Study of Generalized Fibonacci Sequences

Thesis

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In the Faculty of Science

To

Dr. Babasaheb Ambedkar Marathwada University, Aurangabad (M.S.)

By

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Declaration of Authorship

I, Ashok Dnyandeo Godase hereby declare that the thesis entitled "A Study of Generalized Fibonacci Sequences" submitted by me for the award of Degree of Doctor of Philosophy in Mathematics in the faculty of science to Dr. Babasaheb Ambedkar Marathwada University, Aurangabad (M.S.), India, is the original work carried out by me and has not been submitted to any other university, in any form for the fulfilment of any other diploma, degree, fellowship or other similar title by me.

Date: 06^{th} November 2019.
Place: Aurangabad.

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Certificate

This is to certify that the thesis entitled, "A Study of Generalized Fibonacci Sequences" which is being submitted by Ashok Dnyandeo Godase for the award of the degree of Doctor of Philosophy in Mathematics, Under the Faculty of Science to Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, is a record of bonafied research work carried out by him under my supervision, guidance and the same has not been submitted elsewhere for any other degree.

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Contents

Dedicated to

My Family and My Students

Chapter 1

Introduction

This chapter provides introduction to the origin of Fibonacci sequence and Lucas sequence. Herein we list some higher order recurrence sequences namely Tribonacci, Tetranacci, Pentanacci, Hexanacci, Heptanacci, octanacci and Nonanacci and their first few terms. Moreover various types of generalised Fibonacci sequences available in the literature are given.

Introduction

Leonardo Fibonacci, also known as *Leonardo Pisano* or Leonard of Pisa, was the most famous mathematician of the European Middle Ages. Fibonacci was born around (1170) in the *Bonacci* family of Pisa. His father *Guglielmo* (William) was a productive merchant.

Around (1190), he brought Leonardo to Algerian city of *Bugia* to learn the art of computation.

As a teen, Fibonacci frequently visited to Egypt, Syria, France, and Greece where he studied an arithmetic, and shared views with other scholars.

Around (1200), at the age of 30, Fibonacci returned to Pisa. In (1202), Fibonacci published his work, *Liber Abaci* (The Book of the Abacus). *Liber Abaci* was dedicated to arithmetic and algebra. After Liber Abaci, Fibonacci wrote three other famous books. *Practica Geometriae*, written in (1220), next two books, the *Flos* and the *Liber Quadratorum* were published in (1225).

In (1225) Frederick II wanted to check Fibonacci's brilliance, so he invited him for a mathematical tournament. The contest consisted of three problems. In the contest, none of Fibonacci's contender could solve any of three problems other than Fibonacci.

The Emperor acknowledged Fibonacci's contributions as a teacher and as a citizen by forming statue of Fibonacci in a garden near the Leaning Tower of Pisa.

Fibonacci's book, *Liber Abaci*, contains many elementary problems, including rabbit problem. The numbers in the last row of table(1.1) are called Fibonacci numbers, and the natural number sequence 1, 1, 2, 3, 5, 8, . . . is known as the Fibonacci sequence.

Number of Pairs	**Jan**	**Feb**	**March**	**April**	**May**	**June**	**July**	**August**
Adults	0	1	1	2	3	5	8	13
Babies	1	0	1	1	2	3	5	8
Total	1	1	2	3	5	8	13	21

TABLE 1.1: Rabbits after the month April

The sequence is so important that a association of mathematicians, "The Fibonacci Association" is shaped for the study of Fibonacci and connected integer sequences. The association was founded in (1963) it publishes the most precious journal *Fibonacci Quarterly*, dedicated to articles related to integer sequences.

In (1595 – 1632), Dutch mathematician *Albert Girard* demonstrated recursive definition of the n^{th} Fibonacci number, F_n as $F_{n+1} = F_n + F_{n-1}$ with initial conditions $F_1 = 1$, $F_2 = 1$.

In (1870), Edouard Lucas discovered Lucas Sequence or Lucas number having recursive definition of the n^{th} Lucas number, L_n as $L_{n+1} = L_n + L_{n-1}$ with initial conditions $L_1 = 1$, $L_2 = 2$.

Generalised Fibonacci Sequences

In the case of Fibonacci and Lucas numbers, every element, except for the first two, can be obtained by adding its two immediate predecessors. The Fibonacci sequence has been generalized in many ways, some by preserving the initial conditions, and others by preserving the recurrence relation. Suppose we are given three initial conditions and add the three immediate predecessors to compute their successor in a number sequence. Such a sequence is the tribonacci sequence, originally studied by M. Feinberg[14] in(1963).

Generalized Fibonacci sequences are varied sequences similar to the Fibonacci sequence in which initial terms are changed together, or multiplied by a constant. However, the recursive relation is $F_{n+1} = F_n + F_{n-1}$. The new sequence has similar characteristics to that of the Fibonacci, but also has its own patterns and properties.

In (1963), Basin[4] change the initial terms in the Fibonacci sequence by p and q. In a generalized Fibonacci sequence where $G_1 = p$ and $G_2 = p + q$ it is observed that $G_n = pG_n + qG_{n-1}$.

In (1963), Horadam[18] used a similar technique to find the recurrence relation to find n^{th} term.

In (2009), Marcia Edson and Omer Yayenie[22] studied a new generalization $\{q_n\}$, with initial conditions $q_0 = 0$ and $q_1 = 1$ which is generated by the recurrence relation $q_n = aq_{n+1} + q_{n+2}$ (when n is even) or $q_n = bq_{n+1} + q_{n+2}$ (when n is odd), where a and b are non-zero real numbers. Some well-known sequences are special cases of this generalization. The Fibonacci sequence is a special case of $\{q_n\}$ with $a = b = 1$. Pell's sequence is $\{q_n\}$ with $a = b = 2$ and the k-Fibonacci sequence is $\{q_n\}$ with $a = b = k$. Marcia Edson and Omer Yayenie also produced an Binet's formula for the sequence $\{q_n\}$ and identities such as Cassini's, Catalan's, d'Ocagne's, etc for this sequence.

In (2015), Ipek and K. Ari[20] obtained many new relations between the generalizations of Fibonacci and Lucas sequences. The generalized Fibonacci sequence have been rigorously studied for many years and become curious topic in Pure Mathematics as well as Applied Mathematics.

In (2013), Bravo[6] introduced sequences similar to Fibonacci sequence having different initial terms, they can also be lifted to higher powers by summing initial terms. Given that the variable $k \geq 2$ is a whole number, the value of k is used to denote the order of the Fibonacci sequence, or the number of terms summed to generate the next term.

If $k = 2$, the classical Fibonacci sequence is generated, if $k = 3$ the Tribonacci sequence is generated, if $k = 4$ the Tetranacci sequence is generated, if $k = 5$ the Pentanacci sequence is generated and so on.

k	**Name**	**First Few terms**
2	Fibonacci	1, 1, 2, 3, 5, 8, 13, 21, 34,....
3	Tribonacci	1, 1, 2, 4, 7, 13, 24, 44, 81, 149,....
4	Tetranacci	1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490,...
5	Pentanacci	1, 1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, 1793, 3525, 6930, 13624,...
6	Hexanacci	1, 1, 2, 4, 8, 16, 32, 63, 125, 248, 492, 976, 1936, 3840, 7617, 15109,...
7	Heptanacci	1, 1, 2, 4, 8, 16, 32, 64, 127, 253, 504, 1004, 2000, 3984, 7936, 15808,...
8	Octanacci	1, 1, 2, 4, 8, 16, 32, 64, 128, 255, 509, 1016, 2028, 4048, 8080, 16128,...
9	Nonanacci	1, 1, 2, 4, 8, 16, 32, 64, 128, 256, 511, 1021, 2040, 4076, 8144, 16272,....

TABLE 1.2: From left to right, the order, name, and first few terms of each higher order sequence

Coupled Fibonacci Sequences

In (1995), Peter Hope[19] gave the idea for a Fibonacci like sequence with the form $X_0 = 0,\ X_1 = 1,\ X_{n+2} = a_n X_{n+1} + b_n X_n \quad (n \geq 0)$, where $\{a_n\}$ and $\{b_n\}$ are known sequences with positive numbers.

Coupled Fibonacci sequences contain two sequences of integers in which the elements of one sequence are part of the generalization of the other and vice versa. In (1986) Atanassov K. T.[2] first introduced coupled Fibonacci sequences of second order in additive form and also discussed many curious properties and new direction of generalization of Fibonacci sequence in his series of papers on coupled Fibonacci sequences. He defined and studied four different ways to generate coupled sequences and called them coupled Fibonacci sequences (or 2-F sequences). In (2003), P. Glaister[16] studied multiplicative Fibonacci Sequences. The parallel of the standard Fibonacci sequence in this form is $F_0 = a,\ F_1 = b,\ F_{n+2} = F_{n+1} \cdot F_n \quad (n \geq 0)$.

In (1985), Attanasov K. T.[1] introduced a new perspective of generalized Fibonacci sequences by taking a pair of sequences $\{X_n\}_{n=0}^{n=\infty}$ and $\{Y_n\}_{n=0}^{n=\infty}$ and which can be generated by famous Fibonacci formula and gave various identities involving Fibonacci sequence called the coupled Fibonacci sequences. With some initial values x_0, x_1, y_0, y_1 and for $n \geq 0$ four different schemes were defined. He also introduced a new perspective of generalized Fibonacci sequences by taking a pair of sequences $\{X_n\}_{n=0}^{n=\infty}$ and $\{Y_n\}_{n=0}^{n=\infty}$ and which can be generated by famous Tribonacci formula and gave various identities involving Fibonacci sequence called the coupled Fibonacci sequences of third order. With some initial values x_0, x_1, x_2, y_0, y_1 and y_2 eight different schemes were defined.

In (2014), Krishna Kumar Sharma, Vikas Panwar and Sumitkumar Sharma[21] introduced a new perspective of generalized Fibonacci sequences by taking a pair of sequences $\{X_n\}_{n=0}^{n=\infty}$ and $\{Y_n\}_{n=0}^{n=\infty}$ and which can be generated by r^{th} order recurrence relation called the coupled Fibonacci sequences of r^{th} order. With some initial values x_0, x_1, x_2, x_3......, x_{r-1} and y_0, y_1, y_2, y_3......, y_{r-1}, 2^r different schemes were defined.

In (2010), Singh-Sikhwal [23] defined coupled and multiplicative coupled Fibonacci sequences by varying initial conditions and recurrence relation under different schemes for higher order. Herein chapter (2), we defined multiplicative coupled Fibonacci sequences of r^{th} order by varying recurrence relation and some identities for these mentioned sequences are also obtained under four different schemes out of 2^{r-1} schemes.

k-Fibonacci and k-Lucas Sequences

In (2007), Sergio Falcon [7] defined k Generalised Fibonacci sequence with the integer k. The k-Fibonacci numbers can be represented by the recurrence relation $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$, with initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$. The converging ratios are the positive solutions to the equation $r^2 - kr - 1 = 0$, which is $\dfrac{k \mp \sqrt{k^2+4}}{2}$. If $k = 1$, then the traditional Fibonacci sequence is produced and the ratio is $\dfrac{1 \mp \sqrt{5}}{2}$, also known as the Golden Ratio or ϕ. If $k = 2$, then the sequence converges to the Silver Ratio, $1+\sqrt{2}$, and if $k=3$ then the Bronze Ratio is produced, $\dfrac{(3+\sqrt{13})}{2}$.

Fibonacci sequence is a source of many nice and interesting identities. A similar interpretation exists for k-Fibonacci and k-Lucas numbers. Many of these identities have been documented in the work of Falcon and Plaza, where they are proved by algebraic means.

In another papers [8–13] Falcon and Plaza proved many properties of k-Fibonacci and k-Lucas sequences by simple matrix algebra.

In (2014), Yashwant K. Panwar, G. P. S. Rathore and Richa Chawla[25] deduced many properties of these numbers and relate with the so-called Pascal 2-triangle. In addition, the generating functions for these k-Fibonacci sequences have been given. Falcon presented Lucas triangle and its relationship with the k-Lucas numbers, combinatorial formula for k-Lucas numbers, generating function and defined Properties of the diagonals of the Lucas triangle and the rows of the Lucas triangle. In his third paper S. Falcon, study the properties of the k-Lucas numbers and proved these properties are related with the k-Fibonacci numbers. From a special sequence of squares of k-Fibonacci numbers, the k-Lucas sequences are obtained in a natural form. In S. Falcon examine some of the interesting properties of the k-Lucas numbers themselves as well as looking at its close relationship with the k-Fibonacci numbers. The k-Lucas numbers have lots of properties, similar to those of k-Fibonacci numbers and often occur in various formulae simultaneously in papers on k-Fibonacci numbers.

In, (2013) Yazlik, Yilmaz and Taskara[26] investigated some properties of k-Fibonacci and k-Lucas sequences and obtained new identities on sums of powers of these sequences and obtained the recurrence relations for powers of k-Fibonacci and k-Lucas sequences. Also they give new formulas for the powers of k-Fibonacci and k-Lucas sequences.

Plan of The Thesis

In this thesis, the work is presented in seven chapters. A brief account of the work done in each chapter is mentioned below.

Chapter 1 provides introduction to the origin of Fibonacci sequence and Lucas sequence. Herein we list some higher order recurrence sequences namely Tribonacci, Tetranacci, Pentanacci, Hexanacci, Heptanacci, octanacci and Nonanacci and their first few terms. Moreover various types of generalised Fibonacci sequences available in the literature are given.

In chapter 2, coupled Fibonacci sequences of lower order have been generalized in number of ways. In this chapter the Multiplicative Coupled Fibonacci Sequence has been generalized for r^{th} order and some new interesting properties under two specific schemes are given.

In this chapter many results are obtained for multiplicative coupled Fibonacci sequences X_n and Y_n of r^{th} order, some of these are listed below.

Theorem 1.1. *For every integer* $n, r \geq 0 \quad X_{2n(r+1)} \cdot Y_0 = Y_{2n(r+1)} \cdot X_0$.

Theorem 1.2. *For every integer* $n, r \geq 0 \quad X_{2n(r+1)+1} \cdot Y_1 = Y_{2n(r+1)+1} \cdot X_1$.

Theorem 1.3. *For every integer* $n, r \geq 0 \quad X_{2n(r+1)+2} \cdot Y_2 = Y_{2n(r+1)+2} \cdot X_2$.

Theorem 1.4. *For every integer* $n, r \geq 0 \quad X_{2n(r+1)+3} \cdot Y_3 = Y_{2n(r+1)+3} \cdot X_3$

Theorem 1.5. *For every integer* $n, r \geq 0 \quad X_{2n(r+1)+m} \cdot Y_m = Y_{2n(r+1)+m} \cdot X_m$

Theorem 1.6. *For every integer* $n, r \geq 0 \quad \prod_{i=1}^{i=n} X_{ri+1} \cdot Y_{ri+1} = \prod_{i=1}^{i=rn} Y_i \cdot X_i$.

Theorem 1.7. *For every integer* $n, r \geq 0 \quad X_{n(r+1)} \cdot Y_0 = Y_{n(r+1)} \cdot X_0$.

Theorem 1.8. *For every integer* $n, r \geq 0 \quad X_{n(r+1)+1} \cdot Y_1 = Y_{n(r+1)+1} \cdot X_1$

Theorem 1.9. *For every integer* $n, r \geq 0 \quad X_{n(r+1)+2} \cdot Y_2 = Y_{n(r+1)+2} \cdot X_2$.

Theorem 1.10. *For every integer* $n, r \geq 0 \quad X_{n(r+1)+3} \cdot Y_3 = Y_{n(r+1)+3} \cdot X_3$.

Theorem 1.11. *For every integer* $n, r, m \geq 0 \quad X_{n(r+1)+m} \cdot Y_m = Y_{n(r+1)+m} \cdot X_m$.

Theorem 1.12. *For every integer* $n, r \geq 0 \quad \prod_{i=1}^{i=n} X_{ri+1} = \prod_{i=1}^{i=rn} Y_i, \prod_{i=1}^{i=n} Y_{ri+1} = \prod_{i=1}^{i=rn} X_i$.

In chapter 3, some properties of $k-$Fibonacci and $k-$Lucas sequences are derived and proved by using matrix methods. We defined matrices S, M, $M_k(n, m)$, $T_k(n)$, $S_k(n, m)$, A_n, E, Y_n, W_n, G_n and H_n for k-Fibonacci and k-Lucas sequences, using these matrices many interesting identities for $k-$Fibonacci and $k-$Lucas sequences are derived.

In this chapter different interesting results for k- Fibonacci sequences are derived, some of these are listed below.

Theorem 1.13. *For all* $x, y, z \in Z$

$$L^2_{k,x+y} - (k^2+4)(-1)^{x+y+1} F_{k,z-x} L_{k,x+y} F_{k,y+z} - (k^2+4)(-1)^{x+z} F^2_{k,y+z} = (-1)^{y+z} L^2_{k,z}$$

Theorem 1.14. *For all $x, y, z \in Z$, $x \neq z$*

$$L_{k,x+y}^2-(-1)^{x+z}L_{k,z-x}L_{k,x+y}L_{k,y+z}+(-1)^{x+z}L_{k,y+z}^2 = (-1)^{y+z+1}(k^2+4)F_{k,z-x}^2.$$

Theorem 1.15. *For all $x, y, z \in Z$, $x \neq z$*

$$F_{k,x+y}^2 - L_{k,x-z}F_{k,x+y}F_{k,y+z} + (-1)^{x+z}F_{k,y+z}^2 = (-1)^{y+z}F_{k,z-x}^2.$$

In chapter 4, we investigate the binomial sums, congruence properties, telescoping series of k-Fibonacci and k-Lucas sequences. Also, we defined new relationship between k-Fibonacci and k-Lucas using continued fractions and series of fractions. In this chapter different identities for k- Fibonacci sequence $\mathcal{F}_{k,n}$ and k- Lucas sequence $\mathcal{L}_{k,n}$ are derived, some of these are listed below.

Theorem 1.16. $\mathcal{F}_{k,s+2t}+\dfrac{\mathcal{L}_{k,2t-1}}{k}\mathcal{F}_{k,s} = \dfrac{\mathcal{F}_{k,2t}}{k}\mathcal{L}_{k,s+1}, \mathcal{L}_{k,s+10}+\dfrac{\mathcal{L}_{k,2t-1}}{k}\mathcal{L}_{k,s} = \dfrac{\mathcal{F}_{k,2t}}{k}\delta\mathcal{F}_{k,s+1}.$

Theorem 1.17. $\mathcal{F}_{k,s+2t+1} + \dfrac{\mathcal{F}_{k,2t}}{k}\mathcal{L}_{k,s} = \dfrac{\mathcal{L}_{k,2t+1}}{k}\mathcal{F}_{k,s+1}, \mathcal{L}_{k,s+2t+1} + \delta\dfrac{\mathcal{F}_{k,2t}}{k}\mathcal{F}_{k,s} = \dfrac{\mathcal{L}_{k,2t+1}}{k}\mathcal{L}_{k,s+1}.$

Theorem 1.18. *For $n, s, t \geq 1$,*

$$\sum_{i=0}^{n}\binom{n}{i}k^{(i-n)}(\mathcal{L}_{k,2t-1})^{(n-i)}\mathcal{F}_{k,2ti+s} = \begin{cases} k^{-n}(\mathcal{F}_{k,2t})^n\delta^{\frac{n}{2}}\mathcal{F}_{k,n+s}, & \textit{if } n \textit{ is even;} \\ k^{-n}(\mathcal{F}_{k,2t})^n\delta^{\frac{n-1}{2}}\mathcal{L}_{k,n+s}, & \textit{if } n \textit{ is odd,} \end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}k^{(i-n)}(\mathcal{L}_{k,2t-1})^{(n-i)}\mathcal{L}_{k,2ti+s}=\begin{cases}k^{-n}(\mathcal{F}_{k,2t})^{n}\delta^{\frac{n}{2}}\mathcal{L}_{k,n+s}, & \textit{if } n \textit{ is even;}\\ k^{-n}(\mathcal{F}_{k,2t})^{n}\delta^{\frac{n+1}{2}}\mathcal{F}_{k,n+s}, & \textit{if } n \textit{ is odd.}\end{cases}$$

Theorem 1.19. *For $n, s, t \geq 1$,*

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}k^{-i}(L_{k,2t+1})^{i}\mathcal{F}_{k,2t(n-i)+n}=\begin{cases}0, & \textit{if } n \textit{ is even;}\\ 2(k)^{-n}(\mathcal{F}_{k,2t})^{n}\delta^{\frac{n-1}{2}}, & \textit{if } n \textit{ is odd,}\end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}k^{-i}(L_{k,2t+1})^{i}\mathcal{L}_{k,2t(n-i)+n}=\begin{cases}2(k)^{-n}(\mathcal{F}_{k,2t})^{n}\delta^{\frac{n-1}{2}}, & \textit{if } n \textit{ is even;}\\ 0, & \textit{if } n \textit{ is odd.}\end{cases}$$

In chapter 5, we introduce a new generalisation $\mathcal{M}_{k,n}$ of k-Lucas sequence. We present generating functions and Binet formulas for generalized k-Lucas sequence, and state some binomial and congruence sums containing these sequences.

In this chapter many identities for generalized k-Lucas sequence $\mathcal{M}_{k,n}$ are established, some of these are listed below.

Theorem 1.20. ***(Catalan's Identity)***. *For $n, r \geq 1$, $\mathcal{M}_{k,n-r}\mathcal{M}_{k,n+r} - {\mathcal{M}_{k,n}}^2 = (-1)^{n-r}\delta(1-\delta)F_{k,r}^2$.*

Theorem 1.21. ***(Cassini's Identity)***. *For $n \geq 1$, $\mathcal{M}_{k,n-1}\mathcal{M}_{k,n+1} - {\mathcal{M}_{k,n}}^2 = (-1)^{n+1}\delta(1-\delta)$.*

Theorem 1.22. ***(d'Ocagene's Identity)***. *Let n be any non-negative integer and r a natural number. If $n \geq r+1$, then $\mathcal{M}_{k,r}\mathcal{M}_{k,n+1} - \mathcal{M}_{k,r+1}\mathcal{M}_{k,n} = (-1)^{n}\delta(1-\delta)F_{k,r-n}$.*

Theorem 1.23. ***(Convolution Theorem)***. *For* $n, r \geq 1$, $\mathcal{M}_{k,r}\mathcal{M}_{k,n+1}+$ $\mathcal{M}_{k,r-1}\mathcal{M}_{k,n} = \mathcal{M}_{k,n+r} + (\delta^2 + \delta - \sqrt{\delta})F_{k,n+r} + (2\delta + \sqrt{\delta})L_{k,n+r}$.

Theorem 1.24. ***(Asymptotic Behaviour)***. *For* $n, r \geq 1$, $\lim_{n\to\infty} \dfrac{\mathcal{M}_{k,n}}{\mathcal{M}_{k,n-r}} = r_1^r$.

Theorem 1.25. *The generating function for the generalized k-Fibonacci sequence* $\mathcal{M}_{k,tn}$ *is*

$$\sum_{n=0}^{\infty} \mathcal{M}_{k,tn}x^n = \frac{x\mathcal{M}_{k,t} - 2xL_{k,t} + 2}{1 - xL_{k,t} + x^2(-1)^t}.$$

In chapter 6, we demonstrate two new generalizations of Fibonacci polynomial. We produce an extended Binet's formula for these generalized polynomials and thereby identities such as Simpson's, Catalan's, d'Ocagene's, etc. using matrix algebra. Moreover, we derived some identities of $M_n(x)$, $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$ using matrix and vector methods. Some results of these polynomials are listed below.

Theorem 1.26. *For* $n \geq 1$,

$$M_{n-1}(x).M_{n+1}(x) - {M_n}^2(x) = (-1)^{n+1}[k^2(x) - m^2(x) + 4].$$

Theorem 1.27. *For* $n, r \geq 1$,

$$M_r(x)M_{n+1}(x) - M_{r+1}(x)M_n(x) = [k^2(x) - m^2(x) + 4](-1)^n F_{r-n}(x).$$

Theorem 1.28. *For* $n \geq 1$,

$$(-1)^n M_{-n}(x) = M_n(x) - 2m(x)F_n(x).$$

Theorem 1.29. *(Sum of the first n-terms)*

$$\sum_{i=0}^{i=n} M_i(x) = \frac{1}{k(x)}[M_{n+1}(x) + M_n(x) - m(x) + k(x) + 2].$$

Theorem 1.30. *(Sum of the first n-terms with odd indices)*

$$\sum_{i=0}^{i=(n-1)} M_{2i+1}(x) = \frac{1}{k(x)}[M_{2n}(x) + 2].$$

Theorem 1.31.

$$\sum_{i=0}^{i=n-1} M_{2i}(x) = \frac{1}{k(x)}[M_{2n-1}(x) + k(x) - m(x)].$$

Theorem 1.32. *For an integer* $n \geq 0$,

$$\sum_{i=0}^{i=n} \binom{n}{i} k^i(x) M_{2i}(x) = M_{2n}(x).$$

Theorem 1.33. *For arbitrary integers* $p, q \geq 1$,

$$\sum_{i=1}^{p} M_{qi}(x) = \frac{M_{pq+q}(x) - (-1)^q M_{pq}(x) - M_q(x) + 2(-1)^q}{r_1^q + r_2^q - (-1)^q - 1}.$$

Theorem 1.34. *For arbitrary integers $p, q, j \geq 1$ with $j \geq q$*

$$\sum_{i=1}^{p} M_{qi+j}(x) = \frac{M_{pq+q+j}(x) - (-1)^q M_{pq+j}(x) - M_{q+j}(x) + (-1)^q M_j(x)}{r_1^q + r_2^q - (-1)^q - 1}.$$

Theorem 1.35. *For arbitrary integers $n, j \geq 1$, we have*

$$\sum_{i=1}^{n} M_{i+j}(x) = \frac{1}{k(x)}[M_{n+j+1}(x) + M_{n+j}(x) - M_j(x) - M_{j-1}(x)].$$

Theorem 1.36. *(Sum of square)*

$$\sum_{i=1}^{i=n} M_i^2(x) = \frac{M_n(x)M_{n+1}(x) - 2M_1(x)}{k(x)}.$$

In chapter 7, we defined the hyperbolic k-Fibonacci quaternions $\bar{\mathcal{H}^{\mathcal{F}}}_{k,n}$, hyperbolic k-Lucas quaternions $\bar{\mathcal{H}^{\mathcal{L}}}_{k,n}$ and hyperbolic k-Fibonacci octonions $\mathcal{O}^{\mathcal{F}}{}_{k,n}$, hyperbolic k-Lucas octonions $\mathcal{O}^{\mathcal{L}}{}_{k,n}$. We present generating functions and Binet formulas for the k-Fibonacci and k-Lucas hyperbolic quaternions, and establish binomial and congruence sums of hyperbolic k-Fibonacci and k-Lucas quaternions and octonions. Furthermore, we present several well-known identities such as Catalan's, Cassini's and d'Ocagne's identities for k-Fibonacci and k-Lucas hyperbolic octonions. Some main results of this chapter are listed below.

Theorem 1.37. ***(Binet Formulas)***. *For all $n \geq 0$,*

1. $\bar{\mathcal{H}}^{\mathcal{F}}{}_{k,n} = \dfrac{\bar{r_1} {r_1}^n - \bar{r_2} {r_2}^n}{r_1 - r_2}$,
2. $\bar{\mathcal{H}}^{\mathcal{L}}{}_{k,n} = \bar{r_1} {r_1}^n + \bar{r_2} {r_2}^n$.

Theorem 1.38. ***(Catalan's Identity)***. *For any integer t and s,*

(i) $\bar{\mathcal{H}}^{\mathcal{F}}{}_{k,n-t} \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,n+t} - \bar{\mathcal{H}}^{\mathcal{F}}{}^2_{k,n} = (-1)^{n-t} F_{k,t}\big(0, -2F_{k,t+1}, -2F_{k,t+2}, -2F_{k,t+3} + F_{k,t-3} + F_{k,t+1} + F_{k,t-1}\big)$,

(ii) $\bar{\mathcal{H}}^{\mathcal{L}}{}_{k,n-t} \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,n+t} - \bar{\mathcal{H}}^{\mathcal{L}}{}^2_{k,n} = \delta(-1)^{n-t+1} F_{k,t}\big(0, -2F_{k,t+1}, -2F_{k,t+2}, -2F_{k,t+3} + F_{k,t-3} + F_{k,t+1} + F_{k,t-1}\big)$.

Theorem 1.39. ***(Cassini's Identity)***. *For all $n \geq 1$,*

(i) $\bar{\mathcal{H}}^{\mathcal{F}}{}_{k,n-1} \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,n+1} - \bar{\mathcal{H}}^{\mathcal{F}}{}^2_{k,n} = (-1)^n \big(0, -2F_{k,2}, 2F_{k,3}, F_{k,4}\big)$,

(ii) $\bar{\mathcal{H}}^{\mathcal{L}}{}_{k,n-1} \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,n+1} - \bar{\mathcal{H}}^{\mathcal{L}}{}^2_{k,n} = \delta(-1)^{n-1} \big(0, -2F_{k,2}, 2F_{k,3}, F_{k,4}\big)$.

Theorem 1.40. ***(d'Ocagene's Identity)***. *Let n be any non-negative integer and t a natural number. If $t \geq n+1$,*

(i) $\bar{\mathcal{H}}^{\mathcal{F}}{}_{k,t} \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,n+1} - \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,t+1} \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,n} = (-1)^n \big(0, -2F_{k,t-n-1}, 2F_{k,t-n-2}, F_{k,t-n+3} + F_{k,t-n-3} + F_{k,t-n+1} + F_{k,t-n-1}\big)$,

(ii) $\bar{\mathcal{H}}^{\mathcal{L}}{}_{k,t} \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,n+1} - \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,t+1} \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,n} = (-1)^{n+1} \delta \big(0, -2F_{k,t-n-1}, 2F_{k,t-n-2}, F_{k,t-n+3} + F_{k,t-n-3} + F_{k,t-n+1} + F_{k,t-n-1}\big)$.

Theorem 1.41. *For all $n \geq 0$,*

(i) $\mathcal{O}^{\mathcal{F}}{}_{k,n+2} = k\mathcal{O}^{\mathcal{F}}{}_{k,n+1} + \mathcal{O}^{\mathcal{F}}{}_{k,n}$,

(ii) $\mathcal{O}^{\mathcal{L}}{}_{k,n+2} = k\mathcal{O}^{\mathcal{L}}{}_{k,n+1} + \mathcal{O}^{\mathcal{L}}{}_{k,n}$

(iii) $\mathcal{O}^{\mathcal{L}}{}_{k,n} = \mathcal{O}^{\mathcal{F}}{}_{k,n+1} + \mathcal{O}^{\mathcal{F}}{}_{k,n-1}$

(iv) $\bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n+2} = k\bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n+1} + \bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n}$,

(v) $\bar{\mathcal{O}}^{\mathcal{L}}{}_{k,n+2} = k\bar{\mathcal{O}}^{\mathcal{L}}{}_{k,n+1} + \bar{\mathcal{O}}^{\mathcal{L}}{}_{k,n}$

(vi) $\bar{\mathcal{O}}^{\mathcal{L}}{}_{k,n} = \bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n+1} + \bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n-1}.$

Chapter 2

On the Properties of Generalized Multiplicative Coupled Fibonacci Sequence

"An equation means nothing to me unless it expresses a thought of God".

- *SRINIVASA RAMANUJAN*.

Coupled Fibonacci sequences of lower order have been generalized in number of ways. In this chapter the Multiplicative Coupled Fibonacci Sequence has been generalized for r^{th} order and some new interesting properties under two specific schemes are given.

The content of this chapter is published in the following papers.
On the properties of generalized multiplicative coupled Fibonacci sequence of r^{th} order, Int. J. Adv. Appl. Math. and Mech. 2(2015), 252-257.
Identities of Multiplicative Coupled Fibonacci Sequences of r^{th} order, Journal of New Theory, 15(2017), 48-60.

2.1 Introduction

Coupled Fibonacci Sequences

Peter Hope (1977) give the idea for a Fibonacci-type sequence with the form $x_0 = 0$, $x_1 = 1$, $x_{n+2} = a_n x_{n+1} + b_n x_n \quad (n \geq 0)$, where $\{a_n\}$ and $\{b_n\}$ are given sequences with positive numbers. Coupled Fibonacci sequences involve two sequences of integers in which the elements of one sequence are part of the generalization of the other and vice versa. K. T. Atanassov (1982) first introduced coupled Fibonacci sequences of second order in additive form and also discussed many curious properties and new direction of generalization of Fibonacci sequence in his series of papers on coupled Fibonacci sequences. He defined and studied about four different ways to generate coupled sequences and called them coupled Fibonacci sequences (or 2-F sequences). The multiplicative Fibonacci Sequences studied by P. Glaister (2003) and generalized by P. Hope (2005). K. T. Atanassov (2005) notifies four different schemes in multiplicative form for coupled Fibonacci sequences. The analog of the standard Fibonacci sequence in this form is $x_0 = a$, $x_1 = b$, $x_{n+2} = x_{n+1} \cdot x_n \quad (n \geq 0)$.

Generalized multiplicative coupled Fibonacci sequences of second order

Definition 2.1. Let $\{X_i\}_0^\infty$ and $\{Y_i\}_0^\infty$ be two infinite sequences and four arbitrary real numbers a, b, c, d are given. The Multiplicative Coupled Fibonacci Sequence of 2^{nd} order is generated by the following four different ways:

First Scheme:

$$X_{n+2} = X_{n+1} \cdot X_n, \quad Y_{n+2} = Y_{n+1} \cdot Y_n, \quad n \geq 0.$$

Second Scheme:

$$X_{n+2} = Y_{n+1} \cdot X_n, \quad Y_{n+2} = X_{n+1} \cdot Y_n, \quad n \geq 0.$$

Third Scheme:

$$X_{n+2} = X_{n+1} \cdot Y_n, \quad Y_{n+2} = Y_{n+1} \cdot X_n, \quad n \geq 0.$$

Fourth Scheme:

$$X_{n+2} = Y_{n+1} \cdot Y_n, \quad Y_{n+2} = X_{n+1} \cdot X_n, \quad n \geq 0.$$

Generalized multiplicative coupled Fibonacci sequences of third order

Definition 2.2. The Multiplicative Coupled Fibonacci Sequence of 3^{rd} order is defined as, Let $\{X_i\}_0^\infty$ and $\{Y_i\}_0^\infty$ be two infinite sequences and six arbitrary real numbers a, b, c, d, e, f are given. The Multiplicative Coupled Fibonacci Sequence of third order is generated by the following eight different ways:

First scheme:

$$X_{n+3} = Y_{n+2} \cdot Y_{n+1} \cdot Y_n, \quad Y_{n+3} = X_{n+2} \cdot X_{n+1} \cdot X_n, \quad n \geq 0.$$

Second scheme:

$$X_{n+3} = X_{n+2} \cdot X_{n+1} \cdot X_n, \quad Y_{n+3} = Y_{n+2} \cdot Y_{n+1} \cdot Y_n, \quad n \geq 0.$$

Third scheme:

$$X_{n+3} = Y_{n+2} \cdot Y_{n+1} \cdot X_n, \quad Y_{n+3} = X_{n+2} \cdot X_{n+1} \cdot Y_n, \quad n \geq 0.$$

Fourth scheme:

$$X_{n+3} = Y_{n+2} \cdot X_{n+1} \cdot Y_n, \quad Y_{n+3} = X_{n+2} \cdot Y_{n+1} \cdot X_n, \quad n \geq 0.$$

Fifth scheme:

$$X_{n+3} = Y_{n+2} \cdot X_{n+1} \cdot X_n, \quad Y_{n+3} = X_{n+2} \cdot Y_{n+1} \cdot Y_n, \quad n \geq 0.$$

Sixth scheme:

$$X_{n+3} = X_{n+2} \cdot X_{n+1} \cdot Y_n, \quad Y_{n+3} = Y_{n+2} \cdot Y_{n+1} \cdot X_n, \quad n \geq 0.$$

Seventh scheme:

$$X_{n+3} = X_{n+2} \cdot Y_{n+1} \cdot Y_n, \quad Y_{n+3} = Y_{n+2} \cdot X_{n+1} \cdot X_n, \quad n \geq 0.$$

Eigth scheme:

$$X_{n+3} = X_{n+2} \cdot Y_{n+1} \cdot X_n, \quad Y_{n+3} = Y_{n+2} \cdot X_{n+1} \cdot Y_n, \quad n \geq 0.$$

In recent years many authors have been generalized Coupled Fibonacci sequences of lower order in number of ways. In this chapter the multiplicative Coupled Fibonacci sequence has been generalized for r^{th} order.

Generalized multiplicative coupled Fibonacci sequence of r^{th} order

Definition 2.3. Let $\{X_i\}_0^\infty$ and $\{Y_i\}_0^\infty$ be two infinite sequences and $2r$ arbitrary real numbers $x_0, x_1, x_2, x_3 ......, x_{r-1}$ and $y_0, y_1, y_2, y_3 ......, y_{r-1}$ are given. The Multiplicative Coupled Fibonacci Sequence of r^{th} order is generated by the following 2^r different ways:

First Scheme:

$$X_{n+r} = Y_{n+r-1} \cdot Y_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n,$$

$$Y_{n+r} = X_{n+r-1} \cdot X_{n+r-2} \cdot X_{n+r-3} \cdots X_n, \quad n \geq 0.$$

Second Scheme:

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n,$$

$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot X_{n+r-3} \cdots X_n, \quad n \geq 0.$$

$\vdots$

$(2^{r-1})^{th}$ Scheme:

(a) If r is an even,

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots Y_n,$$

$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots X_n, \quad n \geq 0.$$

(b) If r is an odd,

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots X_n,$$

$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n, \quad n \geq 0.$$

$\vdots$

$(2^r)^{th}$ Scheme:

$$X_{n+r} = X_{n+r-1} \cdot X_{n+r-2} \cdot X_{n+r-3} \cdots X_n,$$

$$Y_{n+r} = Y_{n+r-1} \cdot Y_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n, \quad n \geq 0$$

n	Y_{n+r}	X_{n+r}
0	$y_0y_1y_2y_3......\ y_{r-1}$	$x_0x_1x_2x_3......\ x_{r-1}$
1	$x_0x_1x_2x_3......\ x_{r-1}$	
	$y_1y_2y_3......\ y_{r-1}$	$x_1x_2x_3......\ x_{r-1}$
		$y_0y_1y_2y_3......\ y_{r-1}$
2	$x_0x_1^2x_2^2x_3^2......\ x_{r-1}^2$	
	$y_0y_1y_2^2y_3^2......\ y_{r-1}^2$	$x_0x_1x_2^2x_3^2......\ x_{r-1}^2$
		$y_0y_1^2y_2^2y_3^2......\ y_{r-1}^2$
3	$x_0^2x_1^3x_2^4x_3^4......\ x_{r-1}^4$	
	$y_0^2y_1^3y_2^3y_3^4......\ y_{r-1}^4$	$x_0^2x_1^3x_2^3x_3^4......\ x_{r-1}^4$
		$y_0^2y_1^3y_2^4y_3^4......\ y_{r-1}^4$
4	$x_0^4x_1^5x_2^7x_3^8......\ x_{r-1}^8$	
	$y_0^4y_1^6y_2^7y_3^7y_4^8......\ y_{r-1}^8$	$x_0^4x_1^5x_2^7x_3^7x_4^8......\ x_{r-1}^8$
		$y_0^4y_1^6y_2^7y_3^8......\ y_{r-1}^8$

TABLE 2.1: **First few terms of the sequences under $(2^{r-1})^{th}$(a) scheme.**

n	X_{n+r}	Y_{n+r}
0	$y_0y_1y_2y_3......\ y_{r-1}$	$x_0x_1x_2x_3......\ x_{r-1}$
1	$x_0x_1x_2x_3......\ x_{r-1}$	
	$y_1y_2y_3......\ y_{r-1}$	$x_1x_2x_3......\ x_{r-1}$
		$y_0y_1y_2y_3......\ y_{r-1}$
2	$x_0x_1^2x_2^2x_3^2......\ x_{r-1}^2$	
	$y_0y_1y_2^2y_3^2......\ y_{r-1}^2$	$x_0x_1x_2^2x_3^2......\ x_{r-1}^2$
		$y_0y_1^2y_2^2y_3^2......\ y_{r-1}^2$
3	$x_0^2x_1^3x_2^4x_3^4......\ x_{r-1}^4$	
	$y_0^2y_1^3y_2^3y_3^4......\ y_{r-1}^4$	$x_0^2x_1^3x_2^3x_3^4......\ x_{r-1}^4$
		$y_0^2y_1^3y_2^4y_3^4......\ y_{r-1}^4$
4	$x_0^4x_1^5x_2^7x_3^8......\ x_{r-1}^8$	
	$y_0^4y_1^6y_2^7y_3^7y_4^8......\ y_{r-1}^8$	$x_0^4x_1^5x_2^7x_3^7x_4^8......\ x_{r-1}^8$
		$y_0^4y_1^6y_2^7y_3^8......\ y_{r-1}^8$

TABLE 2.2: **First few terms of the sequences under $(2^{r-1})^{th}$(b) scheme.**

n	X_{n+r}	Y_{n+r}
0	$y_0 \cdot y_1 \cdot y_2 \cdot y_3 ...... \cdot y_{r-1}$	$x_0 \cdot x_1 \cdot x_2 \cdot x_3 ...... \cdot x_{r-1}$
1	$x_0 \cdot x_1 \cdot x_2 \cdot x_3 ...... \cdot x_{r-1}$	
	$\cdot y_1 \cdot y_2 \cdot y_3 ...... \cdot y_{r-1}$	$x_1 \cdot x_2 \cdot x_3 ...... \cdot x_{r-1}$
		$\cdot y_0 \cdot y_1 \cdot y_2 \cdot y_3 ...... \cdot y_{r-1}$
2	$x_0 \cdot x_1^2 \cdot x_2^2 \cdot x_3^2 ...... \cdot x_{r-1}^2$	
	$\cdot y_0 \cdot y_1 \cdot y_2^2 \cdot y_3^2 ...... \cdot y_{r-1}^2$	$x_0 \cdot x_1 \cdot x_2^2 \cdot x_3^2 ...... \cdot x_{r-1}^2$
		$\cdot y_0 \cdot y_1^2 \cdot y_2^2 \cdot y_3^2 ...... \cdot y_{r-1}^2$
3	$x_0^2 \cdot x_1^3 \cdot x_2^4 \cdot x_3^4 ...... \cdot x_{r-1}^4$	
	$\cdot y_0^2 \cdot y_1^3 \cdot y_2^3 \cdot y_3^4 ...... \cdot y_{r-1}^4$	$x_0^2 \cdot x_1^3 \cdot x_2^3 \cdot x_3^4 ...... \cdot x_{r-1}^4$
		$\cdot y_0^2 \cdot y_1^3 \cdot y_2^4 \cdot y_3^4 ...... \cdot y_{r-1}^4$
4	$x_0^4 \cdot x_1^5 \cdot x_2^7 \cdot x_3^8 ...... \cdot x_{r-1}^8$	
	$\cdot y_0^4 \cdot y_1^6 \cdot y_2^7 \cdot y_3^7 \cdot y_4^8 ...... \cdot y_{r-1}^8$	$x_0^4 \cdot x_1^5 \cdot x_2^7 \cdot x_3^7 \cdot x_4^8 ...... \cdot x_{r-1}^8$
		$\cdot y_0^4 \cdot y_1^6 \cdot y_2^7 \cdot y_3^8 ...... \cdot y_{r-1}^8$

TABLE 2.3: First few terms of the sequence X_n and Y_n under $(2^r)^{th}$ scheme.

2.2 Properties of generalized multiplicative coupled Fibonacci sequence of r^{th} order under $(2^{r-1})^{th}$ scheme

In this section different properties of generalized multiplicative coupled Fibonacci sequence of r^{th} order under $(2^{r-1})^{th}$ scheme are established.

Theorem 2.4. *For every integer* $n \geq 0, \quad r \geq 0$

$$X_{2n(r+1)}Y_0 = Y_{2n(r+1)}X_0 \tag{2.2.1}$$

Proof. **Case:(a)** If r is an even positive integer then, we have

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots Y_n, \quad n \geq 0,$$

$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots X_n, \quad n \geq 0.$$

If $n = 0$, then result is true because

$$X_o \cdot Y_0 = Y_0 \cdot X_0.$$

Assume that the result is true for some integer $n \geq 1$, now consider

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots Y_n, \quad n \geq 0$$

$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots X_n, n \geq 0.$$

Using induction method, we have

$$\begin{aligned}
&X_{2n(r+1)+2r+2} \cdot Y_0 \\
&= \left[X_{2n(r+1)+2r+1} \cdot Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots Y_{2n(r+1)+r+2}\right] \cdot Y_0, \\
&= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+1}\right] \\
&\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots Y_{2n(r+1)+r+2}\right] \cdot Y_0,
\end{aligned}$$

$$= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+2}\right]$$

$$\cdot \left[Y_{2n(r+1)+r} \cdot X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdots Y_{2n(r+1)+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots Y_{2n(r+1)+r+2}\right] \cdot Y_0,$$

$$= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots X_{2n(r+1)+r+2}\right]$$

$$\cdot \left[Y_{2n(r+1)+r-1} \cdot X_{2n(r+1)+r-2} \cdot Y_{2n(r+1)+r-3} \cdots Y_{2n(r+1)}\right]$$

$$\cdot \left[X_{2n(r+1)+r-1} Y_{2n(r+1)+r-1} X_{2n(r+1)+r-2} \cdots X_{2n(r+1)+r+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+2r} X_{2n(r+1)+2r-1} \cdots Y_{2n(r+1)+r+2}\right] Y_0,$$

$$= [X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots X_{2n(r+1)+r+2}]$$

$$\cdot \left[Y_{2n(r+1)+r-1} \cdot X_{2n(r+1)+r-2} \cdot Y_{2n(r+1)+r-3} \cdots Y_{2n(r+1)+1}\right]$$

$$\cdot \left[X_{2n(r+1)+r-1} Y_{2n(r+1)+r-1} X_{2n(r+1)+r-2} \cdots X_{2n(r+1)+r+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+2r} X_{2n(r+1)+2r-1} \cdots Y_{2n(r+1)+r+2}\right] \left[Y_0 X_{2n(r+1)}\right].$$

Using induction hypothesis, we have

$$X_{2n(r+1)+2r+2} \cdot Y_0$$

$$= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots X_{2n(r+1)+r+2}\right]$$

$$\cdot \left[Y_{2n(r+1)+r-1} \cdot X_{2n(r+1)+r-2} \cdot Y_{2n(r+1)+r-3} \cdots Y_{2n(r+1)+1}\right]$$

$$\cdot \left[X_{2n(r+1)+r-1} Y_{2n(r+1)+r-1} X_{2n(r+1)+r-2} \cdots X_{2n(r+1)+r+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+2r} X_{2n(r+1)+2r-1} \cdots Y_{2n(r+1)+r+2}\right] \left[X_0 Y_{2n(r+1)}\right],$$

$$= \left[X_{2n(r+1)+2r} Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots X_{2n(r+1)+r+2}\right]$$

$$\cdot \left[Y_{2n(r+1)+r-1} \cdot X_{2n(r+1)+r-2} \cdot Y_{2n(r+1)+r-3} \cdots Y_{2n(r+1)+1}\right]$$

$$\cdot \left[X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-1} \cdot X_{2n(r+1)+r-2} \cdots X_{2n(r+1)+1} \cdot Y_{2n(r+1)}\right]$$

$$\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots Y_{2n(r+1)+r+2}\right] \cdot X_0,$$

$$= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots X_{2n(r+1)+r+2}\right]$$

$$\cdot \left[X_{2n(r+1)+r} \cdot Y_{2n(r+1)+r-1} \cdot X_{2n(r+1)+r-2} \cdot Y_{2n(r+1)+r-3} \cdots Y_{2n(r+1)+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots Y_{2n(r+1)+r+2}\right] \cdot X_0,$$

$$= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots X_{2n(r+1)+r+2}\right]$$

$$\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots Y_{2n(r+1)+r+2} \cdot X_{2n(r+1)+r+1}\right] \cdot X_0,$$

$$= \left[Y_{2n(r+1)+2r+1} \cdot X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots X_{2n(r+1)+r+2}\right] \cdot X_0$$

$$= Y_{2n(r+1)+2r+2} \cdot X_0.$$

Case:(b) If r is an odd, we have

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots X_n,$$

$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n, \quad n \geq 0.$$

If $n = 0$, then result is true because

$$X_o \cdot Y_0 = Y_0 \cdot X_0.$$

Assume that the result is true for some integer $n \geq 1$, now consider

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots X_n,$$

$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n, \quad n \geq 0.$$

Using induction method, we have

$$X_{2n(r+1)+2r+2} \cdot Y_0$$

$$= \left[X_{2n(r+1)+2r+1} \cdot Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots X_{2n(r+1)+r+2}\right] \cdot Y_0,$$

$$= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots X_{2n(r+1)+r+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots X_{2n(r+1)+r+2}\right] \cdot Y_0,$$

$$= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+2}\right]$$

$$\cdot \left[X_{2n(r+1)+r} \cdot X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdots X_{2n(r+1)+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots X_{2n(r+1)+r+2}\right] \cdot Y_0,$$

$$= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots X_{2n(r+1)+r+2}\right]$$

$$\cdot \left[X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdot X_{2n(r+1)+r-3} \cdots X_{2n(r+1)}\right]$$

$$\cdot \left[Y_{2n(r+1)+r-1} \cdot X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdots X_{2n(r+1)+r+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots X_{2n(r+1)+r+2}\right] \cdot Y_0,$$

$$= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+2}\right]$$

$$\cdot \left[X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdot X_{2n(r+1)+r-3} \cdots X_{2n(r+1)+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+r-1} \cdot X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdots X_{2n(r+1)+r+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots X_{2n(r+1)+r+2}\right] \cdot \left[Y_0 \cdot X_{2n(r+1)}\right].$$

Using induction hypothesis, we have

$$X_{2n(r+1)+2r+2} \cdot Y_0$$

$$= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+2}\right]$$

$$\cdot \left[X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdot X_{2n(r+1)+r-3} \cdots Y_{2n(r+1)+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+r-1} \cdot X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdots X_{2n(r+1)+r+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots Y_{2n(r+1)+r+2}\right] \cdot \left[X_0 \cdot Y_{2n(r+1)}\right],$$

$$= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+2}\right]$$

$$\cdot \left[X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdot X_{2n(r+1)+r-3} \cdots Y_{2n(r+1)+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+r-1} \cdot X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdots X_{2n(r+1)+1} \cdot Y_{2n(r+1)}\right]$$

$$\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots Y_{2n(r+1)+r+2}\right] \cdot X_0,$$

$$= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+2}\right]$$

$$\cdot \left[Y_{2n(r+1)+r} \cdot X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdot X_{2n(r+1)+r-3} \cdots Y_{2n(r+1)+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots Y_{2n(r+1)+r+2}\right] \cdot X_0,$$

$$= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots Y_{2n(r+1)+r+2} \cdot X_{2n(r+1)+r+1}\right] \cdot X_0,$$

$$= \left[Y_{2n(r+1)+2r+1} \cdot X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+2}\right] \cdot X_0,$$

$$= Y_{2n(r+1)+2r+2} \cdot X_0.$$

$\square$

Theorem 2.5. *For every integer $n, r \geq 0$, we have*

$$i) \quad X_{2n(r+1)+1}Y_1 = Y_{2n(r+1)+1}X_1, \tag{2.2.2}$$

$$ii) \quad X_{2n(r+1)+2} \cdot Y_2 = Y_{2n(r+1)+2} \cdot X_2, \tag{2.2.3}$$

$$iii) \quad X_{2n(r+1)+3} \cdot Y_3 = Y_{2n(r+1)+3} \cdot X_3, \tag{2.2.4}$$

$$iv) \quad X_{2n(r+1)+m} \cdot Y_m = Y_{2n(r+1)+m} \cdot X_m. \tag{2.2.5}$$

Proof. $i)$ **Case:(a)** If r is an even, then we have

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots Y_n,$$

$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots X_n, \quad n \geq 0.$$

If $n = 0$, then result is true because

$$X_m \cdot Y_m = Y_m \cdot X_m.$$

Assume that the result is true for some integer $n \geq 1$.

Now,

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots Y_n,$$

$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots X_n, \quad n \geq 0.$$

Using induction method, we have

$$X_{2n(r+1)+m+2r+2} \cdot Y_m$$

$$= \left[X_{2n(r+1)+m+2r+1} \cdot Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+r+2}\right] \cdot Y_m,$$

$$= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots Y_{2n(r+1)+m+r+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\right] \cdot Y_m,$$

$$= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots Y_{2n(r+1)+m+r+2}\right]$$

$$\cdot \left[Y_{2n(r+1)+m+r} \cdot X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdots Y_{2n(r+1)+m+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\right] \cdot Y_m,$$

$$= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots X_{2n(r+1)+m+r+2}\right]$$

$$\cdot \left[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdot Y_{2n(r+1)+m+r-3} \cdots Y_{2n(r+1)+m}\right]$$

$$\cdot \left[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdots X_{2n(r+1)+m+r+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\right] \cdot Y_m,$$

$$= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots X_{2n(r+1)+m+r+2}\right]$$

$$\cdot \left[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdot Y_{2n(r+1)+m+r-3} \cdots Y_{2n(r+1)+m+1}\right]$$

$$\cdot \left[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdots X_{2n(r+1)+m+r+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\right] \cdot \left[Y_m \cdot X_{2n(r+1)+m}\right].$$

, Using induction hypothesis, we have

$$X_{2n(r+1)+m+2r+2} \cdot Y_m$$

$$= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots X_{2n(r+1)+m+r+2}\right]$$

$$\cdot \left[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdot Y_{2n(r+1)+m+r-3} \cdots Y_{2n(r+1)+m+1}\right]$$

$$\cdot \left[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdots X_{2n(r+1)+m+r+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\right] \cdot \left[X_m \cdot Y_{2n(r+1)} + m\right],$$

$$= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots X_{2n(r+1)+m+r+2}\right]$$

$$\cdot \left[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdot Y_{2n(r+1)+m+r-3} \cdots Y_{2n(r+1)+m+1}\right]$$

$$\cdot \left[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdots X_{2n(r+1)+m+1} \cdot Y_{2n(r+1)+m}\right]$$

$$\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\right] \cdot X_m,$$

$$= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots X_{2n(r+1)+m+r+2}\right]$$

$$\cdot \left[X_{2n(r+1)+m+r} \cdot Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdot Y_{2n(r+1)+m+r-3} \cdots Y_{2n(r+1)+m+}$$

$$\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\right] \cdot X_m,$$

$$= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots X_{2n(r+1)+m+r+2}\right]$$

$$\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2} \cdot X_{2n(r+1)+m+r+1}\right] \cdot X_m,$$

$$= Y_{2n(r+1)+m+2r+1} \cdot X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots$$

$$\cdot X_{2n(r+1)+m+r+2} \cdot X_m$$

$$= Y_{2n(r+1)+m+2r+2} \cdot X_m.$$

Case:(b) If r is an odd, then we have

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots X_n,$$

$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n, \quad n \geq 0.$$

If $n = 0$, then result is true because

$$X_m \cdot Y_m = Y_m \cdot X_m.$$

Assume that the result is true for some integer $n \geq 1$.

