Final Project

Texas A&M University
AERO-430-500 Numerical Simulation
Andrew Hollister

December 10, 2021

Due: December 12, 2021

Contents

1 Abstract

The purpose of this project was to formulate and utilize the analytical and Finite Difference Method (FDM) solutions to the problem of forced vibration within a rod. The FDM solution will be derived to a second order. The rod will consist of a singular material and be analyzed for several material property values. The deflection of the rod within the length of the rod will be analyzed. In addition, the resonance frequency will also be analyzed. The convergence and absolute error of the solutions will be determined to verify the results. The extrapolated solution will also be calculated, and its convergence will also be determined.

2 Analytical Solution

2.1 Vibration Analytical Solution

For this analysis, the rod will have a length of 1 cm. The problem of forced vibrations can be modelled with the following boundary problem (BVP) equation.

$$u"(x) + k^2 u(x) = 0$$

For this problem the rod possesses the following boundary conditions.

$$u(0) = 0$$
$$u(L) = u(1) = 100$$

The general solution to this differential equation can be derived with the following equation and application of the boundary conditions.

$$u(x) = Csin(k) + Dcos(kx)$$
$$u(0) = Csin(0) + Dcos(0)$$
$$D = 0$$
$$u(1) = 100 = Csin(kx)$$
$$C = \frac{100}{\sin(k)}$$
$$u(x) = \frac{100\sin(kx)}{\sin(k)}$$

This exact solution applies for the entire length of the bar since the bar consists of only a singular material.

3 Numerical Solution

3.1 Second Order Central Difference Scheme FDM

The second order FDM solution derivation commences with an Ordinary Taylor Series Expansion.

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f''' + \cdots$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f''' + \cdots$$

Adapting this equation for a point along the bar.

$$U_{i-1} = U_i - \Delta x U_i' + \frac{\Delta x^2}{2} U_i'' + \cdots$$

$$U_{i+1} = U_i + \Delta x U_i' + \frac{\Delta x^2}{2} U_i'' + \cdots$$

Adding these equations yields:

$$U_{i+1} + U_{i-1} = 2U_i + \Delta x^2 U_i'' + \cdots$$

Neglecting the higher order terms and solving for the derivative terms produces:

$$U''(x_i) = \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2}$$

Substituting this equation into the governing fundamental equation produces the following:

$$-U_{i-1} + (2 - k^2 \Delta x^2) U_i - U_{i+1} = 0$$

This result can be applied to solve each node of the mesh applied to the bar. Letting $\kappa = 2 - k^2 \Delta x^2$, the following matrix represents the FDM solution:

$$\begin{bmatrix} \kappa & -1 & 0 & \cdots & 0 & 0 \\ -1 & \kappa & -1 & \cdots & 0 & 0 \\ 0 & -1 & \kappa & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & -1 & 0 \\ 0 & 0 & 0 & -1 & \kappa & -1 \\ 0 & 0 & 0 & 0 & -1 & \kappa \end{bmatrix} \begin{Bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_L \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 100 \end{Bmatrix}$$

This set of equations can be applied to any sized mesh.

4 Derivative Solution at the End of the Rod

4.1 Exact Derivative Solution

The calculation of the derivative equation can be derived by taking the derivative of the exact solution derived in the exact solution section. This derivative is then evaluated at the length of the rod. The results of this derivative are presented below:

$$u'(1) = \frac{100k}{\sin(k)} \cos(k)$$

4.2 Approximate Derivative solution

The approximate derivative solution can be obtained through another application of the Taylor series expansion. Solving for the first derivative term produces:

$$U_{i-1} = U_i - \Delta x U_i' + \frac{\Delta x^2}{2} U_i'' + \cdots$$

$$U'(x_i) = \frac{U(x_i) - U(x_{i-1})}{\Delta x} + \frac{\Delta x}{2} U''(x_i)$$

From forced vibrations, the second derivative is known to be $u''(x_i) = k^2 u(x)$. Substituting this expression into the previous and solving for x = L yields:

$$u'(L) = \frac{U(x_N) - U(x_{N-1})}{\Delta x} - \frac{\Delta x}{2} k^2 u(x_N)$$

This expression can be applied to any mesh to calculate the value of the derivative at the end of the bar.

5 Performance Analysis

5.1 Error

The errors of the numerical solutions are calculated using the equations below:

$$e_h = u(x) - u_h(x)$$

5.2 Percent Error

The percent error of an estimated quantity $Q_{Estimated}$ (calculated using FDM) against its exact values Q_{Exact} is calculated using the equation below:

$$\%Error = \left| \frac{Q_{Exact} - Q_{Estimated}}{Q_{Exact}} \right| \times 100\%$$

5.3 Extrapolation and Convergence

Richardson's Extrapolation was used to extrapolate an approximate of the exact value from a series of approximated values. In general, error is modeled as:

$$Q_{ex} - Q_h = Ch^{\beta}$$

Here Q is the quantity of interest, Q_h is the approximate value at some mesh size h, C is some constant, and β is the convergence rate. In general, it is rare for the exact value to be known, and it is often difficult or impossible to obtain analytical solutions. In this case it is possible to use Richardson's Extrapolation to obtain reasonably accurate approximate value of the exact solution. If we write this equation at another mesh size, say h/2, the two can be divided and the unknown β can be found.

$$Q_{ex} - Q_h = C(h)^\beta$$
$$Q_{ex} - Q_{\frac{h}{2}} = C\left(\frac{h}{2}\right)^\beta$$
$$\frac{Q_{ex} - Q_h}{Q_{ex} - Q_{\frac{h}{2}}} = \frac{C(h)^\beta}{C(\frac{h}{2})^\beta}$$
$$log\left(\frac{Q_{ex} - Q_h}{Q_{ex} - Q_{\frac{h}{2}}}\right) = log\left(\frac{C(h)^\beta}{C(\frac{h}{2})^\beta}\right)$$
$$log(Q_{ex} - Q_h) - log(Q_{ex} - Q_{\frac{h}{2}}) = \beta log(h) - \beta log\left(\frac{h}{2}\right)$$
$$\boxed{\beta = \frac{log(Q_{ex} - Q_h) - log(Q_{ex} - Q_{\frac{h}{2}})}{log(h) - log(\frac{h}{2})}}$$

Again, Richardson's Extrapolation will be used to derive an expression for an extrapolated value. Here we will have to utilize three mesh sizes rather than the previous two.

$$Q_{ex} - Q_h = C(h)^\beta \approx 2^\beta$$
$$Q_{ex} - Q_{\frac{h}{2}} = C\left(\frac{h}{2}\right)^\beta \approx 2^\beta$$
$$Q_{ex} - Q_{\frac{h}{4}} = C\left(\frac{h}{4}\right)^\beta \approx 2^\beta$$
$$\frac{Q_{ex} - Q_h}{Q_{ex} - Q_{\frac{h}{2}}} \approx 2^\beta \approx \frac{Q_{ex} - Q_{\frac{h}{2}}}{Q_{ex} - Q_{\frac{h}{4}}}$$
$$\frac{Q_{extr} - Q_h}{Q_{extr} - Q_{\frac{h}{2}}} = 2^\beta = \frac{Q_{extr} - Q_{\frac{h}{2}}}{Q_{extr} - Q_{\frac{h}{4}}}$$
$$(Q_{extr} - Q_h)(Q_{extr} - Q_{\frac{h}{4}}) = (Q_{extr} - Q_{\frac{h}{2}})^2$$
$$\boxed{Q_{extr} = \frac{Q_{\frac{h}{2}}^2 - Q_h * Q_{\frac{h}{4}}}{2Q_{\frac{h}{2}} - Q_h - Q_{\frac{h}{4}}}}$$

From here Q_{extra} can be substituted to solve for β, which yields:

$$\boxed{\frac{log\left(\frac{Q_{extr} - Q_h}{Q_{extr} - Q_{\frac{h}{2}}}\right)}{log(2)} = \beta = \frac{log\left(\frac{Q_{extr} - Q_{\frac{h}{2}}}{Q_{extr} - Q_{\frac{h}{4}}}\right)}{log(2)}}$$

6 Results

6.1 Vibration Exact Solution Results

The following section outlines the analytical solution results plotted for various material property (k) values.

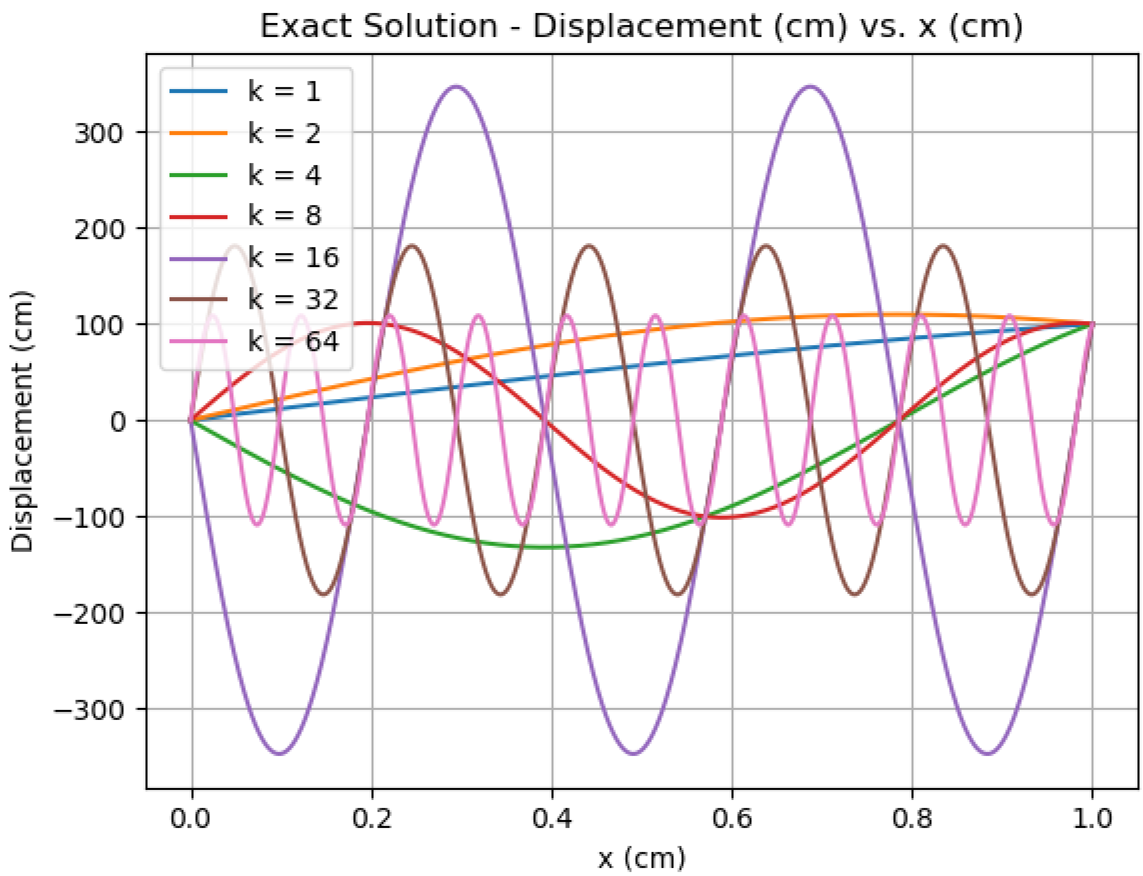

Figure 6.1.1 above shows the vibration results for several values of k

It is interesting to note from the figure above that the frequency of the vibrations increases with the value of k. It is also interesting to note that the amplitude of these vibrations is maximized at k = 16. This is because the vibrations approach the resonance frequency of this material property for the rod analyzed.

6.2 Vibration 2nd Order FDM Results

The following section outlines the results of the 2nd Order FDM solution plotted for various material property (k) values. In addition, the FDM solution is plotted against the exact solution for several number of node analyses.

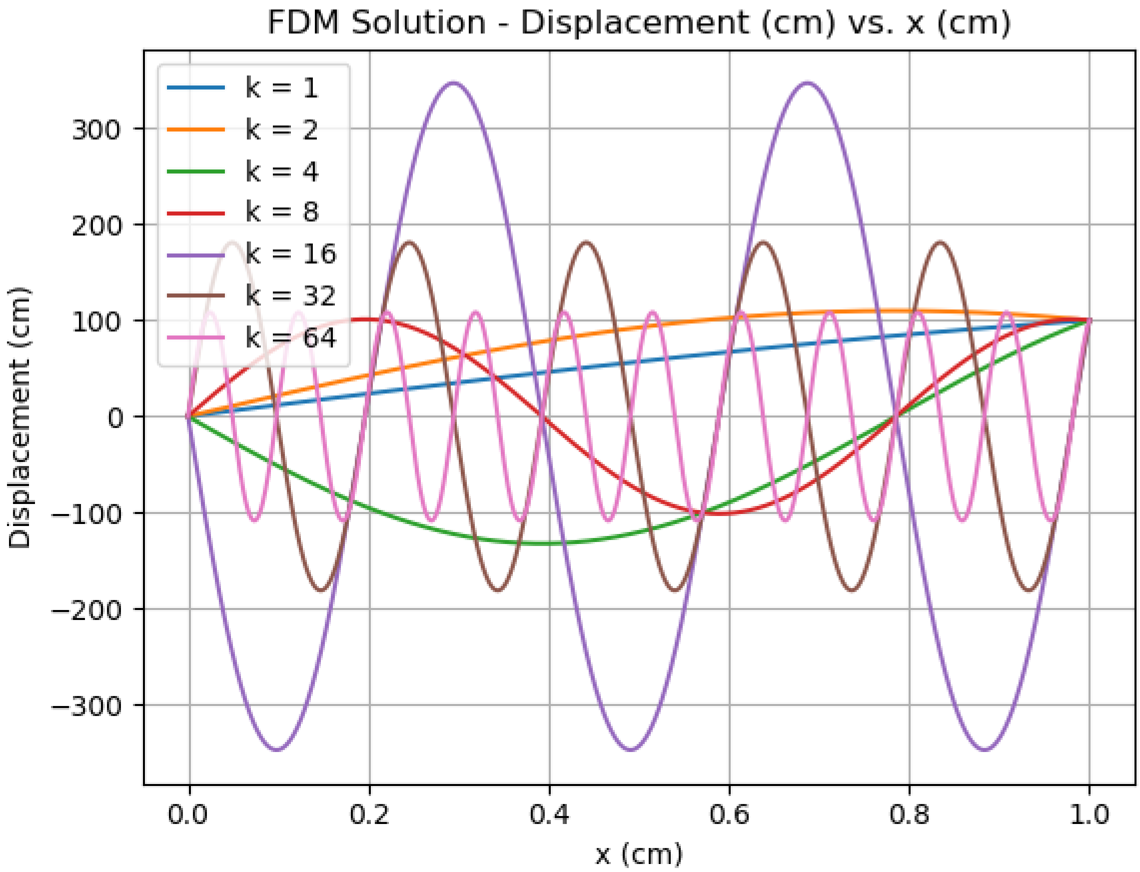

Figure 6.2.1 above shows the vibration results for several values of k.
The plot above shows the results for a 1024 node FDM solution.

It is interesting to note from the figure above that the frequency of the vibrations increases with the value of k. It is also interesting to note that the amplitude of these vibrations is maximized at k = 16. This is because the vibrations approach the resonance frequency of this material property for the rod analyzed.

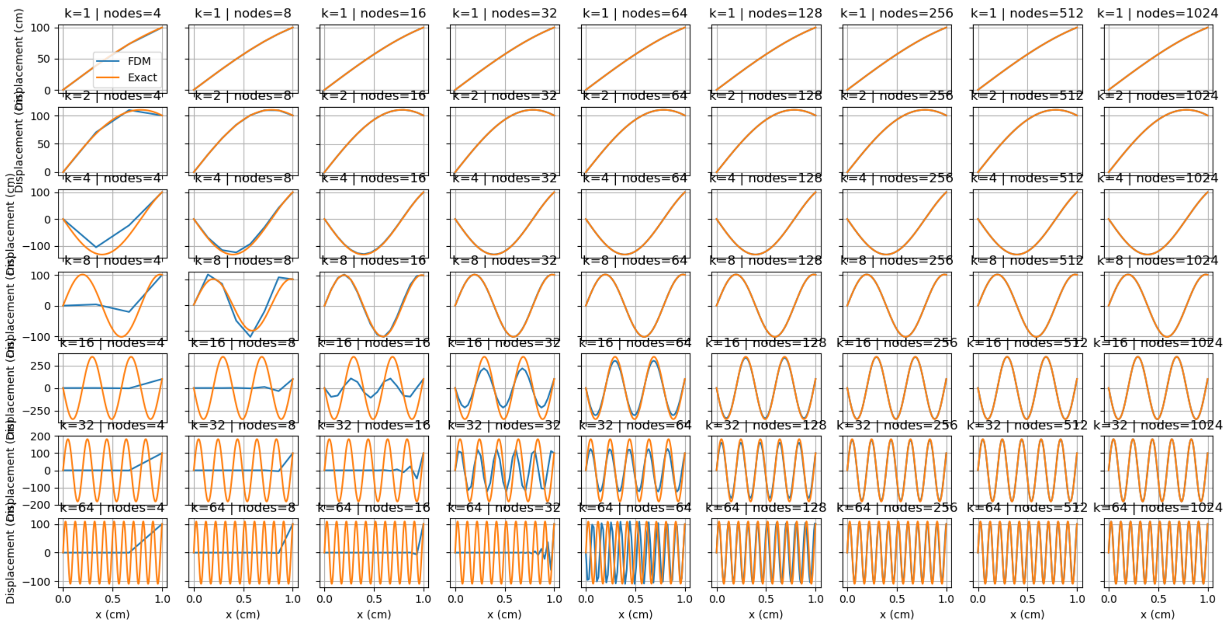

Figure 6.2.2 above shows the vibration results of the FDM solution plotted against the corresponding exact solution for various k values and mesh sizes.

It is interesting to note in the figure above that the accuracy of the FDM solution decreases as the value of k increases. This is expected as there are not enough nodes to represent the higher frequency vibrations that occur within the rod.

6.3 Vibration 2nd Order FDM Error

The following section outlines the error of the approximate FDM solution against the exact solution. It is important to note that the errors listed within this section represent absolute errors.

