cap B| \quad \text{—— (III)}$$

Now, substitute eq (II) & (III) in (I), we get

$$|A \cup B| = |A| - |A \cap B| + |A \cap B| + |B| - |A \cap B|.$$

$$\Rightarrow |A \cup B| = |A| + |B| - |A \cap B|.$$

Similarly, if A, B and C are non-empty finite subsets of U then

$$|A\cup B\cup C| = |A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|.$$

Problem: Suppose that 200 faculty members can speak French and 50 can speak Russian, while only 20 can speak both French and Russian. How many faculty members can speak either French or Russian?

Sol: Let U be the set of all faculty members.

Let F & R be the collection of all faculty members who speak French and Russian respectively.

Then from the given data

$$|F\cap R| = 20, \quad |F| = 200, \quad |R| = 50.$$

By Principle of inclusion and exclusion, we have

$$|F\cup R| = |F| + |R| - |F\cap R|$$

$$= 200 + 50 - 20$$

$$\Rightarrow |F\cup R| = 230.$$

Problem: If there are 200 faculty members that speak French, 50 speak Russian, 100 that speak Spanish, 20 that speak French and Russian, 60 that speak French and Spanish, 35 that speak Russian and Spanish while only 10 speak French, Russian, and spanish. How many speak either French or Russian or Spanish ?.

Sol: Let U be the collection of all faculty members.

Let F, R and S be the collection of faculty members who speak French, Russian and Spanish respectively.

Then from the given data, we have

$|F| = 200$, $|R| = 50$, $|S| = 100$, $|F \cap R| = 20$,

$|F \cap S| = 60$, $|R \cap S| = 35$ and $|F \cap R \cap S| = 10$.

By Principle of inclusion and exclusion, we have

$$|F \cup R \cup S| = |F| + |R| + |S| - |F \cap R| - |F \cap S| - |R \cap S| + |F \cap R \cap S|$$

$$= 200 + 50 + 100 - 20 - 60 - 35 + 10$$

$$\Rightarrow |F \cup R \cup S| = 245.$$

Problem: In a Survey of students at Florida state university the following information was obtained.

260 were taking a statistics course, 208 were taking a Mathematics course, 160 were taking a computer programming course, 76 were Taking statistics and mathematics, 48 were taking statistics and computer programming, 62 were taking mathematics course and computer programming, 30 were taking all 3 kinds of courses, and 150 were taking none of the 3 courses. Find

(a) How many students were surveyed?

(b) How many students were Taking a statistics and a mathematics course but not a computer programming course?

(c) How many were taking a statistics and a computer course but not a mathematics course?

(d) How many were taking a computer programming and a mathematics course, but not a statistics course?

(e) How many were Taking a statistics course but not Taking a course in mathematics or a computer programming?

Sol:- Let U be the set of all students at Florida university.

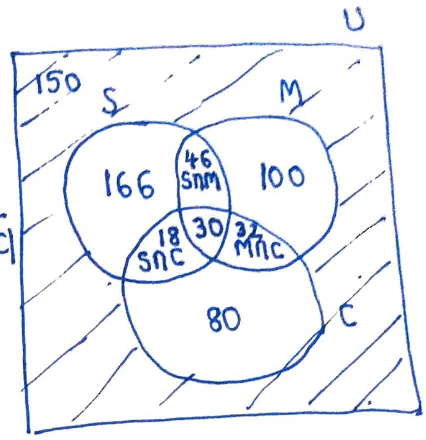
Let S, M and C be the collection of all students who are taking Statistics, Mathematics and computer programming resp.

Given that,

$|S| = 260$, $|M| = 208$, $|C| = 160$, $|S \cap M| = 76$,

$|S \cap C| = 48$, $|M \cap C| = 62$, $|S \cap M \cap C| = 30$,

$|\overline{S \cup M \cup C}| = 150$.



(a) From the venn diagram,

$|U| = 260 + 100 + 32 + 80 + |\overline{S \cup M \cup C}|$

$\Rightarrow |U| = 472 + 150 = 662$

$\Rightarrow |U| = 662$.

(b) From the venn diagram,

$|S \cap M \cap \bar{C}| = 46$.

(c) From the venn diagram,

$|S \cap C \cap \bar{M}| = 18$.

(d) From the venn diagram,

$|M \cap C \cap \bar{S}| = 32$.

(e) From the venn diagram,

$|S \cap (\overline{M \cup C})|$

$= |S \cap \bar{M} \cap \bar{C}|$

$= 166$.

