Assignment 1

Texas A&M University
AERO-430-500 Numerical Simulation
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Contents

1 Problem 1

1.1 Abstract

The purpose of this assignment is to formulate the analytical solution to the problem of heat conduction in a rod, a behavior governed by a second order ODE, as well as finding the total heat loss to the environment. The solution to the problem is hereby presented for different values of α and for three cases with different specified boundary conditions.

1.2 Analytical Solution

In order to formulate the problem of heat conduction through a rod, we have to consider the Conservation of Heat Flux law, Fourier's Law of Heat Conduction, and Newton's Law of Cooling.

1.2.1 Conservation of Heat Flux

The conservation of heat flux states that the heat flux into the system must be equal to the heat flux leaving the system as shown in the equation below:

$$-\dot{q}(x) + \dot{q}(x + dx) + \dot{q}_c(x) = 0$$

Simplifying, yields:

$$\dot{q}_c(x) + \frac{d}{dx}[\dot{q}(x)]dx = 0$$

Here, $\dot{q}(x)$ is the rate of heat transfer through the rod due to conduction in $\frac{cal}{s}$ and $\dot{q}_c(x)$ is the rate of heat transfer to the environment due convection in $\frac{cal}{s}$.

1.2.2 Fourier's Law of Conduction

Fourier's Law of Conduction states that the rate of heat transfer through a material is proportional to the negative gradient of the temperature and to the area as shown in the equation below:

$$\dot{q}_c(x) = -kA\frac{dT}{dx}$$

Here, k is the thermal conductivity of the rod in $\frac{cal}{scm℃}$, A is the cross-sectional area of the rod in cm^2 and $\frac{dT}{dx}$ is the temperature gradient in $\frac{℃}{cm}$.

1.2.3 Newton's Law of Cooling

Newton's Law of Cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and the temperature of its surroundings, as shown in the equation below:

$$\dot{q}_c(x) = hA_s(T - T_a)$$

Where A_s is the surface area of the rod in cm^2, h is the empirical heat transfer coefficient in $\frac{cal}{scm^2°C}$, T is the temperature in °C, and T_a is the temperature in °C.

1.3 General Solution

By combining and simplifying the previous three equations, it is possible to derive the governing differential equation which is shown in the equations below:

$$\frac{dT}{dx}[-kA\frac{dT}{dx}]dx + h(Pdx)(T - T_a)$$

$$T" - \alpha^2 T = -\alpha^2 T_a$$

Where $\alpha^2 = \frac{hP}{kA}$ in cm^{-2} and Pdx is the surface area A_s in cm^2. The general solution to the equation above is shown below. The constants C and D are solved once the proper boundary conditions are applied.

$$T(x) = Csinh(\alpha x) + Dcosh(\alpha x) + T_a$$

1.4 Case 1 (Zero Temperature)

The rod considered has constant cross-sectional area. The boundary conditions for the rod are such that its temperature is kept at 0°C at $x = 0cm$ and 100°C at $x = Lcm$. The ambient temperature, T_a, is specified to be $T_a = 0$°C. The total length of the rod, L, is $L = 1cm$ and its radius, r, is $r = 0.1cm$. Given these boundary conditions, the constants C and D are obtained for this particular case and are shown below:

$$T(x) = Csinh(\alpha x) + Dcosh(\alpha x) + T_a$$

$$T(0) = 0 = Csinh(\alpha * 0) + Dcosh(\alpha * 0)$$

$$\boxed{D = 0}$$

$$T(L) = 100 = Csinh(\alpha * L) + Dcosh(\alpha * L)$$

$$T(L) = 100 = Csinh(\alpha * L) + (0)cosh(\alpha * L)$$

$$\boxed{C = \frac{100}{sinh(\alpha * L)}}$$

1.5 Case 2 (Insulated End)

The rod considered has again constant cross-sectional area. The boundary conditions for the rod are such its temperature is kept at 100°C at $x = Lcm$. The ambient temperature, T_a, is specified to be $T_a = 0$°C. The total length of the rod, L, is $L = 1cm$ and its radius, r, is $r = 0.1cm$. In this case, however an insulated end is applied at $x = 0cm$ such that only $T'(0) = 0$ is known. By considering the given boundary, the constants C and D are obtained for this particular case and are shown below:

$$T(x) = Csinh(\alpha x) + Dcosh(\alpha x)$$
$$T'(x) = \alpha Ccosh(\alpha x) + \alpha Dsinh(\alpha x)$$
$$T'(0) = \alpha Ccosh(\alpha * 0) + \alpha Dsinh(\alpha * 0) = 0$$
$$\boxed{C = 0}$$
$$T(L) = 100 = 0 * sinh(\alpha * L) + Dcosh(\alpha * L)$$
$$\boxed{D = \frac{100}{cosh(\alpha * L)}}$$

