Chapter 3

The Renormalization Group in Momentum Space

The alternative to the real-space RG methods discussed in the preceding chapter is the momentum space formulation, where the degrees of freedom of a system are represented in terms of Fourier modes. The partition function is now expressed as a summation over fluctuations over the full range of Fourier modes, so the partial summation of the partition function is carried out by integrating over momentum shells that correspond to the short-wavelength modes. This step corresponds to coarse-graining in the real-space RG. The spatial variables are then rescaled to restore the range of the momentum space integration to its original value, which necessitates a concomitant rescaling of the degrees of freedom to maintain the spatial dependence of the fluctuations. Momentum space RG is essentially a perturbative method in which the central assumption is that physical quantities can be described as a (renormalized) perturbation around the linear theory.

We begin this chapter by casting the Ising model as a field theory. The mathematical machinery of the momentum space RG is then illustrated for the linear version of this theory. While making no attempt at a quantitative description of the original model, the linear theory can be solved exactly and the results compared with the results of the RG calculation. We then describe a calculation that includes the leading order linearity to show the nature of the perturbation expansion and discuss several issues that arise in the context of this expansion.

3.1 Field Theory for the Ising Model

The application of momentum space RG techniques to the Ising model necessitates recasting the discrete spins on a lattice as a field theory for continuous spins on a continuous space. We will carry out this process in several stages, highlighting those that are exact and those that are inherently perturbative.

3.1.1 The Hubbard–Stratonovich Transformation

The first step in transforming the Ising model into a field theory is motivated by the following integral:

$$e^{\frac{1}{2}Ks^2} = \left(\frac{K}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}K\phi^2+Ks\phi}\, d\phi\,. \tag{3.1}$$

For our purposes, the key point about this relation is that equality holds for *any* value of s. In particular, s need not be a continuous variable.

To apply the integral in Eq. (3.1) to the Ising model, we write the Hamiltonian for this model as

$$\mathcal{H} = -\frac{1}{2}\sum_{i,j=1}^{N} J_{ij}s_is_j\,, \tag{3.2}$$

where, for reasons that will be made clear below, the sums over i and j run over all N sites, which is the reason for the factor of $\frac{1}{2}$, but the coupling constant has the values

$$J_{ij} = \begin{cases} J, & \text{if } i \text{ and } j \text{ are nearest neighbors,} \\ 0, & \text{otherwise,} \end{cases} \tag{3.3}$$

which preserves the standard form of the Ising Hamiltonian. We now invoke the matrix analogue of Eq. (3.1) to write

$$\begin{aligned} &\exp\left(\frac{1}{2}\sum_{ij} K_{ij}s_is_j\right) \\ &\quad = \left[\frac{\det \mathsf{K}}{(2\pi)^N}\right]^{1/2} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{k=1}^{N} d\phi_k\, \exp\left(-\frac{1}{2}\sum_{ij}\phi_iK_{ij}\phi_j + \sum_{ij}s_iK_{ij}\phi_j\right), \end{aligned} \tag{3.4}$$

where K is the matrix with entries $K_{ij} = \beta J_{ij}$. This is the **Hubbard–Stratonovich** transformation[1,2].

The partition function,

$$\mathcal{Z} = \sum_{\{s_i=\pm 1\}} e^{\frac{1}{2}\sum_{ij} K_{ij} s_i s_j}\,, \tag{3.5}$$

is calculated by observing that only the second factor in the exponential on the right-hand side of Eq. (3.4) has a dependence on the s_i. Hence,

$$\begin{aligned}
\sum_{\{s_i=\pm 1\}} \exp\biggl(\sum_{ij} s_i K_{ij}\phi_j\biggr) &= \sum_{\{s_i=\pm 1\}} \prod_i \exp\biggl(s_i \sum_j K_{ij}\phi_j\biggr) \\
&= \prod_i \biggl[\exp\biggl(\sum_j K_{ij}\phi_j\biggr) + \exp\biggl(-\sum_j K_{ij}\phi_j\biggr)\biggr] \\
&= 2^N \prod_i \cosh\biggl(\sum_j K_{ij}\phi_j\biggr) \\
&= 2^N \exp\biggl\{\sum_i \ln\biggl[\cosh\biggl(\sum_j K_{ij}\phi_j\biggr)\biggr]\biggr\}\,.
\end{aligned} \tag{3.6}$$

We thereby obtain

$$\begin{aligned}
\mathcal{Z} = \biggl[\frac{\det \mathsf{K}}{(\frac{1}{2}\pi)^N}\biggr]^{1/2} &\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{k=1}^{N} d\phi_k \\
&\times \exp\biggl\{-\frac{1}{2}\sum_{ij} \phi_i K_{ij}\phi_j + \sum_i \ln\biggl[\cosh\biggl(\sum_j K_{ij}\phi_j\biggr)\biggr]\biggr\}\,,
\end{aligned} \tag{3.7}$$

as the partition function for the Ising model. This is an exact transcription of the original model, but with the degrees of freedom on the lattice sites expressed in terms of the continuous quantities ϕ_i.

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[1] R. L. Stratonovich, "On a Method of Calculating Quantum Distribution Functions", *Soviet Phys. Doklady* **2** (1958), 416–419 (1958).

[2] J. Hubbard, "Calculation of Partition Functions", *Phys. Rev. Lett.* **3**, 77–78 (1959)

3.1.2 Regularization

The next step is to replace the lattice of the Ising model by a continuum and make the corresponding changes in the partition function. We first transform the expression $\sum_j K_{ij}\phi_j$ because it appears in every term on the right-hand side Eq. (3.7). Our analysis can be carried for a lattice of any dimension, so we introduce the notation $\mathbf{i} = (i_1, i_2, \ldots, i_d)$, where the i_k are integers, for the sites of a d-dimensional lattice. The nearest-neighbor directions are denoted by $\mathbf{a}_n$. On a d-dimensional cubic lattice there are $2d$ such vectors:

$$(\pm a, 0, \ldots, 0)\,, \quad (0, \pm a, \ldots, 0)\,, \ldots, (0, 0, \ldots, \pm a)\,, \tag{3.8}$$

in which a is the nearest-neighbor spacing. Thus, we have

$$\sum_j K_{ij}\phi_j = \sum_n K_{\mathbf{i},\mathbf{i}+\mathbf{a}_n}\phi_{\mathbf{i}+\mathbf{a}_n} = K\sum_n \phi_{\mathbf{i}+\mathbf{a}_n}\,, \tag{3.9}$$

where we have invoked Eq. (3.3) for K_{ij}.