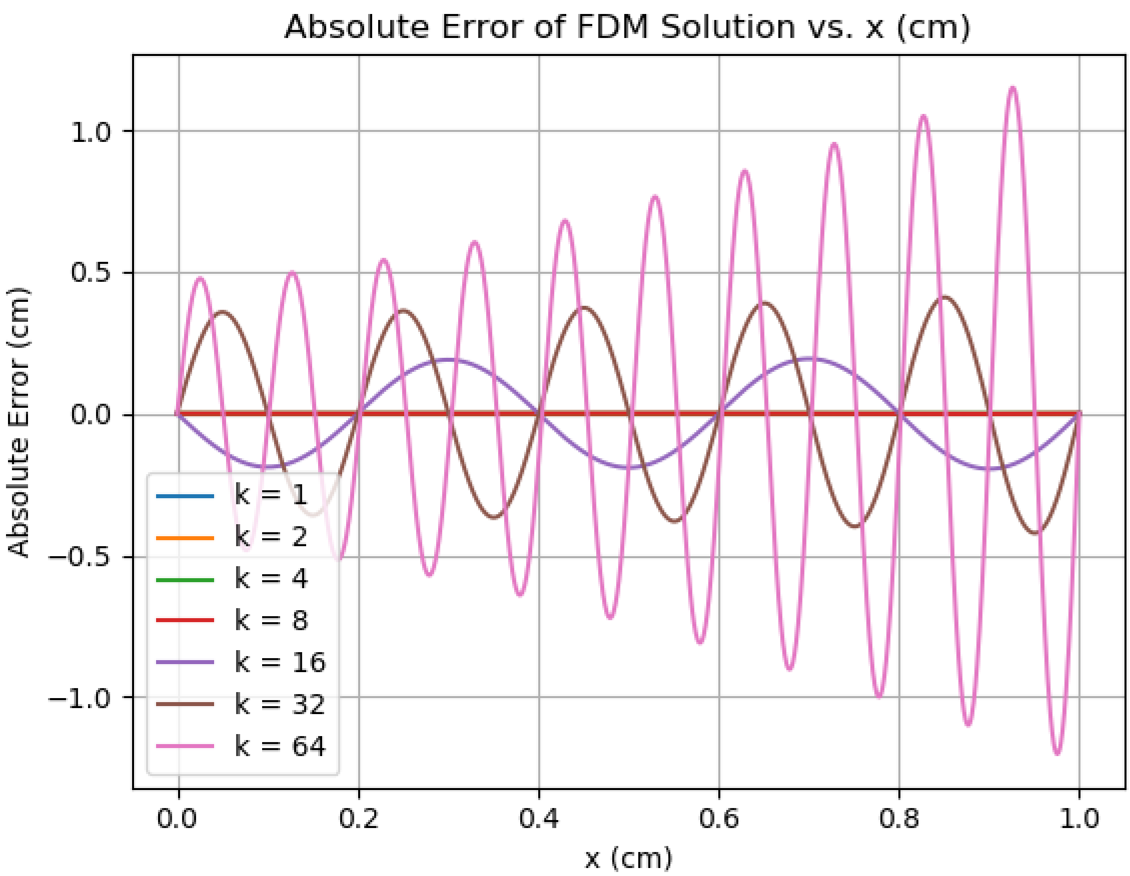

Figure 6.3.1 above represents the absolute error of the FDM solution against The exact solution. This plot represents several values of k

As can be noted from the figure above, the accuracy of the solution decreases as the value of k increases. This is, again, expected as there are higher frequency vibrations within the rod that are more difficult to be captured by the FDM mesh.

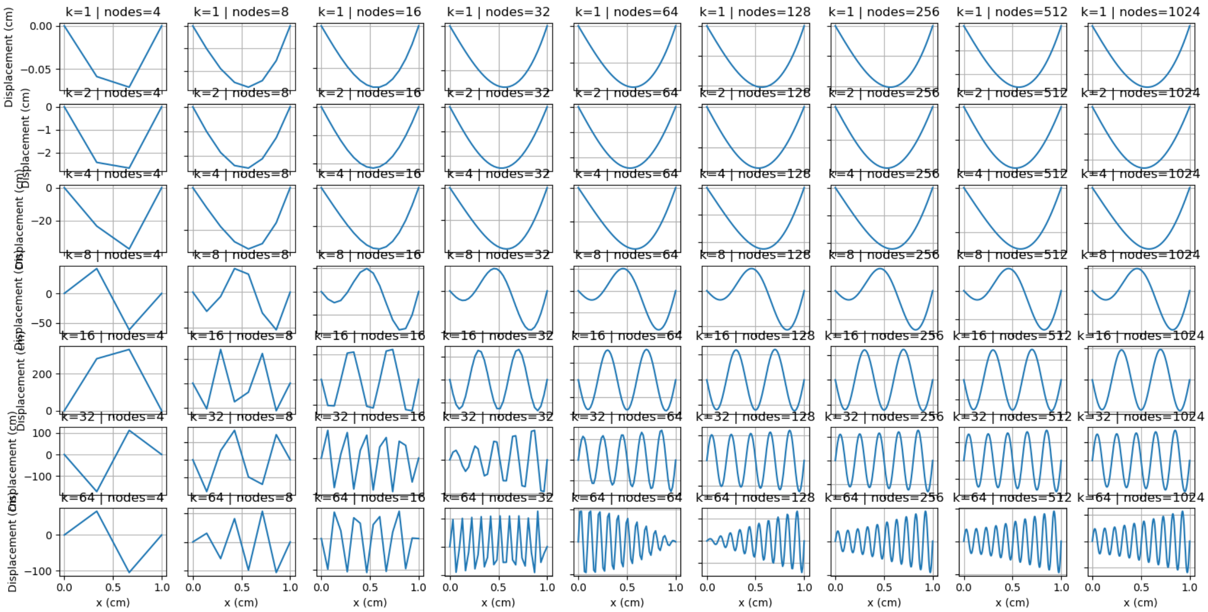

Figure 6.3.2 above shows the absolute error of the FDM solution against the exact solution for various mesh sizes and values of k.

It can once againe be noted from the figure above that the accuracy of the FDM solution decreases as the value of k creates higher frequency vibrations. The following tables in the remainder of this section outline the data captured in the above plots in tabular form.

Error of 2nd Order FDM Solution, k = 1, nodes = 64

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.015873	1.88626	1.88627	-7.08589e-06
0.031746	3.77205	3.77206	-1.416e-05
0.047619	5.65689	5.65691	-2.12106e-05
0.0634921	7.5403	7.54033	-2.82259e-05
0.0793651	9.42181	9.42184	-3.51943e-05
0.0952381	11.3009	11.301	-4.21039e-05
0.111111	13.1772	13.1773	-4.89431e-05
0.126984	15.0502	15.0503	-5.57003e-05
0.142857	16.9194	16.9194	-6.23639e-05
0.15873	18.7843	18.7844	-6.89222e-05
0.174603	20.6445	20.6446	-7.53638e-05
0.190476	22.4995	22.4995	-8.16772e-05
0.206349	24.3488	24.3489	-8.7851e-05
0.222222	26.192	26.1921	-9.38739e-05
0.238095	28.0285	28.0286	-9.97345e-05
0.253968	29.8581	29.8582	-0.000105422
0.269841	31.6801	31.6802	-0.000110925
0.285714	33.4941	33.4942	-0.000116232
0.301587	35.2996	35.2998	-0.000121333
0.31746	37.0963	37.0964	-0.000126216
0.333333	38.8837	38.8838	-0.000130871
0.349206	40.6612	40.6613	-0.000135288
0.365079	42.4285	42.4286	-0.000139456
0.380952	44.1851	44.1852	-0.000143364
0.396825	45.9306	45.9307	-0.000147002
0.412698	47.6645	47.6646	-0.00015036
0.428571	49.3864	49.3865	-0.000153428
0.444444	51.0958	51.096	-0.000156196
0.460317	52.7924	52.7925	-0.000158655
0.47619	54.4757	54.4758	-0.000160794
0.492063	56.1452	56.1454	-0.000162604
0.507937	57.8006	57.8008	-0.000164077
0.52381	59.4414	59.4416	-0.000165203
0.539683	61.0673	61.0675	-0.000165972
0.555556	62.6778	62.678	-0.000166377
0.571429	64.2725	64.2726	-0.000166408
0.587302	65.851	65.8511	-0.000166057
0.603175	67.4129	67.413	-0.000165316
0.619048	68.9578	68.958	-0.000164177
0.634921	70.4853	70.4855	-0.000162631
0.650794	71.9951	71.9953	-0.000160672
0.666667	73.4868	73.4869	-0.000158291
0.68254	74.9599	74.9601	-0.000155482
0.698413	76.4141	76.4143	-0.000152237
0.714286	77.8491	77.8493	-0.00014855
0.730159	79.2645	79.2647	-0.000144413
0.746032	80.6599	80.6601	-0.00013982
0.761905	82.035	82.0351	-0.000134766
0.777778	83.3894	83.3896	-0.000129244
0.793651	84.7228	84.723	-0.000123248
0.809524	86.0349	86.035	-0.000116773
0.825397	87.3253	87.3254	-0.000109813
0.84127	88.5937	88.5938	-0.000102364
0.857143	89.8397	89.8398	-9.44197e-05
0.873016	91.0632	91.0633	-8.59768e-05
0.888889	92.2637	92.2637	-7.70306e-05
0.904762	93.4409	93.441	-6.75769e-05
0.920635	94.5946	94.5947	-5.76118e-05
0.936508	95.7245	95.7245	-4.71319e-05
0.952381	96.8302	96.8303	-3.61337e-05
0.968254	97.9116	97.9116	-2.46141e-05
0.984127	98.9683	98.9683	-1.25704e-05
1	100	100	0

Error of 2nd Order FDM Solution, k = 2, nodes = 64

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.015873	3.49068	3.49096	-0.000280755
0.031746	6.97785	6.97841	-0.000560931
0.047619	10.458	10.4588	-0.000839952
0.0634921	13.9276	13.9287	-0.00111724
0.0793651	17.3831	17.3845	-0.00139222
0.0952381	20.8212	20.8228	-0.00166433
0.111111	24.2382	24.2402	-0.001933
0.126984	27.6309	27.6331	-0.00219768
0.142857	30.9957	30.9981	-0.00245779
0.15873	34.3292	34.3319	-0.00271281
0.174603	37.6282	37.6312	-0.00296219
0.190476	40.8892	40.8924	-0.00320539
0.206349	44.1091	44.1125	-0.00344191
0.222222	47.2845	47.2881	-0.00367122
0.238095	50.4122	50.4161	-0.00389284
0.253968	53.4891	53.4932	-0.00410626
0.269841	56.5122	56.5165	-0.00431101
0.285714	59.4783	59.4828	-0.00450664
0.301587	62.3844	62.3891	-0.0046927
0.31746	65.2277	65.2325	-0.00486874
0.333333	68.0052	68.0103	-0.00503436
0.349206	70.7142	70.7194	-0.00518914
0.365079	73.352	73.3573	-0.00533271
0.380952	75.9158	75.9213	-0.0054647
0.396825	78.4032	78.4088	-0.00558476
0.412698	80.8115	80.8172	-0.00569255
0.428571	83.1384	83.1442	-0.00578777
0.444444	85.3815	85.3874	-0.00587011
0.460317	87.5386	87.5445	-0.00593932
0.47619	89.6074	89.6134	-0.00599513
0.492063	91.586	91.592	-0.00603731
0.507937	93.4723	93.4783	-0.00606565
0.52381	95.2643	95.2704	-0.00607998
0.539683	96.9604	96.9665	-0.00608011
0.555556	98.5588	98.5648	-0.00606591
0.571429	100.058	100.064	-0.00603725
0.587302	101.456	101.462	-0.00599404
0.603175	102.752	102.758	-0.0059362
0.619048	103.944	103.95	-0.00586368
0.634921	105.032	105.038	-0.00577646
0.650794	106.014	106.02	-0.00567452
0.666667	106.889	106.894	-0.0055579
0.68254	107.656	107.662	-0.00542662
0.698413	108.315	108.32	-0.00528077
0.714286	108.865	108.87	-0.00512042
0.730159	109.305	109.309	-0.0049457
0.746032	109.634	109.639	-0.00475675
0.761905	109.854	109.858	-0.00455372
0.777778	109.962	109.967	-0.0043368
0.793651	109.96	109.964	-0.00410621
0.809524	109.847	109.851	-0.00386217
0.825397	109.623	109.627	-0.00360494
0.84127	109.289	109.292	-0.0033348
0.857143	108.845	108.848	-0.00305206
0.873016	108.291	108.294	-0.00275702
0.888889	107.628	107.63	-0.00245004
0.904762	106.856	106.858	-0.00213147
0.920635	105.977	105.979	-0.00180172
0.936508	104.991	104.992	-0.00146118
0.952381	103.899	103.9	-0.00111029
0.968254	102.702	102.703	-0.000749479
0.984127	101.402	101.403	-0.000379222
1	100	100	0

Error of 2nd Order FDM Solution, k = 4, nodes = 64

Position (cm)	Exact	FDM	Abs. Error
0	-0	0	-0
0.015873	-8.38388	-8.38042	-0.00345643
0.031746	-16.734	-16.7271	-0.00691028
0.047619	-25.0166	-25.0063	-0.0103589
0.0634921	-33.1985	-33.1847	-0.0137997
0.0793651	-41.2465	-41.2293	-0.0172298
0.0952381	-49.1284	-49.1077	-0.0206463
0.111111	-56.8122	-56.7882	-0.024046
0.126984	-64.2671	-64.2397	-0.0274258
0.142857	-71.4631	-71.4323	-0.030782
0.15873	-78.371	-78.3369	-0.0341109
0.174603	-84.9631	-84.9257	-0.0374084
0.190476	-91.2128	-91.1721	-0.0406702
0.206349	-97.0949	-97.0511	-0.0438915
0.222222	-102.586	-102.539	-0.0470673
0.238095	-107.663	-107.613	-0.0501924
0.253968	-112.307	-112.254	-0.0532608
0.269841	-116.498	-116.442	-0.0562666
0.285714	-120.219	-120.16	-0.0592034
0.301587	-123.456	-123.394	-0.0620642
0.31746	-126.196	-126.131	-0.0648421
0.333333	-128.427	-128.359	-0.0675294
0.349206	-130.14	-130.07	-0.0701184
0.365079	-131.329	-131.257	-0.0726009
0.380952	-131.989	-131.914	-0.0749686
0.396825	-132.117	-132.04	-0.0772128
0.412698	-131.712	-131.633	-0.0793246
0.428571	-130.777	-130.696	-0.0812951
0.444444	-129.315	-129.231	-0.0831148
0.460317	-127.331	-127.246	-0.0847746
0.47619	-124.834	-124.748	-0.0862651
0.492063	-121.835	-121.747	-0.0875768
0.507937	-118.344	-118.255	-0.0887005
0.52381	-114.376	-114.287	-0.0896269
0.539683	-109.948	-109.858	-0.0903468
0.555556	-105.076	-104.985	-0.0908514
0.571429	-99.7813	-99.6901	-0.091132
0.587302	-94.0841	-93.9929	-0.0911803
0.603175	-88.0078	-87.9168	-0.0909885
0.619048	-81.5768	-81.4863	-0.0905491
0.634921	-74.8171	-74.7273	-0.0898551
0.650794	-67.7559	-67.667	-0.0889001
0.666667	-60.4217	-60.334	-0.0876785
0.68254	-52.8439	-52.7577	-0.0861853
0.698413	-45.0532	-44.9688	-0.0844163
0.714286	-37.0809	-36.9986	-0.0823679
0.730159	-28.9592	-28.8792	-0.0800376
0.746032	-20.7208	-20.6434	-0.077424
0.761905	-12.3989	-12.3244	-0.0745263
0.777778	-4.02705	-3.95571	-0.0713449
0.793651	4.36105	4.42893	-0.0678814
0.809524	12.7316	12.7957	-0.0641383
0.825397	21.0508	21.1109	-0.0601195
0.84127	29.2852	29.341	-0.0558297
0.857143	37.4015	37.4528	-0.0512753
0.873016	45.3672	45.4137	-0.0464635
0.888889	53.15	53.1914	-0.041403
0.904762	60.7186	60.7548	-0.0361036
0.920635	68.0426	68.0732	-0.0305764
0.936508	75.0923	75.1172	-0.0248339
0.952381	81.8395	81.8583	-0.0188895
0.968254	88.2568	88.2695	-0.0127582
0.984127	94.3184	94.3249	-0.00645598
1	100	100	0

Error of 2nd Order FDM Solution, k = 8, nodes = 64

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.015873	12.8005	12.8194	-0.0189072
0.031746	25.3949	25.4322	-0.0372324
0.047619	37.5804	37.6348	-0.0544072
0.0634921	49.1607	49.2306	-0.0698908
0.0793651	59.9493	60.0325	-0.0831828
0.0952381	69.7726	69.8664	-0.0938353
0.111111	78.4723	78.5737	-0.101464
0.126984	85.9083	86.0141	-0.105756
0.142857	91.9609	92.0674	-0.106483
0.15873	96.5327	96.6362	-0.103502
0.174603	99.5499	99.6467	-0.0967609
0.190476	100.964	101.05	-0.0863037
0.206349	100.752	100.825	-0.0722684
0.222222	98.9182	98.9731	-0.0548858
0.238095	95.4912	95.5257	-0.034476
0.253968	90.5264	90.5378	-0.0114423
0.269841	84.1039	84.0901	0.0137363
0.285714	76.327	76.2864	0.0405149
0.301587	67.3209	67.2526	0.0682931
0.31746	57.2308	57.1344	0.096428
0.333333	46.2191	46.0949	0.124247
0.349206	34.4631	34.3121	0.151064
0.365079	22.1522	21.976	0.176191
0.380952	9.48447	9.28551	0.198957
0.396825	-3.33595	-3.55467	0.218721
0.412698	-16.1027	-16.3375	0.234885
0.428571	-28.61	-28.857	0.246912
0.444444	-40.6567	-40.9111	0.254339
0.460317	-52.0487	-52.3055	0.256784
0.47619	-62.6025	-62.8565	0.253962
0.492063	-72.1482	-72.3939	0.245688
0.507937	-80.5321	-80.764	0.23189
0.52381	-87.6191	-87.8317	0.212609
0.539683	-93.2952	-93.4832	0.188002
0.555556	-97.469	-97.6273	0.158344
0.571429	-100.073	-100.197	0.124021
0.587302	-101.066	-101.151	0.0855308
0.603175	-100.431	-100.474	0.043473
0.619048	-98.1788	-98.1773	-0.00146072
0.634921	-94.3457	-94.2972	-0.0484971
0.650794	-88.9933	-88.8965	-0.0967946
0.666667	-82.2078	-82.0623	-0.145459
0.68254	-74.0985	-73.9049	-0.193557
0.698413	-64.7959	-64.5558	-0.24014
0.714286	-54.45	-54.1657	-0.284253
0.730159	-43.2272	-42.9022	-0.324962
0.746032	-31.3083	-30.9469	-0.361367
0.761905	-18.8852	-18.4926	-0.392623
0.777778	-6.15807	-5.74011	-0.417957
0.793651	6.66827	7.10496	-0.436685
0.809524	19.3872	19.8355	-0.448227
0.825397	31.794	32.2461	-0.452121
0.84127	43.6888	44.1368	-0.448037
0.857143	54.88	55.3158	-0.435782
0.873016	65.1875	65.6028	-0.415311
0.888889	74.4452	74.832	-0.386731
0.904762	82.5042	82.8545	-0.350304
0.920635	89.2345	89.5409	-0.306441
0.936508	94.5279	94.7836	-0.255704
0.952381	98.299	98.4978	-0.198797
0.968254	100.487	100.624	-0.136556
0.984127	101.057	101.127	-0.0699359
1	100	100	0