1.6 Case 3 (Newton Cooling)

The rod considered has again constant cross-sectional area. The boundary conditions for the rod are such its temperature is kept at 100°C at $x = Lcm$. The ambient temperature, T_a, is specified to be $T_a = 0$°C. The total length of the rod, L, is $L = 1cm$ and its radius, r, is $r = 0.1cm$. In this case, however Newton's Cooling Law is applied at such that only $T'(0) = \frac{h}{k}(T(0) - T_a)$ is known. By considering the given boundary conditions, the constants C and D are obtained for this particular case and are shown below:

$$T(x) = Csinh(\alpha x) + Dcosh(\alpha x)$$
$$T'(x) = \alpha Ccosh(\alpha x) + \alpha Dsinh(\alpha x)$$
$$T(0) = Csinh(\alpha * 0) + Dcosh(\alpha * 0) = D$$
$$T'(0) = \frac{h}{k}T(0) = \alpha Ccosh(\alpha * 0) + \alpha Dsinh(\alpha * 0)$$
$$\frac{h}{k}T(0) = hkD = \alpha C$$
$$C = \frac{h}{k\alpha}D$$

$$T(L) = 100 = \frac{h}{k\alpha}Dsinh(\alpha * L) + Dcosh(\alpha * L)$$

$$\boxed{D = \frac{100}{\frac{h}{k\alpha}sinh(\alpha * L) + cosh(\alpha * L)}}$$

$$\boxed{C = \frac{h}{k\alpha}(\frac{100}{\frac{h}{k\alpha}sinh(\alpha * L) + cosh(\alpha * L)})}$$

Here, $h = \frac{\alpha^2 kr}{2}$ in $\frac{cal}{scm^2℃}$ obtained by rearranging the definition of α^2.

1.7 Heat Loss to the Environment

The heat loss to the environment, $\dot{q}_c(x = L)$, was calculated by finding the heat entering the bar, which can be found by utilizing Fourier's Law of Heat Conduction as seen in previous sections. Thus, the governing equation is obtained by taking the spatial gradient of the temperature $T(x)$, and plugging it in as shown below:

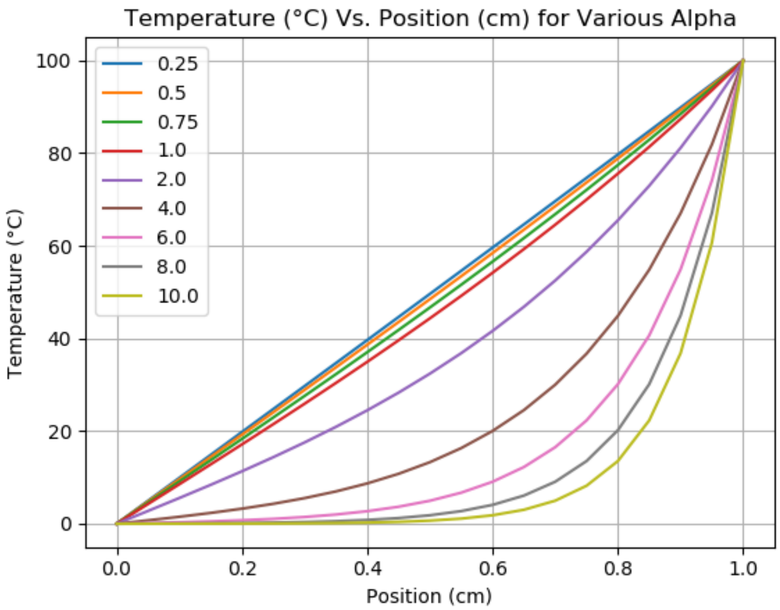
$$\dot{q}_c(L) = -kA\frac{dT(L)}{dx} = -kA\alpha(Ccosh(\alpha * L) + Dsinh(\alpha * L))$$

Here, C and D are the constants calculated in the three cases above and $k = 0.5\frac{\frac{cal}{s}}{\frac{cm^2℃}{cm}}$.

1.8 Results

Firstly, temperature as a function of position, $T(x)$, was calculated for Case 1, Case 2, and Case 3, for values of $\alpha = [0.25,0.5,0.75,1,2,4,6,8,10]$. The figures and tables for each case are shown below respectively.