The continuous lattice variable $\phi_{\mathbf{i}}$ is now replaced by a smooth function $\phi(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2, \ldots, x_d)$. We can now expand the factor $\phi_{\mathbf{i}+\mathbf{a}_n}$ on the right-hand side of Eq. (3.9) as a Taylor series about $\mathbf{x}$:

$$\begin{aligned}
\phi_{\mathbf{i}+\mathbf{a}_n} &\to \phi(\mathbf{x}+\mathbf{a}_n) \\
&= \phi(\mathbf{x}) + \sum_\alpha \frac{\partial\phi}{\partial x_\alpha}a_{n,\alpha} + \frac{1}{2}\sum_{\alpha,\beta}\frac{\partial^2\phi}{\partial x_\alpha \partial x_\beta}a_{n,\alpha}a_{n,\beta} + \cdots\,,
\end{aligned} \tag{3.10}$$

in which we have used Greek indices for vector components. Substitution of this expansion into the right-hand side of Eq. (3.9) yields

$$\begin{aligned}
K\sum_n \phi_{\mathbf{i}+\mathbf{a}_n} &\to K\sum_n \phi(\mathbf{x}+\mathbf{a}_n) \\
&= K\sum_n \phi(\mathbf{x}) + K\sum_\alpha \frac{\partial\phi}{\partial x_\alpha}\left(\sum_n a_{n,\alpha}\right) + \\
&\quad + \frac{K}{2}\sum_{\alpha,\beta}\frac{\partial^2\phi}{\partial x_\alpha \partial x_\beta}\left(\sum_n a_{n,\alpha}a_{n,\beta}\right) + \cdots\,.
\end{aligned} \tag{3.11}$$

Referring to Eq. (3.8), we have that

$$\sum_n a_{n,\alpha} = 0\,, \qquad \sum_n a_{n,\alpha} a_{n,\beta} = 2a^2 \delta_{\alpha,\beta}\,. \tag{3.12}$$

Hence, the right-hand side of Eq. (3.11) reduces to

$$K \sum_n \phi_{\mathbf{i}+\mathbf{a}_n} \to Kz\,\phi(\mathbf{x}) + Ka^2\,\nabla^2\phi(\mathbf{x}) + \cdots\,, \tag{3.13}$$

where $z = 2d$ is the number of nearest neighbors in d dimensions and

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \cdots + \frac{\partial^2\phi}{\partial x_d^2}\,. \tag{3.14}$$

Thus, the continuum limit of the first term on the right-hand side of Eq. (3.7) is

$$\sum_{ij} \phi_i K_{ij} \phi_j \to Kz \int \phi^2(\mathbf{x})\, d\mathbf{x} + Ka^2 \int \phi(\mathbf{x}) \nabla^2 \phi(\mathbf{x})\, d\mathbf{x} + \cdots\,. \tag{3.15}$$

The other term on the right-hand side of Eq. (3.7) is regularized by using the expansion

$$\ln(\cosh x) = \frac{x^2}{2} - \frac{x^4}{12} + \cdots\,, \tag{3.16}$$

to write

$$\begin{aligned} &\sum_i \ln\left[\cosh\left(\sum_j K_{ij}\phi_j\right)\right] \\ &\qquad = \frac{1}{2}\sum_i \left(\sum_j K_{ij}\phi_j\right)^2 - \frac{1}{12}\sum_i \left(\sum_j K_{ij}\phi_j\right)^4 + \cdots \end{aligned} \tag{3.17}$$

For the quadratic term on the right-hand side of this equation, we have from Eq. (3.13) that

$$\begin{aligned} \sum_i \left(\sum_j K_{ij}\phi_j\right)^2 &= \sum_{\mathbf{i}} \left(K \sum_n \phi_{\mathbf{i}+\mathbf{a}_n}\right)^2 \\ &\to (Kz)^2 \int \phi^2(\mathbf{x})\, d\mathbf{x} + 2K^2 a^2 z \int \phi(\mathbf{x}) \nabla^2 \phi(\mathbf{x})\, d\mathbf{x} + \cdots\,, \end{aligned} \tag{3.18}$$

where we have explicitly retained only terms that contain up to two derivatives.

For the quartic term on the right-hand side of Eq. (3.17), we retain only the leading term:

$$\sum_i \Big(\sum_j K_{ij}\phi_j\Big)^4 = \sum_{\mathbf{i}} \Big(K\sum_n \phi_{\mathbf{i}+\mathbf{a}_n}\Big)^4 \to (Kz)^4 \int \phi(\mathbf{x})^4\, d\mathbf{x} + \cdots . \tag{3.19}$$

Thus, collecting terms, we obtain the continuum expression of the partition function as

$$\mathcal{Z} = \left[\frac{\det \mathsf{K}}{(\frac{1}{2}\pi)^N}\right]^{1/2} \int \mathcal{D}\phi(\mathbf{x}) \exp\bigg\{-\int \big[r\phi^2(\mathbf{x}) - D\phi(\mathbf{x})\nabla^2\phi(\mathbf{x}) + u\phi^4(\mathbf{x}) + \cdots\big]\, d\mathbf{x}\bigg\}, \tag{3.20}$$

where

$$r = \tfrac{1}{2}Kz(1-Kz)\,, \qquad D = -\tfrac{1}{2}Ka^2(1-2Kz)\,, \qquad u = \tfrac{1}{12}(Kz)^4\,. \tag{3.21}$$

In writing this expression, we have made the identification

$$\prod_k d\phi_k \to \mathcal{D}\phi(\mathbf{x})\,, \tag{3.22}$$

whereby the N-fold product of integration elements for the ϕ is transformed into a **functional integral** over the $\phi(\mathbf{x})$. Our explicit inclusion of only the leading-order terms in the continuum limit will find justification in the RG calculations we will carry out later in this chapter.

