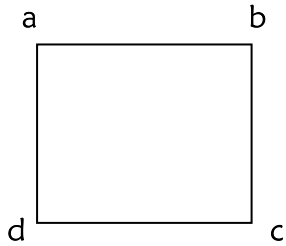
GRAPH THEORY

GRAPH :

It is an ordered pair (V, E), where V is a non-empty finite set, whose elements are called **vertices** and E is a set of Unordered pair of distinct elements of V, whose elements are called e**dges**.

Ex :



Here, V = {a, b, c, d} and

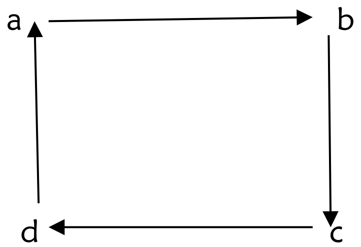
E = {{a,b}, {b,c}, {c,d}, {d,a}} (or) E = {{b,a}, {c,b}, {d,c}, {a,d}}.

Let e_1= {a,b}, e_2= {b,c}, e_3= {c,d}, e_4={d,a}. Then E ={e_1, e_2, e_3, e_4}.

DI- GRAPH :

It is an ordered pair (V, A), where V is a non-empty finite set, whose elements are called **vertices** and A is the of set of ordered pair of distinct elements of V, whose element are called **arcs** .

Ex :



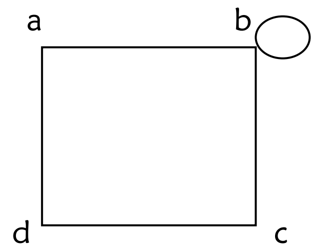
Here, V = {a, b, c, d} and

A = {{a,b}, {b,c}, {c,d}, {d,a}}.

PSEUDO GRAPH :

It is an ordered pair (V, E), where V is a non-empty finite set, whose elements are called **vertices** and E is a set of Unordered pair of elements of V, whose elements are called **edges.**

Ex :



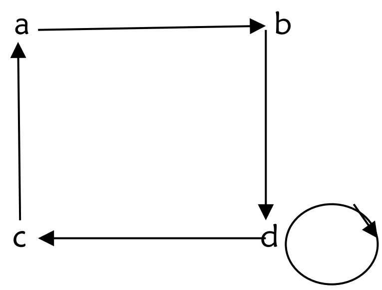
Here, V = {a, b, c, d} and

E = {{a,b}, {b,c}, {c,d}, {d,a}, {b,b}}. Here the edge {b,b} is called a "**loop** ".

PSEUDO DIGRAPH :

It is an ordered pair (V, A), where V is a non-empty finite set, whose elements are called **vertices** and A is a set of ordered pair of elements of V, whose elements are called **arcs.**

Ex :



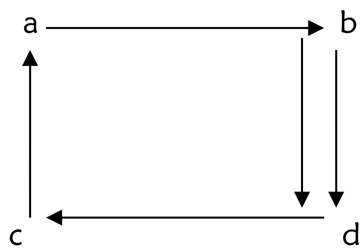
Here, V = {a, b, c, d} and

E = {{a,b}, {b,d}, {d,c}, {c,a}, {d,d}}.

MULTI DIGRAPH :

It is an ordered pair (V, A), where V is a non-empty finite set, whose elements are called **vertices** and A is the of set of ordered pair of distinct elements of V, whose element are called **arcs** .

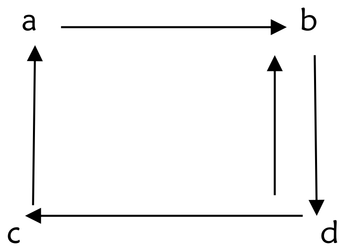
Ex :



Here, V = {a, b, c, d} and

E = {{a,b}, {b,d}, {b,d}, {d,c}, {c,a}}

Ex : :



Here, V = {a, b, c, d}

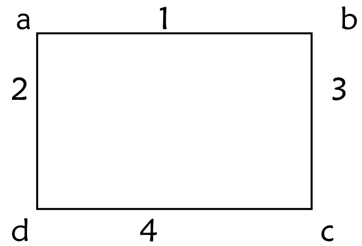
E = {{a,b}, {b,d}, {d,a}, {d,c}, {c,a}} is not a Multi Digraph.

ADJACENT :

Let G = (V,E) be a Graph.

1. Let e_1, e_2 be two edges in G. Then e_1, e_2 are said to be **adjacent** if these two edges have a common end vertex.
2. Let u, v be two vertices in G. Then , u and v are said to be **adjacent** if u and v are connected by an edge.

Ex :



Clearly, the edges 1,3 ; 3,4 ; 4,2 and 2,1 are adjacent and the edges 1,4 and 2,3 are not adjacent.

Clearly, the vertices a,b ; b,c ; c,d and d,a are adjacent and the vertices a,c and b,d are not adjacent.

INCIDENT :

Let G = (V,E) be a graph. Let e be an edge and v be a vertex of G. Then the vertex v is said to be incident to an edge e if v is one of the end vertex of e.

Ex : In the above graph, edge 1 is incident to the vertices a and b and not incident to the vertices c and d (since c and d are not the one of the end vertex of 1).

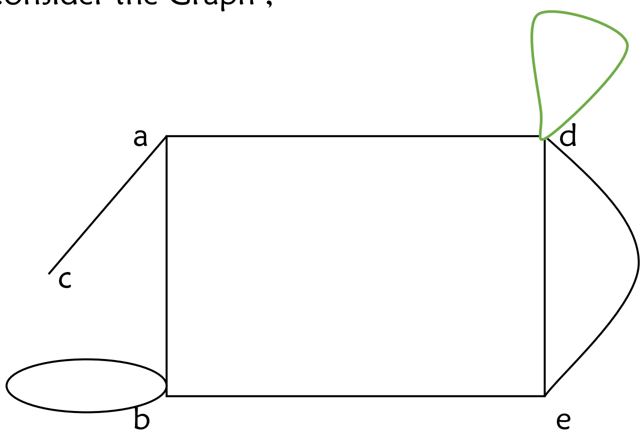
DEGREE OF A VERTEX :

Let G=(V,E) be an undirected graph and v be a vertex of G. Then, the degree of a vertex v is denoted by d(v) and is defined by the number of edges incident to the vertex v.

Note : If there is a loop at a vertex v, then the loop will give the degree 2 to the vertex v.

Ex : In the above graph, the degree d(a) = 2, d(b) = 2, d(c) = 2, d(d) = 2 .