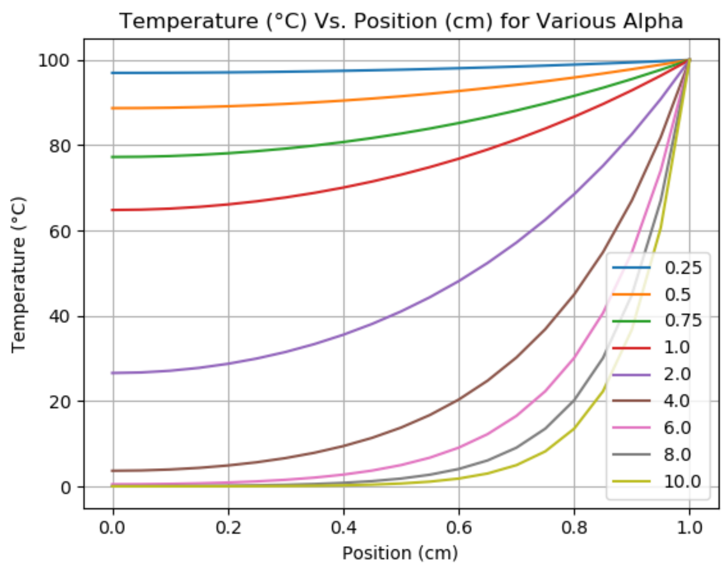
1.8.1 Case 1



Case 1: Zero Temperature

x(cm)	0.25	0.5	0.75	1.0	2.0	4.0	6.0	8.0	10.0
0	0	0	0	0	0	0	0	0	0
0.05	4.94842	4.79809	4.56136	4.25636	2.7618	0.737767	0.150967	0.0275584	0.00473154
0.1	9.89762	9.59917	9.12913	8.52337	5.55125	1.50514	0.315623	0.0595853	0.0106708
0.15	14.8484	14.4063	13.7097	12.8117	8.39625	2.33293	0.508899	0.101274	0.0193338
0.2	19.8014	19.2223	18.3096	17.132	11.3253	3.25434	0.748321	0.159383	0.0329318
0.25	24.7576	24.0505	22.9353	21.4952	14.3677	4.30636	1.0556	0.243335	0.0549358
0.3	29.7176	28.8936	27.5932	25.9122	17.5538	5.53121	1.45859	0.366743	0.0909622
0.35	34.6823	33.7548	32.2899	30.3939	20.9157	6.97804	1.99385	0.549617	0.150207
0.4	39.6524	38.6371	37.032	34.9517	24.4869	8.70493	2.7099	0.821607	0.247792
0.45	44.6286	43.5435	41.8262	39.5968	28.3032	10.7812	3.67168	1.22682	0.408627
0.5	49.6119	48.4772	46.6792	44.3409	32.4027	13.2901	4.9664	1.83095	0.673764
0.55	54.6029	53.4411	51.5979	49.196	36.8265	16.3324	6.71145	2.73196	1.11088
0.6	59.6025	58.4385	56.5891	54.174	41.619	20.0302	9.06508	4.07594	1.83155
0.65	64.6113	63.4724	61.66	59.2875	46.8279	24.5319	12.2407	6.08082	3.01973
0.7	69.6303	68.546	66.8175	64.5493	52.5055	30.0181	16.5263	9.07167	4.9787
0.75	74.6601	73.6624	72.0691	69.9724	58.7086	36.7091	22.3104	13.5334	8.2085
0.8	79.7016	78.8248	77.4219	75.5705	65.4993	44.8733	30.1176	20.1896	13.5335
0.85	84.7555	84.0365	82.8837	81.3576	72.9455	54.8384	40.6557	30.1194	22.313
0.9	89.8227	89.3008	88.4621	87.3482	81.1218	67.0044	54.8804	44.9329	36.7879
0.95	94.9039	94.6208	94.1648	93.5571	90.11	81.8596	74.0814	67.032	60.6531
1	100	100	100	100	100	100	100	100	100