Finally, a word about the signs of the coefficients r, D, and u. Clearly, $u \geq 0$. For r and D, we use the mean-field estimate of the transition temperature, $k_BT_c = Jz$, so that we can write $Kz = T_c/T$, in which case

$$r = \frac{T_c}{2T^2}(T-T_c)\,, \qquad D = \frac{a^2T_c}{2T^2}(2T_c - T)\,. \tag{3.23}$$

Thus, the sign of r is the sign of $T - T_c$, and $r \to 0$ as $T \to T_c$. Similarly, as $T \to T_c$, $D \to \frac{1}{2}a^2$ and, sufficiently near T_c, $D > 0$. The overall stability of the theory is guaranteed by u being positive.

3.2 The Gaussian Model

If only the terms quadratic in $\phi(\mathbf{k})$ are retained in Eq. (3.20), the result is called the **Gaussian theory**:

$$\mathcal{Z} = \left[\frac{\det \mathsf{K}}{(\frac{1}{2}\pi)^N}\right]^{1/2} \int \mathcal{D}\phi(\mathbf{x}) \exp\left\{-\int \left[r\phi^2(\mathbf{x}) - D\phi(\mathbf{x})\nabla^2\phi(\mathbf{x})\right] d\mathbf{x}\right\}, \tag{3.24}$$

The absence of the terms of order ϕ^4 renders this theory meaningful only for $r \geq 0$, so the approach to the critical point at $t = 0$ is from the disordered side of the transition.

The functional Gaussian integrals in Eq. (3.24) are carried out by transforming to a Fourier representation of decoupled modes. For a finite volume $V = L^d$, the Fourier transform $\phi_{\mathbf{k}}$ of $\phi(\mathbf{x})$ is given by

$$\phi_{\mathbf{k}} = \int_V d\mathbf{x}\phi(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}, \tag{3.25}$$

where, since $\phi(\mathbf{x})$ is real, we have that $\phi_{-\mathbf{k}} = \phi^*_{\mathbf{k}}$. The inverse Fourier transform is

$$\phi(\mathbf{x}) = \frac{1}{V}\sum_{\mathbf{k}} \phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{3.26}$$

The largest wavevector in this summation is $\Lambda \equiv 2\pi/a$ and the smallest is $2\pi/L$, which approaches zero as $L \to \infty$. These are referred to as **ultraviolet** and **infrared** cutoffs, respectively. Critical quantities should not depend on the values of these cutoffs. Since there is one wavevector per volume $(2\pi/L)^d$ in $\mathbf{k}$-space, summations over $\mathbf{k}$ are converted into integrals according to

$$\sum_{\mathbf{k}} = \int d\mathbf{k}\left(\frac{L}{2\pi}\right)^d = V\int \frac{d\mathbf{k}}{(2\pi)^d}. \tag{3.27}$$

This transcription is exact only in the thermodynamic limit ($V \to \infty$). The transformed partition function thereby reads

$$\mathcal{Z} = \left[\frac{\det \mathsf{K}}{(\frac{1}{2}\pi)^N}\right]^{1/2} \int \prod_{\mathbf{k}} d\phi_{\mathbf{k}} \exp\left[-\frac{1}{V}\sum_{\mathbf{k}}(r + Dk^2)|\phi_{\mathbf{k}}|^2\right]. \tag{3.28}$$

The Gaussian integrals can be evaluated as

$$\begin{aligned}
&\int \prod_{\mathbf{k}} d\phi_{\mathbf{k}} \exp\left[-\frac{1}{V}\sum_{\mathbf{k}}(r+Dk^2)|\phi_{\mathbf{k}}|^2\right] \\
&= \prod_{\mathbf{k}} \int \exp\left[-\frac{1}{V}(r+Dk^2)|\phi_k|^2\right] d\phi_{\mathbf{k}} \\
&= \prod_{\mathbf{k}} \left(\frac{\pi V}{r+Dk^2}\right)^{1/2} . \qquad (3.29)
\end{aligned}$$

Thus, the Helmholtz free energy $F = -k_B T \ln \mathcal{Z}$ of the Gaussian model is given by

$$\begin{aligned}
F &= -\tfrac{1}{2}k_B T \ln\left[\frac{\det \mathsf{K}}{(\frac{1}{2}\pi)^N}\right] - k_B T \ln\left[\prod_{\mathbf{k}}\left(\frac{\pi V}{r+Dk^2}\right)^{1/2}\right] \\
&= -\tfrac{1}{2}k_B T \ln\left[\frac{\det \mathsf{K}}{(\frac{1}{2}\pi)^N}\right] - \tfrac{1}{2}k_B T \sum_{\mathbf{k}} \ln(\pi V) + \tfrac{1}{2}k_B T \sum_{\mathbf{k}} \ln(r+Dk^2) . \qquad (3.30)
\end{aligned}$$

Thus, by invoking Eq. (3.27), the singular part of the free energy f_s per unit volume is obtained from the last term in this expression:

$$f_s = \frac{1}{2}k_B T \int \frac{d\mathbf{k}}{(2\pi)^d} \ln(r+Dk^2) . \qquad (3.31)$$

Note that when $r \to 0$, then the $\mathbf{k}=0$ mode causes the free energy to diverge, while the free energy for all other modes remains finite.