Consider the Graph ,



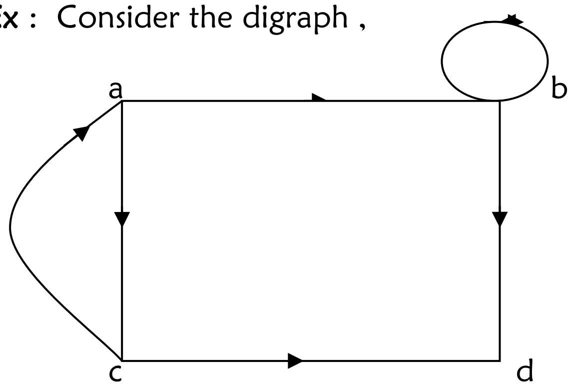
d(a) = 3 ; d(b) = 4 : d(c) = 1 ; d(d) = 5 ; d(e) = 3.

INDEGREE OF THE VERTEX:

Let G = (V, A) be a digraph and v be a vertex of G. Then, the in degree of v is defined as the number of edges towards to the vertex v and the out degree of v is defined as the number of edges away from the vertex v.

Note: The in degree of v is denoted by id(v) (or) $d^{-}(v)$ and out degree of v is denoted by od(v) (or) $d^{+}(v)$.

Ex : Consider the digraph ,



id (a)=1, od (a)=2, id(b) =2, od(b) = 2, id(c)=1, od(c)= 2, id(d) = 2, od(d)=0.

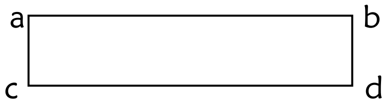
COMPLETE GRAPH :

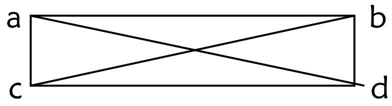
Let G = (V,E) be an undirected graph. Then G is said to be **complete** if every pair of vertices in the graph are adjacent.

(or)

A Graph G = (V,E) is said to be a c**omplete** graph if between every pair of vertices there is an edge.

Ex :





In the first graph ,

the vertices a & d , b & c are not adjacent.

So, the first graph is not complete.

In the second graph,

every pair of vertices are adjacent.

So, the second graph is complete.

Note :

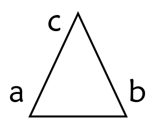
1. In a complete graph G with n vertices, the degree of each vertex is n – 1

   and the number of edges in a complete graph is $\frac{n(n-1)}{2}$

2. Complete graphs are denoted by K_n, where n denotes the number of vertices in G.

Ex : The above Complete Graph is K_4

K_1 = a . K_2 = a ———————— b K_3 =



INTERNAL VERTEX :

A vertex v in a graph G=(V,E) is said to be an internal vertex if d(v) >1.

PENDANT VERTEX :

A vertex v in a graph G=(V,E) is said to be a pendant vertex if d(v) = 1.

ISOLATED VERTEX :

A vertex v in a graph G = (V,E) is said to be an isolated vertex if d(v) = 0.

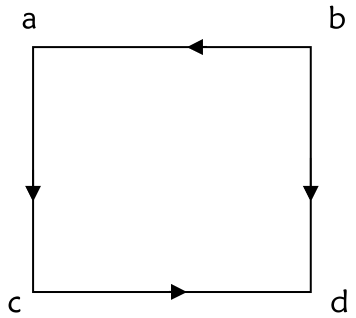
SOURCE :

A non-isolated vertex v in a digraph G=(V,A) is called a **source** , if **id(v)= 0** .

SINK :

A non-isolated vertex v in a digraph G = (V,A) is called a **sink** , if **od(v)=0 .**

Ex :



In the above graph, we have od(d) =0 and hence d is a sink, and

id(b) = 0 and hence b is a source.

Theorem 1 : Let G = (V,E) be any undirected graph. Then $\sum_{v \in V} d(v)$ is even and is equal to '2e' , where 'e' denotes the number of edges in G.

Proof : Let G = (V,E) be an Undirected Graph.

Since each edge in E is incident to exactly two vertices and hence each edge contributes the value 2 to the degree sum $\sum_{v \in V} d(v)$.

Therefore $\sum_{v \in V} d(v) = 2e$, where e denotes the number of edges in G.

Hence $\sum_{v \in V} d(v)$ is even.

Theorem 2 : Let G = (V,E) be any undirected graph. Then there is an even number of vertices having odd degree.

Proof : Let G = (V,E) be an undirected graph.

Let $V_1 = \left\{ {}^{v \in V}\!/_{d(v) = odd\ number} \right\}$ and

$V_2 = \left\{ {}^{v \in V}\!/_{d(v) = even\ number} \right\}$.

Then, $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$.

Clearly, $\sum_{v \in V} d(v) = \sum_{v \in V_1} d(v) + \sum_{v \in V_2} d(v)$. ------ (1)

By a Known result, $\sum_{v \in V} d(v)$ = even. ----------(2)

Since degree of every vertex in V_2 is even and hence

the degree sum $\sum_{v \in V_2} d(v)$ is even. ----------(3)

From (1), (2) & (3)

$\sum_{v \in V_1} d(v)$ = even - even = even. ----------(4)

Since degree of every vertex in V_1 is odd, from (4) we have

the number of vertices in V_1 is even.

That is, the number of vertices having odd degree is even.

Theorem 3 : Let G = (V,A) be a directed graph. Then prove that

$\sum_{v \in V} \boldsymbol{d^+(v)} = \sum_{v \in V} \boldsymbol{d^-(v)} = |\boldsymbol{E}| =$ **the number of arcs in G.**

Proof : Let G = (V,A) be a directed Graph.

Let e = pq be an arc in G from the vertex p to the vertex q.

Clearly, the arc e contributes the value '1' to the vertex p as out degree

and the value '1' to the vertex q as in degree.

So, each arc e contributes the value '1' to the in degree sum $\sum_{v \in V} d^+(v)$

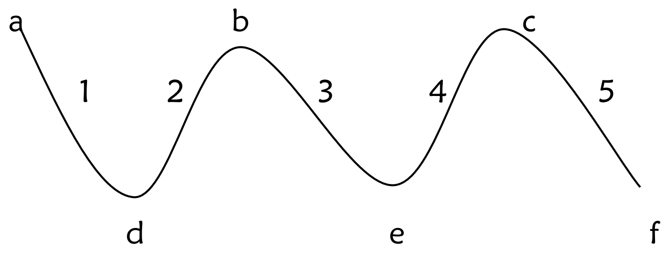
and the value ‘1’ to the out degree sum $\sum_{v \in V} d^{-}(v)$.

Therefore, $\sum_{v \in V} d^{+}(v) = \sum_{v \in V} d^{-}(v) = |E|$ = the number of arcs in G.

WALK :

A sequence of vertices and edges is called a **walk.**

Ex :



Walk = {a,1,d,2,b,3,e,4,c,5,f} and is called a-f walk.