Error of 2nd Order FDM Solution, k = 16, nodes = 64

Position (cm)	Exact	FDM	Abs. Error
0	-0	0	-0
0.015873	-87.2678	-76.5455	-10.7223
0.031746	-168.937	-148.154	-20.7832
0.047619	-239.768	-210.206	-29.562
0.0634921	-295.217	-258.7	-36.517
0.0793651	-331.727	-290.508	-41.2188
0.0952381	-346.955	-303.579	-43.3767
0.111111	-339.925	-297.068	-42.8569
0.126984	-311.087	-271.397	-39.6904
0.142857	-262.292	-228.22	-34.0715
0.15873	-196.669	-170.324	-26.3458
0.174603	-118.43	-101.441	-16.9887
0.190476	-32.5927	-26.0158	-6.57691
0.206349	55.3356	51.0877	4.24786
0.222222	139.714	124.896	14.8178
0.238095	215.129	190.649	24.4803
0.253968	276.743	244.104	32.6382
0.269841	320.602	281.815	38.7868
0.285714	343.894	301.349	42.5445
0.301587	345.123	301.446	43.677
0.31746	324.212	282.1	42.1118
0.333333	282.501	244.558	37.9425
0.349206	222.666	191.243	31.4237
0.365079	148.547	125.592	22.9551
0.380952	64.8977	51.8405	13.0573
0.396825	-22.915	-25.2547	2.33968
0.412698	-109.258	-100.721	-8.53673
0.428571	-188.591	-169.691	-18.9003
0.444444	-255.825	-227.715	-28.1101
0.460317	-306.648	-271.052	-35.5953
0.47619	-337.797	-296.907	-40.8906
0.492063	-347.276	-303.611	-43.6654
0.507937	-334.475	-290.731	-43.7439
0.52381	-300.217	-259.1	-41.1166
0.539683	-246.698	-210.757	-35.9412
0.555556	-177.353	-148.82	-28.5329
0.571429	-96.6301	-77.2844	-19.3456
0.587302	-9.70775	-0.763782	-8.94396
0.603175	77.8374	75.8061	2.03122
0.619048	160.389	147.487	12.9023
0.634921	232.651	209.654	22.9967
0.650794	289.987	258.299	31.6883
0.666667	328.72	290.284	38.4363
0.68254	346.364	303.545	42.8189
0.698413	341.787	297.228	44.5595
0.714286	315.284	271.739	43.5443
0.730159	268.553	228.724	39.8295
0.746032	204.594	170.956	33.6386
0.761905	127.51	102.161	25.3489
0.777778	42.2448	26.7767	15.4682
0.793651	-45.7302	-50.3346	4.60445
0.809524	-130.771	-124.199	-6.5721
0.825397	-207.423	-190.053	-17.37
0.84127	-270.768	-243.649	-27.1193
0.857143	-316.742	-281.529	-35.213
0.873016	-342.396	-301.25	-41.1451
0.888889	-346.083	-301.541	-44.5418
0.904762	-327.568	-282.383	-45.1853
0.920635	-288.039	-245.011	-43.0277
0.936508	-230.03	-191.836	-38.1944
0.952381	-157.264	-126.287	-30.9771
0.968254	-74.4091	-52.5929	-21.8163
0.984127	13.2195	24.4935	-11.274
1	100	100	0

Error of 2nd Order FDM Solution, k = 32, nodes = 64

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.015873	88.2031	60.9042	27.2989
0.031746	154.135	106.095	48.0397
0.047619	181.148	123.914	57.234
0.0634921	162.421	109.762	52.6582
0.0793651	102.682	67.2926	35.3898
0.0952381	17.0168	7.46128	9.55555
0.111111	-72.9454	-54.295	-18.6504
0.126984	-144.489	-102.043	-42.4458
0.142857	-179.549	-123.464	-56.0847
0.15873	-169.273	-113.032	-56.2413
0.174603	-116.256	-73.437	-42.8185
0.190476	-33.8835	-14.8955	-18.988
0.206349	57.0441	47.489	9.55512
0.222222	133.568	97.6213	35.9467
0.238095	176.366	122.567	53.7986
0.253968	174.632	115.891	58.7405
0.269841	128.803	79.3151	49.4877
0.285714	50.4512	22.2757	28.1754
0.301587	-40.6394	-40.5108	-0.12861
0.31746	-121.468	-92.8455	-28.623
0.333333	-171.627	-121.226	-50.4006
0.349206	-178.449	-118.331	-60.1188
0.365079	-140.213	-84.9057	-55.3077
0.380952	-66.5736	-29.5752	-36.9984
0.396825	23.876	33.3857	-9.50964
0.412698	108.297	87.733	20.5639
0.428571	165.373	119.445	45.9276
0.444444	180.692	120.341	60.3514
0.460317	150.387	90.1884	60.1983
0.47619	82.1086	36.7674	45.3412
0.492063	-6.90197	-26.1395	19.2375
0.507937	-94.1698	-82.3025	-11.8673
0.52381	-157.66	-117.231	-40.4283
0.539683	-181.341	-121.915	-59.4258
0.555556	-159.233	-95.1441	-64.0888
0.571429	-96.919	-43.8263	-53.0926
0.587302	-10.133	18.7986	-28.9316
0.603175	79.2116	76.5735	2.63808
0.619048	148.555	114.592	33.9627
0.634921	180.389	123.047	57.342
0.650794	166.674	99.7549	66.9191
0.666667	110.874	50.7264	60.1478
0.68254	27.0785	-11.3895	38.468
0.698413	-63.5544	-70.5669	7.01252
0.714286	-138.14	-111.538	-26.6018
0.730159	-177.845	-123.732	-54.1125
0.746032	-172.644	-104.004	-68.6403
0.761905	-123.851	-57.4425	-66.4084
0.777778	-43.7851	3.93914	-47.7243
0.793651	47.3364	64.3045	-16.9681
0.809524	126.505	108.079	18.4261
0.825397	173.732	123.97	49.7622
0.84127	177.091	107.876	69.215
0.857143	135.735	63.9503	71.7843
0.873016	60.1053	3.52551	56.5798
0.888889	-30.7006	-57.8089	27.1083
0.904762	-113.755	-104.229	-9.52604
0.920635	-168.086	-123.757	-44.3282
0.936508	-179.975	-111.357	-68.6181
0.952381	-146.421	-70.2263	-76.1943
0.968254	-75.8951	-10.9774	-64.9177
0.984127	13.7939	51.1037	-37.3097
1	100	100	0

Error of 2nd Order FDM Solution, k = 64, nodes = 64

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.015873	92.3823	-95.54	187.922
0.031746	97.3484	-92.4829	189.831
0.047619	10.1992	6.01635	4.18285
0.0634921	-86.6009	98.3067	-184.908
0.0793651	-101.455	89.1448	-190.6
0.0952381	-20.3084	-12.0144	-8.29399
0.111111	80.0553	-100.775	180.83
0.126984	104.667	-85.5357	190.203
0.142857	30.2384	17.976	12.2624
0.15873	-72.8033	102.937	-175.74
0.174603	-106.955	81.6668	-188.622
0.190476	-39.9016	-23.8829	-16.0186
0.206349	64.9088	-104.785	169.694
0.222222	108.3	-77.5497	185.849
0.238095	49.2126	29.7173	19.4953
0.253968	-56.4416	106.316	-162.758
0.269841	-108.688	73.1969	-181.885
0.285714	-58.0893	-35.4613	-22.628
0.301587	47.4762	-107.523	155
0.31746	108.118	-68.6216	176.739
0.333333	66.4535	41.0976	25.3559
0.349206	-38.0919	108.404	-146.496
0.365079	-106.593	63.8379	-170.431
0.380952	-74.2312	-46.609	-27.6222
0.396825	28.3715	-108.956	137.327
0.412698	104.128	-58.8601	162.988
0.428571	81.3539	51.9788	29.3751
0.444444	-18.4007	109.176	-127.576
0.460317	-100.744	53.7035	-154.447
0.47619	-87.7586	-57.1906	-30.5681
0.492063	8.26753	-109.064	117.332
0.507937	96.4706	-48.3837	144.854
0.52381	93.389	62.2286	31.1604
0.539683	1.93861	108.621	-106.682
0.555556	-91.3462	42.9169	-134.263
0.571429	-98.1952	-67.0775	-31.1177
0.587302	-12.1276	-107.848	95.7204
0.603175	85.4156	-37.3196	122.735
0.619048	102.135	71.7226	30.4123
0.634921	22.2096	106.747	-84.5375
0.650794	-78.7313	31.6089	-110.34
0.666667	-105.173	-76.1497	-29.0236
0.68254	-32.0957	-105.322	73.2263
0.698413	71.3523	-25.8022	97.1544
0.714286	107.284	80.3454	26.9382
0.730159	41.6985	103.577	-61.8782
0.746032	-63.3435	19.917	-83.2606
0.761905	-108.447	-84.297	-24.1502
0.777778	-50.9333	-101.517	50.5834
0.793651	54.7759	-13.9714	68.7472
0.809524	108.654	87.9923	20.6613
0.825397	59.7186	99.1481	-39.4295
0.84127	-45.7248	7.98327	-53.7081
0.857143	-107.901	-91.4203	-16.4811
0.873016	-67.977	-96.4783	28.5013
0.888889	36.2702	-1.97089	38.2411
0.904762	106.197	94.5705	11.6265
0.920635	75.6355	93.5153	-17.8798
0.936508	-26.4956	-4.04748	-22.4481
0.952381	-103.555	-97.4333	-6.12211
0.968254	-82.6266	-90.2681	7.64159
0.984127	16.4872	10.0535	6.43361
1	100	100	0

6.4 Derivative Equation Exact Solution Results

The following section outlines the results obtained from the exact solution of the derivative equation.

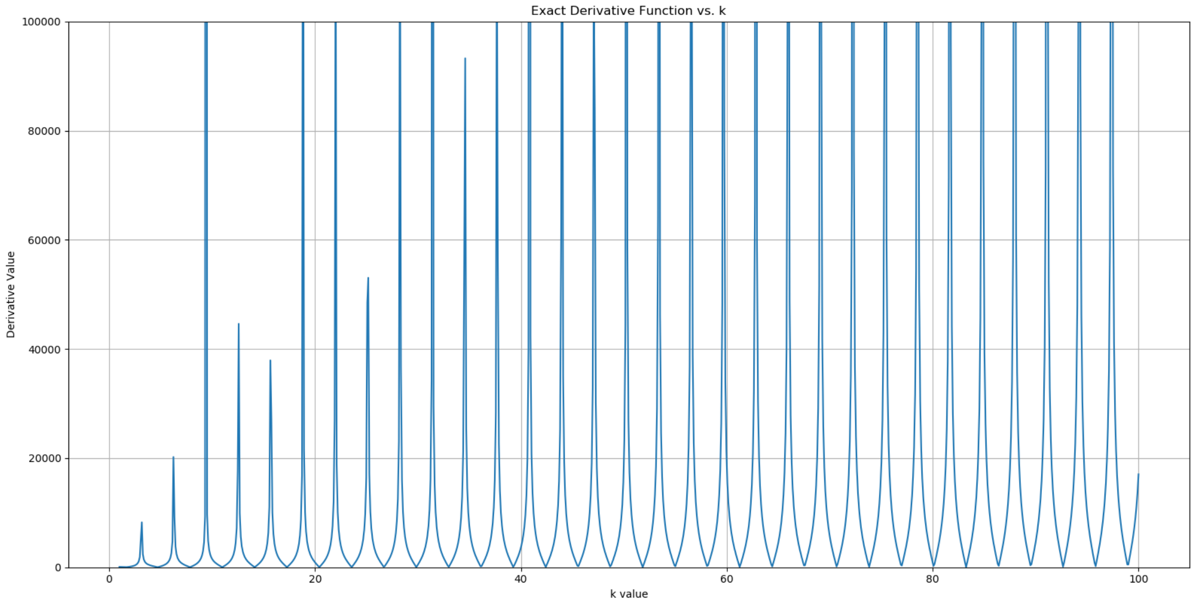

Figure 6.4.1 above shows the derivative function plotted with k values ranging from 1 to 100

It is interesting to note that the value of the derivative function spikes at specific values of k. This represents resonance frequencies that are achieved by the forced vibrations at that specific value of k.

6.5 Derivative Equation 2nd Order FDM Results

The following section outlines the results obtained by the 2nd order FDM solution of the derivative function. The results of the FDM solution are plotted against the exact solution in order to capture the improved accuracy of the solution as the mesh size increases.

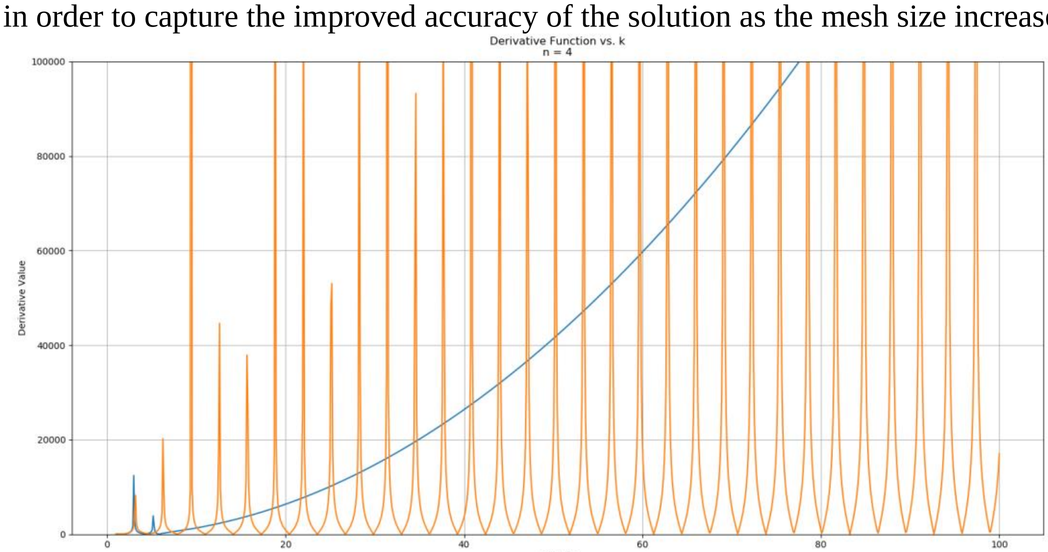

Figure 6.5.1 above shows the derivative function of the FDM solution against the exact solution for 4 nodes

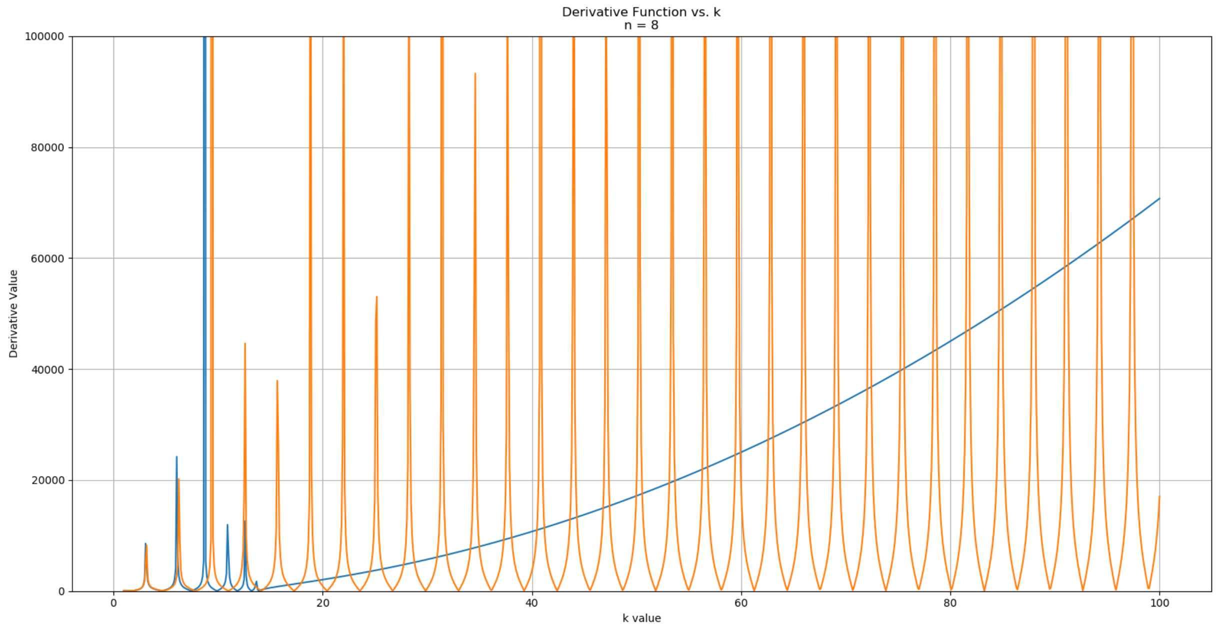

Figure 6.5.1 above shows the derivative function of the FDM solution against the exact solution for 8 nodes

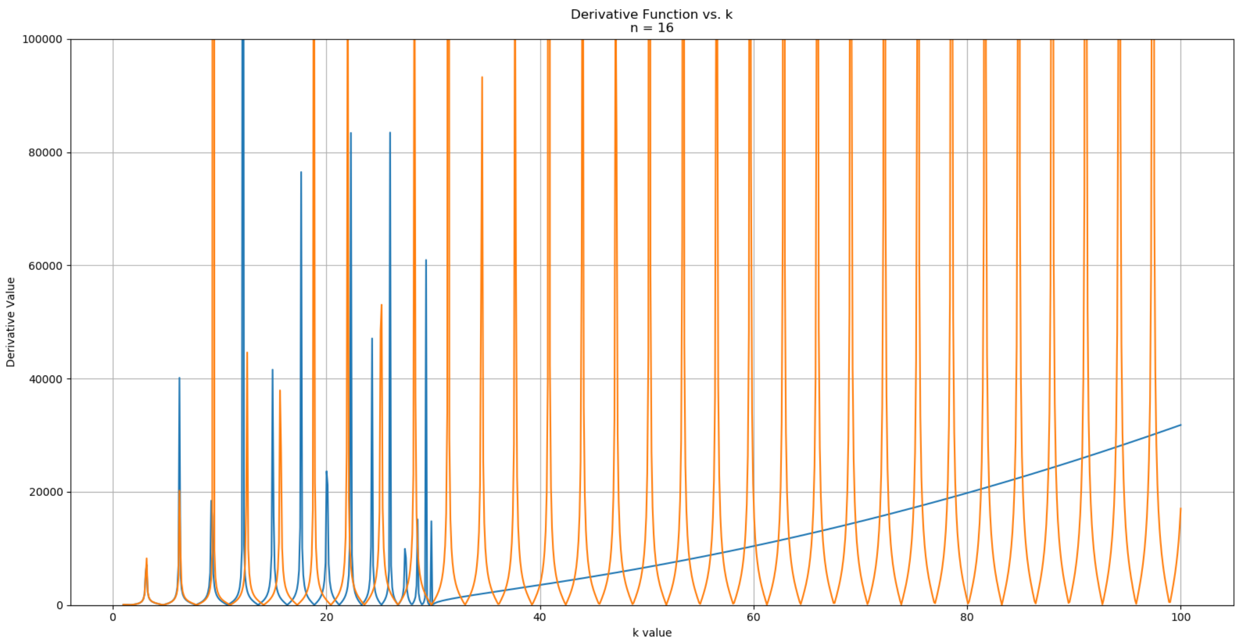

Figure 6.5.1 above shows the derivative function of the FDM solution against the exact solution for 16 nodes

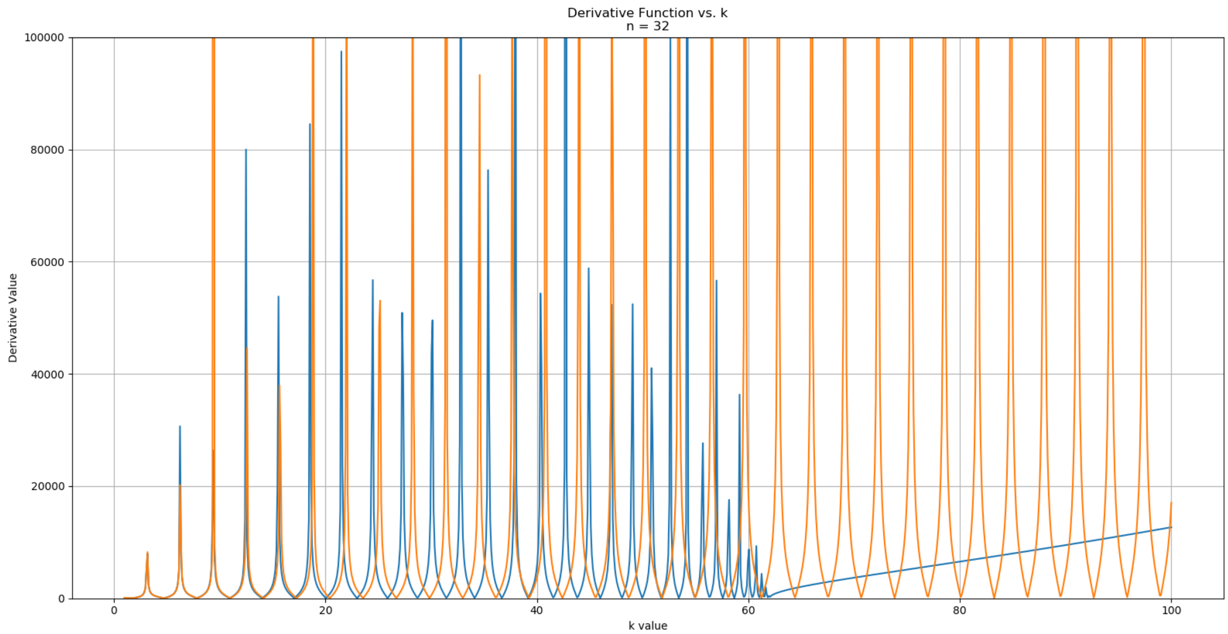

Figure 6.5.1 above shows the derivative function of the FDM solution against the exact solution for 32 nodes