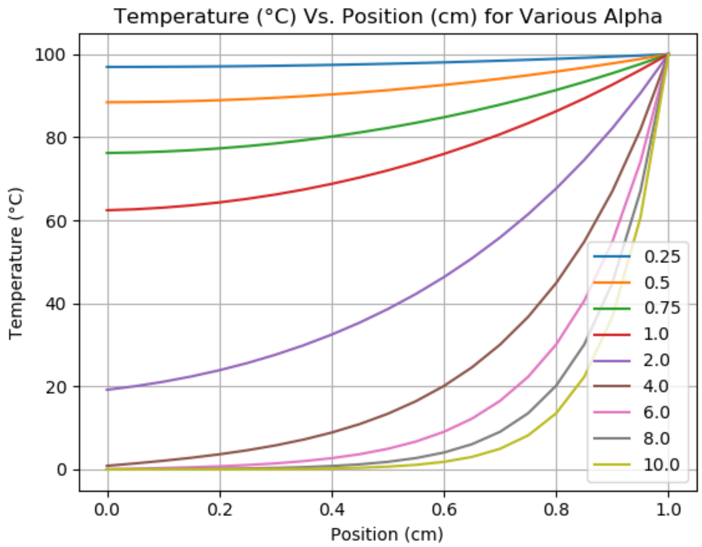
1.8.2 Case 2



Case 2: Insulated at the Left End

x(cm)	0.25	0.5	0.75	1.0	2.0	4.0	6.0	8.0	10.0
0	96.9544	88.6819	77.239	64.8054	26.5802	3.6619	0.495747	0.0670925	0.00907999
0.05	96.9619	88.7096	77.2933	64.8865	26.7132	3.73538	0.518224	0.0725319	0.0102388
0.1	96.9847	88.7928	77.4563	65.1297	27.1136	3.95878	0.587691	0.0897319	0.0140112
0.15	97.0225	88.9314	77.7283	65.5359	27.7853	4.34105	0.710449	0.121481	0.0213598
0.2	97.0756	89.1257	78.1095	66.1059	28.7351	4.89755	0.897628	0.172929	0.0341607
0.25	97.1438	89.3756	78.6007	66.8412	29.9725	5.65061	1.1662	0.252415	0.0556811
0.3	97.2272	89.6814	79.2023	67.7436	31.5099	6.63044	1.54052	0.37283	0.0914142
0.35	97.3258	90.0433	79.9154	68.8154	33.3627	7.87637	2.05453	0.553696	0.150481
0.4	97.4395	90.4614	80.7409	70.0594	35.5493	9.43842	2.75484	0.824342	0.247958
0.45	97.5685	90.9361	81.6799	71.4785	38.0918	11.3793	3.70495	1.22865	0.408728
0.5	97.7128	91.4677	82.7338	73.0763	41.0154	13.7768	4.99102	1.83218	0.673825
0.55	97.8723	92.0564	83.904	74.8568	44.3496	16.7272	6.72965	2.73278	1.11092
0.6	98.0471	92.7026	85.1923	76.8246	48.1276	20.349	9.07851	4.0765	1.83158
0.65	98.2373	93.4068	86.6003	78.9844	52.3873	24.7874	12.2506	6.08119	3.01975
0.7	98.4428	94.1693	88.1302	81.3418	57.1714	30.2207	16.5335	9.07192	4.97871
0.75	98.6636	94.9907	89.784	83.9025	62.5276	36.8668	22.3156	13.5336	8.2085
0.8	98.8999	95.8715	91.5641	86.673	68.5096	44.9925	30.1213	20.1897	13.5335
0.85	99.1517	96.8123	93.473	89.6603	75.1773	54.9239	40.6582	30.1195	22.313
0.9	99.4189	97.8135	95.5133	92.8718	82.5973	67.0596	54.8819	44.9329	36.7879
0.95	99.7017	98.8758	97.688	96.3155	90.8441	81.8866	74.0822	67.032	60.6531
1	100	100	100	100	100	100	100	100	100