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots X_n,$$

$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n, \quad n \geq 0.$$

Using induction method, we have

$$X_{2n(r+1)+m+2r+2} \cdot Y_m$$

$$= \left[X_{2n(r+1)+m+2r+1} \cdot Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots X_{2n(r+1)+m+r+2}\right] \cdot Y_m,$$

$$= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots X_{2n(r+1)+m+r+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots X_{2n(r+1)+m+r+2}\right] \cdot Y_m,$$

$$= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots Y_{2n(r+1)+m+r+2}\right]$$

$$\cdot \left[X_{2n(r+1)+m+r} \cdot X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdots X_{2n(r+1)+m+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots X_{2n(r+1)+m+r+2}\right] \cdot Y_m,$$

$$= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots X_{2n(r+1)+m+r+2}\right]$$

$$\cdot \left[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdot X_{2n(r+1)+m+r-3} \cdots X_{2n(r+1)+m}\right]$$

$$\cdot \left[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdots X_{2n(r+1)+m+r+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots X_{2n(r+1)+m+r+2}\right] \cdot Y_m,$$

$$= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots Y_{2n(r+1)+m+r+2}\right]$$

$$\cdot \left[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdot X_{2n(r+1)+m+r-3} \cdots X_{2n(r+1)+m+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdots X_{2n(r+1)+m+r+1}\right]$$

$$\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots X_{2n(r+1)+m+r+2}\right] \cdot \left[Y_m \cdot X_{2n(r+1)+m}\right].$$

Using induction hypothesis, we have

$$
\begin{aligned}
& X_{2n(r+1)+m+2r+2} \cdot Y_m \\
& = \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots Y_{2n(r+1)+m+r+2}\right] \\
& \cdot \left[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdot X_{2n(r+1)+m+r-3} \cdots Y_{2n(r+1)+m+1}\right] \\
& \cdot \left[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdots X_{2n(r+1)+m+r+1}\right] \\
& \cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\right] \cdot \left[X_m \cdot Y_{2n(r+1)+m}\right], \\
& = \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots Y_{2n(r+1)+m+r+2}\right] \\
& \cdot \left[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdot X_{2n(r+1)+m+r-3} \cdots Y_{2n(r+1)+m+1}\right] \\
& \cdot \left[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdots X_{2n(r+1)+m+1} \cdot Y_{2n(r+1)+m}\right] \\
& \cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\right] \cdot X_m, \\
& = \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots Y_{2n(r+1)+m+r+2}\right] \\
& \cdot \left[Y_{2n(r+1)+m+r} \cdot X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdot X_{2n(r+1)+m+r-3} \cdots Y_{2n(r+1)+m+} \right. \\
& \cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\right] \cdot X_m, \\
& = \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots Y_{2n(r+1)+m+r+1}\right] \\
& \cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2} \cdot X_{2n(r+1)+m+r+1}\right] \cdot X_m, \\
& = Y_{2n(r+1)+m+2r+1} \cdot X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots \\
& \cdot Y_{2n(r+1)+m+r+2} \cdot X_m \\
& = Y_{2n(r+1)+m+2r+2} \cdot X_m.
\end{aligned}
$$

The proof of $ii), iii)$ and $iv)$ is same as $i)$. □

Theorem 2.6. *For every integer $n, r \geq 0$, we have*

$$\prod_{i=1}^{i=n} X_{ri+1} Y_{ri+1} = \prod_{i=1}^{i=rn} Y_i X_i. \tag{2.2.6}$$

Proof. For $n = 1$, the result is true because

$$\begin{aligned} X_{r+1} \cdot Y_{r+1} &= [Y_r \cdot Y_{r-1} \cdot Y_{r-2} \cdot \cdots \cdot Y_1] \cdot [X_r \cdot X_{r-1} \cdot X_{r-2} \cdot \cdots \cdot X_1], \\ &= [Y_r \cdot X_r] \cdot [Y_{r-1} \cdot X_{r-1}] \cdot [Y_{r-2} \cdot X_{r-2}] \cdot \cdots \cdot [Y_1 \cdot X_1], \\ &= \prod_{i=1}^{i=r} Y_i \cdot X_i. \end{aligned}$$

Assume that the result is true for some integer $n \geq 1$. Then if r is an even, we have

$$\begin{aligned} X_{n+r} &= X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdot \cdots \cdot Y_n, \\ Y_{n+r} &= Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdot \cdots \cdot X_n, \quad n \geq 0 \end{aligned}$$

and for r is an odd, we have

$$\begin{aligned} X_{n+r} &= X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdot \cdots \cdot X_n, \\ Y_{n+r} &= Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdot \cdots \cdot Y_n, \quad n \geq 0. \end{aligned}$$

Using induction method for $n + 1$, we have

$$\prod_{1}^{n+1} X_{ri+1} \cdot Y_{ri+1} = \prod_{1}^{i=n} [X_{ri+1} \cdot Y_{ri+1}] \cdot \left[X_{r(n+1)+1} \cdot X_{r(n+1)+1}\right].$$

Using induction hypothesis, we have

$$\prod_{1}^{n+1} X_{ri+1} \cdot Y_{ri+1} = \prod_{1}^{rn} [X_i \cdot Y_i] \cdot [X_{rn+r+1} \cdot Y_{rn+r+1}]$$

$$= \prod_{1}^{rn} [X_i \cdot Y_i] \cdot [Y_{rn+r} \cdot X_{rn+r}] \cdot [Y_{rn+r-1} \cdot X_{rn+r-1}] \cdot [Y_{rn+r-2} \cdot X_{rn+r-2}] \cdots$$

$$\cdot [Y_{rn+1} \cdot X_{rn+1}],$$

$$= \prod_{1}^{rn+r} [Y_i \cdot X_i].$$

□

2.3 Properties of generalized multiplicative coupled Fibonacci sequence of r^{th} order under $(2^r)^{th}$ scheme

In this section, we derive many properties of generalized multiplicative coupled Fibonacci sequence of r^{th} order under $(2^r)^{th}$ scheme.

Theorem 2.7. *For every integer $n, r \geq 0$, we have*

$$X_{n(r+1)} \cdot Y_0 = Y_{n(r+1)} \cdot X_0. \tag{2.3.1}$$

Proof. We use induction method to prove this theorem. If $n = 0$ then result is true because $X_o \cdot Y_0 = Y_0 \cdot X_0$. Assume that the result is true for some integer $n \geq 1$

$$X_{n+r} = Y_{n+r-1} \cdot Y_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n,$$

$$Y_{n+r} = X_{n+r-1} \cdot X_{n+r-2} \cdot X_{n+r-3} \cdots X_n.$$

Using induction method for $n + 1$, we have

$$X_{(n+1)(r+1)} \cdot Y_0 = \left[Y_{n(r+1)+r} \cdot Y_{n(r+1)+r-1} \cdot Y_{n(r+1)+r-2} \cdots Y_{n(r+1)+1}\right] \cdot Y_0,$$

$$= \left[X_{n(r+1)+(r-1)} \cdot X_{n(r+1)+(r-2)} \cdot X_{n(r+1)+(r-3)} \cdots X_{n(r+1)}\right]$$

$$\cdot \left[Y_{n(r+1)+r-1} \cdot Y_{n(r+1)+r-2} \cdots Y_{n(r+1)+1}\right] \cdot Y_0,$$

$$= \left[X_{n(r+1)+(r-1)} \cdot X_{n(r+1)+(r-2)} \cdot X_{n(r+1)+(r-3)} \cdots X_{n(r+1)+1}\right]$$

$$\cdot \left[Y_{n(r+1)+r-1} \cdot Y_{n(r+1)+r-2} \cdots Y_{n(r+1)+1}\right] \left[X_{n(r+1)} \cdot Y_0\right].$$

Using induction hypothesis, we have

$$X_{(n+1)(r+1)} \cdot Y_0 = \left[X_{n(r+1)+(r-1)} \cdot X_{n(r+1)+(r-2)} \cdot X_{n(r+1)+(r-3)} \cdots X_{n(r+1)+1}\right]$$

$$\cdot \left[Y_{n(r+1)+r-1} \cdot Y_{n(r+1)+r-2} \cdots Y_{n(r+1)+1}\right] \left[Y_{n(r+1)} \cdot X_0\right],$$

$$= \left[X_{n(r+1)+(r-1)} \cdot X_{n(r+1)+(r-2)} \cdot X_{n(r+1)+(r-3)} \cdots X_{n(r+1)+1}\right] \cdot \left[X_{n(r+1)+r} \cdot X_0\right],$$

$$= \left[Y_{n(r+1)+r+1} \cdot X_0\right],$$

$$= \left[Y_{(n+1)(r+1)} \cdot X_0\right].$$

$\square$

Theorem 2.8. *For every integer $n, r \geq 0$, we have*

$$i) \quad X_{n(r+1)+1} \cdot Y_1 = Y_{n(r+1)+1} \cdot X_1,$$

$$ii) \quad X_{n(r+1)+2} \cdot Y_2 = Y_{n(r+1)+2} \cdot X_2,$$

$$iii) \quad X_{n(r+1)+3} \cdot Y_3 = Y_{n(r+1)+3} \cdot X_3.$$

Proof. $i)$ We use induction method to prove this result. If $n = 0$ then statement is true because

$$X_1 \cdot Y_1 = Y_1 \cdot X_1.$$

Assume that the result is true for some integer $n \geq 1$

$$X_{n+r} = Y_{n+r-1} \cdot Y_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n,$$

$$Y_{n+r} = X_{n+r-1} \cdot X_{n+r-2} \cdot X_{n+r-3} \cdots X_n.$$

Using induction method for $n + 1$, we have

$$X_{(n+1)(r+1)+1} \cdot Y_1 = X_{n(r+1)+(r+2)} \cdot Y_1,$$

$$= \left[Y_{n(r+1)+r+1} \cdot Y_{n(r+1)+r} \cdot Y_{n(r+1)+r-1} \cdots Y_{n(r+1)+2}\right] \cdot Y_1,$$

$$= \left[X_{n(r+1)+r} \cdot X_{n(r+1)+(r-1)} \cdot X_{n(r+1)+(r-2)} \cdots X_{n(r+1)+1}\right]$$

$$\cdot \left[Y_{n(r+1)+r} \cdot Y_{n(r+1)+r-1} \cdots Y_{n(r+1)+2}\right] \cdot Y_1,$$

$$= \left[X_{n(r+1)+r} \cdot X_{n(r+1)+(r-1)} \cdot X_{n(r+1)+(r-2)} \cdots \cdot X_{n(r+1)+2}\right]$$

$$\cdot \left[Y_{n(r+1)+r} \cdot Y_{n(r+1)+r-1} \cdots \cdot Y_{n(r+1)+2}\right] \left[X_{n(r+1)+1} \cdot Y_1\right].$$

Using induction hypothesis, we have

$$X_{(n+1)(r+1)+1} \cdot Y_1 = \left[X_{n(r+1)+r} \cdot X_{n(r+1)+(r-1)} \cdot X_{n(r+1)+(r-2)} \cdots \cdot X_{n(r+1)+2}\right]$$

$$\cdot \left[Y_{n(r+1)+r} \cdot Y_{n(r+1)+r-1} \cdots \cdot Y_{n(r+1)+2}\right] \left[Y_{n(r+1)+1} \cdot X_1\right],$$

$$= \left[X_{n(r+1)+r} \cdot X_{n(r+1)+(r-1)} \cdot X_{n(r+1)+(r-2)} \cdots \cdot X_{n(r+1)+2}\right]$$

$$\cdot \left[X_{n(r+1)+(r+1)} \cdot X_1\right],$$

$$= \left[Y_{n(r+1)+(r+1)+1} \cdot X_1\right],$$

$$= \left[Y_{(n+1)(r+1)+1} \cdot X_1\right].$$

The proof of *ii*) and *iii*) is same as *i*).

In next theorem we generalised theorem (2.8). □

Theorem 2.9. *For every integer $n, r, m \geq 0$, we have*

$$X_{n(r+1)+m} \cdot Y_m = Y_{n(r+1)+m} \cdot X_m.$$

Proof. We use induction method to prove this result. If $n = 0$ then result is true because

$$X_m \cdot Y_m = Y_m \cdot X_m.$$

Assume that the result is true for some integer $n \geq 1$

$$X_{n+r} = Y_{n+r-1} \cdot Y_{n+r-2} \cdot Y_{n+r-3} \cdots \cdot Y_{n},$$

$$Y_{n+r} = X_{n+r-1} \cdot X_{n+r-2} \cdot X_{n+r-3} \cdots \cdot X_{n}.$$

Using induction method for $n+1$, we have

$$X_{(n+1)(r+1)+m} \cdot Y_{m} = X_{n(r+1)+(r+m+1)} \cdot Y_{m},$$

$$= \left[Y_{n(r+1)+r+m} \cdot Y_{n(r+1)+r+m-1} \cdot Y_{n(r+1)+r+m-2} \cdots \cdot Y_{n(r+1)+m+1}\right] \cdot Y_{m},$$

$$= \left[X_{n(r+1)+r+m-1} \cdot X_{n(r+1)+(r+m-2)} \cdot X_{n(r+1)+(r+m-3)} \cdots \cdot X_{n(r+1)+m}\right]$$

$$\cdot \left[Y_{n(r+1)+r+m-1} \cdot Y_{n(r+1)+r+m-2} \cdots \cdot Y_{n(r+1)+m+1}\right] \cdot Y_{m},$$

$$= \left[X_{n(r+1)+r+m-1} \cdot X_{n(r+1)+(r+m-2)} \cdot X_{n(r+1)+(r+m-3)} \cdots \cdot X_{n(r+1)+m+1}\right]$$

$$\cdot \left[Y_{n(r+1)+r+m-1} \cdot Y_{n(r+1)+r+m-2} \cdots \cdot Y_{n(r+1)+m+1}\right] \left[X_{n(r+1)+m} \cdot Y_{m}\right].$$

Using induction hypothesis, we have

$$X_{(n+1)(r+1)+m} \cdot Y_{m}$$

$$= \left[X_{n(r+1)+r+m-1} \cdot X_{n(r+1)+(r+m-2)} \cdot X_{n(r+1)+(r+m-3)} \cdots \cdot X_{n(r+1)+m+1}\right]$$

$$\cdot \left[Y_{n(r+1)+r+m-1} \cdot Y_{n(r+1)+r+m-2} \cdots \cdot Y_{n(r+1)+m+1}\right] \left[Y_{n(r+1)+m} \cdot X_{m}\right],$$

$$= \left[X_{n(r+1)+r+m-1} \cdot X_{n(r+1)+(r+m-2)} \cdot X_{n(r+1)+(r+m-3)} \cdots \cdot X_{n(r+1)+m+1}\right]$$

$$\left[X_{n(r+1)+(r+m)} \cdot X_{m}\right],$$

$$= \left[Y_{n(r+1)+(r+m)+1} \cdot X_{m}\right],$$

$$= \left[Y_{(n+1)(r+1)+m} \cdot X_m \right].$$

□

Theorem 2.10. *For every integer $n, r \geq 0$, we have*

$$i) \quad \prod_{1}^{n} X_{ri+1} = \prod_{1}^{rn} Y_i, \tag{2.3.2}$$

$$ii) \quad \prod_{1}^{n} Y_{ri+1} = \prod_{1}^{rn} X_i. \tag{2.3.3}$$

Proof. $i)$ We use induction method to prove this result. If $n = 1$ then result is true because

$$X_{r+1} = Y_r \cdot Y_{r-1} \cdot Y_{r-2} \cdots Y_1.$$

Assume that the result is true for some integer $n \geq 1$

$$X_{n+r} = Y_{n+r-1} \cdot Y_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n,$$

$$Y_{n+r} = X_{n+r-1} \cdot X_{n+r-2} \cdot X_{n+r-3} \cdots X_n.$$

Using induction method for $n + 1$, we have

$$\prod_{1}^{n+1} X_{ri+1} = \prod_{1}^{n} X_{ri+1} \cdot X_{r(n+1)+1}.$$

Using induction hypothesis, we have

$$\begin{aligned}\prod_{1}^{n+1} X_{ri+1} &= \prod_{i=1}^{i=rn} Y_i \cdot X_{rn+r+1},\\ &= \prod_{i=1}^{i=rn} Y_i \cdot Y_{rn+r} \cdot Y_{rn+r-1} \cdot Y_{rn+r-2} \cdots Y_{rn+1},\\ &= \prod_{i=1}^{i=rn+r} Y_i.\end{aligned}$$

The proof of $ii)$ is same as $i)$. $\square$

2.4 Conclusion

The identities of generalized multiplicative coupled Fibonacci sequence of r^{th} order under two specific schemes are derived in this chapter, this idea can be extended for multiplicative coupled Fibonacci sequence of different order with negative integers.

Chapter 3

Identities Involving k- Fibonacci and k- Lucas Sequences

In this chapter, we investigate the binomial sums, congruence properties, telescoping series of k-Fibonacci and k-Lucas sequences. Also, we defined new relationship between k-Fibonacci and k-Lucas using continued fractions and series of fractions.

3.1 Introduction

The well known Fibonacci sequence is an integer sequence, which is defined by the numbers that satisfy the second order recurrence relation $\mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2}$ with the initial conditions $\mathcal{F}_0 = 0$ and $\mathcal{F}_1 = 1$.

The content of this chapter is published in the following papers.
Identities Involving k-Fibonacci and k-Lucas Sequences, Mathematics Today 34(2018), 125-143.
Fibonacci and k Lucas Sequences as Series of Fractions, Mathematical Journal of Interdisciplinary Sciences, 04(2016), 107-119.

Fibonacci numbers have many interesting properties and applications in various research areas such as Architecture, Engineering, Nature and Art. The Lucas sequence is companion sequence of Fibonacci sequence defined with the Lucas numbers which are defined with the recurrence relation $\mathcal{L}_n = \mathcal{L}_{n-1} + \mathcal{L}_{n-2}$ with the initial conditions $\mathcal{L}_0 = 2$ and $\mathcal{L}_1 = 1$. Binet's formulas for the Fibonacci and Lucas numbers are

$$\mathcal{F}_n = \frac{{r_1}^n - {r_2}^n}{r_1 - r_2}$$

and

$$\mathcal{L}_n = {r_1}^n + {r_2}^n$$

respectively, where $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $x^2 - x - 1 = 0$. The positive root r_1 is known as the golden ratio.

The Fibonacci and Lucas sequences are generalised by changing the initial conditions or changing the recurrence relation. One of the famous generalization of the Fibonacci sequence is k- Fibonacci sequence first introduced by Falcon and Plaza in [**?**]. The k- Fibonacci sequence is defined by the numbers which satisfy the second order recurrence relation $\mathcal{F}_{k,n} = k\mathcal{F}_{k,n-1} + \mathcal{F}_{k,n-2}$ with the initial conditions $\mathcal{F}_{k,0} = 0$ and $\mathcal{F}_{k,1} = 1$. Falcon [**?**] defined the k- Lucas sequence which is companion sequence of k- Fibonacci sequence defined with the k- Lucas numbers which are defined with the recurrence relation $\mathcal{L}_{k,n} = k\mathcal{L}_{k,n-1} + \mathcal{L}_{k,n-2}$ with the initial conditions $\mathcal{L}_{k,0} = 2$ and $\mathcal{L}_{k,1} = k$. Binet's formulas for the k- Fibonacci and k- Lucas numbers are

$$\mathcal{F}_{k,n} = \frac{{r_1}^n - {r_2}^n}{r_1 - r_2}$$

and

$$\mathcal{L}_{k,n} = {r_1}^n + {r_2}^n$$

respectively, where $r_1 = \frac{k+\sqrt{k^2+4}}{2}$ and $r_2 = \frac{k-\sqrt{k^2+4}}{2}$ are the roots of the characteristic equation $x^2 - kx - 1 = 0$. The characteristic roots r_1 and r_2 satisfy the properties

$$r_1 - r_2 = \sqrt{k^2+4} = \sqrt{\delta}, \quad r_1 + r_2 = k, \quad r_1 r_2 = -1.$$

Many properties of k- Fibonacci and k- Lucas numbers are appeared in [**? ?**], some of these are listed below.

(1) $\mathcal{F}_{k,n-r}\mathcal{F}_{k,n+r} - {\mathcal{F}_{k,n}}^2 = (-1)^{n+1-r}{\mathcal{F}_{k,r}}^2$ **(Catalan's Identity)**,

(2) $\mathcal{F}_{k,n-1}\mathcal{F}_{k,n+1} - {\mathcal{F}_{k,n}}^2 = (-1)^n$ **(Cassini's Identity)** ,

(3) $\mathcal{F}_{k,r+1}\mathcal{F}_{k,n} = (-1)^n\mathcal{F}_{k,r-n}$ **(d'Ocagene's Identity)** ,

(4) $\mathcal{F}_{k,r}\mathcal{F}_{k,n+1} + \mathcal{F}_{k,r-1}\mathcal{F}_{k,n} = \mathcal{F}_{k,n+r}$ **(Convolution Theorem)** ,

(5) $\lim\limits_{n\to\infty} \frac{\mathcal{F}_{k,n}}{\mathcal{F}_{k,n-r}} = r_1^r$ **(Asymptotic Behaviour)** .

The generating functions for the subsequence of k-Fibonacci and k-Lucas sequences are

$$(6)\quad \sum_{n=0}^{\infty} \mathcal{F}_{k,tn} x^n = \frac{x\mathcal{F}_{k,t}}{1 - xL_{k,t} + x^2(-1)^t},$$

$$(7)\quad \sum_{n=0}^{\infty} \mathcal{L}_{k,tn} x^n = \frac{2 - x\mathcal{L}_{k,t}}{1 - xL_{k,t} + x^2(-1)^t},$$

$$(8)\quad \sum_{n=0}^{\infty} \mathcal{F}_{k,tn}\mathcal{L}_{k,tn} x^n = \frac{x\mathcal{F}_{k,2t}}{1 - xL_{k,2t} + x^2(-1)^{2t}}.$$

The sums of k-Fibonacci and k-Lucas sequences are

$$(9)\quad \sum_{i=1}^{n} \mathcal{F}_{k,i} = \frac{\mathcal{F}_{k,n+1} + \mathcal{F}_{k,n} - (2+k)}{k},$$

$$(10)\quad \sum_{i=1}^{n} \mathcal{L}_{k,i} = \frac{\mathcal{L}_{k,n+1} + \mathcal{L}_{k,n} - 1}{k},$$

$$(11)\quad \sum_{i=1}^{n} \mathcal{F}_{k,2i} = \frac{\mathcal{F}_{k,2n+1} - 1}{k},$$

$$(12)\quad \sum_{i=1}^{n} \mathcal{F}_{k,2i-1} = \frac{\mathcal{F}_{k,2n}}{k},$$

$$(13)\quad \sum_{i=1}^{n} \mathcal{L}_{k,2i} = \frac{\mathcal{L}_{k,2n+1} - k}{k},$$

$$(14)\quad \sum_{i=1}^{n} \mathcal{L}_{k,2i-1} = \frac{\mathcal{L}_{k,2n} - 2}{k},$$

$$(15)\quad \sum_{i=1}^{n} \mathcal{F}_{k,i}^2 = \frac{\mathcal{F}_{k,n+1}\mathcal{F}_{k,n} - k^2}{k},$$

$$(16)\quad \sum_{i=1}^{n} \mathcal{L}_{k,i}^2 = \frac{k\mathcal{L}_{k,n+1}\mathcal{L}_{k,n} + k^2}{k},$$

$$(17) \quad \sum_{i=1}^{n} \mathcal{F}_{k,i}\mathcal{L}_{k,i} = \frac{\mathcal{F}_{k,2n+1} - 1}{k}.$$

3.2 Binomial and Congruence Identities of k-Fibonacci and k-Lucas Sequences

In this section, we have adapted the techniques of Carlitz[61] and Zhizheng Zhang[62] to k-Fibonacci and k-Lucas sequences and derived many interesting binomial and congruence identities for k-Fibonacci and k-Lucas sequences.

3.2.1 Binomial Identities of the k-Fibonacci and k-Lucas Sequences

In this section, we explore certain binomial identities of the k-Fibonacci and k-Lucas sequences.

Lemma 3.1. *Let $u = r_1$ or r_2, then*

(a) $u^2 = ku + 1.$

(b) $u^n = u\mathcal{F}_{k,n} + \mathcal{F}_{k,n-1}.$

(c) $u^{2n} = u^n\mathcal{L}_{k,n} - (-1)^n.$

(d) $u^{tn} = u^n\dfrac{\mathcal{F}_{k,tn}}{\mathcal{F}_{k,n}} - (-1)^n - \dfrac{\mathcal{F}_{k,(t-1)n}}{\mathcal{F}_{k,n}}.$

(e) $u^{sn}\mathcal{F}_{k,rn} - u^{rn}\mathcal{F}_{k,sn} = (-1)^{sn}\mathcal{F}_{k,(r-s)n}$.

Theorem 3.2. *For $n, r, s, t \geq 1$, we have*

(a) $\mathcal{F}_{k,n+t} = \mathcal{F}_{k,n}\mathcal{F}_{k,t+1} + \mathcal{F}_{k,n-1}\mathcal{F}_{k,t}$.

(b) $\mathcal{F}_{k,2n+t} = \mathcal{L}_{k,n}\mathcal{F}_{k,n+t} - (-1)^n\mathcal{F}_{k,t}$.

(c) $\mathcal{F}_{k,sn+t} = \dfrac{\mathcal{F}_{k,sn}}{\mathcal{F}_{k,n}}\mathcal{F}_{k,n+t} - (-1)^n\dfrac{\mathcal{F}_{k,(s-1)n}}{\mathcal{F}_{k,n}}\mathcal{F}_{k,t}$.

(d) $\mathcal{F}_{k,sn+t}\mathcal{F}_{k,rn} - \mathcal{F}_{k,rn+t}\mathcal{F}_{k,sn} = (-1)^{sn}\mathcal{F}_{k,t}\mathcal{F}_{k,(r-s)n}$.

Theorem 3.3. *For $n, r, s, t \geq 1$ and $\mathcal{D}_{k,n} = \mathcal{F}_{k,n}$ or $\mathcal{L}_{k,n}$, we have*

1. $\mathcal{D}_{k,2n} = \sum_{i=0}^{n}\binom{n}{i} k^i\mathcal{D}_{k,i}$.
2. $\mathcal{D}_{k,2n+t} = \sum_{i=0}^{n}\binom{n}{i} k^i\mathcal{D}_{k,i+t}$.
3. $\mathcal{D}_{k,rn+t} = \sum_{i=0}^{n}\binom{n}{i} \mathcal{F}^i_{k,r}\mathcal{F}^{n-i}_{k,r-1}\mathcal{D}_{k,i+t}$.
4. $\mathcal{D}_{k,2rn+t} = \sum_{i=0}^{n}\binom{n}{i} (-1)^{(n-i)(r+1)}\mathcal{L}^i_{k,r}\mathcal{D}_{k,ri+t}$.
5. $\mathcal{D}_{k,trn+l} = \dfrac{1}{\mathcal{F}^n_{k,r}}\sum_{i=0}^{n}\binom{n}{i} (-1)^{(n-i)(r+1)}\mathcal{F}^{n-i}_{k,(t-1)r}\mathcal{F}^i_{k,tr}\mathcal{D}_{k,ri+l}$.
6. $\sum_{i=0}^{n}\binom{n}{i} (-1)^i\mathcal{D}_{k,r(n-i)+i+t}\mathcal{F}^i_{k,r} = \mathcal{D}_{k,t}\mathcal{F}^n_{k,r-1}$.
7. $\sum_{i=0}^{n}\binom{n}{i} (-1)^{(n-i)}\mathcal{D}_{k,ri+t}\mathcal{F}^{(n-i)}_{k,r-1} = \mathcal{D}_{k,n+t}\mathcal{F}^n_{k,r}$.
8. $\sum_{i=0}^{n}\binom{n}{i} (-1)^{(n-i)}\mathcal{F}^{(n-i)}_{k,sm}\mathcal{F}^{(i)}_{k,rm}\mathcal{D}_{k,m[rn+i(s-r)]+t} = (-1)^{smn}\mathcal{D}_{k,t}\mathcal{F}^n_{k,(r-s)m}$.

Lemma 3.4. *Let $u = r_1$ or r_2, then*

1. $k + (k^2 + 1)u = u^3$.
2. $1 + ku + u^6 = \mathcal{L}_{k,2}u^4$.
3. $1 + ku + u^{10} = \mathcal{L}_{k,4}u^6$.
4. $1 + ku + u^{18} = \mathcal{L}_{k,8}u^{10}$.
5. $1 + ku + u^{34} = \mathcal{L}_{k,16}u^{18}$.
6. $1 + ku + u^{66} = \mathcal{L}_{k,32}u^{34}$.
7. $1 + ku + u^{130} = \mathcal{L}_{k,64}u^{66}$.
8. $1 + ku + u^{258} = \mathcal{L}_{k,128}u^{130}$.
9. $1 + ku + u^{514} = \mathcal{L}_{k,256}u^{258}$.
10. $1 + ku + u^{1026} = \mathcal{L}_{k,512}u^{514}$.
11. $1 + ku + u^{2050} = \mathcal{L}_{k,1024}u^{1026}$.

In general, if $\mathcal{L}_{k,n}$ *is* n^{th} *k-Lucas sequence and* $u = r_1$ *or* r_2*, then*

$$1 + ku + u^{2(2^{n+1}+1)} = \mathcal{L}_{k,2^{n+1}}u^{2(2^n+1)}.$$

Theorem 3.5. *For* $t \geq 1$ *and* $\mathcal{D}_{k,n} = \mathcal{F}_{k,n}$ *or* $\mathcal{L}_{k,n}$*, we have*

1. $\mathcal{D}_{k,t+3} = (k^2 + 1)\mathcal{D}_{k,t+1} + k\mathcal{D}_{k,t}$.
2. $\mathcal{D}_{k,t+4} = \dfrac{\mathcal{D}_{k,t} + k\mathcal{D}_{k,t+1} + \mathcal{D}_{k,t+6}}{\mathcal{L}_{k,2}}$.

3. $\mathcal{D}_{k,t+6} = \dfrac{\mathcal{D}_{k,t} + k\mathcal{D}_{k,t+1} + \mathcal{D}_{k,t+10}}{\mathcal{L}_{k,4}}.$

4. $\mathcal{D}_{k,t+10} = \dfrac{\mathcal{D}_{k,t} + k\mathcal{D}_{k,t+1} + \mathcal{D}_{k,t+18}}{\mathcal{L}_{k,8}}.$

5. $\mathcal{D}_{k,t+18} = \dfrac{\mathcal{D}_{k,t} + k\mathcal{D}_{k,t+1} + \mathcal{D}_{k,t+34}}{\mathcal{L}_{k,16}}.$

6. $\mathcal{D}_{k,t+34} = \dfrac{\mathcal{D}_{k,t} + k\mathcal{D}_{k,t+1} + \mathcal{D}_{k,t+66}}{\mathcal{L}_{k,32}}.$

7. $\mathcal{D}_{k,t+66} = \dfrac{\mathcal{D}_{k,t} + k\mathcal{D}_{k,t+1} + \mathcal{D}_{k,t+130}}{\mathcal{L}_{k,64}}.$

8. $\mathcal{D}_{k,t+130} = \dfrac{\mathcal{D}_{k,t} + k\mathcal{D}_{k,t+1} + \mathcal{D}_{k,t+258}}{\mathcal{L}_{k,128}}.$

9. $\mathcal{D}_{k,t+258} = \dfrac{\mathcal{D}_{k,t} + k\mathcal{D}_{k,t+1} + \mathcal{D}_{k,t+514}}{\mathcal{L}_{k,256}}.$

10. $\mathcal{D}_{k,t+514} = \dfrac{\mathcal{D}_{k,t} + k\mathcal{D}_{k,t+1} + \mathcal{D}_{k,t+1026}}{\mathcal{L}_{k,512}}.$

11. $\mathcal{D}_{k,t+1026} = \dfrac{\mathcal{D}_{k,t} + k\mathcal{D}_{k,t+1} + \mathcal{D}_{k,t+2050}}{\mathcal{L}_{k,1024}}.$

In general, for $t \geq 1$, we have

$$\mathcal{D}_{k,t+2^{n+1}+2} = \frac{\mathcal{D}_{k,t} + k\mathcal{D}_{k,t+1} + \mathcal{D}_{k,t+2^{n+2}+2}}{\mathcal{L}_{k,2^{n+1}}}.$$

Theorem 3.6. *For $n, t \geq 1$ and $\mathcal{D}_{k,n} = \mathcal{F}_{k,n}$ or $\mathcal{L}_{k,n}$, we have*

1. $\mathcal{D}_{k,n+t} = \sum\limits_{i+j+s=n} \dbinom{n}{i,j} k^{-n}(-1)^{j+s}\mathcal{L}_{k,2}{}^{i}\mathcal{D}_{k,4i+6j+t}.$

2. $\mathcal{D}_{k,n+t} = \sum\limits_{i+j+s=n} \dbinom{n}{i,j} k^{-n}(-1)^{j+s}\mathcal{L}_{k,4}{}^{i}\mathcal{D}_{k,6i+10j+t}.$

3. $\mathcal{D}_{k,n+t} = \sum\limits_{i+j+s=n} \dbinom{n}{i,j} k^{-n}(-1)^{j+s}\mathcal{L}_{k,8}{}^{i}\mathcal{D}_{k,10i+18j+t}.$

4. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n}(-1)^{j+s}\mathcal{L}_{k,16}{}^{i}\mathcal{D}_{k,18i+34j+t}$.

5. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n}(-1)^{j+s}\mathcal{L}_{k,32}{}^{i}\mathcal{D}_{k,34i+66j+t}$.

6. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n}(-1)^{j+s}\mathcal{L}_{k,64}{}^{i}\mathcal{D}_{k,66i+130j+t}$.

7. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n}(-1)^{j+s}\mathcal{L}_{k,128}{}^{i}\mathcal{D}_{k,130i+258j+t}$.

8. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n}(-1)^{j+s}\mathcal{L}_{k,256}{}^{i}\mathcal{D}_{k,258i+514j+t}$.

9. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n}(-1)^{j+s}\mathcal{L}_{k,512}{}^{i}\mathcal{D}_{k,514i+1026j+t}$.

10. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n}(-1)^{j+s}\mathcal{L}_{k,1024}{}^{i}\mathcal{D}_{k,1026i+2050j+t}$.

In general, for $r, n, t \geq 1$, we have

$$\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n}(-1)^{j+s}\mathcal{L}_{k,2^{r+1}}{}^{i}\mathcal{D}_{k,2^{r+1}(i+2j)+2(i+j)+t}.$$

Theorem 3.7. *For $n, t \geq 1$ and $\mathcal{D}_{k,n} = \mathcal{F}_{k,n}$ or $\mathcal{L}_{k,n}$, we have*

1. $\mathcal{D}_{k,6n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j}(-1)^{j+s}\mathcal{L}_{k,2}{}^{i}\mathcal{D}_{k,4i+j+t}$.

2. $\mathcal{D}_{k,10n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j}(-1)^{j+s}\mathcal{L}_{k,4}{}^{i}\mathcal{D}_{k,6i+j+t}$.

3. $\mathcal{D}_{k,18n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j}(-1)^{j+s}\mathcal{L}_{k,8}{}^{i}\mathcal{D}_{k,10i+j+t}$.

4. $\mathcal{D}_{k,34n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j}(-1)^{j+s}\mathcal{L}_{k,16}{}^{i}\mathcal{D}_{k,18i+j+t}$.

5. $\mathcal{D}_{k,66n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j}(-1)^{j+s}\mathcal{L}_{k,32}{}^{i}\mathcal{D}_{k,34i+j+t}$.

6. $\mathcal{D}_{k,130n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} \mathcal{L}_{k,64}{}^i \mathcal{D}_{k,66i+j+t}.$

7. $\mathcal{D}_{k,258n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} \mathcal{L}_{k,128}{}^i \mathcal{D}_{k,130i+j+t}.$

8. $\mathcal{D}_{k,514n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} \mathcal{L}_{k,256}{}^i \mathcal{D}_{k,258i+j+t}.$

9. $\mathcal{D}_{k,1026n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} \mathcal{L}_{k,512}{}^i \mathcal{D}_{k,514i+j+t}.$

10. $\mathcal{D}_{k,2050n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} \mathcal{L}_{k,1024}{}^i \mathcal{D}_{k,1026i+j+t}.$

In general, for $r, n, t \geq 1$, we have

$$\mathcal{D}_{k,(2^{r+2}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} \mathcal{L}_{k,2^{r+1}}{}^i \mathcal{D}_{k,(2^{r+1}+2)i+j+t}.$$

Theorem 3.8. *For $n, t \geq 1$ and $\mathcal{D}_{k,n} = \mathcal{F}_{k,n}$ or $\mathcal{L}_{k,n}$, we have*

1. $\mathcal{D}_{k,4n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j \mathcal{L}_{k,2}{}^{-n} \mathcal{D}_{k,6i+j+t}.$

2. $\mathcal{D}_{k,6n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j \mathcal{L}_{k,4}{}^{-n} \mathcal{D}_{k,10i+j+t}.$

3. $\mathcal{D}_{k,10n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j \mathcal{L}_{k,8}{}^{-n} \mathcal{D}_{k,18i+j+t}.$

4. $\mathcal{D}_{k,18n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j \mathcal{L}_{k,16}{}^{-n} \mathcal{D}_{k,34i+j+t}.$

5. $\mathcal{D}_{k,34n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j \mathcal{L}_{k,32}{}^{-n} \mathcal{D}_{k,66i+j+t}.$

6. $\mathcal{D}_{k,66n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j \mathcal{L}_{k,64}{}^{-n} \mathcal{D}_{k,130i+j+t}.$

7. $\mathcal{D}_{k,130n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j \mathcal{L}_{k,128}{}^{-n} \mathcal{D}_{k,258i+j+t}.$

8. $\mathcal{D}_{k,258n+t} = \sum\limits_{i+j+s=n} \binom{n}{i,j} k^j \mathcal{L}_{k,256}{}^{-n} \mathcal{D}_{k,514i+j+t}.$

9. $\mathcal{D}_{k,514n+t} = \sum\limits_{i+j+s=n} \binom{n}{i,j} k^j \mathcal{L}_{k,512}{}^{-n} \mathcal{D}_{k,1026i+j+t}.$

10. $\mathcal{D}_{k,1026n+t} = \sum\limits_{i+j+s=n} \binom{n}{i,j} k^j \mathcal{L}_{k,1024}{}^{-n} \mathcal{D}_{k,2050i+j+t}.$

In general, for $r, n, t \geq 1$, we have

$$\mathcal{D}_{k,(2^{r+1}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j \mathcal{L}_{k,2^{r+1}}{}^{-n} \mathcal{D}_{k,(2^{r+1}+2)i+j+t}.$$

Lemma 3.9. *Let $u = r_1$ or r_2, then for $l_n = \sum\limits_{i=1}^{n} \mathcal{L}_{k,2^i}$ and $n, t \geq 1$, we have*

1. $1 + u^4 = l_1 u^2.$

2. $1 + u^8 = \dfrac{l_2}{l_1} u^4 = l_2 u^2 - \dfrac{l_2}{l_1}.$

3. $1 + u^{16} = \dfrac{l_3}{l_2} u^8 = \dfrac{l_3}{l_1} u^4 - \dfrac{l_3}{l_2} = l_3 u^2 - \dfrac{l_3}{l_1} - \dfrac{l_3}{l_2}.$

4. $1 + u^{32} = \dfrac{l_4}{l_3} u^{16} = \dfrac{l_4}{l_2} u^8 - \dfrac{l_4}{l_3} = \dfrac{l_4}{l_1} u^4 - l_4 \left[\dfrac{1}{l_2} + \dfrac{1}{l_3}\right] = l_4 u^2 - l_4 \left[\dfrac{1}{l_1} + \dfrac{1}{l_2} + \dfrac{1}{l_3}\right].$

5. $1 + u^{64} = \dfrac{l_5}{l_4} u^{32} = \dfrac{l_5}{l_3} u^{16} - \dfrac{l_5}{l_4} = \dfrac{l_5}{l_2} u^8 - l_5 \left[\dfrac{1}{l_3} + \dfrac{1}{l_4}\right] = \dfrac{l_5}{l_1} u^4 - l_5 \left[\dfrac{1}{l_2} + \dfrac{1}{l_3} + \dfrac{1}{l_4}\right]$
$= l_5 u^2 - l_5 \left[\dfrac{1}{l_1} + \dfrac{1}{l_2} + \dfrac{1}{l_3} + \dfrac{1}{l_4}\right].$

In general, we have

$$1+u^{2^n}=\begin{cases}\dfrac{l_{n-1}}{l_{n-2}}u^{2^{n-1}}; \\ \dfrac{l_{n-1}}{l_{n-t-1}}u^{2^{n-t}}-l_{n-1}\sum\limits_{i=2}^{t}\dfrac{1}{l_{n-i}}, \quad \textit{If } t=2,3,4,\ldots,n-2\ ; \\ l_{n-1}u^2-l_{n-1}\sum\limits_{i=2}^{n-1}\dfrac{1}{l_{n-i}}.\end{cases}$$

Theorem 3.10. *For* $l_n=\sum\limits_{i=1}^{n}\mathcal{L}_{k,2^i}$, $n,t\geq 1$ *and* $\mathcal{D}_{k,n}=\mathcal{F}_{k,n}$ *or* $\mathcal{L}_{k,n}$, *we have*

1. $\mathcal{D}_{k,t+4}=l_1\mathcal{D}_{k,t+2}-\mathcal{D}_{k,t}$.

2. $\mathcal{D}_{k,t+8}=\dfrac{l_2}{l_1}\mathcal{D}_{k,t+4}-\mathcal{D}_{k,t}=l_2\mathcal{D}_{k,t+2}-(1+\dfrac{l_2}{l_1})\mathcal{D}_{k,t}$.

3. $$\begin{aligned}\mathcal{D}_{k,t+16}&=\frac{l_3}{l_2}\mathcal{D}_{k,t+8}-\mathcal{D}_{k,t},\\ &=\frac{l_3}{l_1}\mathcal{D}_{k,t+4}-(1+\frac{l_3}{l_2})\mathcal{D}_{k,t},\\ &=l_3\mathcal{D}_{k,t+2}-(1+\frac{l_3}{l_1}+\frac{l_3}{l_2})\mathcal{D}_{k,t}.\end{aligned}$$

4. $$\begin{aligned}\mathcal{D}_{k,t+32}&=\frac{l_4}{l_3}\mathcal{D}_{k,t+16}-\mathcal{D}_{k,t},\\ &=\frac{l_4}{l_2}\mathcal{D}_{k,t+8}-(1+\frac{l_4}{l_3})\mathcal{D}_{k,t},\\ &=\frac{l_4}{l_1}\mathcal{D}_{k,t+4}-(1+\frac{l_4}{l_2}+\frac{l_4}{l_3})\mathcal{D}_{k,t},\\ &=l_4\mathcal{D}_{k,t+2}-(1+\frac{l_4}{l_1}+\frac{l_4}{l_2}+\frac{l_4}{l_3})\mathcal{D}_{k,t}.\end{aligned}$$

5. $$\begin{aligned}\mathcal{D}_{k,t+64}&=\frac{l_5}{l_4}\mathcal{D}_{k,t+32}-\mathcal{D}_{k,t},\\ &=\frac{l_5}{l_3}\mathcal{D}_{k,t+16}-(1+\frac{l_5}{l_4})\mathcal{D}_{k,t},\\ &=\frac{l_5}{l_2}\mathcal{D}_{k,t+8}-(1+\frac{l_5}{l_3}+\frac{l_5}{l_4})\mathcal{D}_{k,t},\end{aligned}$$

$$= \frac{l_5}{l_1}\mathcal{M}_{k,t+4} - (1 + \frac{l_5}{l_2} + \frac{l_5}{l_3} + \frac{l_5}{l_4})\mathcal{D}_{k,t},$$
$$= l_5\mathcal{D}_{k,t+2} - (1 + \frac{l_5}{l_1} + \frac{l_5}{l_2} + \frac{l_5}{l_3} + \frac{l_5}{l_4})\mathcal{D}_{k,t}.$$

In general, we have

$$\mathcal{D}_{k,t+2^n} = \begin{cases} \frac{l_{n-1}}{l_{n-2}}\mathcal{D}_{k,t+2^{n-1}} - \mathcal{D}_{k,t}; \\ \frac{l_{n-1}}{l_{n-t-1}}\mathcal{D}_{k,t+2^{n-s}} - l_{n-1}\sum_{i=2}^{s}(1 + \frac{1}{l_{n-i}})\mathcal{D}_{k,t}, \quad \text{If } s = 2, 3, 4, \ldots, n-2 \ ; \\ l_{n-1}\mathcal{D}_{k,t+2} - l_{n-1}\sum_{i=2}^{n-1}(\frac{1}{l_{n-i}} + 1)\mathcal{D}_{k,t}. \end{cases}$$

Theorem 3.11. *For* $l_n = \sum_{i=1}^{n} \mathcal{L}_{k,2^i}$, $n, t \geq 1$ *and* $\mathcal{D}_{k,n} = \mathcal{F}_{k,n}$ *or* $\mathcal{L}_{k,n}$, *we have*

1. $\mathcal{D}_{k,4n+t} = \sum_{i+j=n} \binom{n}{i} l_1^i(-1)^j\mathcal{D}_{k,2i+t}.$

2. $\mathcal{D}_{k,8n+t} = \sum_{i+j=n} \binom{n}{i} (\frac{l_2}{l_1})^i(-1)^j\mathcal{D}_{k,4i+t},$
$$= \sum_{i+j=n} \binom{n}{i} l_2^i(-1)^j(\frac{l_1 + l_2}{l_1})\mathcal{D}_{k,2i+t}.$$

3. $\mathcal{D}_{k,16n+t} = \sum_{i+j=n} \binom{n}{i} (\frac{l_3}{l_2})^i(-1)^j\mathcal{D}_{k,8i+t},$
$$= \sum_{i+j=n} \binom{n}{i} (\frac{l_3}{l_1})^i(-1)^j(1 + \frac{l_3}{l_2})\mathcal{D}_{k,4i+t},$$
$$= \sum_{i+j=n} \binom{n}{i} {l_3}^i(-1)^j(1 + \frac{l_3}{l_1} + \frac{l_3}{l_2})\mathcal{D}_{k,2i+t}.$$

4. $\mathcal{D}_{k,32n+t} = \sum_{i+j=n} \binom{n}{i} (\frac{l_4}{l_3})^i(-1)^j\mathcal{D}_{k,16i+t},$
$$= \sum_{i+j=n} \binom{n}{i} (\frac{l_4}{l_2})^i(-1)^j(1 + \frac{l_4}{l_3})\mathcal{D}_{k,8i+t},$$

$$= \sum_{i+j=n} \binom{n}{i} (\frac{l_4}{l_1})^i (-1)^j (1 + \frac{l_4}{l_2} + \frac{l_4}{l_3}) \mathcal{D}_{k,4i+t},$$
$$= \sum_{i+j=n} \binom{n}{i} {l_4}^i (-1)^j (1 + \frac{l_4}{l_1} + \frac{l_4}{l_2} + \frac{l_4}{l_3}) \mathcal{D}_{k,2i+t}.$$

5. $\mathcal{D}_{k,64n+t} = \sum_{i+j=n} \binom{n}{i} (\frac{l_5}{l_4})^i (-1)^j \mathcal{D}_{k,32i+t},$
$$= \sum_{i+j=n} \binom{n}{i} (\frac{l_5}{l_3})^i (-1)^j (1 + \frac{l_5}{l_4}) \mathcal{D}_{k,16i+t},$$
$$= \sum_{i+j=n} \binom{n}{i} (\frac{l_5}{l_2})^i (-1)^j (1 + \frac{l_5}{l_3} + \frac{l_5}{l_4}) \mathcal{D}_{k,8i+t},$$
$$= \sum_{i+j=n} \binom{n}{i} (\frac{l_5}{l_1})^i (-1)^j (1 + \frac{l_5}{l_2} + \frac{l_5}{l_3} + \frac{l_5}{l_4}) \mathcal{D}_{k,4i+t},$$
$$= \sum_{i+j=n} \binom{n}{i} {l_5}^i (-1)^j (1 + \frac{l_5}{l_1} + \frac{l_5}{l_2} + \frac{l_5}{l_3} + \frac{l_5}{l_4}) \mathcal{D}_{k,2i+t}.$$

In general, we have

$$\mathcal{D}_{k,2^r n+t} = \begin{cases} \sum_{i+j=n} \binom{n}{i} (\frac{l_{r-1}}{l_{r-2}})^i (-1)^j \mathcal{D}_{k,2^{r-1}i+t}; \\ \sum_{i+j=n} \binom{n}{i} (\frac{l_{r-1}}{l_{r-s-1}})^i (-1)^j (\sum_{h=2}^{s} (1 + \frac{l_{r-1}}{l_{r-h}})^j \mathcal{D}_{k,2^{n-s}i+t}, \\ \qquad \text{If } s = 2, 3, 4, \ldots, n-2 \text{ ;} \\ \sum_{i+j=n} \binom{n}{i} (l_{r-1})^i (-1)^j (\sum_{h=2}^{s} (1 + \frac{l_{r-1}}{l_{r-h}})^j \mathcal{D}_{k,2i+t}. \end{cases}$$

Lemma 3.12. *For $t \geq 1$, we have*

(1) $r_1^2 = r_1\sqrt{\delta} - 1,$

$r_2^2 = -r_2\sqrt{\delta} - 1.$

(2) $r_1^4 = (k^2 + 2)r_1\sqrt{\delta} - (k^2 + 3),$

$r_2^4 = -(k^2 + 2)r_2\sqrt{\delta} - (k^2 + 3).$

(3) $r_1^6 = (k^2+1)(k^2+3)r_1\sqrt{\delta} - (k^4+5k^2+5),$

$r_2^6 = -(k^2+1)(k^2+3)r_2\sqrt{\delta} - (k^4+5k^2+5).$

(4) $r_1^8 = (k^2+2)(k^4+4k^2+2)r_1\sqrt{\delta} - (k^6+7k^4+14k^2+7),$

$r_2^8 = -(k^2+2)(k^4+4k^2+2)r_2\sqrt{\delta} - (k^6+7k^4+14k^2+7).$

(5) $r_1^{10} = (k^4+3k^2+1)(k^4+5k^2+5)r_1\sqrt{\delta} - (k^2+3)(k^6+6k^4+9k^2+3),$

$r_2^{10} = -(k^4+3k^2+1)(k^4+5k^2+5)r_2\sqrt{\delta} - (k^2+3)(k^6+6k^4+9k^2+3).$

In general, we have

$$r_1^{2t} = \frac{\mathcal{F}_{k,2t}}{k}r_1\sqrt{\delta} - \frac{\mathcal{L}_{k,2t-1}}{k},$$
$$r_2^{2t} = -\frac{\mathcal{F}_{k,2t}}{k}r_2\sqrt{\delta} - \frac{\mathcal{L}_{k,2t-1}}{k}.$$

Lemma 3.13. *For $t \geq 1$, we have*

(1) $r_1^3 = (k^2+3)r_1 - \sqrt{\delta},$

$r_2^3 = (k^2+3)r_2 + \sqrt{\delta}.$

(2) $r_1^5 = (k^4+5k^2+5)r_1 - (k^2+2)\sqrt{\delta},$

$r_2^5 = (k^4+5k^2+5)r_2 + (k^2+2)\sqrt{\delta}.$

(3) $r_1^7 = (k^6+7k^4+14k^2+7)r_1 - (k^2+1)(k^2+3)\sqrt{\delta},$

$r_2^7 = (k^6+7k^4+14k^2+7)r_2 + (k^2+1)(k^2+3)\sqrt{\delta}.$

(4) $r_1^9 = (k^2+3)(k^6+6k^4+9k^2+3)r_1 - (k^2+2)(k^4+4k^2+2)\sqrt{\delta},$

$r_2^9 = (k^2+3)(k^6+6k^4+9k^2+3)r_2 + (k^2+2)(k^4+4k^2+2)\sqrt{\delta},.$

(5) $r_1^{11} = (k^{10} + 11k^8 + 44k^6 + 77k^4 + 55k^2 + 11)r_1 + (k^4 + 3k^2 + 1)(k^4 + 5k^2 + 5)\sqrt{\delta}$,

$r_2^{11} = (k^{10} + 11k^8 + 44k^6 + 77k^4 + 55k^2 + 11)r_2 - (k^4 + 3k^2 + 1)(k^4 + 5k^2 + 5)\sqrt{\delta}$.

In general, we have

$$r_1^{2t+1} = \frac{\mathcal{L}_{k,2t+1}}{k} r_1 - \frac{\mathcal{F}_{k,2t}}{k}\sqrt{\delta},$$
$$r_2^{2t+1} = \frac{\mathcal{L}_{k,2t+1}}{k} r_2 + \frac{\mathcal{F}_{k,2t}}{k}\sqrt{\delta}.$$

Theorem 3.14. *For $s, t \geq 1$, we have*

1. $\mathcal{F}_{k,s+2} + \mathcal{F}_{k,s} = \mathcal{L}_{k,s+1}$,

   $\mathcal{L}_{k,s+2} + \mathcal{L}_{k,s} = \delta \mathcal{F}_{k,s+1}$.

2. $\mathcal{F}_{k,s+4} + (k^2+3)\mathcal{F}_{k,s} = (k^2+2)\mathcal{L}_{k,s+1}$,

   $\mathcal{L}_{k,s+4} + (k^2+3)\mathcal{L}_{k,s} = (k^2+2)\delta \mathcal{F}_{k,s+1}$.

3. $\mathcal{F}_{k,s+6} + (k^4+5k^2+5)\mathcal{F}_{k,s} = (k^2+1)(k^2+3)\mathcal{L}_{k,s+1}$,

   $\mathcal{L}_{k,s+6} + (k^4+5k^2+5)\mathcal{L}_{k,s} = (k^2+1)(k^2+3)\delta \mathcal{F}_{k,s+1}$.

4. $\mathcal{F}_{k,s+8} + (k^6+7k^4+14k^2+7)\mathcal{F}_{k,s} = (k^2+2)(k^4+4k^2+2)\mathcal{L}_{k,s+1}$,

   $\mathcal{L}_{k,s+8} + (k^6+7k^4+14k^2+7)\mathcal{L}_{k,s} = (k^2+2)(k^4+4k^2+2)\delta \mathcal{F}_{k,s+1}$.

5. $\mathcal{F}_{k,s+10} + (k^2+3)(k^6+6k^4+9k^2+3)\mathcal{F}_{k,s} = (k^4+3k^2+1)(k^4+5k^2+5)\mathcal{L}_{k,s+1}$,

$\mathcal{L}_{k,s+10} + (k^2+3)(k^6+6k^4+9k^2+3)\mathcal{L}_{k,s} = (k^4+3k^2+1)(k^4+5k^2+5)\delta\mathcal{F}_{k,s+1}$.

In general, we have

$$\mathcal{F}_{k,s+2t} + \frac{\mathcal{L}_{k,2t-1}}{k}\mathcal{F}_{k,s} = \frac{\mathcal{F}_{k,2t}}{k}\mathcal{L}_{k,s+1}, \tag{3.2.1}$$

$$\mathcal{L}_{k,s+10} + \frac{\mathcal{L}_{k,2t-1}}{k}\mathcal{L}_{k,s} = \frac{\mathcal{F}_{k,2t}}{k}\delta\mathcal{F}_{k,s+1}. \tag{3.2.2}$$

Remark 3.15. Using $\mathcal{L}_{k,2t-1} - \mathcal{F}_{k,2t} = \mathcal{F}_{k,2t-2}$ in (5.2.1), we get

$$\mathcal{F}_{k,s+2t} - \frac{\mathcal{F}_{k,2t}}{k}\mathcal{F}_{k,s+2} + \frac{\mathcal{F}_{k,2t-2}}{k}\mathcal{F}_{k,s} = 0,$$
$$\mathcal{L}_{k,s+2t} - \frac{\mathcal{F}_{k,2t}}{k}\mathcal{L}_{k,s+2} + \frac{\mathcal{F}_{k,2t-2}}{k}\mathcal{L}_{k,s} = 0.$$

Theorem 3.16. *For $s, t \geq 1$, we have*

1. $\mathcal{F}_{k,s+3} + \mathcal{L}_{k,s} = (k^2+3)\mathcal{F}_{k,s+1}$,

   $\mathcal{L}_{k,s+3} + \delta\mathcal{F}_{k,s} = (k^2+3)\mathcal{L}_{k,s+1}$.

2. $\mathcal{F}_{k,s+5} + (k^2+2)\mathcal{L}_{k,s} = (k^4+5k^2+5)\mathcal{F}_{k,s+1}$,

   $\mathcal{L}_{k,s+5} + \delta(k^2+2)\mathcal{F}_{k,s} = (k^4+5k^2+5)\mathcal{L}_{k,s+1}$.

3. $\mathcal{F}_{k,s+7} + (k^2+1)(k^2+3)\mathcal{L}_{k,s} = (k^6+7k^4+14k^2+7)\mathcal{F}_{k,s+1}$,

   $\mathcal{L}_{k,s+7} + \delta(k^2+1)(k^2+3)\mathcal{F}_{k,s} = (k^6+7k^4+14k^2+7)\mathcal{L}_{k,s+1}$.

4. $\mathcal{F}_{k,s+9} + (k^2+2)(k^4+4k^2+2)\mathcal{L}_{k,s} = (k^2+3)(k^6+6k^4+9k^2+3)\mathcal{F}_{k,s+1}$,

$\mathcal{L}_{k,s+9} + \delta(k^2+2)(k^4+4k^2+2)\mathcal{F}_{k,s} = (k^2+3)(k^6+6k^4+9k^2+3)\mathcal{L}_{k,s+1}$.

5. $\mathcal{F}_{k,s+11} + (k^4+3k^2+1)(k^4+5k^2+5)\mathcal{L}_{k,s} = (k^{10}+11k^8+44k^6+77k^4+55k^2+11)\mathcal{F}_{k,s+1}$,

   $\mathcal{L}_{k,s+11} + \delta(k^4+3k^2+1)(k^4+5k^2+5)\mathcal{F}_{k,s} = (k^{10}+11k^8+44k^6+77k^4+55k^2+11)\mathcal{L}_{k,s+1}$.

