RELATIONS

Cartesian product:- Let A and B be any two sets. The set of all ordered pairs such that the first member of the order pair is an element of A and the second member is an element of B is called the cartesian product of A and B and is written as $A \times B$.

i-e $A \times B = \{ (x, y) / x \in A \text{ and } y \in B \}$

Eg ① If $A = \{\alpha, \beta\}$ and $B = \{1, 2, 3\}$, find $A \times B$, $B \times A$, $A \times A$, $B \times B$, and $(A \times B) \cap (B \times A)$

Sol: $A \times B = \{(\alpha, 1) (\alpha, 2), (\alpha, 3), (\beta, 1) (\beta, 2) (\beta, 3)\}$

$B \times A = \{(1, \alpha), (2, \alpha), (3, \alpha), (1, \beta) (2, \beta), (3, \beta)\}$

$A \times A = \{(\alpha, \alpha), (\alpha, \beta), (\beta, \alpha), (\beta, \beta)\}$

$B \times B = \{(1,1) (1,2), (1,3), (2,1), (2,2), (2,3)$
$(3,1) (3,2) (3,3)\}$

$(A \times B) \cap (B \times A) = \phi$.

② If $A = \phi$ and $B = \{\alpha, \beta, \gamma\}$, find $A \times B$ and $B \times A$

Sol. $A \times B = \phi$ and $B \times A = \phi$.

Note: $A \times (B \times C) = \{ (a, (b, c)) / a \in A \wedge (b, c) \in B \times C \}$

$(A \times B) \times C = \{ ((a, b), c) / a \times b \in A \times B \wedge c \in C \}$

Note: $A \times (B \times C) \neq (A \times B) \times C$

Note: $A \times (B \cup C) = (A \times B) \cup (A \times C)$

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

~~A × B ...~~

Eg 3 If $A = \{1\}$, $B = \{a, b\}$, $C = \{2, 3\}$; then

find: $A \times B$, $B \times A$, $A \times B \times C$, B^2, A^3, $B^2 \times A$

T, $A \times B = \{ (1, a) (1, b) \}$

$B \times A = \{ (a, 1) (b, 1) \}$

$A \times B \times C = (1, a, 2), (1, b, 2) (1, a, 3) (1, b, 3) \}$

$B^2 = B \times B = \{ (a, b) (b, a) (a, a) (b, b) \}$

$A^3 = \cancel{A \times A \times A} = \{ (1, 1, 1) \}$

$B^2 \times A = B \times B \times A = \{ (a, a, 1) (a, b, 1), (b, a, 1)$
$(b, b, 1) \}$.

Relations :-

Def: Let A, B are two sets. A subset $A \times B$ is called a binary relation & Relation from A to B.

Note:- 1. If $R \subseteq A \times B$, then R is Relation on from A to B.

2. If $B = A$, then R is a relation on A.

3. If $(a,b) \in R$ is also can be written as aRb, we read as a relates b.

Domain of Relation: and Range of a Relation:

Let 'S' be a binary relation. The domain of the relation 'S' defined as the set of all first elements of the ordered pairs, that belong to S and it denoted by D(S) or D

The Range of a Relation 'S' is defined as the set of all second elements of the ordered pairs that belongs to 'S' is denoted by R(S).

$$D(S) = \text{domain of } S = \{x \mid \exists y ((x,y) \in S)\}$$

$$R(S) = \text{Range of } S = \{y \mid \exists x ((x,y) \in S)\}$$

Eg: Let $A = \{2,3,4\}$, $B = \{3,4,5,6,7\}$

Define a relation R from A to B by $(a,b) \in R$ if $a \mid b$.

Then $R = \{(2,4), (2,6), (3,3), (3,6), (4,4)\}$

Domain $= \{2, 3, 4\}$

Range $= \{3, 4, 6\}$.

Definition:- Let 'R' be the relation from A to B. The inverse of Relation R from $B \to A$ is denoted by R^{-1} and it is defined as $R^{-1} = \{(b, a) / (a, b) \in R\}$

Operations on Relations:

If R and S are any two relations, then $(R \cup S)$ defines a relation such that

$x (R \cup S) y \Leftrightarrow x R y$ or $x S y$

Similarly $x (R \cap S) \Leftrightarrow x R y$ and $x S y$

$x (R - S) \Leftrightarrow x R y$ and $x \not S y$

$x (\sim R) y \Leftrightarrow x \not R y$.

Problem (1) Let $P = \{(1,2), (2,4), (3,3)\}$ and $Q = \{(1,3), (2,4), (4,2)\}$. Find $P \cup Q$, $P \cap Q$, $D(P)$, $D(Q)$, $D(P \cup Q)$, $R(P)$, $R(Q)$ and $R(P \cup Q)$. S.T $D(P \cup Q) = D(P) \cup D(Q)$

and $R(P \cap Q) \subseteq R(P) \cap R(Q)$

Sol: $P = \{(1,2), (2,4), (3,3)\}$

$Q = \{(1,3), (2,4), (4,2)\}$

$P \cup Q = \{(1,2)\ (1,3), (2,4), (3,3)\ (4,2)\}$

$P \cap Q = \{(2,4)\}$

$D(P) = \{1,2,3\}$

$D(Q) = \{2,3,4\}$

$D(P \cup Q) = \{1,2,3,4\}$ —— (1)

$D(P) \cup D(Q) = \{1,2,3,4\}$ —— (2)

from (1) and (2) $D(P \cup Q) = D(P) \cup D(Q)$

Now $R(P) = \{2,3,4\}$

$R(Q) = \{2,3,4\}$

$R(P \cap Q) = \{4\}$ —— (3)

Also $R(P) \cap R(Q) = \{2,3,4\}$ —— (4)

from (3) and (4),

clearly $\{4\} \subseteq \{2,3,4\}$

$R(P \cap Q) \subseteq R(P) \cap R(Q)$.

(2) what is the range of the relation

$S = \{(x, x^2) / x \in N\}$ and $T = \{(x, 2x) / x \in N\}$

where $N = \{0,1,2,\dots\}$? . Find $S \cup T$, $S \cap T$.

Sol: $S = \{(x, x^2) / x \in N\}$, where $N = \{0,1,2,\dots\}$

$S = \{(0,0), (1,1), (2,4), (3,9), \dots\}$

$T = \{(x, 2x) / x \in N\}$, where $N = \{0,1,2,\dots\}$

$T = \{ (0,0), (1,2) (2,4) (3,6) \ldots \}$

$S \cup T = \{ (0,0) (1,1) (1,2) (2,4) (3,6) (3,9) \ldots \}$

$S \cap T = \{ (0,0) (2,4) \ldots \}$

(3) Let L denote the relation "less than or equal to" and D denote the relation "divides", where $x D y$ means 'x divides y'. Both L and D are defined on the set $\{1,2,3,6\}$. Write L and D as sets and find $L \cap D$.

Sol Let $X = \{1, 2, 3, 6\}$

$L = \leq = \{ (1,1) (1,2) (1,3) (1,6) (2,3), (2,6) (3,6) (2,2) (3,3) (6,6) \}$

$D = | = \{ (1,1) (1,2), (1,3), (1,6), (2,6), (3,6) (2,2) (3,3) (6,6) \}$

$L \cap D = \{ (1,1) (1,2), (1,3), (1,6), (2,2) (2,6), (3,3), (3,6), (6,6) \}$

Properties of Relations In a set (- properties of Binary Relations)

1. Reflexive Relations :- A binary relation 'R' in a set X is said to be reflexive if for every $x \in X$ xRx, that is for every $x \in X$, the ordered pair $(x,x) \in R$.

(or)

$\forall x \in X, (x,x) \in R$.

Symmetric Relation :-

A relation 'R' in the set 'X' is said to be symmetric if given xRy, then yRx, for every x and y in X. (or) If $(x,y) \in R$, then $(y,x) \in R$ for every $x,y \in X$

(or)

Transitive Relation :-

A relation 'R' in a set 'X' is said to be transitive if given xRy and yRz then xRz for every x,y,z in X. (or)

If $(x,y) \in R$, and $(y,z) \in R$ then $(x,z) \in R$. for every $x,y,z \in X$.

Irreflexive Relation :-

A relation 'R' in a set 'x' is said to be irreflexive if for every $x \in X$, the ordered pair $(x,x) \notin R$.

Anti-Symmetric Relation :-

A relation 'R' in a set X is said to be antisymmetric, if for every xRy and yRx, then $x=y$ (or)

If $(x,y) \in R$, and $(y,x) \in R$, then $x=y$.

Assymmetric :-

A relation 'R' in a set X is said to be Assymmetric if for every xRy then $y \not{R} x$. or, If $(x,y) \in R$ then $(y,x) \notin R$

Equivalence Relations:

A relation 'R' on a set 'x' is said to be an equivalence relation on x, If

i) R is reflexive on x

ii) R is symmetric on x

iii) R is Transitive on x.

① Given an example of a relation which is transitive but neither reflexive nor symmetric nor anti-symmetric

Sol: Let $A = \{1, 2, 3\}$.

$R = \{(1,1)\ (2,2)\ (1,2)\ (1,3)\ (2,1)\ (2,3)\}$

Reflexive: $(3,3) \notin R$, hence R is not reflexive

Symmetric: $(1,3) \in R$ but $(3,1) \notin R$, hence 'R' is not symmetric

Transitive: Since $(1,2) \in R$, $(2,3) \in R \Rightarrow (1,3) \in R$

$(1,2) \in R$, $(2,1) \in R \Rightarrow (1,1) \in R$.