OPEN WALK :

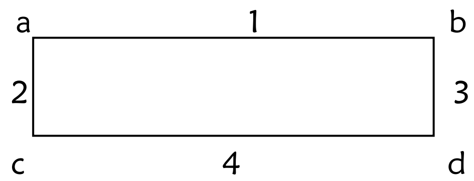
A walk is said to be an **open walk** if the end vertices in the walk are distinct.

Ex : In the above graph, the a-d walk is an example of an open walk.

CLOSED WALK :

A walk is said to be a **closed walk** if the end vertices in the walk are equal .

Ex :



In the graph, the close walk is {a,1,b,3,d,4,c,2,a }

TRIAL :

An open walk is said to be a **trail** if all the edges in the walk are distinct but vertices may be repeated.

Ex : The above open walk is an example of trial.

PATH :

An open walk is said to be a **path** if all the vertices in the walk are distinct.

Ex : The above open walk is an example of path.

CIRCUIT :

A closed walk is said to be a **circuit** if all the edges in the walk are distinct but vertices may be repeated.

Ex : The above closed walk is an example of circuit.

CYCLE :

A closed walk is said to be a **cycle** if all the vertices and edges are distinct in the walk except the end vertices.

Ex : The above closed walk is an example of a cycle.

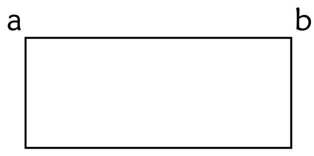
Note :

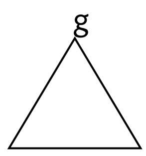
1. The number of edges in a cycle is called the l**ength of the cycle .**
2. The number of edges in a cycle is an even number, then the cycle is called an **even cycle.**
3. The number of edges in a cycle is an odd number, then the cycle is called an **odd cycle.**

CONNECTED :

Let $G = (V,E)$ be an undirected graph. Then G is said to be **connected** if between every pair of vertices there is a path.

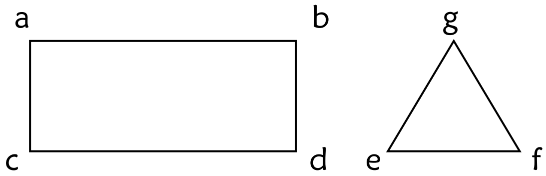
Ex : The following graphs are connected.

1). 

2). 

c d e f

Consider, the following graph (combination of above two graphs)



Since, there is no path between the vertices d and e and hence it is not connected.

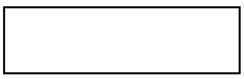
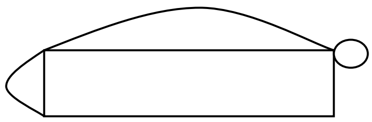
Note :

A graph G is not connected, then the graph G is called disconnected.

Ex : The above graph is an example of disconnected graphs.

SIMPLE GRAPH :

Let G = (V,E) be an undirected graph. Then G is said to be simple if G has no loops and parallel edges.

Ex :  

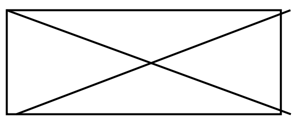
simple graph not a simple graph

REGULAR GRAPH:

Let G = (V,E) be an undirected graph. Then G is said to be a **regular** graph if every vertex in G have same degree.

Note : If every vertex in G have same degree ‘k’ (say), then the graph G is called

‘ **k-regular** ’ graph.

Ex :  

1-regular 2-regular 3-regular

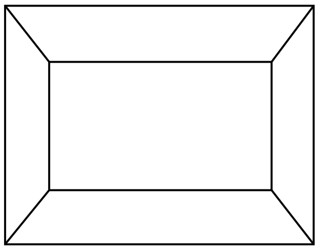
ISOMORPHISM :

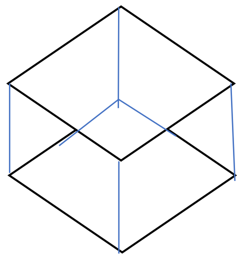
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then G_1 is said to be **isomorphic** to G_2 if there exists a function f : $V_1 \longrightarrow V_2$ such that

1). f is bijective, and

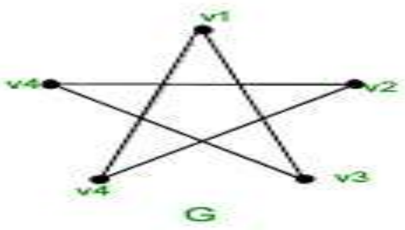
2). $uv \in E_1 \ iff \ f(u)f(v) \in E_2, \ \forall \ u, v \in V_1$.

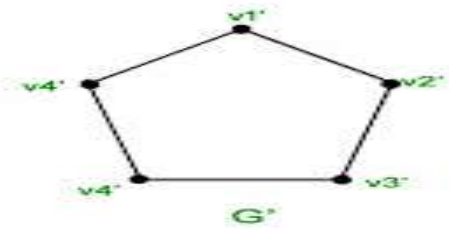
Ex :

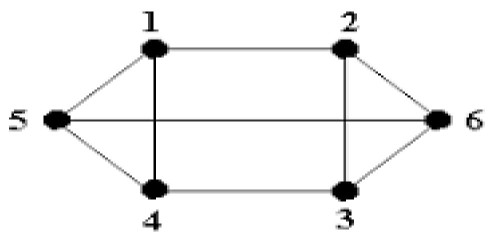


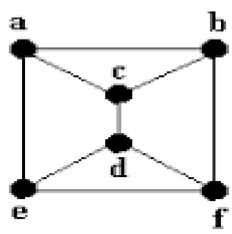


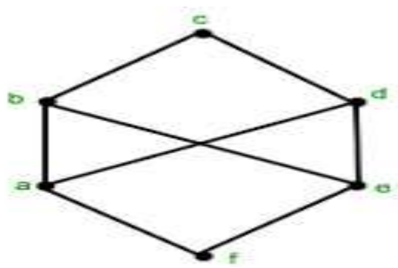
These above two graphs are isomorphic graphs.