1.8.3 Case 3



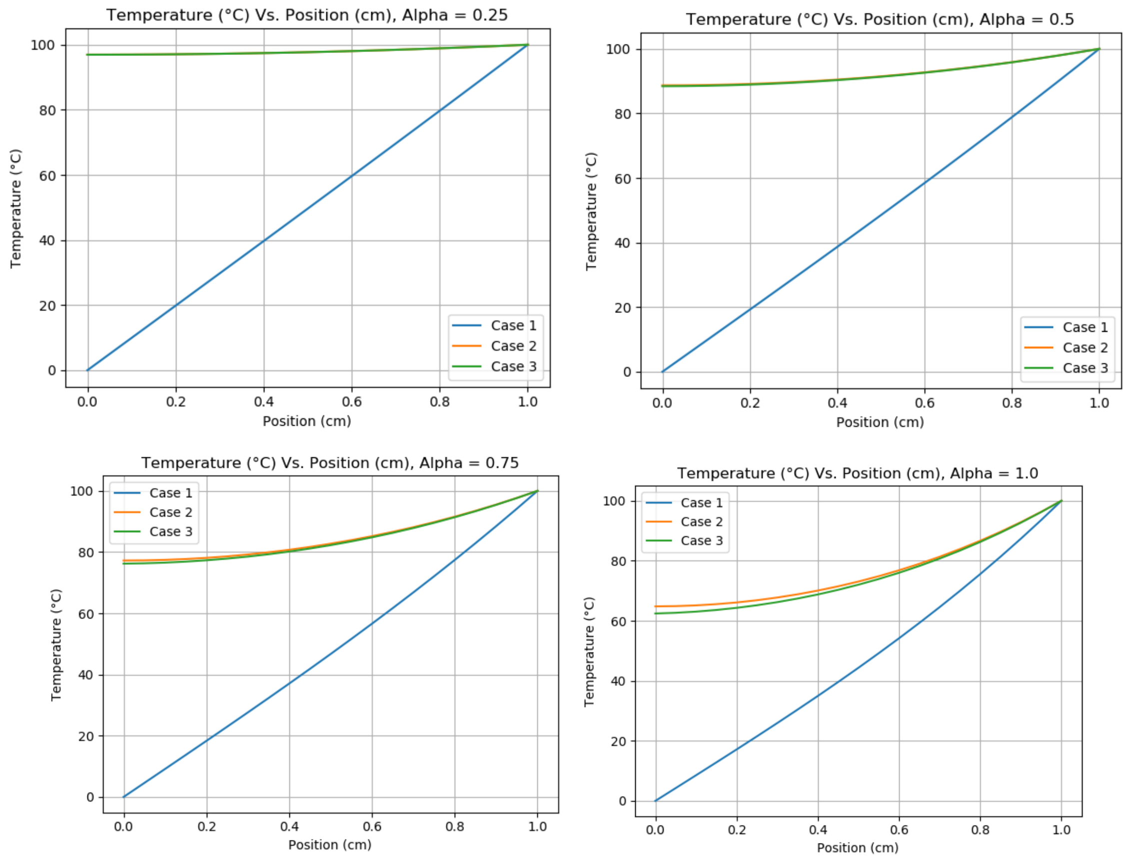
Case 3: Newton's Cooling

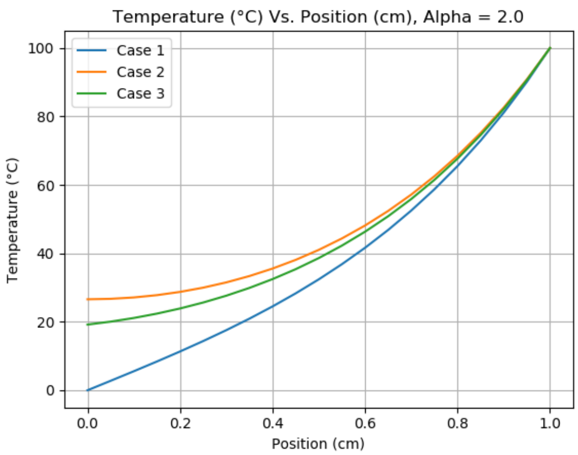
x(cm)	0.25	0.5	0.75	1.0	2.0	4.0	6.0	8.0	10.0
0	96.9358	88.4265	76.2178	62.4282	19.183	0.872327	0.042013	0.00252228	0.000178039
0.05	96.9443	88.4679	76.3317	62.6624	20.0476	1.45185	0.182091	0.0292491	0.00483952
0.1	96.968	88.5647	76.553	63.0532	21.1129	2.08964	0.33868	0.0607186	0.0107363
0.15	97.0068	88.7168	76.8819	63.6018	22.3894	2.8113	0.52598	0.102033	0.0193736
0.2	97.0608	88.9244	77.3189	64.3094	23.89	3.64578	0.760974	0.159892	0.0329559
0.25	97.1299	89.1875	77.8647	65.1777	25.6298	4.62658	1.06497	0.243677	0.0549504
0.3	97.2143	89.5064	78.52	66.2091	27.626	5.79306	1.46554	0.366972	0.090971
0.35	97.3138	89.8812	79.2858	67.406	29.8987	7.19204	1.99899	0.54977	0.150212
0.4	97.4285	90.3122	80.163	68.7715	32.4707	8.87966	2.71371	0.82171	0.247795
0.45	97.5584	90.7996	81.153	70.3089	35.3676	10.9236	3.6745	1.22689	0.408629
0.5	97.7036	91.3439	82.2571	72.0222	38.6185	13.406	4.96848	1.831	0.673765
0.55	97.8641	91.9451	83.4769	73.9155	42.256	16.4265	6.71299	2.73199	1.11088
0.6	98.0398	92.6039	84.8141	75.9937	46.3163	20.1062	9.06622	4.07597	1.83155
0.65	98.2309	93.3206	86.2706	78.2619	50.8402	24.5928	12.2415	6.08084	3.01973
0.7	98.4373	94.0955	87.8484	80.7258	55.8729	30.0664	16.5269	9.07168	4.9787
0.75	98.659	94.9293	89.5498	83.3915	61.4648	36.7466	22.3108	13.5335	8.2085
0.8	98.8962	95.8224	91.3771	86.2658	67.6718	44.9017	30.1179	20.1896	13.5335
0.85	99.1489	96.7755	93.333	89.3557	74.5562	54.8588	40.6559	30.1194	22.313
0.9	99.4171	97.789	95.4201	92.6692	82.1867	67.0176	54.8805	44.9329	36.7879
0.95	99.7007	98.8636	97.6414	96.2143	90.6398	81.866	74.0815	67.032	60.6531
1	100	100	100	100	100	100	100	100	100

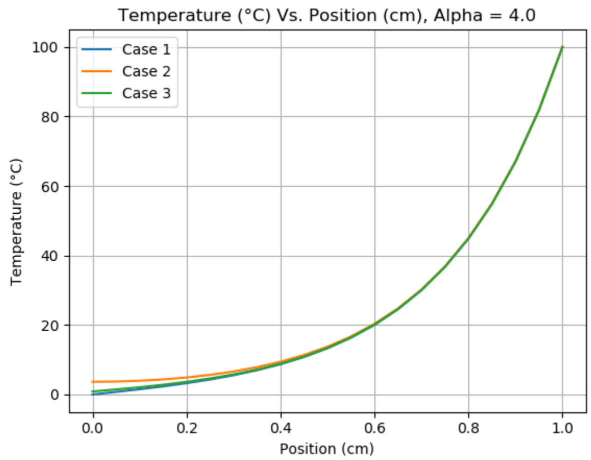
1.8.4 Analytical Solution Comparisons

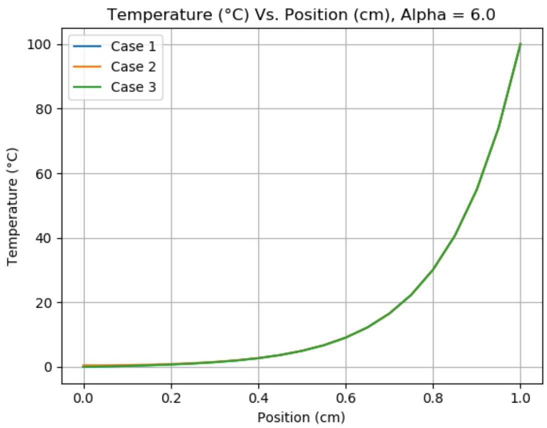
The comparisons for the analytical solutions of Case 1, Case 2, and Case 3 as plots of Temperature vs. Position, T(x), are shown below:

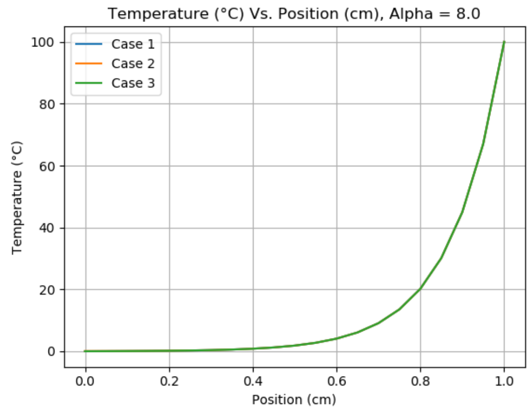
Alpha	Case 1	Case 2	Case 3
0.25	-1.60339	-0.0961793	-0.0964677
0.5	-1.69957	-0.362946	-0.366795
0.75	-1.85484	-0.748267	-0.762897
1	-2.06251	-1.19631	-1.22808
2	-3.25882	-3.02858	-3.09266
4	-6.2874	-6.27897	-6.28539
6	-9.42489	-9.42466	-9.42487
8	-12.5664	-12.5664	-12.5664
10	-15.708	-15.708	-15.708

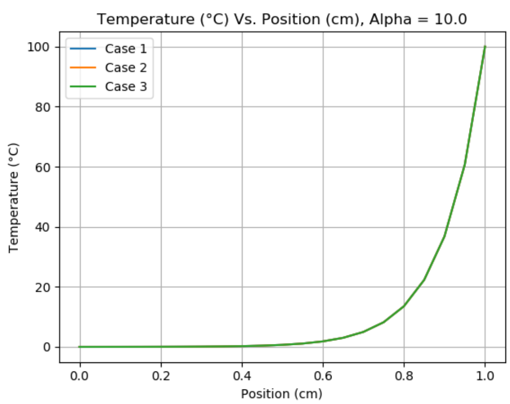



Temperature (°C) Vs. Position (cm), Alpha = 2.0
Case 1
Case 2
Case 3
Temperature (°C)
Position (cm)
0
20
40
60
80
100
0.0
0.2
0.4
0.6
0.8
1.0


Temperature (°C) Vs. Position (cm), Alpha = 4.0
Case 1
Case 2
Case 3
Temperature (°C)
Position (cm)
0
20
40
60
80
100
0.0
0.2
0.4
0.6
0.8
1.0


Temperature (°C) Vs. Position (cm), Alpha = 6.0
Case 1
Case 2
Case 3
Temperature (°C)
Position (cm)
0
20
40
60
80
100
0.0
0.2
0.4
0.6
0.8
1.0


Temperature (°C) Vs. Position (cm), Alpha = 8.0
Case 1
Case 2
Case 3
Temperature (°C)
Position (cm)
0
20
40
60
80
100
0.0
0.2
0.4
0.6
0.8
1.0


Temperature (°C) Vs. Position (cm), Alpha = 10.0
Case 1
Case 2
Case 3
Temperature (°C)
Position (cm)
0
20
40
60
80
100
0.0
0.2
0.4
0.6
0.8
1.0

2 Problem 2

2.1 Abstract

The purpose of this assignment was to utilize the Finite Differences Method (FDM) to obtain an approximate solution to the problem of heat conduction in a rod and compare it to the analytical model developed in the previous problem. Furthermore, the convergence of the quantities of interest was investigated such that it may be possible to learn how to validate the results obtained by the FDM in the case where an analytical solution might not be available.

2.2 The Finite Differences Method

To develop the FDM model, let's first consider the ODE shown in Problem 1 of this report, and recall again below that:

$$T" - \alpha^2 T = -\alpha^2 T_a$$

To utilize the FDM it is necessary to divide the domain into N points such that the equation above can be evaluated at $x_i(i) = [1, 2, 3, \ldots (N-1)]$. Doing so creates a system of differential equations that can be seen below:

$$\begin{gathered} T"(x_1) - \alpha^2 T(x_1) = 0 \\ T"(x_2) - \alpha^2 T(x_2) = 0 \\ T"(x_3) - \alpha^2 T(x_3) = 0 \\ \vdots \\ T"(x_{N-1}) - \alpha^2 T(x_{N-1}) = 0 \end{gathered}$$

It is possible to approximate $T"(x_i)$ through a Taylor Series Expansion such that the approximate value of $T"(x_i)$ is $\bar{T}(x_i)$, where $\sigma(dx^2)$ is the truncation error of the second order expansion.

$$T"(x_i) \approx \bar{T}"(x_i) = \frac{(\bar{T}_{i-1}) - (2\bar{T}_i) + (\bar{T}_{i+1})}{dx^2} + \mathcal{O}(dx^2)$$

However, from this point forward, we will remove the truncation error term which results in the approximate solution shown below:

$$\bar{T}"(x_i) = \frac{(\bar{T}_{i-1}) - (2\bar{T}_i) + (\bar{T}_{i+1})}{dx^2}$$