Hence it is transitive

Anti-symmetric: $(1,2) \in R$, $(2,1) \in R$ but $1 \neq 2$, hence it is not anti-symmetric.

② Given an example of a relation which is symmetric but neither reflexive nor Anti-symmetric nor Transitive.

Sol: Take $A = \{1, 2, 3\}$

Consider the relation $R = \{(1,1), (1,2), (2,1), (3,2), (2,3)\}$ on A.

Here, i) $(2,2) \notin R \Rightarrow R$ is not Reflexive

ii) $(1,2) \in R \Rightarrow (2,1) \in R$

$(2,3) \in R \Rightarrow (3,2) \in R$,

$(1,1) \in R \Rightarrow (1,1) \in R$, then R is symmetric on A

iii) $(1,2) \in R, (2,1) \in R \Rightarrow 1 \neq 2$

$\therefore R$ is not Anti-symmetric on A

iv) $(1,2) \in R, (2,3) \in R \Rightarrow (1,3) \notin R$

$\therefore R$ is not transitive.

(3) Given $S = \{1, 2, 3, \cdots 10\}$ and relation R on S where $R = \{(x,y) \mid x+y = 10\}$. What are the properties of the relation R?

Sol: $R = \{(1,9), (2,8), (3,7), (4,6), (5,5), (6,4), (7,3), (8,2), (9,1)\}$

i) For any $x \in X$ and $(x,x) \notin R$;

Here $1 \in X$, but $(1,1) \notin R$.

$\Rightarrow$ The relation R is not Reflexive, but it is irreflexive.

ii) $(1,9) \in R \Rightarrow (9,1) \in R$

$(2,8) \in R \Rightarrow (8,2) \in R$

⋮

$\Rightarrow$ The relation 'R' is symmetric, but is not anti-symmetric.

$\because$ $(1,9) \in R$ and $(9,1) \in R$

$\Rightarrow (1,1) \notin R$.

(iii) $(2,8) \in R, (8,2) \in R$

But $(2,2) \notin R$.

$\therefore$ The relation 'R' is not transitive.

Hence, R is reflexive and symmetric.

(4) P.T the relation congruence modulo 'm' is given by $x \equiv y \pmod m = \{(x,y) \mid x-y$ is divisible by $m\}$ is an equivalence relation and also S.T

$x_1 \equiv y_1, x_2 \equiv y_2 \Rightarrow x_1 + x_2 \equiv y_1 + y_2$

Sol: $x \equiv y \pmod m$

$R = \{(x,y) \mid x-y$ is divisible by $m\}$

Reflexive:-

W.K.T $x-x=0$ is divisible by 'm'

$\Rightarrow x \equiv x \pmod m$

$\Rightarrow x R x$

$\Rightarrow (x,x) \in R$

$\therefore \equiv$ is Reflexive.

Symmetric:- let $(x,y) \in R$

let $x \equiv y \pmod m$

$\Rightarrow x-y$ is divisible by m

$\Rightarrow -(x-y)$ is " " m

$\Rightarrow \quad y \equiv x \pmod{m}$

$\therefore \equiv$ is Symmetric.

Transitive: let $(x, y) \in R$ and $(y, z) \in R$

let $x \equiv y \pmod{m}$ and $y \equiv z \pmod{m}$

$\Rightarrow$ $x-y$ is divisible by m and $y-z$ is divisible by 'm'

$\Rightarrow$ $x-\cancel{y}+\cancel{y}-z$ is divisible by 'm'

$\Rightarrow$ $x-z$ is divisible by 'm'

$x \equiv z \pmod{m}$

$\therefore \equiv$ is Transitive

$\therefore$ Congruence is Reflexive, Symmetric, & Transitive

Hence Congruence is Equivalence relation

Given that $x_1 \equiv y_1$ and $x_2 \equiv y_2$

$x_1 - y_1$ divisible 'm' and $x_2 - y_2$ is divisible by 'm'

$(x_1 - y_1) + (x_2 - y_2)$ is divisible by 'm'

$(x_1 + x_2) - (y_1 + y_2)$ " " "

$x_1 + x_2 \equiv y_1 + y_2 \pmod{m}$.

$x_1 + x_2 \equiv y_1 + y_2$

Relation Matrix:-

A relation matrix R from a finite set A to a finite set B can be represented by a matrix called the relation matrix of R.

Let $A = \{a_1, a_2 \ldots a_m\}$ and $B = \{b_1, b_2 \ldots b_n\}$ be finite sets containing m and n elements, respectively, and R be the relation from A to B. Then 'R' can be represented by an $m \times n$ matrix $M_R = [r_{ij}]$ which is defined as follows:

$$r_{ij} = \begin{cases} 1 & \text{if } a_i R b_j \\ 0 & \text{if } a_i \not R b_j \end{cases}$$

Note: The Matrix ~~real~~ M_R has the elements as 1's and 0's.

Eg ① Let $A = \{1, 2, 3, 4\}$ and $B = \{b_1, b_2, b_3\}$. Consider the relation $R = \{(1, b_2), (1, b_3), (3, b_2), (4, b_1) (4, b_3)\}$ Determine the matrix of the relation

Sol: $A = \{1, 2, 3, 4\}$; $B = \{b_1, b_2, b_3\}$

Relation $R = \{(1, b_2), (1, b_3), (3, b_2), (4, b_1), (4, b_3)\}$

Matrix of the relation R is written as

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

That is $M_R = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

(2) Let $A = \{1, 2, 3, 4\}$. Find the relation R on A determined by the matrix $M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

Sol: Given $M_R =$

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

The Relation $R = \{(1,1), (1,3), (2,3), (3,1), (4,1) (4,2) (4,4)\}$

Properties of a relation on a set :-

1) If a relation is reflexive, then all the diagonal entries must be '1'.
2) If a relation is symmetric, then the relation matrix is symmetric, i.e $r_{ij} = r_{ji}$ for $\forall$ i and j
3) If a relation is anti-symmetric, then its matrix is such that if $r_{ij} = 1$, then $r_{ji} = 0$ for $i \neq j$

Graph of a Relation :-

A relation defined in a finite set can ~~be~~ also be represented pictorially with the help of a graph.
Let 'R' be a relation in a finite set $A = \{a_1, a_2, \ldots a_n\}$
The elements of A are represented by points or circles called nodes. These nodes are called vertices.

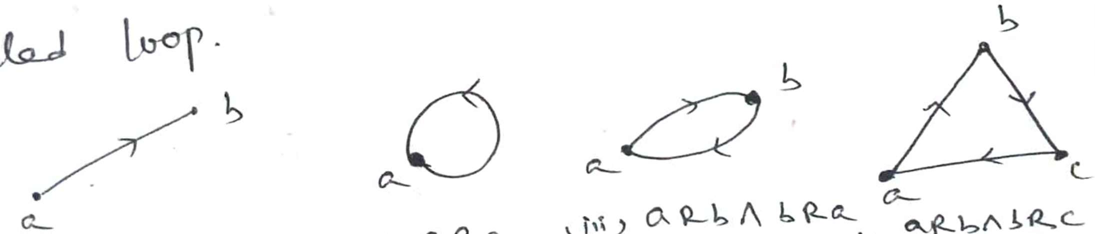
If $(a_i, a_j) \in R$, then we connect the nodes a_i and a_j by means of an arc and put an arrow on the arc in the direction from a_i to a_j. This is called an 'edge'

* If all nodes corresponding to the ordered pairs in R are connected by arcs with proper arrows, then we get a graph of the relation R.

Note: If $a_i R a_j$ and $a_j R a_i$, then we draw two arcs between a_i and a_j with arrows pointing in both directions.

Note: If $a_i R a_i$, then we get an arc which starts from node a_i and returns to node a_i, this arc is called loop.

Ex:



(i) aRb (ii) aRa (iii) $aRb \wedge bRa$ (iv) $aRb \wedge bRc \wedge cRa$.

Properties of Graph of relations:-

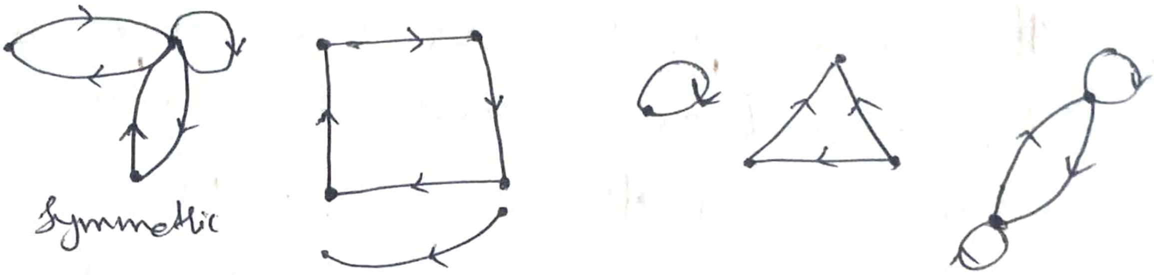
(1) If a relation is reflexive, then there is a loop at any node

(2) If a relation is symmetric and if one node is connected to another, then there must be ~~return~~ return arc from the second node to the first.

(3) For any anti-symmetric relation, no direct return paths should exist.

(4) If a relation is transitive, then the situation is not so simple.

Sol (1)

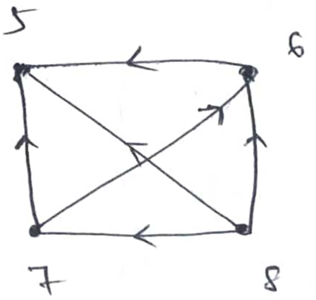


Symmetric   Anti-symmetric   Transitive.