To study the behavior of the free energy at the critical point, we will simplify the integration in Eq. (3.31) by approximating the integration over the Brillouin zone by a d-dimensional hypersphere of radius $2\pi/a$. The $\mathbf{k}$-space integration thereby factors into a radial integral over $k = |\mathbf{k}|$ and an angular integration over d-dimensional hypersphere:

$$\int \frac{d\mathbf{k}}{(2\pi)^d} = \frac{1}{(2\pi)^d}\int k^{d-1}\, dk \int d\Omega_d , \qquad (3.32)$$

where $d\Omega_d$ is the element of surface area of a d-dimensional hypersphere. This expression generalizes circular polar coordinates in 2D and spherical polar coordinates in 3D to any dimension. The integral over k ranges from $k = 2\pi/L$ to $k = 2\pi/a$. For an infinite system, $L \to \infty$ and the lower limit of the integral is zero. Thus, Eq. (3.31) becomes

$$\begin{aligned} f_s &= \frac{1}{2} k_B T K_d \int_0^{\Lambda} \ln(r + Dk^2)\, k^{d-1}\, dk \\ &= \frac{1}{2} k_B T K_d \int_0^{\Lambda} \left[\ln r + \ln\left(1 + \frac{D}{r} k^2\right)\right] k^{d-1}\, dk\,, \end{aligned} \tag{3.33}$$

where $K_d = (S_d/(2\pi)^d$ and S_d is the area of unite sphere in d dimensions. Changing variables to $x = k(D/r)^{1/2}$ yields

$$f_s = \frac{1}{2} k_B T K_d \left(\frac{r}{D}\right)^{d/2} \int_0^{\Lambda(D/r)^{1/2}} \left[\ln r + \ln(1 + x^2)\right] x^{d-1}\, dx\,, \tag{3.34}$$

from which we see that the dominant singular contribution to the free energy is $f_s \sim t^{d/2}$, so that the specific heat exponent

$$\alpha_+ = 2 - \frac{d}{2}\,, \tag{3.35}$$

where the subscript "+" indicates that the exponent is defined only from the disordered side of the transition, i.e. where $T > T_c$. Notice that α_+ depends on the spatial dimension. For $d = 1, 2, 3$, $\alpha_+ > 0$, while $d \geq 4$, $\alpha_+ \leq 0$. This suggests that $d = 4$ plays a special role in this theory. We will return to this point later.

3.3 Renormalization Group Analysis of the Gaussian Model

The RG will be applied to the momentum space representation of the partition function of the Gaussian model, which we write as

$$\mathcal{Z} = \int \mathcal{D}\phi(\mathbf{k}) \exp\left[-\int_0^{\Lambda} \frac{d\mathbf{k}}{(2\pi)^d} (r + Dk^2)|\phi(\mathbf{k})|^2\right], \tag{3.36}$$

in which we have omitted the prefactor of the partition function, which does not affect the singular behavior of the partition function. We have also changed the partition function to a functional integral by replacing the integration measure in Eq. (3.28) with $\mathcal{D}\phi(\mathbf{k})$ to account for the fact that we are now considering an infinite system. As in the preceding section, we have replaced the Brillouin zone integration with an integration over a d-dimensional hypersphere.

The RG transformation in momentum space consists of three steps: (i) coarse-graining, by integrating over degrees of freedom corresponding to large k, (ii) rescaling of the wavevector to restore the original range of the degrees of freedom, and (iii) renormalization of the spin variables to restore the spatial dependence of the fluctuations. We consider the application of each step in turn to Eq. (3.36) in the following sections.

3.3.1 Coarse Graining

In the coarse graining step of the RG, Fourier modes corresponding to the largest values of k, $\Lambda/b < k < \Lambda$, where $b > 1$, are removed by integration into the partition function (Fig. 3.1). In real space, this eliminates fluctuations over length scales $a < x < ba$, and is therefore analogous to the decimation of the Ising model.

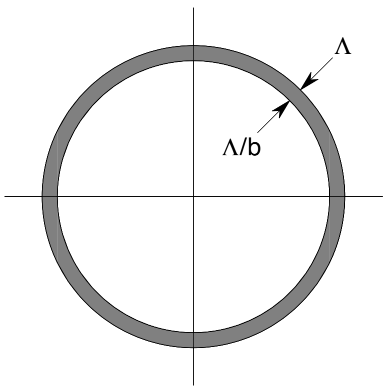

Figure 3.1: The shell in momentum space (indicated by shading) that is integrated out during the coarse graining step of the RG.

The coarse graining of the partition function in Eq. (3.36) is carried out by first separating $\phi(\mathbf{k})$ according to whether $k < \Lambda/b$ or $k > \Lambda/b$:

$$\phi(\mathbf{k}) = \begin{cases} \phi_<(\mathbf{k}), & 0 < k < \Lambda/b; \\ \phi_>(\mathbf{k}), & \Lambda/b < k < \Lambda\,. \end{cases} \tag{3.37}$$

The partition function can be expressed in terms of $\phi_<$ and $\phi_>$ as

$$\begin{aligned}\mathcal{Z} &= \int \mathcal{D}\phi_<(\mathbf{k}) \int \mathcal{D}\phi_>(\mathbf{k}) \exp\left[-\int_0^{\Lambda/b} \frac{d\mathbf{k}}{(2\pi)^d}(r + Dk^2)|\phi_<(\mathbf{k})|^2\right] \\ &\quad \times \exp\left[-\int_{\Lambda/b}^{\Lambda} \frac{d\mathbf{k}}{(2\pi)^d}(r + Dk^2)|\phi_>(\mathbf{k})|^2\right] \\ &= \mathcal{Z}_> \int \mathcal{D}\phi_<(\mathbf{k}) \exp\left[-\int_0^{\Lambda/b} \frac{d\mathbf{k}}{(2\pi)^d}(r + Dk^2)|\phi_<(\mathbf{k})|^2\right], \qquad (3.38)\end{aligned}$$

where $\mathcal{Z}_>$ represents the degrees of freedom that have been integrated out and is given by an expression similar to that in Eq. (3.29).