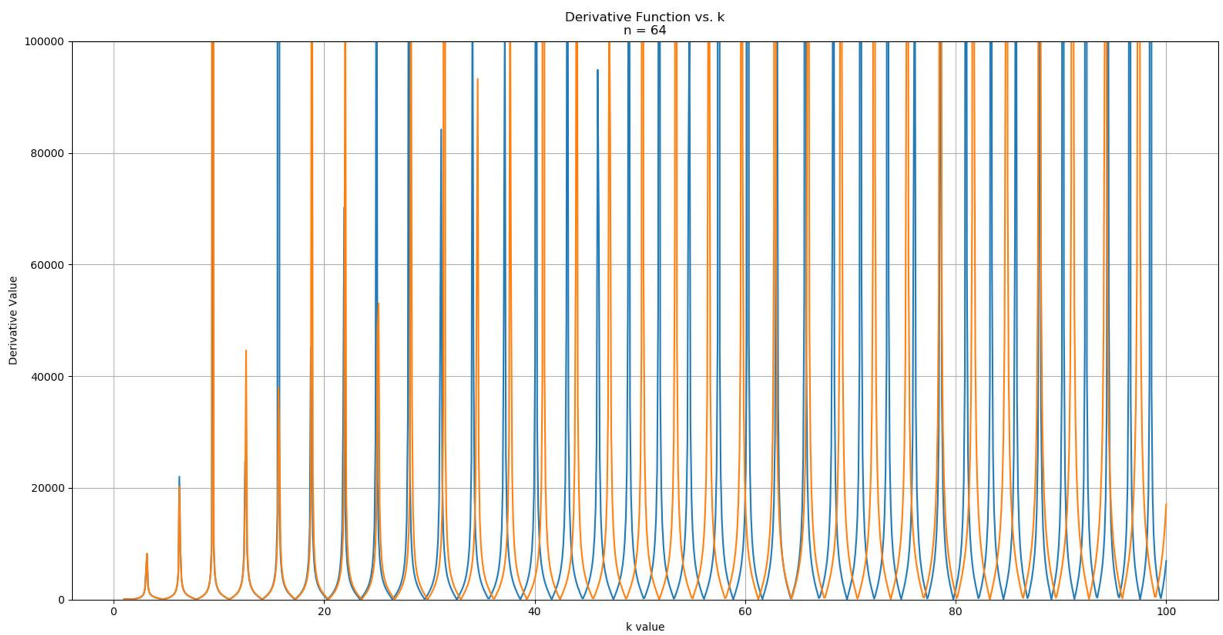

Figure 6.5.1 above shows the derivative function of the FDM solution against the exact solution for 64 nodes

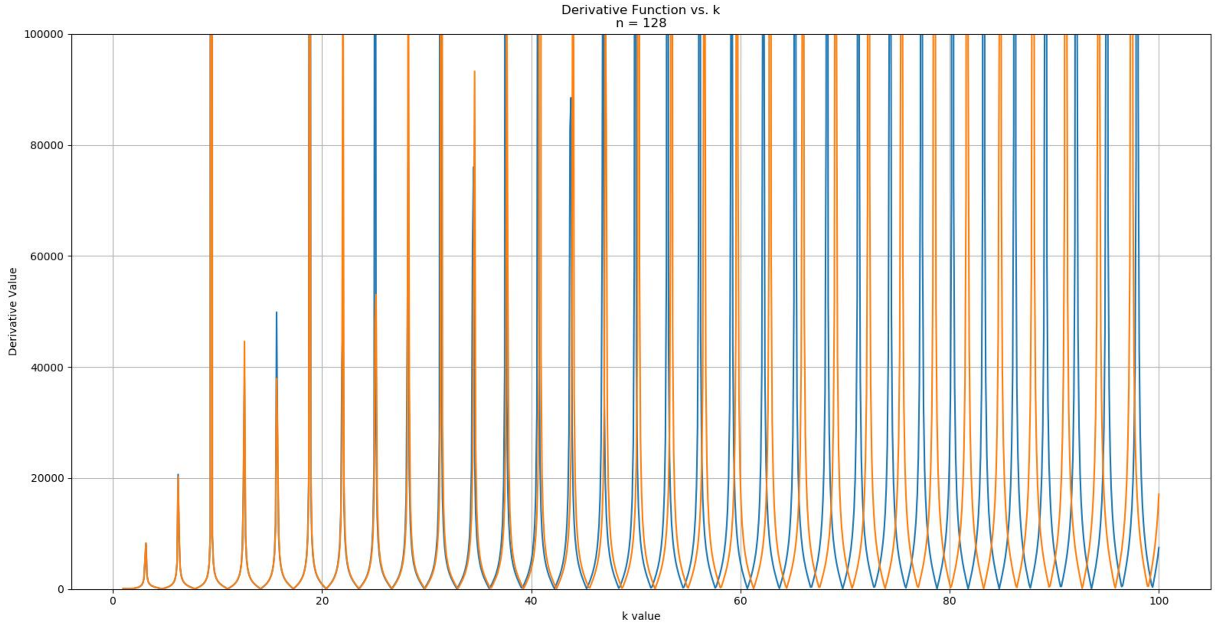

Figure 6.5.1 above shows the derivative function of the FDM solution against the exact solution for 128 nodes

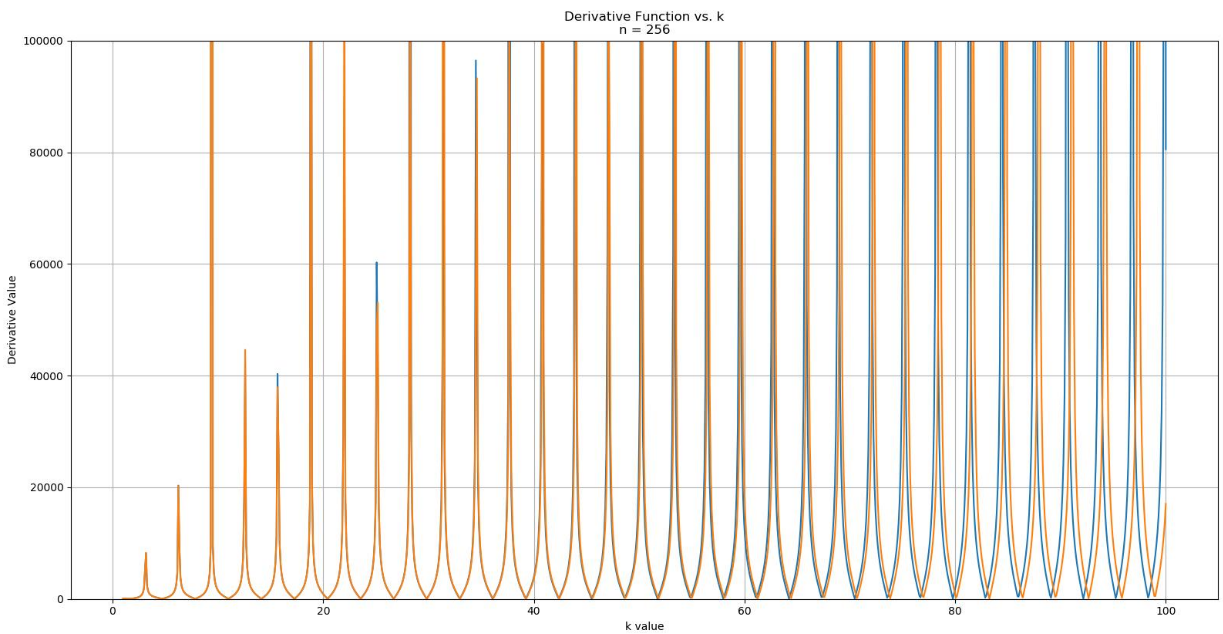

Figure 6.5.1 above shows the derivative function of the FDM solution against the exact solution for 256 nodes

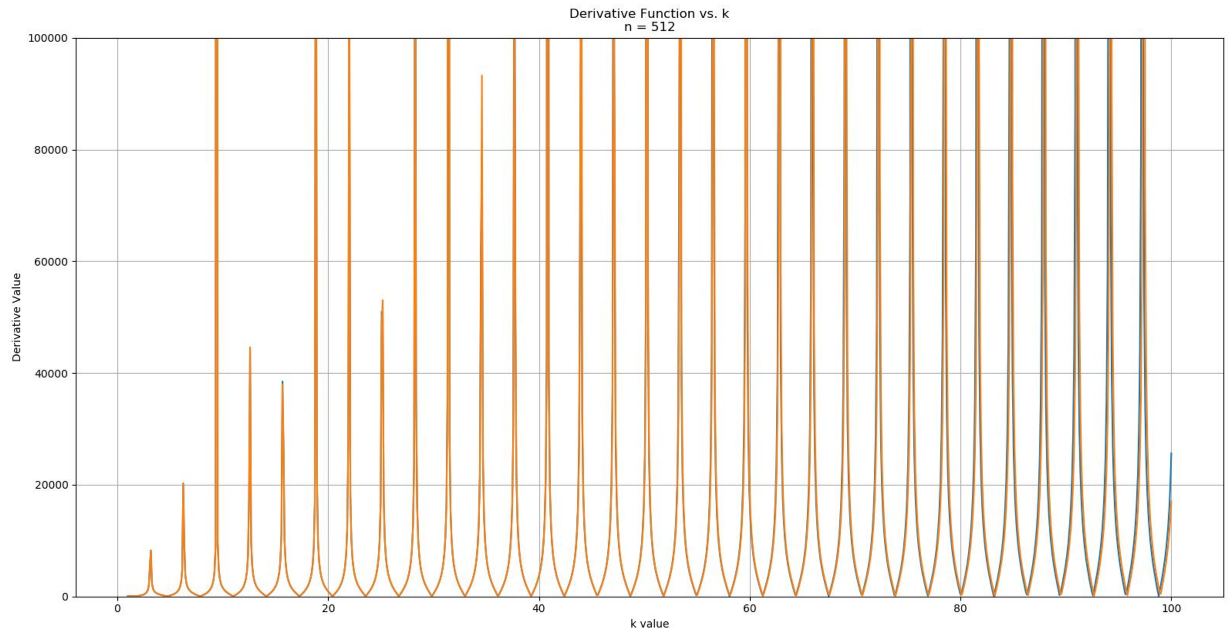

Figure 6.5.1 above shows the derivative function of the FDM solution against the exact solution for 512 nodes

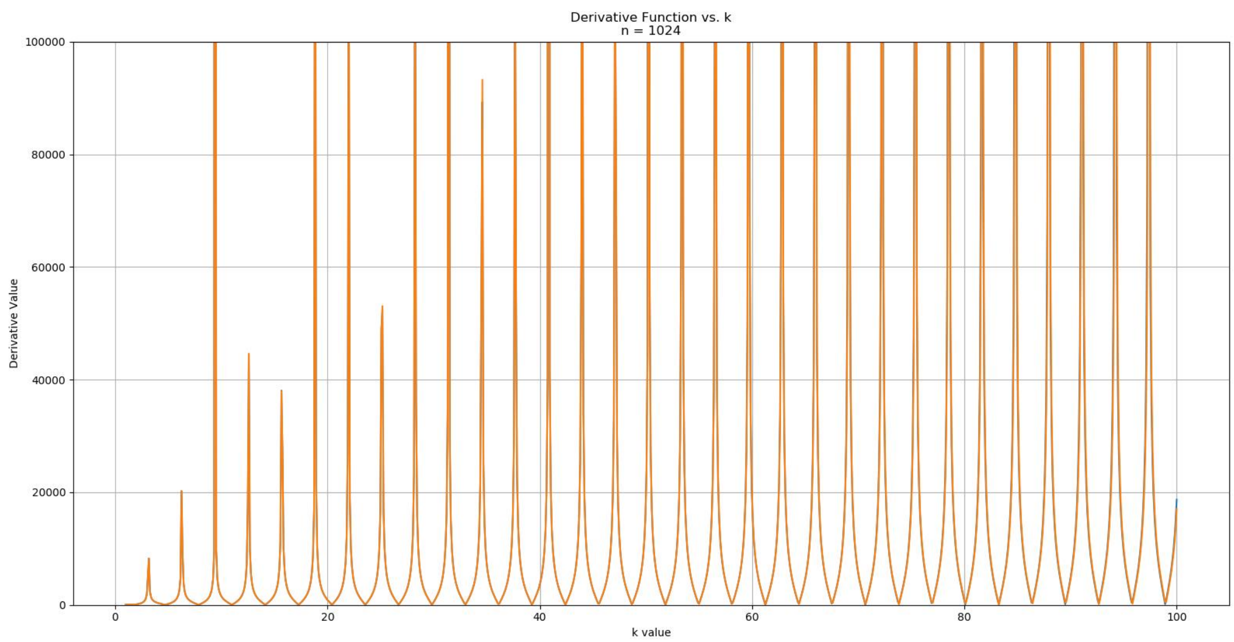

Figure 6.5.1 above shows the derivative function of the FDM solution against the exact solution for 1024 nodes

It can be noted from the figures above that, due to the sporadic nature of the exact solution, accurate results do not begin to emerge for greater values of k until mush larger mesh sizes are utilized.

6.6 Derivative Equation 2nd Order FDM Error and Convergence

The following section outlines the error and convergence rates of the FDM solution against the exact solution.

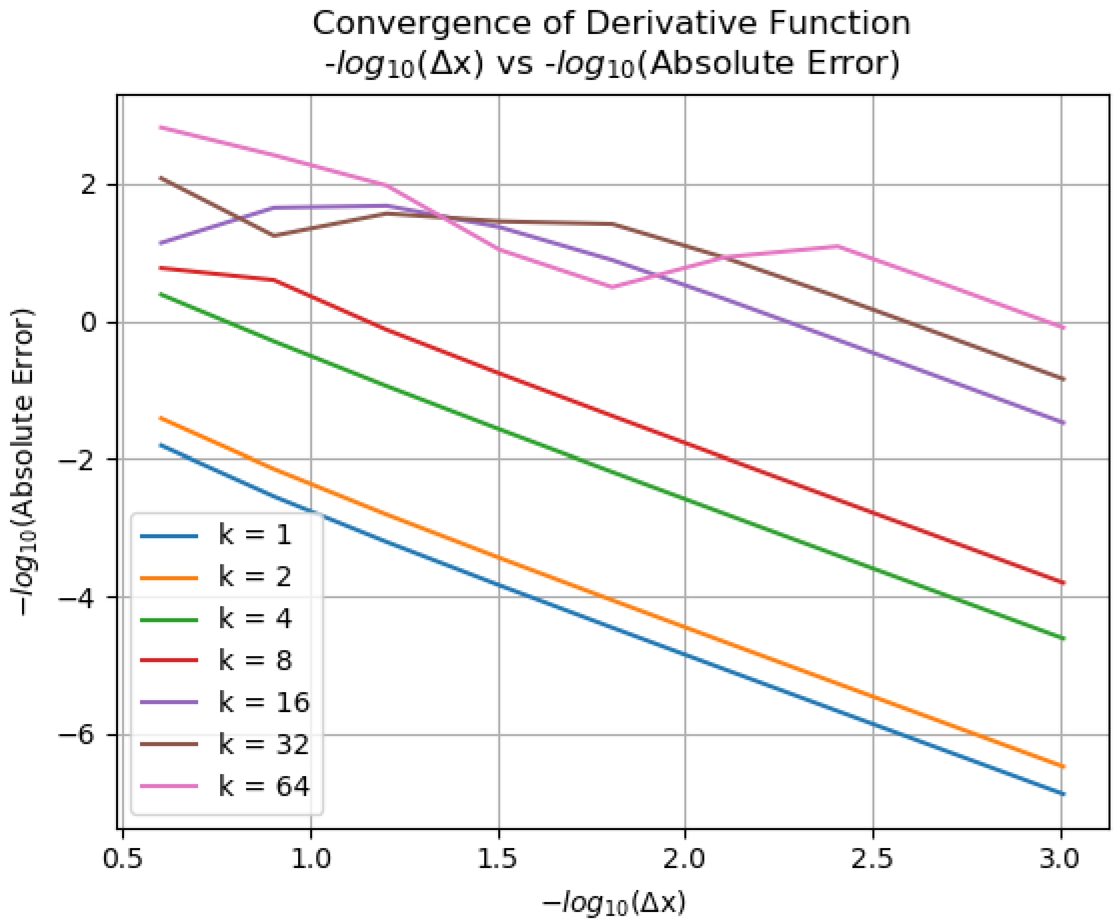

Figure 6.6.1 above shows the convergence of the FDM solution for various values of k

It is interesting to note that for larger values of k, the convergence of the solution is unstable. As, the number of nodes increases, however, the convergence ultimately converges with a rate of 2. This is expected as the FDM solution was derived from a 2nd order Taylor Series Expansion. The following tables display the exact and approximate derivative values, the absolute error, and the beta values for various mesh sizes and k values.