(2) Let $A = \{5, 6, 7, 8\}$ and $R = \{(x, y) \mid x > y\}$.
Draw the graph of R and also give its matrix.

Sol $R = \{(8,5), (8,6) (8,7), (7,5) (7,6), (6,5)\}$.

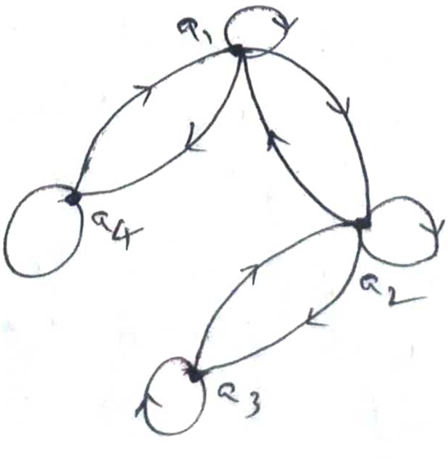
Graph of R



Matrix of R

$$\begin{array}{c|cccc} & 5 & 6 & 7 & 8 \\ \hline 5 & 0 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 & 0 \\ 7 & 1 & 1 & 0 & 0 \\ 8 & 1 & 1 & 1 & 0 \end{array}$$

(3) Find the relation determined by the graph given below and corresponding relation matrix, and also find the properties of the relation given by the graph.



Sol: Relation $R = \{ (a_1, a_1), (a_1, a_2), (a_1, a_4) \ (a_2, a_2)$
$(a_2, a_1), (a_2, a_3), (a_3, a_3), (a_4, a_4) \ (a_4, a_1)$
$(a_3, a_2) \}$

The corresponding Matrix of the relation is

$$\begin{array}{c|cccc} & a_1 & a_2 & a_3 & a_4 \\ \hline a_1 & 1 & 1 & 0 & 1 \\ a_2 & 1 & 1 & 1 & 0 \\ a_3 & 0 & 1 & 1 & 0 \\ a_4 & 1 & 0 & 0 & 1 \end{array}$$

This Relation is reflexive (Since every vertex has loop) and symmetric (Since whenever $a_i R a_j$ then $a_j R a_i$)

Equivalence Relations:-

Def: A relation R on a set 'A' is said to be an equivalence if it is Reflexive, Symmetric and transitive.

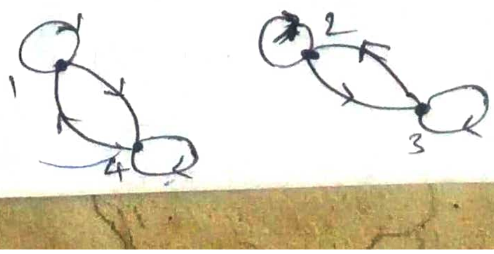
(1) Let $A = \{1, 2, 3, 4\}$ and $R = \{(1,1), (1,4), (4,1), (4,4)$
$(2,2), (2,3), (3,2), (3,3)\}$

P.T 'R' is an equivalence relation.

Sol: The Matrix $M_R =$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 1 & 0 \\ 3 & 0 & 1 & 1 & 0 \\ 4 & 1 & 0 & 0 & 1 \end{array}$$

Graph of R:



from above, the Relation 'R' is reflexive, symmetric and transitive.

Hence 'R' is an equivalence relation.

(2) ~~Let A = {1, 2, ... 7} R = {(x, y)}~~

~~[illegible]~~

Partial ordering Relation:

Def: A relation 'R' on a set 'P' is called a partial order relation or partial ordering on P iff 'R' is reflexive, Anti-symmetric and Transitive

Covering and partition of a set:-

Covering of a set:- Let 'S' be a given set and $A = \{A_1, A_2, \ldots A_n\}$, where each A_i, $i = 1, 2, \ldots m$ is a subset of S and $\bigcup_{i=1}^{n} A_i = S$. Then the set 'A' is called a covering of 'S' and the sets $A_1, A_2 \ldots A_n$ are said to cover 'S'.

Eg (1) Let $S = \{a, b, c\}$ and consider the following collections of subsets of 'S'

$A_1 = \{\{a, b\}, \{b, c\}\}$; $A_2 = \{\{a\}, \{a, c\}\}$

$A_3 = \{\{a\}, \{b, c\}\}$ $A_4 = \{\{a, b, c\}\}$

$A_5 = \{\{a\}, \{b\}, \{c\}\}$ $A_6 = \{\{a\}, \{a, b\}, \{a, c\}\}$

which of the above sets are covering?

Sol: The sets A_1, A_3, A_4, A_5 and A_6 are covering of S. But A_2 is not covering of 'S', since their union is not 'S'.

Partition of a set :-

A partition of a set S is a collection of disjoint non-empty subsets of 'S' that have 'S' as their union.

Eg ① Let $S = \{1, 2, 3, 4, 5, 6\}$.

The collection of sets $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$ and $A_3 = \{6\}$ forms a partition of 'S'. Since these sets are disjoint and their union is 'S'

Note: Every partition is also a covering. But the converse need not be ~~true~~ true.

Eg ① Let $S = \{a, b, c\}$. consider the following collection of subsets of 'S'.

$A_1 = \{\{a, b\}, \{b, c\}\}$, $A_2 = \{\{a\}, \{b, c\}\}$, $A_3 = \{\{a, b, c\}\}$

$A_4 = \{\{a\}, \{b\}, \{c\}\}$ $A_5 = \{\{a\}, \{a, c\}\}$.

Sol: The sets A_2, A_3, A_4, are partitions of 'S' and also they are covering.

Hence every partition is a covering. But the set A_1 covering, but it is not a partition of a set, since the set $\{a, b\}$ and $\{b, c\}$ are not ~~disjoint~~ disjoint.

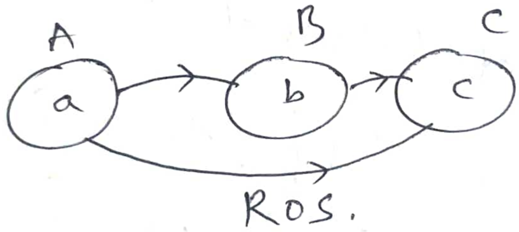
Operations on Relations

Composition of relations :- (Composition of Binary Relation

Let 'R' be a binary relation from the set A to the set B and 'S' be the binary relation from the set B to C, then the ordered pair (R, S) is said to be composable. If (R, S) is a composable pair of binary relations, the composite ROS is a binary relation from A to C such that for $a \in A$, $c \in C$

$a\,(ROS)\,c$ is for some $b \in B$, both aRb, bSc.

i-e $$ROS = \{ (a, c) \mid a \in A,\ c \in C\ \exists\ b \in B \text{ with } (a, b) \in R,\ (b, c) \in S \}.$$



Eg: Let $A = \{1, 2, 3, 4\}$: $B = \{1, 3, 9, 10\}$, $C = \{5, 6, 7, 8\}$

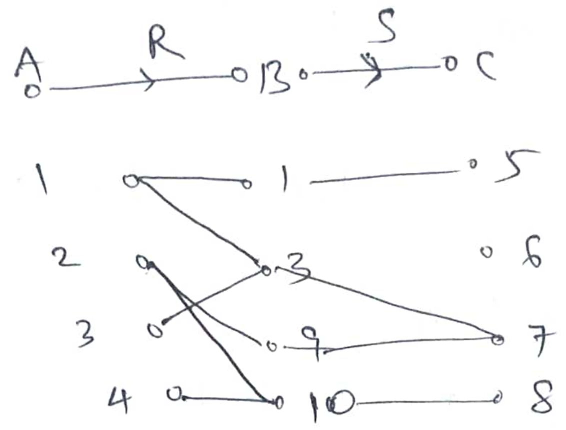
$R = \{ (1,1), (1,3), (2,9), (2,10), (3,3), (4,10) \}$

$S = \{ (1,5), (3,7), (9,7), (10,8) \}$. Find ROS

its relation graph and relation matrix

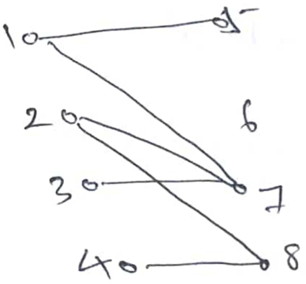
Sol

2.



$R \circ S = \{(1,5), (1,7), (2,7), (2,8), (4,8), (3,7)\}$

Ros Relation graph:



M_{Ros} (Relation Matrix of Ros)

$$M_{Ros} = \begin{array}{c|cccc} & 5 & 6 & 7 & 8 \\ \hline 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 1 \\ 3 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 1 \end{array}$$

Converse of Relation:-

Given relation R from a set A to set B, the converse of R is defined by R^c & $\tilde{R}$ and is defined as

$(a,b) \in R^c$ iff $(b,a) \in R$

Note:- The relation Matrix $M_{\tilde{R}}$ or M_{R^c} of $\tilde{R}$ can be obtained by ~~take take~~ transpose of M_R.

i.e $M_{\tilde{R}} = (M_R)^T$

Eg: ① Given that the relation matrix $M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

Then find $M_{\tilde{R}}$

$$M_{\tilde{R}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Eg: let $R = \{(1,2), (3,4), (2,2)\}$ and

$S = \{(4,2), (2,5), (3,1), (1,3)\}$

find $R \circ S$, $S \circ R$, $R \circ (S \circ R)$, $(R \circ S) \circ R$

$R \circ R$, ~~$S \circ R$~~ $S \circ S$ and $R \circ R \circ R$.

sol

$R \circ S = \{(1,5), (3,2), (2,5)\}$

$S \circ R = \{(4,2), (3,2), (1,4)\}$

Here $R \circ S \neq S \circ R$

$(R \circ S) \circ R = \{(1,5) (3,2) (2,5)\} \circ \{(1,2), (3,4), (2,2)\}$

$= \{(3,2)\}$

$R \circ (S \circ R) = \{(3,2)\} = (R \circ S) \circ R$

$R \circ R = \{(1,2) (2,2)\}$

$S \circ S = \{(4,5), (3,3), (1,1)\}$

$R \circ R \circ R = \{(1,2), (2,2)\}$.

