27/09/23 UNIT-I: MATHEMATICAL LOGIC

FUNDAMENTALS OF LOGIC

· PREPOSITION

→ It is a declarative sentence which is meaningful and assigned one and only one truth value (T/F).

Ex 1. All the CSM-A students gets placed in Amazon with 22 LPA

2. The sun rises in the East.

Note: The prepositions are denoted by lower case letters viz p, q, r, s, t and so on.

→ Prepositions are connected by means of connectives like conjunction (∧/and), disjunction (∨ (or), implies (→ / If...then), double implies (↔ / If..... only if), negation (~ / ¬)

· CONSTRUCTION OF TRUTH TABLES

1. Conjunction (∧)

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

p, q are two prepositions ~~truth positions~~ (statement

p and q is T if both p and q are true, otherwise false.

2. Disjunction (∨)

P.T.O. →

2. Disjunction ($\vee$)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p, q are two propositions (statements).

$p \vee q$ is false, if both p and q are false, otherwise true

3. Implication ($\rightarrow$)

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p, q are two propositions (statements)

$p \rightarrow q$ is false if p is true and q is false, otherwise true

4. Biimplication ($\leftrightarrow$)

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

p, q are two propositions (statements)

$p \leftrightarrow q$ is true if both p and q are same truth values, otherwise false

5. Negation ($\sim$/$\neg$)

p	$\sim p$
T	F
F	T

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TAUTOLOGY

→ It is a preposition function whose truth value is true for all possible values of preposition variables (i.e., in the resultant truth table, the last column consists of all are true).

CONTRADICTION (ABSURDITY)

→ It is a preposition function whose truth value is false for all possible values of preposition variables (i.e., in the resultant truth table, the last column consists of all are false).

Q. Verify the following prepositions are tautology or not

(i) $\sim(p \vee q) \vee [\sim p \wedge q] \vee p$

(ii) $[p \rightarrow [(q \vee r) \wedge \sim q]] \rightarrow (p \rightarrow r)$

(iii) $[(p \rightarrow q) \wedge (q \rightarrow r)] \leftrightarrow [(p \vee q) \rightarrow r]$

(iv) $[(p \vee q) \rightarrow r] \wedge (\neg p)] \rightarrow (q \rightarrow r)$

Sol:

(i)

p	q	$\sim p$	$p \vee q$ (a)	$\sim p \wedge q$ (b)	$b \vee p$ (c)	$\sim(p \vee q)$ $\sim a$	$\sim a \vee c$
T	T	F	T	F	T	F	T
T	F	F	T	F	T	F	T
F	T	T	T	T	T	F	T
F	F	T	F	F	F	T	T

→ The above preposition is a tautology

(ii)

p	q	r	$\sim q$	$q \vee r$ (a)	$a \wedge \sim q$ (b)	$p \to b$ (c)	$p \to r$ (d)	$c \to d$
T	T	T	F	T	F	F	T	T
T	T	F	F	T	F	F	F	T
T	F	T	T	T	T	T	T	T
T	F	F	T	F	F	F	F	T
F	T	T	F	T	F	T	T	T
F	T	F	F	T	F	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	F	F	T	T	T

→ The above preposition is a tautology

(iii)

p	q	r	$p \to q$ (a)	$q \to r$ (b)	$a \wedge b$ (c)	$p \vee q$ (d)	$d \to r$ (e)	$c \leftrightarrow e$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	F	T
T	F	T	F	T	F	T	T	F
T	F	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	F	F	T	F	T
F	F	T	T	T	T	F	T	T
F	F	F	T	T	T	F	T	T

→ The above preposition is not a tautology

→

(iv)

p	q	r	$p \vee q$ (a)	$a \to r$ (b)	$\neg q$	$b \wedge (\neg q)$ (c)	$q \to r$ (d)	$c \to d$
T	T	T	T	T	F	F	T	T
T	T	F	T	F	F	F	F	T
T	F	T	T	T	T	T	T	T
T	F	F	T	F	T	F	T	T
F	T	T	T	T	F	F	T	T
F	T	F	T	F	F	F	F	T
F	F	T	F	T	T	T	T	T
F	F	F	F	T	T	T	T	T

→ The above expression is a tautology

TRANSLATION OF SENTENCES TO THE SYMBOLIC FORM

Q Construct the following prepositions

(a) $p \to$ Vandhana is playing cricket

$q \to$ Vandhana is inside a home

$r \to$ Vandhana is doing her homework

$s \to$ Vandhana is listening to music.

Translate the following sentences into symbolic form by using above prepositions.

(i) Either Vandhana playing cricket or inside her home. [$p \vee q$]

(ii) Neither Vandhana play cricket nor doing her homework. [$\sim p \wedge \sim \cancel{q}$ (or) $\sim(p \vee r)$]

$\boxed{\text{Either... or} \rightarrow \text{or} (\vee) \quad \text{Neither... nor} \rightarrow \text{and} (\wedge)}$

(iii) Vandhana is playing cricket and not doing her homework. $[p \wedge \sim r]$

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(iv) Vandhana is inside doing her homework and not playing cricket. $[(q \wedge r) \wedge (\sim p)]$

(v) Vandhana is inside doing her homework while listening to music. ~~E~~ and not playing cricket.
$[(q \wedge r) \wedge s] \wedge (\neg p)]$

$\boxed{\text{while} \rightarrow \text{and} (\wedge)}$

(vi) Neither Vandhana listening to music nor playing cricket. ~~E~~ and she is doing her homework inside. $[\neg(s \vee p) \wedge (q \wedge r)]$

(vii) Neither Vandhana listening to music nor playing cricket and she is either doing her homework or she is inside. $[\neg(s \vee p) \wedge (q \vee r)]$

Q. Use the above prepositions p, q, r, s and translate the following prepositions into acceptable English.

(i) $(\neg p) \wedge (\sim q)$

(ii) $p \vee (q \wedge r)$

(iii) $p \vee (\sim r)$

(iv) $(\neg p \vee q) \wedge (\neg r \vee s)$

(v) $(\sim p \wedge q) \vee (\neg p \wedge s)$

Sol:

(i) Neither Vandhana is playing cricket nor inside her home.

(ii) Either Vandhana is playing cricket or she is inside her home and ~~playing~~ doing homework.

(iii) Either Vandhana is playing cricket or she is not doing her homework.

(iv) Either Vandhana is not playing cricket or she is inside her house and not doing her homework or listening to music

(v) Neither Vandhana is playing cricket nor inside her home or not playing cricket and listening to music.

INFERENCE

→ It is an idea or conclusion that is drawn from evidence and proper reason.

Ex:

1. A person with a bag having a drafter and laptop
- Therefore the person is an Engineering student.

2. A person with a camera and tripod stand is a photographer.

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Q. If Sai reads a daily worker, then he is a Communist.

$(p \rightarrow q)$

Sai reads daaly worker, therefore he is a Communist.

$$\left.\begin{array}{l} \frac{p}{\therefore q} \rightarrow \text{Premise} \\ \phantom{\frac{p}{q}} \rightarrow \text{Conclusion} \end{array}\right\} \text{Argument}$$

Note: All the premises are connected by conjunction [and (∧)] and the straight line replaced by implication and the conclusion should be written after implication. The whole thing is called an argument.