3.3.2 Rescaling

The partition function in Eq. (3.38) for the modes $\phi_<(\mathbf{k})$ is similar to the original partition function Eq. (3.36), except that the upper cutoff has decreased to Λ/b. The rescaling $\mathbf{k}' = b\,\mathbf{k}$, which corresponds to the rescaling $\mathbf{x}' = \mathbf{x}/b$ in real space, restores the cutoff to its original value. The transformed partition function becomes

$$\mathcal{Z} = \mathcal{Z}_> \int \mathcal{D}\phi_<(\mathbf{k}') \exp\left[-\int_0^{\Lambda} \frac{d\mathbf{k}'}{(2\pi)^d} b^{-d}(r + b^{-2}Dk'^2)|\phi_<(\mathbf{k}')|^2\right]. \qquad (3.39)$$

3.3.3 Renormalization

To motivate the necessity of the renormalization of the $\phi_<(\mathbf{k}')$, consider the behavior of the correlations $\langle\phi(\mathbf{q})\phi(\mathbf{q}')\rangle$:

$$\begin{aligned}\langle\phi(\mathbf{q})\phi(\mathbf{q}')\rangle &= \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi(\mathbf{k})\phi(\mathbf{q})\phi(\mathbf{q}') \exp\left[-\int_0^{\Lambda} \frac{d\mathbf{k}}{(2\pi)^d}(r + Dk^2)|\phi(\mathbf{k})|^2\right] \\ &= \frac{\delta(\mathbf{q} + \mathbf{q}')}{r + Dq^2}. \qquad (3.40)\end{aligned}$$

Under rescaling of the momenta, this quantity rescales according to

$$\langle\phi(b\mathbf{q})\phi(b\mathbf{q}')\rangle = \frac{\delta(b\mathbf{q} + b\mathbf{q}')}{r + Db^2q^2} = \frac{b^{-d}\delta(\mathbf{q} + \mathbf{q}')}{r + Db^2q^2}. \qquad (3.41)$$

As $r \to 0$,

$$\langle \phi(b\mathbf{q})\phi(b\mathbf{q}')\rangle \to b^{-(d+2)}\frac{\delta(\mathbf{q}+\mathbf{q}')}{Dq^2}\,. \tag{3.42}$$

Hence, to preserve the scale of the fluctuations, we must renormalize the ϕ according to

$$\phi_<(\mathbf{k}') = b^{(d+2)/2}\phi'(\mathbf{k}')\,, \tag{3.43}$$

in which case the transformed partition function reads

$$\begin{aligned}\mathcal{Z} &= \mathcal{Z}_> \int \mathcal{D}\phi'(\mathbf{k}')\exp\left[-\int_0^\Lambda \frac{d\mathbf{k}'}{(2\pi)^d} b^{-d}(r + b^{-2}Dk'^2)|b^{d+2}\phi'(\mathbf{k}')|^2\right] \\ &= \mathcal{Z}_> \int \mathcal{D}\phi'(\mathbf{k}')\exp\left[-\int_0^\Lambda \frac{d\mathbf{k}'}{(2\pi)^d}(b^2 r + Dk'^2)|\phi'(\mathbf{k}')|^2\right], \end{aligned} \tag{3.44}$$

where we have neglected the Jacobian factor induced by the renormalization of ϕ. Thus, the net effect of the RG transformation is a rescaling of r.

3.3.4 Recursion Relations

The coefficients in the Hamiltonian of the Gaussian model transform under the RG as

$$r' = b^2 r\,, \qquad D' = D\,.$$

Note that the critical point $r^* = 0$ is unstable. Any initial deviation is sent toward $r^* \to \infty$, at which which point the spins become uncorrelated.

According to Eq. (3.23), r is proportional to the temperature difference from the critical temperature, $r \sim T - T_c$, the RG transformation of the singular part of the free energy is

$$f_s(t) \sim b^{-d} f_s(b^2 t)\,, \tag{3.45}$$

or, by choosing $b^2 t = 1$,

$$f_s(t) \sim t^{d/2}\,, \tag{3.46}$$

which agrees with Eq. (3.34). In particular, we find the same specific heat exponent α_+ as in Eq. (3.35).

3.3.5 Scaling Exponents of Perturbations

The rescaling and renormalization that was used in the preceding section can be used to assess the scale dimensions of various terms that have been neglected in the passage from the Ising model to its field-theoretic representation. Consider the following term:

$$u_p^{n_1\cdots n_p}\int \nabla^{n_1}\phi(\mathbf{x})\cdots\nabla^{n_p}\phi(\mathbf{x})\,d\mathbf{x}\,. \tag{3.47}$$

The behavior of this quantity under rescaling and renormalization can be determined from the following transformations for $\mathbf{x}$,

$$\mathbf{x}\to b\,\mathbf{x}'\,,\qquad d\mathbf{x}\to b^d\,d\mathbf{x}'\,, \tag{3.48}$$

the corresponding changes for derivatives,

$$\frac{\partial}{\partial x}\to\frac{1}{b}\frac{\partial}{\partial x'}\,,\qquad \nabla^{n_k}\to b^{-n_k}\nabla'^{\,n_k}\,, \tag{3.49}$$

and renormalization,

$$\phi'(\mathbf{x}')\to b^{(d-2)/2}\phi(\mathbf{x}')\,,\qquad \phi(\mathbf{x}')\to b^{-(d-2)/2}\phi'(\mathbf{x}')\,. \tag{3.50}$$

Thus, $u_p^{n_1\cdots n_p}$ scales according to

$$u_p'^{\,n_1\cdots n_p}=b^d b^{-n_1}\cdots b^{-n_p}b^{-p(d-2)/2}u_p^{n_1\cdots n_p}\,. \tag{3.51}$$

In particular, the coefficient of the leading polynomial correction to the Gaussian theory,

$$u_4\int\phi^4(\mathbf{x})\,d\mathbf{x}\,, \tag{3.52}$$

transforms as

$$u_4'=b^d b^{-2(d-2)}u_4=b^{4-d}u_4\,. \tag{3.53}$$

Thus, for $d>4$, u_4 has a negative scale dimension and is therefore an *irrelevant* variable, but for $d<4$ is a *relevant* variable.