Eg: let R and S be two relations on a set of positive integers I :

$R = \{(x, 2x) / x \in I\}$: $S = \{(x, 7x) / x \in I\}$

find $R \circ S$, $R \circ R$. $R \circ R \circ R$ and $R \circ S \circ R$

$R \circ S = \{(x, 14x) \mid x \in I\}$

$\left[\begin{array}{l} R\,(1,2),\,(2,4)\,(3,6)\ \ldots \\ S\,\{(1,7)\,(2,14)\,(3,21)\,(4,28)\ \ldots \end{array}\right]$

$R \circ R = \{(x, 4x) \mid x \in I\}$

$R \circ R \circ R = \{(x, 8x) \mid x \in I\}$

$R \circ S \circ R = \{(x, 28x) \mid x \in I\}.$

Q: $R = \{(1,2)\ (3,4),\ (2,2)\}$

$S = \{(4,2),\ (2,5),\ (3,1)\ (1,3)\}$

find the Relation Matrices for $R \circ S$ and $S \circ R$ on the Set $\{1, 2, 3, 4, 5\}$

Sol

$$M_R = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 \end{array} \qquad M_S = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$$M_{R \circ S} = M_R \circ M_S \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \ \& \ M_{S \circ R} = M_S \circ S_R = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Q) Given the relation matrices M_R and M_S, find

$M_{R\circ S}$, $M_{\tilde{R}}$, $M_{\tilde{S}}$, $M_{R\tilde{\circ}S}$ and S.T

$$M_{R\tilde{\circ}S} = M_{\tilde{S}\circ\tilde{R}}$$

Given $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $M_S = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$

$$M_{\tilde{R}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \text{Transpose of } M_R$$

$$M_{\tilde{S}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \text{Transpose of } M_S$$

$$M_{R\circ S} = [M_R][M_S] = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$M_{R\tilde{\circ}S} = \widetilde{[[M_R][M_S]]} = (\text{Transpose of } M_{R\circ S})$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$M_{\tilde{S}\circ\tilde{R}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Results: (1) $(\tilde{\tilde{R}}) = R$ (2) $R = S \Leftrightarrow \tilde{R} = \tilde{S}$

(3) $R \subseteq S \Leftrightarrow \tilde{R} \subseteq \tilde{S}$, (4) $R \tilde{\cap} S = \tilde{R} \cap \tilde{S}$

(5) $R \tilde{\cup} S \Leftrightarrow \tilde{R} \cup \tilde{S}$

(Definition) - The Transitive closure :-

The transitive closure of a relation 'R' is the smallest transitive relation containing R. we denote transitive closure of R by R^+. Let 'X' be any finite set containing 'n' elements and 'R' be a relation on 'X'

The relation $R^+ = R \cup R^2 \cup R^3 \cup \cdots \cup R^n$ on 'X'

is called transitive closure of 'R' on 'X'.

Note:1 $R_1^+ = R_1 \cup R_1^2 \cup \cdots \cup R_1^n$.

Note:2 $R^2 = R \circ R$, $R^3 = R^2 \circ R$, $R^4 = R^3 \circ R$, -----

Eg (1) Let $X = \{1, 2, 3, 4\}$ and $R = \{(1,2), (2,3), (3,4)\}$ be a relation on X. Find R^+.

Sol: Given $R = \{(1,2), (2,3), (3,4)\}$

$$R^2 = R \circ R = \{(1,3), (2,4)\}$$
$$R^3 = R^2 \circ R = \{(1,4)\}$$
$$R^4 = R^3 \circ R = \emptyset$$
$$\therefore R^+ = R \cup R^2 \cup R^3 \cup R^4 = \{(1,2), (2,3)(3,4)\ (1,3)(2,4)(1,4)\}$$

Eg ② Given the relation matrix M_R of a relation R on the set $\{a, b, c\}$ find the relation matrices of $\tilde{R}$, $R^2 = R \circ R$, $R^3 = R \circ R \circ R$, where $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Sol: Given $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

i) $M_{\tilde{R}} = (M_R)^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

ii) $M_{R^2} = M_{R \circ R} = M_R \odot M_R$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

iii) $M_{R^3} = M_{R \circ R \circ R} = M_{R^2 \circ R} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

iv) $M_{R \circ \tilde{R}} = M_{\tilde{R}} \odot M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Definition:- ~~Find~~

Let M_R be the (zero-one) matrix of the relation R on a set X with 'n' elements. Then the matrix of the transitive closure R^+ is given by

$$R^+ = M_R \vee M_R^{(2)} \vee M_R^{(3)} \vee \dots \vee M_R^{(n)}, \text{ where}$$
$$M_R^{(n)} = M_{R^n}.$$

Eg: Find the matrix of the transitive closure of the Relation R, where $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

Sol: Matrix of the transitive closure is

$$M_{R^+} = M_R \vee M_R^{(2)} \vee M_R^{(3)}$$

Now $M_R^{(2)} = M_R \odot M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$M_R^{(3)} = M_R^{(2)} \odot M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$\therefore M_{R^+} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Compatibility Relations :-

Definition:- A relation R in X is said to be ~~Compatib~~ Compatibility relation if it is reflexive and Symmetric.

Eg: let $X = \{$ball, bed, dog, hut, egg$\}$ and the relation 'R' be given by $R = \{(x,y) \mid x, y \in X, xRy$ if x and y contains some ~~common~~ common letter

The compatibility is denoted with symbole "$\approx$"

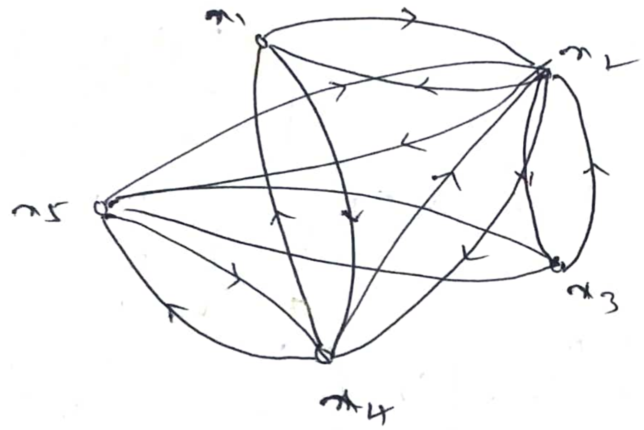
Sol : Let denoting ball by x_1

" bed by x_2

dog by x_3

hut by x_4

egg by x_5, then the graph of compatibility ($\approx$) as follows.



Here ball $\approx$ bed

bed $\approx$ egg

but ball $\not\approx$ egg

(since no common letter).

PARTIAL ORDERING RELATIONS:-

Definition:- A relation 'R' on a set 'P' is called a partial order relation or a ~~pos~~ partial ordering in P iff R is reflexive, anti symmetric and transitive. We denote the partial ordering by the symbol "$\leq$"

Definition:- A set 'P' on which a partial ordering $\leq$ is defined is called partial ordered set or a poset and it is denoted by $(P, \leq)$ (or) $[P, \leq]$.

Eg_2: Let 'A' be any set and P(A) be collection of all subsets of A. Then $(P(A), \subseteq)$ is a poset.

Ex ② $(Z, <)$ is not a poset because $<$ is not reflexive

Sol. i.e $2 \in Z$ but $2 \nless 2$.

Definition :- Let $(P, \leq)$ be a partial ordering. Elements a, b in 'P' are said to be comparable under '$\leq$' if either $a \leq b$ & $b \leq a$, otherwise they are incomparable.

Definition: Let $(P, \leq)$ be a poset (partially ordering). If every pair of elements of A are comparable, then $(P, \leq)$ is called a totally ordered set (or) a chain (or) simply ordered set. Here the relation '$\leq$' is called a totally ordered (or) linear ordered (or) simple ordered in 'P'.

Ex: ① Let 'Z' be set of integers and $\leq$ is the usual ordering on Z, then $(Z, \leq)$ is a poset and also chain.

Ex ② Z, N, Q, R are chains under partial ordering relation '$\leq$'.

Ex ③ What is a partial ordering relation? Let $S = \{x, y, z\}$ and consider power set $P(S)$ with relation 'R' given by set inclusion. Is R a partial order?

Sol: Write the definition of partial ordering relation.

Let $S = \{x, y, z\}$

$\Rightarrow P(S) = \{ \phi, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, \{x,y,z\} \}$

Define a relation 'R' by ARB iff $A \subseteq B$ $\forall$ $A, B \in P(S)$

i) we have $A \subseteq A$ for any $A \in P(S)$

$\Rightarrow$ $\subseteq$ is reflexive on $P(S)$.

ii) For any $A, B \in P(S)$, $A \subseteq B$, $B \subseteq A$ then $A = B$

$\Rightarrow$ R is antisymmetric on $P(S)$.

iii) For any $A, B, C \in P(S)$, $A \subseteq B$, $B \subseteq C \Rightarrow A \subseteq C$

$\therefore$ R is transitive on $P(S)$.