Ex: $\begin{array}{c} p \\ q \\ \hline \therefore r \end{array}$ The tautological form is $(p \wedge q) \rightarrow r$

→ There are two types of arguments (Inferences).
They are: (a) Valid Argument
(b) Invalid Argument

RULES OF INFERENCES & REFERRED TO THE LANGUAGE OF PROPOSITION

Rule of Inference	Tautological Form	Name
1. $\frac{p}{\therefore p \vee q}$ (or) $\frac{q}{\therefore p \vee q}$	$p \rightarrow (p \vee q)$ (or) $q \rightarrow (p \vee q)$	Addition
2. $\frac{p \wedge q}{\therefore p}$ (or) $\frac{p \wedge q}{\therefore q}$	$(p \wedge q) \rightarrow p$ (or) $(p \wedge q) \rightarrow q$	Simplification

3. $\begin{array}{l} p \\ p \to q \\ \hline \therefore q \end{array}$	$(p \wedge (p \to q)) \to q$	Modus Ponneoss
4. $\begin{array}{l} \sim q \\ p \to q \\ \hline \therefore \sim p \end{array}$	$[(\sim q) \wedge (p \to q)] \to \sim p$	Modus Tollens
5. $\begin{array}{l} p \vee q \\ \sim p \\ \hline \therefore q \end{array}$	$[(p \vee q) \wedge (\sim p)] \to q$	Disjunctive Syllogism
6. $\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical Syllogism
7. $\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$	$(p \wedge q) \to (p \wedge q)$	Conjunction
8. $\begin{array}{l} (p \to q) \wedge (r \to s) \\ p \vee r \\ \hline \therefore q \vee s \end{array}$	$[(p \to q) \wedge (r \to s)] \wedge (p \vee r) \to (q \vee s)$	Constructive Delemma
9. $\begin{array}{l} (p \to q) \wedge (r \to s) \\ \sim q \vee \sim s \\ \hline \therefore (\sim p \vee \sim r) \end{array}$	$[(p \to q) \wedge (r \to s)] \wedge (\sim q \vee \sim s) \to (\sim p \vee \sim r)$	Destructive Dilemma

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EXAMPLES ON INFERENCE RULES

1. Addition

p: Sravan is studying very hard.

q: Sravan is a good student.

$\dfrac{p}{\therefore p \vee q}$ i.e.,

Sravan is studying very hard, therefore Sravan is either studying very hard or a good student.

2. Simplification

p: Vardhan is a very good student.

q: Vardhan got first class with distinction.

$\dfrac{p \wedge q}{\therefore p}$

Vardhan is a very good student and got first class with distinction. Therefore Vardhan is a very good student.

3. Modus Ponens

p: Mahesh Babu has password.

q: Mahesh Babu can login to Twitter.

~~$(p \wedge (p \to q)) \to q$~~

$$\begin{array}{c} p \\ \underline{p \to q} \\ \therefore q \end{array}$$

Mahesh Babu has password and if Mahesh Babu has password then he can login to Twitter. Therefore Mahesh Babu can login to Twitter.

4. <u>Modus Tollens</u>

$$\begin{array}{l} \sim q \\ p \to q \\ \hline \therefore \sim p \end{array}$$

Mahesh Babu cannot login to Twitter and if he has password then he can login to Twitter. Therefore Mahesh Babu does not have password.

8. If Mahesh Baby has password then he can login to Twitter but he cannot login to Twitter. Therefore, Mahesh Babu does not have password.

5. <u>Hypothetical Syllogism</u>

p: Harshini is studing very hard.

q: Harshini writes exams very well.

r: Harshini is toppers of the college.

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

If Harshini is studying very hard then she will writes exams very well and if she writes exams very well, then she will be the topper of the college. Therefore if Harshini is studying very hard, then she will be the topper of the college.

6. <u>Disjunctive Syllogism</u>

p: Ice cream is Vanilla flavour.

q: Ice cream is chocolate flavour.

$$\begin{array}{l} p \vee q \\ \underline{\sim p} \\ \therefore q \end{array}$$

Ice cream is either Vanilla or chocolate flavour and not Vanilla flavour. Therefore ice cream is chocolate flavour.

7. <u>Constructive Dilemma</u>

p: It is raining

q: I will take leave

r: It is too hot outside

s: I will go to shower.

$$\begin{array}{l} (p \to q) \wedge (r \to s) \\ \underline{p \vee r} \\ \therefore q \vee s \end{array}$$

If it is raining then I will take leave and if it is too hot outside then I will go to shower and it is either raining or it is too hot ~~as~~ outside. Therefore, I will take leave and I will go to shower.

8. **Destructive Dilemma**

$$(p \to q) \land (r \to s)$$
$$\underline{\sim q \lor \sim s}$$
$$\therefore (\sim p \lor \sim r)$$

If it is raining then I will take leave and if it is too hot outside then I will go to shower and ~~it is~~ either I will not take leave or will not go to shower. Therefore, I it is not raining or it is not too hot outside.

EQUIVALENT FORMULAE

1. $\sim(\sim p) \Leftrightarrow p$ - Double Negation
2. $p \land q \Leftrightarrow q \land p$
3. $p \lor q \Leftrightarrow q \lor p$ } Commutative
4. $p \land (q \land r) \Leftrightarrow (p \land q) \land r$
5. $p \lor (q \lor r) \Leftrightarrow (p \lor q) \lor r$ } Associative
6. $p \lor (q \land r) \Leftrightarrow (p \lor q) \land (p \lor r)$
7. $p \land (q \lor r) \Leftrightarrow (p \land q) \lor (p \land r)$ } Distributive
8. $\sim(p \lor q) \Leftrightarrow (\sim p) \land (\sim q)$
9. $\sim(p \land q) \Leftrightarrow (\sim p) \lor (\sim q)$ } De-Morgan's Law

10. $p \vee p \Leftrightarrow p$

11. $p \wedge p \Leftrightarrow p$

12. $r \vee (p \wedge \sim p) \Leftrightarrow r$

13. $r \wedge (p \vee \sim p) \Leftrightarrow r$

14. $r \wedge (p \wedge \sim p) \Leftrightarrow F$

15. $r \vee (p \vee \sim p) \Leftrightarrow T$

☆ 16. $p \to q \Leftrightarrow \sim p \vee q$

☆ 17. $\sim (p \to q) \Leftrightarrow p \wedge \sim q$

☆ 18. $p \to q \Leftrightarrow (\sim q) \to (\sim p)$

19. $p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$

20. $\sim (p \leftrightarrow q) \Leftrightarrow p \leftrightarrow (\sim q)$

21. $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$

22. $(p \to q) \Leftrightarrow (p \wedge q) \vee (\sim p \wedge \sim q)$

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Q. If the baby is hungry, then baby cries.

If the baby is not mad, then he doesn't cry.

If the baby is mad, then has red face.