Convergence of Derivative Function for k = 1:

Num. Elements	dx	Exact Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	64.2093	62.6603	1.54901	n/a
8	0.125	64.2093	63.9253	0.284004	2.4473580459295463
16	0.0625	64.2093	64.1474	0.0618305	2.199520328933848
32	0.03125	64.2093	64.1948	0.0144755	2.094707014882682
64	0.015625	64.2093	64.2058	0.00350485	2.0461893634892987
128	0.0078125	64.2093	64.2084	0.000862464	2.0228149651665492
256	0.00390625	64.2093	64.209	0.000213928	2.0113400890180184
512	0.00195312	64.2093	64.2092	5.32722e-05	2.0056695201509336
1024	0.000976562	64.2093	64.2092	1.32922e-05	2.0028069305003244

Convergence of Derivative Function for k = 2:

Num. Elements	dx	Exact Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	91.5315	95.3623	3.83081	n/a
8	0.125	91.5315	92.2417	0.710172	2.4314076982919777
16	0.0625	91.5315	91.6864	0.154889	2.1969303835194687
32	0.03125	91.5315	91.5678	0.0362757	2.0941624580199716
64	0.015625	91.5315	91.5403	0.00878394	2.0460635778457164
128	0.0078125	91.5315	91.5337	0.00216158	2.0227845314952604
256	0.00390625	91.5315	91.532	0.000536167	2.011330107560257
512	0.00195312	91.5315	91.5316	0.000133518	2.0056526850320933
1024	0.000976562	91.5315	91.5315	3.33131e-05	2.002867787008173

Convergence of Derivative Function for k = 4:

Num. Elements	dx	Exact Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	345.476	103.463	242.013	n/a
8	0.125	345.476	294.97	50.5069	2.260533661076323
16	0.0625	345.476	334.206	11.2702	2.163970616211407
32	0.03125	345.476	342.824	2.65264	2.0870087309717524
64	0.015625	345.476	344.833	0.643062	2.0443997002907595
128	0.0078125	345.476	345.318	0.15829	2.022383450015597
256	0.00390625	345.476	345.437	0.0392657	2.011232906205285
512	0.00195312	345.476	345.467	0.00977822	2.0056261920676226
1024	0.000976562	345.476	345.474	0.00243978	2.002818581480921

Convergence of Derivative Function for k = 8:

Num. Elements	dx	Exact Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	117.652	705.635	587.983	n/a
8	0.125	117.652	511.063	393.411	0.5797350055699937
16	0.0625	117.652	191.958	74.306	2.40448797239015
32	0.03125	117.652	134.859	17.2072	2.110466257812202
64	0.015625	117.652	121.81	4.15824	2.048966515010708
128	0.0078125	117.652	118.675	1.02281	2.0234383699304286
256	0.00390625	117.652	117.906	0.253674	2.0114891950460283
512	0.00195312	117.652	117.715	0.0631688	2.0056895165848956
1024	0.000976562	117.652	117.668	0.0157612	2.0028314593811527

Convergence of Derivative Function for k = 16:

Num. Elements	dx	Exact Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	5322.12	3955.31	1366.81	n/a
8	0.125	5322.12	885.254	4436.86	-1.6987260366425194
16	0.0625	5322.12	576.497	4745.62	-0.09705661758595073
32	0.03125	5322.12	3004.53	2317.58	1.0339752416575294
64	0.015625	5322.12	4553.74	768.381	1.5927281030157192
128	0.0078125	5322.12	5114.28	207.833	1.8863937277430793
256	0.00390625	5322.12	5269.27	52.8422	1.9756653318267865
512	0.00195312	5322.12	5308.88	13.2411	1.9966698794929922
1024	0.000976562	5322.12	5318.81	3.30894	2.0005774042509294

Convergence of Derivative Function for k = 32:

Num. Elements	dx	Exact Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	4841.11	16764	11922.9	n/a
8	0.125	4841.11	6577.14	1736.03	2.779864591115283
16	0.0625	4841.11	1187.79	3653.32	-1.073410557644296
32	0.03125	4841.11	2029.81	2811.29	0.3779719734337922
64	0.015625	4841.11	2267.77	2573.34	0.1275952093369561
128	0.0078125	4841.11	4012.41	828.699	1.6347194978425632
256	0.00390625	4841.11	4617.42	223.682	1.8893980212309631
512	0.00195312	4841.11	4784.14	56.9659	1.973280444814613
1024	0.000976562	4841.11	4826.81	14.2947	1.994620709092206

Convergence of Derivative Function for k = 64:

Num. Elements	dx	Exact Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	2725.89	67966	65240.1	n/a
8	0.125	2725.89	28548.6	25822.7	1.3371208804578716
16	0.0625	2725.89	12060.4	9334.52	1.4679900485668476
32	0.03125	2725.89	1638.66	1087.23	3.101922956478659
64	0.015625	2725.89	2415.83	310.053	1.8100664626559644
128	0.0078125	2725.89	1879.76	846.127	-1.4483586182104327
256	0.00390625	2725.89	1509.94	1215.95	-0.5231338613089307
512	0.00195312	2725.89	2409.69	316.198	1.9431794293876172
1024	0.000976562	2725.89	2645.99	79.8981	1.9845955409696945

6.7 Derivative Equation Richardson Extrapolation Error and Convergence

The following section outlines the error and convergence rates of the extrapolated FDM solution against the exact solution.

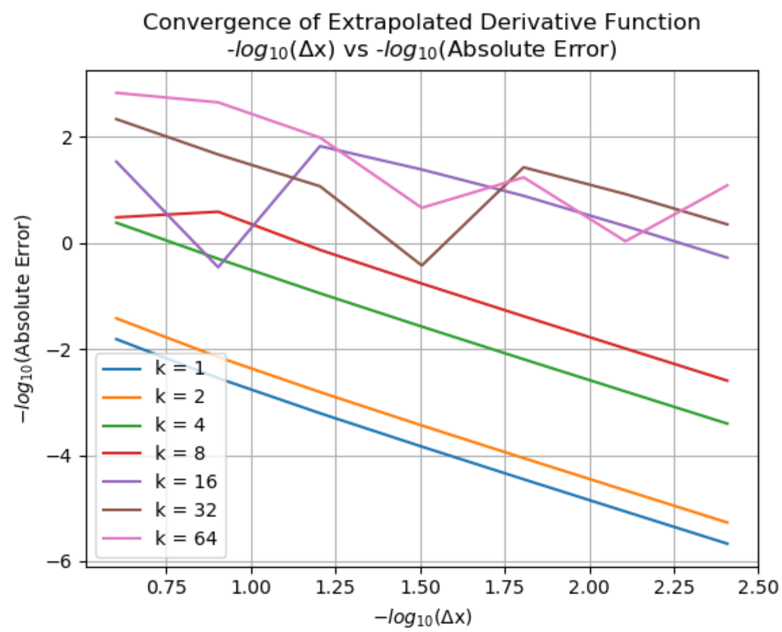

Figure 6.6.2 above shows the convergence of the extrapolated FDM solution for various values of k

It is interesting to note that for larger values of k, the convergence of the solution is unstable. As, the number of nodes increases, however, the convergence ultimately converges with a rate of 2. This is expected as the FDM solution was derived from a 2^{nd} order Taylor Series Expansion. The following tables display the exact and approximate derivative values, the absolute error, and the beta values for various mesh sizes and k values.

Convergence of Extrapolated Derivative Function k = 1:

Num. Elements	dx	Extrapolated Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	64.1948	62.6603	1.53451	2.50938
8	0.125	64.2076	63.9253	0.282356	2.2301
16	0.0625	64.2091	64.1474	0.0616336	2.10987
32	0.03125	64.2092	64.1948	0.0144514	2.05374
64	0.015625	64.2093	64.2058	0.00350187	2.02658
128	0.0078125	64.2093	64.2084	0.000862095	2.01322
256	0.00390625	64.2093	64.209	0.000213878	2.00669

Convergence of Extrapolated Derivative Function k = 2:

Num. Elements	dx	Extrapolated Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	91.5662	95.3623	3.79611	2.49055
8	0.125	91.5356	92.2417	0.706116	2.22695
16	0.0625	91.532	91.6864	0.1544	2.1092
32	0.03125	91.5316	91.5678	0.0362156	2.05358
64	0.015625	91.5315	91.5403	0.00877649	2.02654
128	0.0078125	91.5315	91.5337	0.00216065	2.01321
256	0.00390625	91.5315	91.532	0.000536055	2.00656

Convergence of Extrapolated Derivative Function k = 4:

Num. Elements	dx	Extrapolated Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	344.317	103.463	240.854	2.28712
8	0.125	345.249	294.97	50.2796	2.18686
16	0.0625	345.445	334.206	11.2383	2.10038
32	0.03125	345.472	342.824	2.64847	2.05152
64	0.015625	345.476	344.833	0.64253	2.02604
128	0.0078125	345.476	345.318	0.158223	2.01309
256	0.00390625	345.476	345.437	0.0392573	2.00656

Convergence of Extrapolated Derivative Function k = 8:

Num. Elements	dx	Extrapolated Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	1009.63	705.635	303.997	-0.713734
8	0.125	122.416	511.063	388.648	2.4825
16	0.0625	117.945	191.958	74.0133	2.12953
32	0.03125	117.683	134.859	17.1761	2.0572
64	0.015625	117.656	121.81	4.15457	2.02736
128	0.0078125	117.652	118.675	1.02236	2.01341
256	0.00390625	117.652	117.906	0.253619	2.00664

Convergence of Extrapolated Derivative Function k = 16:

Num. Elements	dx	Extrapolated Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	541.973	3955.31	3413.33	3.31372
8	0.125	850.421	885.254	34.8331	-2.97525
16	0.0625	7284.66	576.497	6708.17	0.648264
32	0.03125	5432.1	3004.53	2427.57	1.46662
64	0.015625	5328.51	4553.74	774.772	1.85465
128	0.0078125	5322.47	5114.28	208.183	1.96857
256	0.00390625	5322.13	5269.27	52.8582	1.99537

Convergence of Extrapolated Derivative Function k = 32:

Num. Elements	dx	Extrapolated Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	-4866.44	16764	21630.4	0.918524
8	0.125	1916.03	6577.14	4661.11	2.67818
16	0.0625	2361.51	1187.79	1173.72	1.82314
32	0.03125	1992.23	2029.81	37.5825	-2.87414
64	0.015625	4938.62	2267.77	2670.85	1.52788
128	0.0078125	4847.55	4012.41	835.147	1.85958
256	0.00390625	4841.49	4617.42	224.066	1.96606

Convergence of Extrapolated Derivative Function k = 64:

Num. Elements	dx	Extrapolated Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	203.998	67966	67762	1.2574
8	0.125	-16265.4	28548.6	44813.9	0.661832
16	0.0625	2361.9	12060.4	9698.51	3.74522
32	0.03125	2098.59	1638.66	459.928	0.535807
64	0.015625	687.304	2415.83	1728.53	0.53561
128	0.0078125	1772.03	1879.76	107.727	-1.2827
256	0.00390625	2730.15	1509.94	1220.21	1.9289

7 Discussion

This report investigated the application of a FDM to solve a forced vibration problem applied to the case of 1 dimensional rod. In addition, the exact solution was derived and utilized to be compared to the FDM solution, in order to determine its error and convergence. The derivative function was also obtained from the exact solution and approximated from the FDM results. The impact that different values of material properties and different mesh sizes had on the results of the exact and FDM solutions was also investigated.

One of the initial findings from the exact solution was that increasing values of k also resulted in increasing vibrational frequency experienced by the bar. Additionally, the amplitude of the vibrations would occasionally increase as the results approached a resonance frequency.

It was also found that the value of k had an impact on the convergence and accuracy of the FDM solution. As the value of k increased, more and more nodes within the mesh were required in order to get the FDM solution to converge. This is expected as higher values of k result in higher frequency vibrations which are more difficult to be captured by smaller mesh sizes. A similar result was discovered with the derivative results. Since the exact solution was found to be rather sporadic, it was found that vary large mesh sizes were needed in order to sufficiently capture this near random nature of the exact solution.

8 References

The following reports and codes were used to assist in the creation of this report and the code utilized within.

- Bryan Caleb's Report and Code

9 Appendix A: Code

```
"""
AERO 430
Final Project Code

The purpose of this code is to evaluate the forced vibrations upon a bar by utilizing the
exact solution and 2nd order finite difference method schemes

Andrew Hollister
127008398
"""

import numpy as np
import matplotlib.pyplot as plt
import tabulate as tbl

"""
Data Class Structure
"""


class DataStructure:
    def __init__(self):
        self.data = []

    def add_data(self, nodes, k, x, vals):
        self.data.append((nodes, k, x, vals))

    def return_data(self, nodes, k):
        x = next(item[2] for item in self.data if item[0] == nodes and item[1] == k)
        vals = next(item[3] for item in self.data if item[0] == nodes and item[1] == k)
        return x, vals


"""
Analytical Solution
"""


def exact_vibe(x, k):
    return v_l/np.sin(k)*np.sin(k*x)


"""
2nd Order FDM Solution
"""


def approx_vibe(k, nodes):

    # Defining Size of mesh
    dim = nodes-2

    # Defining dx
    dx = 1 / (nodes-1)

    # Defining kappa
    kappa = 2 - k**2*dx**2

    # Generating A Matrix
    a = np.zeros([dim, dim])
    a += kappa*np.eye(dim)
    a -= np.eye(dim, k=-1)
    a -= np.eye(dim, k=1)

    for i in range(dim):
        if i % nodes == 0 and i != 0:
            a[i-1][i] = 0
            a[i][i-1] = 0

    # Generating B Matrix
    b = np.zeros(dim)
    b[-1] = v_l

    vibes = np.linalg.solve(a, b)
    vibes = np.concatenate([[0], vibes])
```

```
    vibes = np.append(vibes, v_l)

    return vibes


"""
Analytical Derivative Solution
"""


def exact_dev(k):
    return abs(k*v_l/np.sin(k)*np.cos(k*length))


"""
2nd Order FDM Derivative Solution
"""


def approx_dev(k, nodes, vibes):
    dx = 1/(nodes-1)
    u_prime = (vibes[-1] - vibes[-2])/dx - dx/2*k**2*vibes[-1]
    return abs(u_prime)


"""
Richardson Extrapolation Function
"""


def q_rich_extr(q, q2, q4):
    q_extr = (q2**2 - q*q4)/(2*q2-q-q4)
    perc_err = abs(q_extr-q)
    return q_extr, perc_err


"""
Convergence Rate Function
"""


def get_beta(exact, approx, approx_2, h, h_2):
    a = np.log(abs(exact - approx))
    b = np.log(abs(exact - approx_2))
    c = np.log(h) - np.log(h_2)
    return -(a - b) / c


"""
Main Function
"""


def main():

    # Vibration Boundary Condition at Wall
    global v_l
    v_l = 100

    # Length of Bar
    global length
    length = 1

    # Vibration material property list
    k_list = [2**val for val in range(0, 7)]

    # Defining Number of Nodes List
    node_list = range(2, 11)

    """
    FDM Displacement Data Generation
    """

    # Initializing data structure
    fdm_vibe_data = DataStructure()

    for ki in k_list:
        for n in node_list:
```

```
        # Creating mesh
        n_nodes = 2**n
        x_vals = np.linspace(0, length, n_nodes)

        # Generating Data
        fdm_vibe_sol = approx_vibe(ki, n_nodes)

        # Storing Data
        fdm_vibe_data.add_data(n_nodes, ki, x_vals, fdm_vibe_sol)

"""
FDM Resonance Data Generation
"""

pass

"""
Displacement Absolute Error Calculations
"""

# Initializing variables
x_vals = None
err = None
fig, axs = plt.subplots(len(k_list), len(node_list))

for k_index, ki in enumerate(k_list):
    for node_index, n in enumerate(node_list):

        # Error tables
        vibe_err_tables = []

        # Recreating mesh
        n_nodes = 2**n

        # Displacement absolute error
        x_vals, approx_vals = fdm_vibe_data.return_data(n_nodes, ki)
        exact_vals = exact_vibe(x_vals, ki)
        err = exact_vals - approx_vals

        # Writing data to tables
        if n_nodes < 70:
            vibe_err_tables = np.transpose(np.array([x_vals, exact_vals, approx_vals, err]))
            print(f'\nError of 2nd Order FDM Solution, k = {ki}, nodes = {n_nodes}')
            print(tbl.tabulate(vibe_err_tables, headers=['Position (cm)', 'Exact', 'FDM', 'Abs. Error']))

        # Plotting data
        plt.figure(1)
        axs[k_index, node_index].plot(x_vals, err, label='FDM')
        axs[k_index, node_index].set_title(f'k={round(ki, 3)} | nodes={n_nodes}')
        axs[k_index, node_index].grid()

        # Plotting for highest number of nodes
    plt.figure(2)
    plt.plot(x_vals, err, label=f'k = {ki}')

# Configuring FDM plot
plt.figure(2)
plt.title('Absolute Error of FDM Solution vs. x (cm)')
plt.ylabel('Absolute Error (cm)')
plt.xlabel('x (cm)')
plt.grid()
plt.legend()

# Adds x and y labels
plt.figure(1)
for ax in axs.flat:
    ax.set(xlabel='x (cm)', ylabel='Displacement (cm)')

# Keeps labels that are on the outer parts of the assembly
for ax in axs.flat:
    ax.label_outer()

# Displaying FDM plots
plt.show()

"""
Resonance Error and Convergence
"""
```

```
    pass

    """
    Exact Displacement Plotting
    """

    # Points for Exact Solution
    x_vals = np.linspace(0, length, 1000)

    # Looping through material property list
    for ki in k_list:

        # Vibration values from exact solution
        v_vals = exact_vibe(x_vals, ki)

        # Plotting
        plt.plot(x_vals, v_vals, label=f'k = {ki}')

    # Configuring and displaying plot
    plt.title('Exact Solution - Displacement (cm) vs. x (cm)')
    plt.ylabel('Displacement (cm)')
    plt.xlabel('x (cm)')
    plt.grid()
    plt.legend()
    plt.show()

    """
    FDM Displacement Plotting
    """

    fig, axs = plt.subplots(len(k_list), len(node_list))
    for k_index, ki in enumerate(k_list):
        vibes = 'n/a'
        for node_index, n in enumerate(node_list):

            # Creating mesh
            n_nodes = 2**n

            # Extracting data
            x_vals, vibes = fdm_vibe_data.return_data(n_nodes, ki)
            e_x_vals = np.linspace(0, length, 1000)
            v_vals = exact_vibe(e_x_vals, ki)

            # Plotting data
            plt.figure(1)
            axs[k_index, node_index].plot(x_vals, vibes, label='FDM')
            axs[k_index, node_index].plot(e_x_vals, v_vals, label='Exact')
            axs[k_index, node_index].set_title(f'k={round(ki, 3)} | nodes={n_nodes}')
            axs[k_index, node_index].grid()

        # Plotting for highest number of nodes
        plt.figure(2)
        plt.plot(x_vals, vibes, label=f'k = {ki}')

    # Configuring FDM plot
    plt.figure(2)
    plt.title('FDM Solution - Displacement (cm) vs. x (cm)')
    plt.ylabel('Displacement (cm)')
    plt.xlabel('x (cm)')
    plt.grid()
    plt.legend()

    # Adds x and y labels
    plt.figure(1)
    for ax in axs.flat:
        ax.set(xlabel='x (cm)', ylabel='Displacement (cm)')

    # Keeps labels that are on the outer parts of the assembly
    for ax in axs.flat:
        ax.label_outer()

    # Adding legend to multi-plot
    axs[0, 0].legend()

    # Displaying FDM plots
    plt.show()

    """
    Exact Derivative Function
```

```
    """

    # Resonance Frequency Data
    rf_vals = []

    # Looping through material property list
    for ki in k_list:

        # Vibration values from exact solution
        rf_vals.append(exact_dev(np.sqrt(ki)))

    """
    Convergence and Richardson Extrapolation
    """

    rf_log_err = []
    rf_dx_err = []

    re_rf_log_err = []
    re_rf_dx_err = []

    for k_index, ki in enumerate(k_list):

        rf_error_data = []
        re_rf_error_data = []

        for node_index, n in enumerate(node_list):

            # Recreating mesh
            n_nodes = 2**n

            # Displacement absolute error
            x_vals, vibe_vals = fdm_vibe_data.return_data(n_nodes, ki)
            approx_vals = approx_dev(ki, n_nodes, vibe_vals)
            exact_vals = exact_dev(ki)
            err = abs(exact_vals - approx_vals)
            rf_error_data.append([n_nodes, length / n_nodes, exact_vals, approx_vals, err])

        # Resonance Frequency Convergence
        rf_error_data[0].append('n/a')
        for i in range(len(rf_error_data) - 1):
            rf_error_data[i + 1].append(
                get_beta(rf_error_data[i][2], rf_error_data[i][3], rf_error_data[i + 1][3],
                         rf_error_data[i][0], rf_error_data[i + 1][0]))
        print(f'\nConvergence of Derivative Function for k = {ki}:')
        print(tbl.tabulate(rf_error_data, headers=['Num. Elements', 'dx', 'Exact Resonance Freq.',
                                                   'Approx. Resonance Freq.', 'Absolute Error', 'Beta']))
        rf_log_err.append([np.log10(item[4] / 100) for item in rf_error_data])
        rf_dx_err.append([-np.log10(item[1]) for item in rf_error_data])

        # Richardson Extrapolation and Convergence
        for i in range(len(rf_error_data) - 2):
            q_extr, perc_err = q_rich_extr(rf_error_data[i][3], rf_error_data[i + 1][3],
                                           rf_error_data[i + 2][3])
            beta = get_beta(q_extr, rf_error_data[i][3], rf_error_data[i + 1][3],
                            rf_error_data[i][0], rf_error_data[i + 1][0])
            re_rf_error_data.append([rf_error_data[i][0], rf_error_data[i][1], q_extr, rf_error_data[i][3],
                                     perc_err, beta])
        print(f'\nConvergence of Extrapolated Derivative Function k = {ki}:')
        print(tbl.tabulate(re_rf_error_data,
                           headers=['Num. Elements', 'dx', 'Extrapolated Resonance Freq.',
                                    'Approx. Resonance Freq.', 'Absolute Error', 'Beta']))
        re_rf_log_err.append([np.log10(item[4] / 100) for item in re_rf_error_data])
        re_rf_dx_err.append([-np.log10(item[1]) for item in re_rf_error_data])

    # Plotting Convergence Graphs
    for i in range(len(k_list)):
        plt.plot(rf_dx_err[i], rf_log_err[i], label=f'k = {k_list[i]}')
    plt.title('Convergence of Derivative Function\n' + r'-$log_{10}$($\Delta$x) vs -$log_{10}$(Absolute
Error)')
    plt.xlabel(r'$-log_{10}$($\Delta$x)')
    plt.ylabel(r'$-log_{10}$(Absolute Error)')
    plt.grid()
    plt.legend()
    plt.show()

    for i in range(len(k_list)):
        plt.plot(re_rf_dx_err[i], re_rf_log_err[i], label=f'k = {k_list[i]}')
    plt.title('Convergence of Extrapolated Derivative Function\n' +
```

```
              r'-$log_{10}$($\Delta$x) vs -$log_{10}$(Absolute Error)')
    plt.xlabel(r'$-log_{10}$($\Delta$x)')
    plt.ylabel(r'$-log_{10}$(Absolute Error)')
    plt.grid()
    plt.legend()
    plt.show()

    """
    Exact Resonance Frequency Plotting
    """

    # Redefining Material Property List
    k_list = np.linspace(1, 100, 1000)

    # Resonance Frequency Data
    rf_vals = []

    # Looping through material property list
    for ki in k_list:

        # Vibration values from exact solution
        rf_vals.append(exact_dev(np.sqrt(ki)))

    # Plotting
    plt.plot(k_list, rf_vals)
    plt.title(f'Exact Derivative Function vs. k')
    plt.ylabel('Derivative Value')
    plt.xlabel('k value')
    plt.ylim(0, 1e5)
    plt.grid()
    plt.show()

    """
    FDM Resonance Frequency Plotting
    """

    # Looping through material property list
    for n in node_list:
        rf_fdm_vals = []
        rf_vals = []
        n_nodes = 2**n
        for ki in k_list:

            # Resonance freq. values from exact solution
            rf_vals.append(exact_dev(ki))

            # Resonance freq. values from approx. solution
            vibe_vals = approx_vibe(ki, n_nodes)
            rf_fdm_vals.append(approx_dev(ki, n_nodes, vibe_vals))

        # Plotting
        plt.plot(k_list, rf_fdm_vals)
        plt.plot(k_list, rf_vals)
        plt.title(f'Derivative Function vs. k\nn = {n_nodes}')
        plt.ylabel('Derivative Value')
        plt.xlabel('k value')
        plt.ylim(0, 1e5)
        plt.grid()
        plt.show()


if __name__ == "__main__":
    main()
```

