The Hubbard-Stratonovich Transformation

When is mean field theory exact?

In mean field theory the fluctuations of the total interaction with the neighboring spins is ignored compared with the mean. This is good if there is a **large number** of interactions, either because there are a large number of neighbors (e.g. high spatial dimension), or because the interaction is long range. I will consider the case of **infinite range** interaction to introduce a powerful formal technique known as the **Hubbard-Stratonovich** transformation. We will use the method to show that mean field theory is exact in this case.

Consider the Ising model with infinite range interactions i.e. each spin interacts with the same strength to all the other spins in the lattice. To get a sensible answer we must scale the interaction with N^{-1} with N the total number of spins. This gives the Hamiltonian

$$H = -\frac{J}{2N}\sum_{i,j} s_i s_j - b\sum_i s_i \tag{1}$$

with the ij sums running over all the spins, $s_i = \pm 1$, and $b = \mu B$ gives the coupling to the magnetic field. We assume the ferromagnetic case, $J > 0$.

The partition function in the canonical ensemble is

$$Q_N = \sum_{\{s_i\}} \exp\left[\frac{\beta J}{2N}\sum_{i,j} s_i s_j + \beta b \sum_i s_i\right] \tag{2}$$

with the sum $\sum_{\{s_i\}}$ running over all spin configurations. Note that

$$\sum_{i,j} s_i s_j = \left(\sum_i s_i\right)^2 = S^2 \tag{3}$$

with S the total spin, so that

$$Q_N = \sum_{\{s_i\}} \exp\left[\frac{\beta J}{2N} S^2 + \beta b S\right]. \tag{4}$$

The Hubbard-Stratonovich identity

The key identity in the Hubbard-Stratonovich method is simply an observation of the result of a Gaussian integral. In the present case it takes the form

$$\exp\left[\frac{\beta J}{2N} S^2\right] = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{N\beta J}{2}\mu^2 + \beta J \mu S\right] d\mu. \tag{5}$$

This is readily shown by completing the square

$$-\frac{N\beta J}{2}\mu^2 + \beta J \mu S = -\frac{N\beta J}{2}(\mu - \frac{S}{N})^2 + \frac{\beta J S^2}{2N} \tag{6}$$

and then shifting the integral to one over $x = \mu - S/N$. The usefulness of this identity is that if we substitute Eq. (5) into Eq. (2) the "difficult variable" S, containing all the degrees of freedom s_i that have to be summed over, only occurs linearly in the exponential, not quadratically.

Manipulating the Partition Function

The partition function now takes the form

$$Q_N = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} d\mu \left\{ \exp\left(-\frac{N\beta J}{2}\mu^2\right) \sum_{\{s_i\}} \exp\left[\beta(b + J\mu)S\right] \right\}. \tag{7}$$

This can be interpreted as an average of partition functions of **noninteracting** spins in a field with a fluctuating component $J\mu$, with a Gaussian probability distribution of μ $P(\mu) \propto \exp(-N\beta J\mu^2/2)$:

$$Q_N = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{N\beta J}{2}\mu^2\right) Q_\mu \, d\mu. \tag{8}$$

For each μ the integrand Q_μ is the partition function for a free spin system in an external field

$$Q_\mu = \sum_{\{s_i\}} \exp\left[\beta(b + J\mu)S\right] \tag{9}$$

that may be evaluated in the usual way.

$$Q_\mu = \prod_i \sum_{s_i=\pm 1} \exp[\beta(J\mu + b)s_i] \tag{10}$$

$$= \{2\cosh[\beta(J\mu + b)]\}^N . \tag{11}$$

The full partition function can now be written. This is conveniently written in the form

$$Q_N = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} d\mu \exp[Nq(\beta J, \beta b, \mu)] \tag{12}$$

with

$$q(\beta J, \beta b, \mu) = \ln\{2\cosh[\beta(J\mu + b)]\} - \frac{\beta J}{2}\mu^2 \tag{13}$$

independent of N.

Mean Field

For large N (remember $N \to \infty$ in the thermodynamic limit) we can use the simplest form of the "method of steepest descents" to evaluate the μ-integral, namely we find the value μ_s that maximizes q and argue that, because of the exponential form, this region dominates the integral for large N, and it is sufficient to write

$$Q_N \simeq \exp[Nq((\beta J, \beta b, \mu_s)]. \tag{14}$$

Maximizing q with respect to μ by setting $\partial q/\partial\mu = 0$ gives

$$\mu_s = \tanh[\beta(J\mu_s + b)]. \tag{15}$$

We can find the average spin by differentiating Q_N with respect to the field b in the usual way

$$s = \frac{1}{N\beta}\frac{\partial \ln Q}{\partial b} = \frac{1}{\beta}\frac{\partial q}{\partial b}, \tag{16}$$

which gives

$$s = \tanh[\beta(J\mu_s + b)], \tag{17}$$

showing that μ_s (or more strictly $J\mu_s$) determined by Eq. (15) plays the role of an effective field acting on each spin. Comparing Eqs. (15) and (17) then gives us the self consistency condition for m

$$s = \tanh[\beta(Js + b)]. \tag{18}$$

This is the usual "mean field" result.

Free Energy

We can also derive an expression for the free energy

$$A = -kT \ln Q_N. \tag{19}$$

From Eqs. (14) and (13),

$$A = -NkT \ln\{2\cosh[\beta(Js + b)]\} + \frac{NJ}{2}s^2. \tag{20}$$

Notice that this has the form introduced phenomenologically in class: the free energy of noninteracting spins in the effective field, with the subtraction of the direct interaction energy $-NJm^2/2$ which is otherwise double counted.

Remarks

The a Hubbard-Stratonovich transformation is generally useful for transforming an interacting problem to a sum or integration over noninteracting problems. For example for the nearest neighbor interaction $\sum_{i,\delta} s_i s_{i+\delta}$ the transformation leads to a problem with noninteracting spins and a site dependent effective field μ_i (which cannot however then be evaluated by steepest descents). It is also useful in quantum problems. For example, it is used in the bosonization of interacting fermions — the partition function of a system of interacting fermions is transformed to that of a sum over noninteracting bosons which is more amenable to theoretical attack.