Therefore, the baby is hungry, then he has red face.

Prove this argument is valid by using Inference Rules

Sol: Let p: Baby is hungry

q: Baby cries

r: Baby is mad

s: Baby has red face.

Ex $(p \to q) \wedge \cancel{(\sim r \to \sim q) \wedge (r \to s) \to (p \to s)}$

$\sim r \to \sim q$

$r \to s$

$\therefore p \to s$

[1] $\sim r \to \sim q$ Premise 2

(1)[2] $q \to r$ From equivalent formulae

[3] $r \to s$ Premise 3

(2,3) [4] $q \to s$ Hypothetical Syllogism

[5] $p \to q$ Premise 1

(5,4) [6] $p \to s$ Hypothetical Syllogism

$\therefore$ The given argument is valid.

2. Determine whether the following argument is valid or not.

$p \to (r \to s)$

$\sim r \to \sim p$

p

$\therefore s$

Sol: [1] $\sim r \to \sim p$ Premise 2

(1) [2] $p \to r$ From Equivalent formulae

[3] p Premise 3

(3,2) [4] r Modus Ponnens

[5] $p \to (r \to s)$ Premise 1

(3, 5) [6] $\neg r \to s$ Modus Ponnens

(4, 6) [7] s Modus Ponnens

$\therefore$ The given argument is valid.

3.
$$\begin{array}{l} \neg r \\ p \to r \\ \underline{q \to r} \\ \therefore \sim p \end{array}$$

Determine whether the argument is valid or not.

Sol: [1] $\neg r$ Premise 1

[2] $p \to r$ Premise 2

~~(1, 2) [3] $\sim p$ Modus Tollens~~

[3] $q \to r$ Premise 3

[4] $\sim r \to \sim p$ From Equivalent Formula

(1, 4) [5] $\sim p$ Modus Ponnens

(3) [6] $\sim r \to \sim q$ From Equivalent Formula

(1, 3) [7] $\sim q$ Modus Ponnens

(4, 7) [8] $\sim p \wedge \sim q$ Conjunction

[9] $\sim p$ Simplification

Hence given argument is valid.

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4. $\neg r$; Determine whether the argument is valid or not.

$p \to q$

$q \to r$

$\therefore \sim p$

Sol: [1] $\neg r$ Premise 1

[2] $p \to q$ Premise 2

[3] $q \to r$ Premise 3

(2,3) [4] $p \to r$ Hypothetical Syllogism

(1,4) [5] $\sim p$ Modus Tollens

Hence given argument is valid.

5. $\neg r \to (s \to \neg t)$ Determine whether the argument is valid or not.

$\sim r \vee w$

$\neg p \to s$

$\neg w$

$\therefore t \to p$

Sol: [1] $\sim r \vee w$ Premise 2

(1) [2] $r \to w$ From Equivalent Formulae

[3] $\neg w$ Premise 4

(3,1) [4] $\sim r$ Modus Tollens

[5] $\neg r \to (s \to \neg t)$ Premise 1

[6] $(s \wedge t) \to r$ From Equivalent Formulae

(4,6) [7] $\sim(s \wedge t)$ $[\sim s \vee \sim t]$ Modus Tollens

(7) [8] $s \to \sim t$ From Equivalent Formulae

[9] $\neg p \to s$ Premise 3

(6,5) [10] $\neg p \to \neg t$ Hypothetical Syllogism

(10) [11] $t \to p$ From Equivalent Formulae

Hence given argument is valid.

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6. Check the validity of the following argument by using inference rules.

p: Madhavi is a Mathematician.

q: Madhavi is ambitious.

r: Madhavi is an early riser.

s: Madhavi takes oat meal.

$$\begin{array}{l} p \to q \\ r \to \sim s \\ q \to r \\ \hline \therefore p \to \sim s \end{array}$$

Sol: [1] $p \to q$ Premise 1

[2] $q \to r$ Premise 3

(1,2) [3] $p \to r$ Hypothetical Syllogism

[4] $r \to \sim s$ Premise 2

(3,4) [5] $p \to \sim s$ Hypothetical Syllogism

$\therefore$ If Madhavi is a Mathematician, then she does not take oat meal.

7.

I: If Thrishank is not re-elected, then Shabana will be lose her air base.

II: Thrishank will be re-elected if and only if Shabana votes for him.

III: If Shabana gives her airbase, then Thrishank will be re-elected.

Convert the above statements to mathematical form.

Sol: p: Thrishank will be re-elected

q: Shabana will lose her airbase.

r: Shabana votes for Thrishank.

[1] $\sim p \to q$ Premise 1

[2] $p \leftrightarrow r$ Premise 2

[3] $q \to p$ Premise 3

(1,3) [4] $\sim p \to p$ Hypothetical Syllogism

(2) [5] $(p \to r) \wedge (r \to p)$ From Equivalent Formula

(4) [6] $\sim(\sim p) \vee p$ From Equivalent Formulae

(6) [7] p

(2,7) [8] $p \wedge (p \to r) \wedge (r \to p)$

(8) [9] $r \wedge (r \to p)$ Simplification

(9) [10] p Modus Ponnens

∴ Thrishanu will be re-elected

∴ The given argument is valid.

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METHODS OF PROOF OF IMPLICATION [6 M Q]

1. DIRECT METHOD

→ There are 9 methods of proof of an implication. Out of them, three are given below:

1. Direct Method
2. Indirect / Contrapositive Method
3. Contradiction Method

1. DIRECT METHOD

→ The construction of direct proof of p implies q ($p \to q$) begins p is true and then from the available information, from the frame of inferences conclusion q is shown to be true. by a valid inference.

2. INDIRECT / CONTRAPOSITIVE METHOD

→ The implication $p \to q$ is equivalent to $\sim q \to \sim p$. We can establish the truth values of p and q by establishing $\sim q \to \sim p$.

→ This can be shown by a direct method proceedin from ~~taking~~ assuming that $\sim q$ is true and then from the available information, show that $\sim p$ is true (or) if we assume that q is false and then from the available information, show that p is false.

3. CONTRADICTION METHOD

→ Contradiction method exploits the fact

$p \to q \Leftrightarrow \sim p \vee q$ / q v p.

→ Thus the proof of contradiction is constructed as follows:

1. Assume $p \wedge \sim q$ is true ($\because \sim(\sim p \vee q) \Leftrightarrow p \wedge \sim q$)
2. Discover the basis of the assumption, some Conclusion or patently false ~~~~ ors while its some other facts are already established (contradicted).
3. Hence our assumption is wrong. Therefore $\sim p \vee q$ is true.

P.T.O. →

1. Suppose x is a number that $x^2-5x+6=0$ then $x=2$ ~~and~~ or $x=3$ by direct method/proof.

Sol: Let

Let p: x is a number that $x^2-5x+6=0$.

q: $x=2$ ~~and~~ or $x=3$

Let p is true

$x^2-5x+6=0$ is ~~also~~ true

$(x-2)(x-3)=0$

Now, according to the algebraic system ~~of the~~

i.e, $x-2=0$ or $x-3=0$

$\therefore x=2$ or $x=3$

Hence Proved

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2. If a is an integer such that $a-2$ is divisible by 3, then a^2-1 is divisible by 3. Prove this by direct method.