Problem: Let U be the set of integers x such that $1 \leq x \leq 1000$. (6)

Let A_2, A_3, A_5, A_7 be the set of elements of U divisible by $2, 3, 5$ and 7 respectively. Then find $|\overline{A_2} \cap \overline{A_3} \cap \overline{A_5} \cap \overline{A_7}|$.

Sol: Let $U = \{x / 1 \leq x \leq 1000,\ x \text{ is an integer}\}$.

Let A_2, A_3, A_5, A_7 be the set of elements of U divisible by $2, 3, 5$ and 7 respectively.

Then
$$|A_2| = \left[\frac{1000}{2}\right] = 500,$$
$$|A_3| = \left[\frac{1000}{3}\right] = 333,$$
$$|A_5| = \left[\frac{1000}{5}\right] = 200,$$
$$|A_7| = \left[\frac{1000}{7}\right] = 142,$$

By principal of inclusion and exclusion, we have

$$|\overline{A_2} \cap \overline{A_3} \cap \overline{A_5} \cap \overline{A_7}| = |\overline{A_2 \cup A_3 \cup A_5 \cup A_7}|$$
$$= 1000 - |A_2 \cup A_3 \cup A_5 \cup A_7|.$$
$$= 1000 - [|A_2| + |A_3| + |A_5| + |A_7|$$
$$- |A_2 \cap A_5| - |A_2 \cap A_3| - |A_3 \cap A_5|$$
$$- |A_3 \cap A_7| - |A_5 \cap A_7| - |A_7 \cap A_2|$$
$$+ |A_2 \cap A_3 \cap A_7| + |A_2 \cap A_3 \cap A_5| + |A_3 \cap A_5 \cap A_7|$$
$$- |A_2 \cap A_3 \cap A_5 \cap A_7| + |A_2 \cap A_5 \cap A_7|] \quad \text{— (I)}$$

Clearly,

$$|A_2 \cap A_3| = \left[\frac{1000}{\text{lcm}(2,3)}\right] = \left[\frac{1000}{6}\right] = 166,$$

$$|A_2 \cap A_5| = \left[\frac{1000}{\text{lcm}(2,5)}\right] = \left[\frac{1000}{10}\right] = 100,$$

$$|A_2 \cap A_7| = \left[\frac{1000}{\text{lcm}(2,7)}\right] = \left[\frac{1000}{14}\right] = 71,$$

$$|A_3 \cap A_5| = \left[\frac{1000}{\text{lcm}(3,5)}\right] = \left[\frac{1000}{15}\right] = 66,$$

$$|A_3 \cap A_7| = \left[\frac{1000}{\text{lcm}(3,7)}\right] = \left[\frac{1000}{21}\right] = 47,$$

$$|A_5 \cap A_7| = \left[\frac{1000}{\text{lcm}(5,7)}\right] = \left[\frac{1000}{35}\right] = 28,$$

$$|A_2 \cap A_3 \cap A_7| = \left[\frac{1000}{\text{lcm}(2,3,7)}\right] = \left[\frac{1000}{42}\right] = 23,$$

$$|A_2 \cap A_3 \cap A_5| = \left[\frac{1000}{\text{lcm}(2,3,5)}\right] = \left[\frac{1000}{30}\right] = 33,$$

$$|A_3 \cap A_5 \cap A_7| = \left[\frac{1000}{105}\right] = 9,$$

$$|A_2 \cap A_5 \cap A_7| = \left[\frac{1000}{70}\right] = 14, \text{ and}$$

$$|A_2 \cap A_5 \cap A_3 \cap A_7| = \left[\frac{1000}{210}\right] = 4$$

From (I), we have

$$|\overline{A_2} \cap \overline{A_3} \cap \overline{A_5} \cap \overline{A_7}] = 1000 - [500 + 333 + 200 + 142 - 166 - 100 - 71 - 66 - 4 - 47 - 28 + 23 + 33 + 9 + 14]$$

$$= 1000 - 772 = 228.$$

Euler's function / Euler's totient function :

1. Let m and n be two positive integers. Then they are said to be relatively prime if $(m, n) = 1$. ((m, n) means g.c.d of m & n).

2. Let n be a positive integer. Then the Euler's function is denoted by $\phi(n)$ and defined by the number of positive integers not exceeding n and which are relatively prime to n.

Ex: $\phi(2) = 1$. $\phi(5) = 4$. $\phi(9) = 6$ $\phi(8) = 4$.

$\phi(3) = 2$. $\phi(6) = 2$ $\phi(10) = 4$

Note: $\phi(mn) = \phi(m) \cdot \phi(n) \cdot \dfrac{d}{\phi(d)}$. where $d = (m, n)$