$\therefore$ $(R, \subseteq)$ is a partial order on $P(S)$.

Hasse Diagram (or) poset Diagram :-

A partial ordering $\leq$ on a set 'P' can be represented by means of a diagram known as Hasse diagram or poset diagram of $(P, \leq)$.

Procedure for drawing Hasse diagram for poset P :-

Step 1. Each element is represented by a small circles.

Step 2. The circle for $x \in P$ is drawn below the circle for $y \in P$ if $x < y$.

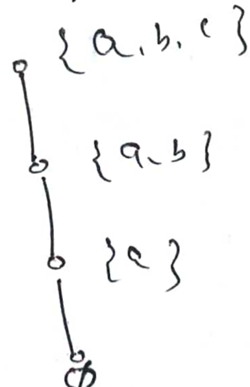
Step 3: A line drawn between x and y if y covers x and if $x < y$. y does not cover x, then x and y are not connected directly by single line. However, they are connected through one or more elements of P. It is possible to obtain the set of ordered pairs in $\leq$ from such a diagram.

Note:- The totally ordered set $(P, \leq)$, the Hasse diagram consists of circles one below the other, thus the poset is called a chain.

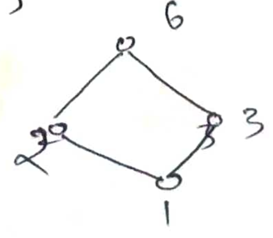
Cover: Let 'P' be a partial order set and let $(a, b) \in R$ we say that 'b' cover 'a' if $a < b$ and there is no $c \in P$ such that $a \leq c$, $c \leq b$.

Ex: If b covers a, we can represent graphically (Hasse diagram) as follows:

Ex: ① Consider $P = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}\}$ where $\subseteq$ is set inclusion, $(P, \subseteq)$ be a poset then find Hasse diagram

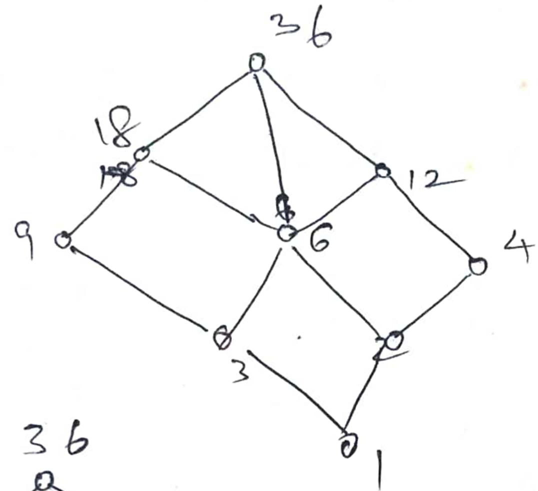


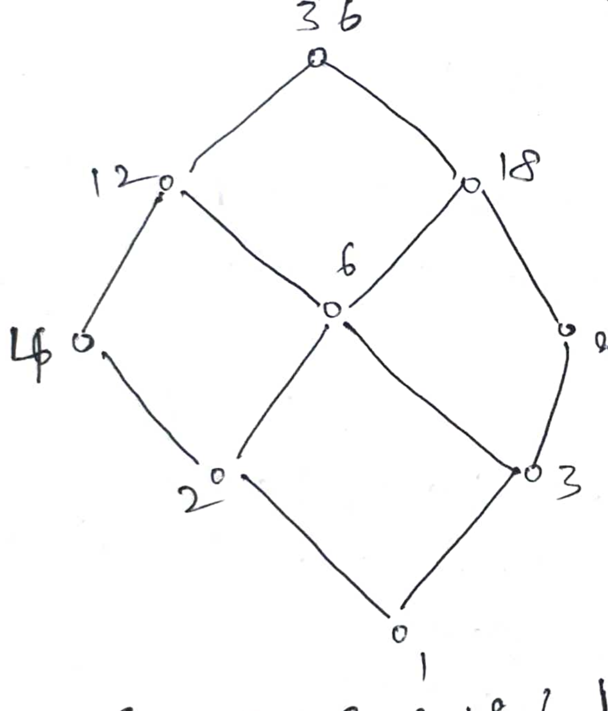
② Let $D_6 = \{1, 2, 3, 6\}$. Draw Hasse diagram of $(D_6, |)$



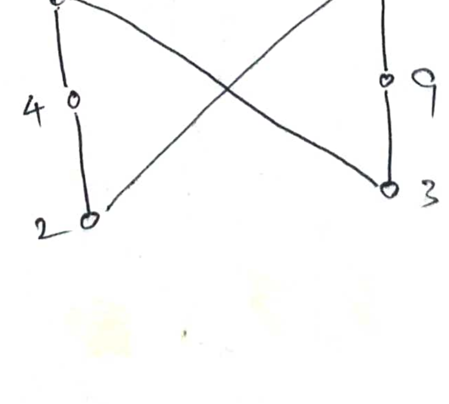
(3) Draw the poset diagram representing the +ve divisors of 36 and ~~[illegible]~~

Divisors of 36 are 36, 18, 12, 9, 6, 4, 3, 2, 1





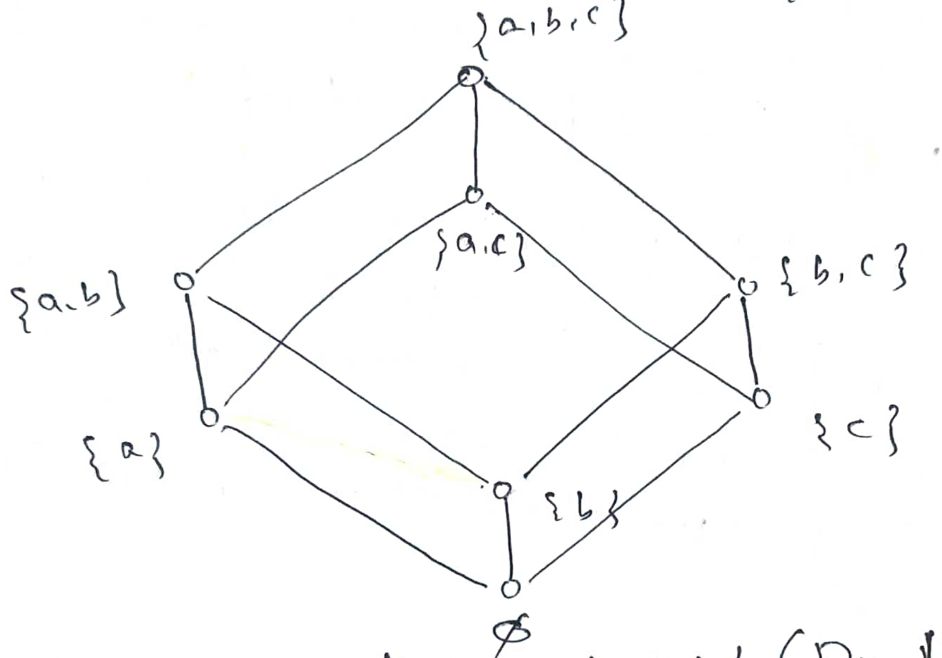
(4) Let ({2, 3, 4, 9, 12, 18}, |) be a poset. Draw its ~~Hasse~~ Hasse diagram.



(29)

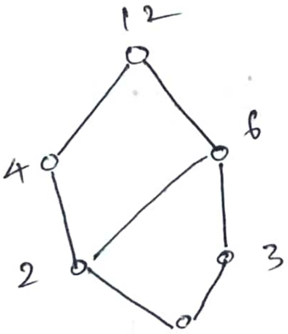
(4) Given $A = \{a, b, c\}$, Draw the Hasse diagram of $(P(A), \subseteq)$

$P(A) = \{ \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \}$



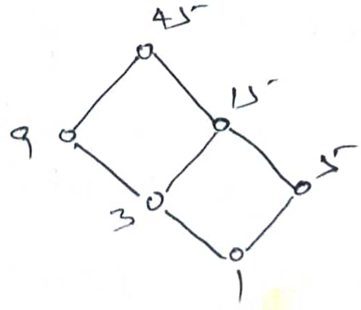
(5) Draw the Hasse diagram of poset $(D_{12}, |)$

Divisiors of 12 are $\{1, 2, 3, 4, 6, 12\}$



(6) Draw the Hasse diagram of poset $(D_{45}, |)$

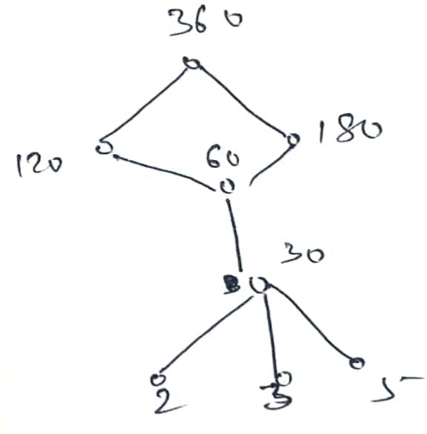
sol $D_{45} = \{1, 3, 5, 9, 15, 45\}$



Definition:- Let $(P, \leq)$ be a poset, and $A \subseteq P$, then $a \in A$ is called a lower bound of A if $a \leq x \; \forall \; x \in A$ and if there are no lower bounds of A which are greater than 'a', then 'a' is called greatest lower bound (g.l.b) of A (or) infimum (inf) of A.