Substituting this approximation into our previous equations yields the following set of equations:

$$\begin{gathered} -\frac{\bar{T}_0 - 2\bar{T}_1 + \bar{T}_2}{dx^2} + \alpha^2 \bar{T}_1 = 0 \\ -\frac{\bar{T}_1 - 2\bar{T}_2 + \bar{T}_3}{dx^2} + \alpha^2 \bar{T}_2 = 0 \\ -\frac{\bar{T}_2 - 2\bar{T}_3 + \bar{T}_4}{dx^2} + \alpha^2 \bar{T}_3 = 0 \\ \vdots \\ -\frac{\bar{T}_{N-2} - 2\bar{T}_{N-1} + \bar{T}_N}{dx^2} + \alpha^2 \bar{T}_{N-1} = 0 \end{gathered}$$

If we now multiply out the $\frac{1}{dx^2}$ term, our equations take on the form:

$$-\bar{T}_0 + (2+\alpha^2 dx^2)\bar{T}_1 - \bar{T}_2 = 0$$
$$-\bar{T}_1 + (2+\alpha^2 dx^2)\bar{T}_2 - \bar{T}_3 = 0$$
$$-\bar{T}_2 + (2+\alpha^2 dx^2)\bar{T}_3 - \bar{T}_4 = 0$$
$$\vdots$$
$$-\bar{T}_{N-2} + (2+\alpha^2 dx^2)\bar{T}_{N-1} - \bar{T}_N = 0$$

2.3 Case 1

In this case, the specified boundary conditions are of the Dirichlet type, which means that the temperature at node 0 and N are known. With $T_a = 0°\text{C}, T_0 = 0°\text{C}, T_L = 100°\text{C}$, and if we let $\kappa = 2 + \alpha^2 dx^2$, then the expressions above can be represented by a matrix as shown below. This matrix can then calculate the unknowns $\bar{T}_1, \bar{T}_2, \bar{T}_3, \dots, \bar{T}_{N-1}$.

$$\begin{bmatrix} \kappa & -1 & 0 & \dots & 0 & 0 \\ -1 & \kappa & -1 & \dots & 0 & 0 \\ 0 & -1 & \kappa & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & -1 & 0 \\ 0 & 0 & 0 & -1 & \kappa & -1 \\ 0 & 0 & 0 & 0 & -1 & \kappa \end{bmatrix} \begin{Bmatrix} \bar{T}_1 \\ \bar{T}_2 \\ \bar{T}_3 \\ \vdots \\ \bar{T}_{N-2} \\ \bar{T}_{N-1} \end{Bmatrix} = \begin{Bmatrix} \bar{T}_0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \bar{T}_L \end{Bmatrix}$$

2.4 Case 2

In this case, the insulated boundary condition at x = 0cm requires an additional equation to be added into the system since T_0 is not known. To solve for this additional unknown, we can utilize the "Ghost Point" method as follows Firstly, we can approximate $\bar{T}'_0$ as shown in Equation 15, shown below:

$$\bar{T}'_0 = \frac{\bar{T}_1 - \bar{T}_{-1}}{2dx}$$

Solving for $\bar{T}_1$, and substituting into the general equation applied at node 0, yields the equation shown below:

$$\frac{\kappa}{2}\bar{T}_0 - \bar{T}_1 = dx\bar{T}'_0 \quad \text{(Eq. 16)}$$

By applying Newton's Law of Cooling as presented in Section 1.2.3 of this report, it is possible to solve for $\bar{T}'_0$ and thus obtain an expression for $\bar{T}_0$ such as that shown in the equation below:

$$-\bar{T}_1 + \left(\frac{dx\, h}{k} + \frac{\kappa}{2}\right)\bar{T}_0 = 0$$

Where $\kappa' = \frac{dx\, h}{k} + \frac{\kappa}{2}$. Therefore, it is now possible to solve for the approximate temperature using a system of linear equations such as that shown in the equation reported below.

$$\begin{bmatrix} \kappa' & -1 & 0 & 0 & 0 & 0 \\ -1 & \kappa & -1 & 0 & 0 & 0 \\ 0 & -1 & \kappa & -1 & 0 & 0 \\ 0 & 0 & -1 & \kappa & -1 & 0 \\ 0 & 0 & 0 & -1 & \kappa & -1 \\ 0 & 0 & 0 & 0 & -1 & \kappa \end{bmatrix} \begin{bmatrix} \bar{T}_0 \\ \bar{T}_1 \\ \vdots \\ \vdots \\ \vdots \\ \bar{T}_{N-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ T_L \end{bmatrix}$$

Where again, $T_L = 100\ °C$.