Consider now the leading derivative correction in the Gaussian theory:

$$u_2^{0,4} \int \phi(\mathbf{x}) \nabla^4 \phi(\mathbf{x})\, d\mathbf{x}\,. \tag{3.54}$$

The coefficient of this term transforms as

$$u_2'^{0,4} = b^d b^{-4} b^{-(d-2)} u_2^{0,4} = b^{-2} u_2^{0,4}\,, \tag{3.55}$$

which is an *irrelevant* variable in *all* spatial dimensions. Since adding additional fields and/or derivatives to the terms considered here produces greater negative scaling exponents, we conclude that the Gaussian theory is stable in $d > 4$ dimensions in that all perturbations calculated in Sec. 3.1.2 are irrelevant. We now consider the perturbative RG for $d < 4$.

3.4 Perturbative Renormalization Group

We will calculate the correction to the Gaussian theory based upon the quartic term in Eq. (3.20) whose partition function, in the momentum representation, is written as

$$\mathcal{Z} = \int \mathcal{D}\phi(\mathbf{k}) e^{-(\mathcal{H}_0 + U)}\,, \tag{3.56}$$

where we introduced the abbreviations

$$\mathcal{H}_0 = \int_0^\Lambda \frac{d\mathbf{k}}{(2\pi)^d} (r + Dk^2) |\phi(\mathbf{k})|^2\,. \tag{3.57}$$

$$U = u \int_0^\Lambda \frac{d\mathbf{k}_1}{(2\pi)^d} \cdots \int_0^\Lambda \frac{d\mathbf{k}_4}{(2\pi)^d} \phi(\mathbf{k}_1) \cdots \phi(\mathbf{k}_4) (2\pi)^d \delta(\mathbf{k}_1 + \cdots \mathbf{k}_4)\,. \tag{3.58}$$

The implementation of the perturbative RG proceeds as in the preceding section by carrying out coarse graining, rescaling, and renormalization.

3.4.1 Coarse Graining

We first partition the range of wavevectors as in Eq. (3.37) and label the ϕ accordingly as $\phi_<(\mathbf{k})$ or $\phi_>(\mathbf{k})$. The effect on the partition function in Eq. (3.56) of carrying out the integration over the $\phi_>(\mathbf{k})$ can be represented as

$$\begin{aligned}
\mathcal{Z} &= \int \mathcal{D}\phi_<(\mathbf{k}) e^{-\mathcal{H}_0\{\phi_<(\mathbf{k})\}} \int \mathcal{D}\phi_>(\mathbf{k}) e^{-\mathcal{H}_0\{\phi_>(\mathbf{k})\} - U\{\phi(\mathbf{k})\}} \\
&= \mathcal{Z}_0^> \int \mathcal{D}\phi_<(\mathbf{k}) e^{-\mathcal{H}_0\{\phi_<(\mathbf{k})\}} \left[\frac{1}{\mathcal{Z}_0^>} \int \mathcal{D}\phi_>(\mathbf{k}) e^{-\mathcal{H}_0\{\phi_>(\mathbf{k})\} - U\{\phi(\mathbf{k})\}} \right] \\
&= \mathcal{Z}_0^> \int \mathcal{D}\phi_<(\mathbf{k}) e^{-\mathcal{H}_0\{\phi_<(\mathbf{k})\}} \big\langle e^{-U\{\phi(\mathbf{k})\}} \big\rangle_{\{\phi_>(\mathbf{k})\}} , \qquad (3.59)
\end{aligned}$$

where we have used $\mathcal{Z}_0^>$ to denote the contribution of the $\phi_>(\mathbf{k})$ to the Gaussian part of the partition function as in Eq. (3.38), and $\langle \cdot \rangle$ represents the average with respect to these Gaussian fields. We can write the right-hand side of this equation in a more useful form as

$$\mathcal{Z} = \mathcal{Z}_0^> \int \mathcal{D}\phi_<(\mathbf{k}) \exp\left[-\mathcal{H}_0\{\phi_<(\mathbf{k})\} + \ln \big\langle e^{-U\{\phi(\mathbf{k})\}} \big\rangle_{\{\phi_>(\mathbf{k})\}} \right] . \qquad (3.60)$$

The renormalization of the Hamiltonian necessitates the expansion of the second term in the exponential, which is a cumulant expansion:

$$\begin{aligned}
\ln \big\langle e^{-U} \big\rangle &= -\langle U \rangle + \tfrac{1}{2}\big(\langle U^2 \rangle - \langle U \rangle^2 \big) + \cdots \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \big\langle U^k \big\rangle_c , \qquad (3.61)
\end{aligned}$$

where $\langle \cdot \rangle_c$ denotes cumulant. In general, each of the terms in this expansion will contribute to the renormalized Hamiltonian, but we will consider only the first term, which is

$$-\langle U \rangle_{\{\phi_>(\mathbf{k})\}} = -\frac{1}{Z_0^>} \int \mathcal{D}\phi_>(\mathbf{k}) U e^{-\mathcal{H}_0\{\phi_>(\mathbf{k})\}} . \qquad (3.62)$$

Only terms that have an even number of factors of $\phi_>(\mathbf{k})$ contribute to this average. There are three possibilities to consider and we will denote by U_n the contribution to U with n factors of $\phi_>(\mathbf{k})$. If there are no such factors, then the corresponding term ,

$$U_0 = u\int_0^{\Lambda/b}\frac{d\mathbf{k}_1}{(2\pi)^d}\cdots\int_0^{\Lambda/b}\frac{d\mathbf{k}_4}{(2\pi)^d}$$
$$\times\,\phi_<(\mathbf{k}_1)\phi_<(\mathbf{k}_2)\phi_<(\mathbf{k}_3)\phi_<(\mathbf{k}_4)(2\pi)^d\delta(\mathbf{k}_1+\cdots+\mathbf{k}_4)\,, \quad (3.63)$$

is unaffected by the averaging and therefore will simply lead to the presence of the bare interaction u. The term with four factors of $\phi_>(\mathbf{k})$,

$$U_4 = u\int_{\Lambda/b}^{\Lambda}\frac{d\mathbf{k}_1}{(2\pi)^d}\cdots\int_{\Lambda/b}^{\Lambda}\frac{d\mathbf{k}_4}{(2\pi)^d}$$
$$\times\,\phi_>(\mathbf{k}_1)\phi_>(\mathbf{k}_2)\phi_>(\mathbf{k}_3)\phi_>(\mathbf{k}_4)(2\pi)^d\delta(\mathbf{k}_1+\cdots+\mathbf{k}_4)\,, \quad (3.64)$$

is integrated out and thereby provides a contribution to the partial partition function. For the purposes of examining the critical properties, we do not need to consider this term further in our discussion.