Definition:- Let $(P, \leq)$ be a poset, $A \subseteq P$, then $a \in A$ is called an upper bound of A if $x \leq a \; \forall x \in A$ and if there are no upper ~~bounds~~ bounds of A which are less than 'a' then 'a' is called least upper bound (l.u.b) of A & supremum (sup) of A.

Eg: Consider the poset $(\{2, 3, 5, 30, 60, 120, 180, 360\}, |)$ having the Hasse diagram.



~~120, 180~~, 360, ~~60~~ ~~are~~ upper bounds

Special Elements in posets:-

Def: Let $(P, \leq)$ be a poset and $A \subseteq P$, then
(i) An element $a \in A$ is called least element of A if $a \leq x \; \forall x \in A$
(2) An element $a \in A$ is called the greatest element of A if $x \leq a \; \forall x \in A$

Note: The least and greatest elements are unique if exist.

Definition:- Let $(P, \leq)$ be a poset, $A \subseteq P$. An element $a \in A$ is said to be a minimal element of A if there exist no 'x' in 'A' such that $x < a$

Def: Let $(P, \leq)$ be a poset. $A \subseteq P$. An element $a \in A$ is said to be a maximal element of A if there exist no 'x' such that $a < x$.

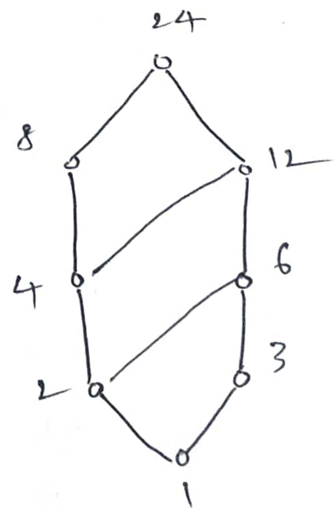
Note: Minimal & Maximal elements are not unique.

Eg ① Let $D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$ and relation divides '|' be a partial ordering on D_{24}. Then draw the Hasse diagram $(D_{24}, |)$ and also find

(i) all lower bounds of 8, 12 (ii) all upper bounds of 8, 12

(iii) g.l.b of 8, 12 (iv) l.u.b of 8, 12

(v) greatest and least elements of this poset if exist.



Sol:

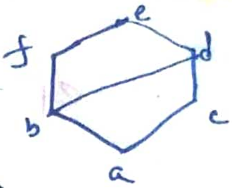
(i) the lower bounds of 8, 12 are ~~1, 2, 4~~ 1, 2, 4

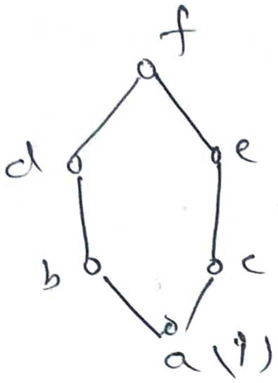
(ii) upper bounds of 8, 12

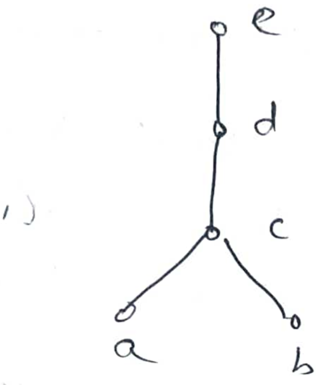
(iii) g.l.b of 8, 12 is 4

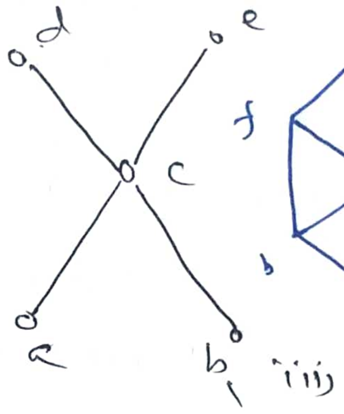
(iv) l.u.b of 8, 12 is 24.

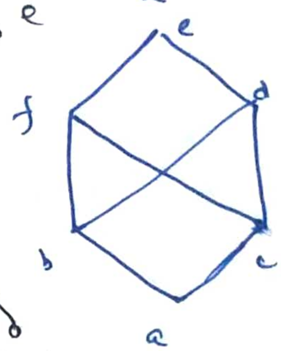
(2) Determine the greatest and least elements of the following posets if they exist.









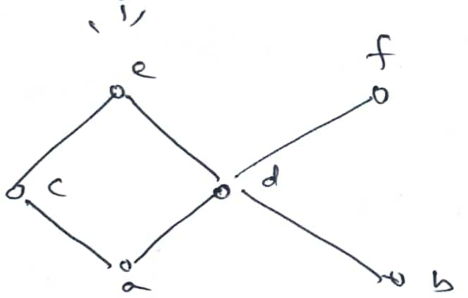


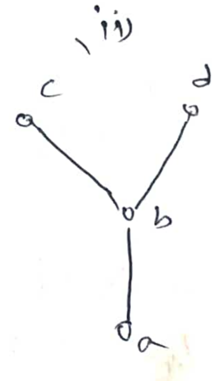
Sol: (i) The greatest of (i) is f and least is 'a'

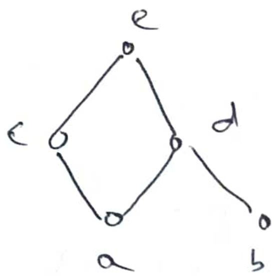
(2) The greatest element of (ii) is e but there is no least element.

(3) No greatest and least elements.

(3) Find the Maximal and Minimal elements of the posets







Sol (i) Maximal elements are e, f
Minimal elements are a, b

(ii) Maximal elements are c, d
~~Mini~~ Minimal element is 'a'

(iii) Maximal element is e
Minimal elements are a, b.

(4) Draw the poset diagram of ~~36~~ all the divisors of 36 and determine all maximal, minimal elements and greatest, least elements if they exist

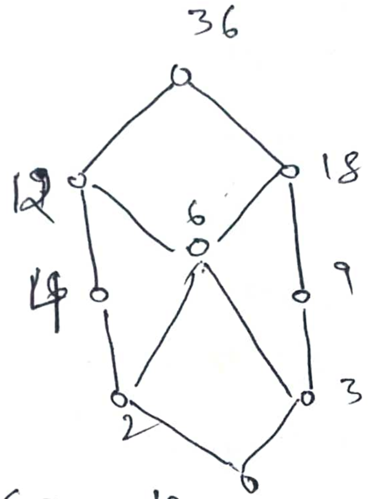
Sol: $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

Maximal element = 36

Minimal element = 1

Greatest element = 36

Least element = 1



(5) Draw the poset diagram of $(D_{30}, |)$ and find Maximal and minimal elements, and least and greatest elements.

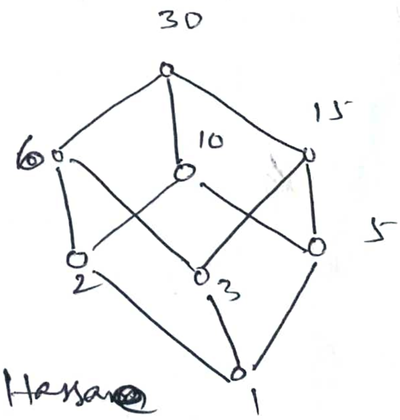
$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

Maximal element - 30

Greatest element - 30

Minimal Element - 1

least element - 1



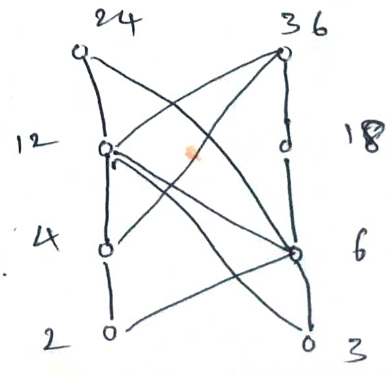
(6) find poset diagram (or) Hassa diagram of $(A, |)$, where $A = \{2, 3, 4, 6, 12, 18, 24, 36$

Minimal elements are 2, 3

Maximal elements are 24, 36

least element does not exist

greatest element does not exist.



ALGEBRAIC SYSTEMS

Def:- A mapping $f: A \times A \times \dots \times A \to A$, of $f: A^n \to A$ is called an n-ary operation and 'n' is called the order of the operation.

* If $f: A \to A$ is called a unary operation

$f: A \times A \to A$ - - " Binary "

Def:- A system consisting of a set and one or more n-ary operations defined on the set is called an algebraic system or algebra.

If $n = 1$ such operation is called unary operation

$n = 2$ " " Binary "

Note: A binary operation will be denoted by means of a symbol such as $*, \Delta, +, \oplus, \circledast, \vee, \wedge$ etc and the result of binary operation on the elements $x_1, x_2 \in X$ is expressed by writting $x_1 * x_2$.

Eg: An algebraic system of a set 'G' is denoted by $(G, *)$ & (G, Δ) & $(G, +)$ ...

Eg: $(N, +)$ is an algebraic system.

Since $\forall a, b \in N$ such that $a + b \in N$.

i.e $1, 2 \in N \Rightarrow 1 + 2 \in N$.

Eg $(N, -)$ is not an algebraic system.