In general, we have

$$\mathcal{F}_{k,s+2t+1} + \frac{\mathcal{F}_{k,2t}}{k}\mathcal{L}_{k,s} = \frac{\mathcal{L}_{k,2t+1}}{k}\mathcal{F}_{k,s+1}, \tag{3.2.3}$$

$$\mathcal{L}_{k,s+2t+1} + \delta\frac{\mathcal{F}_{k,2t}}{k}\mathcal{F}_{k,s} = \frac{\mathcal{L}_{k,2t+1}}{k}\mathcal{L}_{k,s+1}. \tag{3.2.4}$$

Remark 3.17. Using $(k^2+3)\mathcal{F}_{k,2t} - \mathcal{L}_{k,2t-1} = \mathcal{F}_{k,2t-2}$ in (5.2.3), we obtain

$$\mathcal{F}_{k,s+2t+1} - \frac{\mathcal{L}_{k,2t+1}}{k(k^2+3)}\mathcal{F}_{k,s+3} + \frac{\mathcal{F}_{k,2t-2}}{k(k^2+3)}\mathcal{L}_{k,s} = 0,$$

$$\mathcal{L}_{k,s+2t+1} - \frac{\mathcal{L}_{k,2t+1}}{k(k^2+3)}\mathcal{L}_{k,s+3} + \frac{\mathcal{F}_{k,2t-2}}{k(k^2+3)}\delta\mathcal{F}_{k,s} = 0.$$

Theorem 3.18. *For $n, s, t \geq 1$, we have*

1. $$\sum_{i=0}^{n}\binom{n}{i}\mathcal{F}_{k,2i+s}=\begin{cases}\delta^{\frac{n}{2}}\mathcal{F}_{k,n+s}, & \text{if } n \text{ is even;}\\[2ex] \delta^{\frac{n-1}{2}}\mathcal{L}_{k,n+s}, & \text{if } n \text{ is odd,}\end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}\mathcal{L}_{k,2i+s}=\begin{cases}\delta^{\frac{n}{2}}\mathcal{L}_{k,n+s}, & \text{if } n \text{ is even;}\\[2ex] \delta^{\frac{n+1}{2}}\mathcal{F}_{k,n+s}, & \text{if } n \text{ is odd.}\end{cases}$$

2. $$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}\mathcal{F}_{k,4i+s}=\begin{cases}(k^2+2)^n\delta^{\frac{n}{2}}\mathcal{F}_{k,n+s}, & \text{if } n \text{ is even;}\\[2ex] (k^2+2)^n\delta^{\frac{n-1}{2}}\mathcal{L}_{k,n+s}, & \text{if } n \text{ is odd,}\end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}\mathcal{L}_{k,4i+s}=\begin{cases}(k^2+2)^n\delta^{\frac{n}{2}}\mathcal{L}_{k,n+s}, & \text{if } n \text{ is even;}\\[2ex] (k^2+2)^n\delta^{\frac{n+1}{2}}\mathcal{F}_{k,n+s}, & \text{if } n \text{ is odd.}\end{cases}$$

3. $$\sum_{i=0}^{n}\binom{n}{i}(k^4+5k^2+5)^{(n-i)}\mathcal{F}_{k,6i+s}$$
$$=\begin{cases}(k^2+1)^n(k^2+3)^n\delta^{\frac{n}{2}}\mathcal{F}_{k,n+s}, & \text{if } n \text{ is even;}\\[2ex] (k^2+1)^n(k^2+3)^n\delta^{\frac{n-1}{2}}\mathcal{L}_{k,n+s}, & \text{if } n \text{ is odd,}\end{cases}$$
$$\sum_{i=0}^{n}\binom{n}{i}(k^4+5k^2+5)^{(n-i)}\mathcal{L}_{k,6i+s}$$
$$=\begin{cases}(k^2+1)^n(k^2+3)^n\delta^{\frac{n}{2}}\mathcal{L}_{k,n+s}, & \text{if } n \text{ is even;}\\[2ex] (k^2+1)^n(k^2+3)^n\delta^{\frac{n+1}{2}}\mathcal{F}_{k,n+s}, & \text{if } n \text{ is odd.}\end{cases}$$

4. $$\sum_{i=0}^{n}\binom{n}{i}(k^6+7k^4+14k^2+7)^{(n-i)}\mathcal{F}_{k,8i+s}$$
$$=\begin{cases}(k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n}{2}}\mathcal{F}_{k,n+s}, & \text{if } n \text{ is even;}\\[2ex] (k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n-1}{2}}\mathcal{L}_{k,n+s}, & \text{if } n \text{ is odd,}\end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}(k^6+7k^4+14k^2+7)^{(n-i)}\mathcal{L}_{k,8i+s}$$
$$=\begin{cases}(k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n}{2}}\mathcal{L}_{k,n+s}, & \textit{if } n \textit{ is even;}\\ (k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n+1}{2}}\mathcal{F}_{k,n+s}, & \textit{if } n \textit{ is odd.}\end{cases}$$

5. $$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(k^6+6k^4+9k^2+3)^{(n-i)}\mathcal{F}_{k,10i+s}$$
$$=\begin{cases}(k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n}{2}}\mathcal{F}_{k,n+s}, & \textit{if } n \textit{ is even;}\\ (k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n-1}{2}}\mathcal{L}_{k,n+s}, & \textit{if } n \textit{ is odd,}\end{cases}$$
$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(k^6+6k^4+9k^2+3)^{(n-i)}\mathcal{L}_{k,10i+s}$$
$$=\begin{cases}(k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n}{2}}\mathcal{L}_{k,n+s}, & \textit{if } n \textit{ is even;}\\ (k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n+1}{2}}\mathcal{F}_{k,n+s}, & \textit{if } n \textit{ is odd.}\end{cases}$$

In general, for $n, s, t \geq 1$, we have

$$\sum_{i=0}^{n}\binom{n}{i}k^{(i-n)}(\mathcal{L}_{k,2t-1})^{(n-i)}\mathcal{F}_{k,2ti+s}=\begin{cases}k^{-n}(\mathcal{F}_{k,2t})^n\delta^{\frac{n}{2}}\mathcal{F}_{k,n+s}, & \textit{if } n \textit{ is even;}\\ k^{-n}(\mathcal{F}_{k,2t})^n\delta^{\frac{n-1}{2}}\mathcal{L}_{k,n+s}, & \textit{if } n \textit{ is odd,}\end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}k^{(i-n)}(\mathcal{L}_{k,2t-1})^{(n-i)}\mathcal{L}_{k,2ti+s}=\begin{cases}k^{-n}(\mathcal{F}_{k,2t})^n\delta^{\frac{n}{2}}\mathcal{L}_{k,n+s}, & \textit{if } n \textit{ is even;}\\ k^{-n}(\mathcal{F}_{k,2t})^n\delta^{\frac{n+1}{2}}\mathcal{F}_{k,n+s}, & \textit{if } n \textit{ is odd.}\end{cases}$$

Theorem 3.19. *For $n, s, t \geq 1$, we have*

1. $$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^2+3)^i\mathcal{F}_{k,2(n-i)+n}=\begin{cases}0, & \textit{if } n \textit{ is even;}\\ 2\delta^{\frac{n-1}{2}}, & \textit{if } n \textit{ is odd,}\end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^2+3)^i\mathcal{L}_{k,2(n-i)+n}=\begin{cases}2\delta^{\frac{n-1}{2}}, & \textit{if } n \textit{ is even;}\\ 0, & \textit{if } n \textit{ is odd.}\end{cases}$$

2. $$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^4+5k^2+5)^i\mathcal{F}_{k,4(n-i)+n}=\begin{cases}0, & \textit{if } n \textit{ is even;}\\ 2(k^2+2)^n\delta^{\frac{n-1}{2}}, & \textit{if } n \textit{ is odd,}\end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^4+5k^2+5)^i\mathcal{N}_{k,4(n-i)+n}=\begin{cases}2(k^2+2)^n\delta^{\frac{n-1}{2}}, & \textit{if } n \textit{ is even;}\\ 0, & \textit{if } n \textit{ is odd.}\end{cases}$$

3. $$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^6+7k^4+14k^2+7)^i\mathcal{F}_{k,6(n-i)+n}$$
$$=\begin{cases}0, & \textit{if } n \textit{ is even;}\\ 2(k^2+1)^n(k^2+3)^n\delta^{\frac{n-1}{2}}, & \textit{if } n \textit{ is odd,}\end{cases}$$
$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^6+7k^4+14k^2+7)^i\mathcal{L}_{k,6(n-i)+n}$$
$$=\begin{cases}2(k^2+1)^n(k^2+3)^n\delta^{\frac{n-1}{2}}, & \textit{if } n \textit{ is even;}\\ 0, & \textit{if } n \textit{ is odd.}\end{cases}$$

4. $$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^2+3)^i(k^6+6k^4+9k^2+3)^i\mathcal{F}_{k,8(n-i)+n}$$
$$=\begin{cases}0, & \textit{if } n \textit{ is even;}\\ 2(k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n-1}{2}}, & \textit{if } n \textit{ is odd,}\end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^2+3)^i(k^6+6k^4+9k^2+3)^i\mathcal{L}_{k,8(n-i)+n}$$
$$=\begin{cases}2(k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n-1}{2}}, & \textit{if } n \textit{ is even;}\\ 0, & \textit{if } n \textit{ is odd.}\end{cases}$$

5. $$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^{10}+11k^8+44k^6+44k^4+55k^2+11)^i\mathcal{F}_{k,10(n-i)+n}$$
$$=\begin{cases}0, & \textit{if } n \textit{ is even;}\\ 2(k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n-1}{2}}, & \textit{if } n \textit{ is odd,}\end{cases}$$
$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^{10}+11k^8+44k^6+44k^4+55k^2+11)^i\mathcal{L}_{k,10(n-i)+n}$$
$$=\begin{cases}2(k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n-1}{2}}, & \textit{if } n \textit{ is even;}\\ 0, & \textit{if } n \textit{ is odd.}\end{cases}$$

In general, for $n, s, t \geq 1$, we have

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}k^{-i}(L_{k,2t+1})^i\mathcal{F}_{k,2t(n-i)+n}=\begin{cases}0, & \textit{if } n \textit{ is even;}\\ 2(k)^{-n}(\mathcal{F}_{k,2t})^n\delta^{\frac{n-1}{2}}, & \textit{if } n \textit{ is odd,}\end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}k^{-i}(L_{k,2t+1})^i\mathcal{L}_{k,2t(n-i)+n}=\begin{cases}2(k)^{-n}(\mathcal{F}_{k,2t})^n\delta^{\frac{n-1}{2}}, & \textit{if } n \textit{ is even;}\\ 0, & \textit{if } n \textit{ is odd.}\end{cases}$$

In next section, we prove some elementary and binomial properties of k-Fibonacci and k-Lucas sequences.

The Proofs of the Main Results

Proof of Lemma(5.14): We prove only (a), (c) and (d) since the proofs of (b) and (e) are similar.

Proof of (a): Since r_1 and r_2 are roots of $r^2 - kr - 1 = 0$, then

$$r_1^2 = kr_1 + 1, \tag{3.2.5}$$

$$r_2^2 = kr_2 + 1. \tag{3.2.6}$$

This completes the proof of (a).

Proof of (c): From (b), we have

$$\begin{aligned}
u^{2n} &= \mathcal{F}_{k,n} u^{n+1} + u^n \mathcal{F}_{k,n-1} \\
&= \mathcal{F}_{k,n}(u\mathcal{F}_{k,n+1} + \mathcal{F}_{k,n}) + u^n \mathcal{F}_{k,n-1} \\
&= u\mathcal{F}_{k,n}\mathcal{F}_{k,n+1} + \mathcal{F}_{k,n-1}u^n + \mathcal{F}_{k,n}^2 \\
&= (u^n - \mathcal{F}_{k,n-1})\mathcal{F}_{k,n+1} + \mathcal{F}_{k,n-1}u^n + \mathcal{F}_{k,n}^2 \\
&= u^n(\mathcal{F}_{k,n+1} + \mathcal{F}_{k,n-1}) + \mathcal{F}_{k,n}^2 - \mathcal{F}_{k,n}F_{k,n-1}.
\end{aligned}$$

Using $\mathcal{F}_{k,n-1}\mathcal{F}_{k,n+1} - \mathcal{F}_{k,n}^2 = (-1)^n$ and $\mathcal{F}_{k,n+1} + \mathcal{F}_{k,n-1} = \mathcal{L}_{k,n}$, we obtain

$$u^{2n} = \mathcal{L}_{k,n}u^n - (-1)^n.$$

This completes the proof of (c).

Proof of (d): If $u = r_1$, then we have

$$\begin{aligned}\mathcal{F}_{k,tn}r_1^n - (-1)^n\mathcal{F}_{k,(t-1)n} &= (\frac{r_1^{tn} - r_2^{tn}}{r_1 - r_2})r_1^n - (r_1r_2)^n(\frac{r_1^{(t-1)n} - r_2^{(t-1)n}}{r_1 - r_2})\\ &= (\frac{r_1^n - r_2^n}{r_1 - r_2})r_1^{tn}\\ &= \mathcal{F}_{k,n}r_1^{tn}.\end{aligned}$$

This completes the proof of (d).

The proofs of Theorems (5.18), (5.23) are similar. Hence, we prove only Theorem (5.15).

Proof of Theorem(5.15): We prove only (a), since the proofs of (b), (c) and (d) are similar.

Proof of (1): From 5.14(b), we have

$$r_1^n = \mathcal{F}_{k,n}r_1 + \mathcal{F}_{k,n-1}, \tag{3.2.7}$$

$$r_2^n = \mathcal{F}_{k,n}r_2 + \mathcal{F}_{k,n-1}. \tag{3.2.8}$$

Multiplying (5.2.7) by r_1^t, (5.2.8) by r_2^t and subtracting, we obtain

$$\frac{r_1^{n+t} - r_2^{n+t}}{r_1 - r_2} = \mathcal{F}_{k,n}(\frac{r_1^{t+1} - r_2^{t+1}}{r_1 - r_2}) + \mathcal{F}_{k,n-1}(\frac{r_1^t - r_2^t}{r_1 - r_2}).$$

Hence, it gives that

$$\mathcal{F}_{k,n+t} = \mathcal{F}_{k,n}\mathcal{F}_{k,t+1} + \mathcal{F}_{k,n-1}\mathcal{F}_{k,t}.$$

This completes the proof of (a).

The proofs of Theorems (5.19)-(5.21) and (5.24) are similar. Hence, we prove only Theorem (5.16).

Proof of Theorem(5.16): We prove only (3), since the proofs of (1), (2) and (4)-(8) are similar.

*Proof of (3):*From 5.14(b), we have

$$r_1^r = \mathcal{F}_{k,r}r_1 + \mathcal{F}_{k,r-1}, \tag{3.2.9}$$

$$r_2^r = \mathcal{F}_{k,r}r_2 + \mathcal{F}_{k,r-1}. \tag{3.2.10}$$

Now, by the binomial theorem, we have

$$r_1^{rn} = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^{i} F_{k,r-1}^{n-i} r_1^{i}, \tag{3.2.11}$$

$$r_2^{rn} = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^{i} F_{k,r-1}^{n-i} r_2^{i}. \tag{3.2.12}$$

Now, by subtracting (5.2.11) from (5.2.12), we obtain

$$\frac{r_1^{rn+t} - r_2^{rn+t}}{r_1 - r_2} = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^{i} \mathcal{F}_{k,r-1}^{n-i} \left(\frac{r_1^{i+t} - r_2^{i+t}}{r_1 - r_2}\right).$$

Hence, it gives that

$$\mathcal{F}_{k,rn+t} = \sum_{i=0}^{n} \binom{n}{i} \mathcal{F}_{k,r}^{i} \mathcal{F}_{k,r-1}^{n-i} \mathcal{F}_{k,i+t}.$$

Now, by adding (5.2.11) and (5.2.12), we get

$$r_1^{rn+t} + r_2^{rn+t} = \sum_{i=0}^{n} \binom{n}{i} \mathcal{F}_{k,r}^{i} \mathcal{F}_{k,r-1}^{n-i} (r_1^{i+t} + r_2^{i+t}).$$

Hence, it gives that

$$\mathcal{L}_{k,rn+t} = \sum_{i=0}^{n} \binom{n}{i} \mathcal{F}_{k,r}^{i} \mathcal{F}_{k,r-1}^{n-i} \mathcal{L}_{k,i+t}.$$

This completes the proof of (3).

Proof of Lemma(5.17): We prove only (1) and (2) since the proofs of (3)-(11) are similar.

Proof of (1): Using (5.2.5) and (5.2.6), we have

$$\begin{aligned} u^3 &= u^2 u \\ &= (ku+1)u \\ &= ku^2 + u \\ &= k(ku+1) + u \end{aligned}$$

$$
\begin{aligned}
&= k^2u + k + u \\
&= k + (k^2 + 1)u.
\end{aligned}
$$

This completes the proof of (1).

Proof of (2): Using (5.2.5) and (5.2.6), we have

$$
\begin{aligned}
1 + ku + +u^6 &= u^2 + u^6 \\
&= u^2 + u^4(ku + 1) \\
&= u^2 + ku^5 + u^4 \\
&= u^2 + ku^3(ku + 1) + u^4 \\
&= u^2 + k^2u^4 + ku^3 + u^4 \\
&= (k^2 + 1)u^4 + ku^3 + u^2 \\
&= (k^2 + 1)u^4 + u^2(ku + 1) \\
&= (k^2 + 1)u^4 + u^4 \\
&= (k^2 + 2)u^4 \\
&= F_{k,2}u^4.
\end{aligned}
$$

This completes the proof of (2). The proofs of lemma (5.26) are similar. Hence, we prove only Lemma (5.25).

Proof of Lemma(5.25): We prove only (1) and (2) since the proofs of (3) - (5) are similar.

Proof of (1): Using $r_1 - r_2 = \sqrt{\delta}$, we have

$$\begin{aligned} r_1\sqrt{\delta} - 1 &= r_1(r_1 - r_2) - 1 \\ &= r_1^2 - r_1r_2 - 1 \\ &= r_1^2 + 1 - 1 \\ &= r_1^2. \end{aligned}$$

This completes the proof of (1).

Proof of (2): Using (5.2.5) and (5.2.6), we have

$$\begin{aligned} (k^2+2)r_1\sqrt{\delta} - (k^2+3) &= (k^2+2)r_1(r_1 - r_2) - (k^2+3) \\ &= (k^2+2)(r_1^2 - r_1r_2) - (k^2+3) \\ &= (k^2+2)(r_1^2+1) - (k^2+3) \\ &= r_1^2(k^2+2) + (k^2+2) - (k^2+3) \\ &= r_1^2k^2 + 2r_1^2 - 1 \\ &= r_1^2k^2 + 2(kr_1+1) - 1 \\ &= r_1^2k^2 + 2kr_1 + 1 \\ &= r_1^2k^2 + kr_1 + kr_1 + 1 \\ &= (kr_1+1)(kr_1+1) \\ &= r_1^2r_1^2 \end{aligned}$$

$$= r_1^4.$$

This completes the proof of (2).

The proofs of Theorems (5.27) and (5.29) are similar. Hence, we prove only Theorem (5.27).

Proof of Theorem(5.27): We prove only (2), since the proofs of (1) and (3)-(5) are similar.

*Proof of (2):*From 5.25(2), we have

$$r_1^4 + (k^2+3) = (k^2+2)r_1\sqrt{\delta}, \tag{3.2.13}$$

$$r_2^4 + (k^2+3) = -(k^2+2)r_2\sqrt{\delta}. \tag{3.2.14}$$

Multiplying (5.2.13) by r_1^s, (5.2.14) by r_2^s and subtracting, we obtain

$$\frac{r_1^{s+4} - r_2^{s+4}}{r_1 - r_2} + (k^2+3)\frac{r_1^s - r_2^s}{r_1 - r_2} = (k^2+2)(r_1^{s+1} + r_2^{s+1})$$

Hence, it gives that

$$\mathcal{F}_{k,s+4} + (k^2+3)\mathcal{F}_{k,s} = (k^2+2)\mathcal{L}_{k,s+1}.$$

Multiplying (5.2.13) by r_1^s, (5.2.14) by r_2^s and adding, we obtain

$$r_1^{s+4} + r_2^{s+4} + (k^2+3)(r_1^s + r_2^s) = (k^2+2)\delta(\frac{r_1^{s+1} - r_2^{s+1}}{r_1 - r_2})$$

Hence, it gives that

$$\mathcal{L}_{k,s+4} + (k^2+3)\mathcal{L}_{k,s} = (k^2+2)\delta\mathcal{F}_{k,s+1}.$$

This completes the proof of (3).

The proofs of Theorems (5.31) and (5.32) are similar. Hence, we prove only Theorem (5.31).

Proof of Theorem(5.31): We prove only (2), since the proofs of (1)and (3)-(5) are similar.

*Proof of (2):*From 5.25(2), we have

$$r_1^4 + (k^2+3) = (k^2+2)r_1\sqrt{\delta},$$

$$r_2^4 + (k^2+3) = -(k^2+2)r_2\sqrt{\delta}.$$

Now, by the binomial theorem, we have

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(r_1^{4i+s}) = (k^2+2)^n\delta^{\frac{n}{2}}(r_1^{n+s}), \qquad (3.2.15)$$

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(r_2^{4i+s}) = (-1)^n(k^2+2)^n\delta^{\frac{n}{2}}(r_2^{n+s}). \qquad (3.2.16)$$

Now, by subtracting (5.2.15) from (5.2.16), we obtain

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(\frac{r_1^{4i+s}-r_2^{4i+s}}{r_1-r_2}) = (k^2+2)^n\delta^{\frac{n}{2}}(\frac{r_1^{n+s}-(-1)^n r_2^{n+s}}{r_1-r_2}).$$

Hence, it gives that

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}\mathcal{F}_{k,4i+s} = \begin{cases}(k^2+2)^n\delta^{\frac{n}{2}}\mathcal{F}_{k,n+s}, & \text{if } n \text{ is even;}\\[2ex] (k^2+2)^n\delta^{\frac{n-1}{2}}\mathcal{L}_{k,n+s}, & \text{if } n \text{ is odd.}\end{cases}$$

Now, by adding (5.2.15) and (5.2.16), we get

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(r_1^{4i+s}+r_2^{4i+s}) = (k^2+2)^n\delta^{\frac{n}{2}}(\bar{r}_1 r_1^{n+s}+(-1)^n r_2^{n+s}).$$

Hence, it gives that

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}\mathcal{L}_{k,4i+s} = \begin{cases}(k^2+2)^n\delta^{\frac{n}{2}}\mathcal{L}_{k,n+s}, & \text{if } n \text{ is even;}\\[2ex] (k^2+2)^n\delta^{\frac{n+1}{2}}\mathcal{F}_{k,n+s}, & \text{if } n \text{ is odd.}\end{cases}$$

This completes the proof of (3).

In next section, we investigate certain congruence properties of k-Fibonacci and k-Lucas sequences.

3.2.2 Some Congruence Properties of the Generalized k-Lucas Sequence

Theorem 3.20. *For $n, t \geq 1$ and $\mathcal{D}_{k,n} = \mathcal{F}_{k,n}$ or $\mathcal{L}_{k,n}$, we have*

1. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,6j+t} \equiv 0 \pmod{L_{k,2}}$.

2. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,10j+t} \equiv 0 \pmod{L_{k,4}}$.

3. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,18j+t} \equiv 0 \pmod{L_{k,8}}$.

4. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,34j+t} \equiv 0 \pmod{L_{k,16}}$.

5. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,66j+t} \equiv 0 \pmod{L_{k,32}}$.

6. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,130j+t} \equiv 0 \pmod{L_{k,64}}$.

7. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,258j+t} \equiv 0 \pmod{L_{k,128}}$.

8. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,514j+t} \equiv 0 \pmod{L_{k,256}}$.

9. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,1026j+t} \equiv 0 \pmod{L_{k,512}}$.

10. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,2050j+t} \equiv 0 \pmod{L_{k,1024}}$.

In general, for $r, n, t \geq 1$, we have

$$\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,(2^{r+2}+2)j+t} \equiv 0 \pmod{L_{k,2^{r+1}}}.$$

Theorem 3.21. *For $n, t \geq 1$ and $\mathcal{D}_{k,n} = \mathcal{F}_{k,n}$ or $\mathcal{L}_{k,n}$, we have*

1. $\mathcal{D}_{k,6n+t} - \sum_{j=0}^{n} \binom{n}{j} k^j (-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,2}}$.
2. $\mathcal{D}_{k,10n+t} - \sum_{j=0}^{n} \binom{n}{j} k^j (-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,4}}$.
3. $\mathcal{D}_{k,18n+t} - \sum_{j=0}^{n} \binom{n}{j} k^j (-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,8}}$.
4. $\mathcal{D}_{k,34n+t} - \sum_{j=0}^{n} \binom{n}{j} k^j (-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,16}}$.
5. $\mathcal{D}_{k,66n+t} - \sum_{j=0}^{n} \binom{n}{j} k^j (-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,32}}$.
6. $\mathcal{D}_{k,130n+t} - \sum_{j=0}^{n} \binom{n}{j} k^j (-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,64}}$.
7. $\mathcal{D}_{k,258n+t} - \sum_{j=0}^{n} \binom{n}{j} k^j (-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,128}}$.
8. $\mathcal{D}_{k,514n+t} - \sum_{j=0}^{n} \binom{n}{j} k^j (-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,256}}$.
9. $\mathcal{D}_{k,1026n+t} - \sum_{j=0}^{n} \binom{n}{j} k^j (-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,514}}$.
10. $\mathcal{D}_{k,2050n+t} - \sum_{j=0}^{n} \binom{n}{j} k^j (-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,1024}}$.

In general, we have

$$\mathcal{D}_{k,(2^{r+2}+2)n+t} - \sum_{j=0}^{n} \binom{n}{j} k^j (-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,2^{r+1}}}.$$

The proofs of theorems (5.33) and (5.34) are similar. Hence, we prove only Theorem (5.33).

Proof of Theorem(5.33): We prove only (1), since the proofs of (2)-(10) are similar.

*Proof of (1):*From Theorem (5.19;(1)), For $n, t \geq 1$ and $\mathcal{D}_{k,n} = \mathcal{F}_{k,n}$ or $\mathcal{L}_{k,n}$, we have

$$\mathcal{D}_{k,n+t} = \sum_{i+j+s=n; i\neq 0} \binom{n}{i,j} k^{-n}(-1)^{j+s}\mathcal{L}_{k,2}{}^{i}\mathcal{D}_{k,4i+6j+t}$$

$$+ \sum_{i+j+s=n; i=0} \binom{n}{i,j} k^{-n}(-1)^{j+s}\mathcal{L}_{k,2}{}^{i}\mathcal{D}_{k,4i+6j+t},$$

$$= \sum_{i+j+s=n; i\neq 0} \binom{n}{i,j} k^{-n}(-1)^{j+s}\mathcal{L}_{k,2}{}^{i}\mathcal{D}_{k,4i+6j+t} + \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^{n}\mathcal{D}_{k,6j+t}.$$

$$\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^{n}\mathcal{D}_{k,6j+t} = \sum_{i+j+s=n; i\neq 0} \binom{n}{i,j} k^{-n}(-1)^{j+s}\mathcal{L}_{k,2}{}^{i}\mathcal{D}_{k,4i+6j+t},$$

$$\therefore \mathcal{L}_{k,2} \quad \text{divides} \quad (\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^{n}\mathcal{D}_{k,6j+t}),$$

$$\therefore \mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^{n}\mathcal{D}_{k,6j+t} \equiv 0 \pmod{\mathcal{L}_{k,2}}.$$

This completes the proof of (1).

3.3 Some Telescoping Series for k Fibonacci and k Lucas Sequences

In this section some telescoping series are obtained for k- Fibonacci and k- Lucas sequences.

Theorem 3.22. *For m, $n \geq 0$, we have*

$$\sum_{i=1}^{i=n}(-1)^{im}\frac{1}{F_{k,(i+1)m}F_{k,im}} = \frac{1}{2F_{k,m}}\sum_{i=1}^{i=n}\left[\frac{L_{k,mi}}{F_{k,mi}} - \frac{L_{k,m(i+1)}}{F_{k,m(i+1)}}\right]$$
$$= \frac{1}{2F_{k,m}}\left[\frac{L_{k,m}}{F_{k,m}} - \frac{L_{k,m(n+1)}}{F_{k,m(n+1)}}\right],$$
$$\sum_{i=0}^{i=n}(-1)^{im}\frac{1}{F_{k,(i+1)m}L_{k,im}} = \frac{F_{k,m(n+1)}}{2F_{k,m}L_{k,m(n+1)}},$$
$$\sum_{i=1}^{i=n}(-1)^{im}\frac{F_{k,(2i+1)m}}{F^2_{k,(i+1)m}F^2_{k,im}} = \frac{F_{k,m(n+1)}}{2F_{k,m}L_{k,m(n+1)}},$$
$$\sum_{i=0}^{i=n}(-1)^{im}\frac{F_{k,(2i+1)m}}{L^2_{k,(i+1)m}L^2_{k,im}} = \frac{F_{k,m(n+1)}}{2F_{k,m}L_{k,m(n+1)}},$$
$$\sum_{i=0}^{i=n}(-1)^{im}\frac{F^3_{k,(2i+1)m} + F_{k,(2i+1)m}F^2_{k,m}}{L^4_{k,(i+1)m}L^4_{k,im}} = \frac{F_{k,m(n+1)}}{2F_{k,m}L_{k,m(n+1)}}.$$

Proof. For m, $n \geq 0$, we have

$$\frac{F_{k,(2n+1)m}}{L_{k,(n+1)m}L_{k,nm}} = \frac{1}{2}\left[\frac{F_{k,(n+1)m}}{L_{k,(n+1)m}} + \frac{F_{k,nm}}{L_{k,nm}}\right],$$

$$\frac{F_{k,(2n+1)m}}{F_{k,(n+1)m}F_{k,nm}} = \frac{1}{2}\left[\frac{L_{k,(n+1)m}}{F_{k,(n+1)m}} + \frac{F_{k,nm}}{L_{k,nm}}\right],$$

$$\frac{(-1)^{mn}F_{k,m}}{F_{k,(n+1)m}F_{k,nm}} = \frac{1}{2}\left[\frac{L_{k,nm}}{F_{k,n)m}} - \frac{L_{k,(n+1)m}}{F_{k,(n+1)m}}\right],$$

$$\frac{(-1)^{mn}F_{k,m}}{L_{k,(n+1)m}L_{k,nm}} = \frac{1}{2}\left[\frac{F_{k,(n+1)m}}{L_{k,(n+1)m}} - \frac{F_{k,nm}}{L_{k,nm}}\right],$$

For first sum,

$$\sum_{i=1}^{i=n}(-1)^{im}\frac{1}{F_{k,(i+1)m}F_{k,im}} = \frac{1}{2F_{k,m}}\sum_{i=1}^{i=n}\left[\frac{L_{k,mi}}{F_{k,mi}} - \frac{L_{k,m(i+1)}}{F_{k,m(i+1)}}\right]$$
$$= \frac{1}{2F_{k,m}}\left[\frac{L_{k,m}}{F_{k,m}} - \frac{L_{k,m(n+1)}}{F_{k,m(n+1)}}\right],$$

$$\sum_{i=0}^{i=n}(-1)^{im}\frac{1}{F_{k,(i+1)m}L_{k,im}} = \frac{F_{k,m(n+1)}}{2F_{k,m}L_{k,m(n+1)}}.$$

For second sum,

$$(-1)^{mi}\frac{F_{k,(2i+1)m}F_{k,m}}{F^2_{k,(i+1)m}F^2_{k,im}} = \frac{1}{4}\left[\frac{L^2_{k,(i+1)m}}{F^2_{k,(i+1)m}} - \frac{L^2_{k,im}}{F^2_{k,im}}\right],$$

$$\sum_{i=1}^{i=n}(-1)^{im}\frac{F_{k,(2i+1)m}}{F^2_{k,(i+1)m}F^2_{k,im}} = \frac{F_{k,m(n+1)}}{2F_{k,m}L_{k,m(n+1)}}.$$

For third sum,

$$(-1)^{mi}\frac{F_{k,(2i+1)m}F_{k,m}}{L^2_{k,(i+1)m}L^2_{k,im}} = \frac{1}{4}\left[\frac{F^2_{k,(i+1)m}}{L^2_{k,(i+1)m}} - \frac{F^2_{k,im}}{L^2_{k,im}}\right].$$

For fourth sum,

$$\sum_{i=0}^{i=n}(-1)^{im}\frac{F_{k,(2i+1)m}}{L^2_{k,(i+1)m}L^2_{k,im}} = \frac{F_{k,m(n+1)}}{2F_{k,m}L_{k,m(n+1)}},$$

$$\frac{F^2_{k,(2i+1)m}+F^2_{k,m}}{L^2_{k,(i+1)m}L^2_{k,im}} = \frac{1}{4}\left[(\frac{F^2_{k,(i+1)m}}{L_{k,(i+1)m}}+\frac{F_{k,im}}{L_{k,im}})^2+(\frac{F_{k,(i+1)m}}{L_{k,(i+1)m}}-\frac{F_{k,im}}{L_{k,im}})^2\right]$$
$$= \left[\frac{F^2_{k,(i+1)m}}{L^2_{k,(i+1)m}}+\frac{F^2_{k,im}}{L^2_{k,im}}\right],$$

$$(-1)^{mi}\frac{F_{k,(2i+1)m}(F^2_{k,m}+F^2_{k,(i+1)m}}{L^4_{k,(i+1)m}L^4_{k,im}} = \frac{1}{8}\left[\frac{F^4_{k,(i+1)m}}{L^4_{k,(i+1)m}}-\frac{F^4_{k,im}}{L^4_{k,im}}\right],$$

$$\sum_{i=0}^{i=n}(-1)^{im}\frac{F^3_{k,(2i+1)m}+F_{k,(2i+1)m}F^2_{k,m}}{L^4_{k,(i+1)m}L^4_{k,im}} = \frac{F_{k,m(n+1)}}{2F_{k,m}L_{k,m(n+1)}}.$$

□

Lemma 3.23. *For variable k and non-negative integer r, we have*

$$\sum_{i\geq 0}\binom{r}{i}(1+k)^i = rk^{r-1}(1+k) \tag{3.3.1}$$

Lemma 3.24. *For positive integer m, n the solution of the simultaneous equations*

$$1+xr_1^m = yr_1^{-n},$$

$$1+xr_2^m = yr_2^{-m}$$

for the unknown x and y is

$$x = -\frac{F_{k,n}}{F_{k,m+n}}, \quad y = (-1)^n\frac{F_{k,m}}{F_{k,m+n}}.$$

Proof. We have

$$r_1^n + xr_2^{m+n} = y = r_2^n + xr_2^{m+n}.$$

Since, $m + n \neq 0$, we get

$$x = \frac{r_2^n - r_1^n}{r_1^{m+n} - r_2^{m+n}} = -\frac{F_{k,n}}{F_{k,m+n}}.$$

Similarly,

$$-r_1^{-m} + yr_2^{-(m+n)} = x = -r_2^{-m} + yr_2^{-(m+n)},$$

$$y = \frac{r_1^{-m} - r_2^{-m}}{r_1^{-(m+n)} - r_2^{-(m+n)}} = \frac{F_{k,-m}}{F_{k,-(m+n)}} = \frac{(-1)^{m+1}F_{k,m}}{(-1)^{m+n+1}F_{k,m+n}} = (-1)^{-n}\frac{F_{k,m}}{F_{k,m+n}}.$$

□

Theorem 3.25. *Let r be a non-negative integer and for positive integers p, q, we have*

$$\sum_{i\geq 0}\binom{r}{i} F_{k,p}^{i} F_{p+q}^{r-i} L_{k,qi} = (-1)^{q+1} F_{k,p} F_{k,q}^{r-1} L_{k,pr-(p+q)}.$$

Proof. Since $r_1 \cdot r_2 = -1$,

$$r_1^{-q} = (-1)^q r_2^q$$

It gives that

$$x_1 = xr_1^p = -\frac{F_{k,q}}{F_{k,p+q}} r_1^p$$
$$(1 + x_1) = yr_1^{-q} = (-1)^q yr_2^q = \frac{F_{k,p}}{F_{k,p+q}} r_2^q.$$

$$\sum_{i \geq 0} (-1)^{r-i} i \binom{r}{i} \left(\frac{F_{k,p}}{F_{k,p+q} r_2^q} \right)^i = r \left(-\frac{F_{k,q}}{F_{k,p+q} r_1^p} \right)^{r-1} \left(\frac{F_{k,p}}{F_{k,p+q} r_2^q} \right)$$
$$= (-1)^{q+r-1} r \frac{F_{k,p} F_{k,q}^{r-1}}{F_{k,p+q}^r r_1^{pr-(p+q)}}$$

Again using lemma (3.3) with

$$x_1 = -\frac{F_{k,q}}{F_{p+q}} r_2^p,$$
$$(1 + x_1) = \frac{F_{k,p}}{F_{p+q}} r_1^q.$$

It gives that

$$\sum_{i \geq 0} (-1)^{r-i} i \binom{r}{i} \frac{F_{k,p}^i}{F_{p+q}^i} r_1^{qi} = (-1)^{q+r-1} r \frac{F_{k,p} F_{k,q}^{r-1}}{F_{k,p+q}^r} r_2^{pr-(p+q)},$$

$$\sum_{i \geq 0} \binom{r}{i} F_{k,p}^i F_{p+q}^{r-i} L_{k,qi} = (-1)^{q+1} F_{k,p} F_{k,q}^{r-1} L_{k,pr-(p+q)}.$$

$\square$

Lemma 3.26. *For $n \geq 0$, we have*

$$k\sum_{j=0}^{n-1}\binom{2n-1-j}{j}(k^2+4)^{n-j-1}(-1)^j = F_{k,2n}$$

Theorem 3.27. *For $n \geq 0$, we have*

$$\sum_{i=0}^{\left[\frac{n}{2}\right]}\binom{n-i}{i}k^i F_{k,3i} = \frac{kF_{k,2n+1} - F_{k,2n} + (-k)^{n+2}F_{k,n} + (-k)^{n+1}F_{k,n-1}}{(2k^2-1)},$$

$$\sum_{i=0}^{\left[\frac{n}{2}\right]}\binom{n-i}{i}k^i L_{k,3i} = \frac{kL_{k,2n+1} - L_{k,2n} + (-k)^{n+2}L_{k,n} + (-k)^{n+1}L_{k,n-1}}{(2k^2-1)}.$$

Proof. Using lemma (3.3), proof is same as theorem (3.25). □

3.4 Sequences $F_{k,n}$ and $L_{k,n}$ as Continued Fractions

In this section, we obtained the relationship of the sequences $F_{k,n}$ and $L_{k,n}$ as continued fractions and some new properties for k- Fibonacci and k- Lucas sequences are established using series of fraction. Also, we derived the relationship of the sequences $F_{k,n}$ and $L_{k,n}$ as continued fractions. In general, a (simple) continued fraction is an expression of the form

$$[a_0, a_1, ..., a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \cfrac{1}{a_6 + ...}..........}}}}}.$$

The letters a_1, a_2, . . . denote positive integers and a letter a_0 denotes an integer. The expansion $\frac{F_{k,n+1}}{F_{k,n}}$ in continued fraction is written as

$$\frac{F_{k,n+1}}{F_{k,n}} = k + \cfrac{1}{k + \cfrac{1}{k + \cfrac{1}{k + \cfrac{1}{k + \cfrac{1}{k + \cfrac{1}{k + ...}..........}}}}}.$$

Here n denotes the number of quantities equal to k.

We knew that

$$F_{k,n}^2 - F_{k,n-1}F_{k,n+1} = (-1)^{n-1}.$$

Moreover, in general we have

$$\frac{F_{k,n+1}}{F_{k,n}} = r_1 \frac{1 - (\frac{r_2}{r_1})^{n+1}}{1 - (\frac{r_2}{r_1})^n}.$$

Let, r_1 denote the larger of the root, we have

$$\lim_{n\longrightarrow\infty}\frac{F_{k,n+1}}{F_{k,n}}=r_1.$$

More generally, we can write

$$\frac{F_{k,(n+1)t}}{F_{k,nt}}=L_{k,t}-\cfrac{(-1)^t}{L_{k,t}-\cfrac{(-1)^t}{L_{k,t}-\cfrac{(-1)^t}{L_{k,t}-\cfrac{(-1)^t}{L_{k,t}-\cfrac{(-1)^t}{L_{k,t}-\cfrac{(-1)^t}{L_{k,t}-...}..........}}}}}.$$

Here, n denotes the number of $L_{k,t}$'s. When n increases indefinitely, we have

$$\lim_{n\longrightarrow\infty}\frac{F_{k,(n+1)t}}{F_{k,nt}}=(r_1)^t.$$

The relation for $L_{k,n}$ is

$$\frac{L_{k,(n)t}}{F_{k,(n-)t}}=L_{k,t}-\cfrac{(-1)^t}{L_{k,t}-\cfrac{(-1)^t}{L_{k,t}-\cfrac{(-1)^t}{L_{k,t}-\cfrac{(-1)^t}{L_{k,t}-\cfrac{(-1)^t}{L_{k,t}-\cfrac{(-1)^t}{L_{k,t}-\cdots\cdots\cdots-\cfrac{(-1)^t}{(\frac{L_{k,t}}{2})}}}}}}}.$$

Here n denotes the number of quantities equal to $L_{k,t}$. We knew that

$$L_{k,n}^2 - L_{k,n-1}L_{k,n+1} = (-1)^n \triangle.$$

More generally, above equations can be modified as

$$F_{k,nt}^2 - F_{k,(n-1)t}F_{k,(n+1)t} = (-1)^{(n-1)t}(F_{k,t})^2,$$

$$L_{k,nt}^2 - L_{k,(n-1)t}L_{k,(n+1)t} = -(-1)^{(n-1)t}\triangle(F_{k,t})^2.$$

Moreover, we have

$$\Delta F_{k,nt}^2 = r_1^{2n+2t} + r_2^{2n+2t} - 2(-1)^{n+t},$$

$$\Delta L_{k,n}^2 = r_1^{2n} + r_2^{2n} - 2(-1)^n.$$

Again by subtracting these equations, we have

$$\Delta(F_{k,n+t}^2 - (-1)^t F_{k,n}^2) = (r_1^{2n+t} - r_2^{2n+t})(r_1^t + r_2^t)$$

and

$$F_{k,n+t}^2 - (-1)^t F_{k,n}^2 = F_{k,t}F_{k,2n+t}.$$

Similarly, we obtain

$$L_{k,n+t}^2 - (-1)^t L_{k,n}^2 = \Delta F_{k,t} F_{k,2n+t}.$$

3.5 Sequences $F_{k,n}$ and $L_{k,n}$ as a Series of Fractions:

In this section, we obtain the relationship of the sequences $F_{k,n}$ and $L_{k,n}$ as a series of fractions.

Theorem 3.28. *For* $n, k > 0$

$$\frac{F_{k,n+1}}{F_{k,n}} = \frac{F_{k,2}}{F_{k,1}} - \frac{(-1)}{F_{k,1}F_{k,2}} - \frac{(-1)^2}{F_{k,2}F_{k,3}} - \frac{(-1)^3}{F_{k,3}F_{k,4}} - \ldots\ldots - \frac{(-1)^{n-1}}{F_{k,n-1}F_{k,n}},$$
$$\frac{L_{k,n+1}}{L_{k,n}} = \frac{L_{k,2}}{L_{k,1}} - \frac{(-1)^2\triangle}{L_{k,2}L_{k,1}} - \frac{(-1)^3\triangle}{L_{k,2}L_{k,3}} - \frac{(-1)^4\triangle}{L_{k,3}L_{k,4}} - \ldots\ldots - \frac{(-1)^n}{L_{k,n-1}L_{k,n}}.$$

Proof. We can write expressions of $\dfrac{F_{k,n+1}}{F_{k,n}}$ and $\dfrac{L_{k,n+1}}{L_{k,n}}$ in series as

$$\frac{F_{k,n+1}}{F_{k,n}} = \frac{F_{k,2}}{F_{k,1}} + \left(\frac{F_{k,3}}{F_{k,2}} - \frac{F_{k,2}}{F_{k,1}}\right) + \left(\frac{F_{k,4}}{F_{k,3}} - \frac{F_{k,3}}{F_{k,2}}\right) + \ldots\ldots + \left(\frac{F_{k,n+1}}{F_{k,n}} - \frac{F_{k,n}}{F_{k,n-1}}\right)$$
$$= \frac{F_{k,2}}{F_{k,1}} - \frac{(F_{k,2}^2 - F_{k,3}F_{k,1})}{F_{k,1}F_{k,2}} - \frac{(F_{k,3}^2 - F_{k,2}F_{k,4})}{F_{k,2}F_{k,3}} - \ldots\ldots - \frac{(F_{k,n}^2 - F_{k,n+1}F_{k,n-1})}{F_{k,n-1}F_{k,n}},$$
$$\frac{L_{k,n+1}}{L_{k,n}} = \frac{L_{k,2}}{L_{k,1}} + \left(\frac{L_{k,3}}{L_{k,2}} - \frac{L_{k,2}}{L_{k,1}}\right) + \left(\frac{L_{k,4}}{L_{k,3}} - \frac{L_{k,3}}{L_{k,2}}\right) + \ldots\ldots + \left(\frac{L_{k,n+1}}{L_{k,n}} - \frac{L_{k,n}}{L_{k,n-1}}\right)$$
$$= \frac{L_{k,2}}{L_{k,1}} - \frac{(L_{k,2}^2 - L_{k,3}L_{k,1})}{L_{k,1}L_{k,2}} - \frac{(L_{k,3}^2 - L_{k,2}L_{k,4})}{L_{k,2}L_{k,3}} - \ldots\ldots - \frac{(L_{k,n}^2 - L_{k,n+1}L_{k,n-1})}{L_{k,n-1}L_{k,n}}.$$

Using the equations

$$F_{k,n}^2 - F_{k,n-1}F_{k,n+1} = (-1)^{n-1},$$

$$L_{k,n}^2 - L_{k,n-1}L_{k,n+1} = (-1)^n\triangle.$$

We get

$$\frac{F_{k,n+1}}{F_{k,n}} = \frac{F_{k,2}}{F_{k,1}} - \frac{(-1)}{F_{k,1}F_{k,2}} - \frac{(-1)^2}{F_{k,2}F_{k,3}} - \frac{(-1)^3}{F_{k,3}F_{k,4}} - \ldots\ldots - \frac{(-1)^{n-1}}{F_{k,n-1}F_{k,n}},$$

$$\frac{L_{k,n+1}}{L_{k,n}} = \frac{L_{k,2}}{L_{k,1}} - \frac{(-1)^2\triangle}{L_{k,2}L_{k,1}} - \frac{(-1)^3\triangle}{L_{k,2}L_{k,3}} - \frac{(-1)^4\triangle}{L_{k,3}L_{k,4}} - \ldots\ldots - \frac{(-1)^n}{L_{k,n-1}L_{k,n}}.$$

□

Remark 3.29. Taking limit as $n \to \infty$, we get

$$r_1 = \frac{1+\sqrt{k^2+4}}{2} = k + \frac{1}{1.k} - \frac{1}{k.(k^2+4)} + \ldots\ldots\ldots\ldots\ldots$$

For Fibonacci series

$$\frac{1+\sqrt{5}}{2} = 1 + \frac{1}{1.1} - \frac{1}{1.2} + \frac{1}{2.3} - \frac{1}{3.5} + \frac{1}{5.8} - \frac{1}{8.13} + \ldots\ldots\ldots\ldots$$

Now, we obtain more general relation for $F_{k,n}$ and $L_{k,n}$ as a series of fractions.

Theorem 3.30. *For $n, k > 0$, we have*

$$\frac{F_{k,(n+1)t}}{F_{k,nt}} = \frac{F_{k,2t}}{F_{k,t}} - \frac{(-1)^tF_{k,t}^2}{F_{k,t}F_{k,2t}} - \frac{(-1)^{2t}F_{k,t}^2}{F_{k,2t}F_{k,3t}} - \frac{(-1)^{3t}F_{k,t}^2}{F_{k,3t}F_{k,4t}} - \ldots\ldots - \frac{(-1)^{(n-1)t}F_{k,t}^2}{F_{k,(n-1)t}F_{k,nt}},$$

$$\frac{L_{k,(n+1)t}}{L_{k,nt}}=\frac{L_{k,t}}{L_{k,0}}+\frac{\triangle F_{k,t}^2}{L_{k,0}L_{k,t}}+\frac{(-1)^tF_{k,t}^2}{L_{k,t}L_{k,2t}}+\frac{(-1)^{2t}F_{k,t}^2}{L_{k,2t}L_{k,3t}}+......-\frac{(-1)^{(n-1)t}F_{k,t}^2}{L_{k,(n-1)t}L_{k,nt}}.$$

Proof. We can write expressions of $\frac{F_{k,(n+1)t}}{F_{k,nt}}$ and $\frac{L_{k,(n+1)t}}{L_{k,nt}}$ in series as

$$\frac{F_{k,(n+1)t}}{F_{k,nt}}=\frac{F_{k,2t}}{F_{k,t}}+\left(\frac{F_{k,3t}}{F_{k,2t}}-\frac{F_{k,2t}}{F_{k,t}}\right)+\left(\frac{F_{k,4t}}{F_{k,3t}}-\frac{F_{k,3t}}{F_{k,2t}}\right)+......$$
$$+\left(\frac{F_{k,(n+1)t}}{F_{k,nt}}-\frac{F_{k,nt}}{F_{k,(n-1)t}}\right)=\frac{F_{k,2t}}{F_{k,t}}-\frac{(F_{k,2t}^2-F_{k,3t}F_{k,t})}{F_{k,t}F_{k,2t}}-\frac{(F_{k,3t}^2-F_{k,2t}F_{k,4t})}{F_{k,2t}F_{k,3t}}-....$$
$$-\frac{(F_{k,nt}^2-F_{k,(n+1)t}F_{k,(n-1)t})}{F_{k,(n-1)t}F_{k,nt}},$$
$$\frac{L_{k,(n+1)t}}{L_{k,nt}}=\frac{L_{k,t}}{L_{k,0}}+\left(\frac{L_{k,2t}}{L_{k,t}}-\frac{L_{k,t}}{L_{k,0}}\right)+\left(\frac{L_{k,3t}}{L_{k,2t}}-\frac{L_{k,2t}}{L_{k,t}}\right)+......$$
$$+\left(\frac{L_{k,(n+1)t}}{L_{k,nt}}-\frac{L_{k,nt}}{L_{k,(n-1)t}}\right)=\frac{L_{k,t}}{L_{k,0}}-\frac{(L_{k,t}^2-L_{k,0}L_{k,2t})}{L_{k,0}L_{k,t}}-\frac{(L_{k,2t}^2-L_{k,t}L_{k,3t})}{L_{k,t}L_{k,2t}}-......$$
$$-\frac{(L_{k,nt}^2-L_{k,(n+1)t}L_{k,(n-1)t})}{L_{k,(n-1)t}L_{k,nt}}.$$

Using the equations

$$F_{k,nt}^2-F_{k,(n-1)t}F_{k,(n+1)t}=(-1)^{(n-1)t}(F_{k,t})^2,$$

$$L_{k,nt}^2-L_{k,(n-1)t}L_{k,(n+1)t}=-(-1)^{(n-1)t}\triangle(F_{k,t})^2.$$

We get

$$\frac{F_{k,(n+1)t}}{F_{k,nt}}=\frac{F_{k,2t}}{F_{k,t}}-\frac{(-1)^tF_{k,t}^2}{F_{k,t}F_{k,2t}}-\frac{(-1)^{2t}F_{k,t}^2}{F_{k,2t}F_{k,3t}}-\frac{(-1)^{3t}F_{k,t}^2}{F_{k,3t}F_{k,4t}}-......-\frac{(-1)^{(n-1)t}F_{k,t}^2}{F_{k,(n-1)t}F_{k,nt}},$$
$$\frac{L_{k,(n+1)t}}{L_{k,nt}}=\frac{L_{k,t}}{L_{k,0}}+\frac{\triangle F_{k,t}^2}{L_{k,0}L_{k,t}}+\frac{(-1)^tF_{k,t}^2}{L_{k,t}L_{k,2t}}+\frac{(-1)^{2t}F_{k,t}^2}{L_{k,2t}L_{k,3t}}+......-\frac{(-1)^{(n-1)t}F_{k,t}^2}{L_{k,(n-1)t}L_{k,nt}}.$$

□

Theorem 3.31. *For $n, m, k > 0$, we have*

$$\frac{F_{k,n+mt}}{L_{k,n+mt}} = \frac{F_{k,n}}{L_{k,n}} + 2(-1)^n F_{k,t}(\frac{1}{L_{k,n}L_{k,n+t}} + \frac{(-1)^t}{l_{k,n+t}L_{k,n+2t}} + \frac{(-1)^{2t}}{L_{k,n+2t}L_{k,n+3t}} + ...$$
$$+ \frac{(-1)^{(m-1)t}}{L_{k,n+(m-1)t}L_{k,n+mt}}),$$
$$\frac{F_{k,n+mt}}{L_{k,n+mt}} = \frac{F_{k,n}}{L_{k,n}} + 2(-1)^n F_{k,t}(\frac{1}{L_{k,n}L_{k,n+t}} + \frac{(-1)^t}{l_{k,n+t}L_{k,n+2t}} + \frac{(-1)^{2t}}{L_{k,n+2t}L_{k,n+3t}} + ...$$
$$+ \frac{(-1)^{(m-1)t}}{L_{k,n+(m-1)t}L_{k,n+mt}}).$$

Proof. We can write expressions of $\frac{F_{k,n+mt}}{L_{k,n+mt}}$ and $\frac{L_{k,n+mt}}{L_{k,n+mt}}$ in series as

$$\frac{F_{k,n+mt}}{L_{k,n+mt}} = \frac{F_{k,n}}{L_{k,n}} + \left(\frac{F_{k,n+t}}{L_{k,n+t}} - \frac{F_{k,n}}{L_{k,n}}\right) + \left(\frac{F_{k,n+2t}}{L_{k,n+2t}} - \frac{F_{k,n+t}}{L_{k,n+t}}\right) + ...$$
$$+ \left(\frac{F_{k,n+mt}}{L_{k,n+mt}} - \frac{F_{k,n+(m-1)t}}{L_{k,n+(m-1)t}}\right)$$
$$= \frac{F_{k,n}}{L_{k,n}} + \frac{(F_{k,n+t}L_{k,n} - F_{k,n}L_{k,n+t})}{L_{k,n}L_{k,n+t}} + \frac{(F_{k,n+2t}L_{k,n+t} - F_{k,n+t}L_{k,n+2t})}{L_{k,n+t}L_{k,n+2t}} + ...$$
$$+ \frac{(F_{k,n+mt}L_{k,n+(m-1)t} - F_{k,n+(m-1)t}L_{k,n+mt})}{L_{k,nt}L_{k,n+(m-1)t}},$$
$$\frac{L_{k,n+mt}}{F_{k,n+mt}} = \frac{L_{k,n}}{F_{k,n}} + \left(\frac{L_{k,n+t}}{F_{k,n+t}} - \frac{L_{k,n}}{F_{k,n}}\right) + \left(\frac{L_{k,n+2t}}{F_{k,n+2t}} - \frac{L_{k,n+t}}{F_{k,n+t}}\right) + ...$$
$$+ \left(\frac{L_{k,n+mt}}{F_{k,n+mt}} - \frac{L_{k,n+(m-1)t}}{F_{k,n+(m-1)t}}\right)$$
$$= \frac{L_{k,n}}{F_{k,n}} - \frac{(F_{k,n+t}L_{k,n} - F_{k,n}L_{k,n+t})}{L_{k,n}L_{k,n+t}} - \frac{(F_{k,n+2t}L_{k,n+t} - F_{k,n+t}L_{k,n+2t})}{L_{k,n+t}L_{k,n+2t}} + ...$$
$$\frac{(F_{k,n+mt}L_{k,n+(m-1)t} - F_{k,n+(m-1)t}L_{k,n+mt})}{L_{k,nt}L_{k,n+(m-1)t}}.$$

Using the equations

$$F^2_{k,nt} - F_{k,(n-1)t}F_{k,(n+1)t} = (-1)^{(n-1)t}(F_{k,t})^2,$$

$$L_{k,nt}^2 - L_{k,(n-1)t}L_{k,(n+1)t} = -(-1)^{(n-1)t}\triangle(F_{k,t})^2.$$

We get

$$\frac{F_{k,n+mt}}{L_{k,n+mt}} = \frac{F_{k,n}}{L_{k,n}} + 2(-1)^n F_{k,t}(\frac{1}{L_{k,n}L_{k,n+t}} + \frac{(-1)^t}{l_{k,n+t}L_{k,n+2t}} + \frac{(-1)^{2t}}{L_{k,n+2t}L_{k,n+3t}} + ...$$
$$+ \frac{(-1)^{(m-1)t}}{L_{k,n+(m-1)t}L_{k,n+mt}}),$$
$$\frac{F_{k,n+mt}}{L_{k,n+mt}} = \frac{F_{k,n}}{L_{k,n}} + 2(-1)^n F_{k,t}(\frac{1}{L_{k,n}L_{k,n+t}} + \frac{(-1)^t}{l_{k,n+t}L_{k,n+2t}} + \frac{(-1)^{2t}}{L_{k,n+2t}L_{k,n+3t}} + ...$$
$$+ \frac{(-1)^{(m-1)t}}{L_{k,n+(m-1)t}L_{k,n+mt}}).$$

□

3.6 Concluding Remarks

In this paper, we derived telescoping series for k- Fibonacci and k-Lucas sequences and proved their relationships with k- Fibonacci and k- Lucas sequences, same identities can be derived using M matrices. The relationship between k- Fibonacci and k- Lucas sequences using continued fractions and series of fractions derived is different and never tried in k- Fibonacci sequence literature.

Chapter 4

On the properties of k-Fibonacci and k-Lucas numbers

In this chapter, some properties of $k-$Fibonacci and $k-$Lucas sequences are derived and proved by using matrix methods. We defined matrices S, M, $M_k(n,m)$, $T_k(n)$, $S_k(n,m)$, A_n, E, Y_n, W_n, G_n and H_n for k-Fibonacci and k-Lucas sequences, using these matrices many interesting identities for $k-$Fibonacci and $k-$Lucas sequences are derived.

The content of this chapter is published in the following papers.
Determinantal Identities for k Lucas Sequence, Journal of New Theory, 19(2018), 01-19.
Properties of k- Fibonacci Sequence Using Matrix Method, MAYFEB Journal of Mathematics, 12(2016), 01-07.
Summation identities for k-Fibonacci and k-Lucas numbers using matrix methods, Int. J. Adv. Appl. Math. and Mech. 5(2016), 74-80.

4.1 Introduction

This chapter represents an interesting investigation about some special relations between matrices and k-Fibonacci sequences,k-Lucas sequences. This investigation is valuable to obtain new k-Fibonacci ,k-Lucas identities by different methods. This chapter contributes to k-Fibonacci,k-Lucas numbers literature, and encourage to investigate the properties of such number sequences. The most commonly used matrix in relation to the recurrence relation of k-Fibonacci sequence is

$$M = \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix}, \tag{4.1.1}$$

which for $k = 1$ reduces to the ordinary Q- matrix studied in literature of k-Fibonacci sequence. In next section, we define more general matrices $M_k(n,m), T_k(n), S_k(n,m)$ for Q-matrix. We use these matrices to develop various summation identities involving terms from the numbers $F_{k,n}$ and $L_{k,n}$. Several identities for $F_{k,n}$ and $L_{k,n}$ are proved by many authors using Binet forms, some of these are listed below

$$F_{k,n+1} + F_{k,n-1} = L_{k,n},$$

$$F_{k,n+1} + L_{k,n-1} = \Delta F_{k,n},$$

$$F_{k,2n} - 2(-1)^n = \Delta F_{k,n}^2,$$

$$F_{k,m+n} - (-1)^m L_{k,n-m} = F_{k,m} L_{k,n},$$

$$L_{k,m+n} - (-1)^m L_{k,n-m} = \Delta F_{k,m} F_{k,n},$$

$$F_{k,m+n} F_{k,n-m} - F_{k,n}^2 = (-1)^{n-m+1} F_{k,m}^2,$$

$$L_{k,m+n} L_{k,n-m} - L_{k,n}^2 = (-1)^{n-m} F_{k,m}^2,$$

$$F_{k,m+n} F_{k,r+m} - (-1)^m F_{k,n} F_{k,r} = F_{k,m} F_{k,n+r+m},$$

$$L_{k,mn} L_{k,n} + \Delta F_{k,mn} F_{k,n} = 2L_{k,(m+1)n},$$

$$F_{k,mn} L_{k,n} + L_{k,mn} F_{k,n} = 2F_{k,(m+1)n},$$

$$L_{k,m+n}^2 + (-1)^{m-1} L_{k,n}^2 = \Delta F_{k,2n+m} F_{k,m},$$

$$L_{k,m+n} L_{k,n} + (-1)^{m+1} L_{k,n-m} L_{k,n} = \Delta F_{k,2n} F_{k,m},$$

$$F_{k,m+2rn} F_{k,2n+m} + (-1)^{m+1} F_{k,2rn} F_{k,2n} = F_{k,2(r+1)n+m} F_{k,m}.$$

The matrix M is generalized using principle of mathematical induction as

$$M^n = \begin{bmatrix} F_{k,n+1} & F_{k,n} \\ F_{k,n} & F_{k,n-1} \end{bmatrix}, \quad \text{where } n \text{ is an integer.}$$

4.1.1 Properties of k-Fibonacci and k-Lucas sequences using matrices

Lemma 4.1. *If X is a square matrix with $X^2 = kX + I$, then $X^n = F_{k,n}X + F_{k,n-1}I$, for all $n \in Z$*

Proof. If $n = 0$ then result is obvious. If $n = 1$ then

$$\begin{aligned}(X)^1 &= F_{k,1}X + F_{k,0}I\\ &= 1X + 0I\\ &= X\end{aligned}$$

Hence result is true for $n = 1$. It can be shown by induction that, $X^n = F_{k,n}X + F_{k,n-1}I$, for all $n \in Z$. Assume that $X^n = F_{k,n}X + F_{k,n-1}I$ and prove that $X^{n+1} = F_{k,n+1}X + F_{k,n}I$.