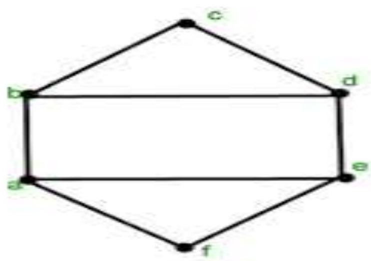












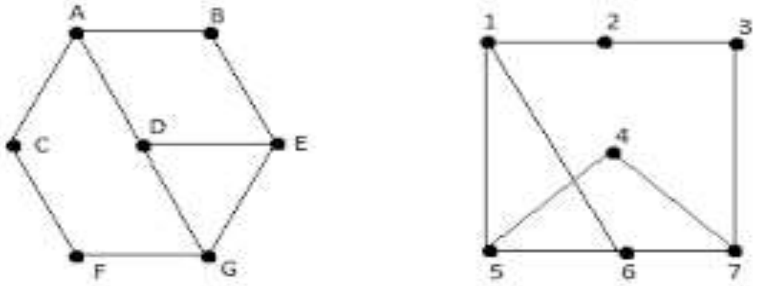
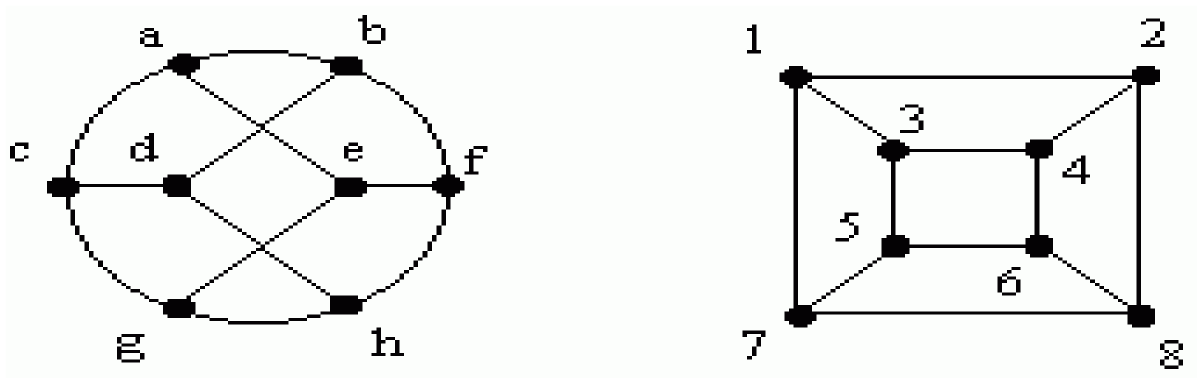


Figure 1.1 – Petersen graph and pentagonal prism.

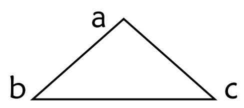


ADJACENT MATRIX :

Let G = (V,E) be an undirected graph. Let V = $\{v_1 , v_2 ,\ldots. v_n\}$. Then the **adjacent matrix** of G is of order n X n and is defined by

$$M = [a_{ij}] = \begin{cases} 1 \text{ if } v_i \text{ is adjacent to } v_j \\ 0 \text{ otherwise .} \end{cases}$$

Ex : Consider the graph



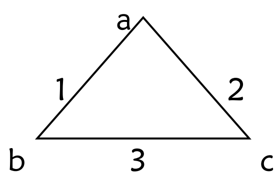
Then the adjacent matrix of G is $M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

INCIDENT MATRIX :

Let G = (V,E) be an undirected graph. Let V = $\{v_1 , v_2 ,\ldots. v_n\}$ and E = $\{e_1 , e_2 ,\ldots. e_m\}$. Then, the **incident matrix** of G is of order n X m and is defined by

$$M = [a_{ij}] = \begin{cases} 1 \text{ if } v_i \text{ is incident to } e_j \\ 0 \text{ otherwise} \end{cases}$$

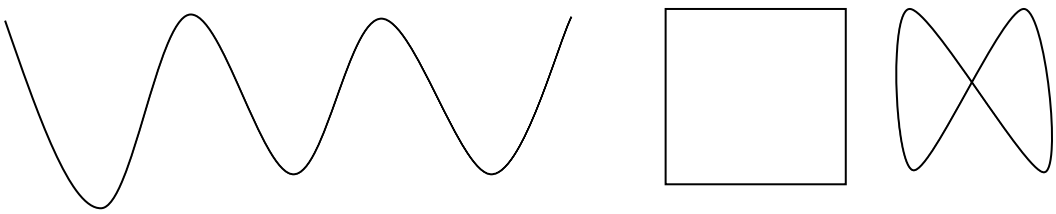
Ex : Consider the Graph



The incident matrix of G is $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

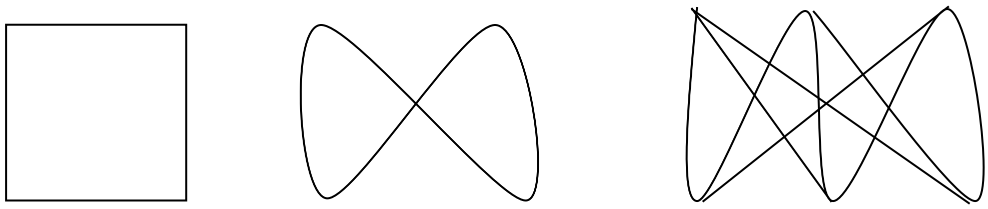
BI-PARTITE GRAPH:

Let G = (V, E) be an undirected graph. Then G is said to be a **bi-partite** graph if V = $V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and every edge in G has one end vertex in V_1 and another end vertex in V_2 .

Ex : 

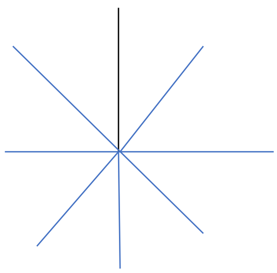
NOTE :

1. In a bi-partite graph G each and every vertex in V_1 is adjacent to each every vertex in V_2 and vice-versa, then the Graph G is said to be a **complete bi-partite graph** is denoted by $\boldsymbol{K_{m,n}}$, where m denotes the number of elements in V_1 and n denotes the number of element in V_2 .

Ex : 

2. The complete bi-partite graph $K_{1,n}$ is called "**Star graph**".

   Ex.:



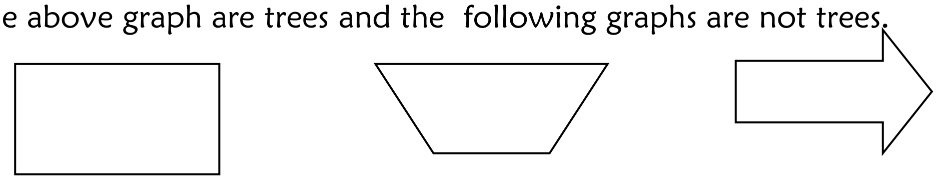
NOTE A graph G = (V,E) is bi-partite if and only if it contains no odd cycles.

TREE :

Let G = (V,E) be an undirected simple graph. If between every pair of vertices there is a unique path , then G is called a **tree.**

Ex :

The above graph are trees and the following graphs are not trees.

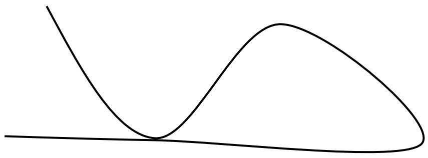


CUT EDGE :

Let G = (V,E) be a simple undirected graph and e be an edge in G . If G – e is disconnected, then the edge e is called **cut edge** (or) **bridge** of the graph G.

Ex : 

In the above graph G has no Cut edges.



The above graph has cut edges.