Since $a \forall a, b \in N$ such that $a - b \notin N$

i.e $1, 2 \in N \Rightarrow 1 - 2 = -1 \notin N$. [redacted]

~~ALGEBR~~

Ex 1. $(Z, +)$, $(Z, -)$, $(R, +)$, $(R, -)$ are all algebraic ~~algebraic~~ System

2. $(Z, \div)$, $(Z^+, \div)$ $(N, \div)$ are not algebraic ~~algebraic~~ System.

since $1, 2 \in Z$, but $1 \div 2 \notin Z$

Properties of algebraic systems:

Consider $(G, +)$ is an algebraic system, then

1) Associativity (or) Associative Property

For $\forall a, b, c \in G \Rightarrow a+(b+c) = (a+b)+c$.

2) Identity element ~~property~~ :-

For (all) $\forall a, \in G$, there exist a unique element $0 \in G$ such that $a+0 = 0+a = a$.

Here '0' (zero) is called additive identity (or) Identity ~~additive~~ w.r.t addition.

3) Inverse element

For all $a \in G$, there exist an unique element '$-a$' such that $a+(-a) = -a+a = 0$, then '$-a$' is called inverse element w.r.t addition.

4) Commutative Property:

For all $a, b \in G$, such that $a+b = b+a$, then $(G, +)$ is commutative.

Consider $(G, \times)$ is an algebraic system, then

(or) $(G, \cdot)$

(1) Associative Property:

For all $a, b, c \in G$, such that

$$a \times (b \times c) = (a \times b) \times c$$

(2) Identity element:

For all $(\forall)$ $a \in G$ ~~such that~~ there exist a unique element $1 \in G$ such that

$$1 \times a = a \times 1 = a.$$

Here '1' (one) is called multiplicative identity

(or) identity w.r.t multiplication.

(3) Inverse element:

For all $a \in G$, there exist $1/a \in G$ such that $a \times \frac{1}{a} = \frac{1}{a} \times a = 1$

Here ~~[illegible]~~ a and $1/a$ are inverse to each other.

(4) Commutative:

For all $(\forall)$ $a, b \in G$ such that

$a \times b = b \times a$, then $(G, \times)$ is commutative.

(5) Cancellation Property:

For $a, b, c \in G$ and $a \neq 0$ such that

$$a \times b = a \times c \Rightarrow b = c.$$

Distributive property

$\forall a, b, c \in G$ such that

$$a \times (b + c) = a \times b + a \times c.$$

Eg: $(N, +)$ is satisfied all property except identity and inverse.

$(N, -)$ is not an algebraic system.

$(N, \times)$ is satisfies Associative and commutative only.

$(Z, +)$ satisfies all property w.r.t '+'.

i.e. Associative, Identity, inverse & commutative.

In general for the algebraic system we can use $\circ$ & $*$. i.e $(G, \circ)$ & $(G, *)$

Property:

Associative: $\forall a, b, c \in G \Rightarrow a \circ (b \circ c) = (a \circ b) \circ c$

Identity: $\forall a \in G$, there exist unique $e \in G$ such that $a \circ e = e \circ a = a$, here '$e$' is identity element w.r.t '$\circ$'

Inverse: $\forall a \in G$, there exist unique $b \in G$ such that $a \circ b = b \circ a = e$,

$\therefore a, b$ are inverse to each other.

Commutative: $\forall a, b \in G$ such that $a \circ b = b \circ a$.

Distributive: $\forall a, b, c \in G$ such that $a \circ (b * c) = (a \circ b) * (a \circ c)$.

Semi group: Let 'S' be a non-empty set and 'o' be a binary operation on 'S'. The algebraic system (S, o) is called a Semi group if the operation satisfies associative property w.r.t 'o'.

i.e., if $\forall a, b, c \in S$ then $a \circ (b \circ c) = (a \circ b) \circ c$.

Eg: $(N, +)$, $(Z, +)$, $(R, +)$ are Semi groups.

$(N, -)$ is not a Semi group,

since $1, 2, 3 \in N$ but

$1-(2-3) \neq (1-2)-3$

Monoid: Let 'M' be a non-empty set and 'o' be a binary operation on M. The algebraic system (M, o) is called a Monoid, if it satisfies associative and it consists identity element.

i.e., 1) $\forall a, b, c \in M \Rightarrow a \circ (b \circ c) = (a \circ b) \circ c$

2) $\forall a \in M$, there exist unique $e \in G$ such that $a \circ e = e \circ a = a$.

(or)

A semi group (M, o) with an identity element w.r.t 'o' is called a monoid.

Eg: $(Z, +)$, $(R, +)$ are Monoids.

$(N, +)$ is not a Monoid, since identity element does not exist w.r.t '+'.

$(Z, \times)$, $(R, \times)$ are monoids. Since these are satisfies associativity & identity element w.r.t multiplication exists.

i.e $1 \in Z$ & $1 \in R$.

Ex: The Composition Table is given by where $G = \{a, b, c\}$.

*	a	b	c
a	a	b	c
b	b	b	c
c	c	b	c

$(G, *)$ is a monoid.

Since it satisfies associative

i.e $\boxed{a*(b*c) = (a*b)*c}$

① $\rightarrow$ $a*(b*c) = a*(c)$ [∵ from table]
$= a*c$
$= c$ [∵ from table]

② $\rightarrow$ $(a*b)*c = (b)*c$ [∵ from table]
$= b*c$
$= c$ [∵ from table]

∴ ① = ②

Also satisfies identity w.r.t '*'

i.e $a*a = a$
$a*b = b$
$a*c = c$
∴ a is identity.

Ex: Let 'S' be a nonempty set and $P(S)$ be a power set. The algebras $(P(S), \cup)$ and $(P(S), \cap)$ are monoids with identities ϕ and S respectively.

Eg: The composition table is given by where $G = \{1, 2, 3\}$.

$\oplus$	1	2	3
1	1	2	1
2	1	2	2
3	1	2	3

Now $(G, \oplus)$ is a monoid.
Since it satisfies Associative & identity property

Associative:

$1 \oplus (2 \oplus 3) = (1 \oplus 2) \oplus 3$

$1 \oplus (2) = (2) \oplus 3$ [$\because$ from table

$2 = 2$ [$\because$ from table]

Identity:

$3 \oplus 1 = 1$
$3 \oplus 2 = 2$ $\quad \therefore$ 3 is identity element w.r.t $\oplus$
$3 \oplus 3 = 3$

Addition modulo 'm' :- "$+_m$"

The addition modulo 'm' of a and b is defined as the least non-negative remainder 'r' obtained when a+b is divided by 'm'.
It is denoted as $a +_m b = r$, where $0 \leq r < m$.

Eg:- 1, $8 +_4 9 = 1$
[$\because$ $8 + 9 = 17$ and when 17 is divided by '4' the remainder is '1']

2, $8 +_3 6 = 2$

[$\because$ $8 + 6 = 14$ and when 14 is divided by '3' the remainder is '2']

3, $8 +_3 7 = 0$
[$\because$ $8 + 7 = 15$, when 15 is divided by '3', the remainder is '0'

4) $2 +_6 3 = 5$

[∵ $2+3=5$, when '5' is divided by '6', the remainder is '5'.]

Multiplication modulo m : '$\times_m$'

The multiplication modulo 'm' of a and b is defined as the least non-negative remainder 'r' obtained when product ab is divided by 'm'. And is denoted by $a \times_m b = r$, $0 \leq r < m$.

Sol 1) $3 \times_2 5 = 1$

[∵ $3 \times 5 = 15$, when 15 is divided by '2' the remainder is '1'.]

2) $6 \times_4 5 = 2$

[∵ $6 \times 5 = 30$, when, 30 is divided by '4', the remainder is '2'.]

3) $9 \times_6 8 = 0$

[∵ $9 \times 8 = 72$, when 72 is divided by '6', the remainder is '0'.]

Eg: If $Z_4 = \{0, 1, 2, 3\}$. Find the composition table for addition modulo '4'.

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

* Here Identity is '0'

$(Z_4, +_4)$ is a monoid

∵ Associative satisfies & identity exist

$(1 +_4 2) +_4 3 = 1 +_4 (2 +_4 3)$

$3 +_4 3 = 1 +_4 1$

$2 = 2$

Eg: If $Z_5 = \{0, 1, 2, 3, 4\}$. Find the composition table for addition modulo '5'

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$(Z_5, +_5)$ is a monoid.

Here identity is '0'

Eg: If $Z_4 = \{1, 2, 3, 4\}$. Find the composition table for multiplication modulo '4'

$\times_4$	1	2	3	4
1	1	2	3	4
2	2	0	2	0
3	3	2	1	0
4	0	0	0	0

$(Z_4, \times_4)$ is not an ~~this is not~~ algebraic structure

∵ $2 \times_4 2 = 0$

$0 \notin Z_4$

Q) $Z_{10} = \{1, 3, 7, 9\}$, find the composition table for multiplication modulo '10' ($\times_{10}$)

$\times_{10}$	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

Here $(Z_{10}, \times_{10})$ is a monoid.

Here identity is '1'

Associative:

$(1 \times_{10} 3) \times_{10} 7 = 1 \times_{10} (3 \times_{10} 7)$

$3 \times_{10} 7 = 1 \times_{10} 1$

$1 = 1$

Q) $Z_6 = \{0, 1, 2, 3, 4, 5\}$.

the composition table for multiplication modulo '6' ($\times_6$)

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

$(Z_6, \times_6)$ is a monoid.

Here identity is '1'

Also satisfies associative property.