Consider,

$$\begin{aligned}F_{k,n+1}X + F_{k,n}I &= (kF_{k,n} + F_{k,n-1})X + F_{k,n}I\\ &= (kX + I)F_{k,n} + XF_{k,n-1}\\ &= X^2F_{k,n} + XF_{k,n-1}\\ &= X(XF_{k,n} + F_{k,n-1})\\ &= X(X^n)\\ &= X^{n+1}\end{aligned}$$

Hence, $X^{n+1} = F_{k,n+1}X + F_{k,n}I$. By first principle of Induction,

$X^n = F_{k,n}X + F_{k,n-1}I$, for all $n \in Z$.

Next, we show that $X^{-(n)} = F_{k,-n}X + F_{k,-n-1}I$, for all $n \in Z^+$. Let

$Y = kI - X$, then $XY = -I$, i. e. $Y = -X^{-1}$, now consider

$$\begin{aligned}
Y^2 &= (kI - X)^2 \\
&= k^2I - 2kX + X^2 \\
&= k^2I - 2kX + kX + I \\
&= k^2I - kX + I \\
&= k(kI - X) + X + I \\
&= kY + I.
\end{aligned}$$

From first part, we obtain

$$Y^n = F_{k,n}Y + F_{k,n-1}I,$$

It gives that

$$\begin{aligned}
(-X^{-1})^n &= F_{k,n}(kI - X) + F_{k,n-1}I \\
(-1)^n X^{-n} &= -F_{k,n}X + F_{k,n+1}I.
\end{aligned}$$

Since, $F_{k,-n} = (-1)^{n+1}F_{k,n}$, $F_{k,-n-1} = (-1)^n F_{k,n+1}$, gives $X^{-(n)} =$

$F_{k,-n}X + F_{k,-n-1}I$, for all $n \in Z^+$.. □

Corollary 4.2. *Let,* $M = \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix}$, *then* $M^n = \begin{bmatrix} F_{k,n+1} & F_{k,n} \\ F_{k,n} & F_{k,n-1} \end{bmatrix}$.

Proof. Since $M^2 = kM + I$, therefore using lemma(4.1), we have

$$\begin{aligned} M^n &= F_{k,n}M + F_{k,n-1}I \\ &= \begin{bmatrix} kF_{k,n} & F_{k,n} \\ F_{k,n} & 0 \end{bmatrix} + \begin{bmatrix} F_{k,n-1} & 0 \\ 0 & F_{k,n-1} \end{bmatrix} \\ &= \begin{bmatrix} F_{k,n+1} & F_{k,n} \\ F_{k,n} & F_{k,n-1} \end{bmatrix}, \text{for all} \quad n \in Z. \end{aligned}$$

□

Corollary 4.3. *If,* $S = \begin{bmatrix} \frac{k}{2} & \frac{k^2+4}{2} \\ \frac{1}{2} & \frac{k}{2} \end{bmatrix}$, *then* $S^n = \begin{bmatrix} \frac{L_{k,n}}{2} & \frac{(k^2+4)F_{k,n}}{2} \\ \frac{F_{k,n}}{2} & \frac{L_{k,n}}{2} \end{bmatrix}$, *for every* $n \in Z$.

Proof. Since

$$S^2 = \begin{bmatrix} \frac{k^2+2}{2} & \frac{k(k^2+4)}{2} \\ \frac{k}{2} & \frac{k^2+2}{2} \end{bmatrix} = kS + I.$$

Using lemma (4.1), we have

$$S^n = F_{k,n}S + F_{k,n-1}I.$$

It gives that $S^n = \begin{bmatrix} \frac{L_{k,n}}{2} & \frac{(k^2+4)F_{k,n}}{2} \\ \frac{F_{k,n}}{2} & \frac{L_{k,n}}{2} \end{bmatrix}$, for every $n \in Z$. □

Lemma 4.4. *For all $n \in Z$, $L_{k,n}^2 - (k^2+4)F_{k,n}^2 = 4(-1)^n$.*

Proof. Since,

$$det(S) = -1,$$

$$det(S^n) = (-1)^n.$$

Moreover since,

$$S^n = \begin{bmatrix} \frac{L_{k,n}}{2} & \frac{(k^2+4)F_{k,n}}{2} \\ \frac{F_{k,n}}{2} & \frac{L_{k,n}}{2} \end{bmatrix}.$$

We get

$$det(S^n) = \frac{L_{k,n}^2}{4} - \frac{(k^2+4)F_{k,n}^2}{4}.$$

Thus it follows that $L_{k,n}^2 - (k^2+4)F_{k,n}^2 = 4(-1)^n$, for all $n \in Z$. □

Lemma 4.5. *For $n, m \in Z$*

$$2L_{k,n+m} = L_{k,n}L_{k,m} + (k^2+4)F_{k,n}F_{k,m}, \tag{4.1.2}$$

$$2F_{k,n+m} = F_{k,n}L_{k,m} + L_{k,n}F_{k,m}. \tag{4.1.3}$$

Proof. Since,

$S^{n+m} = S^n \cdot S^m$

$$= \begin{bmatrix} \dfrac{L_{k,n}}{2} & \dfrac{(k^2+4)F_{k,n}}{2} \\ \dfrac{F_{k,n}}{2} & \dfrac{L_{k,n}}{2} \end{bmatrix} . \begin{bmatrix} \dfrac{L_{k,m}}{2} & \dfrac{(k^2+4)F_{k,m}}{2} \\ \dfrac{F_{k,m}}{2} & \dfrac{L_{k,m}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{L_{k,n}L_{k,m}+(k^2+4)F_{k,n}F_{k,m}}{4} & \dfrac{(k^2+4)[L_{k,n}F_{k,m}+F_{k,n}L_{k,m}}{4} \\ \dfrac{L_{k,n}F_{k,m}+F_{k,n}L_{k,m}}{4} & \dfrac{L_{k,n}L_{k,m}+(k^2+4)F_{k,n}F_{k,m}}{4} \end{bmatrix}.$$

Also, we have

$$S^{n+m} = \begin{bmatrix} \dfrac{L_{k,n+m}}{2} & \dfrac{(k^2+4)F_{k,n+m}}{2} \\ \dfrac{F_{k,n+m}}{2} & \dfrac{L_{k,n+m}}{2} \end{bmatrix}.$$

It gives that

$$2L_{k,n+m} = L_{k,n}L_{k,m} + (k^2+4)F_{k,n}F_{k,m},$$

$$2F_{k,n+m} = F_{k,n}L_{k,m} + L_{k,n}F_{k,m}, \text{for all} \quad n, m \in Z.$$

□

Lemma 4.6. *For $n, m \in Z$*

$$2(-1)^m L_{k,n-m} = L_{k,n}L_{k,m} - (k^2+4)F_{k,n}F_{k,m}, \tag{4.1.4}$$

$$2(-1)^m F_{k,n-m} = F_{k,n}L_{k,m} - L_{k,n}F_{k,m}. \tag{4.1.5}$$

Proof. Since

$S^{n-m} = S^n.S^{-m}$

$$= S^n.(S^m)^{-1}$$

$$= S^n.(-1)^m \begin{bmatrix} \dfrac{L_{k,m}}{2} & \dfrac{-(k^2+4)F_{k,m}}{2} \\ \dfrac{-F_{k,m}}{2} & \dfrac{L_{k,m}}{2} \end{bmatrix}$$

$$= (-1)^m \begin{bmatrix} \dfrac{L_{k,n}}{2} & \dfrac{(k^2+4)F_{k,n}}{2} \\ \dfrac{F_{k,n}}{2} & \dfrac{L_{k,n}}{2} \end{bmatrix} \begin{bmatrix} \dfrac{L_{k,m}}{2} & \dfrac{-(k^2+4)F_{k,m}}{2} \\ \frac{-F_{k,m}}{2} & \frac{L_{k,m}}{2} \end{bmatrix}$$

$$= (-1)^m \begin{bmatrix} \dfrac{L_{k,n}L_{k,m} - (k^2+4)F_{k,n}F_{k,m}}{4} & \dfrac{(k^2+4)[L_{k,n}F_{k,m} - F_{k,n}L_{k,m}}{4} \\ \dfrac{L_{k,n}F_{k,m} - F_{k,n}L_{k,m}}{4} & \dfrac{L_{k,n}L_{k,m} - (k^2+4)F_{k,n}F_{k,m}}{4} \end{bmatrix}.$$

But, we have

$$S^{n-m} = \begin{bmatrix} \dfrac{L_{k,n-m}}{2} & \dfrac{(k^2+4)F_{k,n-m}}{2} \\ \dfrac{F_{k,n-m}}{2} & \dfrac{L_{k,n-m}}{2} \end{bmatrix}.$$

It gives that

$$2(-1)^m L_{k,n-m} = L_{k,n}L_{k,m} - (k^2+4)F_{k,n}F_{k,m},$$

$$2(-1)^m F_{k,n-m} = F_{k,n}L_{k,m} - L_{k,n}F_{k,m}, \text{for all} \quad n,m \in Z.$$

□

Lemma 4.7. *For all $n, m \in Z$*

$$(-1)^m L_{k,n-m} + L_{k,n+m} = L_{k,n}L_{k,m}, \tag{4.1.6}$$

$$(-1)^m F_{k,n-m} + F_{k,n+m} = F_{k,n}L_{k,m}. \tag{4.1.7}$$

Proof. using definition of the matrix S^n, it can be seen that

$$S^{n+m}+(-1)^m S^{n-m} = \begin{bmatrix} \frac{L_{k,n+m}+(-1)^m L_{k,n-m}}{2} & \frac{(k^2+4)[F_{k,n+m}+(-1)^m F_{k,n-m}}{2} \\ \frac{F_{k,n+m}+(-1)^m F_{k,n-m}}{2} & \frac{L_{k,n+m}+(-1)^m L_{k,n-m}}{2} \end{bmatrix}$$

On the other hand, we have

$$\begin{aligned} S^{n+m}+(-1)^m S^{n-m} &= S^n S^m + (-1)^m S^n S^{-m} \\ &= S^n[S^m + (-1)^m S^{-m}] \\ &= \begin{bmatrix} \frac{L_{k,n}}{2} & \frac{(k^2+4)F_{k,n}}{2} \\ \frac{F_{k,n}}{2} & \frac{L_{k,n}}{2} \end{bmatrix} \left\langle \begin{bmatrix} \frac{L_{k,m}}{2} & \frac{(k^2+4)F_{k,m}}{2} \\ \frac{F_{k,m}}{2} & \frac{L_{k,m}}{2} \end{bmatrix} \right. \\ &\left. + (-1)^m \begin{bmatrix} \frac{L_{k,m}}{2} & \frac{-(k^2+4)F_{k,m}}{2} \\ \frac{-F_{k,m}}{2} & \frac{L_{k,m}}{2} \end{bmatrix} \right\rangle \\ &= \begin{bmatrix} \frac{L_{k,n}}{2} & \frac{(k^2+4)F_{k,n}}{2} \\ \frac{F_{k,n}}{2} & \frac{L_{k,n}}{2} \end{bmatrix} . \begin{bmatrix} L_{k,m} & 0 \\ 0 & L_{k,m} \end{bmatrix} \\ &= \begin{bmatrix} \frac{L_{k,m}L_{k,n}}{2} & \frac{(k^2+4)F_{k,n}L_{k,m}}{2} \\ \frac{F_{k,n}L_{k,m}}{2} & \frac{L_{k,m}L_{k,n}}{2} \end{bmatrix}. \end{aligned}$$

It gives that

$$(-1)^m L_{k,n-m} + L_{k,n+m} = L_{k,n}L_{k,m},$$

$$(-1)^m F_{k,n-m} + F_{k,n+m} = F_{k,n}L_{k,m}, \text{for all} \quad n, m \in Z.$$

□

Lemma 4.8. *For all $x, y, z \in Z$*

$$8F_{k,x+y+z} = L_{k,x}L_{k,y}F_{k,z} + F_{k,x}L_{k,y}L_{k,z} + L_{k,x}F_{k,y}L_{k,z} + (k^2+4)F_{k,x}F_{k,y}F_{k,z}, \tag{4.1.8}$$

$$8L_{k,x+y+z} = L_{k,x}L_{k,y}L_{k,z} + (k^2+4)[L_{k,x}F_{k,y}F_{k,z} + F_{k,x}L_{k,y}F_{k,z} + F_{k,x}F_{k,y}L_{k,z}. \tag{4.1.9}$$

Proof. Using definition of the matrix S^n, it can be seen that

$$S^{x+y+z} = \begin{bmatrix} \frac{L_{k,x+y+z}}{2} & \frac{(k^2+4)F_{k,x+y+z}}{2} \\ \frac{F_{k,x+y+z}}{2} & \frac{L_{k,x+y+z}}{2} \end{bmatrix}.$$

On the other hand, we have

$$\begin{aligned} S^{x+y+z} &= S^{x+y}S^z \\ &= \begin{bmatrix} \frac{L_{k,x+y}}{2} & \frac{(k^2+4)F_{k,x+y}}{2} \\ \frac{F_{k,x+y}}{2} & \frac{L_{k,x+y}}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{L_{k,z}}{2} & \frac{(k^2+4)F_{k,z}}{2} \\ \frac{F_{k,z}}{2} & \frac{L_{k,z}}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{L_{k,x+y}L_{k,z} + (k^2+4)F_{k,x+y}F_{k,z}}{4} & \frac{(k^2+4)[L_{k,x+y}F_{k,z} + F_{k,x+y}L_{k,z}}{4} \\ \frac{L_{k,z}F_{k,x+y} + F_{k,z}L_{k,x+y}}{4} & \frac{L_{k,x+y}L_{k,z} + (k^2+4)F_{k,x+y}F_{k,z}}{4} \end{bmatrix} \end{aligned}$$

Using lemma(4.5),

$$2L_{k,x+y} = L_{k,x}L_{k,y} + (k^2+4)F_{k,x}F_{k,y}$$

$$2F_{k,x+y} = L_{k,y}F_{k,x} + (k^2+4)F_{k,y}L_{k,x}.$$

We get

$$8F_{k,x+y+z} = L_{k,x}L_{k,y}F_{k,z} + F_{k,x}L_{k,y}L_{k,z} + L_{k,x}F_{k,y}L_{k,z} + (k^2+4)F_{k,x}F_{k,y}F_{k,z},$$

$$8L_{k,x+y+z} = L_{k,x}L_{k,y}L_{k,z} + (k^2+4)[L_{k,x}F_{k,y}F_{k,z} + F_{k,x}L_{k,y}F_{k,z} + F_{k,x}F_{k,y}L_{k,z},$$

for all$x, y, z \in Z$.

□

Theorem 4.9. *For all $x, y, z \in Z$*

$$L_{k,x+y}^2 - (k^2+4)(-1)^{x+y+1}F_{k,z-x}L_{k,x+y}F_{k,y+z} - (k^2+4)(-1)^{x+z}F_{k,y+z}^2$$

$$= (-1)^{y+z}L_{k,z-x}^2.$$

Proof. Consider the matrix multiplication

$$\begin{bmatrix} \frac{L_{k,x}}{2} & \frac{(k^2+4)F_{k,x}}{2} \\ \frac{F_{k,z}}{2} & \frac{L_{k,z}}{2} \end{bmatrix} . \begin{bmatrix} L_{k,y} \\ \\ F_{k,y} \end{bmatrix} = \begin{bmatrix} L_{k,x+y} \\ \\ F_{k,y+z} \end{bmatrix}.$$

Also

$$det.\begin{bmatrix} \frac{L_{k,x}}{2} & \frac{(k^2+4)F_{k,x}}{2} \\ \frac{F_{k,z}}{2} & \frac{L_{k,z}}{2} \end{bmatrix} = \frac{L_{k,x}L_{k,z} - (k^2+4)F_{k,x}F_{k,z}}{4}$$

$$= \frac{(-1)^x L_{k,z-x}}{2}$$

$$= Q$$

$$\neq 0.$$

Hence, we can write

$$\begin{bmatrix} L_{k,y} \\ \\ F_{k,y} \end{bmatrix} = \begin{bmatrix} \frac{L_{k,x}}{2} & \frac{(k^2+4)F_{k,x}}{2} \\ \frac{F_{k,z}}{2} & \frac{L_{k,z}}{2} \end{bmatrix}^{-1} \cdot \begin{bmatrix} L_{k,x+y} \\ \\ F_{k,y+z} \end{bmatrix}$$
$$= \frac{1}{Q} \begin{bmatrix} \frac{L_{k,z}}{2} & \frac{-(k^2+4)F_{k,x}}{2} \\ \frac{-F_{k,z}}{2} & \frac{L_{k,x}}{2} \end{bmatrix} \cdot \begin{bmatrix} L_{k,x+y} \\ \\ F_{k,y+z} \end{bmatrix}.$$

It gives that

$$L_{k,y} = \frac{(-1)^x [L_{k,z} L_{k,x+y} - (k^2+4) F_{k,x} F_{k,y+z}]}{L_{k,z-x}},$$
$$F_{k,y} = \frac{(-1)^x [L_{k,x} F_{k,z+y} - F_{k,z} L_{k,y+x}]}{L_{k,z-x}}.$$

Since

$$L_{k,y}^2 - (k^2+4)F_{k,y}^2 = 4(-1)^y.$$

We get

$$[L_{k,z}L_{k,x+y} - (k^2+4)F_{k,x}F_{k,y+z}]^2 - (k^2+4)^2[L_{k,x}F_{k,z+y} - F_{k,z}L_{k,y+x}]^2 = 4(-1)^y L_{k,z-x}^2$$

Using lemmas(4.6, 4.7), we get

$$(L_{k,z}^2 L_{k,x+y}^2 - 2(k^2+4)L_{k,z}F_{k,x+y}F_{k,y+z} + (k^2+4)^2 F_{k,x}^2 F_{k,y+z}^2)$$

$$-(k^2+4)(L_{k,x}^2F_{k,y+z}^2-2L_{k,x}F_{k,z}F_{k,y+z}L_{k,x+y}+F_{k,z}^2L_{k,x+y}^2)=4(-1)^yL_{k,z-x}^2.$$

It gives that

$$L_{k,x+y}^2-(k^2+4)(-1)^{x+y+1}F_{k,z-x}L_{k,x+y}F_{k,y+z}-(k^2+4)(-1)^{x+z}F_{k,y+z}^2$$

$$=(-1)^{y+z}L_{k,z-x}^2,\text{for all} x,y,z\in Z.$$

□

Theorem 4.10. *For all $x, y, z \in Z$, $x \neq z$*

$$L_{k,x+y}^2-(-1)^{x+z}L_{k,z-x}L_{k,x+y}L_{k,y+z}+(-1)^{x+z}L_{k,y+z}^2=(-1)^{y+z+1}(k^2+4)F_{k,z-x}^2.$$

Proof. Consider matrix multiplication

$$\begin{bmatrix}\frac{L_{k,x}}{2} & \frac{(k^2+4)F_{k,x}}{2}\\ \frac{L_{k,z}}{2} & \frac{(k^2+4)F_{k,z}}{2}\end{bmatrix}.\begin{bmatrix}L_{k,y}\\ \\ F_{k,y}\end{bmatrix}=\begin{bmatrix}L_{k,x+y}\\ \\ L_{k,y+z}\end{bmatrix}.$$

Also, we have

$$det\begin{bmatrix}\frac{L_{k,x}}{2} & \frac{(k^2+4)F_{k,x}}{2}\\ \frac{L_{k,z}}{2} & \frac{(k^2+4)F_{k,z}}{2}\end{bmatrix}=\frac{(k^2+4)(-1)^xF_{k,z-x}}{2}$$

$$=P$$

$$\neq 0,(\text{if } x\neq z).$$

Therefore, for $x \neq z$, we can write

$$\begin{bmatrix} L_{k,y} \\ \\ F_{k,y} \end{bmatrix} = \begin{bmatrix} \dfrac{L_{k,x}}{2} & \dfrac{(k^2+4)F_{k,x}}{2} \\ \dfrac{L_{k,z}}{2} & \dfrac{(k^2+4)F_{k,z}}{2} \end{bmatrix}^{-1} . \begin{bmatrix} L_{k,x+y} \\ \\ L_{k,y+z} \end{bmatrix}$$
$$= \frac{1}{P} \begin{bmatrix} \dfrac{(k^2+4)F_{k,z}}{2} & \dfrac{-(k^2+4)F_{k,x}}{2} \\ \dfrac{-L_{k,z}}{2} & \dfrac{L_{k,x}}{2} \end{bmatrix} . \begin{bmatrix} L_{k,x+y} \\ \\ L_{k,y+z} \end{bmatrix}.$$

It gives that

$$L_{k,y} = \frac{(-1)^x [F_{k,z} L_{k,x+y} - F_{k,x} L_{k,y+z}]}{F_{k,z-x}},$$
$$F_{k,y} = \frac{(-1)^x [L_{k,x} L_{k,z+y} - L_{k,z} L_{k,y+x}]}{(k^2+4) F_{k,z-x}}.$$

Since

$$L_{k,y}^2 - (k^2+4) F_{k,y}^2 = 4(-1)^y.$$

We get

$$(k^2+4)[F_{k,z} L_{k,x+y} - F_{k,x} L_{k,y+z}]^2 - [L_{k,x} L_{k,z+y} - L_{k,z} L_{k,y+x}]^2$$
$$= 4(k^2+4)(-1)^y F_{k,z-x}^2,$$
$$L_{k,x+y}^2 - (-1)^{x+z} L_{k,z-x} L_{k,x+y} L_{k,y+z} + (-1)^{x+z} L_{k,y+z}^2$$
$$= (-1)^{y+z+1}(k^2+4) F_{k,z-x}^2, \text{for all} x, y, z \in Z, x \neq z.$$

□

Theorem 4.11. *For all $x, y, z \in Z$, $x \neq z$*

$$F^2_{k,x+y} - L_{k,x-z}F_{k,x+y}F_{k,y+z} + (-1)^{x+z}F^2_{k,y+z} = (-1)^{y+z}F^2_{k,z-x}.$$

Proof. Consider the matrix multiplication

$$\begin{bmatrix} \frac{F_{k,x}}{2} & \frac{F_{k,x}}{2} \\ \frac{F_{k,z}}{2} & \frac{L_{k,z}}{2} \end{bmatrix} \begin{bmatrix} L_{k,y} \\ \\ F_{k,y} \end{bmatrix} = \begin{bmatrix} F_{k,x+y} \\ \\ F_{k,y+z} \end{bmatrix}.$$

Also, we have

$$\begin{vmatrix} \frac{F_{k,x}}{2} & \frac{F_{k,x}}{2} \\ \frac{F_{k,z}}{2} & \frac{L_{k,z}}{2} \end{vmatrix} = \frac{(-1)^z F_{k,x-z}}{2}$$

$$= R$$

$$\neq 0, (if x \neq z).$$

Therefore, for $x \neq z$, we get

$$\begin{bmatrix} L_{k,y} \\ \\ F_{k,y} \end{bmatrix} = \begin{bmatrix} \frac{F_{k,x}}{2} & \frac{F_{k,x}}{2} \\ \frac{F_{k,z}}{2} & \frac{L_{k,z}}{2} \end{bmatrix}^{-1} . \begin{bmatrix} F_{k,x+y} \\ \\ F_{k,y+z} \end{bmatrix}$$

$$= \frac{1}{R} \begin{bmatrix} \frac{L_{k,z}}{2} & \frac{-L_{k,x}}{2} \\ \frac{-F_{k,z}}{2} & \frac{F_{k,x}}{2} \end{bmatrix} . \begin{bmatrix} F_{k,x+y} \\ \\ F_{k,y+z} \end{bmatrix}.$$

It gives that

$$L_{k,y} = \frac{(-1)^z[L_{k,z}F_{k,x+y} - L_{k,x}F_{k,y+z}]}{F_{k,x-z}},$$
$$F_{k,y} = \frac{(-1)^z[F_{k,x}F_{k,z+y} - F_{k,z}F_{k,y+x}]}{F_{k,x-z}}.$$

Now consider

$$[L_{k,z}F_{k,x+y} - L_{k,x}F_{k,y+z}]^2 - (k^2+4)[F_{k,x}F_{k,z+y} - F_{k,z}F_{k,y+x}]^2 = 4(-1)^y F^2_{k,x-z}.$$

Hence, we get

$$F^2_{k,x+y} - L_{k,x-z}F_{k,x+y}F_{k,y+z} + (-1)^{x+z}F^2_{k,y+z}$$
$$= (-1)^{y+z}F^2_{k,z-x}, \text{for all} x, y, z \in Z, x \neq z.$$

□

4.1.2 Summation Identities for k-Fibonacci and k-Lucas numbers using matrix methods

In this section, we define general matrices $M_k(n,m)$, $T_{k,n}$ and $S_k(n,m)$ for k-Fibonacci number. Using these matrices we find some new summation properties for k-Fibonacci and k-Lucas numbers.

4.1.2.1 The Matrix $M_k(n, m)$

First we give a generalization of the matrix M and use it to produce summation identities involving terms of the sequences $F_{k,n}$ and $L_{k,n}$.

Definition 4.12.

$$M_k(n,m) = \begin{bmatrix} F_{k,n+m} & (-1)^{m+1}F_{k,n} \\ \\ F_{k,n} & (-1)^{m+1}F_{k,n-m} \end{bmatrix}, \quad \text{where } m \text{ and } n \text{ are integers.} \tag{4.1.10}$$

Theorem 4.13. *Let $M_k(n, m)$ be a matrix as in (4.12) then*

$$M_k(n,m)^r = F_{k,m}^r \begin{bmatrix} F_{k,rn+m} & (-1)^{m+1}F_{k,rn} \\ \\ F_{k,rn} & (-1)^{m+1}F_{k,rn-m} \end{bmatrix}.$$

Proof. We use principle of Mathematical induction (P. M. I.). It is clear that the result is true for $r = 1$. Assume that the result is true for r. i. e.

$$M_k(n,m)^r = F_{k,m}^r \begin{bmatrix} F_{k,rn+m} & (-1)^{m+1}F_{k,rn} \\ \\ F_{k,rn} & (-1)^{m+1}F_{k,rn-m} \end{bmatrix}.$$

Now consider

$M_k(n, m)^{r+1} = M_k(n, m)^r M_k(n, m),$

$$= F_{k,m}^{r}\begin{bmatrix} F_{k,rn+m} & (-1)^{m+1}F_{k,rn} \\ F_{k,rn} & (-1)^{m+1}F_{k,rn-m}\end{bmatrix}\cdot\begin{bmatrix} F_{k,n+m} & (-1)^{m+1}F_{k,n} \\ F_{k,n} & (-1)^{m+1}F_{k,n-m}\end{bmatrix},$$

$$= F_{k,m}^{r}\begin{bmatrix} F_{k,rn+m}F_{k,n+m}+(-1)^{m+1}F_{k,rn}F_{k,n} & (-1)^{m+1}F_{k,rn+m}F_{k,n}+F_{k,rn}F_{k,n-m} \\ F_{k,rn}F_{k,n+m}+(-1)^{m+1}F_{k,rn-m}F_{k,n} & (-1)^{m+1}F_{k,rn}F_{k,n}+F_{k,rn-m}F_{k,n-m}\end{bmatrix}$$

$$= F_{k,m}^{r+1}\begin{bmatrix} F_{k,(r+1)n+m} & (-1)^{m+1}F_{k,(r+1)n} \\ F_{k,(r+1)n} & (-1)^{m+1}F_{k,(r+1)n-m}\end{bmatrix}.$$

Hence proof. □

We find that characteristic equation of $M_k(n,m)$ is $\lambda^2 - F_{k,m}F_{k,n}\lambda + (-1)^n F_{k,}^2$

and by Cauchy-Hamilton theorem $M_k(n,m)^2 - F_{k,m}F_{k,n}M_k(n,m) + (-1)^n F_{k,m}^2$

Multiplying both sides of equation (25) by $M_k(n,m)^t$ gives

$(F_{k,m}F_{k,n}M_k(n,m) - (-1)^n F_{k,m}^2 I)^r M_k(n,m)^t = M_k(n,m)^{2r+t}$,

and expanding gives $\sum_{i=0}^{i=r}\binom{r}{i}(-1)^{(r-1)(n+1)}F_{k,m}^{2r-1}F_{k,n}^{i}M_k(n,m)^{i+t} = M_k(n,m)^{2r+}$

Using (18) to equate upper left entries gives

$$\sum_{i=0}^{i=r}\binom{r}{i}(-1)^{(r-1)(n+1)}L_{k,n}^{i}F_{k,(i+t)n+m} = F_{k,(2r+t)n+m}.$$

In similar way, we can obtain

$$\sum_{i=0}^{i=r}\binom{r}{i}(-1)^{n(r-i)}F_{k,2in+m} = L_{k,n}^{r}F_{k,rn+m},$$

$$\sum_{i=0}^{i=2r}\binom{2r}{i}(-1)^{2nr-i(n-1)}F_{k,2in+m}=\Delta^r F_{k,n}^{2r}F_{k,2rn+m},$$
$$\sum_{i=0}^{i=2r+1}\binom{2r+1}{i}(-1)^{n(2r-i+1)+i+1}F_{k,2in+m}=\Delta^r F_{k,n}^{2r+1}L_{k,(2r+1)n+m},$$
$$\sum_{i=0}^{i=2r}\binom{2r}{i}(-1)^{i}2^{i}L_{k,n}^{2r-i}F_{k,in+m}=\Delta^r F_{k,n}^{2r}F_{k,m}.$$

4.1.2.2 The Matrix $T_{k,n}$

We now give another generalization of the matrix M and use it to produce summation identities involving terms of the sequences $F_{k,n}$ and $L_{k,n}$.

Definition 4.14.

$$T_{k,n}=\begin{bmatrix} L_{k,n} & F_{k,n} \\ \Delta F_{k,n} & L_{k,n}\end{bmatrix},\quad \text{where } n \text{ is an integer.} \tag{4.1.11}$$

Theorem 4.15. *Let $T_{k,n}$ be a matrix as in (4.14) then*

$$T_{k,n}^m=2^{m-1}\begin{bmatrix} L_{k,nm} & F_{k,nm} \\ \Delta F_{k,nm} & L_{k,nm}\end{bmatrix}. \tag{4.1.12}$$

Proof. : We use principle of Mathematical induction on m. It is clear that the result is true for $m=1$. Assume that the result is true for

m.

$$T_{k,n}^{m} = 2^{m-1}\begin{bmatrix} L_{k,nm} & F_{k,nm} \\ \\ \Delta F_{k,nm} & L_{k,nm} \end{bmatrix}.$$

Now consider

$$T_{k,n}^{m+1} = T_{k,n}^{m}T_{k,n} = 2^{m-1}\begin{bmatrix} L_{k,nm} & F_{k,nm} \\ \\ \Delta F_{k,nm} & L_{k,nm} \end{bmatrix} \cdot \begin{bmatrix} L_{k,n} & F_{k,n} \\ \\ \Delta F_{k,n} & L_{k,n} \end{bmatrix}$$

$$= 2^{m-1}\begin{bmatrix} L_{k,mn}L_{k,n} + \Delta F_{k,nm}F_{k,n} & L_{k,mn}F_{k,n} + F_{k,mn}L_{k,n} \\ \\ \Delta(L_{k,mn}F_{k,n}+ \\ F_{k,mn}L_{k,n}) & L_{k,mn}L_{k,n} + \Delta F_{k,nm}F_{k,n} \end{bmatrix}$$

$$= 2^{m}\begin{bmatrix} L_{k,n(m+1)} & F_{k,n(m+1)} \\ \\ \Delta F_{k,n(m+1)} & L_{k,n(m+1)} \end{bmatrix}.$$

Hence proof. □

The characteristic equation of $T_{k,n}$ is

$$\lambda^2 - 2L_{k,n}\lambda + 4(-1)^n = 0.$$

By Cauchy-Hamilton theorem

$$T_{k,n}^2 - 2L_{k,n}T_{k,n} + 4(-1)^n I = 0.$$

It gives that

$$T_{k,n}^{m} T_{k,t} = 2^{m} \begin{bmatrix} L_{k,nm+t} & F_{k,nm+t} \\ \\ \Delta F_{k,nm+t} & L_{k,nm+t} \end{bmatrix}.$$

Consider the case $n = 1$,

$$T_{k,1}^{m} = 2^{m-1}(F_{k,m} T_{k,1} + 2F_{k,m-1} I, \quad \text{where } m \geq 2. \tag{4.1.13}$$

It produces

$$\sum_{i=0}^{i=r} \binom{r}{i} F_{k,n-1}^{r-i} F_{k,n}^{i} F_{k,i+s+t} = F_{k,nr+s+t}. \tag{4.1.14}$$

The methods applied to $M_k(n, m)$ in previous section when applied to $T_{k,n}$ produce most of the summation identities that we have obtained so far.

4.1.2.3 The Matrix $S_k(n, m)$

We now give one more generalization of the matrix M and use it to produce summation identities involving terms of the sequences $F_{k,n}$ and $L_{k,n}$.

Definition 4.16.

$$S_k(n,m) = \begin{bmatrix} L_{k,n+m} & (-1)^{m+1}L_{k,n} \\ L_{k,n} & (-1)^{m+1}L_{k,n-m} \end{bmatrix}, \quad \text{where } n,\ m \text{ are integers.} \tag{4.1.15}$$

Theorem 4.17. *Let $S_k(n,m)$ be a matrix as in (4.1.15) then for all integer r*

$$S_k(n,m)^{2r} = F_{k,m}^{2r-1}\Delta^r \begin{bmatrix} F_{k,2rn+m} & (-1)^{m+1}F_{k,2rn} \\ F_{k,2rn} & (-1)^{m+1}F_{k,2rn-m} \end{bmatrix},$$

$$S_k(n,m)^{2r-1} = F_{k,m}^{2r-2}\Delta^{r-1} \begin{bmatrix} L_{k,2(r-1)n+m} & (-1)^{m+1}L_{k,2(r-1)n} \\ L_{k,2(r-1)n} & (-1)^{m+1}L_{k,2(r-1)n-m} \end{bmatrix}.$$

Proof. Using Principle of Mathematical induction, for $r = 1$.

$$S_k(n,m)^2 = \begin{bmatrix} L_{k,n+m} & (-1)^{m+1}L_{k,n} \\ L_{k,n} & (-1)^{m+1}L_{k,n-m} \end{bmatrix} \cdot \begin{bmatrix} L_{k,n+m} & (-1)^{m+1}L_{k,n} \\ L_{k,n} & (-1)^{m+1}L_{k,n-m} \end{bmatrix}$$

$$= \begin{bmatrix} L_{k,n+m}^2 + (-1)^{m+1}L_{k,n}^2 & (-1)^{m+1}L_{k,n+m}L_{k,n} + L_{k,n}L_{k,n-m} \\ L_{k,n+m}L_{k,n} + (-1)^{m+1}L_{k,n}L_{k,n-m} & (-1)^{m+1}L_{k,n+m}^2 + L_{k,n}^2 \end{bmatrix}$$

$$= F_{k,m}\Delta \begin{bmatrix} F_{k,2n+m} & (-1)^{m+1}F_{k,2n} \\ \\ F_{k,2n} & (-1)^{m+1}F_{k,2n-m} \end{bmatrix}.$$

The result is true for $r = 1$. Assume that the result is true for r (IH).

$$S_k(n,m)^{2r} = F_{k,m}^{2r-1}\Delta^r \begin{bmatrix} F_{k,2rn+m} & (-1)^{m+1}F_{k,2rn} \\ \\ F_{k,2rn} & (-1)^{m+1}F_{k,2rn-m} \end{bmatrix}.$$

Now consider

$$S_k(n,m)^{2r+2} = S_k(n,m)^{2r}S_k(n,m)^2$$

$$= F_{k,m}^{2r-1}\Delta^r \begin{bmatrix} F_{k,2rn+m} & (-1)^{m+1}F_{k,2rn} \\ \\ F_{k,2rn} & (-1)^{m+1}F_{k,2rn-m} \end{bmatrix} \cdot F_{k,m}\Delta \begin{bmatrix} F_{k,2n+m} & (-1)^{m+1}F_{k,2n} \\ \\ F_{k,2n} & (-1)^{m+1}F_{k,2n-m} \end{bmatrix}$$

$$= F_{k,m}^{2r}\Delta^{r+1} \cdot \begin{bmatrix} \begin{array}{c} F_{k,2rn+m}F_{k,2n+m} \\ +(-1)^{m+1}F_{k,2rn}F_{k,2n} \end{array} & \begin{array}{c} (-1)^{m+1}F_{k,2rn+m}F_{k,2n} \\ +F_{k,2rn}F_{k,2rn-m} \end{array} \\ \\ \begin{array}{c} F_{k,2rn}F_{k,2n+m} \\ +(-1)^{m+1}F_{k,2rn-m}F_{k,2n} \end{array} & \begin{array}{c} (-1)^{m+1}F_{k,2rn}F_{k,2n} \\ +F_{k,2rn-m}F_{k,2n-m} \end{array} \end{bmatrix}$$

$$= F_{k,m}^{2r+1}\Delta^{r+1} \begin{bmatrix} F_{k,2(r+1)n+m} & (-1)^{m+1}F_{k,2(r+1)n} \\ \\ F_{k,2(r+1)n} & (-1)^{m+1}F_{k,2(r+1)n-m} \end{bmatrix}.$$

Hence proof. $\square$

The characteristic equation of $S_k(n,m)$ is

$\lambda^2 - \Delta F_{k,n}F_{k,m}\lambda - \Delta(-1)^n F_{k,n}^2 = 0$ and by Cauchy-Hamilton theorem

$S_k(n,m)^2 - \Delta F_{k,n}F_{k,m}S_k(n,m) - \Delta(-1)^n F_{k,n}^2 I = 0.$

Manipulating above equation gives

$\Delta F_{k,m}(F_{k,n}S_k(n,m) + (-1)^n F_{k,m}I) = S_k(n,m)^2,$

$(2S_k(n,m) - \Delta F_{k,n}F_{k,m}I) = \Delta F_{k,m}^2 L_{k,n}^2.$

$\therefore \quad \Delta^r F_{k,m}^r (F_{k,n}S_k(n,m) + (-1)^n F_{k,m}I)^r = S_k(n,m)^{2r},$

$(2S_k(n,m) - \Delta F_{k,n}F_{k,m}I)^{2r} = \Delta^r F_{k,m}^{2r} L_{k,n}^{2r} I,$

$(2S_k(n,m) - \Delta F_{k,n}F_{k,m}I)^{2r+1} = \Delta^r F_{k,m}^{2r} L_{k,n}^{2r}(2S_k(n,m) - \Delta F_{k,n}F_{k,m}I).$

Now expanding previous four equations and equating upper left entries of the relevant matrices gives respectively to

$$\sum_{i=0,i-even}^{i=r} \binom{r}{i}(-1)^{n(r-i)+1}\Delta^{\frac{i-1}{2}}F_{k,n}^i L_{k,in+m} + \sum_{i=0,i-odd}^{i=r} \binom{r}{i}(-1)^{n(r-i)}\Delta^{\frac{i}{2}}F_{k,n}^i F_{k,in+m}$$

$$= F_{k,2nr+m}\sum_{i=0,i-even}^{i=2r}\binom{2r}{i}2^i\Delta^{\frac{2r-i}{2}}F_{k,n}^{2r-i}F_{k,in+m}$$

$$-\sum_{i=0,i-odd}^{i=2r-1}\binom{2r}{i}2^i\Delta^{\frac{2r-1-i}{2}}F_{k,n}^{2r-i}L_{k,in+m}$$

$$= L_{k,n}^{2r}F_{k,m}\sum_{i=0,i-odd}^{i=2r+1}\binom{2r+1}{i}2^i\Delta^{\frac{2r+3-i}{2}}F_{k,n}^{2r+1-i}F_{k,in+m}$$

$$-\sum_{i=0,i-even}^{i=2r-1}\binom{2r+1}{i}2^i\Delta^{\frac{2r+2-i}{2}}F_{k,n}^{2r+1-i}L_{k,in+m}$$

$$= \Delta L_{k,n}^{2r+1} F_{k,m}.$$

Theorem 4.18. *Let $S_k(n,m)$ be a matrix as in (4.1.15) then for all integer r*

$$S_k(n,m)^{2r} = F_{k,m}^{2r-1}\Delta^r \begin{bmatrix} F_{k,2rn+m} & (-1)^{m+1}F_{k,2rn} \\ F_{k,2rn} & (-1)^{m+1}F_{k,2rn-m} \end{bmatrix}$$

$$S_k(n,m)^{2r-1} = F_{k,m}^{2r-2}\Delta^{r-1} \begin{bmatrix} L_{k,2(r-1)n+m} & (-1)^{m+1}L_{k,2(r-1)n} \\ L_{k,2(r-1)n} & (-1)^{m+1}L_{k,2(r-1)n-m} \end{bmatrix}$$

Theorem 4.19. *For $n, m, r \geq 1$, we have*

$$\sum_{i=0,i-even}^{i=r} \binom{r}{i} (-1)^{n(r-i)+1} \Delta^{\frac{i-1}{2}} F_{k,n}^i L_{k,in+m} + \sum_{i=0,i-odd}^{i=r} \binom{r}{i} (-1)^{n(r-i)} \Delta^{\frac{i}{2}} F_{k,n}^i F_{k,in+m}$$

$$= F_{k,2nr+m},$$

$$\sum_{i=0,i-even}^{i=2r} \binom{2r}{i} 2^i \Delta^{\frac{2r-i}{2}} F_{k,n}^{2r-i} F_{k,in+m} - \sum_{i=0,i-odd}^{i=2r-1} \binom{2r}{i} 2^i \Delta^{\frac{2r-1-i}{2}} F_{k,n}^{2r-i} L_{k,in+m}$$

$$= L_{k,n}^{2r} F_{k,m},$$

$$\sum_{i=0,i-odd}^{i=2r+1} \binom{2r+1}{i} 2^i \Delta^{\frac{2r+3-i}{2}} F_{k,n}^{2r+1-i} F_{k,in+m}$$

$$- \sum_{i=0,i-even}^{i=2r-1} \binom{2r+1}{i} 2^i \Delta^{\frac{2r+2-i}{2}} F_{k,n}^{2r+1-i} L_{k,in+m} = \Delta L_{k,n}^{2r+1} F_{k,m}.$$

Definition 4.20. Define 3×3 matrix A as

$$A = \begin{bmatrix} k^2+1 & k^2+1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Definition 4.21. Define 3×3 matrix A_n as

$$A_n = \begin{bmatrix} \sum_{i=0}^{i=n+1} F_{k,i}^2 & F_{k,n}F_{k,n+2} & -\sum_{i=0}^{i=n} F_{k,i}^2 \\ \sum_{i=0}^{i=n} F_{k,i}^2 & F_{k,n-1}F_{k,n+1} & -\sum_{i=0}^{i=n-1} F_{k,i}^2 \\ \sum_{i=0}^{i=n-1} F_{k,i}^2 & F_{k,n-2}F_{k,n} & -\sum_{i=0}^{i=n-2} F_{k,i}^2 \end{bmatrix}.$$

where n, k are integers.

Theorem 4.22. *Let the matrices A and A_n have the form (4.20) and (4.21), respectively. Then for all integer $n > 1$, we have*

$$A^n = A_n.$$

Proof. Using principal of Mathematical induction on n. If $n = 2$, then

$$A^2 = \begin{bmatrix} k^4+3k^2+1 & k^2(k^2+2) & -k^2-1 \\ \\ k^2+1 & k^2+1 & -1 \\ \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=0}^{i=3} F_{k,i}^2 & F_{k,2}F_{k,4} & -\sum_{i=0}^{i=2} F_{k,i}^2 \\ \\ \sum_{i=0}^{i=2} F_{k,i}^2 & F_{k,1}F_{k,3} & -\sum_{i=0}^{i=1} F_{k,i}^2 \\ \\ \sum_{i=0}^{i=1} F_{k,i}^2 & F_{k,0}F_{k,2} & -\sum_{i=0}^{i=0} F_{k,i}^2 \end{bmatrix}$$

$$= A_2.$$

Hence result is true for $n = 2$, now suppose that the result is true for $n, (n > 2)$.

$$A^n = A_n.$$

Now consider

$$A^{n+1} = A^n A$$

$$= \begin{bmatrix} \sum_{i=0}^{i=n+1} F_{k,i}^2 & F_{k,n}F_{k,n+2} & -\sum_{i=0}^{i=n} F_{k,i}^2 \\ \sum_{i=0}^{i=n} F_{k,i}^2 & F_{k,n-1}F_{k,n+1} & -\sum_{i=0}^{i=n-1} F_{k,i}^2 \\ \sum_{i=0}^{i=n-1} F_{k,i}^2 & F_{k,n-2}F_{k,n} & -\sum_{i=0}^{i=n-2} F_{k,i}^2 \end{bmatrix} \begin{bmatrix} k^2+1 & k^2+1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (k^2+1)\sum_{i=0}^{i=n+1} F_{k,i}^2 + F_{k,n}F_{k,n+2} & (k^+1)\sum_{i=0}^{i=n+1} F_{k,i}^2 - \sum_{i=0}^{i=n} F_{k,i}^2 & - \\ (k^2+1)\sum_{i=0}^{i=n} F_{k,i}^2 + F_{k,n-1}F_{k,n+1} & (k^+1)\sum_{i=0}^{i=n} F_{k,i}^2 - \sum_{i=0}^{i=n-1} F_{k,i}^2 & - \\ (k^2+1)\sum_{i=0}^{i=n-1} F_{k,i}^2 + F_{k,n-2}F_{k,n} & (k^+1)\sum_{i=0}^{i=n-1} F_{k,i}^2 - \sum_{i=0}^{i=n-2} F_{k,i}^2 & - \end{bmatrix}$$

Using

$$(k^2+1)F_{k,n}F_{k,n+1} + kF_{k,n-1}F_{k,n+1} = k\sum_{i=0}^{i=n+1} F_{k,i}^2,$$

$$(k^2+1)F_{k,n}F_{k,n+1} - F_{k,n-1}F_{k,n} = kF_{k,n}F_{k,n+2}.$$

$$A^{n+1} = \begin{bmatrix} \sum_{i=0}^{i=n+2} F_{k,i}^2 & F_{k,n+1}F_{k,n+3} & -\sum_{i=0}^{i=n+1} F_{k,i}^2 \\ \sum_{i=0}^{i=n+1} F_{k,i}^2 & F_{k,n}F_{k,n+2} & -\sum_{i=0}^{i=n} F_{k,i}^2 \\ \sum_{i=0}^{i=n} F_{k,i}^2 & F_{k,n-1}F_{k,n+1} & -\sum_{i=0}^{i=n-1} F_{k,i}^2 \end{bmatrix}$$

$$= A_{n+1}.$$

$\square$

Let the matrix A have the form (4.20) then

$$A^n = \begin{bmatrix} \frac{F_{k,n+1}F_{k,n+2}}{k} & F_{k,n}F_{k,n+2} & -\frac{F_{k,n}F_{k,n+1}}{k} \\ \frac{F_{k,n}F_{k,n+1}}{k} & F_{k,n-1}F_{k,n+1} & -\frac{F_{k,n-1}F_{k,n}}{k} \\ \frac{F_{k,n-1}F_{k,n}}{k} & F_{k,n-2}F_{k,n} & -\frac{F_{k,n-2}F_{k,n-1}}{k} \end{bmatrix}.$$

With $det(A) = -1$. The characteristic equation of the matrix A is

$$\lambda^3 + (-k^2 - 1)\lambda^2 + (-k^2 - 1)\lambda + 1 = 0.$$

Thus the eigenvalues of the matrix A are

$$\begin{aligned} \lambda_1 &= \frac{1}{2}k^2 + \frac{1}{2}k\sqrt{\Delta} + 1, \\ \lambda_2 &= \frac{11}{2}k^2 - \frac{1}{2}k\sqrt{\Delta} + 1, \\ \lambda_3 &= -1. \end{aligned}$$

We can easily obtain

$$\begin{aligned} \lambda_1 &= r_1^2, \\ \lambda_2 &= r_2^2. \end{aligned}$$

Definition 4.23. Define the 3×3 matrix B as follows

$$B = \begin{bmatrix} \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ 1 & 1 & 1 \end{bmatrix},$$

$$detB = k(\Delta)^{\frac{3}{2}}.$$

Let

$$C = B^T = \begin{bmatrix} \lambda_1^2 & \lambda_1 & 1 \\ \lambda_2^2 & \lambda_2 & 1 \\ \lambda_3^2 & \lambda_3 & 1 \end{bmatrix}$$

and

$$D_i = \begin{bmatrix} \lambda_1^{n-i+3} \\ \lambda_2^{n-i+3} \\ \lambda_3^{n-i+3} \end{bmatrix}.$$

Theorem 4.24. *Let the matrix $A_n = (a_{i,j})$ have the form (4.21). Then for all i, j such that $1 \leq i, j \leq 3$, we have*

$$a_{i,j} = \frac{det(C_j^{(i)})}{det(C)}.$$

Definition 4.25. Define 4×4 matrix E as follows

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & k^2+1 & k^2+1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

where n, k are integers.

Theorem 4.26. *Let the matrix E as in (4.25), then for $n > 2$*

$$E^n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{k}\sum_{i=0}^{i=n} F_{k,i}F_{k,i+1} & \frac{F_{k,n+1}F_{k,n+2}}{k} & F_{k,n}F_{k,n+2} & -\frac{F_{k,n}F_{k,n+1}}{k} \\ \frac{1}{k}\sum_{i=0}^{i=n-1} F_{k,i}F_{k,i+1} & \frac{F_{k,n}F_{k,n+1}}{k} & F_{k,n-1}F_{k,n+1} & -\frac{F_{k,n-1}F_{k,n}}{k} \\ \frac{1}{k}\sum_{i=0}^{i=n-2} F_{k,i}F_{k,i+1} & \frac{F_{k,n-1}F_{k,n}}{k} & F_{k,n-2}F_{k,n} & -\frac{F_{k,n-2}F_{k,n-1}}{k} \end{bmatrix}.$$

Proof. Theorem can easily proved by using Mathematical induction on n. □

Theorem 4.27. *For $n > 0$, we have*

$$\sum_{i=0}^{i=n} F_{k,i}F_{k,i+1} = \frac{F_{k,n+1}^2 + F_{k,n}F_{k,n+2} - 1}{2k^2}. \tag{4.1.16}$$

Definition 4.28. Define 4×4 matrix

$$Y_n = \begin{bmatrix} F_{k,n}^2 & -F_{k,n}F_{k,n+3} & F_{k,n+3}^2 & F_{k,n}F_{k,n+3} \\ -F_{k,n}F_{k,n+3} & F_{k,n+3}^2 & F_{k,n+3}F_{k,n} & F_{k,n}^2 \\ F_{k,n+3}^2 & F_{k,n}F_{k,n+3} & F_{k,n}^2 & -F_{k,n}F_{k,n+3} \\ F_{k,n}F_{k,n+3} & F_{k,n}^2 & -F_{k,n}F_{k,n+3} & F_{k,n+3}^2 \end{bmatrix}$$

Where n, k are integers.

Theorem 4.29. *For any positive integer n, we have*

$$det(Y_n) = -\left[F_{k,n}^2 + F_{k,n+3}^2\right]^4.$$

Proof. Let

$$x = F_{k,n},$$

$$y = F_{k,n+3}.$$

It gives that

$$det(Y_n) = \begin{vmatrix} x^2 & -xy & y^2 & xy \\ -xy & y^2 & xy & x^2 \\ y^2 & xy & x^2 & -xy \\ xy & x^2 & -xy & y^2 \end{vmatrix}.$$

By interchanging C_2 and C_4, C_3 and C_4, R_3 and R_4, changing the sign of C_3 and R_4, we get

$$det(Y_n) = -\begin{vmatrix} x^2 & xy & xy & y^2 \\ -xy & x^2 & -y^2 & xy \\ xy & y^2 & -x^2 & -xy \\ -y^2 & xy & xy & -x^2 \end{vmatrix}$$

$$= -\left(x^2 + y^2\right)^4.$$

Finally theorem follows

$$det(Y_n) = -\left[F_{k,n}^2 + F_{k,n+3}^2\right]^4.$$

□

Definition 4.30. Define 5×5 matrix

$$W_n = \begin{bmatrix} F_{k,n}F_{k,n+2} + k^2F^2_{k,n+1} & F^2_{k,n} & k^2F^2_{k,n+1} & F^2_{k,n+2} & \begin{matrix} -(F_{k,n} \\ +k^2F^2_{k} \end{matrix} \\ F_{k,n}F_{k,n+2} + k^2F^2_{k,n+1} & F_{k,n}(kF_{k,n+1} + F_{k,n+2} & -kF_{k,n+1}L_{k,n+1} & \begin{matrix} (kF_{k,n+1} \\ -F_{k,n})F_{k,n+2} \end{matrix} & \begin{matrix} F_{k,n}F \\ +k^2F \end{matrix} \\ 0 & 2kF_{k,n+1}F_{k,n+2} & 2F_{k,n}F_{k,n+2} & -2kF_{k,n}F_{k,n+1} & 0 \\ F_{k,n}F_{k,n+2} + k^2F^2_{k,n+1} & -F_{k,n}(kF_{k,n+1} + F_{k,n+2}) & kF_{k,n+1}L_{k,n+1} & \begin{matrix} -F_{k,n+2}(kF_{k,n+1} \\ -F_{k,n}) \end{matrix} & \begin{matrix} F_{k,n}F \\ +k^2F \end{matrix} \\ -(F_{k,n}F_{k,n+2} + k^2F^2_{k,n+1}) & F^2_{k,n} & k^2F^2_{k,n+1} & F^2_{k,n+2} & \begin{matrix} F_{k,n}F \\ +k^2F \end{matrix} \end{bmatrix}$$

where n, k are integers.

Theorem 4.31. *Let W_n be a 5×5 matrix as in (4.30), then*

$$det(W_n) = -32\left[k^2F^2_{k,n+1} + F_{k,n}(kF_{k,n+1} + F_{k,n})\right]^5. \tag{4.1.17}$$

Proof. Put

$$F_{k,n} = x,$$

$$kF_{k,n+1} = y,$$

$$F_{k,n+2} = z.$$

Therefore

$$det(W_n) = \begin{vmatrix} xz+y^2 & x^2 & y^2 & z^2 & -(xz+k^2y^2) \\ xz+y^2 & x(y+z) & -y(x+z) & (y-x)z & xz+y^2 \\ 0 & 2yz & 2xz & -2xy & 0 \\ xz+y^2 & -x(y+z) & y(x+z) & -z(y-x) & xz+y^2 \\ -(xz+y^2) & x^2 & y^2 & z^2 & xz+y^2 \end{vmatrix}$$

$$= (xz+y^2)^2 \begin{vmatrix} 1 & x^2 & y^2 & z^2 & -1 \\ 1 & x(y+z) & -y(x+z) & (y-x)z & 1 \\ 0 & 2yz & 2xz & -2xy & 0 \\ 1 & -x(y+z) & y(x+z) & -z(y-x) & 1 \\ -1 & x^2 & y^2 & z^2 & 1 \end{vmatrix}.$$

Making the row-column transformations $(C_5+C_1 \to C_1)$, $(R_2+R_4 \to R_4)$ and $(-R_1+R_5 \to R_5)$ yields

$$det(W_n) = \left(xz+y^2\right)^2 \begin{vmatrix} 0 & x^2 & y^2 & z^2 & -1 \\ 2 & x(y+z) & -y(x+z) & (y-x)z & 1 \\ 0 & 2yz & 2xz & -2xy & 0 \\ 4 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

$$= -16(xz+y^2)^2 \begin{vmatrix} x^2 & y^2 & z^2 \\ x(y+z) & -y(x+z) & (y-x)z \\ yz & xz & -xy \end{vmatrix}.$$

Using $z = y + x$, we have

$$det(W_n) = -16(xz+y^2)^2 \begin{vmatrix} x^2 & y^2 & (x+y)^2 \\ x^2+2xy & -y^2-2xy & (y^2-x^2)z \\ y^2+xy & x^2+xy & -xy \end{vmatrix}.$$

Making the row transformation $(R_1 + R_3 \rightarrow R_3)$, we obtain

$$det(W_n) = -16(x^2+y^2+xy)^3 \begin{vmatrix} x^2 & y^2 & (x+y)^2 \\ x^2+2xy & -y^2-2xy & (y^2-x^2)z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= -32(x^2+y^2+xy)^5.$$

Finally theorem follows

$$det(W_n) = -32\left[k^2F_{k,n+1}^2 + F_{k,n}(kF_{k,n+1}+F_{k,n})\right]^5.$$

$\square$

Definition 4.32. Define $n \times n$ super-diagonal matrix G_n as

$$G_n = \begin{bmatrix} k^2+1 & k^2+1 & -1 & 0 & 0 & \dots\dots\dots & 0 \\ -1 & k^2+1 & k^2+1 & -1 & 0 & \dots\dots\dots & 0 \\ 0 & -1 & k^2+1 & k^2+1 & -1 & \dots\dots\dots & 0 \\ \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots \\ 0 & \dots\dots\dots & \dots\dots\dots & 0 & -1 & k^2+1 & k^2+1 \\ 0 & \dots\dots\dots & \dots\dots\dots & 0 & 0 & -1 & k^2+1 \end{bmatrix}_{n\times n}.$$

where n, k are integers.