Theorem 4 : A simple undirected graph G is a tree if and only if G is connected and contains no cycles.

Proof: Let G be a simple undirected graph.

Let us suppose G is a tree.

Claim: G is connected and has no cycles.

Since G is tree, we have between every pair of vertices in G there is a unique path. Therefore G is connected.

Now, we prove that G has no cycles.

Let us suppose G has a cycle C (say).

Let u, v be two vertices on C.

Then, there exist two disjoint paths between u and v namely P and Q.

This is a contradiction (since G is a tree).

This contradiction is due to supposing G has cycles.

Therefore G has no cycles.

Conversely, let us suppose G is connected and no cycles.

Claim : G is a tree.

To prove G is a tree, it is enough to prove between every pair of vertices there is a unique path.

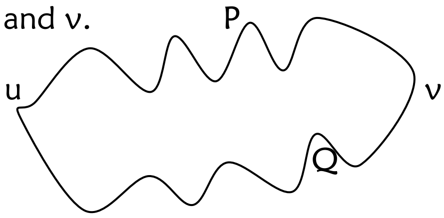
Let u and v be two distinct vertices in G.

Since G is connected we have there is a path between u and v.

Let P and Q be two disjoint paths between u and v.

Then, there exist a cycle from u to u.

This is a contradiction (since G has no cycles).



This contradiction is due to supposing between u and v there are two paths.

Therefore, between u and v there exist a unique path.

Hence G is a tree.

NOTE :

1. From the above theorem, a tree can be defined as a "**connected acyclic graph**".
2. A tree is said to be **non-trivial** , if the number of vertices in that tree is at least 2, otherwise it is said to be **trivial** tree.

Theorem 5 : In a every non-trivial tree , there is at least one vertex of degree 1.

Proof : Let G be a non-trivial tree.

Let $v_1 \in$ G be any vertex in G.

If $d(v_1) = 1$, then the result is true.

Let us suppose, $d(v_1) \neq 1$.

Let v_2 be a vertex in G adjacent to v_1.

If $d(v_2) = 1$, then the result is true.

Let us suppose $d(v_2) \neq 1$.

Let v_3 be a vertex in G adjacent to v_2.

On continuing this process we get a vertex v such that d (v) = 1 .

(since number of vertices in G is finite)

Theorem 6 : A tree with n vertices has exactly n-1 edges.

Proof : Let G be a tree with n vertices.

We prove this result by using strong mathematical induction on the number of vertices in G.

Let n = 1.

Then the number of edges in G is equal to 0 = 1 – 1 = n – 1.

Let n > 1.

Assume that, the result is true for all trees having number of vertices is less than n.

Let H be a tree with n vertices.

Since n > 1 we have H is a non-trivial tree.

By a known theorem, there is a vertex v in H such that d (v) = 1.

Now consider the graph H - v.

Clearly H – v is a tree and the number of vertices in H – v is n – 1.

By induction hypothesis we have H – v contains (n - 1) - 1 = n - 2 edges.

Now we consider

number of edges in H = number of edges in (H – v) + 1

= (n – 2) + 1

= n – 1.

Therefore the number of edges in H is n – 1 and hence the result is true for n.

Hence by strong mathematical induction the result is true for every n.

That is, every tree with n vertices has n – 1 edges.

Theorem 7 : If G is a non-trivial tree, then G contains at least two pendent vertices.

Proof : Let G be a non-trivial tree with n vertices.

By a known result, $\sum_{v \in V} d(v) = 2e$ ------- (1) where e denotes number of edges in G.

Since G is a tree with n vertices we have the number of edges in G is n – 1.

From (1), we get

$$\sum_{v \in V} d(v) = 2(n-1) = 2n-2. \quad \text{--------- (2)}$$

Now, we prove G has at least two pendant vertices.

Let us suppose G has exactly one pendant vertex say v_1.

Then $d(v_1) = 1$.

Clearly

$$\sum_{v \in V} d(v) = d(v_1) + d(v_2) + \ldots\ldots + d(v_n)$$

$$\geq 1 + 2(n-1)$$

$$= 2n - 1.$$

Therefore $\sum_{v \in V} d(v) \geq 2n-1$. ----------- (3)

From (2) and (3) we have

$$2n - 2 \geq 2n - 1.$$

This is a contradiction.

This contradiction is due to supposing G has only one pendant vertex.

Therefore G has at least 2 pendant vertices.

Theorem 8: Prove that if G is a tree then the sum of degrees is equal to 2 $|V|$ – 2.

Proof : Let G be a tree with $|V|$ vertices.

Then by a known result we have

$$\sum_{v \in V} d(v) = 2e \text{ ----- (1)}$$

where e denotes number of edges in G.

Since G is a tree with $|V|$ vertices, we have G has $|V| - 1$ edges.

From (1) we have

$$\sum_{v \in V} d(v) = 2\,(|V| - 1) = 2\,|V| - 2.$$

Theorem 9: A graph G is a tree if and only if G has no cycles and $|E| = |V| - 1$.

Proof : Let G be a graph with n vertices.

Let us suppose G is a tree.

Then by a known result we have G has no cycles and it has n – 1 edges.

That is, $|E| = |V| - 1$.

Conversely, suppose that G has no cycles and $|E| = |V| - 1$.

Claim : G is a tree.

Since G has no cycles, by a known theorem, to prove G is a tree it is sufficient to prove G is connected.

Let us suppose G is disconnected.

Then G can be partitioned in to components.

Let $C_1, C_2, C_3, \ldots . C_k$ be the k components of G with $n_1, n_2, n_3, \ldots . n_k$ vertices respectively.

Since G has no cycles we have each component C_i ($1 \leq i \leq k$) has no cycles.

We know that , components are always connected.

So, each component C_i is a tree. (since they does not contain cycles)

Since the number of vertices in each C_i is n_i and hence

the number of edges in each C_i is $n_i - 1$ ($1 \leq i \leq k$).

Clearly n = $n_1 + n_2 + n_3 + \cdots + n_k$ and

$$n - 1 = n_1 - 1 + n_2 - 1 + n_3 - 1 + \cdots + n_k - 1 .$$

$$\Rightarrow n - 1 = n_1 + n_2 + n_3 + \cdots + n_k - k .$$

$$\Rightarrow n - 1 = n - k .$$

$$\Rightarrow \quad -1 \quad = \quad -k.$$

$$\Rightarrow \quad 1 \quad = \quad k.$$

Therefore G is connected.

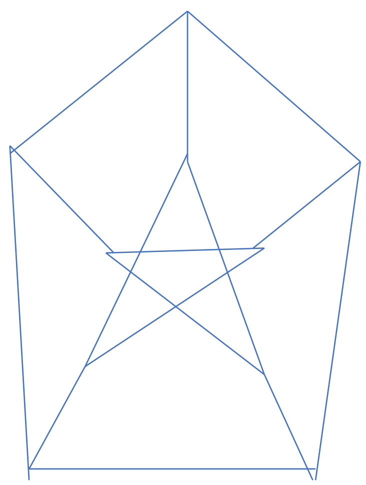
SUBGRAPH:

Let G = (V, E) be a graph. Let H = (V_1, E_1) be another graph.