Group:

Def: A group $(G, *)$ is an algebraic system in which the binary operation $*$ on G satisfies ~~these~~ three conditions:

1. Associative,

   i.e $\forall\ x, y, z \in G$,

   $$x * (y * z) = (x * y) * z$$

2. Identity:

   i.e there exist ($\exists$) an element $e \in G$ such that $x * e = e * x = x$ for $x \in G$.

   Here 'e' is called identity element.

3. Existence of inverse;

   i.e for every $x \in G$, there exist an element $x^{-1} \in G$ such that

   $$x * x^{-1} = x^{-1} * x = e.$$

   x & x^{-1} are said to be inverse to each other. 'e' is an identity.

Eg: $(Z, +)$ is a group

i It is an algebraic system.

Associative: $(0+2)+3 = 0+(2+3)$

$5 = 5$

Identity: '0' is an identity element w.r.t addition

$\because\ 0+4 = 4+0 = 4$

Inverse: $4-4=0$, $2-2=0$ $\therefore$ inverse exist.

$(Z, \times)$ is not a group.

Since inverse does not exist.

i.e for $2 \in Z$,

$\frac{1}{2} \notin Z$. [$\because$ inverse of 2 is $\frac{1}{2}$ w.r.t $\times$.]

$(Q, +)$, $(R, +)$ are groups

$(Q^+, \times)$, $(R^+, \times)$ are groups.

Q^+ = { Set of all number in p/q $(q \neq 0)$, except '0']

R^+ : { set of all +ve real numbers except '0']

Identity is '1' w.r.t $\times$ (multiplication)

inverse element are exist.

$2 \times \frac{1}{2} = 1$

$4 \times \frac{1}{4} = 1$ .....

Ex $G = \{a, e\}$ and Composition table w.r.t $*$ is given by

$*$	a	e
a	e	a
e	a	e

$(G, *)$ is a group

Since, It satisfies association, identity & inverse.

Here identity is 'e'

inverse of a is a. [Self inverse]

" " e is e.

Eg: Let I be a set of integers, then $(I, +)$ is a group.

Def: Order of a group:

The order of a group $(G, *)$ denoted by $|G|$ and is defined as no. of elements in G.

Def: Abelian group: A group $(G, *)$ in which the operation '$*$' is commutative is called abelian group.

Def: when $\forall x, y \in G$, if $x*y = y*x$, then it is called commutative.

Eg: Let I be a set of integers, the algebraic structure $(I, +)$ is an abelian group.

Since,

1, algebraic structure:

i.e. $\forall y, x \in I$, $x+y \in I$

2, Associative:

$\forall x, y, z \in I$,

$x+(y+z) = (x+y)+z$

3, Identity:

for any $x \in I$, there exist an element '0' (identity w.r.t +) is exist such that $x+0 = 0+x = x$.

3, Inverse:

for any $x \in I$, there exist $-x \in I$ such that $x+(-x) = (-x)+(x) = 0$.

4 Commutative:

$\forall x, y \in I$ such that $x+y = y+x$.

Hence it is commutative.

Eg, The set of all Rational numbers excluding '0' is an abelian group (commutative group) under multiplication. (~~time~~ ~~CE~~,

Eg, The set of all Real numbers excluding '0' is an abelian group under multiplication

Eg, $(I, +)$, $(Q, +)$, $(R, +)$, $(C, +)$ are all abelian groups.

Eg, Let $G = \{e, a, b\}$, the operation $*$ on G given in the following composition table, then $(G, *)$ is an abelian group.

$*$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Sol: Since, $(G, *)$ is algebraic system

2, It satisfies associative

3, Identity exist i-e 'e'

4, Inverse exist for all elements e, a, b

i-e $e * e = e$ → e is self inverse

$a * b = e$ → a, b are inverse to each other.

5, Commutative exist

$a * b = e = b * e$.

Eg: $Z_4 = \{1, 2, 3, 4\}$, the operation multiplication modulo 5 ($\times_5$) on Z_4, then $(Z_4, \times_5)$ is an abelian group.

proof: 1, $(Z_4, \times_5)$ is an algebraic structure

2, It satisfies associative

3, Identity exist. i.e '1'. (multiplicative identity).

4, Inverse exist for all elements.

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

i.e $2 \times_5 3 = 1$

$1 \times_5 1 = 1$

$4 \times_5 4 = 1$

$\therefore$ 2, 3 are multiplicative inverse to each other

1 & 4 are self inverses.

5, commutative exist

i.e $2 \times_5 4 = 4 \times_5 2$

$3 = 3$.

Eg 2 $Z_3 = \{0, 1, 2\}$, the operation addition modulo 3 $(+_3)$ on Z_3, then $(Z_3, +_3)$ is an abelian group.

proof: i, $(Z_3, +_3)$ is an algebraic

ii, " is satisfy associative.

iii have additive identity i.e '0'

iv, inverse exist for all elements.

$+_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

i.e $1 +_3 2 = 0 \Rightarrow$ 1 & 2 are inverse to each other

'0' is self inverse.

vi) commutative:

$2 +_3 1 = 1 +_3 2$

$0 = 0$.

Algebraic system with Two Binary operations :-

(Ring) Def: An algebraic system $(S, +, \cdot)$ is called a ring if the binary operations $+$ and $\cdot$ on S satisfy the following three properties.

1. $(S, +)$ is an abelian group
2. $(S, \cdot)$ is a semi group.
3. the operation '$\cdot$' is distributive over '$+$'

i-e for any $a, b, c \in S$,

$a \cdot (b+c) = a \cdot b + a \cdot c$ and

$(b+c) \cdot a = b \cdot a + c \cdot a$

Note: If $(S, \cdot)$ is commutative, then $(S, +, \cdot)$ is called commutative ring & abelian ring.

Note: If $(S, \cdot)$ is monoid, then $(S, +, \cdot)$ is called a ring with identity.

Field : (Def.) : A commutative ring $(S, +, \cdot)$, which has more than one element such that every non-zero element of S has multiplicative inverse in S. is called a field.

2) $(Z, +, \cdot)$ is a ring.

Since $(Z, +)$ is an abelian group, and

$(Z, \cdot)$ is a semi group., also satisfies distributive w.r.t multiplication over addition

3) $(R, +, \cdot)$, $(Q, +, \cdot)$, $(C, +, \cdot)$ are rings.

Since $(R, +)$, $(Q, +)$, $(C, +)$ are abelian groups

$(R, \cdot)$ $(Q, \cdot)$, $(C, \cdot)$ are semi groups

and also satisfies distributive property.

2) Let $R = \{a, b, c, d\}$ and define operations $+$ and $\cdot$ on R as shown in the following composition tables. S.T $(R, +, \cdot)$ is a ring.

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

Table (1)

·	a	b	c	d
a	a	a	a	a
b	a	a	b	a
c	a	b	c	d
d	a	a	d	a

Table (2)

From Table(1): $(R, +)$ is an abelian group

From Table(2) $(R, \cdot)$ is a semi group.

And also distributive.

i. $\underset{LHS}{a \cdot (b + c)} = \underset{RHS}{a \cdot b + a \cdot c}$

from T_1 $a \cdot d = a + a$ (from T_2)

From T_2 $a = a$ (from T_1)

Eg: The Ring $\langle \{a, b, c, d\}, +, \cdot \rangle$ whose operations are given by the following table

+	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	c	a	c
c	a	a	a	a
d	a	c	a	a

Is it a commutative Ring? Does it have an identity? what is the zero of this ring (additive identity)? find the additive inverse of each elements.

Sol: If $(\{a, b, c, d\}, \cdot)$ is commutative ~~ac~~ then ~~it is~~ $(\{a, b, c, d\}, +, \cdot)$ is a commutative ring.

now
$a \cdot b = a$ & $b \cdot a = a$
$a \cdot c = a$ & $c \cdot a = a$ } Hence It is commu
$c \cdot d = a$ & $d \cdot c = a$
$b \cdot d = c$ & $d \cdot b = c$

$\therefore$ It is a Commutative Ring.

Additive Identity:

$a + a = a$
$b + a = a$ hence 'a' is Identity.
$c + a = c$
$d + a = d$

Additive Inverses:

$a + a = a$ $\Rightarrow$ a is self inverse
$b + d = a$ $\Rightarrow$ b is inverse of d
$c + c = a$ $\Rightarrow$ c is self inverse
$d + b = a$ $\Rightarrow$ d is inverse of b.

Def: If $(S, \cdot)$ is a monoid, then $(S, +, \cdot)$ is Called a Ring with identity.

Zero divisor: - In a Ring, a non-zero element $a \in R$ $(a \neq 0)$ is called a zero divisor, if there exist an element $b \in R$, $b \neq 0$ Such that $ab = 0$ & $ba = 0$.

eg: $Z_6 = \{0, 1, 2, 3, 4, 5\}$

$(Z_6, +_6)$

$2 +_6 4 = 0 \Rightarrow 2, 4$ are zero divisors w.r.t $+_6$ ($2 \neq 0$, $4 \neq 0$)

$3 \times_6 4 = 0$ - $3, 4$ " " " $\times_6$.

($3 \neq 0$, $4 \neq 0$)

Integral domain: A commutative Ring $(R, +, \cdot)$ with identity and without zero divisors is called an Integral domain.

eg. $(Z, +, \cdot)$ is an integral domain

Eg The Polynomial Rings $Z[x]$, $R[x]$ are integral domains.

Eg If p is a prime, the Ring Z_p is an Integral domain

Note: Every field is an integral domain.
Not Every finite Integral domain is a field.

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