Theorem 4.33. *For $n > 1$, we have*

$$det(G_n) = \sum_{i=0}^{i=n+1} F_{k,i}^2.$$

where

$$det(G_1) = \sum_{i=0}^{i=2} F_{k,i}^2 = k^2 + 1$$

Proof. Using the Laplace expansion of a determinant gives

$$\begin{aligned}
det(G_n) &= (k^2+1)det(G_n-1) + (k^2+1)det(G_n-2) - det(G_n-3),\\
det(G_1) &= \sum_{i=0}^{i=2} F_{k,i}^2,\\
det(G_2) &= \sum_{i=0}^{i=3} F_{k,i}^2,\\
det(G_3) &= \sum_{i=0}^{i=4} F_{k,i}^2.
\end{aligned}$$

These equations gives

$$\begin{aligned}
(k^2+1)\sum_{i=0}^{i=n+1} F_{k,i}^2 + (k^2+1)\sum_{i=0}^{i=n} F_{k,i}^2 - \sum_{i=0}^{i=n-1} F_{k,i}^2 &= \sum_{i=0}^{i=n+2} F_{k,i}^2,\\
det(G_n) = \sum_{i=0}^{i=n+1} F_{k,i}^2.
\end{aligned}$$

□

Definition 4.34. Define $n \times n$ matrix H_n as

$$H_n = \begin{bmatrix} k^2+1 & k^2+1 & -1 & 0 & 0 & \dots\dots\dots & 0 \\ -1 & k^2+1 & k^2+1 & -1 & 0 & \dots\dots\dots & 0 \\ 0 & -1 & k^2+1 & k^2+1 & -1 & \dots\dots\dots & 0 \\ \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots \\ 0 & \dots\dots\dots & \dots\dots\dots & 0 & -1 & k^2+1 & k^2+1 \\ 0 & \dots\dots\dots & \dots\dots\dots & 0 & 0 & -1 & 0 \end{bmatrix}_{n\times n}.$$

where n, k are integers.

The matrix H_n is obtained from matrix G_n by deleting $(n,n)^{th}$ entry.

Theorem 4.35. *For $n > 1$*

$$det(H_n) = F_{k,n-1}F_{k,n+1}$$

Where $det(H_n) = (k^2+1)$.

Proof. Similar to theorem (4.33), we obtain

$$det(H_n) = (k^2+1)det(H_n-1) + (k^2+1)det(H_n-2) - det(H_n-3).$$

We have

$$det(H_2) = F_{k,1}F_{k,3},$$

$$det(H_3) = F_{k,2}F_{k,4},$$

$$det(H_4) = F_{k,3}F_{k,5}.$$

It gives that

$$(k^2+1)F_{k,n+1}F_{k,n+3} + (k^2+1)F_{k,n}F_{k,n+2} - F_{k,n-1}F_{k,n+1} = F_{k,n+2}F_{k,n+4},$$

$$det(H_n) = F_{k,n-1}F_{k,n+1}.$$

□

4.1.3 Identities for k-Fibonacci and k-Lucas Sequences using determinants of some matrices

In this section, some determinantal techniques are used to obtain many k-Lucas identities.

Theorem 4.36. *If n, i, j, t, m are positive integers with $0 < t < i$, $i + 1 < m$, $j = 1$, then*

$$det\begin{bmatrix} L^2_{k,n+t} + 4L^2_{k,n-i} & L_{k,n+i+j} & L_{k,n+i+j} \\ L_{k,n+t} & 4L^2_{k,n+i} + L^2_{k,n+i+j} & L_{k,n+t} \\ L_{k,n+i} & 2L_{k,n+i} & \dfrac{L^2_{k,n+i+j} + L^2_{k,n+t}}{2L_{k,n+i}} \end{bmatrix}$$

$$= 8L_{k,n+i}L_{k,n+t}L_{k,n+i+j}.$$

Proof. Let

$$\aleph_1 = det\begin{bmatrix} L^2_{k,n+t} + 4L^2_{k,n-i} & L_{k,n+i+j} & L_{k,n+i+j} \\ L_{k,n+t} & 4L^2_{k,n+i} + L^2_{k,n+i+j} & L_{k,n+t} \\ L_{k,n+i} & 2L_{k,n+i} & \dfrac{L^2_{k,n+i+j} + L^2_{k,n+t}}{2L_{k,n+i}} \end{bmatrix}.$$

Assume that

$$L_{k,n+t} = \phi,$$

$$L_{k,n+i} = \varphi.$$

Then, we have

$$L_{k,n+i+j} = k\varphi + \phi.$$

Now

$$\aleph_1 = det\begin{bmatrix} \dfrac{\phi^2+\varphi^2}{k\varphi+\phi} & k\varphi+\phi & k\varphi+\phi \\ \\ \phi & \frac{\varphi^2+(k\varphi+\phi)^2}{\phi} & \phi \\ \\ \varphi & \varphi & \dfrac{\phi^2+(k\varphi+\phi)^2}{\varphi} \end{bmatrix}.$$

Making the row operations $\dfrac{1}{(k\varphi+\phi)}\,[(k\varphi+\phi)R_1],\ \dfrac{1}{\phi}\,[\phi R_2],\ \dfrac{1}{\varphi}\,[\varphi R_3]$, we get

$$\aleph_1 = \frac{1}{\phi\varphi(k\varphi+\phi)}det\begin{bmatrix} \phi^2+\varphi^2 & (k\varphi+\phi)^2 & (k\varphi+\phi)^2 \\ \\ \phi^2 & \varphi^2+(k\varphi+\phi)^2 & \phi^2 \\ \\ \varphi^2 & \varphi^2 & \phi^2+(k\varphi+\phi)^2 \end{bmatrix}.$$

Again making the row operations $R_1+R_2+R_3 \to R_1$, $R_3-R_1 \to R_3$ and $R_2-R_1 \to R_2$, gives that

$$\aleph_1 = \frac{1}{\phi\varphi(k\varphi+\phi)}det\begin{bmatrix} \phi^2+\varphi^2 & \varphi^2+(k\varphi+\phi)^2 & \phi^2+(k\varphi+\phi)^2 \\ \\ -\varphi^2 & 0 & -(k\varphi+\phi)^2 \\ \\ \phi^2 & -(k\varphi+\phi)^2 & 0 \end{bmatrix}.$$

Expanding these, we get

$$\aleph_1 = 8\phi\varphi(k\varphi + \phi).$$

Putting

$$L_{k,n+t} = \phi, L_{k,n+i} = \varphi, L_{k,n+i+j} = k\varphi + \phi.$$

It gives that

$$\aleph_1 = 8L_{k,n+i}L_{k,n+t}L_{k,n+i+j}.$$

$\square$

Theorem 4.37. *If n, i, j, t, m are positive integers with $0 < t < i$, $i + 1 < m$, $j = 1$, then*

$$\begin{vmatrix} L^2_{k,n+t} & 2L_{k,n+i}L_{k,n+i+j} & L_{k,n+t}L_{k,n+i+j} + L_{k,n+i+j} \\ L^2_{k,n+t} + 2L_{k,n+i}L_{k,n+t} & 4L^2_{k,n+i} & L_{k,n+t}L_{k,n+i+j} \\ 2L_{k,n+i}L_{k,n+t} & 4L^2_{k,n+i} + 2L_{k,n+i}L_{k,n+i+j} & L^2_{k,n+i+j} \end{vmatrix}$$

$$= \left[4L_{k,n+i}L_{k,n+i+j}\right]^2.$$

Proof. Let

$\aleph$

$$= \begin{vmatrix} L_{k,n+t}^2 & 2L_{k,n+i}L_{k,n+i+j} & L_{k,n+t}L_{k,n+i+j} + L_{k,n+i+j} \\ L_{k,n+t}^2 + 2L_{k,n+i}L_{k,n+t} & 4L_{k,n+i}^2 & L_{k,n+t}L_{k,n+i+j} \\ 2L_{k,n+i}L_{k,n+t} & 4L_{k,n+i}^2 + 2L_{k,n+i}L_{k,n+i+j} & L_{k,n+i+j}^2 \end{vmatrix}.$$

Assume that

$$L_{k,n+t} = \phi, L_{k,n+i} = \varphi.$$

Then we have

$$\aleph_2 = \begin{vmatrix} \phi^2 & \varphi(k\varphi + \phi) & \phi(k\varphi + \phi) + (k\varphi + \phi)^2 \\ \phi^2 + \phi\varphi & \varphi^2 & \phi(k\varphi + \phi) \\ \phi\varphi & \varphi^2 + \varphi(k\varphi + \phi) & (k\varphi + \phi)^2 \end{vmatrix}.$$

Making the row operations $R_2 \to R_2 - (R_1 + R_3))$, we get

$$\aleph_1 = \frac{1}{\phi\varphi(k\varphi + \phi)} \begin{vmatrix} \phi & (k\varphi + \phi) & \phi + (k\varphi + \phi) \\ 0 & -2(k\varphi + \phi) & -2(k\varphi + \phi) \\ \varphi & \varphi + (k\varphi + \phi) & (k\varphi + \phi) \end{vmatrix}.$$

Making the Column operations $C_2 \to C_2 - C_3$ and expanding it gives

$$\aleph_2 = 4\left[2\phi\varphi(k\varphi + \phi)\right]^2.$$

Putting

$$L_{k,n+t} = \phi, L_{k,n+i} = \varphi, L_{k,n+i+j} = k\varphi + \phi.$$

It gives that

$$\aleph_2 = \left[4L_{k,n+i}L_{k,n+i+j}\right]^2.$$

□

Corollary 4.38. *If* n, i, j, t, m *are positive integers with* $0 < t < i$, $i + 1 < m$, $j = 1$, *then*

$$det \begin{bmatrix} -L_{k,n+t}^2 & 2L_{k,n+i}L_{k,n+t} & L_{k,n+t}L_{k,n+i+j} \\ 2L_{k,n+i}L_{k,n+t} & -4L_{k,n+i}^2 & 2L_{k,n+i}L_{k,n+i+j} \\ L_{k,n+t}L_{k,n+i+j} & 2L_{k,n+i}L_{k,n+i+j} & -L_{k,n+i+j}^2 \end{bmatrix}$$

$$= \left[4L_{k,n+i}L_{k,n+t}L_{k,n+i+j}\right]^2.$$

Corollary 4.39. *If n, i, j, t, m are positive integers with $0 < t < i$, $i+1 < m$, $j = 1$, then*

$$det\begin{bmatrix} 4L_{k,n+i}^2 + L_{k,n+i+j}^2 & 2L_{k,n+i}L_{k,n+t} & L_{k,n+t}L_{k,n+i+j} \\ 2L_{k,n+i}L_{k,n+t} & L_{k,n+t}^2 & 2L_{k,n+i}L_{k,n+i+j} \\ L_{k,n+t}L_{k,n+i+j} & 2L_{k,n+i}L_{k,n+i+j} & 4L_{k,n+i}^2 + L_{k,n+t}^2 \end{bmatrix}$$

$$= \left[4L_{k,n+i}L_{k,n+t}L_{k,n+i+j}\right]^2.$$

Corollary 4.40. *If n, i, j, t, m are positive integers with $0 < t < i$, $i+1 < m$, $j = 1$, then*

$$\begin{vmatrix} (2L_{k,n+i+j} + 2L_{k,n+i} + L_{k,n+t}) & L_{k,n+t} & 2L_{k,n+i} \\ L_{k,n+i+j} & 2L_{k,n+t} + 2L_{k,n+i} + L_{k,n+i+j} & 2L_{k,n+i} \\ L_{k,n+i+j} & 2L_{k,n+t} & 4L_{k,n+i} + L_{k,n+t} + L_{k,n+i+j} \end{vmatrix}$$

$$= 2\left[2L_{k,n+i} + L_{k,n+t} + L_{k,n+i+j}\right]^3.$$

Corollary 4.41. *If n, i, j, t, m are positive integers with $0 < t < i$, $i + 1 < m$, $j = 1$, then*

$$det \begin{bmatrix} 1 + L_{k,n+t} & 1 & 1 \\ 1 & 1 + 2L_{k,n+i} & 1 \\ 1 & 1 & 1 + L_{k,n+i+j} \end{bmatrix}$$

$$= \{2L_{k,n+i}L_{k,n+t}L_{k,n+i+j}\}\{\frac{1}{L_{k,n+t}} + \frac{1}{2L_{k,n+i}} + \frac{1}{L_{k,n+i+j}} + 1\}$$

$$\{2L_{k,n+i}L_{k,n+t}L_{k,n+i+j} + 2L_{k,n+i}L_{k,n+i+j} + L_{k,n+t}L_{k,n+t}L_{k,n+i+j} + 2L_{k,n+i}L_{k,n+t}\}.$$

4.2 Concluding remark

In this chapter, some new identities obtained for the $k-$Fibonacci and $k-$Lucas sequences using matrix methods. Also, we obtained determinantal identities for k Lucas sequence.

Chapter 5

On Some Properties of the Generalized k-Lucas Sequence and Its Companion Sequence

The k-Lucas sequence is companion sequence of k-Fibonacci sequence defined with the k-Lucas numbers which are defined with the recurrence relation $\mathcal{L}_{k,n+1} = k\mathcal{L}_{k,n} + \mathcal{L}_{k,n-1}$, with the initial conditions $\mathcal{L}_{k,0} = 2$, $\mathcal{L}_{k,1} = 2$, for $n \geq 1$. In this paper, we introduce a new generalisation $\mathcal{M}_{k,n}$ of k-Lucas sequence. We present generating functions and Binet formulas for generalized k-Lucas sequence, and state some binomial and congruence sums containing these sequences.

5.1 Introduction

In this chapter, we defined generalised k-Lucas sequence $\mathcal{M}_{k,n}$and derived the relations connecting the generalised k-Lucas sequence $\mathcal{M}_{k,n}$and it's companion sequence $\mathcal{N}_{k,n}$. We have adapted the methods of Carlitz [61] and Zhizheng Zhang [62] to the generalised k-Lucas sequence $\mathcal{M}_{k,n}$and derived some fundamental and congruence identities for these generalised k-Lucas sequences $\mathcal{M}_{k,n}$and $\mathcal{N}_{k,n}$.

Generalized k-Lucas Sequence $\mathcal{M}_{k,n}$

Definition 5.1. For $n \geq 1$, the generalized k-Lucas sequence $\mathcal{M}_{k,n}$ is defined by the recurrence relation $\mathcal{M}_{k,n+1} = k\mathcal{M}_{k,n} + \mathcal{M}_{k,n-1}$, with $\mathcal{M}_{k,0} = 2$ and $\mathcal{M}_{k,1} = k + \delta$.

Definition 5.2. For $n \geq 1$, the companion sequence $\mathcal{N}_{k,n}$ of $\mathcal{N}_{k,n}$ is defined by the relation $\mathcal{N}_{k,n} = \mathcal{N}_{k,n+1} + \mathcal{N}_{k,n-1}$.

The characteristic equation of the initial recurrence relation of $\mathcal{M}_{k,n}$is same as k-Lucas sequence.

It is interesting to observe that if we add δ with k in initial condition of sequence $\mathcal{M}_{k,n}$ then it is multiplied with $F_{k,n}$ and $F_{k,n} + L_{k,n}$ in the identities

$$\mathcal{M}_{k,n} = \delta F_{k,n} + L_{k,n},$$

$$\mathcal{N}_{k,n} = \delta\left(F_{k,n} + L_{k,n}\right),$$

respectively.

Theorem 5.3. ***(Binet Formulas)***. *For $n \geq 1$,*

$$\mathcal{M}_{k,n} = \frac{\bar{r_1}{r_1}^n - \bar{r_2}{r_2}^n}{r_1 - r_2} \tag{5.1.1}$$

and

$$\mathcal{N}_{k,n} = \bar{r_1}{r_1}^n + \bar{r_2}{r_2}^n, \tag{5.1.2}$$

where, $\bar{r_1} = \delta + \sqrt{\delta}$ and $\bar{r_2} = \delta - \sqrt{\delta}$.

Next, we state certain basic properties of the generalized k-Lucas sequence, these properties can proved using (5.1.1) and (5.1.2).

Theorem 5.4. ***(Catalan's Identity)***. *For $n, r \geq 1$, we have*

$$\mathcal{M}_{k,n-r}\mathcal{M}_{k,n+r} - {\mathcal{M}_{k,n}}^2 = (-1)^{n-r}\delta(1-\delta)F_{k,r}^2.$$

Theorem 5.5. ***(Cassini's Identity)***. *For $n \geq 1$, we have*

$$\mathcal{M}_{k,n-1}\mathcal{M}_{k,n+1} - {\mathcal{M}_{k,n}}^2 = (-1)^{n+1}\delta(1-\delta).$$

Theorem 5.6. ***(d'Ocagene's Identity)***. *Let n be any non-negative integer and r a natural number. If $n \geq r+1$, then*

$$\mathcal{M}_{k,r}\mathcal{M}_{k,n+1} - \mathcal{M}_{k,r+1}\mathcal{M}_{k,n} = (-1)^n\delta(1-\delta)F_{k,r-n}.$$

Theorem 5.7. ***(Convolution Theorem)***. *For $n, r \geq 1$, we have*

$$\begin{aligned}\mathcal{M}_{k,r}\mathcal{M}_{k,n+1} + \mathcal{M}_{k,r-1}\mathcal{M}_{k,n} = \mathcal{M}_{k,n+r} &+ (\delta^2 + \delta - \sqrt{\delta})F_{k,n+r} \\ &+ (2\delta + \sqrt{\delta})L_{k,n+r}.\end{aligned}$$

Theorem 5.8. ***(Asymptotic Behaviour)***. *For $n, r \geq 1$, we have*

$$\lim_{n\to\infty}\frac{\mathcal{M}_{k,n}}{\mathcal{M}_{k,n-r}} = r_1^r.$$

Theorem 5.9. *The generating function for the generalized k-Fibonacci sequence $\mathcal{M}_{k,tn}$ is*

$$\sum_{n=0}^{\infty}\mathcal{M}_{k,tn}x^n = \frac{x\mathcal{M}_{k,t} - 2xL_{k,t} + 2}{1 - xL_{k,t} + x^2(-1)^t}.$$

Theorem 5.10. *For $n \geq 3$, we have*

$$r_1^{n-2} < \mathcal{M}_{k,n}.$$

Theorem 5.11. *For $n, k \geq 1$*

1. $\mathcal{M}_{k,2n}\mathcal{M}_{k,2n+1} = \mathcal{M}_{k,4n+1} + \delta(F_{k,4n+1} + L_{k,4n+1}) - k(\delta - 1),$
2. $\sum_{i=1}^{n} \mathcal{M}_{k,i} = \frac{\mathcal{M}_{k,n+1} + \mathcal{M}_{k,n} - (2 + k + \delta)}{k},$
3. $\sum_{i=1}^{n} \mathcal{M}_{k,2i} = \frac{\mathcal{M}_{k,2n+1} - (k + \delta)}{k},$
4. $\sum_{i=1}^{n} \mathcal{M}_{k,2i-1} = \frac{\mathcal{M}_{k,2n} - 2}{k},$
5. $\sum_{i=1}^{n} \mathcal{M}_{k,i}^2 = \frac{k\mathcal{M}_{k,n+1}\mathcal{M}_{k,n} - k^2 - \delta(2k - 1) - 2\delta^2}{k^2}.$

Proposition 5.12. *For $n \geq 0$, the following identities hold for $\mathcal{M}_{k,n}$ and $\mathcal{N}_{k,n}$:*

1. $\mathcal{M}_{k,n} = \delta F_{k,n} + L_{k,n},$
2. $\mathcal{N}_{k,n} = \delta\left[F_{k,n} + L_{k,n}\right],$
3. $\mathcal{M}_{k,n} + \mathcal{M}_{k,n+4} = (k^2 + 2)\mathcal{M}_{k,n+2},$
4. $\mathcal{N}_{k,n} + \mathcal{N}_{k,n+4} = (k^2 + 2)\mathcal{N}_{k,n+2},$
5. $\mathcal{N}_{k,n} + \mathcal{N}_{k,n+2} = \delta\mathcal{M}_{k,n+1},$
6. $\mathcal{M}_{k,n-3} + \mathcal{M}_{k,n+3} = (k^2 + 1)\mathcal{N}_{k,n},$

7. $\mathcal{N}_{k,n-3} + \mathcal{N}_{k,n+3} = \delta(k^2+1)\mathcal{M}_{k,n}$,

8. $\mathcal{N}_{k,n}{}^2 - \delta\mathcal{M}_{k,n}{}^2 = 4(-1)^{n+1}\delta(1-\delta)$.

Theorem 5.13. *For $n, k \geq 1$, the following identity hold for $\mathcal{M}_{k,n}$:*

$$M_{k,n} = \begin{cases} \frac{1}{2^{n-1}}\left[\sum_{i=0}^{\frac{n}{2}}\binom{n}{2i}k^{n-2i}\delta^{i} + \sum_{i=0}^{\frac{n-2}{2}}\binom{n}{2i+1}k^{n-(2i+1)}\delta^{i+1}\right] & \\ & \textit{if } n \textit{ is even,} \\ \frac{1}{2^{n-1}}\left[\sum_{i=0}^{\frac{n-1}{2}}\binom{n}{2i}k^{n-2i}\delta^{2i} + \sum_{i=0}^{\frac{n-1}{2}}\binom{n}{2i+1}k^{n-(2i+1)}\delta^{i+1}\right] & \\ & \textit{if } n \textit{ is odd .} \end{cases}$$

Throughout this paper, the symbol $\binom{n}{i_1, i_2, ..., i_{(n-1)}}$ is defined by $\dfrac{n!}{i_1!i_2!....i_{(n-1)}!s!}$, where $s = n - (i_1 + i_2 + ... + i_{(n-1)})$.

In next section, we explore certain properties of the generalized k-Lucas sequence $\mathcal{M}_{k,n}$.

5.2 Properties of the Generalized k-Lucas Sequence $\mathcal{M}_{k,n}$

Lemma 5.14. *Let $u = r_1$ or r_2, then*

(a) $u^2 = ku + 1,$

(b) $u^n = uF_{k,n} + F_{k,n-1},$

(c) $u^{2n} = u^n L_{k,n} - (-1)^n,$

(d) $u^{tn} = u^n \dfrac{F_{k,tn}}{F_{k,n}} - (-1)^n - \dfrac{F_{k,(t-1)n}}{F_{k,n}},$

(e) $u^{sn}F_{k,rn} - u^{rn}F_{k,sn} = (-1)^{sn}F_{k,(r-s)n}.$

Theorem 5.15. *For $n, r, s, t \geq 1$, we have*

(a) $\mathcal{M}_{k,n+t} = F_{k,n}\mathcal{M}_{k,t+1} + F_{k,n-1}\mathcal{M}_{k,t},$

(b) $\mathcal{M}_{k,2n+t} = L_{k,n}\mathcal{M}_{k,n+t} - (-1)^n\mathcal{M}_{k,t},$

(c) $\mathcal{M}_{k,sn+t} = \dfrac{F_{k,sn}}{F_{k,n}}\mathcal{M}_{k,n+t} - (-1)^n \dfrac{F_{k,(s-1)n}}{F_{k,n}}\mathcal{M}_{k,t},$

(d) $\mathcal{M}_{k,sn+t}F_{k,rn} - \mathcal{M}_{k,rn+t}F_{k,sn} = (-1)^{sn}\mathcal{M}_{k,t}F_{k,(r-s)n}.$

Theorem 5.16. *For $n, r, s, t \geq 1$ and $\mathcal{D}_{k,n} = \mathcal{M}_{k,n}$ or $\mathcal{N}_{k,n}$, we have*

1. $\mathcal{D}_{k,2n} = \sum_{i=0}^{n} \binom{n}{i} k^i \mathcal{D}_{k,i},$
2. $\mathcal{D}_{k,2n+t} = \sum_{i=0}^{n} \binom{n}{i} k^i \mathcal{D}_{k,i+t},$
3. $\mathcal{D}_{k,rn+t} = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^{i} F_{k,r-1}^{n-i} \mathcal{D}_{k,i+t},$
4. $\mathcal{D}_{k,2rn+t} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{(n-i)(r+1)} L_{k,r}^{i} \mathcal{D}_{k,ri+t},$
5. $\mathcal{D}_{k,trn+l} = \dfrac{1}{F_{k,r}^{n}} \sum_{i=0}^{n} \binom{n}{i} (-1)^{(n-i)(r+1)} F_{k,(t-1)r}^{n-i} F_{k,tr}^{i} \mathcal{D}_{k,ri+l},$

6. $\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\mathcal{D}_{k,r(n-i)+i+t}F_{k,r}^{i}=\mathcal{D}_{k,t}F_{k,r-1}^{n}$,

7. $\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}\mathcal{D}_{k,ri+t}F_{k,r-1}^{(n-i)}=\mathcal{D}_{k,n+t}F_{k,r}^{n}$,

8. $\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}F_{k,sm}^{(n-i)}F_{k,rm}^{(i)}\mathcal{D}_{k,m[rn+i(s-r)]+t}=(-1)^{smn}\mathcal{D}_{k,t}F_{k,(r-s)m}^{n}$.

Lemma 5.17. *Let $u=r_1$ or r_2, then*

1. $k+(k^2+1)u=u^3$,

2. $1+ku+u^6=L_{k,2}u^4$,

3. $1+ku+u^{10}=L_{k,4}u^6$,

4. $1+ku+u^{18}=L_{k,8}u^{10}$,

5. $1+ku+u^{34}=L_{k,16}u^{18}$,

6. $1+ku+u^{66}=L_{k,32}u^{34}$,

7. $1+ku+u^{130}=L_{k,64}u^{66}$,

8. $1+ku+u^{258}=L_{k,128}u^{130}$,

9. $1+ku+u^{514}=L_{k,256}u^{258}$,

10. $1+ku+u^{1026}=L_{k,512}u^{514}$,

11. $1+ku+u^{2050}=L_{k,1024}u^{1026}$.

In general, if $L_{k,n}$ is n^{th} k-Lucas sequence and $u=r_1$ or r_2, then

$$1+ku+u^{2(2^{n+1}+1)}=L_{k,2^{n+1}}u^{2(2^n+1)}.$$

Theorem 5.18. *For $t \geq 1$ and $\mathcal{D}_{k,n} = \mathcal{M}_{k,n}$ or $\mathcal{N}_{k,n}$, we have*

1. $\mathcal{D}_{k,t+3} = (k^2+1)\mathcal{D}_{k,t+1} + k\mathcal{D}_{k,t}$,
2. $\mathcal{D}_{k,t+4} = \dfrac{\mathcal{D}_{k,t} + k\mathcal{D}_{k,t+1} + \mathcal{D}_{k,t+6}}{L_{k,2}}$,
3. $\mathcal{D}_{k,t+6} = \dfrac{\mathcal{D}_{k,t} + k\mathcal{D}_{k,t+1} + \mathcal{D}_{k,t+10}}{L_{k,4}}$,
4. $\mathcal{D}_{k,t+10} = \dfrac{\mathcal{D}_{k,t} + k\mathcal{D}_{k,t+1} + \mathcal{D}_{k,t+18}}{L_{k,8}}$,
5. $\mathcal{D}_{k,t+18} = \dfrac{\mathcal{D}_{k,t} + k\mathcal{D}_{k,t+1} + \mathcal{D}_{k,t+34}}{L_{k,16}}$,
6. $\mathcal{D}_{k,t+34} = \dfrac{\mathcal{D}_{k,t} + k\mathcal{D}_{k,t+1} + \mathcal{D}_{k,t+66}}{L_{k,32}}$,
7. $\mathcal{D}_{k,t+66} = \dfrac{\mathcal{D}_{k,t} + k\mathcal{D}_{k,t+1} + \mathcal{D}_{k,t+130}}{L_{k,64}}$,
8. $\mathcal{D}_{k,t+130} = \dfrac{\mathcal{D}_{k,t} + k\mathcal{D}_{k,t+1} + \mathcal{D}_{k,t+258}}{L_{k,128}}$,
9. $\mathcal{D}_{k,t+258} = \dfrac{\mathcal{D}_{k,t} + k\mathcal{D}_{k,t+1} + \mathcal{D}_{k,t+514}}{L_{k,256}}$,
10. $\mathcal{D}_{k,t+514} = \dfrac{\mathcal{D}_{k,t} + k\mathcal{D}_{k,t+1} + \mathcal{D}_{k,t+1026}}{L_{k,512}}$,
11. $\mathcal{D}_{k,t+1026} = \dfrac{\mathcal{D}_{k,t} + k\mathcal{D}_{k,t+1} + \mathcal{D}_{k,t+2050}}{L_{k,1024}}$.

In general, for $t \geq 1$, we have

$$\mathcal{D}_{k,t+2^{n+1}+2} = \frac{\mathcal{D}_{k,t} + k\mathcal{D}_{k,t+1} + \mathcal{D}_{k,t+2^{n+2}+2}}{L_{k,2^{n+1}}}.$$

Theorem 5.19. *For $n, t \geq 1$ and $\mathcal{D}_{k,n} = \mathcal{M}_{k,n}$ or $\mathcal{N}_{k,n}$, we have*

1. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n} (-1)^{j+s} {L_{k,2}}^i \mathcal{D}_{k,4i+6j+t}$,

2. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n} (-1)^{j+s} {L_{k,4}}^i \mathcal{D}_{k,6i+10j+t}$,

3. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n} (-1)^{j+s} {L_{k,8}}^i \mathcal{D}_{k,10i+18j+t}$,

4. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n} (-1)^{j+s} {L_{k,16}}^i \mathcal{D}_{k,18i+34j+t}$,

5. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n} (-1)^{j+s} {L_{k,32}}^i \mathcal{D}_{k,34i+66j+t}$,

6. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n} (-1)^{j+s} {L_{k,64}}^i \mathcal{D}_{k,66i+130j+t}$,

7. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n} (-1)^{j+s} {L_{k,128}}^i \mathcal{D}_{k,130i+258j+t}$,

8. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n} (-1)^{j+s} {L_{k,256}}^i \mathcal{D}_{k,258i+514j+t}$,

9. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n} (-1)^{j+s} {L_{k,512}}^i \mathcal{D}_{k,514i+1026j+t}$,

10. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n} (-1)^{j+s} {L_{k,1024}}^i \mathcal{D}_{k,1026i+2050j+t}$.

In general, for $r, n, t \geq 1$, *we have*

$$\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n} (-1)^{j+s} {L_{k,2^{r+1}}}^i \mathcal{D}_{k,2^{r+1}(i+2j)+2(i+j)+t}.$$

Theorem 5.20. *For* $n, t \geq 1$ *and* $\mathcal{D}_{k,n} = \mathcal{M}_{k,n}$ *or* $\mathcal{N}_{k,n}$, *we have*

1. $\mathcal{D}_{k,6n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j} (-1)^{j+s} {L_{k,2}}^i \mathcal{D}_{k,4i+j+t}$,

2. $\mathcal{D}_{k,10n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j} (-1)^{j+s} {L_{k,4}}^i \mathcal{D}_{k,6i+j+t}$.

3. $\mathcal{D}_{k,18n+t} = \sum\limits_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} {L_{k,8}}^i \mathcal{D}_{k,10i+j+t}$,

4. $\mathcal{D}_{k,34n+t} = \sum\limits_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} {L_{k,16}}^i \mathcal{D}_{k,18i+j+t}$,

5. $\mathcal{D}_{k,66n+t} = \sum\limits_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} {L_{k,32}}^i \mathcal{D}_{k,34i+j+t}$,

6. $\mathcal{D}_{k,130n+t} = \sum\limits_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} {L_{k,64}}^i \mathcal{D}_{k,66i+j+t}$,

7. $\mathcal{D}_{k,258n+t} = \sum\limits_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} {L_{k,128}}^i \mathcal{D}_{k,130i+j+t}$,

8. $\mathcal{D}_{k,514n+t} = \sum\limits_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} {L_{k,256}}^i \mathcal{D}_{k,258i+j+t}$,

9. $\mathcal{D}_{k,1026n+t} = \sum\limits_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} {L_{k,512}}^i \mathcal{D}_{k,514i+j+t}$,

10. $\mathcal{D}_{k,2050n+t} = \sum\limits_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} {L_{k,1024}}^i \mathcal{D}_{k,1026i+j+t}$.

In general, for $r, n, t \geq 1$, we have

$$\mathcal{D}_{k,(2^{r+2}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} {L_{k,2^{r+1}}}^i \mathcal{D}_{k,(2^{r+1}+2)i+j+t}.$$

Theorem 5.21. *For $n, t \geq 1$ and $\mathcal{D}_{k,n} = \mathcal{M}_{k,n}$ or $\mathcal{N}_{k,n}$, we have*

1. $\mathcal{D}_{k,4n+t} = \sum\limits_{i+j+s=n} \binom{n}{i,j} k^j {L_{k,2}}^{-n} \mathcal{D}_{k,6i+j+t}$,

2. $\mathcal{D}_{k,6n+t} = \sum\limits_{i+j+s=n} \binom{n}{i,j} k^j {L_{k,4}}^{-n} \mathcal{D}_{k,10i+j+t}$,

3. $\mathcal{D}_{k,10n+t} = \sum\limits_{i+j+s=n} \binom{n}{i,j} k^j {L_{k,8}}^{-n} \mathcal{D}_{k,18i+j+t}$,

4. $\mathcal{D}_{k,18n+t} = \sum\limits_{i+j+s=n} \binom{n}{i,j} k^j {L_{k,16}}^{-n} \mathcal{D}_{k,34i+j+t}$,

5. $\mathcal{D}_{k,34n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j {L_{k,32}}^{-n} \mathcal{D}_{k,66i+j+t}$,

6. $\mathcal{D}_{k,66n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j {L_{k,64}}^{-n} \mathcal{D}_{k,130i+j+t}$,

7. $\mathcal{D}_{k,130n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j {L_{k,128}}^{-n} \mathcal{D}_{k,258i+j+t}$,

8. $\mathcal{D}_{k,258n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j {L_{k,256}}^{-n} \mathcal{D}_{k,514i+j+t}$,

9. $\mathcal{D}_{k,514n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j {L_{k,512}}^{-n} \mathcal{D}_{k,1026i+j+t}$,

10. $\mathcal{D}_{k,1026n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j {L_{k,1024}}^{-n} \mathcal{D}_{k,2050i+j+t}$.

In general, for $r, n, t \geq 1$, we have

$$\mathcal{D}_{k,(2^{r+1}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j {L_{k,2^{r+1}}}^{-n} \mathcal{D}_{k,(2^{r+1}+2)i+j+t}.$$

Lemma 5.22. *Let $u = r_1$ or r_2, then for $l_n = \sum_{i=1}^{n} L_{k,2^i}$ and $n, t \geq 1$, we have*

1. $1 + u^4 = l_1 u^2$,

2. $1 + u^8 = \frac{l_2}{l_1} u^4 = l_2 u^2 - \frac{l_2}{l_1}$,

3. $1 + u^{16} = \frac{l_3}{l_2} u^8 = \frac{l_3}{l_1} u^4 - \frac{l_3}{l_2} = l_3 u^2 - \frac{l_3}{l_1} - \frac{l_3}{l_2}$,

4. $1 + u^{32} = \frac{l_4}{l_3} u^{16} = \frac{l_4}{l_2} u^8 - \frac{l_4}{l_3} = \frac{l_4}{l_1} u^4 - l_4 \left[\frac{1}{l_2} + \frac{1}{l_3}\right] = l_4 u^2 - l_4 \left[\frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3}\right]$,

5. $1+u^{64} = \dfrac{l_5}{l_4}u^{32} = \dfrac{l_5}{l_3}u^{16} - \dfrac{l_5}{l_4} = \dfrac{l_5}{l_2}u^{8} - l_5\left[\dfrac{1}{l_3}+\dfrac{1}{l_4}\right] = \dfrac{l_5}{l_1}u^{4} - l_5\left[\dfrac{1}{l_2}+\dfrac{1}{l_3}+\dfrac{1}{l_4}\right]$
$= l_5u^2 - l_5\left[\dfrac{1}{l_1}+\dfrac{1}{l_2}+\dfrac{1}{l_3}+\dfrac{1}{l_4}\right].$

In general, we have

$$1+u^{2^n} = \begin{cases} \dfrac{l_{n-1}}{l_{n-2}}u^{2^{n-1}}; \\ \dfrac{l_{n-1}}{l_{n-t-1}}u^{2^{n-t}} - l_{n-1}\sum\limits_{i=2}^{t}\dfrac{1}{l_{n-i}}, \quad \mathit{If}\ t=2,3,4,\ldots,n-2\ ; \\ l_{n-1}u^2 - l_{n-1}\sum\limits_{i=2}^{n-1}\dfrac{1}{l_{n-i}}. \end{cases}$$

Theorem 5.23. *For* $l_n = \sum\limits_{i=1}^{n} L_{k,2^i}$, $n,t \geq 1$ *and* $\mathcal{D}_{k,n} = \mathcal{M}_{k,n}$ *or* $\mathcal{N}_{k,n}$, *we have*

1. $\mathcal{D}_{k,t+4} = l_1\mathcal{D}_{k,t+2} - \mathcal{D}_{k,t}$,

2. $\mathcal{D}_{k,t+8} = \dfrac{l_2}{l_1}\mathcal{D}_{k,t+4} - \mathcal{D}_{k,t} = l_2\mathcal{D}_{k,t+2} - (1+\dfrac{l_2}{l_1})\mathcal{D}_{k,t}$,

3. $\mathcal{D}_{k,t+16} = \dfrac{l_3}{l_2}\mathcal{D}_{k,t+8} - \mathcal{D}_{k,t}$,
$= \dfrac{l_3}{l_1}\mathcal{D}_{k,t+4} - (1+\dfrac{l_3}{l_2})\mathcal{D}_{k,t}$,
$= l_3\mathcal{D}_{k,t+2} - (1+\dfrac{l_3}{l_1}+\dfrac{l_3}{l_2})\mathcal{D}_{k,t}$,

4. $\mathcal{D}_{k,t+32} = \dfrac{l_4}{l_3}\mathcal{D}_{k,t+16} - \mathcal{D}_{k,t}$,
$= \dfrac{l_4}{l_2}\mathcal{D}_{k,t+8} - (1+\dfrac{l_4}{l_3})\mathcal{D}_{k,t}$,
$= \dfrac{l_4}{l_1}\mathcal{D}_{k,t+4} - (1+\dfrac{l_4}{l_2}+\dfrac{l_4}{l_3})\mathcal{D}_{k,t}$,
$= l_4\mathcal{D}_{k,t+2} - (1+\dfrac{l_4}{l_1}+\dfrac{l_4}{l_2}+\dfrac{l_4}{l_3})\mathcal{D}_{k,t}$,

5. $\mathcal{D}_{k,t+64} = \frac{l_5}{l_4}\mathcal{D}_{k,t+32} - \mathcal{D}_{k,t},$
$= \frac{l_5}{l_3}\mathcal{D}_{k,t+16} - (1 + \frac{l_5}{l_4})\mathcal{D}_{k,t},$
$= \frac{l_5}{l_2}\mathcal{D}_{k,t+8} - (1 + \frac{l_5}{l_3} + \frac{l_5}{l_4})\mathcal{D}_{k,t},$
$= \frac{l_5}{l_1}\mathcal{M}_{k,t+4} - (1 + \frac{l_5}{l_2} + \frac{l_5}{l_3} + \frac{l_5}{l_4})\mathcal{D}_{k,t},$
$= l_5\mathcal{D}_{k,t+2} - (1 + \frac{l_5}{l_1} + \frac{l_5}{l_2} + \frac{l_5}{l_3} + \frac{l_5}{l_4})\mathcal{D}_{k,t}.$

In general, we have

$$\mathcal{D}_{k,t+2^n} = \begin{cases} \frac{l_{n-1}}{l_{n-2}}\mathcal{D}_{k,t+2^{n-1}} - \mathcal{D}_{k,t}; \\ \frac{l_{n-1}}{l_{n-t-1}}\mathcal{D}_{k,t+2^{n-s}} - l_{n-1}\sum_{i=2}^{s}(1 + \frac{1}{l_{n-i}})\mathcal{D}_{k,t}, \quad \text{If } s = 2, 3, 4, \ldots, n-2 \ ; \\ l_{n-1}\mathcal{D}_{k,t+2} - l_{n-1}\sum_{i=2}^{n-1}(\frac{1}{l_{n-i}} + 1)\mathcal{D}_{k,t}. \end{cases}$$

Theorem 5.24. *For* $l_n = \sum_{i=1}^{n} L_{k,2^i}$, $n, t \geq 1$ *and* $\mathcal{D}_{k,n} = \mathcal{M}_{k,n}$ *or* $\mathcal{N}_{k,n}$, *we have*

1. $\mathcal{D}_{k,4n+t} = \sum_{i+j=n} \binom{n}{i} l_1^i (-1)^j \mathcal{D}_{k,2i+t},$

2. $\mathcal{D}_{k,8n+t} = \sum_{i+j=n} \binom{n}{i} (\frac{l_2}{l_1})^i (-1)^j \mathcal{D}_{k,4i+t},$
$= \sum_{i+j=n} \binom{n}{i} l_2^i (-1)^j (\frac{l_1 + l_2}{l_1})\mathcal{D}_{k,2i+t}.$

3. $\mathcal{D}_{k,16n+t} = \sum_{i+j=n} \binom{n}{i} (\frac{l_3}{l_2})^i (-1)^j \mathcal{D}_{k,8i+t},$
$= \sum_{i+j=n} \binom{n}{i} (\frac{l_3}{l_1})^i (-1)^j (1 + \frac{l_3}{l_2})\mathcal{D}_{k,4i+t},$
$= \sum_{i+j=n} \binom{n}{i} {l_3}^i (-1)^j (1 + \frac{l_3}{l_1} + \frac{l_3}{l_2})\mathcal{D}_{k,2i+t}.$

4. $\mathcal{D}_{k,32n+t} = \sum_{i+j=n} \binom{n}{i} (\frac{l_4}{l_3})^i(-1)^j\mathcal{D}_{k,16i+t},$

$= \sum_{i+j=n} \binom{n}{i} (\frac{l_4}{l_2})^i(-1)^j(1+\frac{l_4}{l_3})\mathcal{D}_{k,8i+t},$

$= \sum_{i+j=n} \binom{n}{i} (\frac{l_4}{l_1})^i(-1)^j(1+\frac{l_4}{l_2}+\frac{l_4}{l_3})\mathcal{D}_{k,4i+t},$

$= \sum_{i+j=n} \binom{n}{i} {l_4}^i(-1)^j(1+\frac{l_4}{l_1}+\frac{l_4}{l_2}+\frac{l_4}{l_3})\mathcal{D}_{k,2i+t},$

5. $\mathcal{D}_{k,64n+t} = \sum_{i+j=n} \binom{n}{i} (\frac{l_5}{l_4})^i(-1)^j\mathcal{D}_{k,32i+t},$

$= \sum_{i+j=n} \binom{n}{i} (\frac{l_5}{l_3})^i(-1)^j(1+\frac{l_5}{l_4})\mathcal{D}_{k,16i+t},$

$= \sum_{i+j=n} \binom{n}{i} (\frac{l_5}{l_2})^i(-1)^j(1+\frac{l_5}{l_3}+\frac{l_5}{l_4})\mathcal{D}_{k,8i+t},$

$= \sum_{i+j=n} \binom{n}{i} (\frac{l_5}{l_1})^i(-1)^j(1+\frac{l_5}{l_2}+\frac{l_5}{l_3}+\frac{l_5}{l_4})\mathcal{D}_{k,4i+t},$

$= \sum_{i+j=n} \binom{n}{i} {l_5}^i(-1)^j(1+\frac{l_5}{l_1}+\frac{l_5}{l_2}+\frac{l_5}{l_3}+\frac{l_5}{l_4})\mathcal{D}_{k,2i+t}.$

In general, we have

$$\mathcal{D}_{k,2^rn+t} = \begin{cases} \sum_{i+j=n} \binom{n}{i} (\frac{l_{r-1}}{l_{r-2}})^i(-1)^j\mathcal{D}_{k,2^{r-1}i+t}; \\ \sum_{i+j=n} \binom{n}{i} (\frac{l_{r-1}}{l_{r-s-1}})^i(-1)^j(\sum_{h=2}^{s}(1+\frac{l_{r-1}}{l_{r-h}})^j\mathcal{D}_{k,2^{n-s}i+t}, \\ \qquad\qquad \text{If } s = 2,3,4,\ldots,n-2 \;; \\ \sum_{i+j=n} \binom{n}{i} (l_{r-1})^i(-1)^j(\sum_{h=2}^{s}(1+\frac{l_{r-1}}{l_{r-h}})^j\mathcal{D}_{k,2i+t}. \end{cases}$$

Lemma 5.25. *For $t \geq 1$, we have*

(1) $r_1^2 = r_1\sqrt{\delta} - 1,$

$r_2^2 = -r_2\sqrt{\delta} - 1,$

(2) $r_1^4 = (k^2+2)r_1\sqrt{\delta} - (k^2+3)$,

$r_2^4 = -(k^2+2)r_2\sqrt{\delta} - (k^2+3)$.

(3) $r_1^6 = (k^2+1)(k^2+3)r_1\sqrt{\delta} - (k^4+5k^2+5)$,

$r_2^6 = -(k^2+1)(k^2+3)r_2\sqrt{\delta} - (k^4+5k^2+5)$,

(4) $r_1^8 = (k^2+2)(k^4+4k^2+2)r_1\sqrt{\delta} - (k^6+7k^4+14k^2+7)$,

$r_2^8 = -(k^2+2)(k^4+4k^2+2)r_2\sqrt{\delta} - (k^6+7k^4+14k^2+7)$,

(5) $r_1^{10} = (k^4+3k^2+1)(k^4+5k^2+5)r_1\sqrt{\delta} - (k^2+3)(k^6+6k^4+9k^2+3)$,

$r_2^{10} = -(k^4+3k^2+1)(k^4+5k^2+5)r_2\sqrt{\delta} - (k^2+3)(k^6+6k^4+9k^2+3)$.

In general, we have

$$r_1^{2t} = \frac{F_{k,2t}}{k}r_1\sqrt{\delta} - \frac{L_{k,2t-1}}{k},$$
$$r_2^{2t} = -\frac{F_{k,2t}}{k}r_2\sqrt{\delta} - \frac{L_{k,2t-1}}{k}.$$

Lemma 5.26. *For $t \geq 1$, we have*

(1) $r_1^3 = (k^2+3)r_1 - \sqrt{\delta}$,

$r_2^3 = (k^2+3)r_2 + \sqrt{\delta}$,

(2) $r_1^5 = (k^4+5k^2+5)r_1 - (k^2+2)\sqrt{\delta}$,

$r_2^5 = (k^4+5k^2+5)r_2 + (k^2+2)\sqrt{\delta}$,

(3) $r_1^7 = (k^6+7k^4+14k^2+7)r_1 - (k^2+1)(k^2+3)\sqrt{\delta}$,

$r_2^7 = (k^6+7k^4+14k^2+7)r_2 + (k^2+1)(k^2+3)\sqrt{\delta}$, *(4)* $r_1^9 = (k^2+3)(k^6+6k^4+9k^2+3)r_1 - (k^2+2)(k^4+4k^2+2)\sqrt{\delta}$,

$r_2^9 = (k^2+3)(k^6+6k^4+9k^2+3)r_2 + (k^2+2)(k^4+4k^2+2)\sqrt{\delta}$,

(5) $r_1^{11} = (k^{10}+11k^8+44k^6+77k^4+55k^2+11)r_1 + (k^4+3k^2+1)(k^4+5k^2+5)\sqrt{\delta}$,

$r_2^{11} = (k^{10}+11k^8+44k^6+77k^4+55k^2+11)r_2 - (k^4+3k^2+1)(k^4+5k^2+5)\sqrt{\delta}$.

In general, we have

$$r_1^{2t+1} = \frac{L_{k,2t+1}}{k} r_1 - \frac{F_{k,2t}}{k}\sqrt{\delta},$$
$$r_2^{2t+1} = \frac{L_{k,2t+1}}{k} r_2 + \frac{F_{k,2t}}{k}\sqrt{\delta}.$$

Theorem 5.27. *For $s, t \geq 1$, we have*

1. $\mathcal{M}_{k,s+2} + \mathcal{M}_{k,s} = \mathcal{N}_{k,s+1}$,

   $\mathcal{N}_{k,s+2} + \mathcal{N}_{k,s} = \delta\mathcal{M}_{k,s+1}$,

2. $\mathcal{M}_{k,s+4} + (k^2+3)\mathcal{M}_{k,s} = (k^2+2)\mathcal{N}_{k,s+1}$,

   $\mathcal{N}_{k,s+4} + (k^2+3)\mathcal{N}_{k,s} = (k^2+2)\delta\mathcal{M}_{k,s+1}$,

3. $\mathcal{M}_{k,s+6} + (k^4+5k^2+5)\mathcal{M}_{k,s} = (k^2+1)(k^2+3)\mathcal{N}_{k,s+1}$,

   $\mathcal{N}_{k,s+6} + (k^4+5k^2+5)\mathcal{N}_{k,s} = (k^2+1)(k^2+3)\delta\mathcal{M}_{k,s+1}$,

4. $\mathcal{M}_{k,s+8} + (k^6+7k^4+14k^2+7)\mathcal{M}_{k,s} = (k^2+2)(k^4+4k^2+2)\mathcal{N}_{k,s+1}$,

   $\mathcal{N}_{k,s+8} + (k^6+7k^4+14k^2+7)\mathcal{N}_{k,s} = (k^2+2)(k^4+4k^2+2)\delta\mathcal{M}_{k,s+1}$,

5. $\mathcal{M}_{k,s+10}+(k^2+3)(k^6+6k^4+9k^2+3)\mathcal{M}_{k,s}=(k^4+3k^2+1)(k^4+5k^2+5)\mathcal{N}_{k,s+1}$,

   $\mathcal{N}_{k,s+10}+(k^2+3)(k^6+6k^4+9k^2+3)\mathcal{N}_{k,s}=(k^4+3k^2+1)(k^4+5k^2+5)\delta\mathcal{N}_{k,s+1}$.

In general, we have

$$\mathcal{M}_{k,s+2t}+\frac{L_{k,2t-1}}{k}\mathcal{M}_{k,s}=\frac{F_{k,2t}}{k}\mathcal{N}_{k,s+1}, \tag{5.2.1}$$

$$\mathcal{N}_{k,s+10}+\frac{L_{k,2t-1}}{k}\mathcal{N}_{k,s}=\frac{F_{k,2t}}{k}\delta\mathcal{N}_{k,s+1}. \tag{5.2.2}$$

Remark 5.28. Using $L_{k,2t-1}-F_{k,2t}=F_{k,2t-2}$ in (5.2.1), we get

$$\mathcal{M}_{k,s+2t}-\frac{F_{k,2t}}{k}\mathcal{M}_{k,s+2}+\frac{F_{k,2t-2}}{k}\mathcal{M}_{k,s}=0,$$

$$\mathcal{N}_{k,s+2t}-\frac{F_{k,2t}}{k}\mathcal{N}_{k,s+2}+\frac{F_{k,2t-2}}{k}\mathcal{N}_{k,s}=0.$$

Theorem 5.29. *For $s,t\geq 1$, we have*

1. $\mathcal{M}_{k,s+3}+\mathcal{N}_{k,s}=(k^2+3)\mathcal{M}_{k,s+1}$,

   $\mathcal{N}_{k,s+3}+\delta\mathcal{M}_{k,s}=(k^2+3)\mathcal{N}_{k,s+1}$,

2. $\mathcal{M}_{k,s+5}+(k^2+2)\mathcal{N}_{k,s}=(k^4+5k^2+5)\mathcal{M}_{k,s+1}$,

   $\mathcal{N}_{k,s+5}+\delta(k^2+2)\mathcal{M}_{k,s}=(k^4+5k^2+5)\mathcal{N}_{k,s+1}$,

3. $\mathcal{M}_{k,s+7}+(k^2+1)(k^2+3)\mathcal{N}_{k,s}=(k^6+7k^4+14k^2+7)\mathcal{M}_{k,s+1}$,

   $\mathcal{N}_{k,s+7}+\delta(k^2+1)(k^2+3)\mathcal{M}_{k,s}=(k^6+7k^4+14k^2+7)\mathcal{N}_{k,s+1}$,

4. $\mathcal{M}_{k,s+9} + (k^2+2)(k^4+4k^2+2)\mathcal{N}_{k,s} = (k^2+3)(k^6+6k^4+9k^2+3)\mathcal{M}_{k,s+1}$,

   $\mathcal{N}_{k,s+9} + \delta(k^2+2)(k^4+4k^2+2)\mathcal{M}_{k,s} = (k^2+3)(k^6+6k^4+9k^2+3)\mathcal{N}_{k,s+1}$,

5. $\mathcal{M}_{k,s+11} + (k^4+3k^2+1)(k^4+5k^2+5)\mathcal{N}_{k,s} = (k^{10}+11k^8+44k^6+77k^4+55k^2+11)\mathcal{M}_{k,s+1}$,

   $\mathcal{N}_{k,s+11} + \delta(k^4+3k^2+1)(k^4+5k^2+5)\mathcal{M}_{k,s} = (k^{10}+11k^8+44k^6+77k^4+55k^2+11)\mathcal{N}_{k,s+1}$.