Then we say that,

- H is a **subgraph** of G if $V_1 \subseteq V$ *and* $E_1 \subseteq E$.
- H is a **proper subgraph** of G if $V_1 \subset V$ *and* $E_1 \subset E$.
- H is a **spanning subgraph** of G if $V_1 = V$ *and* $E_1 \subset E$.

NOTE: The following graph is called "**Petersen graph**".



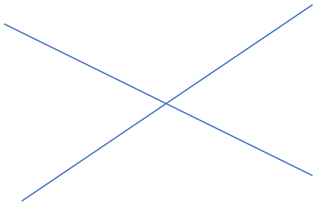
COMPLEMENT OF THE GRAPH:

Let G = (V,E) be a graph. Then the complement of G is denoted by $\bar{G} = (\bar{V}, \bar{E})$ and is defined as follows.

1. $\bar{V} = V$ and
2. Let $p, q \in V$. Then if p, q are adjacent in G then they are not adjacent in $\bar{G}$ and if p, q are not adjacent in G then they are adjacent in $\bar{G}$.

Ex.:

G = 

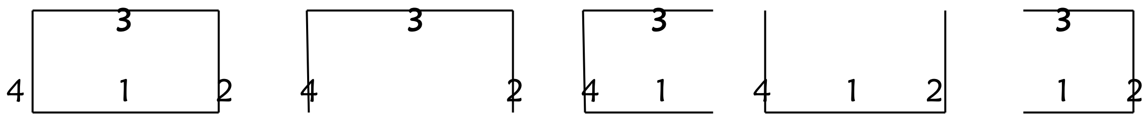
$\overline{G}$ = 

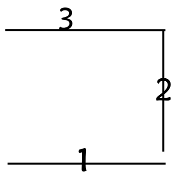
SPANNING TREE :

A Subgraph H of a Graph G is called a **spanning tree** if H is a tree and H contains all the vertices of G.

MINIMAL SPANNING TREE :

Let G = (V,E) be a connected weighted graph . The spanning tree of G with the smallest total weight is called **minimal spanning tree (or) minimum spanning tree.**

Ex : 

The minimal spanning tree is 

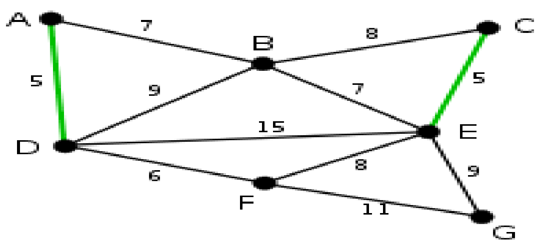
KRUSKAL'S ALGORITHM:

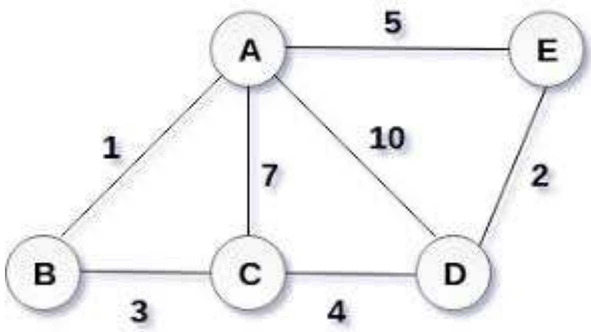
Step 1 : Select any edge of minimal weight that is not a loop. This is the first edge of minimal spanning tree T.

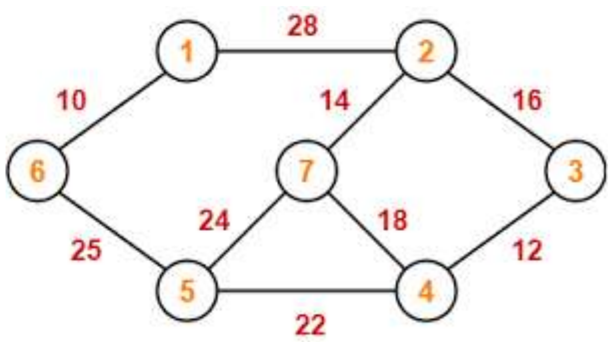
Step 2 : Select any remaining edge of G having minimum value that does not form a cycle with the edges already included in T.

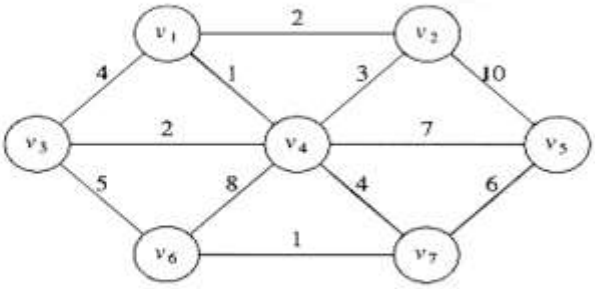
Step 3 : Continue step 2 until T contains n – 1 edges.

PROBLEMS:





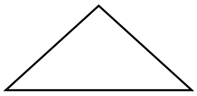




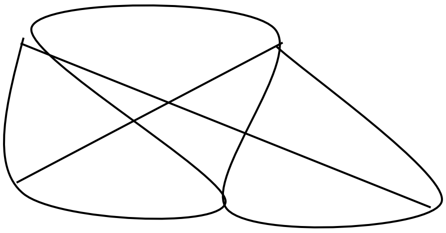
PLANAR GRAPH :

A Graph G is said to be planar (simply we say plane graph), if it can be drawn on a plane without any crossings. Otherwise, it is said to be non-planar.

Ex ;



The above two graphs are planar and the below graph is non-planar.



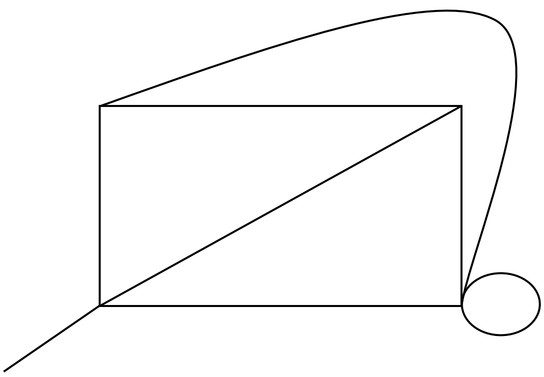
DUAL OF THE GRAPH:

Let G be a plane graph. Then, the dual of G denoted by G^* is a plane graph whose vertices corresponds to the faces (or) regions of G and the edges of G^* correspond to the edges of G as follows.