Sol:

p: $a-2$ is divisible by 3

q: a^2-1 is divisible by 3

Let p is true

i.e, $a-2$ is divisible by 3

According to Division Algorithm Property

Dividend = Divisor × Quotient + Remainder

$\Rightarrow a-2 = 3k$ — ①

Now we have to prove that q is true

To prove a^2-1 is true, it is enough to prove that a^2-1 is scalar multiple of 3.

$$a^2-1 = (a-1)(a+1)$$
$$= (3k+1)(3k+3) \quad [\text{From ①}, a = 3k+2]$$
$$= 3[(3k+1)(k+1)]$$
$$a^2-1 = 3k_1$$

Hence Proved

∴ ~~It is proved that~~

3. If the product of two integers a, b is an even, then either a is even or b is even. Prove this by indirect method (contrapositive method).

Sol: p: a, b are two integers ~~and~~ then their product is even

q: Either a is even or b is even

Let q is false

i.e., Neither a is even nor b is even ⇒

⇒ Both a and b are odd

$a = 2k_1 + 1$

$b = 2k_2 + 1$

Product $ab = (2k+1)(2k-1)$

$= 4k^2 - 1$

$= 2k_1 - 1$ [$\because 4k^2$ is always even]

$=$ odd

~~Hence~~

$\therefore$ p is false

Hence Proved

4. If N is the product of two positive integers a, b, then either $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$. Prove this by indirect method.

Sol: p: n is the product of two positive integers a, b

q: Either $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$

Let q is false

i.e., Neither $a \leq \sqrt{n}$ nor $b \leq \sqrt{n}$

$\Rightarrow$ Both $a > \sqrt{n}$ and $b > \sqrt{n}$ $\Rightarrow ab > n$ — ①

Product $n = ab$ — ②

But $ab > n$ [From ①]

$\therefore$ p is false

Hence Proved

5. Suppose that 10 integers [1, 2, 3, ... , 10] are randomly positioned on the circular wheel.

Show that sum of three consecutive positioned numbers is atleast 17. Prove this argument by contradiction method.

Sol: E:

Let

p: 10 integers are randomly positioned on the circular wheel

q: ~~Sum~~ Some set of three consecutive positioned numbers is atleast 17

Let x_1, x_2, x_3 be three consecutive integers

i.e, $x_1 + x_2 + x_3 \geq 17$ — ①

Assume that negation of ①

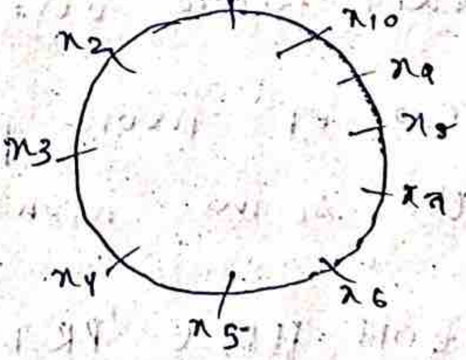
$x_1 + x_2 + x_3 < 17$

$\Rightarrow x_1 + x_2 + x_3 \leq 16$ [Since, given numbers are integers]

$\Rightarrow x_2 + x_3 + x_4 \leq 16$

$\vdots$

$x_{10} + x_1 + x_2 \leq 16$



$3(x_1 + x_2 + \ldots + x_{10}) \leq 16(10)$

$3(1 + 2 + \ldots + 10) \leq 160$

$3(55) \leq 160$

$165 \leq 160$

which is contradiction. Hence our assumption ~~is wrong~~ $x_1 + x_2 + x_3 < 17$ is wrong

∴ The given statement $x_1+x_2+x_3 \geq 17$ is correct

6. Suppose that in a room of 13 people, 2 or more people have their birthdays in the same month. Prove this method by using contradiction method.

Sol: Let

p: There are 13 people in a room

q: Two or more people have their birthdays in the same month.

Let one person have their birthday in one different month. But there are 13 members in that room. Therefore, 13 member have their ~~Therefor~~ birthdays in 13 months which is contradiction (because we have 12 months in a year). Hence our assumption is wrong.

∴ Two or more people have their birthdays in the same month.

PIGEON HOLE PRINCIPLE

Statement

→ If there are more number of pigeons than the pigeon holes, then there must be atleast ~~two~~

pigeons hole with atleast two pigeons.

(or)

If there are 'm' number of ~~pigeons~~ objects placed in 'n' boxes ~~then~~ where $m > n$, then there exists atleast one box consisting of atleast two objects.

Proof:

The pigeon hole principle can be used by using contradiction method.

Let p: No. of pigeons (objects) be m and no. of pigeon holes (boxes) is n $(m > n)$

q: At least one pigeon consists at least two pigeons.

Let p_i be the i^{th} hole where $i = 1, 2, 3, \ldots\ldots, n$

We have to prove that $p_i \geq 2$

Assume that $p_i < 2$ $(\sim(p_i \geq 2)$ is $p_i < 2)$

i.e., $p_i \leq 1$ [$\because$ Pigeons are integers]

$\Rightarrow p_1 \leq 1$

$p_2 \leq 1$

$\vdots$

$p_n \leq 1$

$p_1 + p_2 + p_3 + \cdots + p_n \leq n(1)$

$\Rightarrow m \leq n$

which is contradiction to $m > n$

Hence our assumption is wrong

$\therefore$ ~~One~~ Atleast one pigeon hole consists of atleast two pigeons $[p_i \geq 2]$

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PREDICATE CALCULUS

FIRST ORDER LOGIC

→ It is that part of logic which emphasize the content of the sentence involved in the argument as well as form of arguments.

PREDICATE

Varun is a software engineer
N S

→ This can be symbolised as SC

where S - predicate, - subject

→ Subjects are denoted by small letters and predicate by capital letter. Subject should be in the brackets.

Note: In general any statement of the type s is P can be written as P(s).

QUANTIFIERS

→ Certain declarative sentences involve the words that indicate quantity such as 'all', 'some', ~~'at l~~

~~atmost~~ 'atleast', 'atmost', 'none'; etc.

Types of Quantifiers

→ There are two types of quantifiers:

(a) Universal ($\forall$)

(b) Existency ($\exists$)

(a) Universal Quantifier

→ The quantifier "all" is called universal quantifier and denoted by "$\forall$".

Ex: All natural numbers are integers.
Let 'x' be a number; then we can symbolize the statement as

$$\forall(x)\,[N(x) \rightarrow W(x)]$$

Ex: All squares are rectangles
~~where~~ Let 'x' be a quadrilateral, then

$$\forall(x)\,[S(x) \rightarrow R(x)]$$

Note: 'For all' can also be written as for every
2. for all 'x' which can be written as $\forall(x)$ / $\forall x$.

(b) Existency Quantifier

→ The quantifier "some" is the existency quantifier and is denoted by '$\exists$' and which can ~~be~~ read as "there exist".