In general, we have

$$\mathcal{M}_{k,s+2t+1} + \frac{F_{k,2t}}{k}\mathcal{N}_{k,s} = \frac{L_{k,2t+1}}{k}\mathcal{M}_{k,s+1}, \tag{5.2.3}$$

$$\mathcal{N}_{k,s+2t+1} + \delta\frac{F_{k,2t}}{k}\mathcal{M}_{k,s} = \frac{L_{k,2t+1}}{k}\mathcal{N}_{k,s+1}. \tag{5.2.4}$$

Remark 5.30. Using $(k^2+3)F_{k,2t} - L_{k,2t-1} = F_{k,2t-2}$ in (5.2.3), we obtain

$$\mathcal{M}_{k,s+2t+1} - \frac{L_{k,2t+1}}{k(k^2+3)}\mathcal{M}_{k,s+3} + \frac{F_{k,2t-2}}{k(k^2+3)}\mathcal{N}_{k,s} = 0,$$
$$\mathcal{N}_{k,s+2t+1} - \frac{L_{k,2t+1}}{k(k^2+3)}\mathcal{N}_{k,s+3} + \frac{F_{k,2t-2}}{k(k^2+3)}\delta\mathcal{M}_{k,s} = 0.$$

Theorem 5.31. *For $n, s, t \geq 1$, we have*

1. $$\sum_{i=0}^{n}\binom{n}{i}\mathcal{M}_{k,2i+s}=\begin{cases}\delta^{\frac{n}{2}}\mathcal{M}_{k,n+s}, & \textit{if } n \textit{ is even;}\\ \delta^{\frac{n-1}{2}}\mathcal{N}_{k,n+s}, & \textit{if } n \textit{ is odd,}\end{cases}$$
$$\sum_{i=0}^{n}\binom{n}{i}\mathcal{N}_{k,2i+s}=\begin{cases}\delta^{\frac{n}{2}}\mathcal{N}_{k,n+s}, & \textit{if } n \textit{ is even;}\\ \delta^{\frac{n+1}{2}}\mathcal{M}_{k,n+s}, & \textit{if } n \textit{ is odd}\end{cases},$$

2. $$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}\mathcal{M}_{k,4i+s}=\begin{cases}(k^2+2)^n\delta^{\frac{n}{2}}\mathcal{M}_{k,n+s}, & \textit{if } n \textit{ is even;}\\ (k^2+2)^n\delta^{\frac{n-1}{2}}\mathcal{N}_{k,n+s}, & \textit{if } n \textit{ is odd,}\end{cases}$$
$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}\mathcal{N}_{k,4i+s}=\begin{cases}(k^2+2)^n\delta^{\frac{n}{2}}\mathcal{N}_{k,n+s}, & \textit{if } n \textit{ is even;}\\ (k^2+2)^n\delta^{\frac{n+1}{2}}\mathcal{M}_{k,n+s}, & \textit{if } n \textit{ is odd}\end{cases},$$

3. $$\sum_{i=0}^{n}\binom{n}{i}(k^4+5k^2+5)^{(n-i)}\mathcal{M}_{k,6i+s}$$
$$=\begin{cases}(k^2+1)^n(k^2+3)^n\delta^{\frac{n}{2}}\mathcal{M}_{k,n+s}, & \textit{if } n \textit{ is even;}\\ (k^2+1)^n(k^2+3)^n\delta^{\frac{n-1}{2}}\mathcal{N}_{k,n+s}, & \textit{if } n \textit{ is odd,}\end{cases}$$
$$\sum_{i=0}^{n}\binom{n}{i}(k^4+5k^2+5)^{(n-i)}\mathcal{N}_{k,6i+s}$$
$$=\begin{cases}(k^2+1)^n(k^2+3)^n\delta^{\frac{n}{2}}\mathcal{N}_{k,n+s}, & \textit{if } n \textit{ is even;}\\ (k^2+1)^n(k^2+3)^n\delta^{\frac{n+1}{2}}\mathcal{M}_{k,n+s}, & \textit{if } n \textit{ is odd}\end{cases},$$

4. $$\sum_{i=0}^{n}\binom{n}{i}(k^6+7k^4+14k^2+7)^{(n-i)}\mathcal{M}_{k,8i+s}$$
$$=\begin{cases}(k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n}{2}}\mathcal{M}_{k,n+s}, & \textit{if } n \textit{ is even;}\\ (k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n-1}{2}}\mathcal{N}_{k,n+s}, & \textit{if } n \textit{ is odd,}\end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}(k^6+7k^4+14k^2+7)^{(n-i)}\mathcal{N}_{k,8i+s}$$
$$=\begin{cases}(k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n}{2}}\mathcal{N}_{k,n+s}, & \text{if } n \text{ is even;}\\ (k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n+1}{2}}\mathcal{M}_{k,n+s}, & \text{if } n \text{ is odd}\end{cases},$$

5. $$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(k^6+6k^4+9k^2+3)^{(n-i)}\mathcal{M}_{k,10i+s}$$
$$=\begin{cases}(k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n}{2}}\mathcal{M}_{k,n+s}, & \text{if } n \text{ is even;}\\ (k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n-1}{2}}\mathcal{N}_{k,n+s}, & \text{if } n \text{ is odd,}\end{cases}$$
$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(k^6+6k^4+9k^2+3)^{(n-i)}\mathcal{N}_{k,10i+s}$$
$$=\begin{cases}(k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n}{2}}\mathcal{N}_{k,n+s}, & \text{if } n \text{ is even;}\\ (k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n+1}{2}}\mathcal{M}_{k,n+s}, & \text{if } n \text{ is odd.}\end{cases}$$

In general, for $n,s,t\geq 1$, we have

$$\sum_{i=0}^{n}\binom{n}{i}k^{(i-n)}(L_{k,2t-1})^{(n-i)}\mathcal{M}_{k,2ti+s}=\begin{cases}k^{-n}(F_{k,2t})^n\delta^{\frac{n}{2}}\mathcal{M}_{k,n+s}, & \text{if } n \text{ is even;}\\ k^{-n}(F_{k,2t})^n\delta^{\frac{n-1}{2}}\mathcal{N}_{k,n+s}, & \text{if } n \text{ is odd,}\end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}k^{(i-n)}(L_{k,2t-1})^{(n-i)}\mathcal{N}_{k,2ti+s}=\begin{cases}k^{-n}(F_{k,2t})^n\delta^{\frac{n}{2}}\mathcal{N}_{k,n+s}, & \text{if } n \text{ is even;}\\ k^{-n}(F_{k,2t})^n\delta^{\frac{n+1}{2}}\mathcal{M}_{k,n+s}, & \text{if } n \text{ is odd.}\end{cases}$$

Theorem 5.32. *For $n,s,t\geq 1$, we have*

1. $$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^2+3)^i\mathcal{M}_{k,2(n-i)+n}=\begin{cases}2\delta^{\frac{n}{2}}, & \text{if } n \text{ is even;}\\ 2\delta^{\frac{n+1}{2}}, & \text{if } n \text{ is odd,}\end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^2+3)^i\mathcal{N}_{k,2(n-i)+n}=\begin{cases}2\delta^{\frac{n+2}{2}}, & \text{if } n \text{ is even;}\\ 2\delta^{\frac{n+1}{2}}, & \text{if } n \text{ is odd.}\end{cases}$$

2. $$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^4+5k^2+5)^i\mathcal{M}_{k,4(n-i)+n}=\begin{cases}2(k^2+2)^n\delta^{\frac{n}{2}}, & \text{if } n \text{ is even;}\\ 2(k^2+2)^n\delta^{\frac{n+1}{2}}, & \text{if } n \text{ is odd,}\end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^4+5k^2+5)^i\mathcal{N}_{k,4(n-i)+n}=\begin{cases}2(k^2+2)^n\delta^{\frac{n+2}{2}}, & \text{if } n \text{ is even;}\\ 2(k^2+2)^n\delta^{\frac{n+1}{2}}, & \text{if } n \text{ is odd.}\end{cases}$$

3. $$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^6+7k^4+14k^2+7)^i\mathcal{M}_{k,6(n-i)+n}$$
$$=\begin{cases}2(k^2+1)^n(k^2+3)^n\delta^{\frac{n}{2}}, & \text{if } n \text{ is even;}\\ 2(k^2+1)^n(k^2+3)^n\delta^{\frac{n+1}{2}}, & \text{if } n \text{ is odd,}\end{cases}$$
$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^6+7k^4+14k^2+7)^i\mathcal{N}_{k,6(n-i)+n}$$
$$=\begin{cases}2(k^2+1)^n(k^2+3)^n\delta^{\frac{n+2}{2}}, & \text{if } n \text{ is even;}\\ 2(k^2+1)^n(k^2+3)^n\delta^{\frac{n+1}{2}}, & \text{if } n \text{ is odd.}\end{cases}$$

4. $$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^2+3)^i(k^6+6k^4+9k^2+3)^i\mathcal{M}_{k,8(n-i)+n}$$
$$=\begin{cases}2(k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n}{2}}, & \text{if } n \text{ is even;}\\ 2(k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n+1}{2}}, & \text{if } n \text{ is odd,}\end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^2+3)^i(k^6+6k^4+9k^2+3)^i\mathcal{N}_{k,8(n-i)+n}$$
$$=\begin{cases}2(k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n+2}{2}}, & \textit{if } n \textit{ is even;}\\ 2(k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n+1}{2}}, & \textit{if } n \textit{ is odd.}\end{cases}$$

5. $$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^{10}+11k^8+44k^6+44k^4+55k^2+11)^i\mathcal{M}_{k,10(n-i)+n}$$
$$=\begin{cases}2(k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n}{2}}, & \textit{if } n \textit{ is even;}\\ 2(k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n+1}{2}}, & \textit{if } n \textit{ is odd,}\end{cases}$$
$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^{10}+11k^8+44k^6+44k^4+55k^2+11)^i\mathcal{N}_{k,10(n-i)+n}$$
$$=\begin{cases}2(k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n+2}{2}}, & \textit{if } n \textit{ is even;}\\ 2(k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n+1}{2}}, & \textit{if } n \textit{ is odd.}\end{cases}$$

In general, for $n, s, t \geq 1$, we have

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}k^{-i}(L_{k,2t+1})^i\mathcal{M}_{k,2t(n-i)+n}=\begin{cases}2(k)^{-n}(F_{k,2t})^n\delta^{\frac{n}{2}}, & \textit{if } n \textit{ is even;}\\ 2(k)^{-n}(F_{k,2t})^n\delta^{\frac{n+1}{2}}, & \textit{if } n \textit{ is odd,}\end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}k^{-i}(L_{k,2t+1})^i\mathcal{N}_{k,2t(n-i)+n}=\begin{cases}2(k)^{-n}(F_{k,2t})^n\delta^{\frac{n+2}{2}}, & \textit{if } n \textit{ is even;}\\ 2(k)^{-n}(F_{k,2t})^n\delta^{\frac{n+1}{2}}, & \textit{if } n \textit{ is odd.}\end{cases}$$

In next section, we prove some properties of the generalized k-Lucas sequence.

The Proofs of the Main Results

Proof of Lemma(5.14): We prove only (a), (c) and (d) since the proofs of (b) and (e) are similar.

Proof of (a): Since r_1 and r_2 are roots of $r^2 - kr - 1 = 0$, then

$$r_1^2 = kr_1 + 1, \tag{5.2.5}$$

$$r_2^2 = kr_2 + 1. \tag{5.2.6}$$

This completes the proof of (a).

Proof of (c): From (b), we have

$$\begin{aligned}
u^{2n} &= F_{k,n}u^{n+1} + u^n F_{k,n-1} \\
&= F_{k,n}(uF_{k,n+1} + F_{k,n}) + u^n F_{k,n-1} \\
&= uF_{k,n}F_{k,n+1} + F_{k,n-1}u^n + F_{k,n}^2 \\
&= (u^n - F_{k,n-1})F_{k,n+1} + F_{k,n-1}u^n + F_{k,n}^2 \\
&= u^n(F_{k,n+1} + F_{k,n-1}) + F_{k,n}^2 - F_{k,n}F_{k,n-1}.
\end{aligned}$$

Using $F_{k,n-1}F_{k,n+1} - F_{k,n}^2 = (-1)^n$ and $F_{k,n+1} + F_{k,n-1} = L_{k,n}$, we obtain

$$u^{2n} = L_{k,n}u^n - (-1)^n.$$

This completes the proof of (c).

Proof of (d): If $u = r_1$, then we have

$$\begin{aligned}
F_{k,tn}r_1^n - (-1)^n F_{k,(t-1)n} &= (\frac{r_1^{tn} - r_2^{tn}}{r_1 - r_2})r_1^n - (r_1r_2)^n(\frac{r_1^{(t-1)n} - r_2^{(t-1)n}}{r_1 - r_2}) \\
&= (\frac{r_1^n - r_2^n}{r_1 - r_2})r_1^{tn} \\
&= F_{k,n}r_1^{tn}.
\end{aligned}$$

This completes the proof of (d).

The proofs of Theorems (5.18), (5.23) are similar. Hence, we prove only Theorem (5.15).

Proof of Theorem(5.15): We prove only (a), since the proofs of (b), (c) and (d) are similar.

Proof of (1): From 5.14(b), we have

$$r_1^n = F_{k,n}r_1 + F_{k,n-1}, \tag{5.2.7}$$

$$r_2^n = F_{k,n}r_2 + F_{k,n-1}. \tag{5.2.8}$$

Multiplying (5.2.7) by $\bar{r_1}r_1^t$, (5.2.8) by $\bar{r_2}r_2^t$ and subtracting, we obtain

$$\frac{\bar{r_1}r_1^{n+t} - \bar{r_2}r_2^{n+t}}{r_1 - r_2} = F_{k,n}(\frac{\bar{r_1}r_1^{t+1} - \bar{r_2}r_2^{t+1}}{r_1 - r_2}) + F_{k,n-1}(\frac{\bar{r_1}r_1^t - \bar{r_2}r_2^t}{r_1 - r_2}).$$

Hence, it gives that

$$\mathcal{M}_{k,n+t} = F_{k,n}\mathcal{M}_{k,t+1} + F_{k,n-1}\mathcal{M}_{k,t}.$$

This completes the proof of (a).

The proofs of Theorems (5.19)-(5.21) and (5.24) are similar. Hence, we prove only Theorem (5.16).

Proof of Theorem(5.16): We prove only (3), since the proofs of (1), (2) and (4)-(8) are similar.

Proof of (3): From 5.14(b), we have

$$r_1^r = F_{k,r}r_1 + F_{k,r-1}, \tag{5.2.9}$$

$$r_2^r = F_{k,r}r_2 + F_{k,r-1}. \tag{5.2.10}$$

Now, by the binomial theorem, we have

$$\bar{r_1}r_1^{rn} = \sum_{i=0}^{n}\binom{n}{i}F_{k,r}^{i}F_{k,r-1}^{n-i}\bar{r_1}r_1^{i}, \tag{5.2.11}$$

$$\bar{r_2}r_2^{rn} = \sum_{i=0}^{n}\binom{n}{i}F_{k,r}^{i}F_{k,r-1}^{n-i}\bar{r_2}r_2^{i}. \tag{5.2.12}$$

Now, by subtracting (5.2.11) from (5.2.12), we obtain

$$\frac{\bar{r_1}r_1^{rn+t}-\bar{r_2}r_2^{rn+t}}{r_1-r_2}=\sum_{i=0}^{n}\binom{n}{i}F_{k,r}^{i}F_{k,r-1}^{n-i}(\frac{\bar{r_1}r_1^{i+t}-\bar{r_2}r_2^{i+t}}{r_1-r_2}).$$

Hence, it gives that

$$\mathcal{M}_{k,rn+t}=\sum_{i=0}^{n}\binom{n}{i}F_{k,r}^{i}F_{k,r-1}^{n-i}\mathcal{M}_{k,i+t}.$$

Now, by adding (5.2.11) and (5.2.12), we get

$$\bar{r_1}r_1^{rn+t}+\bar{r_2}r_2^{rn+t}=\sum_{i=0}^{n}\binom{n}{i}F_{k,r}^{i}F_{k,r-1}^{n-i}(\bar{r_1}r_1^{i+t}+\bar{r_2}r_2^{i+t}).$$

Hence, it gives that

$$\mathcal{N}_{k,rn+t}=\sum_{i=0}^{n}\binom{n}{i}F_{k,r}^{i}F_{k,r-1}^{n-i}\mathcal{N}_{k,i+t}.$$

This completes the proof of (3).

Proof of Lemma(5.17): We prove only (1) and (2) since the proofs of (3)-(11) are similar.

Proof of (1): Using (5.2.5) and (5.2.6), we have

$$\begin{aligned}u^3&=u^2u\\&=(ku+1)u\\&=ku^2+u\end{aligned}$$

$$= k(ku+1)+u$$

$$= k^2u+k+u$$

$$= k+(k^2+1)u.$$

This completes the proof of (1).

Proof of (2): Using (5.2.5) and (5.2.6), we have

$$\begin{aligned}
1+ku++u^6 &= u^2+u^6\\
&= u^2+u^4(ku+1)\\
&= u^2+ku^5+u^4\\
&= u^2+ku^3(ku+1)+u^4\\
&= u^2+k^2u^4+ku^3+u^4\\
&= (k^2+1)u^4+ku^3+u^2\\
&= (k^2+1)u^4+u^2(ku+1)\\
&= (k^2+1)u^4+u^4\\
&= (k^2+2)u^4\\
&= F_{k,2}u^4.
\end{aligned}$$

This completes the proof of (2). The proofs of lemma (5.26) are similar. Hence, we prove only Lemma (5.25).

Proof of Lemma(5.25): We prove only (1) and (2) since the proofs of (3) - (5) are similar.

Proof of (1): Using (**??**), we have

$$
\begin{aligned}
r_1\sqrt{\delta} - 1 &= r_1(r_1 - r_2) - 1 \\
&= r_1^2 - r_1 r_2 - 1 \\
&= r_1^2 + 1 - 1 \\
&= r_1^2.
\end{aligned}
$$

This completes the proof of (1).

Proof of (2): Using (5.2.5) and (5.2.6), we have

$$
\begin{aligned}
(k^2+2)r_1\sqrt{\delta} - (k^2+3) &= (k^2+2)r_1(r_1 - r_2) - (k^2+3) \\
&= (k^2+2)(r_1^2 - r_1 r_2) - (k^2+3) \\
&= (k^2+2)(r_1^2 + 1) - (k^2+3) \\
&= r_1^2(k^2+2) + (k^2+2) - (k^2+3) \\
&= r_1^2 k^2 + 2r_1^2 - 1 \\
&= r_1^2 k^2 + 2(kr_1 + 1) - 1 \\
&= r_1^2 k^2 + 2kr_1 + 1 \\
&= r_1^2 k^2 + kr_1 + kr_1 + 1 \\
&= (kr_1 + 1)(kr_1 + 1)
\end{aligned}
$$

$$= r_1^2 r_1^2$$

$$= r_1^4.$$

This completes the proof of (2).

The proofs of Theorems (5.27) and (5.29) are similar. Hence, we prove only Theorem (5.27).

Proof of Theorem(5.27): We prove only (2), since the proofs of (1) and (3)-(5) are similar.

*Proof of (2):*From 5.25(2), we have

$$r_1^4 + (k^2+3) = (k^2+2) r_1 \sqrt{\delta}, \tag{5.2.13}$$

$$r_2^4 + (k^2+3) = -(k^2+2) r_2 \sqrt{\delta}. \tag{5.2.14}$$

Multiplying (5.2.13) by $\bar{r_1} r_1^s$, (5.2.14) by $\bar{r_2} r_2^s$ and subtracting, we obtain

$$\frac{\bar{r_1} r_1^{s+4} - \bar{r_2} r_2^{s+4}}{r_1 - r_2} + (k^2+3) \frac{\bar{r_1} r_1^s - \bar{r_2} r_2^s}{r_1 - r_2} = (k^2+2)(\bar{r_1} r_1^{s+1} + \bar{r_2} r_2^{s+1})$$

Hence, it gives that

$$\mathcal{M}_{k,s+4} + (k^2+3)\mathcal{M}_{k,s} = (k^2+2)\mathcal{N}_{k,s+1}.$$

Multiplying (5.2.13) by $\bar{r_1}r_1^s$, (5.2.14) by $\bar{r_2}r_2^s$ and adding, we obtain

$$\bar{r_1}r_1^{s+4}+\bar{r_2}r_2^{s+4}+(k^2+3)(\bar{r_1}r_1^s+\bar{r_2}r_2^s)=(k^2+2)\delta(\frac{\bar{r_1}r_1^{s+1}-\bar{r_2}r_2^{s+1}}{r_1-r_2})$$

Hence, it gives that

$$\mathcal{N}_{k,s+4}+(k^2+3)\mathcal{N}_{k,s}=(k^2+2)\delta\mathcal{M}_{k,s+1}.$$

This completes the proof of (3).

The proofs of Theorems (5.31) and (5.32) are similar. Hence, we prove only Theorem (5.31).

Proof of Theorem(5.31): We prove only (2), since the proofs of (1)and (3)-(5) are similar.

*Proof of (2):*From 5.25(2), we have

$$r_1^4+(k^2+3)=(k^2+2)r_1\sqrt{\delta},$$

$$r_2^4+(k^2+3)=-(k^2+2)r_2\sqrt{\delta}.$$

Now, by the binomial theorem, we have

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(\bar{r_1}r_1^{4i+s})=(k^2+2)^n\delta^{\frac{n}{2}}(\bar{r_1}r_1^{n+s}), \qquad (5.2.15)$$

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(\bar{r_2}r_2^{4i+s}) = (-1)^n(k^2+2)^n\delta^{\frac{n}{2}}(\bar{r_2}r_2^{n+s}). \quad (5.2.16)$$

Now, by subtracting (5.2.15) from (5.2.16), we obtain

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}\left(\frac{\bar{r_1}r_1^{4i+s}-\bar{r_2}r_2^{4i+s}}{r_1-r_2}\right) = (k^2+2)^n\delta^{\frac{n}{2}}\left(\frac{\bar{r_1}r_1^{n+s}-(-1)^n\bar{r_2}r_2^{n+s}}{r_1-r_2}\right).$$

Hence, it gives that

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}\mathcal{M}_{k,4i+s} = \begin{cases}(k^2+2)^n\delta^{\frac{n}{2}}\mathcal{M}_{k,n+s}, & \text{if } n \text{ is even;}\\[2ex] (k^2+2)^n\delta^{\frac{n-1}{2}}\mathcal{N}_{k,n+s}, & \text{if } n \text{ is odd.}\end{cases}$$

Now, by adding (5.2.15) and (5.2.16), we get

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(\bar{r_1}r_1^{4i+s}+\bar{r_2}r_2^{4i+s}) = (k^2+2)^n\delta^{\frac{n}{2}}(\bar{r_1}r_1^{n+s}+(-1)^n\bar{r_2}r_2^{n+s}).$$

Hence, it gives that

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}\mathcal{N}_{k,4i+s} = \begin{cases}(k^2+2)^n\delta^{\frac{n}{2}}\mathcal{N}_{k,n+s}, & \text{if } n \text{ is even;}\\[2ex] (k^2+2)^n\delta^{\frac{n+1}{2}}\mathcal{M}_{k,n+s}, & \text{if } n \text{ is odd.}\end{cases}$$

This completes the proof of (3).

5.3 Some Congruence Properties of the Generalized k-Lucas Sequences

In this section, we established and proved certain congruence properties of the generalized k-Lucas sequence.

Theorem 5.33. *For $n, t \geq 1$ and $\mathcal{D}_{k,n} = \mathcal{M}_{k,n}$ or $\mathcal{N}_{k,n}$, we have*

1. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,6j+t} \equiv 0 \pmod{L_{k,2}}$.

2. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,10j+t} \equiv 0 \pmod{L_{k,4}}$.

3. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,18j+t} \equiv 0 \pmod{L_{k,8}}$.

4. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,34j+t} \equiv 0 \pmod{L_{k,16}}$.

5. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,66j+t} \equiv 0 \pmod{L_{k,32}}$.

6. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,130j+t} \equiv 0 \pmod{L_{k,64}}$.

7. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,258j+t} \equiv 0 \pmod{L_{k,128}}$.

8. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,514j+t} \equiv 0 \pmod{L_{k,256}}$.

9. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,1026j+t} \equiv 0 \pmod{L_{k,512}}$.

10. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,2050j+t} \equiv 0 \pmod{L_{k,1024}}$.

In general, for $r, n, t \geq 1$, we have

$$\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,(2^{r+2}+2)j+t} \equiv 0 \pmod{L_{k,2^{r+1}}}.$$

Theorem 5.34. *For $n, t \geq 1$ and $\mathcal{D}_{k,n} = \mathcal{M}_{k,n}$ or $\mathcal{N}_{k,n}$, we have*

1. $\mathcal{D}_{k,6n+t} - \sum_{j=0}^{n} \binom{n}{j} k^j(-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,2}}$.
2. $\mathcal{D}_{k,10n+t} - \sum_{j=0}^{n} \binom{n}{j} k^j(-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,4}}$.
3. $\mathcal{D}_{k,18n+t} - \sum_{j=0}^{n} \binom{n}{j} k^j(-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,8}}$.
4. $\mathcal{D}_{k,34n+t} - \sum_{j=0}^{n} \binom{n}{j} k^j(-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,16}}$.
5. $\mathcal{D}_{k,66n+t} - \sum_{j=0}^{n} \binom{n}{j} k^j(-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,32}}$.
6. $\mathcal{D}_{k,130n+t} - \sum_{j=0}^{n} \binom{n}{j} k^j(-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,64}}$.
7. $\mathcal{D}_{k,258n+t} - \sum_{j=0}^{n} \binom{n}{j} k^j(-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,128}}$.
8. $\mathcal{D}_{k,514n+t} - \sum_{j=0}^{n} \binom{n}{j} k^j(-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,256}}$.
9. $\mathcal{D}_{k,1026n+t} - \sum_{j=0}^{n} \binom{n}{j} k^j(-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,514}}$.
10. $\mathcal{D}_{k,2050n+t} - \sum_{j=0}^{n} \binom{n}{j} k^j(-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,1024}}$.

In general, we have

$$\mathcal{D}_{k,(2^{r+2}+2)n+t} - \sum_{j=0}^{n} \binom{n}{j} k^j(-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,2^{r+1}}}.$$

The proofs of Theorems (5.33) and (5.34) are similar. Hence, we prove only Theorem (5.33).

Proof of Theorem(5.33): We prove only (1), since the proofs of (2)-(10) are similar.

*Proof of (1):*From Theorem (5.19;(1)), For $n, t \geq 1$ and $\mathcal{D}_{k,n} = \mathcal{M}_{k,n}$ or $\mathcal{N}_{k,n}$, we have

$$\begin{aligned}
\mathcal{D}_{k,n+t} &= \sum_{i+j+s=n;_{i\neq 0}} \binom{n}{i,j} k^{-n}(-1)^{j+s} {L_{k,2}}^i \mathcal{D}_{k,4i+6j+t} \\
&+ \sum_{i+j+s=n;_{i=0}} \binom{n}{i,j} k^{-n}(-1)^{j+s} {L_{k,2}}^i \mathcal{D}_{k,4i+6j+t}, \\
&= \sum_{i+j+s=n;_{i\neq 0}} \binom{n}{i,j} k^{-n}(-1)^{j+s} {L_{k,2}}^i \mathcal{D}_{k,4i+6j+t} \\
&\qquad + \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^{n} \mathcal{D}_{k,6j+t}.
\end{aligned}$$

$$\begin{aligned}
&\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^{n} \mathcal{D}_{k,6j+t} \\
&= \sum_{i+j+s=n;_{i\neq 0}} \binom{n}{i,j} k^{-n}(-1)^{j+s} {L_{k,2}}^i \mathcal{D}_{k,4i+6j+t}, \\
&\therefore L_{k,2} \quad \text{divides} \quad (\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^{n} \mathcal{D}_{k,6j+t}), \\
&\therefore \mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^{n} \mathcal{D}_{k,6j+t} \equiv 0 \pmod{L_{k,2}}
\end{aligned}$$

This completes the proof of (1).

5.4 Concluding Remarks

In this chapter, we present generating functions and Binet formulas for generalized k-Lucas sequence and its companion sequence. Also, derived some binomial and congruence sums containing these sequences.

Chapter 6

Applications of k- Fibonacci and k- Lucas Sequences in Non-Associative Algebra

The hyperbolic quaternions form a 4-dimensional non-associative and non-commutative algebra over the set of real numbers. In first section of this paper, we introduce the hyperbolic k-Fibonacci and k-Lucas quaternions. We present generating functions and Binet formulas for the k-Fibonacci and k-Lucas hyperbolic quaternions, and establish binomial and congruence sums of hyperbolic k-Fibonacci and k-Lucas quaternions. In second section of this paper, We investigate some binomial and congruence properties for the k-Fibonacci and k-Lucas hyperbolic octonions. In addition, we present several well-known identities such as Catalan's, Cassini's and d'Ocagne's identities for k-Fibonacci and k-Lucas hyperbolic octonions.

6.1 Introduction

The well known integer sequence, Fibonacci sequence is defined by the numbers which satisfy the second order recurrence relation $F_n = F_{n-1} + F_{n-2}$ with the initial conditions $F_0 = 0$ and $F_1 = 1$. Fibonacci numbers have many interesting properties and applications in various research areas such as Architecture, Engineering, Nature and Art. The Lucas sequence is companion sequence of Fibonacci sequence defined with the Lucas numbers which are defined with the recurrence relation $L_n = L_{n-1} + L_{n-2}$ with the initial conditions $L_0 = 2$ and $L_1 = 1$. Binet's formulas for the Fibonacci and Lucas numbers are

$$F_n = \frac{{r_1}^n - {r_2}^n}{r_1 - r_2}$$

and

$$L_n = {r_1}^n + {r_2}^n$$

respectively, where $r_1 = \dfrac{1+\sqrt{5}}{2}$ and $r_2 = \dfrac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $x^2 - x - 1 = 0$. The positive root r_1 is known as

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Properties of k-Fibonacci and k-Lucas Octonions, Indian Journal of Pure and Applied Mathematics

the golden ratio. The Fibonacci and Lucas sequences are generalised by changing the initial conditions or changing the recurrence relation. One of the generalizations of the Fibonacci sequence is k- Fibonacci sequence first introduced by Falcon and Plaza [**?**]. The k- Fibonacci sequence is defined by the numbers which satisfy the second order recurrence relation $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ with the initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$. Falcon [65] defined the k- Lucas sequence which is companion sequence of k- Fibonacci sequence defined with the k- Lucas numbers which are defined with the recurrence relation $L_{k,n} = kL_{k,n-1} + L_{k,n-2}$ with the initial conditions $L_{k,0} = 2$ and $L_{k,1} = k$. Binet's formulas for the k- Fibonacci and k- Lucas numbers are

$$F_{k,n} = \frac{{r_1}^n - {r_2}^n}{r_1 - r_2}$$

and

$$L_{k,n} = {r_1}^n + {r_2}^n$$

respectively, where $r_1 = \dfrac{k + \sqrt{k^2+4}}{2}$ and $r_2 = \dfrac{k - \sqrt{k^2+4}}{2}$ are the roots of the characteristic equation $x^2 - kx - 1 = 0$. The characteristic roots r_1 and r_2 satisfy the properties

$$r_1 - r_2 = \sqrt{k^2+4} = \sqrt{\delta}, \quad r_1 + r_2 = k, \quad r_1 r_2 = -1.$$

The quaternions are generalized numbers. The quaternions first introduced by Irish mathematician William Rowan Hamilton in 1843. Hamilton [79] introduced the set of quaternions form a 4-dimensional real vector space with a multiplicative operation. The quaternions are used in applied sciences such as physics computer science and Clifford algebras in mathematics. In particular, they are important in mechanics [66], chemistry[68], kinematics [67], quantum mechanics[69], differential geometry, pure algebra. A quaternion a, with real components a_0, a_1, a_2, a_3 and basis $1, i, j, k$, is an element of the form

$$a = a_0 + a_1 i + a_2 j + a_3 k = \big(a_0, a_1, a_2, a_3\big),$$

where

$$i^2 = j^2 = k^2 = ijk = -1,$$

$$ij = k = -ji, jk = i = -kj, ki = j == ik.$$

Horadam[70] defined the n^{th} Fibonacci and n^{th} Lucas quaternions as

$$\bar{F_n} = F_n + F_{n+1} i + F_{n+2} j + F_{n+3} k = \big(F_n, F_{n+1}, F_{n+2}, F_{n+3}\big)$$

and

$$\bar{L_n} = L_n + L_{n+1} i + L_{n+2} j + L_{n+3} k = \big(L_n, L_{n+1}, L_{n+2}, L_{n+3}\big)$$

respectively.

Ramirez [71] has defined and studied the k-Fibonacci and k-Lucas quaternions as

$$\bar{F_{k,n}} = F_{k,n}+F_{k,n+1}i+F_{k,n+2}j+F_{k,n+3}k = \big(F_{k,n}, F_{k,n+1}, F_{k,n+2}, F_{k,n+3}\big)$$

and

$$\bar{L_{k,n}} = L_{k,n}+L_{k,n+1}i+L_{k,n+2}j+L_{k,n+3}k = \big(L_{k,n}, L_{k,n+1}, L_{k,n+2}, L_{k,n+3}\big)$$

respectively. where $F_{k,n}$ is the n^{th} k-Fibonacci sequence and $L_{k,n}$ is the n^{th} k-Lucas sequence.

Different quaternions of sequences have been studied by different researchers. For example, Iyer [72, 73] obtained various relations containing the Fibonacci and Lucas quaternions. Halici [74] studied some combinatorial properties of Fibonacci quaternions. Akyigit et al. [75, 76] established and investigated the Fibonacci generalized quaternions and split Fibonacci quaternions. Catarino [77] obtained different properties of the $h(x)$-Fibonacci quaternion polynomials. Polatli and Kesim [78] have introduced quaternions with generalized Fibonacci and Lucas number components.

A hyperbolic quaternion h is an expression of the form

$$h = h_1i_1 + h_2i_2 + h_3i_3 + h_4i_4 = \big(h_1, h_2, h_3, h_4\big),$$

with real components $h_1,\ h_2,\ h_3,\ h_4$ and i_1, i_2, i_3, i_4 are hyperbolic quaternion units which satisfy the non-commutative multiplication rules

$$i_2{}^2 = i_3{}^2 = i_4{}^2 = i_2 i_3 i_4 = +1, i_1 = 1$$

$$i_2 i_3 = i_4 = -i_3 i_2, i_3 i_4 = i_2 = -i_4 i_3, i_4 i_2 = i_3 = -i_2 i_4. \tag{6.1.1}$$

The scalar and the vector part of a hyperbolic quaternion h are denoted by $S_h = h_1$ and $\overrightarrow{V}_h = h_2 i_2 + h_3 i_3 + h_4 i_4$, respectively. Thus, a hyperbolic quaternion h is given by $h = S_h + \overrightarrow{V}_h$. For any two hyperbolic quaternion $h^{(1)} = h_1^{(1)} i_1 + h_2^{(1)} i_2 + h_3^{(1)} i_3 + h_4^{(1)} i_4$ and $h^{(2)} = h_1^{(2)} i_1 + h_2^{(2)} i_2 + h_3^{(2)} i_3 + h_4^{(2)} i_4$. Addition and subtraction of the hyperbolic quaternions is defined by

$$h^{(1)} \pm h^{(2)} = \left(h_1^{(1)} i_1 + h_2^{(1)} i_2 + h_3^{(1)} i_3 + h_4^{(1)} i_4\right)$$
$$\pm \left(h_1^{(2)} i_1 + h_2^{(2)} i_2 + h_3^{(2)} i_3 + h_4^{(2)} i_4\right)$$
$$= \left(h_1^{(1)} \pm h_1^{(2)}\right) i_1 + \left(h_2^{(1)} \pm h_2^{(2)}\right) i_2 + \left(h_3^{(1)} \pm h_3^{(2)}\right) i_3 + \left(h_4^{(1)} \pm h_4^{(2)}\right) i_4\Big)$$

Multiplication of the hyperbolic quaternions is defined by

$$h^{(1)} \cdot h^{(2)} = \left(h_1^{(1)} i_1 + h_2^{(1)} i_2 + h_3^{(1)} i_3 + h_4^{(1)} i_4\right)$$
$$\cdot \left(h_1^{(2)} i_1 + h_2^{(2)} i_2 + h_3^{(2)} i_3 + h_4^{(2)} i_4\right)$$

$$
\begin{aligned}
&= \left(h_1^{(1)}h_1^{(2)} + h_2^{(1)}h_2^{(2)} + h_3^{(1)}h_3^{(2)} + h_4^{(1)}h_4^{(2)}\right) \\
&+ \left(h_1^{(1)}h_2^{(2)} + h_2^{(1)}h_1^{(2)} + h_3^{(1)}h_4^{(2)} - h_4^{(1)}h_3^{(2)}\right)i_2 \\
&+ \left(h_1^{(1)}h_3^{(2)} - h_2^{(1)}h_4^{(2)} + h_3^{(1)}h_1^{(2)} + h_4^{(1)}h_2^{(2)}\right)i_3 \\
&+ \left(h_1^{(1)}h_4^{(2)} + h_2^{(1)}h_3^{(2)} - h_3^{(1)}h_2^{(2)} + h_4^{(1)}h_1^{(2)}\right)i_4.
\end{aligned}
$$

A. Cariow and G. Cariow [80] state low multiplicative complexity algorithm for multiplying two hyperbolic octonions.

The conjugate of hyperbolic quaternion h is denoted by $\widehat{h}$ and it is

$$\widehat{h} = h_1 i_1 - h_2 i_2 - h_3 i_3 - h_4 i_4 = \left(h_1, -h_2, -h_3, -h_4\right).$$

The norm of h is defined as

$$N_h = h \cdot \widehat{h} = {h_1}^2 - {h_2}^2 - {h_3}^2 - {h_4}^2.$$

In the present paper, our main aim is to define hyperbolic k-Fibonacci quaternion $\bar{\mathcal{H}}^{\mathcal{F}}{}_{k,n}$ and hyperbolic k-Lucas quaternion $\bar{\mathcal{H}}^{\mathcal{L}}{}_{k,n}$ and derive the relations connecting the hyperbolic k-Fibonacci and k-Lucas quaternions. We have adapted the methods of Carlitz [61] and Zhizheng Zhang [62] to the hyperbolic k-Fibonacci and k-Lucas quaternions and derived some fundamental and congruence identities for these quaternions.

A hyperbolic octonion $\mathcal{O}$ is an expression of the form

$$\mathcal{O} = h_0 + h_1 i_1 + h_2 i_2 + h_3 i_3 + h_4 e_4 + h_5 e_5 + h_6 e_6 + h_7 e_7$$
$$= \langle h_0, h_1, h_2, h_3, h_4, h_5, h_6, h_7 \rangle,$$

with real components h_0, h_1, h_2, h_3, h_4, h_5, h_6, h_7 and i_1, i_2, i_3 are quaternion imaginary units, $e_4({e_4}^2 = 1)$ is a counter imaginary unit, and the bases of hyperbolic octonions are defined as follows:

$$i_1 e_4 = e_5, i_2 e_4 = e_6, i_3 e_4 = e_7,$$
$${e_4}^2 = {e_5}^2 = {e_6}^2 = {e_7}^2 = 1.$$

The bases of hyperbolic octonion $\mathcal{O}$ have multiplication rules as in Table- (??).

$\cdot$	i_1	i_2	i_3	e_4	e_5	e_6	e_7
i_1	-1	i_3	$-i_2$	e_5	e_4	$-e_7$	e_6
i_2	$-e_3$	-1	i_1	e_6	e_7	e_4	$-e_5$
i_3	i_2	$-i_1$	-1	e_7	$-e_6$	e_5	e_4
e_4	$-e_5$	$-e_6$	$-e_7$	1	i_1	i_2	i_3
e_5	$-e_4$	$-e_7$	e_6	$-i_1$	1	i_3	$-i_2$
e_6	e_7	$-e_4$	$-e_5$	$-i_2$	$-i_3$	1	i_1
e_7	$-e_6$	e_5	$-e_4$	$-i_3$	i_2	$-i_1$	1

TABLE 6.1: Rules for multiplication of hyperbolic octonion bases

A. Cariow and G. Cariow [80] state low multiplicative complexity algorithm for multiplying two hyperbolic octonions.

The scalar and the vector part of a hyperbolic octonion $\mathcal{O}$ are denoted by $S_{\mathcal{O}} = h_0$ and $\overrightarrow{V}_{\mathcal{O}} = h_1 i_1 + h_2 i_2 + h_3 i_3 + h_4 e_4 + h_5 e_5 + h_6 e_6 + h_7 e_7$, respectively. Thus, a hyperbolic octonion $\mathcal{O}$ is given by $\mathcal{O} = S_{\mathcal{O}} + \overrightarrow{V}_{\mathcal{O}}$.

For any two hyperbolic octonions $\mathcal{O}^{(h)} = h_0 + h_1 i_1 + h_2 i_2 + h_3 i_3 + h_4 e_4 + h_5 e_5 + h_6 e_6 + h_7 e_7$ and $\mathcal{O}^{(H)} = H_0 + H_1 i_1 + H_2 i_2 + H_3 i_3 + H_4 e_4 + H_5 e_5 + H_6 e_6 + H_7 e_7$ addition and subtraction of the hyperbolic octonions is defined by

$$\begin{aligned}
\mathcal{O}^{(h)} \pm \mathcal{O}^{(H)} &= \big(h_0 + h_1 i_1 + h_2 i_2 + h_3 i_3 + h_4 e_4 + h_5 e_5 + h_6 e_6 + h_7 e_7\big) \\
&\pm \big(H_0 + H_1 i_1 + H_2 i_2 + H_3 i_3 + H_4 e_4 + H_5 e_5 + H_6 e_6 + H_7 e_7\big) \\
&= \big(h_0 \pm H_0\big) + \big(h_1 \pm H_1\big) i_1 + \big(h_2 \pm H_2\big) i_2\big) + \big(h_3 \pm H_3\big) i_3 \\
&+ \big(h_4 \pm H_4\big) e_4 + \big(h_5 \pm H_5\big) e_5 + \big(h_6 \pm H_6\big) e_6 + \big(h_7 \pm H_7\big) e_7.
\end{aligned}$$

Multiplication of the hyperbolic octonions is defined by

$$\begin{aligned}
\mathcal{O}^{(h)} \cdot \mathcal{O}^{(H)} &= \big(h_0 + h_1 i_1 + h_2 i_2 + h_3 i_3 + h_4 e_4 + h_5 e_5 + h_6 e_6 + h_7 e_7\big) \\
&\cdot \big(H_0 + H_1 i_1 + H_2 i_2 + H_3 i_3 + H_4 e_4 + H_5 e_5 + H_6 e_6 + H_7 e_7\big) \\
&= \mathcal{O}_0 + \mathcal{O}_1 i_1 + \mathcal{O}_2 i_2 + \mathcal{O}_3 i_3 + \mathcal{O}_4 e_4 + \mathcal{O}_5 e_5 + \mathcal{O}_6 e_6 + \mathcal{O}_7 e_7,
\end{aligned}$$

where,

$$\mathcal{O}_0 = h_0 H_0 - h_1 H_1 - h_2 H_2 - h_3 H_3 + h_4 H_4 + h_5 H_5 + h_6 H_6 + h_7 H_7,$$

$$
\begin{aligned}
\mathcal{O}_1 &= h_0H_1 + h_1H_0 + h_2H_3 - h_3H_2 + h_4H_5 - h_5H_4 + h_6H_7 - h_7H_6,\\
\mathcal{O}_2 &= h_0H_2 - h_1H_3 + h_2H_0 + h_3H_1 + h_4H_6 - h_5H_7 - h_6H_4 + h_7H_5,\\
\mathcal{O}_3 &= h_0H_3 + h_1H_2 - h_2H_1 + h_3H_0 + h_4H_7 + h_5H_6 - h_6H_5 - h_7H_4,\\
\mathcal{O}_4 &= h_0H_4 + h_1H_5 + h_2H_6 + h_3H_7 + h_4H_0 - h_5H_1 - h_6H_2 - h_7H_3,\\
\mathcal{O}_5 &= h_0H_5 + h_1H_4 - h_2H_7 + h_3H_6 - h_4H_1 + h_5H_0 - h_6H_3 + h_7H_2,\\
\mathcal{O}_6 &= h_0H_6 + h_1H_7 + h_2H_4 - h_3H_5 - h_4H_2 + h_5H_3 + h_6H_0 - h_7H_1,\\
\mathcal{O}_7 &= h_0H_7 - h_1H_6 + h_2H_5 + h_3H_4 - h_4H_3 - h_5H_2 + h_6H_1 + h_7H_0.
\end{aligned}
$$

The conjugate of hyperbolic octonion $\mathcal{O}$ is denoted by $\bar{\mathcal{O}}$ and it is

$$\bar{\mathcal{O}} = h_0 - h_1i_1 - h_2i_2 - h_3i_3 - h_4e_4 - h_5e_5 - h_6e_6 - h_7e_7.$$

The norm of $\mathcal{O}$ is defined as

$$N_{\mathcal{O}} = \mathcal{O} \cdot \bar{\mathcal{O}} = {h_0}^2 - {h_1}^2 - {h_2}^2 - {h_3}^2 + {h_4}^2 + {h_5}^2 + {h_6}^2 + {h_7}^2.$$

In [**?**], the hyperbolic k-Fibonacci and k-Lucas octonions $\mathcal{O}^{\mathcal{F}}{}_{k,n}$ and $\mathcal{O}^{\mathcal{L}}{}_{k,n}$ are defined as

$$
\begin{aligned}
\mathcal{O}^{\mathcal{F}}{}_{k,n} &= F_{k,n} + F_{k,n+1}i_1 + F_{k,n+2}i_2 + F_{k,n+3}i_3 + F_{k,n+4}e_4 + F_{k,n+5}e_5\\
&\quad + F_{k,n+6}e_6 + F_{k,n+7}e_7
\end{aligned}
$$

$$= \left\langle F_{k,n}, F_{k,n+1}, F_{k,n+2}, F_{k,n+3}, F_{k,n+4}, F_{k,n+5}, F_{k,n+6}, F_{k,n+7} \right\rangle,$$

and

$$\begin{aligned}\mathcal{O}^{\mathcal{L}}{}_{k,n} &= L_{k,n} + L_{k,n+1}i_1 + L_{k,n+2}i_2 + L_{k,n+3}i_3 + L_{k,n+4}e_4 + L_{k,n+5}e_5 \\ &+ L_{k,n+6}e_6 + L_{k,n+7}e_7 \\ &= \left\langle L_{k,n}, L_{k,n+1}, L_{k,n+2}, L_{k,n+3}, L_{k,n+4}, L_{k,n+5}, L_{k,n+6}, L_{k,n+7} \right\rangle,\end{aligned}$$

respectively. The conjugate of hyperbolic k-Fibonacci and k-Lucas octonions $\bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n}$ and $\bar{\mathcal{O}}^{\mathcal{L}}{}_{k,n}$ are defined as

$$\begin{aligned}\bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n} &= F_{k,n} - F_{k,n+1}i_1 - F_{k,n+2}i_2 - F_{k,n+3}i_3 - F_{k,n+4}e_4 - F_{k,n+5}e_5 \\ &- F_{k,n+6}e_6 - F_{k,n+7}e_7 \\ &= \left\langle F_{k,n}, -F_{k,n+1}, -F_{k,n+2}, -F_{k,n+3}, -F_{k,n+4}, -F_{k,n+5}, -F_{k,n+6}, -F_{k,n+7} \right\rangle,\end{aligned}$$

and

$$\begin{aligned}\bar{\mathcal{O}}^{\mathcal{L}}{}_{k,n} &= L_{k,n} - L_{k,n+1}i_1 - L_{k,n+2}i_2 - L_{k,n+3}i_3 - L_{k,n+4}e_4 - L_{k,n+5}e_5 \\ &- L_{k,n+6}e_6 - L_{k,n+7}e_7 \\ &= \left\langle L_{k,n}, -L_{k,n+1}, -L_{k,n+2}, -L_{k,n+3}, -L_{k,n+4}, -L_{k,n+5}, -L_{k,n+6}, -L_{k,n+7} \right\rangle,\end{aligned}$$

respectively, where $F_{k,n}$ is n^{th} k-Fibonacci sequence and $L_{k,n}$ is n^{th} k-Lucas sequence. Here, i_1, i_2, i_3 are quaternion imaginary units,

$e_4({e_4}^2 = 1)$ is a counter imaginary unit, and the bases of hyperbolic octonions $\mathcal{O^F}_{k,n}$ and $\mathcal{O^L}_{k,n}$ are defined as $i_1e_4 = e_5, i_2e_4 = e_6, i_3e_4 = e_7, {e_4}^2 = {e_5}^2 = {e_6}^2 = {e_7}^2 = 1$. The bases of hyperbolic octonions $\mathcal{O^F}_{k,n}$ and $\mathcal{O^L}_{k,n}$ have multiplication rules as in Table-(??).

6.2 Some Fundamental Properties of Hyperbolic k- Fibonacci and k- Lucas Quaternions

In this section, we establish certain elementary properties of the hyperbolic k-Fibonacci and k-Lucas quaternions.

Definition 6.1. For $n \geq 0$, the hyperbolic k-Fibonacci and k-Lucas quaternions $\bar{\mathcal{H}}^{\mathcal{F}}{}_{k,n}$ and $\bar{\mathcal{H}}^{\mathcal{L}}{}_{k,n}$ are defined by

$$\bar{\mathcal{H}}^{\mathcal{F}}{}_{k,n} = F_{k,n}i_1 + F_{k,n+1}i_2 + F_{k,n+2}i_3 + F_{k,n+3}i_4 \tag{6.2.1}$$

$$= \left(F_{k,n}, F_{k,n+1}, F_{k,n+2}, F_{k,n+3}\right)$$

and

$$\bar{\mathcal{H}}^{\mathcal{L}}{}_{k,n} = L_{k,n}i_1 + L_{k,n+1}i_2 + L_{k,n+2}i_3 + L_{k,n+3}i_4 \tag{6.2.2}$$

$$= \left(L_{k,n}, L_{k,n+1}, L_{k,n+2}, L_{k,n+3}\right),$$

respectively, where $F_{k,n}$ is n-th k- Fibonacci sequence and $L_{k,n}$ is n-th k- Lucas sequence. Here, i_1, i_2, i_3, i_4 are hyperbolic quaternion units which satisfy the multiplication rule (6.1.1).

Theorem 6.2. *For all $n \geq 0$, we have*

$$\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+2} = k\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+1} + \bar{\mathcal{H}^{\mathcal{F}}}_{k,n}, \tag{6.2.3}$$

$$\bar{\mathcal{H}^{\mathcal{L}}}_{k,n+2} = k\bar{\mathcal{H}^{\mathcal{L}}}_{k,n+1} + \bar{\mathcal{H}^{\mathcal{L}}}_{k,n}, \tag{6.2.4}$$

$$\bar{\mathcal{H}^{\mathcal{L}}}_{k,n} = \bar{\mathcal{H}^{\mathcal{F}}}_{k,n+1} + \bar{\mathcal{H}^{\mathcal{L}}}_{k,n-1}. \tag{6.2.5}$$

Proof. i. From equations (6.2.1) and (6.2.2), we have

$$\begin{aligned}
k\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+1} + \bar{\mathcal{H}^{\mathcal{F}}}_{k,n} &= k\big[F_{k,n+1}i_1 + F_{k,n+2}i_2 + F_{k,n+3}i_3 + F_{k,n+4}i_4\big] \\
&+ \big[F_{k,n}i_1 + F_{k,n+1}i_2 + F_{k,n+2}i_3 + F_{k,n+3}i_4\big] \\
&= \big[kF_{k,n+1} + F_{k,n}\big]i_1 + \big[kF_{k,n+2} + F_{k,n+1}\big]i_2 \\
&+ \big[kF_{k,n+3} + F_{k,n+2}\big]i_3 + \big[kF_{k,n+4} + F_{k,n+3}\big]i_4 \\
&= F_{k,n+2}i_1 + F_{k,n+3}i_2 + F_{k,n+4}i_3 + F_{k,n+5}i_4 \\
&= \bar{\mathcal{H}^{\mathcal{F}}}_{k,n+2}.
\end{aligned}$$

The proofs of (ii) and (iii) are similar to (i), using equations (6.2.1) and (6.2.2). □

Theorem 6.3. ***(Binet Formulas)***. *For all $n \geq 0$, we have*

$$\bar{\mathcal{H}^{\mathcal{F}}}_{k,n} = \frac{\bar{r_1}{r_1}^n - \bar{r_2}{r_2}^n}{r_1 - r_2} \tag{6.2.6}$$

and

$$\bar{\mathcal{H}^{\mathcal{L}}}_{k,n} = \bar{r_1}{r_1}^n + \bar{r_2}{r_2}^n, \tag{6.2.7}$$

where, $\bar{r_1} = i_1 + r_1 i_2 + {r_1}^2 i_3 + {r_1}^3 i_4 = \left(1, r_1, {r_1}^2, {r_1}^3\right)$, $\bar{r_2} = i_1 + r_2 i_2 + {r_2}^2 i_3 + {r_2}^3 i_4 = \left(1, r_2, {r_2}^2, {r_2}^3\right)$ *and* i_1, i_2, i_3, i_4 *are hyperbolic quaternion units which satisfy the multiplication rule (6.1.1).*

Proof. Using the definition of $\bar{\mathcal{H}^{\mathcal{F}}}_{k,n}$ and the Binet formulas of k-Fibonacci and k- Lucas sequences, we have

$$\begin{aligned}
\bar{\mathcal{H}^{\mathcal{F}}}_{k,n} &= F_{k,n} i_1 + F_{k,n+1} i_2 + F_{k,n+2} i_3 + F_{k,n+3} i_4 \\
&= \Big[\frac{{r_1}^n - {r_2}^n}{r_1 - r_2}\Big] i_1 + \Big[\frac{{r_1}^{n+1} - {r_2}^{n+1}}{r_1 - r_2}\Big] i_2 + \Big[\frac{{r_1}^{n+2} - {r_2}^{n+2}}{r_1 - r_2}\Big] i_3 \\
&+ \Big[\frac{{r_1}^{n+3} - {r_2}^{n+3}}{r_1 - r_2}\Big] i_4 \\
&= \frac{{r_1}^n}{r_1 - r_2}\left(i_1 + r_1 i_2 + {r_1}^2 i_3 + {r_1}^3 i_4\right) - \frac{{r_2}^n}{r_1 - r_2}\left(i_1 + r_2 i_2 + {r_2}^2 i_3 + {r_2}^3 i_4\right) \\
&= \frac{\bar{r_1}{r_1}^n - \bar{r_2}{r_2}^n}{r_1 - r_2}
\end{aligned}$$

and

$$\bar{\mathcal{H}^{\mathcal{L}}}_{k,n} = L_{k,n} i_1 + L_{k,n+1} i_2 + L_{k,n+2} i_3 + L_{k,n+3} i_4$$

$$= \left[{r_1}^n + {r_2}^n\right]i_1 + \left[{r_1}^{n+1} + {r_2}^{n+1}\right]i_2 + \left[{r_1}^{n+2} + {r_2}^{n+2}\right]i_3 + \left[{r_1}^{n+3}{r_2}^{n+3}\right]i_4$$

$$= {r_1}^n\big(i_1 + r_1 i_2 + {r_1}^2 i_3 + {r_1}^3 i_4\big) + {r_2}^n\big(i_1 + r_2 i_2 + {r_2}^2 i_3 + {r_2}^3 i_4\big)$$

$$= \bar{r_1}{r_1}^n + \bar{r_2}{r_2}^n.$$

□

Lemma 6.4. *For $\bar{r_1}$ and $\bar{r_2}$, we have*

(i) $\bar{r_1} - \bar{r_2} = \sqrt{\delta}\bar{\mathcal{H}^{\mathcal{F}}}_{k,0}$,

(ii) $\bar{r_1} + \bar{r_2} = \bar{\mathcal{H}^{\mathcal{L}}}_{k,0}$,

(iii) $\bar{r_1}\bar{r_2} = \big(0, 2r_2, 2{r_2}^2, {r_1}^3 + {r_2}^3 + r_1 - r_2\big)$,

(iv) $\bar{r_2}\bar{r_1} = \big(0, 2r_1, 2{r_1}^2, {r_1}^3 + {r_2}^3 - r_1 + r_2\big)$,

(v) $\bar{r_1}^2 = \big(-1 + {r_1}^2 + {r_1}^4 + {r_1}^6\big) + 2\bar{r_1}$,

(vi) $\bar{r_2}^2 = \big(-1 + {r_2}^2 + {r_2}^4 + {r_2}^6\big) + 2\bar{r_2}$,

(vii) $\bar{r_1}\bar{r_2} + \bar{r_2}\bar{r_1} = 2\big(\bar{\mathcal{H}^{\mathcal{L}}}_{k,0} - 2\big)$,

$(viiii)$ $\bar{r_1}\bar{r_2} - \bar{r_2}\bar{r_1} = 2\sqrt{\delta}\big(0, -1, -k, 1\big)$,

(ix) $\bar{r_1}^2 - \bar{r_2}^2 = \sqrt{\delta}\big(F_{k,2} + F_{k,4} + F_{k,6} + 2\bar{\mathcal{H}^{\mathcal{F}}}_{k,0}\big)$,

(x) $\bar{r_1}^2 + \bar{r_2}^2 = \big(-L_{k,0} + L_{k,2} + L_{k,4} + L_{k,6}2\bar{\mathcal{H}^{\mathcal{L}}}_{k,0}\big)$.