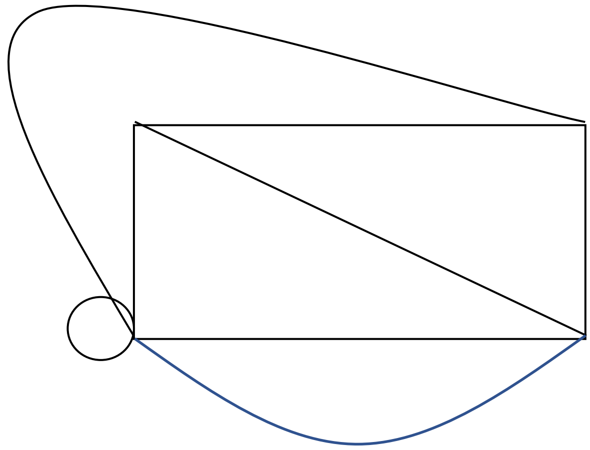
If e is an edge of G with region X on one side and region Y on the other side, then the end vertices of the dual edge e^* are the vertices X, Y of G^*.

That is, if the edge e is a boundary of the regions X and Y, then we adjacent the vertices X and Y in the dual of G .

Ex:



The dual of the above graph is



Theorem 10 : Let G be a plane graph, then the sum of degrees of the regions is 2 $|E|$ where $|E|$ denotes number of edges in G.

Proof : Let G = (V,E) be a graph.

Let $|V|, |E|$ and $|R|$ be the number of vertices, edges and regions in G resp.

Let $G^*=(V^*, E^*)$ be the dual of the graph G.

Let $|V^*|, |E^*|$ and $|R^*|$ be the number of vertices ,edges and regions in G^* resp.

Then by definition of G^* we have

$|R| = |V^*|$ and $|E| = |E^*|$. ---------------(1)

Let $\sum_{r\in R(G)} d(r)$ be the sum of degrees of the regions in G , where R(G) is the set of all regions in G.

Clearly $\sum_{r\in R(G)} d(r) = \sum_{v\in V^*} d(v) = 2\,|E^*| = 2\,|E|$. (by (1)

$\Rightarrow \sum_{r\in R(G)} d(r) = 2\,|E|$.

Theorem 11 : **Euler's Formula**

If G is a connected plane graph then n – e + f = 2, where n , e, and f be number of vertices , edges and regions in G respectively.

(or)

If G is a connected plane graph, then $|\boldsymbol{V}| - |\boldsymbol{E}| + |\boldsymbol{R}| = \mathbf{2}$, where $|\boldsymbol{V}|, |\boldsymbol{E}|$ $\boldsymbol{and}$ $|\boldsymbol{R}|$ denotes number of vertices, edges and regions of G respectively.

Proof : Let G be a connected plane graph.

Let n , e, and f be number of vertices, edges and regions in G respectively.

We prove this theorem, by using strong mathematical induction on f.

Let f = 1.

Then the only face in G is exterior region and which implies that

G has no cycles.

Since G is connected and hence G is a tree.

By a known theorem, we have e = n – 1.

We consider

$$n - e + f = n - (n - 1) + 1$$

$$= n - n + 1 + 1 = 2.$$

$$\Rightarrow n - e + f = 2.$$

Therefore the result is true for f = 1.

Let f > 1.

Let us assume, the result is true for every connected plane graph with less than f regions.

Now we show that the result is true for a connected plane graph with f regions.

Let G be a connected plane graph with n vertices, e edges, and f regions.

Let w be any edge in the graph G.

Consider the graph $G_1 = G - w$.

Clearly G_1 is a connected plane graph.

Let n_1, e_1 *and* f_1 be number of vertices, edges and regions in G_1 respectively.

Then $n_1 = n$, $e_1 = e - 1$ and $f_1 = f - 1 < f$.

Since number of regions in G_1 is less than f, by induction hypothesis, we get $n_1 - e_1 + f_1 = 2$.

This implies that,

$$n - (e - 1) + (f - 1) = 2.$$

That is $n - e + f = 2$.

Therefore the result is true for all connected plane graph G with number of regions f.

Hence, by strong mathematical induction the result is true for every connected plane graph with number of regions f ≥ 1.

Theorem 12 : Let G be a simple plane graph with e > 1, then

1). e ≤ 3 n – 6.

2). If G is triangular free , then e ≤ 2 n – 4.

3). There is a vertex v in G such that d (v) ≤ 5.

Proof :_ Let G be a simple plane graph with n vertices, e edges and f regions.

Then by Euler's theorem we get

n – e + f = 2. ------------- (1)

By a known theorem we have

$$\sum_{r \in R(G)} d(r) = 2e = \sum_{v \in V} d(v) \text{. ------------ (2)}$$

Since G is simple, we have G has no loops and parallel edges.

We know that,

the degree of the region bounded by the loop is 1 and

the degree of the region bounded by the parallel edges is 2.

Since G has no loops and parallel edges we have the degree of every region in the graph G is at least 3.

1). **Claim :** $e \leq 3n - 6$.

From (2) we have

$$2e = \sum_{r \in R(G)} d(r) \geq 3f.$$

$\Rightarrow 2e \geq 3f$.

$\Rightarrow f \leq \frac{2}{3}e$. -------------(3)

From (1) we have

$2 + e = n + f$.

$\Rightarrow 2 + e \leq n + \frac{2}{3}e$.

$\Rightarrow 2 + e - \frac{2}{3}e \leq n$.

$\Rightarrow 6 + 3e - 2e \leq 3n$.

$\Rightarrow e \leq 3n - 6$.

2). Let us suppose G is triangular free.

Then the degree of every region in G is at least 4.

Claim : $e \leq 2n - 4$.

From (2) we have

$$2e = \sum_{r \in R(G)} d(r) \geq 4f.$$

$\Rightarrow 2e \geq 4f$.

$\Rightarrow f \leq \frac{2}{4}e$.

From (1) we have

$2 + e = n + f$.

$\Rightarrow 2 + e \leq n + \frac{2}{4}e$.

$\Rightarrow 2 + e - \frac{2}{4}e \leq n$.

$\Rightarrow 8 + 4\,e - 2\,e \leq 4\,n.$

$\Rightarrow 2e \leq 4\,n - 8.$

$\Rightarrow e \leq 2\,n - 4.$

3). Claim : there is a vertex v of G such that d (v) ≤ 5.

Let us suppose degree of every vertex is at least 6.

That is $d(v) \geq 6 \;\; \forall \;\; v \in V(G)$.

From (2) we have

$$2\,e = \sum_{v \in V} d(v) \geq 6\,n.$$

$\Rightarrow 2\,e \geq 6\,n$.

$\Rightarrow n \leq \frac{1}{3}\,e$. ----------- (4)

Since G is planar, by Euler's theorem we have n – e + f = 2.