10 Appendix B: Code Console Outputs

```
C:\Users\Andrew\Anaconda3\python.exe C:/Users/Andrew/PycharmProjects/TAMU/Fall2021/AERO430/AERO_430_Final_Project_Hollister.py

Error of 2nd Order FDM Solution, k = 1, nodes = 4
  Position (cm)     Exact       FDM    Abs. Error
---------------  --------  --------  ------------
       0            0         0          0
       0.333333  38.8837   38.9423    -0.0586501
       0.666667  73.4868   73.5577    -0.0709276
       1        100       100          0

Error of 2nd Order FDM Solution, k = 1, nodes = 8
  Position (cm)     Exact       FDM    Abs. Error
---------------  --------  --------  ------------
       0            0         0         0
       0.142857  16.9194   16.9245   -0.00506617
       0.285714  33.4941   33.5035   -0.00944212
       0.428571  49.3864   49.3988   -0.0124637
       0.571429  64.2725   64.286    -0.0135179
       0.714286  77.8491   77.8612   -0.0120671
       0.857143  89.8397   89.8474   -0.0076698
       1        100       100         0

Error of 2nd Order FDM Solution, k = 1, nodes = 16
  Position (cm)      Exact         FDM    Abs. Error
---------------  ---------  ---------  ------------
      0              0          0         0
      0.0666667      7.91677    7.91729  -0.000522878
      0.133333     15.7984    15.7994    -0.0010304
      0.2          23.6098    23.6113    -0.00150735
      0.266667     31.3163    31.3182    -0.00193873
      0.333333     38.8837    38.886     -0.00230996
      0.4          46.2783    46.2809    -0.00260693
      0.466667     53.4673    53.4701    -0.00281614
      0.533333     60.4188    60.4217    -0.00292483
      0.6          67.1018    67.1048    -0.00292109
      0.666667     73.4868    73.4896    -0.00279393
      0.733333     79.5452    79.5477    -0.00253339
      0.8          85.2502    85.2524    -0.00213069
      0.866667     90.5765    90.5781    -0.0015782
      0.933333     95.5004    95.5013    -0.000869621
      1           100        100          0

Error of 2nd Order FDM Solution, k = 1, nodes = 32
  Position (cm)      Exact         FDM    Abs. Error
---------------  ---------  ---------  ------------
      0              0          0         0
      0.0322581      3.83287    3.83293  -5.94297e-05
      0.0645161      7.66175    7.66187  -0.000118452
      0.0967742    11.4827    11.4828    -0.000176659
      0.129032     15.2916    15.2918    -0.000233647
      0.16129      19.0847    19.085     -0.000289011
      0.193548     22.8579    22.8582    -0.000342353
      0.225806     26.6073    26.6077    -0.000393276
      0.258065     30.329     30.3294    -0.000441389
      0.290323     34.0192    34.0196    -0.000486307
      0.322581     37.6739    37.6745    -0.000527648
      0.354839     41.2895    41.2901    -0.000565041
      0.387097     44.8621    44.8627    -0.000598121
      0.419355     48.388     48.3887    -0.00062653
      0.451613     51.8636    51.8643    -0.000649921
      0.483871     55.2852    55.2859    -0.000667956
      0.516129     58.6493    58.65      -0.000680308
      0.548387     61.9524    61.9531    -0.000686659
      0.580645     65.191     65.1917    -0.000686706
      0.612903     68.3618    68.3625    -0.000680157
      0.645161     71.4614    71.4621    -0.000666731
      0.677419     74.4867    74.4874    -0.000646163
      0.709677     77.4345    77.4351    -0.000618202
      0.741935     80.3018    80.3023    -0.00058261
      0.774194     83.0854    83.086     -0.000539167
      0.806452     85.7826    85.7831    -0.000487665
      0.83871      88.3906    88.391     -0.000427916
      0.870968     90.9066    90.907     -0.000359746
      0.903226     93.328     93.3283    -0.000282999
      0.935484     95.6523    95.6525    -0.000197537
      0.967742     97.8771    97.8772    -0.000103238
      1           100        100          0
```

Error of 2nd Order FDM Solution, k = 1, nodes = 64

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.015873	1.88626	1.88627	-7.08589e-06
0.031746	3.77205	3.77206	-1.416e-05
0.047619	5.65689	5.65691	-2.12106e-05
0.0634921	7.5403	7.54033	-2.82259e-05
0.0793651	9.42181	9.42184	-3.51943e-05
0.0952381	11.3009	11.301	-4.21039e-05
0.111111	13.1772	13.1773	-4.89431e-05
0.126984	15.0502	15.0503	-5.57003e-05
0.142857	16.9194	16.9194	-6.23639e-05
0.15873	18.7843	18.7844	-6.89222e-05
0.174603	20.6445	20.6446	-7.53638e-05
0.190476	22.4995	22.4995	-8.16772e-05
0.206349	24.3488	24.3489	-8.7851e-05
0.222222	26.192	26.1921	-9.38739e-05
0.238095	28.0285	28.0286	-9.97345e-05
0.253968	29.8581	29.8582	-0.000105422
0.269841	31.6801	31.6802	-0.000110925
0.285714	33.4941	33.4942	-0.000116232
0.301587	35.2996	35.2998	-0.000121333
0.31746	37.0963	37.0964	-0.000126216
0.333333	38.8837	38.8838	-0.000130871
0.349206	40.6612	40.6613	-0.000135288
0.365079	42.4285	42.4286	-0.000139456
0.380952	44.1851	44.1852	-0.000143364
0.396825	45.9306	45.9307	-0.000147002
0.412698	47.6645	47.6646	-0.00015036
0.428571	49.3864	49.3865	-0.000153428
0.444444	51.0958	51.096	-0.000156196
0.460317	52.7924	52.7925	-0.000158655
0.47619	54.4757	54.4758	-0.000160794
0.492063	56.1452	56.1454	-0.000162604
0.507937	57.8006	57.8008	-0.000164077
0.52381	59.4414	59.4416	-0.000165203
0.539683	61.0673	61.0675	-0.000165972
0.555556	62.6778	62.678	-0.000166377
0.571429	64.2725	64.2726	-0.000166408
0.587302	65.851	65.8511	-0.000166057
0.603175	67.4129	67.413	-0.000165316
0.619048	68.9578	68.958	-0.000164177
0.634921	70.4853	70.4855	-0.000162631
0.650794	71.9951	71.9953	-0.000160672
0.666667	73.4868	73.4869	-0.000158291
0.68254	74.9599	74.9601	-0.000155482
0.698413	76.4141	76.4143	-0.000152237
0.714286	77.8491	77.8493	-0.00014855
0.730159	79.2645	79.2647	-0.000144413
0.746032	80.6599	80.6601	-0.00013982
0.761905	82.035	82.0351	-0.000134766
0.777778	83.3894	83.3896	-0.000129244
0.793651	84.7228	84.723	-0.000123248
0.809524	86.0349	86.035	-0.000116773
0.825397	87.3253	87.3254	-0.000109813
0.84127	88.5937	88.5938	-0.000102364
0.857143	89.8397	89.8398	-9.44197e-05
0.873016	91.0632	91.0633	-8.59768e-05
0.888889	92.2637	92.2637	-7.70306e-05
0.904762	93.4409	93.441	-6.75769e-05
0.920635	94.5946	94.5947	-5.76118e-05
0.936508	95.7245	95.7245	-4.71319e-05
0.952381	96.8302	96.8303	-3.61337e-05
0.968254	97.9116	97.9116	-2.46141e-05
0.984127	98.9683	98.9683	-1.25704e-05
1	100	100	0

Error of 2nd Order FDM Solution, k = 2, nodes = 4

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.333333	68.0052	70.4348	-2.42955
0.666667	106.889	109.565	-2.67633
1	100	100	0

Error of 2nd Order FDM Solution, k = 2, nodes = 8

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.142857	30.9957	31.1979	-0.202248
0.285714	59.4783	59.8491	-0.37082
0.428571	83.1384	83.6146	-0.476181
0.571429	100.058	100.554	-0.496627
0.714286	108.865	109.286	-0.421118
0.857143	108.845	109.096	-0.250942
1	100	100	0

Error of 2nd Order FDM Solution, k = 2, nodes = 4

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.333333	68.0052	70.4348	-2.42955
0.666667	106.889	109.565	-2.67633
1	100	100	0

Error of 2nd Order FDM Solution, k = 2, nodes = 8

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.142857	30.9957	31.1979	-0.202248
0.285714	59.4783	59.8491	-0.37082
0.428571	83.1384	83.6146	-0.476181
0.571429	100.058	100.554	-0.496627
0.714286	108.865	109.286	-0.421118
0.857143	108.845	109.096	-0.250942
1	100	100	0

Error of 2nd Order FDM Solution, k = 2, nodes = 16

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.0666667	14.6199	14.6407	-0.0207494
0.133333	28.9803	29.0211	-0.0407451
0.2	42.8263	42.8855	-0.0592537
0.266667	55.912	55.9876	-0.0755815
0.333333	68.0052	68.0943	-0.089094
0.4	78.8912	78.9905	-0.0992326
0.466667	88.3768	88.4824	-0.105531
0.533333	96.2936	96.4012	-0.107626
0.6	102.501	102.606	-0.105274
0.666667	106.889	106.987	-0.0983516
0.733333	109.379	109.466	-0.0868677
0.8	109.928	109.999	-0.0709603
0.866667	108.526	108.576	-0.0508979
0.933333	105.196	105.224	-0.0270741
1	100	100	0

Error of 2nd Order FDM Solution, k = 2, nodes = 32

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.0322581	7.09024	7.0926	-0.00235539
0.0645161	14.151	14.1557	-0.00469074
0.0967742	21.1528	21.1598	-0.00698614
0.129032	28.0667	28.0759	-0.00922193
0.16129	34.8637	34.8751	-0.0113788
0.193548	41.5157	41.5292	-0.013438
0.225806	47.995	48.0104	-0.0153813
0.258065	54.2746	54.2917	-0.0171914
0.290323	60.3283	60.3471	-0.0188515
0.322581	66.131	66.1513	-0.0203461
0.354839	71.6585	71.6802	-0.0216605
0.387097	76.8879	76.9107	-0.0227813
0.419355	81.7974	81.8211	-0.0236963
0.451613	86.3665	86.3909	-0.0243946
0.483871	90.5762	90.6011	-0.0248667
0.516129	94.4091	94.4342	-0.0251045
0.548387	97.8491	97.8742	-0.0251015
0.580645	100.882	100.907	-0.0248529
0.612903	103.495	103.52	-0.0243551
0.645161	105.678	105.701	-0.0236066
0.677419	107.42	107.443	-0.0226072
0.709677	108.716	108.738	-0.0213587
0.741935	109.56	109.58	-0.0198644
0.774194	109.947	109.966	-0.0181292
0.806452	109.878	109.894	-0.0161598
0.83871	109.35	109.364	-0.0139646
0.870968	108.368	108.38	-0.0115534
0.903226	106.935	106.944	-0.00893766
0.935484	105.058	105.064	-0.00613036
0.967742	102.743	102.746	-0.00314588
1	100	100	0

Error of 2nd Order FDM Solution, k = 2, nodes = 64

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.015873	3.49068	3.49096	-0.000280755
0.031746	6.97785	6.97841	-0.000560931
0.047619	10.458	10.4588	-0.000839952
0.0634921	13.9276	13.9287	-0.00111724
0.0793651	17.3831	17.3845	-0.00139222
0.0952381	20.8212	20.8228	-0.00166433
0.111111	24.2382	24.2402	-0.001933
0.126984	27.6309	27.6331	-0.00219768
0.142857	30.9957	30.9981	-0.00245779
0.15873	34.3292	34.3319	-0.00271281
0.174603	37.6282	37.6312	-0.00296219
0.190476	40.8892	40.8924	-0.00320539
0.206349	44.1091	44.1125	-0.00344191
0.222222	47.2845	47.2881	-0.00367122
0.238095	50.4122	50.4161	-0.00389284
0.253968	53.4891	53.4932	-0.00410626
0.269841	56.5122	56.5165	-0.00431101
0.285714	59.4783	59.4828	-0.00450664
0.301587	62.3844	62.3891	-0.0046927
0.31746	65.2277	65.2325	-0.00486874
0.333333	68.0052	68.0103	-0.00503436
0.349206	70.7142	70.7194	-0.00518914
0.365079	73.352	73.3573	-0.00533271
0.380952	75.9158	75.9213	-0.0054647
0.396825	78.4032	78.4088	-0.00558476
0.412698	80.8115	80.8172	-0.00569255
0.428571	83.1384	83.1442	-0.00578777
0.444444	85.3815	85.3874	-0.00587011
0.460317	87.5386	87.5445	-0.00593932
0.47619	89.6074	89.6134	-0.00599513
0.492063	91.586	91.592	-0.00603731
0.507937	93.4723	93.4783	-0.00606565
0.52381	95.2643	95.2704	-0.00607998
0.539683	96.9604	96.9665	-0.00608011
0.555556	98.5588	98.5648	-0.00606591
0.571429	100.058	100.064	-0.00603725
0.587302	101.456	101.462	-0.00599404
0.603175	102.752	102.758	-0.0059362
0.619048	103.944	103.95	-0.00586368
0.634921	105.032	105.038	-0.00577646
0.650794	106.014	106.02	-0.00567452
0.666667	106.889	106.894	-0.0055579
0.68254	107.656	107.662	-0.00542662
0.698413	108.315	108.32	-0.00528077
0.714286	108.865	108.87	-0.00512042
0.730159	109.305	109.309	-0.0049457
0.746032	109.634	109.639	-0.00475675
0.761905	109.854	109.858	-0.00455372
0.777778	109.962	109.967	-0.0043368
0.793651	109.96	109.964	-0.00410621
0.809524	109.847	109.851	-0.00386217
0.825397	109.623	109.627	-0.00360494
0.84127	109.289	109.292	-0.0033348
0.857143	108.845	108.848	-0.00305206
0.873016	108.291	108.294	-0.00275702
0.888889	107.628	107.63	-0.00245004
0.904762	106.856	106.858	-0.00213147
0.920635	105.977	105.979	-0.00180172
0.936508	104.991	104.992	-0.00146118
0.952381	103.899	103.9	-0.00111029
0.968254	102.702	102.703	-0.000749479
0.984127	101.402	101.403	-0.000379222
1	100	100	0

Error of 2nd Order FDM Solution, k = 4, nodes = 4

Position (cm)	Exact	FDM	Abs. Error
0	-0	0	-0
0.333333	-128.427	-105.195	-23.2321
0.666667	-60.4217	-23.3766	-37.045
1	100	100	0

Error of 2nd Order FDM Solution, k = 4, nodes = 8

Position (cm)	Exact	FDM	Abs. Error
0	-0	0	-0
0.142857	-71.4631	-69.0971	-2.36599
0.285714	-120.219	-115.632	-4.58751
0.428571	-130.777	-124.409	-6.36768
0.571429	-99.7813	-92.5633	-7.218
0.714286	-37.0809	-30.4925	-6.58841
0.857143	37.4015	41.535	-4.13341
1	100	100	0

Error of 2nd Order FDM Solution, k = 4, nodes = 16

Position (cm)	Exact	FDM	Abs. Error
0	-0	0	-0
0.0666667	-34.8198	-34.5671	-0.25271
0.133333	-67.1782	-66.6761	-0.502088
0.2	-94.7878	-94.0438	-0.744004
0.266667	-115.697	-114.724	-0.972862
0.333333	-128.427	-127.246	-1.18118
0.4	-132.079	-130.719	-1.35949
0.466667	-126.393	-124.897	-1.49665
0.533333	-111.774	-110.193	-1.58052
0.6	-89.2522	-87.6533	-1.59899
0.666667	-60.4217	-58.8804	-1.54127
0.733333	-27.3198	-25.9205	-1.39936
0.8	7.71326	8.88267	-1.16941
0.866667	42.2011	43.0542	-0.853072
0.933333	73.7057	74.164	-0.458325
1	100	100	0