Theorem 6.5. *For all $s, t \in Z^+$,$s \geq t$ and $n \in N$, the generating functions for the hyperbolic k-Fibonacci and k-Lucas quaternions*

$\bar{\mathcal{H}}^{\mathcal{F}}{}_{k,tn}$ *and* $\bar{\mathcal{H}}^{\mathcal{L}}{}_{k,tn}$ *are*

$$(i)\quad \sum_{n=0}^{\infty} \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,tn} x^n = \frac{\bar{\mathcal{H}}^{\mathcal{F}}{}_{k,0} + \big(\bar{\mathcal{H}}^{\mathcal{L}}{}_{k,0} F_{k,t} - \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,t}\big)x}{1 - xL_{k,t} + x^2(-1)^t},$$

$$(ii)\quad \sum_{n=0}^{\infty} \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,tn} x^n = \frac{\bar{\mathcal{H}}^{\mathcal{L}}{}_{k,0} - \big(\bar{\mathcal{H}}^{\mathcal{L}}{}_{k,0} L_{k,t} - \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,t}\big)x}{1 - xL_{k,t} + x^2(-1)^t},$$

$$(iii)\quad \sum_{n=0}^{\infty} \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,tn+s} x^n = \frac{\bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s} + (-1)^t x \bar{\mathcal{H}}^{\mathcal{F}}{}_{s,s-t}}{1 - xL_{k,t} + x^2(-1)^t},$$

$$(iv)\quad \sum_{n=0}^{\infty} \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,tn+s} x^n = \frac{\bar{\mathcal{H}}^{\mathcal{L}}{}_{k,s} + (-1)^t x \bar{\mathcal{H}}^{\mathcal{L}}{}_{s-t}}{1 - xL_{k,t} + x^2(-1)^t}.$$

Proof. (1). Using theorem (6.30), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,tn} x^n &= \sum_{n=0}^{\infty} \frac{\bar{r_1} {r_1}^{tn} - \bar{r_2} {r_2}^{tn}}{r_1 - r_2} x^n \\
&= \frac{\bar{r_1}}{r_1 - r_2} \sum_{n=0}^{\infty} \big({r_1}^t\big)^n x^n - \frac{\bar{r_2}}{r_1 - r_2} \sum_{n=0}^{\infty} \big({r_2}^t\big)^n x^n \\
&= \frac{\big(\frac{\bar{r_1} - \bar{r_2}}{r_1 - r_2}\big) + \big[(\bar{r_1} + \bar{r_2})\big(\frac{{r_1}^t - {r_2}^t}{r_1 - r_2}\big) - \big(\frac{\bar{r_1}{r_1}^t - \bar{r_2}{r_2}^t}{r_1 - r_2}\big)\big]x}{1 - \big({r_1}^t + {r_2}^t\big)x + x^2(r_1 r_2)^t} \\
&= \frac{\bar{\mathcal{H}}^{\mathcal{F}}{}_{k,0} + \big(\bar{\mathcal{H}}^{\mathcal{L}}{}_{k,0} F_{k,t} - \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,t}\big)x}{1 - xL_{k,t} + x^2(-1)^t}
\end{aligned}$$

The proofs of (ii), (iii) and (iv) are similar to (i), using theorem (6.30). □

Theorem 6.6. *For all* $t \in Z^+$ *and* $n \in N$*, the exponential generating functions for the hyperbolic* k*-Fibonacci and* k*-Lucas quaternions*

$\bar{\mathcal{H}^{\mathcal{F}}}_{k,tn}$ *and* $\bar{\mathcal{H}^{\mathcal{L}}}_{k,tn}$ *are*

$$\sum_{n=0}^{\infty} \frac{\bar{\mathcal{H}^{\mathcal{F}}}_{k,tn}}{n!} x^n = \frac{\bar{r_1} e^{r_1{}^t x} - \bar{r_2} e^{r_2{}^t x}}{r_1 - r_2}$$

and

$$\sum_{n=0}^{\infty} \frac{\bar{\mathcal{H}^{\mathcal{L}}}_{k,tn}}{n!} x^n = \bar{r_1} e^{r_1{}^t x} + \bar{r_2} e^{r_2{}^t x}.$$

Proof. The proof is similar to theorem (6.6). □

Theorem 6.7. *For all $n \in N$, we have*

$$(i) \qquad \sum_{i=0}^{n} \binom{n}{i} k^i \bar{\mathcal{H}^{\mathcal{F}}}_{k,i} = \bar{\mathcal{H}^{\mathcal{F}}}_{k,2n},$$

$$(ii) \qquad \sum_{i=0}^{n} \binom{n}{i} k^i \bar{\mathcal{H}^{\mathcal{L}}}_{k,i} = \bar{\mathcal{H}^{\mathcal{L}}}_{k,2n}.$$

Proof. (i). Using theorem (6.30), we obtain

$$\begin{aligned}
\sum_{i=0}^{n} \binom{n}{i} k^i \bar{\mathcal{H}^{\mathcal{F}}}_{k,i} &= \sum_{i=0}^{n} \binom{n}{i} k^i \left(\frac{\bar{r_1} r_1{}^i - \bar{r_2} r_2{}^i}{r_1 - r_2}\right) \\
&= \frac{\bar{r_1}}{r_1 - r_2} \sum_{i=0}^{n} \binom{n}{i} (kr_1)^i - \frac{\bar{r_2}}{r_1 - r_2} \sum_{i=0}^{n} \binom{n}{i} (kr_2)^i \\
&= \frac{\bar{r_1}}{r_1 - r_2} \big(1 + kr_1\big)^n - \frac{\bar{r_2}}{r_1 - r_2} \big(1 + kr_2\big)^n \\
&= \frac{\bar{r_1} r_1{}^{2n} - \bar{r_2} r_2{}^{2n}}{r_1 - r_2} \\
&= \bar{\mathcal{H}^{\mathcal{F}}}_{k,2n}.
\end{aligned}$$

The proof of (ii) is similar to (i), using theorem (6.30). □

Lemma 6.8. *For all $t \geqslant 0$ and $m \geqslant n$, we have*

$$(i) \quad \frac{\bar{r_1}\bar{r_2}{r_2}^t - \bar{r_2}\bar{r_1}{r_1}^t}{r_1 - r_2} = \big(0, -2F_{k,t+1}, -2F_{k,t+2}, \\ - F_{k,t+3} + F_{k,t-3} + F_{k,t+1} + F_{k,t-1}\big),$$

$$(ii) \quad \frac{{r_1}^{m-n}\bar{r_1}\bar{r_2} - {r_2}^{m-n}\bar{r_2}\bar{r_1}}{r_1 - r_2} = \big(0, -2F_{k,m-n-1}, 2F_{k,m-n-2}, \\ F_{k,m-n+3} - F_{k,m-n-3} + F_{k,m-n+1} + F_{k,m-n-1}\big).$$

Theorem 6.9. ***(Catalan's Identity)***. *For any integer t and s, we have*

$$(i) \quad \bar{\mathcal{H}^{\mathcal{F}}}_{k,n-t}\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+t} - \bar{\mathcal{H}^{\mathcal{F}}}^2_{k,n} = (-1)^{n-t}F_{k,t}\big(0, -2F_{k,t+1}, -2F_{k,t+2}, \\ - 2F_{k,t+3} + F_{k,t-3} + F_{k,t+1} + F_{k,t-1}\big),$$

$$(ii) \quad \bar{\mathcal{H}^{\mathcal{L}}}_{k,n-t}\bar{\mathcal{H}^{\mathcal{L}}}_{k,n+t} - \bar{\mathcal{H}^{\mathcal{L}}}^2_{k,n} = \delta(-1)^{n-t+1}F_{k,t}\big(0, -2F_{k,t+1}, -2F_{k,t+2}, \\ - 2F_{k,t+3} + F_{k,t-3} + F_{k,t+1} + F_{k,t-1}\big).$$

Proof. Using theorem (6.30), we have

$$\bar{\mathcal{H}^{\mathcal{F}}}_{k,n-t}\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+t} - \bar{\mathcal{H}^{\mathcal{F}}}^2_{k,n} = \Big(\frac{\bar{r_1}{r_1}^{n-t} - \bar{r_2}{r_2}^{n-t}}{r_1 - r_2}\Big)\Big(\frac{\bar{r_1}{r_1}^{n+t} - \bar{r_2}{r_2}^{n+t}}{r_1 - r_2}\Big) \\ - \Big(\frac{\bar{r_1}{r_1}^{n} - \bar{r_2}{r_2}^{n}}{r_1 - r_2}\Big)^2.$$

Using lemma (6.34), we obtain

$$= (-1)^{n-t}F_{k,t}\Big(\frac{\bar{r_1}\bar{r_2}{r_1}^t - \bar{r_2}\bar{r_1}{r_2}^t}{r_1 - r_2}\Big)$$

$$= (-1)^{n-t} F_{k,t}\big(0, -2F_{k,t+1}, -2F_{k,t+2},$$

$$- 2F_{k,t+3} + F_{k,t-3} + F_{k,t+1} + F_{k,t-1}\big).$$

The proof of (ii) is similar to (i), using theorem (6.30) and lemma (6.34). □

Theorem 6.10. ***(Cassini's Identity)***. *For all $n \geq 1$, we have*

$$\bar{\mathcal{H}^{\mathcal{F}}}_{k,n-1}\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+1} - \bar{\mathcal{H}^{\mathcal{F}}}^2_{k,n} = (-1)^n\big(0, -2F_{k,2}, 2F_{k,3}, F_{k,4}\big)$$

and

$$\bar{\mathcal{H}^{\mathcal{L}}}_{k,n-1}\bar{\mathcal{H}^{\mathcal{L}}}_{k,n+1} - \bar{\mathcal{H}^{\mathcal{L}}}^2_{k,n} = \delta(-1)^{n-1}\big(0, -2F_{k,2}, 2F_{k,3}, F_{k,4}\big).$$

Theorem 6.11. ***(d'Ocagene's Identity)***. *Let n be any non-negative integer and t a natural number. If $t \geq n+1$, then we have*

$$(i)\quad \bar{\mathcal{H}^{\mathcal{F}}}_{k,t}\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+1} - \bar{\mathcal{H}^{\mathcal{F}}}_{k,t+1}\bar{\mathcal{H}^{\mathcal{F}}}_{k,n} = (-1)^n\big(0, -2F_{k,t-n-1}, 2F_{k,t-n-2},$$

$$F_{k,t-n+3} + F_{k,t-n-3} + F_{k,t-n+1} + F_{k,t-n-1}\big),$$

$$(ii)\quad \bar{\mathcal{H}^{\mathcal{L}}}_{k,t}\bar{\mathcal{H}^{\mathcal{L}}}_{k,n+1} - \bar{\mathcal{H}^{\mathcal{L}}}_{k,t+1}\bar{\mathcal{H}^{\mathcal{L}}}_{k,n} = (-1)^{n+1}\delta\big(0, -2F_{k,t-n-1}, 2F_{k,t-n-2},$$

$$F_{k,t-n+3} + F_{k,t-n-3} + F_{k,t-n+1} + F_{k,t-n-1}\big).$$

Proof. Using theorem (6.30), we get

$$\bar{\mathcal{H}^{\mathcal{F}}}_{k,t}\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+1} - \bar{\mathcal{H}^{\mathcal{F}}}_{k,t+1}\bar{\mathcal{H}^{\mathcal{F}}}_{k,n} = \Big(\frac{\bar{r_1}{r_1}^t - \bar{r_2}{r_2}^t}{r_1 - r_2}\Big)\Big(\frac{\bar{r_1}{r_1}^{n+1} - \bar{r_2}{r_2}^{n+1}}{r_1 - r_2}\Big)$$

$$-\Big(\frac{\bar{r_1}{r_1}^{t+1}-\bar{r_2}{r_2}^{t+1}}{r_1-r_2}\Big)\Big(\frac{\bar{r_1}{r_1}^{n}-\bar{r_2}{r_2}^{n}}{r_1-r_2}\Big).$$

Using lemma (6.34), we obtain

$$\begin{aligned}&=(-1)^n\Big(\frac{\bar{r_1}\bar{r_2}{r_1}^{t-n}-\bar{r_2}\bar{r_1}{r_2}^{t-n}}{r_1-r_2}\Big)\\&=(-1)^n\big(0,-2F_{k,t-n-1},2F_{k,t-n-2},F_{k,t-n+3}+F_{k,t-n-3}\\&+F_{k,t-n+1}+F_{k,t-n-1}\big).\end{aligned}$$

The proof of (ii) is similar to (i), using theorem (6.30) and lemma (6.34). □

Theorem 6.12. *For any integer t, we have*

$$\begin{aligned}(i)\quad \bar{\mathcal{HF}}^2_{k,t}+\bar{\mathcal{HL}}^2_{k,t}&=\frac{2(k^2+5)}{k}\bar{\mathcal{HL}}_{k,2t}+\delta(k^2+5)L_{k,2t+3}+2(-1)^t\frac{(k^2+3)}{\delta}\\&\quad\big(\bar{\mathcal{HL}}_{k,0}-2\big),\\(ii)\quad \bar{\mathcal{HF}}^2_{k,t}-\bar{\mathcal{HL}}^2_{k,t}&=\frac{2(k^2+3)}{\delta}\bar{\mathcal{HL}}_{k,2t}+(k^2+3)(k^2+2)L_{k,2t+3}\\&\quad+2(-1)^{t+1}\frac{(k^2+5)}{\delta}\big(\bar{\mathcal{HL}}_{k,0}-2\big).\end{aligned}$$

Proof. Using theorem (6.30), we get

$$\begin{aligned}\bar{\mathcal{HF}}^2_{k,t}+\bar{\mathcal{HL}}^2_{k,t+1}&=\Big(\frac{\bar{r_1}{r_1}^{t}-\bar{r_2}{r_2}^{t}}{r_1-r_2}\Big)^2+\big(\bar{r_1}{r_1}^{t}+\bar{r_2}{r_2}^{t}\big)^2\\&=\frac{k^2+5}{\delta}\Big[-{r_1}^{2t}-{r_2}^{2t}+{r_1}^{2t+2}+{r_2}^{2t+2}+{r_1}^{2t+4}+{r_2}^{2t+4}\\&+{r_1}^{2t+6}+{r_2}^{2t+6}+2\big(\bar{r_1}{r_1}^{2t}+\bar{r_2}{r_2}^{2t}\big)\Big]\end{aligned}$$

$$+\frac{k^2+3}{\delta}(-1)^t\big[\bar{r_1}\bar{r_2}+\bar{r_2}\bar{r_1}\big].$$

Using lemma (6.31), we obtain

$$=\frac{2(k^2+5)}{k}\bar{\mathcal{H}^{\mathcal{L}}}_{k,2t}+\delta(k^2+5)L_{k,2t+3}+2(-1)^t\frac{(k^2+3)}{\delta}\big(\bar{\mathcal{H}^{\mathcal{L}}}_{k,0}-2\big).$$

The proof of (ii) is similar to (i), using theorem (6.30) and lemma (6.31). □

Theorem 6.13. *For any integer r, $s \geq t$, we have*

$$\bar{\mathcal{H}^{\mathcal{F}}}_{k,r+s}\bar{\mathcal{H}^{\mathcal{L}}}_{k,r+t}-\bar{\mathcal{H}^{\mathcal{F}}}_{k,r+t}\bar{\mathcal{H}^{\mathcal{L}}}_{k,r+s}=2(-1)^{r+t}\big(\bar{\mathcal{H}^{\mathcal{L}}}_{k,0}-2\big)F_{k,s-t}.$$

Proof. Using theorem (6.30), we obtain

$$\begin{aligned}\bar{\mathcal{H}^{\mathcal{F}}}_{k,r+s}\bar{\mathcal{H}^{\mathcal{L}}}_{k,r+t}-\bar{\mathcal{H}^{\mathcal{F}}}_{k,r+t}\bar{\mathcal{H}^{\mathcal{L}}}_{k,r+s}&=\frac{1}{r_1-r_2}\Big[\big(\bar{r_1}{r_1}^{r+s}-\bar{r_2}{r_2}^{r+s}\big)\\&\big(\bar{r_1}{r_1}^{r+t}+\bar{r_2}{r_2}^{r+t}\big)-\big(\bar{r_1}{r_1}^{r+t}-\bar{r_2}{r_2}^{r+t}\big)\\&\big(\bar{r_1}{r_1}^{r+s}+\bar{r_2}{r_2}^{r+s}\big)\Big]\\&=\frac{(-1)^r}{r_1-r_2}\big(\bar{r_1}\bar{r_2}+\bar{r_2}\bar{r_1}\big)\big({r_1}^s{r_2}^t-{r_1}^t{r_2}^s\big).\end{aligned}$$

Using lemma (6.31), we obtain

$$=2(-1)^{r+t}\big(\bar{\mathcal{H}^{\mathcal{L}}}_{k,0}-2\big)F_{k,s-t}.$$

□

Theorem 6.14. *For any integer s, and t, we have*

$$\bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s+t} + (-1)^t \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s-t} = \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s} L_{k,t} \tag{6.2.8}$$

and

$$\bar{\mathcal{H}}^{\mathcal{L}}{}_{k,s+t} + (-1)^t \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,s-t} = \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,s} L_{k,t}. \tag{6.2.9}$$

Proof of (6.2.8): Using theorem (6.30), we get

$$\begin{aligned}
\bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s+t} + (-1)^t \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s-t} &= \frac{1}{r_1 - r_2}\Big[\big(\bar{r_1} {r_1}^{s+t} - \bar{r_2} {r_2}^{s+t}\big) \\
&\quad + (-1)^t \big(\bar{r_1} {r_1}^{s-t} + \bar{r_2} {r_2}^{s-t}\big)\Big] \\
&= \Big(\frac{\bar{r_1} {r_1}^{s} - \bar{r_2} {r_2}^{s}}{r_1 - r_2}\Big)\big({r_1}^t + {r_2}^t\big).
\end{aligned}$$

Using theorem (6.30), we obtain

$$= \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s} L_{k,t}.$$

The proof of (6.2.9) is similar to (6.2.8), using theorem (6.30).

Theorem 6.15. *For any integer $s \leq t$, we have*

$$\bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s} \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,t} - \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,t} \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s} = 2(-1)^s F_{k,t-s}\big(0, -1, -k, 1\big) \tag{6.2.10}$$

and

$$\bar{\mathcal{H}^{\mathcal{L}}}_{k,s}\bar{\mathcal{H}^{\mathcal{L}}}_{k,t} - \bar{\mathcal{H}^{\mathcal{L}}}_{k,t}\bar{\mathcal{H}^{\mathcal{L}}}_{k,s} = 2(-1)^{s+1}F_{k,t-s}\delta\big(0,-1,-k,1\big). \quad (6.2.11)$$

Proof of (6.2.10): Using theorem (6.30), we have

$$\bar{\mathcal{H}^{\mathcal{L}}}_{k,s}\bar{\mathcal{H}^{\mathcal{L}}}_{k,t} - \bar{\mathcal{H}^{\mathcal{L}}}_{k,t}\bar{\mathcal{H}^{\mathcal{L}}}_{k,s} = \frac{1}{(r_1-r_2)^2}\big[{r_1}^t{r_2}^s - {r_1}^s{r_2}^t\big]\big[\bar{r_1}\bar{r_2} - \bar{r_2}\bar{r_1}\big].$$

Using lemma (6.31), we obtain

$$= 2(-1)^s F_{k,t-s}\big(0,-1,-k,1\big).$$

The proof of (6.2.11) is similar to (6.2.10), using theorem (6.30) and lemma (6.31).

Theorem 6.16. *For any integer $s \leq t$, we have*

$$\bar{\mathcal{H}^{\mathcal{F}}}_{k,t}\bar{\mathcal{H}^{\mathcal{L}}}_{k,s} - \bar{\mathcal{H}^{\mathcal{F}}}_{k,s}\bar{\mathcal{H}^{\mathcal{L}}}_{k,t} = 2(-1)^s F_{k,t-s}\big(\bar{\mathcal{H}^{\mathcal{L}}}_{k,0} - 2\big) \quad (6.2.12)$$

and

$$\bar{\mathcal{H}^{\mathcal{F}}}_{k,t}\bar{\mathcal{H}^{\mathcal{L}}}_{k,s} - \bar{\mathcal{H}^{\mathcal{L}}}_{k,t}\bar{\mathcal{H}^{\mathcal{F}}}_{k,s} = 2(-1)^s\bar{r_2}\big[\bar{\mathcal{H}^{\mathcal{F}}}_{k,t-s} - {r_2}^{t-s}\big(0,1,k,k^2+1\big)\big]. \quad (6.2.13)$$

Proof of (6.2.12): Using theorem (6.30), we get

$$\bar{\mathcal{H}}^{\mathcal{F}}{}_{k,t}\bar{\mathcal{H}}^{\mathcal{L}}{}_{k,s} - \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s}\bar{\mathcal{H}}^{\mathcal{L}}{}_{k,t} = \frac{1}{(r_1 - r_2)}\left[{r_1}^t{r_2}^s - {r_1}^s{r_2}^t\right]\left[\bar{r_1}\bar{r_2} + \bar{r_2}\bar{r_1}\right].$$

Using lemma (6.31), we obtain

$$= 2(-1)^s F_{k,t-s}\left(\bar{\mathcal{H}}^{\mathcal{L}}{}_{k,0} - 2\right).$$

The proof of (6.2.13) is similar to (6.2.12), using theorem (6.30) and lemma (6.31). This completes the proof of theorem (6.16).

6.3 Some Binomial and Congruence Properties of Hyperbolic k- Fibonacci and k- Lucas Quaternions

In this section, we explore some binomial and congruence properties of the hyperbolic k-Fibonacci and k-Lucas quaternions.

Lemma 6.17. *Let $u = r_1$ or r_2, then we have*

$$(a) \quad u^n = uF_{k,n} + F_{k,n-1},$$

$$(b) \quad u^{2n} = u^n L_{k,n} - (-1)^n,$$

$$(c) \quad u^{tn} = u^n \frac{F_{k,tn}}{F_{k,n}} - (-1)^n - \frac{F_{k,(t-1)n}}{F_{k,n}},$$

$$(d) \quad u^{sn}F_{k,rn} - u^{rn}F_{k,sn} = (-1)^{sn}F_{k,(r-s)n}.$$

Proof. We prove only (a) and (c) since the proofs of (b) and (d) are similar.

(a): Since r_1 and r_2 are roots of $r^2 - kr - 1 = 0$, then we have $r_1^2 = kr_1 + 1$ and $r_2^2 = kr_2 + 1$. Therefore, we have

$$\begin{aligned}
u^{2n} &= F_{k,n}u^{n+1} + u^n F_{k,n-1} \\
&= F_{k,n}(uF_{k,n+1} + F_{k,n}) + u^n F_{k,n-1} \\
&= uF_{k,n}F_{k,n+1} + F_{k,n-1}u^n + F_{k,n}^2 \\
&= (u^n - F_{k,n-1})F_{k,n+1} + F_{k,n-1}u^n + F_{k,n}^2 \\
&= u^n(F_{k,n+1} + F_{k,n-1}) + F_{k,n}^2 - F_{k,n}F_{k,n-1}.
\end{aligned}$$

Using $F_{k,n-1}F_{k,n+1} - F_{k,n}^2 = (-1)^n$ and $F_{k,n+1} + F_{k,n-1} = L_{k,n}$, we obtain

$$u^{2n} = L_{k,n}u^n - (-1)^n.$$

This completes the proof of (a).

(c): If $u = r_1$, then we have

$$\begin{aligned}
F_{k,tn}r_1^n - (-1)^n F_{k,(t-1)n} &= (\frac{r_1^{tn} - r_2^{tn}}{r_1 - r_2})r_1^n - (r_1r_2)^n(\frac{r_1^{(t-1)n} - r_2^{(t-1)n}}{r_1 - r_2}) \\
&= (\frac{r_1^n - r_2^n}{r_1 - r_2})r_1^{tn} \\
&= F_{k,n}r_1^{tn}.
\end{aligned}$$

This completes the proof of (c). □

Theorem 6.18. *For all $n, r, s, t \geq 1$, we have*

$$(i) \quad \bar{\mathcal{H}^{\mathcal{F}}}_{k,n+t} = F_{k,n}\bar{\mathcal{H}^{\mathcal{F}}}_{k,t+1} + F_{k,n-1}\bar{\mathcal{H}^{\mathcal{F}}}_{k,t},$$

$$(ii) \quad \bar{\mathcal{H}^{\mathcal{F}}}_{k,2n+t} = L_{k,n}\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+t} - (-1)^n\bar{\mathcal{H}^{\mathcal{F}}}_{k,t},$$

$$(iii) \quad \bar{\mathcal{H}^{\mathcal{F}}}_{k,sn+t} = \frac{F_{k,sn}}{F_{k,n}}\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+t} - (-1)^n\frac{F_{k,(s-1)n}}{F_{k,n}}\bar{\mathcal{H}^{\mathcal{F}}}_{k,t},$$

$$(iv) \quad \bar{\mathcal{H}^{\mathcal{F}}}_{k,sn+t}F_{k,rn} - \bar{\mathcal{H}^{\mathcal{F}}}_{k,rn+t}F_{k,sn} = (-1)^{sn}\bar{\mathcal{H}^{\mathcal{F}}}_{k,t}F_{k,(r-s)n}.$$

Theorem 6.19. *For all $n, r, s, t \geq 1$ and $\mathcal{G}_{k,n} = \bar{\mathcal{H}^{\mathcal{F}}}_{k,n}$ or $\bar{\mathcal{H}^{\mathcal{L}}}_{k,n}$, we have*

$$(i) \quad \mathcal{G}_{k,rn+t} = \sum_{i=0}^{n}\binom{n}{i}F_{k,r}^{i}F_{k,r-1}^{n-i}\mathcal{G}_{k,i+t},$$

$$(ii) \quad \mathcal{G}_{k,2rn+t} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)(r+1)}L_{k,r}^{i}\mathcal{G}_{k,ri+t},$$

$$(iii) \quad \mathcal{G}_{k,trn+l} = \frac{1}{F_{k,r}^{n}}\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)(r+1)}F_{k,(t-1)r}^{n-i}F_{k,tr}^{i}\mathcal{G}_{k,ri+l},$$

$$(iv) \quad \sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\mathcal{G}_{k,r(n-i)+i+t}F_{k,r}^{i} = \mathcal{G}_{k,t}F_{k,r-1}^{n},$$

$$(v) \quad \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}\mathcal{G}_{k,ri+t}F_{k,r-1}^{(n-i)} = \mathcal{G}_{k,n+t}F_{k,r}^{n},$$

$$(vi) \quad \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}F_{k,sm}^{(n-i)}F_{k,rm}^{(i)}\mathcal{G}_{k,m[rn+i(s-r)]+t} = (-1)^{smn}\mathcal{G}_{k,t}F_{k,(r-s)m}^{n}.$$

Lemma 6.20. *If $L_{k,n}$ is n^{th} k-Lucas sequence and $u = r_1$ or r_2, then we have*

$$1 + ku + u^{2(2^{n+1}+1)} = L_{k,2^{n+1}} u^{2(2^n+1)}.$$

Theorem 6.21. *For all $t \geq 1$ and $\mathcal{G}_{k,n} = \bar{\mathcal{H}^{\mathcal{F}}}_{k,n}$ or $\bar{\mathcal{H}^{\mathcal{L}}}_{k,n}$, we have*

$$(i) \quad \mathcal{G}_{k,t+2^{n+1}+2} = \frac{\mathcal{G}_{k,t} + k\mathcal{G}_{k,t+1} + \mathcal{G}_{k,t+2^{n+2}+2}}{L_{k,2^{n+1}}},$$

$$(ii) \quad \mathcal{G}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n}(-1)^{j+s} {L_{k,2^{r+1}}}^{i} \mathcal{G}_{k,2^{r+1}(i+2j)+2(i+j)+t},$$

$$(iii) \quad \mathcal{G}_{k,(2^{r+2}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j}(-1)^{j+s} {L_{k,2^{r+1}}}^{i} \mathcal{G}_{k,(2^{r+1}+2)i+j+t},$$

$$(iv) \quad \mathcal{G}_{k,(2^{r+1}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j} {L_{k,2^{r+1}}}^{-n} \mathcal{G}_{k,(2^{r+1}+2)i+j+t}.$$

Lemma 6.22. *Let $u = r_1$ or r_2, then for $l_n = \sum_{i=1}^{n} L_{k,2^i}$ and for every $n, t \geq 1$, we have*

$$1 + u^{2^n} = \begin{cases} \dfrac{l_{n-1}}{l_{n-2}} u^{2^{n-1}}; \\ \dfrac{l_{n-1}}{l_{n-t-1}} u^{2^{n-t}} - l_{n-1} \sum_{i=2}^{t} \dfrac{1}{l_{n-i}}, & \text{If } t = 2, 3, 4, \ldots, n-2 \ ; \\ l_{n-1} u^2 - l_{n-1} \sum_{i=2}^{n-1} \dfrac{1}{l_{n-i}}. \end{cases}$$

Theorem 6.23. *For $l_n = \sum\limits_{i=1}^{n} \mathcal{L}_{k,2^i}$, for every $n, t \geq 1$ and $\mathcal{G}_{k,n} = \mathcal{F}_{k,n}$ or $\mathcal{L}_{k,n}$, we have*

$$(i) \quad \mathcal{G}_{k,t+2^n} = \begin{cases} \dfrac{l_{n-1}}{l_{n-2}}\mathcal{G}_{k,t+2^{n-1}} - \mathcal{G}_{k,t}; \\ \dfrac{l_{n-1}}{l_{n-t-1}}\mathcal{G}_{k,t+2^{n-s}} - l_{n-1}\sum\limits_{i=2}^{s}(1+\dfrac{1}{l_{n-i}})\mathcal{G}_{k,t}, \\ \qquad\qquad\qquad \text{If } s = 2,3,4,\ldots,n-2\ ; \\ l_{n-1}\mathcal{G}_{k,t+2} - l_{n-1}\sum\limits_{i=2}^{n-1}(\dfrac{1}{l_{n-i}}+1)\mathcal{G}_{k,t}. \end{cases}$$

$$(ii) \quad \mathcal{G}_{k,2^r n+t} = \begin{cases} \sum\limits_{i+j=n}\binom{n}{i}(\dfrac{l_{r-1}}{l_{r-2}})^i(-1)^j\mathcal{G}_{k,2^{r-1}i+t}; \\ \sum\limits_{i+j=n}\binom{n}{i}(\dfrac{l_{r-1}}{l_{r-s-1}})^i(-1)^j(\sum_{h=2}^{s}(1+\dfrac{l_{r-1}}{l_{r-h}})^j\mathcal{G}_{k,2^{n-s}i+t}, \\ \qquad\qquad\qquad \text{If } s = 2,3,4,\ldots,n-2\ ; \\ \sum\limits_{i+j=n}\binom{n}{i}(l_{r-1})^i(-1)^j(\sum_{h=2}^{s}(1+\dfrac{l_{r-1}}{l_{r-h}})^j\mathcal{G}_{k,2i+t}. \end{cases}$$

Lemma 6.24. *For all $t \geq 1$, we have*

$$(i) \quad r_1^{2t} = \frac{F_{k,2t}}{k}r_1\sqrt{\delta} - \frac{L_{k,2t-1}}{k}, r_2^{2t} = -\frac{F_{k,2t}}{k}r_2\sqrt{\delta} - \frac{L_{k,2t-1}}{k}.$$

$$(ii) \quad r_1^{2t+1} = \frac{L_{k,2t+1}}{k}r_1 - \frac{F_{k,2t}}{k}\sqrt{\delta}, r_2^{2t+1} = \frac{L_{k,2t+1}}{k}r_2 + \frac{F_{k,2t}}{k}\sqrt{\delta}.$$

Theorem 6.25. *For $s, t \geq 1$, we have*

$$(i) \quad \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s+2t} = \frac{F_{k,2t}}{k}\bar{\mathcal{H}}^{\mathcal{L}}{}_{k,s+1} - \frac{L_{k,2t-1}}{k}\bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s},$$

$$(ii) \quad \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,s+2t} = \frac{F_{k,2t}}{k}\delta\bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s+1} - \frac{L_{k,2t-1}}{k}\bar{\mathcal{H}}^{\mathcal{L}}{}_{k,s},$$

$$(iii) \quad \bar{\mathcal{H}^{\mathcal{F}}}_{k,s+2t} - \frac{F_{k,2t}}{k}\bar{\mathcal{H}^{\mathcal{F}}}_{k,s+2} + \frac{F_{k,2t-2}}{k}\bar{\mathcal{H}^{\mathcal{F}}}_{k,s} = 0,$$

$$(iv) \quad \bar{\mathcal{H}^{\mathcal{L}}}_{k,s+2t} - \frac{F_{k,2t}}{k}\bar{\mathcal{H}^{\mathcal{L}}}_{k,s+2} + \frac{F_{k,2t-2}}{k}\bar{\mathcal{H}^{\mathcal{L}}}_{k,s} = 0.$$

Theorem 6.26. *For all $s, t \geq 1$, we have*

$$(i) \quad \bar{\mathcal{H}^{\mathcal{F}}}_{k,s+2t+1} = \frac{L_{k,2t+1}}{k}\bar{\mathcal{H}^{\mathcal{F}}}_{k,s+1} - \frac{F_{k,2t}}{k}\bar{\mathcal{H}^{\mathcal{L}}}_{k,s},$$

$$(ii) \quad \bar{\mathcal{H}^{\mathcal{L}}}_{k,s+2t+1} = \frac{L_{k,2t+1}}{k}\bar{\mathcal{H}^{\mathcal{L}}}_{k,s+1} - \delta\frac{F_{k,2t}}{k}\bar{\mathcal{H}^{\mathcal{F}}}_{k,s},$$

$$(iii) \quad \bar{\mathcal{H}^{\mathcal{F}}}_{k,s+2t+1} - \frac{L_{k,2t+1}}{k(k^2+3)}\bar{\mathcal{H}^{\mathcal{F}}}_{k,s+3} + \frac{F_{k,2t-2}}{k(k^2+3)}\bar{\mathcal{H}^{\mathcal{L}}}_{k,s} = 0,$$

$$(iv) \quad \bar{\mathcal{H}^{\mathcal{L}}}_{k,s+2t+1} - \frac{L_{k,2t+1}}{k(k^2+3)}\bar{\mathcal{H}^{\mathcal{L}}}_{k,s+3} + \frac{F_{k,2t-2}}{k(k^2+3)}\delta\bar{\mathcal{H}^{\mathcal{F}}}_{k,s} = 0.$$

Theorem 6.27. *For $n, s, t \geq 1$, we have*

$$(i) \quad \sum_{i=0}^{n}\binom{n}{i} k^{(i-n)}(L_{k,2t-1})^{(n-i)}\bar{\mathcal{H}^{\mathcal{F}}}_{k,2ti+s}$$

$$= \begin{cases} k^{-n}(\mathcal{F}_{k,2t})^n\delta^{\frac{n}{2}}\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+s}, & \textit{if } n \textit{ is even;} \\ k^{-n}(\mathcal{F}_{k,2t})^n\delta^{\frac{n-1}{2}}\bar{\mathcal{H}^{\mathcal{L}}}_{k,n+s}, & \textit{if } n \textit{ is odd,} \end{cases}$$

$$(ii) \quad \sum_{i=0}^{n}\binom{n}{i} k^{(i-n)}(L_{k,2t-1})^{(n-i)}\bar{\mathcal{H}^{\mathcal{L}}}_{k,2ti+s}$$

$$= \begin{cases} k^{-n}(F_{k,2t})^n\delta^{\frac{n}{2}}\bar{\mathcal{H}^{\mathcal{L}}}_{k,n+s}, & \textit{if } n \textit{ is even;} \\ k^{-n}(F_{k,2t})^n\delta^{\frac{n+1}{2}}\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+s}, & \textit{if } n \textit{ is odd.} \end{cases}$$

$$(iii) \quad \sum_{i=0}^{n}\binom{n}{i} (-1)^{(n-i)}k^{-i}(L_{k,2t+1})^{i}\bar{\mathcal{H}^{\mathcal{F}}}_{k,2t(n-i)+n}$$

$$= \begin{cases} \delta^{\frac{n}{2}} \bar{\mathcal{H}^{\mathcal{F}}}_{k,0}, & \textit{if } n \textit{ is even;} \\ \delta^{\frac{n-1}{2}} \bar{\mathcal{H}^{\mathcal{L}}}_{k,0}, & \textit{if } n \textit{ is odd,} \end{cases}$$

$$(iv) \quad \sum_{i=0}^{n} \binom{n}{i} (-1)^{(n-i)} k^{-i} (L_{k,2t+1})^{i} \bar{\mathcal{H}^{\mathcal{L}}}_{k,2t(n-i)+n}$$

$$= \begin{cases} \delta^{\frac{n}{2}} \bar{\mathcal{H}^{\mathcal{L}}}_{k,0}, & \textit{if } n \textit{ is even;} \\ \delta^{\frac{n+1}{2}} \bar{\mathcal{H}^{\mathcal{F}}}_{k,0}, & \textit{if } n \textit{ is odd.} \end{cases}$$

The following theorem deals with congruence properties of the hyperbolic k-Fibonacci and k-Lucas quaternions.

Theorem 6.28. *For $n, t \geq 1$ and $\mathcal{G}_{k,n} = \bar{\mathcal{H}^{\mathcal{F}}}_{k,n}$ or $\bar{\mathcal{H}^{\mathcal{L}}}_{k,n}$, we have*

$$(i) \qquad \mathcal{G}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n} (-1)^{n} \mathcal{G}_{k,(2^{r+2}+2)j+t} \equiv 0 \quad (mod L_{k,2^{r+1}}),$$

$$(ii) \qquad \mathcal{G}_{k,(2^{r+2}+2)n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{j} (-1)^{n} \mathcal{G}_{k,j+t} \equiv 0 \quad (mod L_{k,2^{r+1}}).$$

Proof. From theorem (6.21; (ii)), for all $n, t \geq 1$ and $\mathcal{G}_{k,n} = \bar{\mathcal{H}^{\mathcal{F}}}_{k,n}$ or $\bar{\mathcal{H}^{\mathcal{L}}}_{k,n}$, we have

$$\begin{aligned} \mathcal{G}_{k,n+t} = & \sum_{i+j+s=n; i \neq 0} \binom{n}{i,j} k^{-n} (-1)^{j+s} {L_{k,2^{r+1}}}^{i} \mathcal{G}_{k,2^{r+1}(i+2j)+2(i+j)+t} \\ & + \sum_{i+j+s=n; i=0} \binom{n}{i,j} k^{-n} (-1)^{j+s} {L_{k,2^{r+1}}}^{i} \mathcal{G}_{k,2^{r+1}(i+2j)+2(i+j)+t}, \\ = & \sum_{i+j+s=n; i \neq 0} \binom{n}{i,j} k^{-n} (-1)^{j+s} {L_{k,2^{r+1}}}^{i} \mathcal{G}_{k,2^{r+1}(i+2j)+2(i+j)+t} \end{aligned}$$

$$+\sum_{j=0}^{n}\binom{n}{j}k^{-n}(-1)^{n}\mathcal{G}_{k,(2^{r+2}+2)j+t}.$$

$$\mathcal{G}_{k,n+t}-\sum_{j=0}^{n}\binom{n}{j}k^{-n}(-1)^{n}\mathcal{G}_{k,(2^{r+2}+2)j+t}$$

$$=\sum_{i+j+s=n;i\neq 0}\binom{n}{i,j}k^{-n}(-1)^{j+s}L_{k,2^{r+1}}{}^{i}\mathcal{G}_{k,2^{r+1}(i+2j)+2(i+j)+t},$$

$$\therefore L_{k,2} \quad \text{divides} \quad (\mathcal{G}_{k,n+t}-\sum_{j=0}^{n}\binom{n}{j}k^{-n}(-1)^{n}\mathcal{G}_{k,(2^{r+2}+2)j+t}),$$

$$\therefore \mathcal{G}_{k,n+t}-\sum_{j=0}^{n}\binom{n}{j}k^{-n}(-1)^{n}\mathcal{G}_{k,(2^{r+2}+2)j+t}\equiv 0 \pmod{L_{k,2}}.$$

This completes the proof of (i).

The proof of (ii) is similar to (i), using theorem (6.21; (iii)). □

6.4 Some Fundamental Properties of Hyperbolic k- Fibonacci and k- Lucas Octonions

In this section, we establish certain elementary properties of the hyperbolic k-Fibonacci and k-Lucas octonions.

Theorem 6.29. *For all $n \geq 0$, we have*

$$(i) \quad \mathcal{O}^{\mathcal{F}}{}_{k,n+2}=k\mathcal{O}^{\mathcal{F}}{}_{k,n+1}+\mathcal{O}^{\mathcal{F}}{}_{k,n},$$

$$(ii) \quad \mathcal{O}^{\mathcal{L}}{}_{k,n+2}=k\mathcal{O}^{\mathcal{L}}{}_{k,n+1}+\mathcal{O}^{\mathcal{L}}{}_{k,n}$$

$$(iii) \quad \mathcal{O}^{\mathcal{L}}{}_{k,n} = \mathcal{O}^{\mathcal{F}}{}_{k,n+1} + \mathcal{O}^{\mathcal{F}}{}_{k,n-1}$$

$$(iv) \quad \bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n+2} = k\bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n+1} + \bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n},$$

$$(v) \quad \bar{\mathcal{O}}^{\mathcal{L}}{}_{k,n+2} = k\bar{\mathcal{O}}^{\mathcal{L}}{}_{k,n+1} + \bar{\mathcal{O}}^{\mathcal{L}}{}_{k,n}$$

$$(vi) \quad \bar{\mathcal{O}}^{\mathcal{L}}{}_{k,n} = \bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n+1} + \bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n-1}.$$

Proof. (i). Using definition (**??**), we have

$$\begin{aligned}
k\mathcal{O}^{\mathcal{F}}{}_{k,n+1} + \mathcal{O}^{\mathcal{F}}{}_{k,n} &= k\big(F_{k,n+1} + F_{k,n+2}i_1 + F_{k,n+3}i_2 + F_{k,n+4}i_3 + F_{k,n+5}e_4 \\
&\quad + F_{k,n+6}e_5 + F_{k,n+7}e_6 + F_{k,n+8}e_7\big) \\
&\quad + \big(F_{k,n} + F_{k,n+1}i_1 + F_{k,n+2}i_2 + F_{k,n+3}i_3 + F_{k,n+4}e_4 \\
&\quad + F_{k,n+5}e_5 + F_{k,n+6}e_6 + F_{k,n+7}e_7\big) \\
&= \big(kF_{k,n+1} + F_{k,n}\big) + \big(kF_{k,n+2} + F_{k,n+1}\big)i_1 \\
&\quad + \big(kF_{k,n+3} + F_{k,n+2}\big)i_2 + \big(kF_{k,n+4} + F_{k,n+3}\big)i_3 \\
&\quad + \big(kF_{k,n+5} + F_{k,n+4}\big)e_4 + \big(kF_{k,n+6} + F_{k,n+5}\big)e_5 \\
&\quad + \big(kF_{k,n+7} + F_{k,n+6}\big)e_6 + \big(kF_{k,n+8} + F_{k,n+7}\big)e_7 \\
&= F_{k,n+2} + F_{k,n+3}i_1 + F_{k,n+4}i_2 + F_{k,n+5}i_3 + F_{k,n+6}e_4 \\
&\quad + F_{k,n+7}e_5 + F_{k,n+8}e_6 + F_{k,n+9}e_7 \\
&= \mathcal{O}^{\mathcal{F}}{}_{k,n+2}.
\end{aligned}$$

The proofs of (ii), (iii), (iv), (v) and (vi) are similar to (i), using definition (**??**). □

Theorem 6.30. ***(Binet Formulas)***. *For all $n \geq 0$, we have*

$$(i) \quad \mathcal{O^F}_{k,n} = \frac{\bar{r_1}{r_1}^n - \bar{r_2}{r_2}^n}{r_1 - r_2}$$

$$(ii) \quad \mathcal{O^L}_{k,n} = \bar{r_1}{r_1}^n + \bar{r_2}{r_2}^n,$$

$$(iii) \quad \bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n} = \frac{\bar{r_3}{r_1}^n - \bar{r_4}{r_2}^n}{r_1 - r_2}$$

$$(iv) \quad \bar{\mathcal{O}}^{\mathcal{L}}{}_{k,n} = \bar{r_3}{r_1}^n + \bar{r_4}{r_2}^n,$$

where,

$$\bar{r_1} = 1 + r_1 i_1 + {r_1}^2 i_2 + {r_1}^3 i_3 + {r_1}^4 e_4 + {r_1}^5 e_5 + {r_1}^6 e_6 + {r_1}^7 e_7$$
$$= \left\langle 1, r_1, {r_1}^2, {r_1}^3, {r_1}^4, {r_1}^5, {r_1}^6, {r_1}^7 \right\rangle,$$
$$\bar{r_2} = 1 + r_2 i_1 + {r_2}^2 i_2 + {r_2}^3 i_3 + {r_2}^4 e_4 + {r_2}^5 e_5 + {r_2}^6 e_6 + {r_2}^7 e_7$$
$$= \left\langle 1, r_2, {r_2}^2, {r_2}^3, {r_2}^4, {r_2}^5, {r_2}^6, {r_2}^7 \right\rangle,$$
$$\bar{r_3} = 1 - r_1 i_1 - {r_1}^2 i_2 - {r_1}^3 i_3 - {r_1}^4 e_4 - {r_1}^5 e_5 - {r_1}^6 e_6 - {r_1}^7 e_7$$
$$= \left\langle 1, -r_1, -{r_1}^2, -{r_1}^3, -{r_1}^4, -{r_1}^5, -{r_1}^6, -{r_1}^7 \right\rangle,$$
$$\bar{r_4} = 1 + r_2 i_1 - {r_2}^2 i_2 - {r_2}^3 i_3 - {r_2}^4 e_4 - {r_2}^5 e_5 - {r_2}^6 e_6 - {r_2}^7 e_7$$
$$= \left\langle 1, -r_2, -{r_2}^2, -{r_2}^3, -{r_2}^4, -{r_2}^5, -{r_2}^6, -{r_2}^7 \right\rangle.$$

And, i_1, i_2, i_3 are quaternion imaginary units, $e_4({e_4}^2 = 1)$ is a counter imaginary unit, and the bases of hyperbolic octonions $\mathcal{O^F}_{k,n}$ and $\mathcal{O^L}_{k,n}$ are defined as $i_1 e_4 = e_5, i_2 e_4 = e_6, i_3 e_4 = e_7, {e_4}^2 = {e_5}^2 = {e_6}^2 = {e_7}^2 = 1$. The bases of hyperbolic octonions $\mathcal{O^F}_{k,n}$ and $\mathcal{O^L}_{k,n}$ have multiplication rules as in Table-(**??**).

Proof. (i). Using the definition (**??**) and the Binet formulas of k-Fibonacci and k- Lucas sequences, we have

$$\begin{aligned}
\mathcal{O}^{\mathcal{F}}{}_{k,n} &= F_{k,n} + F_{k,n+1}i_1 + F_{k,n+2}i_2 + F_{k,n+3}i_3 + F_{k,n+4}e_4 + F_{k,n+5}e_5 \\
&+ F_{k,n+6}e_6 + F_{k,n+7}e_7 \\
&= [\frac{{r_1}^n - {r_2}^n}{r_1 - r_2}] + [\frac{{r_1}^{n+1} - {r_2}^{n+1}}{r_1 - r_2}]i_1 + [\frac{{r_1}^{n+2} - {r_2}^{n+2}}{r_1 - r_2}]i_2 \\
&+ [\frac{{r_1}^{n+3} - {r_2}^{n+3}}{r_1 - r_2}]i_3 + [\frac{{r_1}^{n+4} - {r_2}^{n+4}}{r_1 - r_2}]e_4 + [\frac{{r_1}^{n+5} - {r_2}^{n+5}}{r_1 - r_2}]e_5 \\
&+ [\frac{{r_1}^{n+6} - {r_2}^{n+6}}{r_1 - r_2}]e_6 + [\frac{{r_1}^{n+7} - {r_2}^{n+7}}{r_1 - r_2}]e_7 \\
&= \frac{{r_1}^n}{r_1 - r_2}\big(1 + r_1 i_1 + {r_1}^2 i_2 + {r_1}^3 i_3 + {r_1}^4 e_4 + {r_1}^5 e_5 + {r_1}^6 e_6 + {r_1}^7 e_7\big) \\
&- \frac{{r_2}^n}{r_1 - r_2}\big(1 + r_2 i_1 + {r_2}^2 i_2 + {r_2}^3 i_3 + {r_2}^4 e_4 + {r_2}^5 e_5 + {r_2}^6 e_6 + {r_2}^7 e_7\big) \\
&= \frac{\bar{r_1}{r_1}^n - \bar{r}_2{r_2}^n}{r_1 - r_2}
\end{aligned}$$

(ii).
$$\begin{aligned}
\mathcal{O}^{\mathcal{L}}{}_{k,n} &= L_{k,n} + L_{k,n+1}i_1 + L_{k,n+2}i_2 + L_{k,n+3}i_3 + L_{k,n+4}e_4 + L_{k,n+5}e_5 \\
&+ L_{k,n+6}e_6 + L_{k,n+7}e_7 \\
&= \big({r_1}^n + {r_2}^n\big) + \big({r_1}^{n+1} + {r_2}^{n+1}\big)i_1 + \big({r_1}^{n+2} + {r_2}^{n+2}\big)i_2 + \big({r_1}^{n+3}{r_2}^{n+3}\big)i_3 \\
&+ \big({r_1}^{n+4}{r_2}^{n+4}\big)e_4 + \big({r_1}^{n+5}{r_2}^{n+5}\big)e_5 + \big({r_1}^{n+6}{r_2}^{n+6}\big)e_6 + \big({r_1}^{n+7}{r_2}^{n+7}\big)e_7 \\
&= {r_1}^n\big(1 + r_1 i_1 + {r_1}^2 i_2 + {r_1}^3 i_3 + {r_1}^4 e_4 + {r_1}^5 e_5 + {r_1}^6 e_6 + {r_1}^7 e_7\big) \\
&+ {r_2}^n\big(1 + r_2 i_1 + {r_2}^2 i_2 + {r_2}^3 i_3 + {r_2}^4 e_4 + {r_2}^5 e_5 + {r_2}^6 e_6 + {r_2}^7 e_7\big) \\
&= \bar{r_1}{r_1}^n + \bar{r}_2{r_2}^n.
\end{aligned}$$

The proofs of (iii) and (iv) are similar to (i) and (ii) using definition (**??**). □

Lemma 6.31. *For* $\bar{r_1} = \left\langle 1, r_1, {r_1}^2, {r_1}^3, {r_1}^4, {r_1}^5, {r_1}^6, {r_1}^7 \right\rangle$, $\bar{r_2} = \left\langle 1, r_2, {r_2}^2, {r_2}^3, {r_2}^4,\right.$ $\left. {r_2}^5, {r_2}^6, {r_2}^7 \right\rangle$, $\bar{r_3} = \left\langle 1, -r_1, -{r_1}^2, -{r_1}^3, -{r_1}^4, -{r_1}^5, -{r_1}^6, -{r_1}^7 \right\rangle$ *and* $\bar{r_4} =$ $\left\langle 1, -r_2, \right.$ $\left. -{r_2}^2, -{r_2}^3, -{r_2}^4, -{r_2}^5, -{r_2}^6, -{r_2}^7 \right\rangle$, *we have*

(1) $\bar{r_1} - \bar{r_2} = \sqrt{\delta}{\mathcal{O}^{\mathcal{F}}}_{k,0}$,

(2) $\bar{r_1} + \bar{r_2} = {\mathcal{O}^{\mathcal{L}}}_{k,0}$,

(3) $$\bar{r_1}\bar{r_2} = \left\langle 2, -2(r_1 - 2r_2), -2({r_1}^2 - 2{r_2}^2), 2(r_1 - r_2 + {r_2}^3), 2{r_1}^4, \right.$$ $$\left. 2({r_1}^3 - {r_2}^3) + 2{r_1}^5, -2({r_1}^2 - {r_2}^2 + {r_1}^6), {r_1}^7 + {r_2}^7 - (r_1 - r_2)({r_1}^4 + {r_2}^4 - 1) \right\rangle$$ $$= \bar{u_1},$$

(4) $$\bar{r_2}\bar{r_1} = \left\langle 2, -2(r_2 - 2r_1), -2({r_2}^2 - 2{r_1}^2), 2(r_2 - r_1 + {r_1}^3), 2{r_2}^4, \right.$$ $$\left. 2({r_2}^3 - {r_1}^3) + 2{r_2}^5, -2({r_2}^2 - {r_1}^2 + {r_2}^6), {r_2}^7 + {r_1}^7 - (r_2 - r_1)({r_1}^4 + {r_2}^4 - 1) \right\rangle$$ $$= \bar{u_2},$$

(5) $\bar{r_1}^2 = \left(-1 - {r_1}^2 - {r_1}^4 - {r_1}^6 + {r_1}^8 + {r_1}^{10} + {r_1}^{12} + {r_1}^{14} \right) + 2\bar{r_1} = \bar{u_3}$,

(6) $\bar{r_2}^2 = \left(-1 - {r_2}^2 - {r_2}^4 - {r_2}^6 + {r_2}^8 + {r_2}^{10} + {r_2}^{12} + {r_2}^{14} \right) + 2\bar{r_2} = \bar{u_4}$,

(7) $\bar{r_1}\bar{r_2} + \bar{r_2}\bar{r_1} = 2{\mathcal{O}^{\mathcal{L}}}_{k,0}$,

(8) $$\bar{r_1}\bar{r_2} - \bar{r_2}\bar{r_1} = 2\sqrt{\delta}\left\langle 0, -3, -3k, (1 - k^2), k(k^2 + 2), \right.$$ $$\left. k^4 + 5k^2 + 3, k^5 + 4k^3 + k, -(k^4 + 4k + 1) \right\rangle = \bar{u_5},$$

(9) $\bar{r_1}^2 - \bar{r_2}^2 = \sqrt{\delta}\big(k^{13} + 13k^{11} + 66k^9 + 165k^7 + 208k^5 + 116k^3 + 16k + 2{\mathcal{O}^{\mathcal{F}}}_{k,}$

$$= \bar{u_6},$$

(10) $\bar{r_1}^2 + \bar{r_2}^2 = \big(k^{14} + 15k^{12} + 90k^{10} + 275k^8 + 448k^6 + 364k^4 + 112k^2 + 2\mathcal{O}^{\mathcal{L}}{}_{k}$

$$= \bar{u_7},$$

(11) $\bar{r_3} = 2 - \bar{r_1} \quad and \quad \bar{r_4} = 2 - \bar{r_2},$

(12) $\bar{r_1}\bar{r_3} = \bar{r_3}\bar{r_1} = \big(1 + {r_1}^2 + {r_1}^4 + {r_1}^6 - {r_1}^8 - {r_1}^{10} - {r_1}^{12} - {r_1}^{14}\big) = u_8,$

(13) $\bar{r_2}\bar{r_4} = \bar{r_4}\bar{r_2} = \big(1 + {r_2}^2 + {r_2}^4 + {r_2}^6 - {r_2}^8 - {r_2}^{10} - {r_2}^{12} - {r_2}^{14}\big) = u_9,$

(14) $\bar{r_1}\bar{r_4} = \big\langle 0, 4(r_1 - r_2), 4({r_1}^2 - {r_2}^2), -2(r_1 - r_2 - {r_1}^3 - {r_2}^3), 0,$

$$-2({r_1}^3 - {r_2}^3), 2({r_1}^2 - {r_2}^2 + 2{r_1}^6), {r_1}^7 - {r_2}^7 + (r_1 - r_2)({r_1}^4 + {r_2}^4 - 1)\big\rangle = \bar{u_{10}},$$

(15) $\bar{r_4}\bar{r_1} = \big\langle 0, -2(r_1 - r_2), -2({r_1}^2 - {r_2}^2), 2(r_1 - r_2), -2({r_1}^4 - {r_2}^4), 2({r_1}^3 - {r_2}^3$

$$-{r_2}^5 + {r_1}^5), -2({r_1}^2 - {r_2}^2 - {r_1}^6 - {r_2}^6), {r_1}^7 - {r_2}^7 - (r_1 - r_2)({r_1}^4 + {r_2}^4 - 1)\big\rangle$$

$$= \bar{u_{11}},$$

(16) $\bar{r_2}\bar{r_3} = \big\langle 0, -2(r_1 - r_2), -2({r_1}^2 - {r_2}^2), 2(r_1 - r_2 - {r_1}^3 + {r_2}^3), -2({r_1}^4 - {r_2}^4),$

$$2({r_1}^3 - {r_2}^3), -2({r_1}^2 - {r_2}^2 - 2{r_2}^6), -{r_1}^7 + {r_2}^7 - (r_1 - r_2)({r_1}^4 + {r_2}^4 - 1)\big\rangle = \bar{u_{12}},$$

(17) $\bar{r_3}\bar{r_2} = \big\langle 0, 2(r_1 - r_2), 2({r_1}^2 - {r_2}^2), -2(r_1 - r_2), -2({r_1}^4 - {r_2}^4), -2({r_1}^3 - {r_2}^3$

$$+{r_1}^5 - {r_2}^5), 2({r_1}^2 - {r_2}^2 + {r_1}^6 + {r_2}^6), -{r_1}^7 + {r_2}^7 + (r_1 - r_2)({r_1}^4 + {r_2}^4 - 1)\big\rangle$$

$$= \bar{u_{13}},$$

(18) $\bar{r_3}\bar{r_4} = 4 - 2\mathcal{O}^{\mathcal{L}}{}_{k,0} + \bar{u_1},,$

(19) $\bar{r_4}\bar{r_3} = 4 - 2\mathcal{O}^{\mathcal{L}}{}_{k,0} + \bar{u_2},$

(20) $\bar{r_3}\bar{r_4} - \bar{r_4}\bar{r_3} = \bar{u_1} - \bar{u_2},$

(21) $\bar{r_3}\bar{r_2} = \big\langle 0, -2(r_1 - r_2), -2({r_1}^2 - {r_2}^2), -2({r_1}^3 - {r_2}^3), -4({r_1}^4 - {r_2}^4),$

$$-2({r_1}^5-{r_2}^5),2({r_1}^6+3{r_2}^6),-2({r_1}^7-{r_2}^7)\big\rangle=\bar{u_{14}},$$

(22) $\bar{r_3}\bar{r_2}-\bar{r_2}\bar{r_3}=\big\langle 0,6(r_1-r_2),6({r_1}^2-{r_2}^2),-2(2r_1-2r_2-{r_1}^3+{r_2}^3),0,$

$-2(2{r_1}^3-2{r_2}^3+{r_1}^5-{r_2}^5),2(2{r_1}^2-2{r_2}^2+{r_1}^6-{r_2}^6),2(r_1-r_2)({r_1}^4+{r_2}^4-$

$=\bar{u_{15}},$

(23) $\bar{r_1}\bar{r_4}-\bar{r_4}\bar{r_1}=\big\langle 0,6(r_1-r_2),6({r_1}^2-{r_2}^2),-2(2r_1-2r_2-{r_1}^3-{r_2}^3),2({r_2}^4-$

$-2(2{r_1}^3-2{r_2}^3+{r_1}^5-{r_2}^5),2(2{r_1}^2-2{r_2}^2+{r_1}^6-{r_2}^6),2(r_1-r_2)({r_1}^4+{r_2}^4-$

$=\bar{u_{16}},$

(24) $\bar{r_3}^2=4\bar{r_3}+\bar{u_3}-4=\bar{u_{17}},$

(25) $\bar{r_4}^2=4\bar{r_4}+\bar{u_4}-4=\bar{u_{18}}.$

Theorem 6.32. *For all $s,t\in Z^+,s\geq t$ and $n\in N$, the generating functions for the hyperbolic k-Fibonacci and k-Lucas quaternions ${\mathcal{O}^{\mathcal{F}}}_{k,tn}$ and ${\mathcal{O}^{\mathcal{L}}}_{k,tn}$ are*

$$(i)\quad \sum_{n=0}^{\infty}{\mathcal{O}^{\mathcal{F}}}_{k,tn}x^n=\frac{{\mathcal{O}^{\mathcal{F}}}_{k,0}+\big({\mathcal{O}^{\mathcal{L}}}_{k,0}F_{k,t}-{\mathcal{O}^{\mathcal{F}}}_{k,t}\big)x}{1-xL_{k,t}+x^2(-1)^t},$$

$$(ii)\quad \sum_{n=0}^{\infty}{\mathcal{O}^{\mathcal{L}}}_{k,tn}x^n=\frac{{\mathcal{O}^{\mathcal{L}}}_{k,0}-\big({\mathcal{O}^{\mathcal{L}}}_{k,0}L_{k,t}-{\mathcal{O}^{\mathcal{L}}}_{k,t}\big)x}{1-xL_{k,t}+x^2(-1)^t},$$

$$(iii)\quad \sum_{n=0}^{\infty}{\mathcal{O}^{\mathcal{F}}}_{k,tn+s}x^n=\frac{{\mathcal{O}^{\mathcal{F}}}_{k,s}+(-1)^t x{\mathcal{O}^{\mathcal{F}}}_{s,s-t}}{1-xL_{k,t}+x^2(-1)^t},$$

$$(iv)\quad \sum_{n=0}^{\infty}{\mathcal{O}^{\mathcal{L}}}_{k,tn+s}x^n=\frac{{\mathcal{O}^{\mathcal{L}}}_{k,s}+(-1)^t x{\mathcal{O}^{\mathcal{L}}}_{s-t}}{1-xL_{k,t}+x^2(-1)^t},$$

the exponential generating functions for the hyperbolic k-Fibonacci and k-Lucas quaternions $\mathcal{O}^{\mathcal{F}}{}_{k,tn}$ *and* $\mathcal{O}^{\mathcal{L}}{}_{k,tn}$ *are*

$$(v) \qquad \sum_{n=0}^{\infty} \frac{\mathcal{O}^{\mathcal{F}}{}_{k,tn}}{n!} x^n = \frac{\bar{r_1} e^{{r_1}^t x} - \bar{r_2} e^{{r_2}^t x}}{r_1 - r_2},$$

$$(vi) \qquad \sum_{n=0}^{\infty} \frac{\mathcal{O}^{\mathcal{L}}{}_{k,tn}}{n!} x^n = \bar{r_1} e^{{r_1}^t x} + \bar{r_2} e^{{r_2}^t x}.$$

Proof. (i). Using theorem (6.30), we obtain

$$\sum_{n=0}^{\infty} \mathcal{O}^{\mathcal{F}}{}_{k,tn} x^n = \sum_{n=0}^{\infty} \Big(\frac{\bar{r_1} {r_1}^{tn} - \bar{r_2} {r_2}^{tn}}{r_1 - r_2}\Big) x^n$$

$$= \frac{\bar{r_1}}{r_1 - r_2} \sum_{n=0}^{\infty} \left({r_1}^t\right)^n x^n - \frac{\bar{r_2}}{r_1 - r_2} \sum_{n=0}^{\infty} \left({r_2}^t\right)^n x^n$$

$$= \frac{\bar{r_1}}{r_1 - r_2} \Big(\frac{1}{1 - {r_1}^t x}\Big) - \frac{\bar{r_2}}{r_1 - r_2} \Big(\frac{1}{1 - {r_2}^t x}\Big)$$

$$= \frac{1}{r_1 - r_2} \Big[\frac{\left(\bar{r_1} - \bar{r_2}\right) + \left[\bar{r_2} {r_1}^t - \bar{r_1} {r_2}^t\right] x}{1 - \left({r_1}^t + {r_2}^t\right) x + x^2 (r_1 r_2)^t}\Big]$$

$$= \frac{1}{r_1 - r_2} \Big[\frac{\left(\bar{r_1} - \bar{r_2}\right) + \left[\bar{r_2} {r_1}^t - \bar{r_2} {r_2}^t + \bar{r_2} {r_2}^t - \bar{r_1} {r_1}^t + \bar{r_1} {r_1}^t - \bar{r_1} {r_2}^t\right] x}{1 - \left({r_1}^t + {r_2}^t\right) x + x^2 (r_1 r_2)^t}\Big]$$

$$= \frac{(\frac{\bar{r_1} - \bar{r_2}}{r_1 - r_2}) + \left[\left(\bar{r_1} + \bar{r_2}\right)(\frac{{r_1}^t - {r_2}^t}{r_1 - r_2}) - (\frac{\bar{r_1}{r_1}^t - \bar{r_2}{r_2}^t}{r_1 - r_2})\right]x}{1 - \left({r_1}^t + {r_2}^t\right)x + x^2(r_1r_2)^t}.$$