Ex: Some birds can fly.

$\exists(x)[B(x) \rightarrow F(x)]$

Some odd numbers are prime numbers

$\exists(x)[O(x) \rightarrow P(x)]$

Note: 1. $\forall(x)[F(x)]$ means for all x, F(x) is true

2. $\exists(x)[F(x)]$ means for some x, F(x) is true

(or)

at least one x exists such that F(x) is true

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3. $\exists!(x)$ means that there is a unique 'x' exists

4. $\exists!(x)[F(x)]$ means there is one and only one 'x' exists such that F(x) is true.

Note: The universal quantifier for all x ($\forall x$) can be connected by conjunction ($\wedge$) whereas existence quantifier there exists ($\exists$) can be connected by disjunction.

~~Some~~ SOME IMPORTANT ABBREVIATIONS

→ Let F(x) is true.

Sentence	Abbreviation
1. $\forall(x)[F(x)]$	All are true (None False)
2. $\exists(x)[F(x)]$	Some are true (Atleast one true
3. $\sim[\exists(x) F(x)]$	All false / None True
4. $\forall(x)[\sim F(x)]$	All are false

5. $\exists(x)[\sim F(x)]$	Some are false / Atleast one false
6. $\sim[\exists(x)[\sim F(x)]]$	All are true / None False
7. $\sim[\forall(x)[F(x)]]$	Not all are true / Some are false / Atleast one false
8. $\sim\forall(x)[\sim F(x)]$	~~S~~ Not all false / Atleast one true
9. $\sim[\forall(x)[\sim F(x)]]$	Some are true

SOME EQUIVALENT STATEMENTS

1. All are true : $\forall(x)[F(x)] \equiv \sim[\exists(x)[\sim F(x)]]$ - None false
2. Atleast one false : $\exists(x)[F(x)] \equiv \sim[\forall(x) F(x)]$ - Not all true
3. All are false : $\forall(x)[\sim F(x)] \equiv \sim[\exists(x) F(x)]$ - None true
4. Not all are false : $\sim\forall(x)[\sim F(x)] \equiv [\forall(x)[\sim F(x)]]$
$\equiv [\exists(x)[F(x)]]$ - Atleast one true

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NEGATION OF SOME STATEMENTS

Statement	Negation
1. All are true - $\forall(x)[F(x)]$	Atleast one false - $\exists(x)[\sim F(x)]$
2. Atleast one false - $\exists(x)[\sim F(x)]$	All are true - $\forall(x)[F(x)]$
3. All are false - $\forall(x)[\sim F(x)]$	Atleast one true - $\exists(x)[F(x)]$
4. Atleast one true - $\exists(x)[F(x)]$	All are false - $\forall(x)[\sim F(x)]$

Note:

1. An open predicate F(x) acquantified by the univers quantifier ($\forall$) is really just the conjunction of many prepositions.

Similarly there exists ("$\exists$") is really just the disjunction of many prepositions

2. The universe consists of a, b, c, d then

$\forall(x)[F(x)]$ means $F(a) \wedge F(b) \wedge F(c) \wedge F(d)$

$\exists(x)[F(x)]$ means $F(a) \vee F(b) \vee F(c) \vee F(d)$.

DE-MORGAN'S LAW FOR QUANTIFIERS

$$1.\ \sim[\forall(x)[F(x)]] \equiv \sim[F(a) \wedge F(b) \wedge F(c) \wedge F(d)]$$
$$\equiv \sim F(a) \vee \sim F(b) \vee \sim F(c) \vee \sim F(d)$$
$$\equiv \exists(x)[\sim F(x)]$$

$$2.\ \sim[\exists(x)F(x)] \equiv \forall(x)[\sim F(x)]$$

→ We have seen that there are four main types of statements involving a single quantifier.

Statement	is true	is false
1. $\forall(x)[F(x)]$	for all 'c' is true, then F(c)	atleast one 'c' is false, then F(c).
2. $\exists(x)[F(x)]$.	atleast one 'c' is true, then F(c)	all 'c' are false, then F(c)
3. $\forall(x)[\sim F(x)]$	for all 'c' are false, then F(c)	atleast one 'c' is true, then F(c)

4. $F(x) \Rightarrow F(x)$	atleast one 'c' is false, then $F(c)$	for all 'c' is true, then $F(c)$

SENTENCES WITH MULTIPLE QUANTIFIERS

→ In general, if $P(x, y)$ is any predicate (open preposition) involving the two variables x and y, then the following possibilities exist.

$(\forall x)(\forall y)\, P(x,y)$ $(\forall y)(\forall x)\, P(x,y)$

$(\exists x)(\forall y)\, P(x,y)$ $(\exists y)(\forall x)\, P(x,y)$

$(\forall x)(\exists y)\, P(x,y)$ $(\forall y)(\exists x)\, P(x,y)$

$(\exists x)(\exists y)\, P(x,y)$ $(\exists y)(\exists x)\, P(x,y)$

→ In a statement, if both universal and existential quantifiers involve, one always works from left to right.

Ex: $(\forall x)(\exists y)\,[x + y = 5]$

for all 'x', there exists 'y' such that sum of x and y is 5, while $y = 5 - x$.

(or)

$(\forall y)(\exists x)\,[x + y = 5]$

→ There are logical relationships b/w sentences with two quantifiers, if the same predicate is involved in the same sentences.

We can depict these relationships in the following diagram:

$$
\begin{tikzcd}
& (\forall x)(\forall y) \arrow[r, leftrightarrow] \arrow[dl] & (\forall y)(\forall x) \arrow[dr] & \\
(\exists y)(\forall x) \arrow[d] & & & (\exists x)(\forall y) \arrow[d] \\
(\exists y)(\forall x) \arrow[dr] & & & (\forall y)(\exists x) \arrow[dl] \\
& (\exists y)(\exists x) \arrow[r, leftrightarrow] & (\exists x)(\exists y) &
\end{tikzcd}
$$

Q Translate each of the following statements into symbols using quantifiers, variables & predicate symbols:

1. All birds can fly.

$\forall x\, [B(x) \rightarrow F(x)]$

$B(x)$: x is a bird

$F(x)$: x can fly

2. Not all birds can fly

$(\exists x)[B(x) \rightarrow F(x)]$

$B(x)$; x is a bird

$F(x)$: x can fly

3. Some men are giants

$(\exists x)[M(x) \rightarrow G(x)]$

4. No men are giants

$\forall(x)\,[M(x) \rightarrow \sim G(x)]$

5. There is a student who likes DMS but not watching movies.

$\exists(x)\,[S(x) \wedge D(x) \wedge \sim M(x)]$

RULES OF INFERENCES FOR QUANTIFYING PROPOSITION

1. UNIVERSAL SPECIFICATION (U.S)

→ If a statement of the form $(\forall x)P(x)$ is assumed to be true, then the Universal Quantifier can be drawn to obtain $P(c)$ is true, here 'c' is true for an arbitrary 'c' in the universe.