Error of 2nd Order FDM Solution, k = 4, nodes = 32

Position (cm)	Exact	FDM	Abs. Error
0	-0	0	-0
0.0322581	-17.0024	-16.9735	-0.0289387
0.0645161	-33.7221	-33.6643	-0.0577882
0.0967742	-49.8811	-49.7947	-0.0864541
0.129032	-65.2108	-65.096	-0.114832
0.16129	-79.4563	-79.3135	-0.142804
0.193548	-92.3807	-92.2105	-0.170232
0.225806	-103.769	-103.572	-0.19696
0.258065	-113.432	-113.21	-0.222803
0.290323	-121.21	-120.962	-0.247556
0.322581	-126.972	-126.701	-0.270986
0.354839	-130.623	-130.33	-0.292835
0.387097	-132.102	-131.789	-0.312825
0.419355	-131.384	-131.054	-0.330656
0.451613	-128.483	-128.137	-0.346015
0.483871	-123.445	-123.086	-0.35858
0.516129	-116.355	-115.987	-0.368024
0.548387	-107.33	-106.956	-0.374028
0.580645	-96.5207	-96.1445	-0.376282
0.612903	-84.1067	-83.7322	-0.374499
0.645161	-70.2943	-69.9258	-0.368424
0.677419	-55.3131	-54.9553	-0.357837
0.709677	-39.4123	-39.0697	-0.342569
0.741935	-22.8562	-22.5337	-0.322508
0.774194	-5.9201	-5.62249	-0.297605
0.806452	11.1144	11.3823	-0.267884
0.83871	27.9642	28.1976	-0.233446
0.870968	44.349	44.5434	-0.194475
0.903226	59.9964	60.1477	-0.151243
0.935484	74.6463	74.7505	-0.104108
0.967742	88.0552	88.1087	-0.053516
1	100	100	0

Error of 2nd Order FDM Solution, k = 4, nodes = 64

Position (cm)	Exact	FDM	Abs. Error
0	-0	0	-0
0.015873	-8.38388	-8.38042	-0.00345643
0.031746	-16.734	-16.7271	-0.00691028
0.047619	-25.0166	-25.0063	-0.0103589
0.0634921	-33.1985	-33.1847	-0.0137997
0.0793651	-41.2465	-41.2293	-0.0172298
0.0952381	-49.1284	-49.1077	-0.0206463
0.111111	-56.8122	-56.7882	-0.024046
0.126984	-64.2671	-64.2397	-0.0274258
0.142857	-71.4631	-71.4323	-0.030782
0.15873	-78.371	-78.3369	-0.0341109
0.174603	-84.9631	-84.9257	-0.0374084
0.190476	-91.2128	-91.1721	-0.0406702
0.206349	-97.0949	-97.0511	-0.0438915
0.222222	-102.586	-102.539	-0.0470673
0.238095	-107.663	-107.613	-0.0501924
0.253968	-112.307	-112.254	-0.0532608
0.269841	-116.498	-116.442	-0.0562666
0.285714	-120.219	-120.16	-0.0592034
0.301587	-123.456	-123.394	-0.0620642
0.31746	-126.196	-126.131	-0.0648421
0.333333	-128.427	-128.359	-0.0675294
0.349206	-130.14	-130.07	-0.0701184
0.365079	-131.329	-131.257	-0.0726009
0.380952	-131.989	-131.914	-0.0749686
0.396825	-132.117	-132.04	-0.0772128
0.412698	-131.712	-131.633	-0.0793246
0.428571	-130.777	-130.696	-0.0812951
0.444444	-129.315	-129.231	-0.0831148
0.460317	-127.331	-127.246	-0.0847746
0.47619	-124.834	-124.748	-0.0862651
0.492063	-121.835	-121.747	-0.0875768
0.507937	-118.344	-118.255	-0.0887005
0.52381	-114.376	-114.287	-0.0896269
0.539683	-109.948	-109.858	-0.0903468
0.555556	-105.076	-104.985	-0.0908514
0.571429	-99.7813	-99.6901	-0.091132
0.587302	-94.0841	-93.9929	-0.0911803
0.603175	-88.0078	-87.9168	-0.0909885
0.619048	-81.5768	-81.4863	-0.0905491
0.634921	-74.8171	-74.7273	-0.0898551
0.650794	-67.7559	-67.667	-0.0889001
0.666667	-60.4217	-60.334	-0.0876785
0.68254	-52.8439	-52.7577	-0.0861853
0.698413	-45.0532	-44.9688	-0.0844163
0.714286	-37.0809	-36.9986	-0.0823679
0.730159	-28.9592	-28.8792	-0.0800376
0.746032	-20.7208	-20.6434	-0.077424
0.761905	-12.3989	-12.3244	-0.0745263
0.777778	-4.02705	-3.95571	-0.0713449
0.793651	4.36105	4.42893	-0.0678814
0.809524	12.7316	12.7957	-0.0641383
0.825397	21.0508	21.1109	-0.0601195
0.84127	29.2852	29.341	-0.0558297
0.857143	37.4015	37.4528	-0.0512753
0.873016	45.3672	45.4137	-0.0464635
0.888889	53.15	53.1914	-0.041403
0.904762	60.7186	60.7548	-0.0361036
0.920635	68.0426	68.0732	-0.0305764
0.936508	75.0923	75.1172	-0.0248339
0.952381	81.8395	81.8583	-0.0188895
0.968254	88.2568	88.2695	-0.0127582
0.984127	94.3184	94.3249	-0.00645598
1	100	100	0

Error of 2nd Order FDM Solution, k = 8, nodes = 4

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.333333	46.2191	3.98034	42.2388
0.666667	-82.2078	-20.344	-61.8638
1	100	100	0

Error of 2nd Order FDM Solution, k = 8, nodes = 8

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.142857	91.9609	118.856	-26.8947
0.285714	76.327	82.4712	-6.14429
0.428571	-28.61	-61.6307	33.0206
0.571429	-100.073	-125.235	25.1623
0.714286	-54.45	-25.2673	-29.1826
0.857143	54.88	107.703	-52.8229
1	100	100	0

```
Error of 2nd Order FDM Solution, k = 8, nodes = 16
  Position (cm)       Exact          FDM     Abs. Error
---------------  ---------  ---------  ------------
      0             0          0            0
      0.0666667    51.3875    52.9711     -1.58363
      0.133333     88.5013    90.8749     -2.3736
      0.2         101.033    102.93       -1.89733
      0.266667     85.5003    85.7069     -0.206598
      0.333333     46.2191    44.1052      2.11393
      0.4          -5.9002   -10.0421      4.14186
      0.466667    -56.3806   -61.3329      4.95224
      0.533333    -91.2005   -95.1779      3.97744
      0.6        -100.688   -101.95        1.26217
      0.666667    -82.2078   -79.7232     -2.4846
      0.733333    -40.8931   -34.8194     -6.07368
      0.8          11.7803    19.9885     -8.20825
      0.866667     61.1815    69.1109     -7.92935
      0.933333     93.5886    98.575      -4.98638
      1           100        100            0

Error of 2nd Order FDM Solution, k = 8, nodes = 32
  Position (cm)        Exact          FDM    Abs. Error
---------------  ----------  ----------  ------------
      0             0           0            0
      0.0322581    25.7955     25.9579      -0.162378
      0.0645161    49.8826     50.187       -0.30443
      0.0967742    70.666      71.0738      -0.407811
      0.129032     86.7693     87.2273      -0.457973
      0.16129      97.126      97.5717      -0.445637
      0.193548    101.05      101.418       -0.367804
      0.225806     98.282      98.5103      -0.228211
      0.258065     89.0048     89.042       -0.0371755
      0.290323     73.8329     73.6437       0.189159
      0.322581     53.7711     53.341        0.430125
      0.354839     30.1481     29.4859       0.662274
      0.387097      4.52854     3.6671       0.861437
      0.419355    -21.391     -22.3959       1.0049
      0.451613    -45.8938    -46.9674       1.07355
      0.483871    -67.3572    -68.411        1.05378
      0.516129    -84.3596    -85.2986       0.938993
      0.548387    -95.775     -96.5055       0.73056
      0.580645   -100.847    -101.285        0.438153
      0.612903    -99.2407    -99.3201       0.0793767
      0.645161    -91.0615    -90.7402      -0.321284
      0.677419    -76.8514    -76.1173      -0.73413
      0.709677    -57.5516    -56.4252      -1.12643
      0.741935    -34.4403    -32.9753      -1.46493
      0.774194     -9.04798    -7.32941     -1.71857
      0.806452     16.9435     18.8047      -1.8611
      0.83871      41.8129     43.6864      -1.87344
      0.870968     63.9131     65.6587      -1.74559
      0.903226     81.7804     83.2583      -1.47791
      0.935484     94.2315     95.3132      -1.08166
      0.967742    100.442     101.02        -0.578617
      1           100         100            0
```

Error of 2nd Order FDM Solution, k = 8, nodes = 64

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.015873	12.8005	12.8194	-0.0189072
0.031746	25.3949	25.4322	-0.0372324
0.047619	37.5804	37.6348	-0.0544072
0.0634921	49.1607	49.2306	-0.0698908
0.0793651	59.9493	60.0325	-0.0831828
0.0952381	69.7726	69.8664	-0.0938353
0.111111	78.4723	78.5737	-0.101464
0.126984	85.9083	86.0141	-0.105756
0.142857	91.9609	92.0674	-0.106483
0.15873	96.5327	96.6362	-0.103502
0.174603	99.5499	99.6467	-0.0967609
0.190476	100.964	101.05	-0.0863037
0.206349	100.752	100.825	-0.0722684
0.222222	98.9182	98.9731	-0.0548858
0.238095	95.4912	95.5257	-0.034476
0.253968	90.5264	90.5378	-0.0114423
0.269841	84.1039	84.0901	0.0137363
0.285714	76.327	76.2864	0.0405149
0.301587	67.3209	67.2526	0.0682931
0.31746	57.2308	57.1344	0.096428
0.333333	46.2191	46.0949	0.124247
0.349206	34.4631	34.3121	0.151064
0.365079	22.1522	21.976	0.176191
0.380952	9.48447	9.28551	0.198957
0.396825	-3.33595	-3.55467	0.218721
0.412698	-16.1027	-16.3375	0.234885
0.428571	-28.61	-28.857	0.246912
0.444444	-40.6567	-40.9111	0.254339
0.460317	-52.0487	-52.3055	0.256784
0.47619	-62.6025	-62.8565	0.253962
0.492063	-72.1482	-72.3939	0.245688
0.507937	-80.5321	-80.764	0.23189
0.52381	-87.6191	-87.8317	0.212609
0.539683	-93.2952	-93.4832	0.188002
0.555556	-97.469	-97.6273	0.158344
0.571429	-100.073	-100.197	0.124021
0.587302	-101.066	-101.151	0.0855308
0.603175	-100.431	-100.474	0.043473
0.619048	-98.1788	-98.1773	-0.00146072
0.634921	-94.3457	-94.2972	-0.0484971
0.650794	-88.9933	-88.8965	-0.0967946
0.666667	-82.2078	-82.0623	-0.145459
0.68254	-74.0985	-73.9049	-0.193557
0.698413	-64.7959	-64.5558	-0.24014
0.714286	-54.45	-54.1657	-0.284253
0.730159	-43.2272	-42.9022	-0.324962
0.746032	-31.3083	-30.9469	-0.361367
0.761905	-18.8852	-18.4926	-0.392623
0.777778	-6.15807	-5.74011	-0.417957
0.793651	6.66827	7.10496	-0.436685
0.809524	19.3872	19.8355	-0.448227
0.825397	31.794	32.2461	-0.452121
0.84127	43.6888	44.1368	-0.448037
0.857143	54.88	55.3158	-0.435782
0.873016	65.1875	65.6028	-0.415311
0.888889	74.4452	74.832	-0.386731
0.904762	82.5042	82.8545	-0.350304
0.920635	89.2345	89.5409	-0.306441
0.936508	94.5279	94.7836	-0.255704
0.952381	98.299	98.4978	-0.198797
0.968254	100.487	100.624	-0.136556
0.984127	101.057	101.127	-0.0699359
1	100	100	0

Error of 2nd Order FDM Solution, k = 16, nodes = 4

Position (cm)	Exact	FDM	Abs. Error
0	-0	0	-0
0.333333	282.501	0.143203	282.358
0.666667	328.72	-3.78693	332.507
1	100	100	0

Error of 2nd Order FDM Solution, k = 16, nodes = 8

Position (cm)	Exact	FDM	Abs. Error
0	-0	0	-0
0.142857	-262.292	0.155072	-262.447
0.285714	343.894	-0.500029	344.394
0.428571	-188.591	1.45727	-190.048
0.571429	-96.6301	-4.19891	-92.4311
0.714286	315.284	12.0821	303.202
0.857143	-316.742	-34.7596	-281.982
1	100	100	0

Error of 2nd Order FDM Solution, k = 16, nodes = 16

Position (cm)	Exact	FDM	Abs. Error
0	-0	0	-0
0.0666667	-304.128	-98.0741	-206.054
0.133333	-293.816	-84.5617	-209.254
0.2	20.2756	25.1632	-4.88755
0.266667	313.404	106.258	207.146
0.333333	282.501	66.4548	216.046
0.4	-40.4821	-48.9591	8.47707
0.466667	-321.61	-108.668	-212.942
0.533333	-270.223	-44.7372	-225.486
0.6	60.5505	70.095	-9.54457
0.666667	328.72	105.175	223.545
0.733333	257.023	20.5889	236.434
0.8	-80.4124	-87.4225	7.01012
0.866667	-334.709	-95.9665	-238.742
0.933333	-242.947	4.67799	-247.625
1	100	100	0

Error of 2nd Order FDM Solution, k = 16, nodes = 32

Position (cm)	Exact	FDM	Abs. Error
0	-0	0	-0
0.0322581	-171.418	-108.996	-62.422
0.0645161	-298.176	-188.956	-109.22
0.0967742	-347.252	-218.581	-128.671
0.129032	-305.858	-189.978	-115.881
0.16129	-184.78	-110.767	-74.0136
0.193548	-15.562	-2.0487	-13.5133
0.225806	157.711	107.215	50.4957
0.258065	289.895	187.918	101.977
0.290323	346.554	218.561	127.993
0.322581	312.926	190.983	121.943
0.354839	197.772	112.528	85.2439
0.387097	31.0927	4.09722	26.9955
0.419355	-143.687	-105.425	-38.262
0.451613	-281.032	-186.863	-94.1691
0.483871	-345.161	-218.523	-126.638
0.516129	-319.365	-191.971	-127.395
0.548387	-210.366	-114.279	-96.0869
0.580645	-46.561	-6.14538	-40.4156
0.612903	129.375	103.626	25.749
0.645161	271.605	185.792	85.8127
0.677419	343.074	218.465	124.609
0.709677	325.163	192.942	132.221
0.741935	222.538	116.021	106.517
0.774194	61.9358	8.19299	53.7428
0.806452	-114.803	-101.817	-12.9853
0.83871	-261.632	-184.705	-76.9272
0.870968	-340.299	-218.389	-121.91
0.903226	-330.308	-193.896	-136.412
0.935484	-234.263	-117.752	-116.511
0.967742	-77.1861	-10.2399	-66.9462
1	100	100	0

Error of 2nd Order FDM Solution, k = 16, nodes = 64

Position (cm)	Exact	FDM	Abs. Error
0	-0	0	-0
0.015873	-87.2678	-76.5455	-10.7223
0.031746	-168.937	-148.154	-20.7832
0.047619	-239.768	-210.206	-29.562
0.0634921	-295.217	-258.7	-36.517
0.0793651	-331.727	-290.508	-41.2188
0.0952381	-346.955	-303.579	-43.3767
0.111111	-339.925	-297.068	-42.8569
0.126984	-311.087	-271.397	-39.6904
0.142857	-262.292	-228.22	-34.0715
0.15873	-196.669	-170.324	-26.3458
0.174603	-118.43	-101.441	-16.9887
0.190476	-32.5927	-26.0158	-6.57691
0.206349	55.3356	51.0877	4.24786
0.222222	139.714	124.896	14.8178
0.238095	215.129	190.649	24.4803
0.253968	276.743	244.104	32.6382
0.269841	320.602	281.815	38.7868
0.285714	343.894	301.349	42.5445
0.301587	345.123	301.446	43.677
0.31746	324.212	282.1	42.1118
0.333333	282.501	244.558	37.9425
0.349206	222.666	191.243	31.4237
0.365079	148.547	125.592	22.9551
0.380952	64.8977	51.8405	13.0573
0.396825	-22.915	-25.2547	2.33968
0.412698	-109.258	-100.721	-8.53673
0.428571	-188.591	-169.691	-18.9003
0.444444	-255.825	-227.715	-28.1101
0.460317	-306.648	-271.052	-35.5953
0.47619	-337.797	-296.907	-40.8906
0.492063	-347.276	-303.611	-43.6654
0.507937	-334.475	-290.731	-43.7439
0.52381	-300.217	-259.1	-41.1166
0.539683	-246.698	-210.757	-35.9412
0.555556	-177.353	-148.82	-28.5329
0.571429	-96.6301	-77.2844	-19.3456
0.587302	-9.70775	-0.763782	-8.94396
0.603175	77.8374	75.8061	2.03122
0.619048	160.389	147.487	12.9023
0.634921	232.651	209.654	22.9967
0.650794	289.987	258.299	31.6883
0.666667	328.72	290.284	38.4363
0.68254	346.364	303.545	42.8189
0.698413	341.787	297.228	44.5595
0.714286	315.284	271.739	43.5443
0.730159	268.553	228.724	39.8295
0.746032	204.594	170.956	33.6386
0.761905	127.51	102.161	25.3489
0.777778	42.2448	26.7767	15.4682
0.793651	-45.7302	-50.3346	4.60445
0.809524	-130.771	-124.199	-6.5721
0.825397	-207.423	-190.053	-17.37
0.84127	-270.768	-243.649	-27.1193
0.857143	-316.742	-281.529	-35.213
0.873016	-342.396	-301.25	-41.1451
0.888889	-346.083	-301.541	-44.5418
0.904762	-327.568	-282.383	-45.1853
0.920635	-288.039	-245.011	-43.0277
0.936508	-230.03	-191.836	-38.1944
0.952381	-157.264	-126.287	-30.9771
0.968254	-74.4091	-52.5929	-21.8163
0.984127	13.2195	24.4935	-11.274
1	100	100	0

Error of 2nd Order FDM Solution, k = 32, nodes = 4

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.333333	-171.627	0.00800431	-171.635
0.666667	110.874	-0.894704	111.769
1	100	100	0

Error of 2nd Order FDM Solution, k = 32, nodes = 8

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.142857	-179.549	2.22645e-06	-179.549
0.285714	50.4512	-4.20754e-05	50.4512
0.428571	165.373	0.000792912	165.372
0.571429	-96.919	-0.0149423	-96.904
0.714286	-138.14	0.281587	-138.421
0.857143	135.735	-5.30648	141.041
1	100	100	0

Error of 2nd Order FDM Solution, k = 32, nodes = 16

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.0666667	153.403	0.00293942	153.4
0.133333	-163.63	-0.0074988	-163.623
0.2	21.1359	0.0161908	21.1197
0.266667	141.085	-0.0338058	141.119
0.333333	-171.627	0.0700516	-171.697
0.4	41.9838	-0.144904	42.1287
0.466667	126.844	0.299614	126.544
0.533333	-177.284	-0.619444	-176.665
0.6	62.2594	1.28066	60.9787
0.666667	110.874	-2.64765	113.522
0.733333	-180.525	5.4738	-185.999
0.8	81.6864	-11.3166	93.003
0.866667	93.393	23.3962	69.9968
0.933333	-181.306	-48.3696	-132.936
1	100	100	0