Using theorem (6.30) and lemma (6.31), we obtain

$$= \frac{{\mathcal{O}^{\mathcal{F}}}_{k,0} + \left({\mathcal{O}^{\mathcal{L}}}_{k,0}F_{k,t} - {\mathcal{O}^{\mathcal{F}}}_{k,t}\right)x}{1 - xL_{k,t} + x^2(-1)^t}.$$

The proofs of (ii), (iii), (iv), (v) and (vi) are similar to (i), using theorem (6.30). □

Theorem 6.33. *For all $n \in N$, we have*

$$(i) \qquad \sum_{i=0}^{n} \binom{n}{i} k^i {\mathcal{O}^{\mathcal{F}}}_{k,i} = {\mathcal{O}^{\mathcal{F}}}_{k,2n},$$

$$(ii) \qquad \sum_{i=0}^{n} \binom{n}{i} k^i {\mathcal{O}^{\mathcal{L}}}_{k,i} = {\mathcal{O}^{\mathcal{L}}}_{k,2n}.$$

Proof. (i). Using theorem (6.30), we obtain

$$\begin{aligned}
\sum_{i=0}^{n} \binom{n}{i} k^i {\mathcal{O}^{\mathcal{F}}}_{k,i} &= \sum_{i=0}^{n} \binom{n}{i} k^i (\frac{\bar{r_1}{r_1}^i - \bar{r_2}{r_2}^i}{r_1 - r_2}) \\
&= \frac{\bar{r_1}}{r_1 - r_2} \sum_{i=0}^{n} \binom{n}{i} (kr_1)^i - \frac{\bar{r_2}}{r_1 - r_2} \sum_{i=0}^{n} \binom{n}{i} (kr_2)^i \\
&= \frac{\bar{r_1}}{r_1 - r_2} \left(1 + kr_1\right)^n - \frac{\bar{r_2}}{r_1 - r_2} \left(1 + kr_2\right)^n \\
&= \frac{\bar{r_1}{r_1}^{2n} - \bar{r_2}{r_2}^{2n}}{r_1 - r_2} \\
&= {\mathcal{O}^{\mathcal{F}}}_{k,2n}.
\end{aligned}$$

The proof of (ii) is similar to (i), using theorem (6.30). □

Lemma 6.34. *For all $t \geq 0$, we have*

$$(i) \quad \frac{\bar{u_1}{r_2}^t - \bar{u_2}{r_1}^t}{r_1 - r_2} = \bar{\mathcal{U}}_{k,t},$$

where

$$\bar{\mathcal{U}}_{k,t} = \Big\langle -2F_{k,t}, -2F_{k,t-1} - 4F_{k,t+1}, 2F_{k,t-2} - 4F_{k,t+2}, 2F_{k,t-2} + 2F_{k,t+1} - 2F_{k,t+3}, -2F_{k,t-4}, 2F_{k,t-3} + 2F_{k,t+3} + 2F_{k,t-5}, 2F_{k,t-2} - 2F_{k,t+2} + 2F_{k,t-6}, F_{k,t-7} - F_{k,t+7} - L_{k,t+4} - L_{k,t-4} + L_{k,t}\Big\rangle,$$

$$(ii) \quad \frac{{r_1}^{m-n}\bar{u_1} - {r_2}^{m-n}\bar{u_2}}{r_1 - r_2} = \bar{\mathcal{V}}_{k,m-n},$$

where

$$\bar{\mathcal{V}}_{k,m-n} = \Big\langle 2F_{k,m-n}, -2F_{k,m-n+1} - 4F_{k,m-n-1}, -2F_{k,m-n+2} + 4F_{k,m-n-2}, 2F_{k,m-n+1} + 2F_{k,m-n-1} - 2F_{k,m-n-3}, 2F_{k,m-n+4}, 2F_{k,m-n+3} + 2F_{k,m-n-3} + F_{k,m-n+5}, -2F_{k,m-n+2} + 2F_{k,m-n-2} - 2F_{k,m-n+6}, F_{k,m-n+7} - F_{k,m-n-7} - L_{k,m-n+4} - L_{k,m-n-4} + L_{k,m-n}\Big\rangle,$$

$$(iii) \quad \bar{u_3}{r_1}^t + \bar{u_4}{r_2}^t = \bar{\mathcal{W}}_{k,t},$$

where

$$\bar{\mathcal{W}}_{k,t} = \big(- L_{k,t} - L_{k,t+2} - L_{k,t+4} - L_{k,t+6} + L_{k,t+8} + L_{k,t+10} + L_{k,t+12} + L_{k,t+14}\big) + 2{\mathcal{O}^{\mathcal{L}}}_{k,t},$$

$$(iv)\quad \frac{\bar{u_3}{r_1}^t-\bar{u_4}{r_2}^t}{r_1-r_2}=\bar{\mathcal{X}}_{k,t},$$

where

$$\bar{\mathcal{X}}_{k,t}=\big(-F_{k,t}-F_{k,t+2}-F_{k,t+4}-F_{k,t+6}+F_{k,t+8}+F_{k,t+10}+Fk,t+12$$

$$+F_{k,t+14}\big)+2{\mathcal{O}^{\mathcal{F}}}_{k,t},$$

$$(v)\quad \frac{\bar{u_8}{r_1}^t-\bar{u_9}{r_2}^t}{r_1-r_2}=L_{k,t+1}+L_{k,t+3}-L_{k,t+9}-L_{k,t+13},$$

$$(vi)\quad \frac{\bar{u_{17}}{r_1}^t-\bar{u_{18}}{r_2}^t}{r_1-r_2}=\bar{\mathcal{Y}}_{k,t},$$

where

$$\bar{\mathcal{Y}}_{k,t}=\big(-F_{k,t}-F_{k,t+2}-F_{k,t+4}-F_{k,t+6}+F_{k,t+8}+F_{k,t+10}+Fk,t+12$$

$$+F_{k,t+14}\big)+4-2{\mathcal{O}^{\mathcal{F}}}_{k,t}.$$

Theorem 6.35. ***(Catalan's Identity)***. *For any integer t and s, we have*

$$(i)\quad {\mathcal{O}^{\mathcal{F}}}_{k,n-t}{\mathcal{O}^{\mathcal{F}}}_{k,n+t}-{{\mathcal{O}^{\mathcal{F}}}_{k,n}}^2=(-1)^{n-t}F_{k,t}\bar{\mathcal{U}}_{k,t},$$

$$(ii)\quad {\mathcal{O}^{\mathcal{L}}}_{k,n-t}{\mathcal{O}^{\mathcal{L}}}_{k,n+t}-{{\mathcal{O}^{\mathcal{L}}}_{k,n}}^2=\delta(-1)^{n-t+1}F_{k,t}\bar{\mathcal{V}}_{k,m-n}.$$

Proof. Using theorem (6.30), we have

$${\mathcal{O}^{\mathcal{F}}}_{k,n-t}{\mathcal{O}^{\mathcal{F}}}_{k,n+t}-{{\mathcal{O}^{\mathcal{F}}}_{k,n}}^2=\Big(\frac{\bar{r_1}{r_1}^{n-t}-\bar{r_2}{r_2}^{n-t}}{r_1-r_2}\Big)\Big(\frac{\bar{r_1}{r_1}^{n+t}-\bar{r_2}{r_2}^{n+t}}{r_1-r_2}\Big)$$
$$-\Big(\frac{\bar{r_1}{r_1}^{n}-\bar{r_2}{r_2}^{n}}{r_1-r_2}\Big)^2$$
$$=\frac{1}{(r_1-r_2)^2}\big[\bar{r_1}^2{r_1}^{2n}-\bar{r_1}\bar{r_2}{r_1}^{n-t}{r_2}^{n+t}-\bar{r_2}\bar{r_1}{r_2}^{n-t}{r_1}^{n+t}+\bar{r_2}^2{r_2}^{2n}$$

$$-\bar{r_1}^2{r_1}^{2n}+\bar{r_1}\bar{r_2}(r_1r_2)^n+\bar{r_2}\bar{r_1}(r_1r_2)^n-\bar{r_2}^2{r_2}^{2n}\big]$$

$$\begin{aligned}
&=\frac{(r_1r_2)^n}{(r_1-r_2)^2}\big[\bar{r_1}\bar{r_2}{r_1}^t{r_1}^{-t}-\bar{r_2}\bar{r_1}{r_1}^t{r_2}^{-t}-\bar{r_1}\bar{r_2}{r_2}^t{r_2}^{-t}+\bar{r_2}\bar{r_1}{r_2}^t{r_2}^{-t}\big]\\
&=\frac{(r_1r_2)^n}{(r_1-r_2)^2}\big[{r_1}^t[(\bar{r_1}\bar{r_2}){r_1}^{-t}-(\bar{r_2}\bar{r_1}){r_2}^{-t}]-{r_2}^t[(\bar{r_1}\bar{r_2}){r_1}^{-t}-(\bar{r_2}\bar{r_1}){r_2}^{-t}]\big]\\
&=(r_1r_2)^n\big(\frac{{r_1}^t-{r_2}^t}{r_1-r_2}\big)\big(\frac{(\bar{r_1}\bar{r_2}){r_1}^{-t}-(\bar{r_2}\bar{r_1}){r_2}^{-t}}{r_1-r_2}\big)\\
&=(r_1r_2)^{n-t}\big(\frac{{r_1}^t-{r_2}^t}{r_1-r_2}\big)\big(\frac{(\bar{r_1}\bar{r_2}){r_1}^{t}-(\bar{r_2}\bar{r_1}){r_2}^{t}}{r_1-r_2}\big).
\end{aligned}$$

Using lemma (6.34) and $r_1r_2=-1$, we obtain

$$=(-1)^{n-t}F_{k,t}\bar{\mathcal{U}}_{k,t}.$$

The proof of (ii) is similar to (i), using theorem (6.30) and lemma (6.34). $\square$

Theorem 6.36. ***(Cassini's Identity)***. *For all $n\geq 1$, we have*

$$(i)\quad {\mathcal{O}^{\mathcal{F}}}_{k,n-1}{\mathcal{O}^{\mathcal{F}}}_{k,n+1}-{{\mathcal{O}^{\mathcal{F}}}_{k,n}}^2=(-1)^{n-1}F_{k,t}\bar{\mathcal{U}}_{k,1},$$

$$(ii)\quad {\mathcal{O}^{\mathcal{L}}}_{k,n-1}{\mathcal{O}^{\mathcal{L}}}_{k,n+1}-{{\mathcal{O}^{\mathcal{L}}}_{k,n}}^2=\delta(-1)^{n}F_{k,t}\bar{\mathcal{V}}_{k,1}.$$

Theorem 6.37. ***(d'Ocagene's Identity)****. Let n be any non-negative integer and t a natural number. If $t \geq n+1$, then we have*

$$(i) \quad {\mathcal{O}^{\mathcal{F}}}_{k,t}{\mathcal{O}^{\mathcal{F}}}_{k,n+1} - {\mathcal{O}^{\mathcal{F}}}_{k,t+1}{\mathcal{O}^{\mathcal{F}}}_{k,n} = (-1)^n \bar{\mathcal{V}}_{k,t-n},$$

$$(ii) \quad {\mathcal{O}^{\mathcal{L}}}_{k,t}{\mathcal{O}^{\mathcal{L}}}_{k,n+1} - {\mathcal{O}^{\mathcal{L}}}_{k,t+1}{\mathcal{O}^{\mathcal{L}}}_{k,n} = (-1)^{n+1} \delta \bar{\mathcal{V}}_{k,t-n}.$$

Proof. Using theorem (6.30), we get

$$\begin{aligned}
{\mathcal{O}^{\mathcal{F}}}_{k,t}{\mathcal{O}^{\mathcal{F}}}_{k,n+1} - {\mathcal{O}^{\mathcal{F}}}_{k,t+1}{\mathcal{O}^{\mathcal{F}}}_{k,n} &= \Big(\frac{\bar{r_1}{r_1}^t - \bar{r_2}{r_2}^t}{r_1 - r_2}\Big)\Big(\frac{\bar{r_1}{r_1}^{n+1} - \bar{r_2}{r_2}^{n+1}}{r_1 - r_2}\Big) \\
&- \Big(\frac{\bar{r_1}{r_1}^{t+1} - \bar{r_2}{r_2}^{t+1}}{r_1 - r_2}\Big)\Big(\frac{\bar{r_1}{r_1}^{n} - \bar{r_2}{r_2}^{n}}{r_1 - r_2}\Big) \\
&= \frac{1}{(r_1 - r_2)^2}\Big[\bar{r_1}^2{r_1}^{n+t+1} - \bar{r_1}\bar{r_2}{r_1}^t{r_2}^{n+1} - \bar{r_2}\bar{r_1}{r_2}^t{r_1}^{n+1} + \bar{r_2}^2{r_2}^{n+t+1} \\
&- \bar{r_1}^2{r_1}^{n+t+1} + \bar{r_1}\bar{r_2}({r_1}^{t+1}{r_2}^n) + \bar{r_2}\bar{r_1}({r_1}^n{r_2}^{t+1}) - \bar{r_2}^2{r_2}^{n+t+1}\Big] \\
&= \frac{(r_1 r_2)^n}{(r_1 - r_2)^2}\Big[\bar{r_1}\bar{r_2}{r_1}^{t-n}(r_1 - r_2) - \bar{r_2}\bar{r_1}{r_2}^{t-n}(r_1 - r_2)\Big] \\
&= (r_1 r_2)^n\Big[\frac{\bar{r_1}\bar{r_2}{r_1}^{t-n} - \bar{r_2}\bar{r_1}{r_2}^{t-n}}{r_1 - r_2}\Big].
\end{aligned}$$

Using lemma (6.34) and $r_1 r_2 = -1$, we obtain

$$= (-1)^n \bar{\mathcal{V}}_{k,t-n}.$$

The proof of (ii) is similar to (i), using theorem (6.30) and lemma (6.34). □

Theorem 6.38. *For any integer t, we have*

$$(i) \quad {\mathcal{O}^{\mathcal{F}}_{k,t}}^2 + {\mathcal{O}^{\mathcal{L}}_{k,t}}^2 = \frac{1}{\delta}\big[(1+\delta)\bar{\mathcal{W}}_{k,2t} + (\delta-1)(-1)^t \mathcal{O}^{\mathcal{L}}_{k,0}\big],$$

$$(ii) \quad {\mathcal{O}^{\mathcal{F}}_{k,t}}^2 - {\mathcal{O}^{\mathcal{L}}_{k,t}}^2 = \frac{1}{\delta}\big[(1-\delta)\bar{\mathcal{W}}_{k,2t} - (1+\delta)(-1)^t \mathcal{O}^{\mathcal{L}}_{k,0}\big].$$

Proof. Using theorem (6.30), we get

$$\begin{aligned}
{\mathcal{O}^{\mathcal{F}}_{k,t}}^2 + {\mathcal{O}^{\mathcal{L}}_{k,t+1}}^2 &= \Big(\frac{\bar{r_1}{r_1}^t - \bar{r_2}{r_2}^t}{r_1 - r_2}\Big)^2 + \big(\bar{r_1}{r_1}^t + \bar{r_2}{r_2}^t\big)^2 \\
&= \frac{1}{(r_1-r_2)^2}\big[(\bar{r_1})^2{r_1}^{2t} + (\bar{r_2})^2{r_2}^{2t} - \bar{r_1}\bar{r_2}r_1{r_2}^t - \bar{r_2}\bar{r_1}r_1{r_2}^t\big] \\
&+ \big[(\bar{r_1})^2{r_1}^{2t} + (\bar{r_2})^2{r_2}^{2t} + \bar{r_1}\bar{r_2}r_1{r_2}^t + \bar{r_2}\bar{r_1}r_1{r_2}^t\big] \\
&= \frac{(1+\delta)}{\delta}\big[(\bar{r_1})^2{r_1}^{2t} + \bar{r_2})^2{r_2}^{2t}\big) + \frac{(\delta-1)(-1)^t}{\delta}\big[\bar{r_1}\bar{r_2} + \bar{r_2}\bar{r_1}\big].
\end{aligned}$$

Using lemma (6.31), we obtain

$$= \frac{1}{\delta}\big[(1+\delta)\bar{\mathcal{W}}_{k,2t} + (\delta-1)(-1)^t \mathcal{O}^{\mathcal{L}}_{k,0}\big].$$

The proof of (ii) is similar to (i), using theorem (6.30) and lemma (6.31). □

Theorem 6.39. *For any integer r, $s \geq t$, we have*

$$\mathcal{O}^{\mathcal{F}}_{k,r+s}\mathcal{O}^{\mathcal{L}}_{k,r+t} - \mathcal{O}^{\mathcal{F}}_{k,r+t}\mathcal{O}^{\mathcal{L}}_{k,r+s} = 2(-1)^{r+t}\big(\mathcal{O}^{\mathcal{L}}_{k,0} - 2\big)F_{k,s-t}.$$

Proof. Using theorem (6.30) and $r_1 r_2 = -1$, we get

$$\mathcal{O}^{\mathcal{F}}_{k,r+s}\mathcal{O}^{\mathcal{L}}_{k,r+t} - \mathcal{O}^{\mathcal{F}}_{k,r+t}\mathcal{O}^{\mathcal{L}}_{k,r+s} = \frac{1}{r_1 - r_2}\big[\big(\bar{r_1}{r_1}^{r+s} - \bar{r_2}{r_2}^{r+s}\big)\cdot$$

$$\left(\bar{r_1}r_1{}^{r+t}+\bar{r_2}r_2{}^{r+t}\right)-\left(\bar{r_1}r_1{}^{r+t}-\bar{r_2}r_2{}^{r+t}\right)\left(\bar{r_1}r_1{}^{r+s}+\bar{r_2}r_2{}^{r+s}\right)\Big]$$
$$=\frac{(r_1r_2)^r}{r_1-r_2}\Big[(\bar{r_1}\bar{r_2}+\bar{r_2}\bar{r_1})r_1{}^sr_2{}^t-(\bar{r_1}\bar{r_2}+\bar{r_2}\bar{r_1})r_1{}^tr_2{}^s\Big]$$
$$=\frac{(r_1r_2)^r}{r_1-r_2}\left(\bar{r_1}\bar{r_2}+\bar{r_2}\bar{r_1}\right)\left(r_1{}^sr_2{}^t-r_1{}^tr_2{}^s\right).$$

Using lemma (6.31), we obtain

$$=2(-1)^{r+t}\left(\mathcal{O}^{\mathcal{L}}{}_{k,0}-2\right)F_{k,s-t}.$$

□

Theorem 6.40. *For any integer s, and t, we have*

$$(i)\quad \mathcal{O}^{\mathcal{F}}{}_{k,s+t}+(-1)^t\mathcal{O}^{\mathcal{F}}{}_{k,s-t}=\mathcal{O}^{\mathcal{F}}{}_{k,s}L_{k,t},$$

$$(ii)\quad \mathcal{O}^{\mathcal{L}}{}_{k,s+t}+(-1)^t\mathcal{O}^{\mathcal{L}}{}_{k,s-t}=\mathcal{O}^{\mathcal{L}}{}_{k,s}L_{k,t}.$$

Proof. Using theorem (6.30), we get

$$\mathcal{O}^{\mathcal{F}}{}_{k,s+t}+(-1)^t\mathcal{O}^{\mathcal{F}}{}_{k,s-t}=\frac{1}{r_1-r_2}\Big[\left(\bar{r_1}r_1{}^{s+t}-\bar{r_2}r_2{}^{s+t}\right)$$
$$+(-1)^t\left(\bar{r_1}r_1{}^{s-t}+\bar{r_2}r_2{}^{s-t}\right)\Big]$$
$$=\frac{1}{r_1-r_2}\Big[\bar{r_1}r_1{}^{s+t}-\bar{r_2}r_2{}^{s+t}+\bar{r_1}r_1{}^sr_2{}^t-\bar{r_2}r_1{}^tr_2{}^s\Big]$$
$$=\frac{1}{r_1-r_2}\Big[\bar{r_1}r_1{}^s(r_1{}^t+r_2{}^t)-\bar{r_2}r_2{}^s(r_1{}^t+r_2{}^t)$$
$$=\Big(\frac{\bar{r_1}r_1{}^s-\bar{r_2}r_2{}^s}{r_1-r_2}\Big)\left(r_1{}^t+r_2{}^t\right).$$

Using theorem (6.30), we obtain

$$=\mathcal{O}^{\mathcal{F}}{}_{k,s}L_{k,t}.$$

The proof of (ii) is similar to (i), using theorem (6.30). □

Theorem 6.41. *For any integer $s \leq t$, we have*

$$(i) \quad \mathcal{O}^{\mathcal{F}}{}_{k,s}\mathcal{O}^{\mathcal{F}}{}_{k,t} - \mathcal{O}^{\mathcal{F}}{}_{k,t}\mathcal{O}^{\mathcal{F}}{}_{k,s} = (-1)^s\delta^{-\frac{1}{2}}\bar{u_5}F_{k,t-s},$$

$$(ii) \quad \mathcal{O}^{\mathcal{L}}{}_{k,s}\mathcal{O}^{\mathcal{L}}{}_{k,t} - \mathcal{O}^{\mathcal{L}}{}_{k,t}\mathcal{O}^{\mathcal{L}}{}_{k,s} = (-1)^t\delta^{\frac{1}{2}}\bar{u_5}F_{k,s-t},$$

$$(iii) \quad \mathcal{O}^{\mathcal{F}}{}_{k,t}\mathcal{O}^{\mathcal{L}}{}_{k,s} - \mathcal{O}^{\mathcal{F}}{}_{k,s}\mathcal{O}^{\mathcal{L}}{}_{k,t} = 2(-1)^sF_{k,t-s}\mathcal{O}^{\mathcal{L}}{}_{k,0},$$

$$(iv) \quad \mathcal{O}^{\mathcal{F}}{}_{k,t}\mathcal{O}^{\mathcal{L}}{}_{k,s} - \mathcal{O}^{\mathcal{L}}{}_{k,t}\mathcal{O}^{\mathcal{F}}{}_{k,s} = 2(-1)^s\bar{\mathcal{V}}_{k,t-s}.$$

Proof. (i). Using theorem (6.30), we have

$$\begin{aligned}
\mathcal{O}^{\mathcal{L}}{}_{k,s}\mathcal{O}^{\mathcal{L}}{}_{k,t} - \mathcal{O}^{\mathcal{L}}{}_{k,t}\mathcal{O}^{\mathcal{L}}{}_{k,s} &= \Big(\frac{\bar{r_1}{r_1}^s - \bar{r_2}{r_2}^s}{r_1 - r_2}\Big)\Big(\frac{\bar{r_1}{r_1}^t - \bar{r_2}{r_2}^t}{r_1 - r_2}\Big) \\
&- \Big(\frac{\bar{r_1}{r_1}^t - \bar{r_2}{r_2}^t}{r_1 - r_2}\Big)\Big(\frac{\bar{r_1}{r_1}^s - \bar{r_2}{r_2}^s}{r_1 - r_2}\Big) \\
&= \frac{1}{(r_1 - r_2)^2}\big({r_1}^t{r_2}^s - {r_1}^s{r_2}^t\big)\big(\bar{r_1}\bar{r_2} - \bar{r_2}\bar{r_1}\big) \\
&= (r_1r_2)^s(r_1 - r_2)^{-1}\big(\bar{r_1}\bar{r_2} - \bar{r_2}\bar{r_1}\big)\Big(\frac{{r_1}^{t-s} - {r_2}^{t-s}}{r_1 - r_2}\Big)
\end{aligned}$$

Using lemma (6.31), we obtain

$$= (-1)^s\delta^{-\frac{1}{2}}\bar{u_5}F_{k,t-s}.$$

The proof of (ii), (iii) and (iv) is similar to (i), using theorem (6.30) and lemma (6.31). □

Theorem 6.42. *For any integer $n \geq 0$, we have*

$$(i) \quad \mathcal{O}^{\mathcal{F}}{}_{k,n}\bar{\mathcal{O}}^{\mathcal{L}}{}_{k,n} - \bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n}\mathcal{O}^{\mathcal{F}}{}_{k,n} = (-1)^n\delta^{-\frac{1}{2}}(2\bar{u_{10}} - \bar{u_{14}}),$$

$$(ii) \quad \mathcal{O}^{\mathcal{F}}{}_{k,n}\bar{\mathcal{O}}^{\mathcal{L}}{}_{k,n} + \bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n}\mathcal{O}^{\mathcal{F}}{}_{k,n} = 2\delta^{-\frac{1}{2}}(L_{k,t+1} + L_{k,t+3} - L_{k,t+9} - L_{k,t+13})$$

$$+ (-1)^n\delta^{-\frac{1}{2}}(2\bar{u_{15}} - \bar{u_{16}}),$$

$$(iii) \mathcal{O}^{\mathcal{F}}{}_{k,n}\mathcal{O}^{\mathcal{L}}{}_{k,n} - \bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n}\bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n} = \bar{\mathcal{X}}_{k,2n} + \delta^{-\frac{1}{2}}\big((-1)^n\bar{u_5} - \bar{u_1} + \bar{u_2}\big) - \bar{\mathcal{Y}}_{k,2n}.$$

Proof. (i). Using theorem (6.30), we have

$$\mathcal{O}^{\mathcal{F}}{}_{k,n}\bar{\mathcal{O}}^{\mathcal{L}}{}_{k,n} - \bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n}\mathcal{O}^{\mathcal{F}}{}_{k,n} = \Big(\frac{\bar{r_1}{r_1}^n - \bar{r_2}{r_2}^n}{r_1 - r_2}\Big)\big(\bar{r_3}{r_1}^n + \bar{r_4}{r_2}^n\big)$$

$$- \Big(\frac{\bar{r_3}{r_1}^n - \bar{r_4}{r_2}^n}{r_1 - r_2}\Big)\big(\bar{r_1}{r_1}^n + \bar{r_2}{r_2}^n\big)$$

$$= \frac{(r_1r_2)^n}{(r_1 - r_2)}\big(2\bar{r_1}\bar{r_4} - \bar{r_2}\bar{r_3} - \bar{r_3}\bar{r_2}\big)$$

Using lemma (6.31), we obtain

$$= (-1)^n\delta^{-\frac{1}{2}}(2\bar{u_{10}} - \bar{u_{14}}).$$

The proof of (ii) and (iii) is similar to (i), using theorem (6.30) and lemma (6.31). □

6.5 Some Binomial and Congruence Properties of Hyperbolic k- Fibonacci and k- Lucas Octonions

In this section, we explore some binomial and congruence properties of the hyperbolic k-Fibonacci and k-Lucas octonions.

Lemma 6.43. *For $n \geq 0$, we have*

(i) $\quad {r_1}^n = r_1 F_{k,n} + F_{k,n-1},$

(ii) $\quad {r_2}^n = r_2 F_{k,n} + F_{k,n-1},$

(iii) $\quad {r_1}^{2n} = {r_1}^n L_{k,n} - (-1)^n,$

(iv) $\quad {r_2}^{2n} = {r_2}^n L_{k,n} - (-1)^n,$

(v) $\quad {r_1}^{tn} = {r_1}^n \dfrac{F_{k,tn}}{F_{k,n}} - (-1)^n - \dfrac{F_{k,(t-1)n}}{F_{k,n}},$

(vi) $\quad {r_2}^{tn} = {r_2}^n \dfrac{F_{k,tn}}{F_{k,n}} - (-1)^n - \dfrac{F_{k,(t-1)n}}{F_{k,n}},$

(vii) $\quad {r_1}^{sn} F_{k,rn} - {r_1}^{rn} F_{k,sn} = (-1)^{sn} F_{k,(r-s)n},$

$(viii)$ $\quad {r_2}^{sn} F_{k,rn} - {r_2}^{rn} F_{k,sn} = (-1)^{sn} F_{k,(r-s)n},$

(ix) $\quad 1 + kr_1 + {r_1}^{2(2^{n+1}+1)} = L_{k,2^{n+1}} {r_1}^{2(2^n+1)},$

(x) $\quad 1 + kr_2 + {r_2}^{2(2^{n+1}+1)} = L_{k,2^{n+1}} {r_1}^{2(2^n+1)},$

(xi) $\quad$ *For every $n, t \geq 1$ and $l_n = \sum_{i=1}^{n} L_{k,2^i}$, we have*

$$(a) \quad 1+{r_1}^{2^n}=\begin{cases}\dfrac{l_{n-1}}{l_{n-2}}{r_1}^{2^{n-1}}; \\ \dfrac{l_{n-1}}{l_{n-t-1}}{r_1}^{2^{n-t}}-l_{n-1}\sum\limits_{i=2}^{t}\dfrac{1}{l_{n-i}}, \quad \textit{for } t=2,3,4,\ldots,n-2\ ;, \\ l_{n-1}{r_1}^{2}-l_{n-1}\sum\limits_{i=2}^{n-1}\dfrac{1}{l_{n-i}}\end{cases}$$

$$(b) \quad 1+{r_2}^{2^n}=\begin{cases}\dfrac{l_{n-1}}{l_{n-2}}{r_2}^{2^{n-1}}; \\ \dfrac{l_{n-1}}{l_{n-t-1}}{r_2}^{2^{n-t}}-l_{n-1}\sum\limits_{i=2}^{t}\dfrac{1}{l_{n-i}}, \quad \textit{for } t=2,3,4,\ldots,n-2\ ;, \\ l_{n-1}{r_2}^{2}-l_{n-1}\sum\limits_{i=2}^{n-1}\dfrac{1}{l_{n-i}}.\end{cases}$$

$(xii) \quad r_1^{2t}=\dfrac{F_{k,2t}}{k}r_1\sqrt{\delta}-\dfrac{L_{k,2t-1}}{k},$

$(xiii) \quad r_2^{2t}=-\dfrac{F_{k,2t}}{k}r_2\sqrt{\delta}-\dfrac{L_{k,2t-1}}{k}.$

$(xiv) \quad r_1^{2t+1}=\dfrac{L_{k,2t+1}}{k}r_1-\dfrac{F_{k,2t}}{k}\sqrt{\delta},$

$(xv) \quad r_2^{2t+1}=\dfrac{L_{k,2t+1}}{k}r_2+\dfrac{F_{k,2t}}{k}\sqrt{\delta}.$

Proof. (i). We use induction principle on n, for $n=1$, we have

$$ {r_1}^1=1\cdot r_1+0=r_1F_{k,1}+F_{k,0}.$$

For $n=2$, since r_1 is root of $r^2-kr-1=0$ therefore we have

$${r_1}^2=kr_1+1=r_1F_{k,2}+F_{k,1}.$$

Now, consider

$$r_1{}^{n+1} = r_1{}^n \cdot r_1 = (r_1 F_{k,n} + F_{k,n-1})r_1 = r_1{}^2 F_{k,n} + r_1 F_{k,n-1}.$$

Using $r_1{}^2 = kr_1 + 1$, we have

$$= (kr_1 + 1)F_{k,n} + r_1 F_{k,n-1} = r_1(kF_{k,n} + F_{k,n-1}) + F_{k,n} = r_1 F_{k,n+1} + F_{k,n}.$$

This complete the proof of (i).

(iii). Using (i), we have

$$\begin{aligned}
r_1{}^{2n} &= F_{k,n} r_1{}^{n+1} + r_1{}^n F_{k,n-1} \\
&= F_{k,n}(r_1 F_{k,n+1} + F_{k,n}) + r_1{}^n F_{k,n-1} \\
&= r_1 F_{k,n} F_{k,n+1} + F_{k,n-1} r_1{}^n + F_{k,n}^2 \\
&= (r_1{}^n - F_{k,n-1})F_{k,n+1} + F_{k,n-1} r_1{}^n + F_{k,n}^2 \\
&= r_1{}^n(F_{k,n+1} + F_{k,n-1}) + F_{k,n}^2 - F_{k,n} F_{k,n-1}.
\end{aligned}$$

Using $F_{k,n-1}F_{k,n+1} - F_{k,n}^2 = (-1)^n$ and $F_{k,n+1} + F_{k,n-1} = L_{k,n}$, we obtain

$$r_1{}^{2n} = L_{k,n} r_1{}^n - (-1)^n.$$

This complete the proof of (iii).

The proofs of (ii), (iv), (v), (vii), (viii), (ix), (x), (xi),(xii), (xiii), (xiv) and (xv) are similar to (i) and (iii). □

Theorem 6.44. *For all $n, r, s, t \geq 1$, we have*

$$(i) \quad {\mathcal{O}^{\mathcal{F}}}_{k,n+t} = F_{k,n}{\mathcal{O}^{\mathcal{F}}}_{k,t+1} + F_{k,n-1}{\mathcal{O}^{\mathcal{F}}}_{k,t},$$

$$(ii) \quad {\mathcal{O}^{\mathcal{F}}}_{k,2n+t} = L_{k,n}{\mathcal{O}^{\mathcal{F}}}_{k,n+t} - (-1)^n{\mathcal{O}^{\mathcal{F}}}_{k,t},$$

$$(iii) \quad {\mathcal{O}^{\mathcal{F}}}_{k,sn+t} = \frac{F_{k,sn}}{F_{k,n}}{\mathcal{O}^{\mathcal{F}}}_{k,n+t} - (-1)^n\frac{F_{k,(s-1)n}}{F_{k,n}}{\mathcal{O}^{\mathcal{F}}}_{k,t},$$

$$(iv) \quad {\mathcal{O}^{\mathcal{F}}}_{k,sn+t}F_{k,rn} - {\mathcal{O}^{\mathcal{F}}}_{k,rn+t}F_{k,sn} = (-1)^{sn}{\mathcal{O}^{\mathcal{F}}}_{k,t}F_{k,(r-s)n},$$

$$(v) \quad {\mathcal{O}^{\mathcal{F}}}_{k,t+2^{n+1}+2} = \frac{{\mathcal{O}^{\mathcal{F}}}_{k,t} + k{\mathcal{O}^{\mathcal{F}}}_{k,t+1} + {\mathcal{O}^{\mathcal{F}}}_{k,t+2^{n+2}+2}}{L_{k,2^{n+1}}},$$

$$(vi) \quad {\mathcal{O}^{\mathcal{L}}}_{k,t+2^{n+1}+2} = \frac{{\mathcal{O}^{\mathcal{L}}}_{k,t} + k{\mathcal{O}^{\mathcal{L}}}_{k,t+1} + {\mathcal{O}^{\mathcal{L}}}_{k,t+2^{n+2}+2}}{L_{k,2^{n+1}}},$$

(vii) *For every $n, t \geq 1$ and $l_n = \sum_{i=1}^{n} L_{k,2^i}$, we have*

$$(a) \quad {\mathcal{O}^{\mathcal{F}}}_{k,t+2^n} = \begin{cases} \dfrac{l_{n-1}}{l_{n-2}}{\mathcal{O}^{\mathcal{F}}}_{k,t+2^{n-1}} - {\mathcal{O}^{\mathcal{F}}}_{k,t}; \\ \dfrac{l_{n-1}}{l_{n-t-1}}{\mathcal{O}^{\mathcal{F}}}_{k,t+2^{n-s}} - l_{n-1}\sum_{i=2}^{s}(1+\dfrac{1}{l_{n-i}}){\mathcal{O}^{\mathcal{F}}}_{k,t}, \\ \qquad\qquad\qquad \text{If } s = 2,3,4,\ldots,n-2\ ; \\ l_{n-1}{\mathcal{O}^{\mathcal{F}}}_{k,t+2} - l_{n-1}\sum_{i=2}^{n-1}(\dfrac{1}{l_{n-i}}+1){\mathcal{O}^{\mathcal{F}}}_{k,t}. \end{cases},$$

$$(b) \quad {\mathcal{O}^{\mathcal{L}}}_{k,t+2^n} = \begin{cases} \dfrac{l_{n-1}}{l_{n-2}}{\mathcal{O}^{\mathcal{L}}}_{k,t+2^{n-1}} - {\mathcal{O}^{\mathcal{L}}}_{k,t}; \\ \dfrac{l_{n-1}}{l_{n-t-1}}{\mathcal{O}^{\mathcal{L}}}_{k,t+2^{n-s}} - l_{n-1}\sum_{i=2}^{s}(1+\dfrac{1}{l_{n-i}}){\mathcal{O}^{\mathcal{L}}}_{k,t}, \\ \qquad\qquad\qquad \text{If } s = 2,3,4,\ldots,n-2\ ; \\ l_{n-1}{\mathcal{O}^{\mathcal{L}}}_{k,t+2} - l_{n-1}\sum_{i=2}^{n-1}(\dfrac{1}{l_{n-i}}+1){\mathcal{O}^{\mathcal{L}}}_{k,t}. \end{cases},$$

$$(c) \quad \mathcal{O}^{\mathcal{F}}{}_{k,2^r n+t} = \begin{cases} \sum_{i+j=n} \binom{n}{i} (\frac{l_{r-1}}{l_{r-2}})^i (-1)^j \mathcal{O}^{\mathcal{F}}{}_{k,2^{r-1}i+t}; \\ \sum_{i+j=n} \binom{n}{i} (\frac{l_{r-1}}{l_{r-s-1}})^i (-1)^j (\sum_{h=2}^{s} (1 + \frac{l_{r-1}}{l_{r-h}})^j \mathcal{O}^{\mathcal{F}}{}_{k,2^{n-s}i+t}, \\ \qquad\qquad \text{If } s = 2,3,4,\ldots,n-2 \ ; \\ \sum_{i+j=n} \binom{n}{i} (l_{r-1})^i (-1)^j (\sum_{h=2}^{s} (1 + \frac{l_{r-1}}{l_{r-h}})^j \mathcal{O}^{\mathcal{F}}{}_{k,2i+t}. \end{cases},$$

$$(d) \quad \mathcal{O}^{\mathcal{L}}{}_{k,2^r n+t} = \begin{cases} \sum_{i+j=n} \binom{n}{i} (\frac{l_{r-1}}{l_{r-2}})^i (-1)^j \mathcal{O}^{\mathcal{L}}{}_{k,2^{r-1}i+t}; \\ \sum_{i+j=n} \binom{n}{i} (\frac{l_{r-1}}{l_{r-s-1}})^i (-1)^j (\sum_{h=2}^{s} (1 + \frac{l_{r-1}}{l_{r-h}})^j \mathcal{O}^{\mathcal{L}}{}_{k,2^{n-s}i+t}, \\ \qquad\qquad \text{If } s = 2,3,4,\ldots,n-2 \ ; \\ \sum_{i+j=n} \binom{n}{i} (l_{r-1})^i (-1)^j (\sum_{h=2}^{s} (1 + \frac{l_{r-1}}{l_{r-h}})^j \mathcal{O}^{\mathcal{L}}{}_{k,2i+t}. \end{cases},$$

$$(vii) \quad \mathcal{O}^{\mathcal{F}}{}_{k,s+2t} = \frac{F_{k,2t}}{k}\mathcal{O}^{\mathcal{L}}{}_{k,s+1} - \frac{L_{k,2t-1}}{k}\mathcal{O}^{\mathcal{F}}{}_{k,s},$$

$$(viii) \quad \mathcal{O}^{\mathcal{L}}{}_{k,s+2t} = \frac{F_{k,2t}}{k}\delta\mathcal{O}^{\mathcal{F}}{}_{k,s+1} - \frac{L_{k,2t-1}}{k}\mathcal{O}^{\mathcal{L}}{}_{k,s},$$

$$(ix) \quad \mathcal{O}^{\mathcal{F}}{}_{k,s+2t} - \frac{F_{k,2t}}{k}\mathcal{O}^{\mathcal{F}}{}_{k,s+2} + \frac{F_{k,2t-2}}{k}\mathcal{O}^{\mathcal{F}}{}_{k,s} = 0,$$

$$(x) \quad \mathcal{O}^{\mathcal{L}}{}_{k,s+2t} - \frac{F_{k,2t}}{k}\mathcal{O}^{\mathcal{L}}{}_{k,s+2} + \frac{F_{k,2t-2}}{k}\mathcal{O}^{\mathcal{L}}{}_{k,s} = 0,$$

$$(xi) \quad \mathcal{O}^{\mathcal{F}}{}_{k,s+2t+1} = \frac{L_{k,2t+1}}{k}\mathcal{O}^{\mathcal{F}}{}_{k,s+1} - \frac{F_{k,2t}}{k}\mathcal{O}^{\mathcal{L}}{}_{k,s},$$

$$(xii) \quad \mathcal{O}^{\mathcal{L}}{}_{k,s+2t+1} = \frac{L_{k,2t+1}}{k}\mathcal{O}^{\mathcal{L}}{}_{k,s+1} - \delta\frac{F_{k,2t}}{k}\mathcal{O}^{\mathcal{F}}{}_{k,s},$$

$$(xiii) \quad \mathcal{O}^{\mathcal{F}}{}_{k,s+2t+1} - \frac{L_{k,2t+1}}{k(k^2+3)}\mathcal{O}^{\mathcal{F}}{}_{k,s+3} + \frac{F_{k,2t-2}}{k(k^2+3)}\mathcal{O}^{\mathcal{L}}{}_{k,s} = 0,$$

$$(xiv) \quad \mathcal{O}^{\mathcal{L}}{}_{k,s+2t+1} - \frac{L_{k,2t+1}}{k(k^2+3)}\mathcal{O}^{\mathcal{L}}{}_{k,s+3} + \frac{F_{k,2t-2}}{k(k^2+3)}\delta\mathcal{O}^{\mathcal{F}}{}_{k,s} = 0.$$

Proof. (i). From (i) and (ii) of lemma (6.43), we have

$$r_1{}^n = r_1 F_{k,n} + F_{k,n-1}, \tag{6.5.1}$$

$$r_2{}^n = r_2 F_{k,n} + F_{k,n-1}. \tag{6.5.2}$$

Multiplying (6.5.1) by $\frac{\bar{r_1} r_1{}^t}{r_1 - r_2}$ and (6.5.2) by $\frac{\bar{r_2} r_2{}^t}{r_1 - r_2}$, we get

$$\frac{\bar{r_1} r_1{}^{n+t}}{r_1 - r_2} = \frac{\bar{r_1} r_1{}^{t+1}}{r_1 - r_2} F_{k,n} + \frac{\bar{r_1} r_1{}^t}{r_1 - r_2} F_{k,n-1}, \tag{6.5.3}$$

$$\frac{\bar{r_2} r_2{}^{n+t}}{r_1 - r_2} = \frac{\bar{r_2} r_2{}^{t+1}}{r_1 - r_2} F_{k,n} + \frac{\bar{r_2} r_2{}^t}{r_1 - r_2} F_{k,n-1}. \tag{6.5.4}$$

Subtracting (6.5.3) and (6.5.4), we obtain

$$\frac{\bar{r_1} r_1{}^{n+t} - \bar{r_2} r_2{}^{n+t}}{r_1 - r_2} = \frac{\bar{r_1} r_1{}^{t+1} - \bar{r_2} r_2{}^{t+1}}{r_1 - r_2} F_{k,n} + \frac{\bar{r_1} r_1{}^t - \bar{r_2} r_2{}^t}{r_1 - r_2} F_{k,n-1}.$$

Using Binet formula of the hyperbolic k-Fibonacci octonion $\mathcal{O}^{\mathcal{F}}{}_{k,n}$, we get

$$\mathcal{O}^{\mathcal{F}}{}_{k,n+t} = F_{k,n}\mathcal{O}^{\mathcal{F}}{}_{k,t+1} + F_{k,n-1}\mathcal{O}^{\mathcal{F}}{}_{k,t}.$$

The proofs of (ii)-(xiv) are similar to (i) using lemma (6.43). □

Theorem 6.45. *For all $n, r, s, t \geq 1$, we have*

$$(i) \quad \mathcal{O^F}_{k,rn+t} = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^{i} F_{k,r-1}^{n-i} \mathcal{O^F}_{k,i+t},$$

$$(ii) \mathcal{O^L}_{k,rn+t} = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^{i} F_{k,r-1}^{n-i} \mathcal{O^L}_{k,i+t},$$

$$(iii) \quad \mathcal{O^F}_{k,2rn+t} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{(n-i)(r+1)} L_{k,r}^{i} \mathcal{O^F}_{k,ri+t},$$

$$(iv) \quad \mathcal{O^L}_{k,2rn+t} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{(n-i)(r+1)} L_{k,r}^{i} \mathcal{O^L}_{k,ri+t},$$

$$(v) \quad \mathcal{O^F}_{k,trn+l} = \frac{1}{F_{k,r}^{n}} \sum_{i=0}^{n} \binom{n}{i} (-1)^{(n-i)(r+1)} F_{k,(t-1)r}^{n-i} F_{k,tr}^{i} \mathcal{O^F}_{k,ri+l},$$

$$(vi) \quad \mathcal{O^L}_{k,trn+l} = \frac{1}{F_{k,r}^{n}} \sum_{i=0}^{n} \binom{n}{i} (-1)^{(n-i)(r+1)} F_{k,(t-1)r}^{n-i} F_{k,tr}^{i} \mathcal{O^L}_{k,ri+l},$$

$$(vii) \quad \sum_{i=0}^{n} \binom{n}{i} (-1)^{i} \mathcal{O^F}_{k,r(n-i)+i+t} F_{k,r}^{i} = \mathcal{O^F}_{k,t} F_{k,r-1}^{n},$$

$$(viii) \quad \sum_{i=0}^{n} \binom{n}{i} (-1)^{i} \mathcal{O^L}_{k,r(n-i)+i+t} F_{k,r}^{i} = \mathcal{O^L}_{k,t} F_{k,r-1}^{n},$$

$$(ix) \quad \sum_{i=0}^{n} \binom{n}{i} (-1)^{(n-i)} \mathcal{O^F}_{k,ri+t} F_{k,r-1}^{(n-i)} = \mathcal{O^F}_{k,n+t} F_{k,r}^{n},$$

$$(x) \quad \sum_{i=0}^{n} \binom{n}{i} (-1)^{(n-i)} \mathcal{O^L}_{k,ri+t} F_{k,r-1}^{(n-i)} = \mathcal{O^L}_{k,n+t} F_{k,r}^{n},$$

$$(xi) \quad \sum_{i=0}^{n} \binom{n}{i} (-1)^{(n-i)} F_{k,sm}^{(n-i)} F_{k,rm}^{(i)} \mathcal{O^F}_{k,m[rn+i(s-r)]+t} = (-1)^{smn} \mathcal{O^F}_{k,t} F_{k,(r-s)m}^{n},$$

$$(xii) \quad \sum_{i=0}^{n} \binom{n}{i} (-1)^{(n-i)} F_{k,sm}^{(n-i)} F_{k,rm}^{(i)} \mathcal{O^L}_{k,m[rn+i(s-r)]+t} = (-1)^{smn} \mathcal{O^L}_{k,t} F_{k,(r-s)m}^{n},$$

$$(xiv) \quad \mathcal{O^F}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n} (-1)^{j+s} L_{k,2^{r+1}}{}^{i} \mathcal{O^F}_{k,2^{r+1}(i+2j)+2(i+j)+t},$$

$$(xv) \quad \mathcal{O^L}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n} (-1)^{j+s} L_{k,2^{r+1}}{}^{i} \mathcal{O^L}_{k,2^{r+1}(i+2j)+2(i+j)+t},$$

$$(xvi)\quad {\mathcal{O}^{\mathcal{F}}}_{k,(2^{r+2}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} {L_{k,2^{r+1}}}^i {\mathcal{O}^{\mathcal{F}}}_{k,(2^{r+1}+2)i+j+t},$$

$$(xvii)\quad {\mathcal{O}^{\mathcal{L}}}_{k,(2^{r+2}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} {L_{k,2^{r+1}}}^i {\mathcal{O}^{\mathcal{L}}}_{k,(2^{r+1}+2)i+j+t},$$

$$(xviii)\quad {\mathcal{O}^{\mathcal{F}}}_{k,(2^{r+1}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j {L_{k,2^{r+1}}}^{-n} {\mathcal{O}^{\mathcal{F}}}_{k,(2^{r+1}+2)i+j+t},$$

$$(xix)\quad {\mathcal{O}^{\mathcal{F}}}_{k,(2^{r+1}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j})k^j {L_{k,2^{r+1}}}^{-n} {\mathcal{O}^{\mathcal{F}}}_{k,(2^{r+1}+2)i+j+t},$$

$$(xx)\quad \sum_{i=0}^{n} \binom{n}{i} k^{(i-n)} (L_{k,2t-1})^{(n-i)} {\mathcal{O}^{\mathcal{F}}}_{k,2ti+s}$$

$$= \begin{cases} k^{-n}(\mathcal{F}_{k,2t})^n \delta^{\frac{n}{2}} {\mathcal{O}^{\mathcal{F}}}_{k,n+s}, & \textit{if } n \textit{ is even;} \\ k^{-n}(\mathcal{F}_{k,2t})^n \delta^{\frac{n-1}{2}} {\mathcal{O}^{\mathcal{L}}}_{k,n+s}, & \textit{if } n \textit{ is odd,} \end{cases},$$

$$(xxi)\quad \sum_{i=0}^{n} \binom{n}{i} k^{(i-n)} (L_{k,2t-1})^{(n-i)} {\mathcal{O}^{\mathcal{L}}}_{k,2ti+s}$$

$$= \begin{cases} k^{-n}(F_{k,2t})^n \delta^{\frac{n}{2}} {\mathcal{O}^{\mathcal{L}}}_{k,n+s}, & \textit{if } n \textit{ is even;} \\ k^{-n}(F_{k,2t})^n \delta^{\frac{n+1}{2}} {\mathcal{O}^{\mathcal{F}}}_{k,n+s}, & \textit{if } n \textit{ is odd.} \end{cases},$$

$$(xxii)\quad \sum_{i=0}^{n} \binom{n}{i} (-1)^{(n-i)} k^{-i} (L_{k,2t+1})^i {\mathcal{O}^{\mathcal{F}}}_{k,2t(n-i)+n}$$

$$= \begin{cases} \delta^{\frac{n}{2}} {\mathcal{O}^{\mathcal{F}}}_{k,0}, & \textit{if } n \textit{ is even;} \\ \delta^{\frac{n-1}{2}} {\mathcal{O}^{\mathcal{L}}}_{k,0}, & \textit{if } n \textit{ is odd,} \end{cases},$$

$$(xxiii)\quad \sum_{i=0}^{n} \binom{n}{i} (-1)^{(n-i)} k^{-i} (L_{k,2t+1})^i {\mathcal{O}^{\mathcal{L}}}_{k,2t(n-i)+n}$$

$$= \begin{cases} \delta^{\frac{n}{2}} \mathcal{O}^{\mathcal{L}}{}_{k,0}, & \text{if } n \text{ is even;} \\ \delta^{\frac{n+1}{2}} \mathcal{O}^{\mathcal{F}}{}_{k,0}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. (i). From (i) and (ii) of lemma (6.43), we have

$$r_1^r = F_{k,r} r_1 + F_{k,r-1},$$

$$r_2^r = F_{k,r} r_2 + F_{k,r-1}.$$

Now, by the binomial theorem, we have

$$r_1^{rn} = (F_{k,r} r_1 + F_{k,r-1})^n = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^i F_{k,r-1}^{n-i} r_1^i, \tag{6.5.5}$$

$$r_2^{rn} = (F_{k,r} r_2 + F_{k,r-1})^n = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^i F_{k,r-1}^{n-i} r_2^i. \tag{6.5.6}$$

Multiplying (6.5.5) by $\dfrac{\bar{r_1}}{r_1 - r_2}$ and (6.5.6) by $\dfrac{\bar{r_2}}{r_1 - r_2}$ and subtracting, we obtain

$$\frac{\bar{r_1} r_1^{rn+t} - \bar{r_2} r_2^{rn+t}}{r_1 - r_2} = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^i F_{k,r-1}^{n-i} \left(\frac{\bar{r_1} r_1^{i+t} - \bar{r_2} r_2^{i+t}}{r_1 - r_2}\right).$$

Using Binet foemula of $\mathcal{O}^{\mathcal{F}}{}_{k,rn+t}$ and $\mathcal{O}^{\mathcal{F}}{}_{k,i+t}$, we get

$$\mathcal{O}^{\mathcal{F}}{}_{k,rn+t} = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^i F_{k,r-1}^{n-i} \mathcal{O}^{\mathcal{F}}{}_{k,i+t}.$$

(ii). Multiplying (6.5.5) by $\bar{r_1}$ and (6.5.6) by $\bar{r_2}$ and adding, we obtain

$$\bar{r_1}r_1^{rn+t}+\bar{r_2}r_2^{rn+t}=\sum_{i=0}^{n}\binom{n}{i}F_{k,r}^{i}F_{k,r-1}^{n-i}(\bar{r_1}r_1^{i+t}+\bar{r_2}r_2^{i+t}).$$

Using Binet foemula of ${\mathcal{O}^{\mathcal{L}}}_{k,rn+t}$ and ${\mathcal{O}^{\mathcal{L}}}_{k,i+t}$, we get

$${\mathcal{O}^{\mathcal{L}}}_{k,rn+t}=\sum_{i=0}^{n}\binom{n}{i}F_{k,r}^{i}F_{k,r-1}^{n-i}{\mathcal{O}^{\mathcal{L}}}_{k,i+t}.$$

The proofs of (iii)-(xxiii) are similar to (i) and (ii) using lemma (6.43). □

Next theorem deals with congruence properties of the hyperbolic k-Fibonacci and k-Lucas octonions.

Theorem 6.46. *For $n,t\geq 1$ and $\mathcal{G}_{k,n}={\mathcal{O}^{\mathcal{F}}}_{k,n}$ or ${\mathcal{O}^{\mathcal{L}}}_{k,n}$, we have*

(i) $${\mathcal{O}^{\mathcal{F}}}_{k,n+t}-\sum_{j=0}^{n}\binom{n}{j}k^{-n}(-1)^{n}{\mathcal{O}^{\mathcal{F}}}_{k,(2^{r+2}+2)j+t}\equiv 0 \pmod{L_{k,2^{r+1}}},$$

(ii) $${\mathcal{O}^{\mathcal{L}}}_{k,n+t}-\sum_{j=0}^{n}\binom{n}{j}k^{-n}(-1)^{n}{\mathcal{O}^{\mathcal{L}}}_{k,(2^{r+2}+2)j+t}\equiv 0 \pmod{L_{k,2^{r+1}}},$$

(iii) $${\mathcal{O}^{\mathcal{F}}}_{k,(2^{r+2}+2)n+t}-\sum_{j=0}^{n}\binom{n}{j}k^{j}(-1)^{n}{\mathcal{O}^{\mathcal{F}}}_{k,j+t}\equiv 0 \pmod{L_{k,2^{r+1}}},$$

(iv) $${\mathcal{O}^{\mathcal{L}}}_{k,(2^{r+2}+2)n+t}-\sum_{j=0}^{n}\binom{n}{j}k^{j}(-1)^{n}{\mathcal{O}^{\mathcal{L}}}_{k,j+t}\equiv 0 \pmod{L_{k,2^{r+1}}}.$$

Proof. (i). From (xiv) of theorem (6.45), for all $n, t \geq 1$, we have

$$\mathcal{O}^{\mathcal{F}}{}_{k,n+t} = \sum_{i+j+s=n; i\neq 0} \binom{n}{i,j} k^{-n}(-1)^{j+s} L_{k,2^{r+1}}{}^{i} \mathcal{O}^{\mathcal{F}}{}_{k,2^{r+1}(i+2j)+2(i+j)+t}$$
$$+ \sum_{i+j+s=n; i=0} \binom{n}{i,j} k^{-n}(-1)^{j+s} L_{k,2^{r+1}}{}^{i} \mathcal{O}^{\mathcal{F}}{}_{k,2^{r+1}(i+2j)+2(i+j)+t},$$

$$= \sum_{i+j+s=n; i\neq 0} \binom{n}{i,j} k^{-n}(-1)^{j+s} L_{k,2^{r+1}}{}^{i} \mathcal{O}^{\mathcal{F}}{}_{k,2^{r+1}(i+2j)+2(i+j)+t}$$
$$+ \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^{n} \mathcal{O}^{\mathcal{F}}{}_{k,(2^{r+2}+2)j+t}.$$
$$\mathcal{O}^{\mathcal{F}}{}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^{n} \mathcal{O}^{\mathcal{F}}{}_{k,(2^{r+2}+2)j+t}$$
$$= \sum_{i+j+s=n; i\neq 0} \binom{n}{i,j} k^{-n}(-1)^{j+s} L_{k,2^{r+1}}{}^{i} \mathcal{O}^{\mathcal{F}}{}_{k,2^{r+1}(i+2j)+2(i+j)+t},$$
$$\therefore L_{k,2} \quad \text{divides} \quad (\mathcal{O}^{\mathcal{F}}{}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^{n} \mathcal{O}^{\mathcal{F}}{}_{k,(2^{r+2}+2)j+t}),$$
$$\therefore \mathcal{O}^{\mathcal{F}}{}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^{n} \mathcal{O}^{\mathcal{F}}{}_{k,(2^{r+2}+2)j+t} \equiv 0 \pmod{L_{k,2}}.$$

The proofs of (ii), (iii) and (iv) are similar to (i), using theorem (6.45). □

6.6 Concluding Remarks

In this paper, we derived telescoping series for k- Fibonacci and k- Lucas sequences and proved their relationships with k- Fibonacci and k- Lucas sequences, same identities can be derived using M matrices. The relationship between k- Fibonacci and k- Lucas sequences using continued fractions and series of fractions derived is different and never tried in k- Fibonacci sequence literature.

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