$\Rightarrow 2 + e = n + f$.

$\Rightarrow 2 + e \leq \frac{1}{3}\,e + \frac{2}{3}\,e.$

$\Rightarrow 2 + e \leq e$.

$\Rightarrow 2 \leq 0.$

This is a contradiction.

This contradiction is due to supposing degree of every vertex is at least 6.

Therefore , there is a vertex v of G such that d (v) ≤ 5,

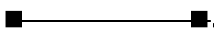
Theorem 13 : A complete graph K_n is planar if and only if n ≤ 4.

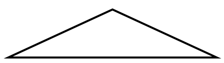
Proof :_ Let K_n be a Complete Graph with n vertices.

Let us suppose n ≤ 4.

Then the possible values of n are 1, 2, 3 and 4.

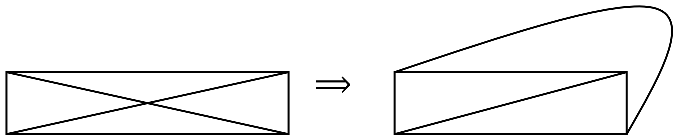
If n = 1 , then K_n = ■.

If n = 2 , then K_n = .



If n = 3 , then K_n =

If n = 4, then K_n =



From the above cases

If n ≤ 4 , then K_n is planar.

Conversely, let us suppose K_n is planar.

Claim : n ≤ 4.

Now we prove this result by using contradiction method.

That is if n ≥ 5, then K_n is non – planar.

Let n ≥ 5. Now we prove that K_n is non- planar.

Let us suppose K_n is planar.

With out loss of generality, let us take n = 5.

The number of edges in K_n is $\frac{5(5-1)}{2}$ = 10.

Since K_n is planar, by a known result we have e ≤ 3 n – 6.

⟹ 10 ≤ 3×5 – 6.

⟹ 10 ≤ 9.

This is a contradiction.

This contradiction is due to supposing K_n is planar.

Therefore K_n is non – planar.

This is true for every n ≥ 5.

Theorem 14 : A Complete Bi-partite graph $K_{m,n}$ is planar if and only if m ≤ 2 (or) n≤ 2 .

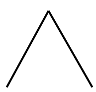
<u>**Proof :**</u> Let $K_{m,n}$ be a complete bi-partite graph.

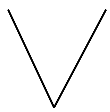
Let us suppose m ≤ 2 (or) n≤ 2.

Then the possible cases are

m = 1 , n = 1 -----------

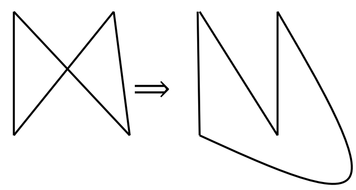
m = 1 , n = 2 ------------------------





$m = 2, n = 1$ --

and $m = 2, n = 2$ --



From the above case we have

If $m \leq 2$ (or) $n \leq 2$ then $K_{m,n}$ is planar.

Conversely let us suppose $K_{m,n}$ is planar.

Claim : $m \leq 2$ (or) $n \leq 2$.

We prove this result by using contrapositive way.

That is, if $m \geq 3$ and $n \geq 3$, then $K_{m,n}$ is non-planar.

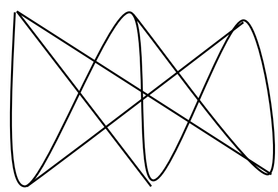
Let us suppose, $m \geq 3$ and $n \geq 3$.

Without loss of generality, we assume that $m = 3$ and $n = 3$.

Now we prove that $K_{3,3}$ is non-planar.

Let us suppose $K_{3,3}$ is planar.

Then, the graph $K_{3,3}$ is



Clearly, total number of vertices is 6, edges is 9

and $K_{3,3}$ is triangular free.

By a known theorem we have

$$e \leq 2n - 4.$$

$$\Rightarrow 9 \leq 2 \times 6 - 4.$$

$$\Rightarrow 9 \leq 8.$$

This is a Contradiction.

This contradiction is due to supposing $K_{3,3}$ is planar.

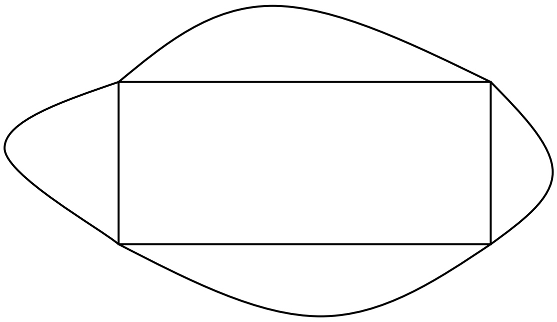
Therefore, $K_{3,3}$ is non – planar.

This is true for every $m \geq 3$ and $n \geq 3$.

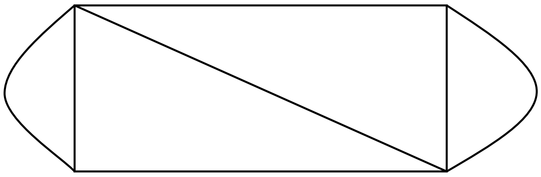
EULERIAN GRAPH : A closed trail that contains each and every edge exactly once, then that trail is called **Euler trail (or) Euler circuit** .

A Multigraph G is said to be **Eulerian** , if it has an **Euler trail.**

Ex :



Eulerian Graph



Not an Eulerian Graph

NOTE : A multigraph to be Eulerian, if degree of every vertex is even.

HAMILTONIAN GRAPH :

A cycle that contains each and every vertex in the graph is called **Hamilton cycle** .

A Graph G is said to be **Hamiltonian** , if G has Hamilton cycle

Ex :

Hamiltonian

Not Hamiltonian

NOTE :

1). A pair of vertices in a digraph are **weakly connected** if there is a nondirected path between them.

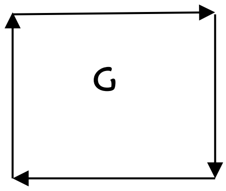
2). A pair of vertices in a digraph are **unilaterally connected** if there is a directed path between them.

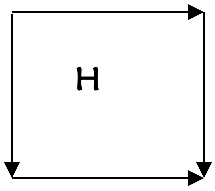
3). A pair of vertices in a digraph are **strongly connected** if there is a directed path from x to y and a directed path from y to x .

STRONGLY CONNECTED :

A directed graph G is said to be **strongly connected** if every pair of vertices in the graph is **strongly connected .**

Ex :





In the above examples, G is strongly connected and H is not strongly connected.

UNILATERALLY CONNECTED :

A directed graph G is said to be **unilaterally connected** if every pair of vertices in the graph is **unilaterally connected .**

WEAKLY CONNECTED :

A directed graph G is said to be **weakly connected** if every pair of vertices in the graph is **weakly connected .**