Error of 2nd Order FDM Solution, k = 32, nodes = 32

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.0322581	155.68	110.02	45.6598
0.0645161	159.691	102.807	56.8832
0.0967742	8.12501	-13.9523	22.0773
0.129032	-151.356	-115.845	-35.5112
0.16129	-163.381	-94.2983	-69.0824
0.193548	-16.2337	27.7286	-43.9623
0.225806	146.729	120.209	26.5197
0.258065	166.743	84.6	82.1427
0.290323	24.3098	-41.1552	65.4649
0.322581	-141.807	-123.057	-18.7494
0.354839	-169.77	-73.8348	-95.935
0.387097	-32.337	54.0627	-86.3998
0.419355	136.6	124.353	12.2463
0.451613	172.456	62.1384	110.318
0.483871	40.2994	-66.2885	106.588
0.516129	-131.118	-124.081	-7.03703
0.548387	-174.796	-49.6584	-125.137
0.580645	-48.1807	77.6783	-125.859
0.612903	125.374	122.244	3.1293
0.645161	176.784	36.5521	140.232
0.677419	55.9654	-88.0885	144.054
0.709677	-119.377	-118.866	-0.511428
0.741935	-178.418	-22.9849	-155.433
0.774194	-63.6376	97.3878	-161.025
0.806452	113.141	113.988	-0.84714
0.83871	179.694	9.12774	170.566
0.870968	71.182	-105.459	176.641
0.903226	-106.678	-107.673	0.995395
0.935484	-180.608	4.84449	-185.453
0.967742	-78.5835	112.2	-190.783
1	100	100	0

Error of 2nd Order FDM Solution, k = 32, nodes = 64

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.015873	88.2031	60.9042	27.2989
0.031746	154.135	106.095	48.0397
0.047619	181.148	123.914	57.234
0.0634921	162.421	109.762	52.6582
0.0793651	102.682	67.2926	35.3898
0.0952381	17.0168	7.46128	9.55555
0.111111	-72.9454	-54.295	-18.6504
0.126984	-144.489	-102.043	-42.4458
0.142857	-179.549	-123.464	-56.0847
0.15873	-169.273	-113.032	-56.2413
0.174603	-116.256	-73.437	-42.8185
0.190476	-33.8835	-14.8955	-18.988
0.206349	57.0441	47.489	9.55512
0.222222	133.568	97.6213	35.9467
0.238095	176.366	122.567	53.7986
0.253968	174.632	115.891	58.7405
0.269841	128.803	79.3151	49.4877
0.285714	50.4512	22.2757	28.1754
0.301587	-40.6394	-40.5108	-0.12861
0.31746	-121.468	-92.8455	-28.623
0.333333	-171.627	-121.226	-50.4006
0.349206	-178.449	-118.331	-60.1188
0.365079	-140.213	-84.9057	-55.3077
0.380952	-66.5736	-29.5752	-36.9984
0.396825	23.876	33.3857	-9.50964
0.412698	108.297	87.733	20.5639
0.428571	165.373	119.445	45.9276
0.444444	180.692	120.341	60.3514
0.460317	150.387	90.1884	60.1983
0.47619	82.1086	36.7674	45.3412
0.492063	-6.90197	-26.1395	19.2375
0.507937	-94.1698	-82.3025	-11.8673
0.52381	-157.66	-117.231	-40.4283
0.539683	-181.341	-121.915	-59.4258
0.555556	-159.233	-95.1441	-64.0888
0.571429	-96.919	-43.8263	-53.0926
0.587302	-10.133	18.7986	-28.9316
0.603175	79.2116	76.5735	2.63808
0.619048	148.555	114.592	33.9627
0.634921	180.389	123.047	57.342
0.650794	166.674	99.7549	66.9191
0.666667	110.874	50.7264	60.1478
0.68254	27.0785	-11.3895	38.468
0.698413	-63.5544	-70.5669	7.01252
0.714286	-138.14	-111.538	-26.6018
0.730159	-177.845	-123.732	-54.1125
0.746032	-172.644	-104.004	-68.6403
0.761905	-123.851	-57.4425	-66.4084
0.777778	-43.7851	3.93914	-47.7243
0.793651	47.3364	64.3045	-16.9681
0.809524	126.505	108.079	18.4261
0.825397	173.732	123.97	49.7622
0.84127	177.091	107.876	69.215
0.857143	135.735	63.9503	71.7843
0.873016	60.1053	3.52551	56.5798
0.888889	-30.7006	-57.8089	27.1083
0.904762	-113.755	-104.229	-9.52604
0.920635	-168.086	-123.757	-44.3282
0.936508	-179.975	-111.357	-68.6181
0.952381	-146.421	-70.2263	-76.1943
0.968254	-75.8951	-10.9774	-64.9177
0.984127	13.7939	51.1037	-37.3097
1	100	100	0

Error of 2nd Order FDM Solution, k = 64, nodes = 4

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.333333	66.4535	0.000487071	66.453
0.666667	-105.173	-0.220697	-104.953
1	100	100	0

Error of 2nd Order FDM Solution, k = 64, nodes = 8

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.142857	30.2384	3.39192e-10	30.2384
0.285714	-58.0893	-2.76753e-08	-58.0893
0.428571	81.3539	2.25774e-06	81.3539
0.571429	-98.1952	-0.000184186	-98.195
0.714286	107.284	0.0150258	107.269
0.857143	-107.901	-1.2258	-106.676
1	100	100	0

Error of 2nd Order FDM Solution, k = 64, nodes = 16

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.0666667	-98.0733	1.22107e-15	-98.0733
0.133333	84.5607	-1.97868e-14	84.5607
0.2	25.1634	3.19412e-13	25.1634
0.266667	-106.257	-5.15611e-12	-106.257
0.333333	66.4535	8.32325e-11	66.4535
0.4	48.9595	-1.34358e-09	48.9595
0.466667	-108.667	2.16887e-08	-108.667
0.533333	44.7356	-3.50111e-07	44.7356
0.6	70.0955	5.65166e-06	70.0955
0.666667	-105.173	-9.12319e-05	-105.173
0.733333	20.587	0.00147271	20.5855
0.8	87.4228	-0.0237732	87.4466
0.866667	-95.9646	0.383759	-96.3483
0.933333	-4.68026	-6.19483	1.51457
1	100	100	0

Error of 2nd Order FDM Solution, k = 64, nodes = 32

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.0322581	95.7121	1.59637e-05	95.7121
0.0645161	-90.7168	-3.61135e-05	-90.7168
0.0967742	-9.72983	6.57332e-05	-9.72989
0.129032	99.9388	-0.00011259	99.939
0.16129	-84.9932	0.000188971	-84.9933
0.193548	-19.3815	-0.000314905	-19.3812
0.225806	103.363	0.000523416	103.363
0.258065	-78.587	-0.00086918	-78.5862
0.290323	-28.8776	0.00144287	-28.879
0.322581	105.958	-0.00239491	105.96
0.354839	-71.5499	0.00397497	-71.5539
0.387097	-38.1418	-0.00659736	-38.1352
0.419355	107.701	0.0109498	107.69
0.451613	-63.9383	-0.0181735	-63.9201
0.483871	-47.0998	0.0301628	-47.1299
0.516129	108.58	-0.0500616	108.63
0.548387	-55.8133	0.0830879	-55.8964
0.580645	-55.6795	-0.137902	-55.5416
0.612903	108.587	0.228878	108.358
0.645161	-47.2401	-0.379872	-46.8603
0.677419	-63.8122	0.630478	-64.4427
0.709677	107.722	-1.04641	108.768
0.741935	-38.2877	1.73674	-40.0244
0.774194	-71.4326	-2.8825	-68.5501
0.806452	105.992	4.78412	101.208
0.83871	-29.0278	-7.94026	-21.0875
0.870968	-78.4793	13.1786	-91.6579
0.903226	103.411	-21.8726	125.284
0.935484	-19.5348	36.3023	-55.8371
0.967742	-84.8959	-60.2514	-24.6446
1	100	100	0

Error of 2nd Order FDM Solution, k = 64, nodes = 64

Position (cm)	Exact	FDM	Abs. Error
0	0	0	0
0.015873	92.3823	-95.54	187.922
0.031746	97.3484	-92.4829	189.831
0.047619	10.1992	6.01635	4.18285
0.0634921	-86.6009	98.3067	-184.908
0.0793651	-101.455	89.1448	-190.6
0.0952381	-20.3084	-12.0144	-8.29399
0.111111	80.0553	-100.775	180.83
0.126984	104.667	-85.5357	190.203
0.142857	30.2384	17.976	12.2624
0.15873	-72.8033	102.937	-175.74
0.174603	-106.955	81.6668	-188.622
0.190476	-39.9016	-23.8829	-16.0186
0.206349	64.9088	-104.785	169.694
0.222222	108.3	-77.5497	185.849
0.238095	49.2126	29.7173	19.4953
0.253968	-56.4416	106.316	-162.758
0.269841	-108.688	73.1969	-181.885
0.285714	-58.0893	-35.4613	-22.628
0.301587	47.4762	-107.523	155
0.31746	108.118	-68.6216	176.739
0.333333	66.4535	41.0976	25.3559
0.349206	-38.0919	108.404	-146.496
0.365079	-106.593	63.8379	-170.431
0.380952	-74.2312	-46.609	-27.6222
0.396825	28.3715	-108.956	137.327
0.412698	104.128	-58.8601	162.988
0.428571	81.3539	51.9788	29.3751
0.444444	-18.4007	109.176	-127.576
0.460317	-100.744	53.7035	-154.447
0.47619	-87.7586	-57.1906	-30.5681
0.492063	8.26753	-109.064	117.332
0.507937	96.4706	-48.3837	144.854
0.52381	93.389	62.2286	31.1604
0.539683	1.93861	108.621	-106.682
0.555556	-91.3462	42.9169	-134.263
0.571429	-98.1952	-67.0775	-31.1177
0.587302	-12.1276	-107.848	95.7204
0.603175	85.4156	-37.3196	122.735
0.619048	102.135	71.7226	30.4123
0.634921	22.2096	106.747	-84.5375
0.650794	-78.7313	31.6089	-110.34
0.666667	-105.173	-76.1497	-29.0236
0.68254	-32.0957	-105.322	73.2263
0.698413	71.3523	-25.8022	97.1544
0.714286	107.284	80.3454	26.9382
0.730159	41.6985	103.577	-61.8782
0.746032	-63.3435	19.917	-83.2606
0.761905	-108.447	-84.297	-24.1502
0.777778	-50.9333	-101.517	50.5834
0.793651	54.7759	-13.9714	68.7472
0.809524	108.654	87.9923	20.6613
0.825397	59.7186	99.1481	-39.4295
0.84127	-45.7248	7.98327	-53.7081
0.857143	-107.901	-91.4203	-16.4811
0.873016	-67.977	-96.4783	28.5013
0.888889	36.2702	-1.97089	38.2411
0.904762	106.197	94.5705	11.6265
0.920635	75.6355	93.5153	-17.8798
0.936508	-26.4956	-4.04748	-22.4481
0.952381	-103.555	-97.4333	-6.12211
0.968254	-82.6266	-90.2681	7.64159
0.984127	16.4872	10.0535	6.43361
1	100	100	0

Convergence of Derivative Function for k = 1:

Num. Elements	dx	Exact Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	64.2093	62.6603	1.54901	n/a
8	0.125	64.2093	63.9253	0.284004	2.4473580459295463
16	0.0625	64.2093	64.1474	0.0618305	2.199520328933848
32	0.03125	64.2093	64.1948	0.0144755	2.094707014882682
64	0.015625	64.2093	64.2058	0.00350485	2.0461893634892987
128	0.0078125	64.2093	64.2084	0.000862464	2.0228149651665492
256	0.00390625	64.2093	64.209	0.000213928	2.0113400890180184
512	0.00195312	64.2093	64.2092	5.32722e-05	2.0056695201509336
1024	0.000976562	64.2093	64.2092	1.32922e-05	2.0028069305003244

Convergence of Extrapolated Derivative Function k = 1:

Num. Elements	dx	Extrapolated Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	64.1948	62.6603	1.53451	2.50938
8	0.125	64.2076	63.9253	0.282356	2.2301
16	0.0625	64.2091	64.1474	0.0616336	2.10987
32	0.03125	64.2092	64.1948	0.0144514	2.05374
64	0.015625	64.2093	64.2058	0.00350187	2.02658
128	0.0078125	64.2093	64.2084	0.000862095	2.01322
256	0.00390625	64.2093	64.209	0.000213878	2.00669

Convergence of Derivative Function for k = 2:

Num. Elements	dx	Exact Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	91.5315	95.3623	3.83081	n/a
8	0.125	91.5315	92.2417	0.710172	2.4314076982919777
16	0.0625	91.5315	91.6864	0.154889	2.1969303835194687
32	0.03125	91.5315	91.5678	0.0362757	2.0941624580199716
64	0.015625	91.5315	91.5403	0.00878394	2.0460635778457164
128	0.0078125	91.5315	91.5337	0.00216158	2.0227845314952604
256	0.00390625	91.5315	91.532	0.000536167	2.011330107560257
512	0.00195312	91.5315	91.5316	0.000133518	2.0056526850320933
1024	0.000976562	91.5315	91.5315	3.33131e-05	2.002867787008173

Convergence of Extrapolated Derivative Function k = 2:

Num. Elements	dx	Extrapolated Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	91.5662	95.3623	3.79611	2.49055
8	0.125	91.5356	92.2417	0.706116	2.22695
16	0.0625	91.532	91.6864	0.1544	2.1092
32	0.03125	91.5316	91.5678	0.0362156	2.05358
64	0.015625	91.5315	91.5403	0.00877649	2.02654
128	0.0078125	91.5315	91.5337	0.00216065	2.01321
256	0.00390625	91.5315	91.532	0.000536055	2.00656

Convergence of Derivative Function for k = 4:

Num. Elements	dx	Exact Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	345.476	103.463	242.013	n/a
8	0.125	345.476	294.97	50.5069	2.260533661076323
16	0.0625	345.476	334.206	11.2702	2.163970616211407
32	0.03125	345.476	342.824	2.65264	2.0870087309717524
64	0.015625	345.476	344.833	0.643062	2.0443997002907595
128	0.0078125	345.476	345.318	0.15829	2.022383450015597
256	0.00390625	345.476	345.437	0.0392657	2.011232906205285
512	0.00195312	345.476	345.467	0.00977822	2.0056261920676226
1024	0.000976562	345.476	345.474	0.00243978	2.002818581480921

Convergence of Extrapolated Derivative Function k = 4:

Num. Elements	dx	Extrapolated Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	344.317	103.463	240.854	2.28712
8	0.125	345.249	294.97	50.2796	2.18686
16	0.0625	345.445	334.206	11.2383	2.10038
32	0.03125	345.472	342.824	2.64847	2.05152
64	0.015625	345.476	344.833	0.64253	2.02604
128	0.0078125	345.476	345.318	0.158223	2.01309
256	0.00390625	345.476	345.437	0.0392573	2.00656

Convergence of Derivative Function for k = 8:

Num. Elements	dx	Exact Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	117.652	705.635	587.983	n/a
8	0.125	117.652	511.063	393.411	0.5797350055699937
16	0.0625	117.652	191.958	74.306	2.40448797239015
32	0.03125	117.652	134.859	17.2072	2.110466257812202
64	0.015625	117.652	121.81	4.15824	2.048966515010708
128	0.0078125	117.652	118.675	1.02281	2.0234383699304286
256	0.00390625	117.652	117.906	0.253674	2.0114891950460283
512	0.00195312	117.652	117.715	0.0631688	2.0056895165848956
1024	0.000976562	117.652	117.668	0.0157612	2.0028314593811527

Convergence of Extrapolated Derivative Function k = 8:

Num. Elements	dx	Extrapolated Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	1009.63	705.635	303.997	-0.713734
8	0.125	122.416	511.063	388.648	2.4825
16	0.0625	117.945	191.958	74.0133	2.12953
32	0.03125	117.683	134.859	17.1761	2.0572
64	0.015625	117.656	121.81	4.15457	2.02736
128	0.0078125	117.652	118.675	1.02236	2.01341
256	0.00390625	117.652	117.906	0.253619	2.00664

Convergence of Derivative Function for k = 16:

Num. Elements	dx	Exact Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	5322.12	3955.31	1366.81	n/a
8	0.125	5322.12	885.254	4436.86	-1.6987260366425194
16	0.0625	5322.12	576.497	4745.62	-0.09705661758595073
32	0.03125	5322.12	3004.53	2317.58	1.0339752416575294
64	0.015625	5322.12	4553.74	768.381	1.5927281030157192
128	0.0078125	5322.12	5114.28	207.833	1.8863937277430793
256	0.00390625	5322.12	5269.27	52.8422	1.9756653318267865
512	0.00195312	5322.12	5308.88	13.2411	1.9966698794929922
1024	0.000976562	5322.12	5318.81	3.30894	2.0005774042509294

Convergence of Extrapolated Derivative Function k = 16:

Num. Elements	dx	Extrapolated Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	541.973	3955.31	3413.33	3.31372
8	0.125	850.421	885.254	34.8331	-2.97525
16	0.0625	7284.66	576.497	6708.17	0.648264
32	0.03125	5432.1	3004.53	2427.57	1.46662
64	0.015625	5328.51	4553.74	774.772	1.85465
128	0.0078125	5322.47	5114.28	208.183	1.96857
256	0.00390625	5322.13	5269.27	52.8582	1.99537

Convergence of Derivative Function for k = 32:

Num. Elements	dx	Exact Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	4841.11	16764	11922.9	n/a
8	0.125	4841.11	6577.14	1736.03	2.779864591115283
16	0.0625	4841.11	1187.79	3653.32	-1.073410557644296
32	0.03125	4841.11	2029.81	2811.29	0.3779719734337922
64	0.015625	4841.11	2267.77	2573.34	0.1275952093369561
128	0.0078125	4841.11	4012.41	828.699	1.6347194978425632
256	0.00390625	4841.11	4617.42	223.682	1.8893980212309631
512	0.00195312	4841.11	4784.14	56.9659	1.973280444814613
1024	0.000976562	4841.11	4826.81	14.2947	1.994620709092206

Convergence of Extrapolated Derivative Function k = 32:

Num. Elements	dx	Extrapolated Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	-4866.44	16764	21630.4	0.918524
8	0.125	1916.03	6577.14	4661.11	2.67818
16	0.0625	2361.51	1187.79	1173.72	1.82314
32	0.03125	1992.23	2029.81	37.5825	-2.87414
64	0.015625	4938.62	2267.77	2670.85	1.52788
128	0.0078125	4847.55	4012.41	835.147	1.85958
256	0.00390625	4841.49	4617.42	224.066	1.96606

Convergence of Derivative Function for k = 64:

Num. Elements	dx	Exact Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	2725.89	67966	65240.1	n/a
8	0.125	2725.89	28548.6	25822.7	1.3371208804578716
16	0.0625	2725.89	12060.4	9334.52	1.4679900485668476
32	0.03125	2725.89	1638.66	1087.23	3.101922956478659
64	0.015625	2725.89	2415.83	310.053	1.8100664626559644
128	0.0078125	2725.89	1879.76	846.127	-1.4483586182104327
256	0.00390625	2725.89	1509.94	1215.95	-0.5231338613089307
512	0.00195312	2725.89	2409.69	316.198	1.9431794293876172
1024	0.000976562	2725.89	2645.99	79.8981	1.9845955409696945

Convergence of Extrapolated Derivative Function k = 64:

Num. Elements	dx	Extrapolated Resonance Freq.	Approx. Resonance Freq.	Absolute Error	Beta
4	0.25	203.998	67966	67762	1.2574
8	0.125	-16265.4	28548.6	44813.9	0.661832
16	0.0625	2361.9	12060.4	9698.51	3.74522
32	0.03125	2098.59	1638.66	459.928	0.535807
64	0.015625	687.304	2415.83	1728.53	0.53561
128	0.0078125	1772.03	1879.76	107.727	-1.2827
256	0.00390625	2730.15	1509.94	1220.21	1.9289

Process finished with exit code 0