The remaining term contains two factors of $\phi_>(\mathbf{k})$:

$$U_2 = 6u\int_{\Lambda/b}^{\Lambda}\frac{d\mathbf{k}_1}{(2\pi)^d}\int_{\Lambda/b}^{\Lambda}\frac{d\mathbf{k}_1}{(2\pi)^d}\int_0^{\Lambda/b}\frac{d\mathbf{k}_1}{(2\pi)^d}\int_0^{\Lambda/b}\frac{d\mathbf{k}_4}{(2\pi)^d}$$
$$\times\,\phi_>(\mathbf{k}_1)\phi_>(\mathbf{k}_2)\phi_<(\mathbf{k}_3)\phi_<(\mathbf{k}_4)(2\pi)^d\delta(\mathbf{k}_1+\cdots+\mathbf{k}_4)\,, \quad (3.65)$$

where the factor of 6 accounts for the equivalent permutations of the ϕ. Using the fact that

$$\langle\phi_>(\mathbf{k}_1)\phi_>(\mathbf{k}_2)\rangle = \frac{(2\pi)^d\delta(\mathbf{k}_1+\mathbf{k}_2)}{r+Dk_1^2}\,, \quad (3.66)$$

we obtain

$$\langle U_2\rangle_{\{\phi_>(\mathbf{k})\}} = 6u\left(\int_{\Lambda/b}^{\Lambda}\frac{d\mathbf{k}}{(2\pi)^d}\frac{1}{r+Dk^2}\right)\int_0^{\Lambda/b}\frac{d\mathbf{k}}{(2\pi)^d}|\phi_<(\mathbf{k})|^2\,. \quad (3.67)$$

To summarize, the perturbative coarse graining of the partition function in Eq. (3.56) has resulted in a quartic term [Eq. (3.63)], which was present in the original Hamiltonian, and an additional (constant) quadratic term [Eq. (3.67)], which will provide a correction to r. Had we retained higher-order terms in the cumulant expansion in Eq. (3.61) higher-order products of the ψ would have been generated along with corrections to D.

3.4.2 Rescaling

The rescaling $\mathbf{k}' = b\,\mathbf{k}$ leads to the following changes to $\mathcal{H}_0$, U_0, and $\langle U_2\rangle_{\{\phi_>(\mathbf{k})\}}$:

$$\mathcal{H}_0 \to b^{-d}\int_0^\Lambda \frac{d\mathbf{k}'}{(2\pi)^d}(r+b^{-2}Dk'^2)|\phi_<(\mathbf{k}')|^2\,, \tag{3.68}$$

$$\begin{aligned} U_0 \to b^{-3d}u&\int_0^\Lambda \frac{d\mathbf{k}_1'}{(2\pi)^d}\cdots\int_0^\Lambda \frac{d\mathbf{k}_4'}{(2\pi)^d} \\ &\times \phi_<(\mathbf{k}_1')\phi_<(\mathbf{k}_2')\phi_<(\mathbf{k}_3')\phi_<(\mathbf{k}_4')(2\pi)^d\delta(\mathbf{k}_1'+\cdots+\mathbf{k}_4')\,, \end{aligned} \tag{3.69}$$

where the rescaling of the arguments of the δ-function cancels one of the factors obtained from the rescaling of the volume elements $d\mathbf{k}_i$, and

$$\langle U_2\rangle_{\{\phi_>(\mathbf{k})\}} \to 6b^{-d}u\left(\int_{\Lambda/b}^\Lambda \frac{d\mathbf{k}}{(2\pi)^d}\frac{1}{r+Dk^2}\right)\int_0^\Lambda \frac{d\mathbf{k}'}{(2\pi)^d}|\phi_<(\mathbf{k}')|^2\,. \tag{3.70}$$

The factor in parenthesis is a pure number and is therefore unaffected by this rescaling.

3.4.3 Renormalization

The final step in the RG transformation is the renormalization of the ϕ according to Eq. (3.43):

$$\phi_<(\mathbf{k}') = b^{(d+2)/2}\phi'(\mathbf{k}')\,, \tag{3.71}$$

We therefore obtain the following expressions for the terms in the transformed Hamiltonian:

$$\mathcal{H}_0 \to \int_0^\Lambda \frac{d\mathbf{k}'}{(2\pi)^d}(b^2r+Dk'^2)|\phi'(\mathbf{k}')|^2\,, \tag{3.72}$$

$$\begin{aligned} U_0 \to b^{4-d}u&\int_0^\Lambda \frac{d\mathbf{k}_1'}{(2\pi)^d}\cdots\int_0^\Lambda \frac{d\mathbf{k}_4'}{(2\pi)^d} \\ &\times \phi'(\mathbf{k}_1')\phi'(\mathbf{k}_2')\phi'(\mathbf{k}_3')\phi'(\mathbf{k}_4')(2\pi)^d\delta(\mathbf{k}_1'+\cdots+\mathbf{k}_4')\,, \end{aligned} \tag{3.73}$$

and

$$\langle U_2 \rangle_{\{\phi_>(\mathbf{k})\}} \to 6b^2 u \left(\int_{\Lambda/b}^{\Lambda} \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{r + Dk^2} \right) \int_0^{\Lambda} \frac{d\mathbf{k}'}{(2\pi)^d} |\phi'(\mathbf{k}')|^2 . \quad (3.74)$$