The rule may be represented as:

$$\frac{(\forall x)\,P(x)}{\therefore P(c) \text{ for all } c}$$

2. UNIVERSAL GENERALISATION (U.G)

→ If a statement $P(c)$ is true, for each element 'c' in the universe, then the universal quantifier may be prefix for all 'x', $P(x)$.

$\forall x\, P(x)$

$$\Rightarrow \frac{P(c) \text{ for all 'c'}}{\therefore (\forall x)\,P(x)}$$

3. EXISTENTIAL SPECIFICATION (E.S)

→ If there exists 'x', $P(x)$ is assumed to true, then there is an element 'c' in the universe, $P(c)$ is true. This can be represented as:

$$\Rightarrow \frac{(\exists x)\,P(x)}{\therefore P(c) \text{ for atleast one c/ some 'c'}}$$

2. EXISTENTIAL GENERALISATION (E.G.)

⇒ If $P(c)$ is true, for some 'c' in the universe, then there exists 'x', such that $P(x)$ is true and can be represented as,

$$\Rightarrow \frac{P(c) \text{ for some 'c'}}{\therefore (\exists x)\, P(x)}$$

Q. All men are mortal.

All kings are men.

Therefore all kings are mortal.

Represent with the help of quantifier as well as the predicates.

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Q1. All birds can fly.

All parrots are birds.

All parrots can fly.

Prove this argument using inference rules.

(or)

Verify the validity of the argument.

Sol:

$B(x)$: x is a bird

$F(x)$: x can fly

$P(x)$: x is a parrot

$$\begin{array}{l} \forall(x)\,[B(x) \rightarrow F(x)] \\ \forall(x)\,[P(x) \rightarrow B(x)] \\ \hline \therefore \forall(x)\,[P(x) \rightarrow F(x)] \end{array}$$

[1] $\forall(x)[B(x) \to F(x)]$ Premise 1

[2] $B(c) \to F(c)$ Universal Specification

[3] $\forall(x)[P(x) \to B(x)]$ Premise 2

[4] $P(c) \to B(c)$ Universal Specification

[4,2) [5] $P(c) \to F(c)$ Hypothetical Syllogism

[6] $\forall(x)[P(x) \to F(x)]$ Universal Generalisation

Q.2. All men are mortal.
All kings are men.
Therefore all kings are mortal.
~~Represent will~~ Check the validity of the argument.

Sol:

$M_1(x)$: x is man
$M_2(x)$: x is mortal
$K(x)$: x is a king

$$\begin{array}{l} \forall(x)[M_1(x) \to M_2(x)] \\ \forall(x)[K(x) \to M_1(x)] \\ \hline \therefore \forall(x)[K(x) \to M_2(x)] \end{array}$$

[1] $\forall(x)[M_1(x) \to M_2(x)]$ Premise 1

[2] $M_1(c) \to M_2(c)$ Universal Specification

[3] $\forall(x)\,[K(x) \rightarrow M_1(x)]$ Premise 2

[4] $K(c) \rightarrow M_1(c)$ Universal Specificati

(4, 2) [5] $K(c) \rightarrow M_2(c)$ Hypothetical Syllogism

[6] $\forall(x)\,[K(x) \rightarrow M_2(x)]$ Universal Generaliza

Q. Prove or disprove the validity of the following argument by rules of inference.

Some dogs are animals

Some cats are animals

Therefore some dogs are cats.

Sol:

$D(x)$: x is a dog

$C(x)$: x is a cat

$A(x)$: x is a animal

$$\exists(x)\,[D(x) \rightarrow A(x)]$$
$$\exists(x)\,[C(x) \rightarrow A(x)]$$
$$\therefore \exists(x)\,[D(x) \rightarrow C(x)]$$

[1] $\exists(x)\,[D(x) \rightarrow A(x)]$ Premise 1

[2] $D(c) \rightarrow A(c)$ Existential Specification

[3] $\exists(x)\,[C(x) \rightarrow A(x)]$ Premise 2

[4] $C(c) \rightarrow A(c)$ Existential Specificatio

$\therefore$ from 2, and 4, we cannot conclude $D(c) \rightarrow E(c)$

4. One student in CSM-A knows Python programming.

Everyone who knows Python programming can get high package job.

Therefore someone in CSM-A gets high package job.

Check the validity of argument using inference rules.

<u>Sol</u>:

$C(x)$: x is a student of CSM-A.

$P(x)$: x knows Python programming.

$J(x)$: x gets high package job.

$$\boxed{1} \quad \begin{array}{l} \exists(x)[C(x) \rightarrow P(x)] \\ \underline{\forall(x)[P(x) \rightarrow J(x)]} \\ \therefore \exists(x)[C(x) \rightarrow J(x)] \end{array}$$

[1] $\exists(x)[C(x) \rightarrow P(x)]$ Premise 1

[2] $C(c) \rightarrow P(c)$ Existential Specification

[3] $\forall(x)[P(x) \rightarrow J(x)]$ Premise 2

[4] $P(c) \rightarrow J(c)$ Existential Specification

(2,4) [5] $C(c) \to J(c)$ Hypothetical Syllogism

[6] $\exists(x)[C(x) \to J(x)]$ Existential Generalization.

Hence,
$\therefore$ The argument is valid.

5. Every living thing is a plant or animal.

David's dog is alive and it is not a plant.

All animals have heart.

Hence, David's dog has heart.

Sol: Let (x) is a living thing

$P(x)$: x is a plant

$H(x)$: x has heart

$A(x)$: x is an animal

d : d is David's dog.

$$\forall(x)[P(x) \vee A(x)]$$
$$\sim P(d)$$
$$\forall(x)[A(x) \to H(x)]$$
$$\therefore H(d)$$

[1] $\forall(x)[P(x) \vee A(x)]$ Premise 1

[2] $P(d) \vee A(d)$ Universal Specification

[3] $\sim P(d)$ Premise 2

(2,3) [4] $A(d)$ Disjunctive Syllogism

[5] $\forall(x)[A(x) \to H(x)]$ Premise 3

[6] $A(d) \to H(d)$ Universal Specification

(4, 6) [7] $H(d)$ Modus Ponnens

$\therefore$ The given argument is valid.

6. Lions are dangerous animals.
   There are lions.
   Therefore ~~they~~ there are dangerous lions.

Sol: $L(x)$: x is a lion

$D(x)$: x is a dangerous ~~animal~~ animal

$$\forall(x)[L(x) \to D(x)]$$
$$\exists(x)[L(x)]$$
$$\overline{\therefore \exists(x)[D(x)]}$$

[1] $\forall(x)[L(x) \to D(x)]$ Premise 1

[2] $L(c) \to D(c)$ Universal Specification

[3] $\exists(x)[L(x)]$ Premise 2

[4] $\cancel{\exists(x)}\ L(c)$ Existential Specification

(2, 4) [5] $D(c)$ Modus Ponnens

[6] $\exists(x)[D(x)]$ Existential Generalization

$\therefore$ The given argument is valid.

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MATHEMATICAL INDUCTION

→ Let P(n) be a statement for which every integer n, P(n) may be either True or False.

→ To p prove P(n) is true, we can use Mathematical Induction. In the Mathematical Induction, there are three steps involved: They are:

Step I: Basis of Induction

→ We have to prove P(n) is true for n = 1 i.e; we have to prove P(1) is true

Step II: Inductive Hypothesis

→ Assume that P(n) is true for n = K $\forall K \geq 1$, i.e, assume P(K) is true $\forall K \geq 1$.

Step III: Inductive Step

→ We have to prove P(n) is true for n = K+1, i.e; we have to prove P(K+1) is true.

1. Prove that sum of squares of n natural number is $\frac{n(n+1)(2n+1)}{6}$.

(OR)

Sol:

$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$. Prove it.

Sol:

Step I: Basis of Induction

Let $S(n) = \frac{n(n+1)(2n+1)}{6}$ — ①

Now, $S(1) = \frac{1(1+1)(2+1)}{6} = 1$

$1 = 1$

Hence Proved

Step II: Inductive Hypothesis

Assume that $S(k)$ is true $\forall\ k \geq 1$

i.e; $1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$ — ②

Step III: Inductive Step

Prove that $S(k+1)$ is true.

~~Consider, S(k+1) = (k+1)(k+1~~

Consider,

$1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$

Add $(k+1)^{th}$ term ~~to~~ on b.s.

$1^2 + 2^2 + \dots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$= \cancel{(k^2+k)(2k+1)}\ (k+1)\left[\frac{2k^2+k}{6} + k+1\right]$

$= \frac{(k+1)}{6}[2k^2 + k + 6k + 6]$

$= \frac{(k+1)}{6}[2k^2 + 7k + 6]$

$= \frac{k+1}{6}[2k^2 + 4k + 3k + 6]$

$$= \frac{(k+1)}{6}\left[2k(k+2) + 3(k+2)\right]$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$\therefore S(k+1) = \frac{(k+1)(k+2)(2k+3)}{6}$$

$\therefore S(k+1)$ is true $\forall$ $k+1$

Hence, by mathematical induction ~~is true~~ $S(n)$ is true for $\forall$ $k \geq 1$

2 Prove that $11^{n+2} + 12^{2n+1}$ is divisible by 133.

Sol:

Step I: Basis of Induction

Let $D(n) = 11^{n+2} + 12^{2n+1}$ — ①

Now, $D(0) = 11^{0+2} + 12^{0+1}$
$= 11^2 + 12^1$
$= 121 + 12$
$= 133$

Here, $D(0) = 133$ which is divisible by 133

$\therefore D(0)$ is true

Hence Proved

Step II: Inductive Hypothesis

Assume that $D(k)$ is true $\forall$ $k \geq 0$

i.e; $11^{k+2}+12^{2k+1}$ is divisible by 133

i.e; $D(k) = 133m$ [By using Division Algorithm rule]

$\Rightarrow 11^{k+2}+12^{2k+1} = 133m$ — ②

$\Rightarrow 12^{2k+1} = 133m - 11^{k+2}$ — ③

Step III: Inductive Step

We have to prove that $D(k+1)$ is true

i.e; $D(k+1)$ is divisible by 133

Now,

$D(k+1) = 11^{(k+1)+2} + 12^{2(k+1)+1}$

$= 11^{k+3} + 12^{2k+3}$

$= 11^{k+3} + 12^{2k+1+2}$

$= 11^{k+3} + 12^{(2k+1)} \cdot 12^2$

$= 11^{k+3} + (133m - 11^{k+2}) \cdot 12^2$ [From ③]

$= 11^{k+3} - 11^{k+2} \cdot 12^2 + 133m \cdot 12^2$

$= 11^{k+2}[11 - 12^2] + 133m \cdot 12^2$

$= 11^{k+2}[-133] + 133m \cdot 12^2$

$= 133[-11^{k+2} + 12^2 m]$

$\therefore D(k+1) = 133[q]$ [Let $q = 12^2 \cdot m - 11^{k+2}$]

i.e; $D(k+1)$ is divisible by 133

Hence by Mathematical Induction, $D(n)$ is true for $\forall\ k \geq 0$

P.T.O. $\rightarrow$

3. Prove that $6^{n+2} + 7^{2n+1}$ is divisible by 43

(CoE)

4. Prove that $1^2 - 2^2 + 3^2 - \ldots\ldots + (-1)^{n+1} = \frac{(-1)^{n-1} \cdot n(n+1)}{2}$

5. Prove that $\forall$ integers $n > 4$, $3^n > n^3$.

~~02~~ 02/11/23

3. Sol:

Step I: Basis of Induction

Let $D(n) = 6^{n+2} + 7^{2n+1}$ — ①

Now, $D(0) = 6^{0+2} + 7^{0+1}$

$= 6^2 + 7^1$

$= 43$

Here, $D(0) = 43$ which is divisible by 43.

$\therefore D(0)$ is true

Hence Proved

Step II: Inductive Hypothesis

Assume that $D(k)$ is true $\forall k \geq 0$

i.e., $6^{k+2} + 7^{2k+1}$ is divisible by 43

i.e., ~~$6^{k+2} + 7^{2k+1}$~~ $D(k) = 43m$ [By using Division Algorithm rule]

$\Rightarrow 6^{k+2} + 7^{2k+1} = 43m$ — ②

$\Rightarrow 7^{2k+1} = 43m - 6^{k+2}$ — ③

Step III: Inductive Step

We have to prove that $D(k+1)$ is true

i.e; $D(k+1)$ is divisible by 43

Now,

$D(k+1) = 6^{(k+1)+2} + 7^{2(k+1)+1}$

$= 6^{k+3} + 7^{2k+3}$

$= 6^{k+3} + 7^{2k+1} \cdot 7^2$

$= 6^{k+3} + [43m - 6^{k+2}] \cdot 7^2$ — [From ③]

$= 6^{k+3} + 43m \cdot 7^2 - 6^{k+2} \cdot 7^2$

$= 6^{k+2}[6 - 7^2] + 43m \cdot 7^2$

$= 6^{k+2}[-43] + 43m \cdot 7^2$

$= 43[49m - 6^{k+2}]$

$\therefore D(k+1) = 43[q]$

$\boxed{\text{Let } q = 49m - 6^{k+2}}$

i.e; $D(k+1)$ is divisible by 43

Hence By Mathematical Induction, $D(n)$ is true for $\forall\, k \geq 0$.