3.4.4 Recursion Relations

By comparing the original Hamiltonian in Eqs. (3.57) and (3.58) with the renormalized terms in Eqs. (3.72)-(3.74), the recursion relations for the coefficients r, D, and u are obtained as

$$r' = b^2 r + 6b^2 u \left(\int_{\Lambda/b}^{\Lambda} \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{r + Dk^2} \right), \quad (3.75)$$

$$D' = D , \quad (3.76)$$

$$u' = b^{4-d} u . \quad (3.77)$$

The analysis of these equations is simplified if we convert these discrete recursion relations, obtained from a finite value of b, to differential equations obtained from an *infinitesimal* renormalization parameter $b = e^{\delta\ell}$, in which $\delta\ell$ is a small quantity. Thus,

$$b^2 = e^{2\delta\ell} = 1 + 2\delta\ell + \cdots , \quad (3.78)$$

$$b^{4-d} = e^{4-d)\delta\ell} = 1 + (4-d)\delta\ell + \cdots . \quad (3.79)$$

and we obtain immediately

$$u' = b^{4-d} u = \big[1 + (4-d)\delta\ell + \cdots \big] u , \quad (3.80)$$

which can be rearranged to obtain

$$\lim_{\delta\ell \to 0} \left(\frac{u' - u}{\delta\ell} \right) = \frac{du}{d\ell} = (4-d) u , \quad (3.81)$$

as the differential recursion relation for u.

To determine the differential recursion relation for r, we first evaluate the integral on the right-hand side of Eq. (3.75):

$$\begin{aligned}
\int_{\Lambda/b}^{\Lambda} \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{r+Dk^2} &= \int_{\Lambda e^{-\delta\ell}}^{\Lambda} \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{r+Dk^2} \\
&= K_d \int_{\Lambda e^{-\delta\ell}}^{\Lambda} \frac{k^{d-1}dk}{r+Dk^2} \\
&= K_d \left(\frac{k^{d-1}}{r+Dk^2} \right) \bigg|_{k=\Lambda} \left(\Lambda - \Lambda e^{-\delta\ell} \right) \\
&= \frac{K_d \Lambda^d}{r+D\Lambda^2} \delta\ell \,.
\end{aligned} \tag{3.82}$$

Proceeding as in Eq. (3.81), the differential recursion relation for r is obtained as

$$\frac{dr}{d\ell} = 2r + \frac{12uK_d\Lambda^d}{r+D\Lambda^2} \,. \tag{3.83}$$

3.4.5 Fixed Points and Trajectories

We are now in a position to analyze fixed points and RG trajectories. The fixed points are determined by

$$2r^* + \frac{12u^* K_d \Lambda^d}{r+\Lambda^2} = 0 \,, \qquad (4-d)u^* = 0 \,. \tag{3.84}$$

which has the solution $r^* = 0$ and $u^* = 0$, which is the same as the Gaussian model. The recursion relations can be linearized about this fixed point by setting $r = r^* + \delta r$ and $u = u^* + \delta u$ and retaining terms only to first order in the deviations from the fixed-point values:

$$\frac{d}{d\ell} \begin{pmatrix} \delta r \\ \delta u \end{pmatrix} = \begin{pmatrix} 2 & 12K_d\Lambda^{d-2}s^{-1} \\ 0 & 4-d \end{pmatrix} \begin{pmatrix} \delta r \\ \delta u \end{pmatrix} \,. \tag{3.85}$$

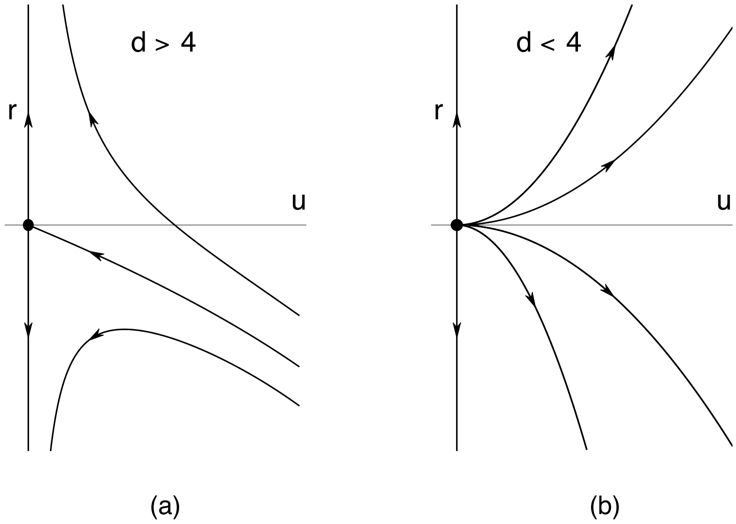

Figure 3.2: RG trajectories generated by the differential recursion relations in Eqs. (3.81 and (3.83). For $d > 4$ the Gaussian fixed point $(u^* = 0, r^* = 0)$ has only one unstable direction, which is associated with y_r and therefore correctly describes the phase transition. For $d < 4$, however, both eigenvalues are positive and the Gaussian is therefore unstable.

Since the lower left entry of the matrix is zero, the eigenvalues are diagonal entries: $y_r = 2$ and $y_u = 4 - d$. The corresponding eigendirections are $u = 0$ and

$$r = \frac{12uK_d\Lambda^{d-2}}{D(2-d)}, \tag{3.86}$$

respectively.

The RG trajectories, obtained by integration the differential recursion relations in Eqs. (3.81 and (3.83) are shown in Fig. 3.2. For spatial dimensions $d > 4$, the Gaussian fixed point $(u^* = 0, r^* = 0)$ has an unstable direction associated with the positive eigenvalue $y_r = 2$. The critical "surface" of this model is the trajectory that approaches the Gaussian fixed and separates the trajectories into those for which $r \to \infty$ (the disordered phase of the Ising ferromagnet) and $r \to \infty$, which is the ordered phase. Therefore, this provides the correct framework for describing the phase transition.

For $d < 4$ spatial dimensions, however, an altogether different scenario emerges. There are now no stable directions near the fixed point, since $y_r = 2$ and $y_u = 4 - d$ are both positive. Thus, all RG trajectories emanate away from the Gaussian fixed point so that this fixed point is unstable. The recursion relations have no other fixed points with finite values of r and u, so we must look to higher-order terms in the cumulant expansion (3.61) to obtain a description of critical phenomena in $d < 4$ dimensions.