5. Sol:

Step I: Basis of Induction

Let $I(n) = 3^n > n^3$ — ①

$I(4) = 3^4 > 4^3$

$= 81 > 64$

$\therefore I(n)$ is true for $n = 4$

Step II: Inductive Hypothesis

Assume that $I(n)$ is true ~~∀ k ≥ 0~~ for $n = k$

i.e; $3^k > k^3$ — ②

Step III: Inductive Step

We have to prove that $P(n)$ is true for $n=k+1$

i.e; $3^{k+1} > (k+1)^3$

Consider, $3^{k+1} > (k+1)^3$

$$\Rightarrow 3^k \cdot 3 > k^3+1+3k^2+3k$$

$$\Rightarrow 3^k \cdot 3 > k^3\left[1+\frac{1}{k^3}+\frac{3}{k}+\frac{3}{k^2}\right] \quad \text{— (3)}$$

$$\Rightarrow 3^k \cdot 3 > k^3\left[1+\frac{3}{k}+\frac{3}{k^2}+\frac{1}{k^3}\right] \quad \text{— (3)}$$

From (2), $3^k > k^3$

~~From the is~~ To prove the inequality (3), it is enough to prove that $3 > \left(1+\frac{3}{k}+\frac{3}{k^2}+\frac{1}{k^3}\right)$ — (4)

$\cancel{3 \Rightarrow}$

$3 > f(k)$, where $f(k) = 1+\frac{3}{k}+\frac{3}{k^2}+\frac{1}{k^3}$

Now, $f(k)$ ~~is~~ decreases when k increases

$\therefore f(k)$ is maximum for $k=4$

Now, $f(4) = 1+\frac{3}{4}+\frac{3}{16}+\frac{1}{64}$

$= 1.95\ldots$

$\Rightarrow 3 > f(4)$

$\Rightarrow 3 >$ Max. value of $f(k)$

$\Rightarrow 3 > f(k) \quad \forall\ k \geq 4$ — (5)

$\therefore$ From (5)

(3) is true ~~for~~ [i.e; $3^{k+1} > (k+1)^3$]

RECURSION

→ 1, 3, 9, 27, 81, ... = 3^n ∀ non-negative integers ($n \geq 0$). But the same sequence can be represented recursively.

$T(0) = 1$

$T(1) = 3 = 3 \cdot T(0)$

$T(2) = 9 = 3 \cdot T(1)$

$\vdots$

$T(n) = 3 \cdot T(n-1)$

$T(n+1) = 3 \cdot T_n$

→ An equation that express "a_n" in terms of one or more of the previous terms of the sequence $a_0, a_1, a_2, \ldots\ldots, a_{n-1}$ ∀ integers $n \geq n_0$

[n_0 – starting value / initial value of the sequence where n_0 is non-negative integer and the equation is called Recurrence Relation (R R) (or) Difference Equation of $\{a_n\}$.

Ex:

The ~~binomial~~ Fibonacci series can be expressed in the form $F_{n+1} = F_n + F_{n-1}, \forall n \geq 1$

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STRONG MATHEMATICAL INDUCTION

BASIS OF INDUCTION

→ Let $P(n)$ be a statement, prove that $P(a)$ is true $\forall a \geq 1$, i.e., $P(1), P(2), \ldots$ are all true.

STRONG INDUCTIVE HYPOTHESIS

→ Assume that $P(i)$ is true $\forall 1 \leq i \leq k$.

INDUCTIVE STEP

→ Prove that $P(n)$ is true for $n = k+1$.

1. If f_n is the Fibonacci number then show that $f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]$.

Sol: Here,

$f_{n+1} = f_n + f_{n-1}, \forall n \geq 1$ — ①

where $f_0 = 0, f_1 = 1$

Basis of Induction

Prove that $f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$ is true for $n=1$

$$= \frac{1}{\sqrt{5}} \left[\frac{(1+\sqrt{5})^2}{4} + \frac{(1-\sqrt{5})^2}{4} \right]$$

$$= \frac{1}{4\sqrt{5}} \left[(1+\sqrt{5})^2 + (1-\sqrt{5})^2 \right]$$

$$= \frac{1}{\cancel{4}\cancel{\sqrt{5}}} \left[\cancel{4}(1)(\cancel{\sqrt{5}}) \right] = 1$$

$$= \frac{3}{\sqrt{5}} \Rightarrow f_1 = 1$$

Hence f_n is true for $n = 1$

~~SE~~ <u>Strong Inductive Hypothesis</u>

Assume that $P(i)$ is true $\forall\ 1 \le i \le k$

$$f_k = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{k+1} \right] \quad \text{—} (2)$$

<u>Inductive Step</u>

Prove that f_n is true for $n = k+1$

$$f_{k+1} = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k+2} - \left(\frac{1-\sqrt{5}}{2} \right)^{k+2} \right]$$

From the Fibonacci Relation:

$$f_{k+1} = f_k + f_{k-1}$$

Now,

$$f_k + f_{k-1} = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{k+1} \right]$$

$$+ \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k-\cancel{1}+\cancel{1}} - \left(\frac{1-\sqrt{5}}{2} \right)^{k-\cancel{1}+\cancel{1}} \right] \quad \text{[From (2)]}$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} + \left(\frac{1+\sqrt{5}}{2}\right)^{k}\right] - \frac{1}{\sqrt{5}}\left[\left(\frac{1-\sqrt{5}}{2}\right)^{k+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right]$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left[\frac{1+\sqrt{5}}{2}+1\right] - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{k}\left[\frac{1-\sqrt{5}}{2}+1\right]$$

$$= \frac{1}{\sqrt{5}}\left\{\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\left(\frac{3-\sqrt{5}}{2}\right)\right]\right.$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left(\frac{1+\sqrt{5}}{2}\right)^{2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\left(\frac{1-\sqrt{5}}{2}\right)^{2}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+2}\right]$$

2) For $\forall\, n \geq 1$, prove that for each positive integer, the n^{th} Fibonacci number $f_n < \left(\frac{7}{4}\right)^n$

Sol: We know that the Fibonacci relation

$f_{n+1} = f_n + f_{n-1} \quad \forall\, n \geq 1 \quad - ①$

$f_0 = 0 \quad f_1 = 1$

Basis of Induction

$f_1 = \left(\frac{7}{4}\right)^1$ for $n = 1$

$1 < \frac{7}{4}$, which is true

$\therefore f_n$ is true for $n = 1$

Strong Mathematical Induction

Assume that f_i is true $\forall\, 1 \leq i \leq k$

i.e; $f_k < \left(\frac{7}{4}\right)^k \quad - ②$

Inductive Step.

We have to prove that $f_{k+1} < \left(\frac{7}{4}\right)^{k+1}$

From the Fibonacci Recurrence Relation,

$f_{k+1} = f_k + f_{k-1} \quad \forall\, k \geq 1$

Now,

$$f_k + f_{k-1} < \left(\frac{7}{4}\right)^{k} + \left(\frac{7}{4}\right)^{k-1}$$

$$< \left(\frac{7}{4}\right)^{k} \left[1 + \left(\frac{7}{4}\right)^{-1}\right]$$

$$< \left(\frac{7}{4}\right)^{k} \left(\frac{11}{7}\right)$$

$$< \left(\frac{7}{4}\right)^{k} \left(\frac{7}{4}\right) \qquad \left[\because \frac{11}{7} < \frac{7}{4}\right]$$

$$\Rightarrow f_k + f_{k-1} < \left(\frac{7}{4}\right)^{k+1}$$

$$\Rightarrow \quad f_{k+1} < \left(\frac{7}{4}\right)^{k+1}$$

Hence Proved

Hence, by Mathematical Induction for each positive integer, the n^{th} Fibonacci number $f_n < \left(\frac{7}{4}\right)^n$