2.5 Case 3

In this case, the Newton boundary at x = 0cm requires an additional equation to be added into the system since $\bar{T}_0$ is also unknown. To solve for this additional unknown, we can utilize the "ghost point" method as follows. With $T_L = 100$ ℃ and T_0 as an unknown, let us observe the equation that will solve for T_0.

$$-\bar{T}_{-1} + (2 + \alpha^2 dx^2)\bar{T}_0 - \bar{T}_1 = 0$$

Here, $\bar{T}_{-1}$ means that we introduced a ghost point at i = -1. We can avoid this point by introducing the first order FDA,

$$T'_i = \frac{T_{i+1} - T_{i-1}}{2dx} + \mathcal{O}(dx^2)$$

Which at node 0 can be approximated as:

$$\bar{T}'_0 = \frac{\bar{T}_1 - \bar{T}_{-1}}{2dx}$$

Finally, if we substitute this result into our previously found 0 equation, we find that:

$$-(\bar{T}_1 - 2dx\bar{T}'_0) + (2 + \alpha^2 dx^2)\bar{T}_0 - \bar{T}_1 = 0$$

From here, we can apply the Newton Cooling boundary conditions that states

$$T'(0) = \frac{h}{\kappa}(T(0) - T_a)$$

We can recall that $T_a = 0$℃ and then substitute this value into the node 0 equation as shown below:

$$-(\bar{T}_1 - 2dx\frac{h}{\kappa}\bar{T}_0) + (2 + \alpha^2 dx^2)\bar{T}_0 - \bar{T}_1 = 0$$

Rearranging the above equations yields:

$$(2dx\frac{h}{\kappa} + 2 + \alpha^2 dx^2)\bar{T}_0 - 2\bar{T}_1 = (2dx\frac{h}{\kappa} + \kappa)\bar{T}_0 - 2\bar{T}_1 = 0$$

If we divide this equation by 2, we can return to the symmetry in our matrix.

$$(\frac{dxh}{\kappa} + \frac{\kappa}{2})\bar{T}_0 - \bar{T}_1 = 0$$

If we define $\kappa' = \frac{dxh}{\kappa} + \frac{\kappa}{2}$, it is now possible to solve for the approximate temperature using a system of linear equations like the one shown below:

$$\begin{bmatrix} \kappa' & -1 & 0 & \dots & 0 & 0 \\ -1 & \kappa & -1 & \dots & 0 & 0 \\ 0 & -1 & \kappa & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & -1 & 0 \\ 0 & 0 & 0 & -1 & \kappa & -1 \\ 0 & 0 & 0 & 0 & -1 & \kappa \end{bmatrix} \begin{Bmatrix} \bar{T}_0 \\ \bar{T}_1 \\ \bar{T}_2 \\ \vdots \\ \bar{T}_{N-2} \\ \bar{T}_{N-1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \bar{T}_L \end{Bmatrix}$$

2.6 Approximation of the Heat Loss

As before, the total heat loss is calculated by finding the heat entering the domain at the interface as articulated for the analytical case in Section 1.7 of this report. Numerically, it can be calculated by approximating $\overline{T}'(L)$ through a Taylor Series expansion as shown below:

$$\overline{T}(L - dx) = \overline{T}(L) - dx\overline{T}^{'}(L) + \frac{dx^2}{2}\overline{T}^{''}(L)$$

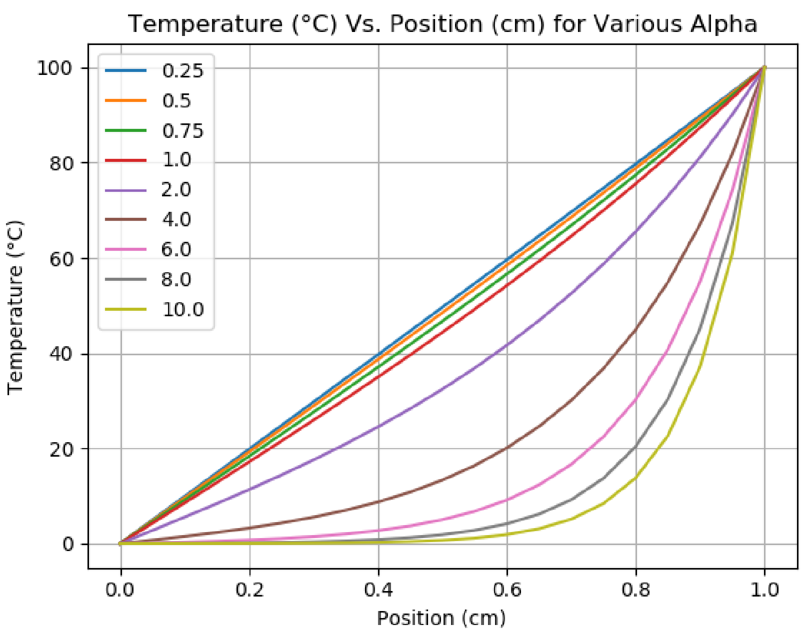
Where $\overline{T}''(L)$ is also unknown but can be found utilizing Equation 5 and is therefore defined as $\overline{T}^{''}(L) = \alpha^2 T(L)$. Thus, an expression for the heat loss can be obtained and is shown below in Equation 20, which allows for a rate of convergence to the second order.

$$\overline{q_c} = -kA\,\overline{T}^{'}(L) = -kA\,\left[\frac{\overline{T}_N - \overline{T}_{N-1}}{dx} + \frac{\alpha^2 dx\,\overline{T}_N}{2}\right]$$

2.7 Results

Firstly, temperature as a function of position, T(x), was calculated for Case 1, Case 2, and Case 3, for values of α = [0.25, 0.5, 0.75, 2, 4, 6, 8, 10]. The figures and tables for each case are shown below respectively.

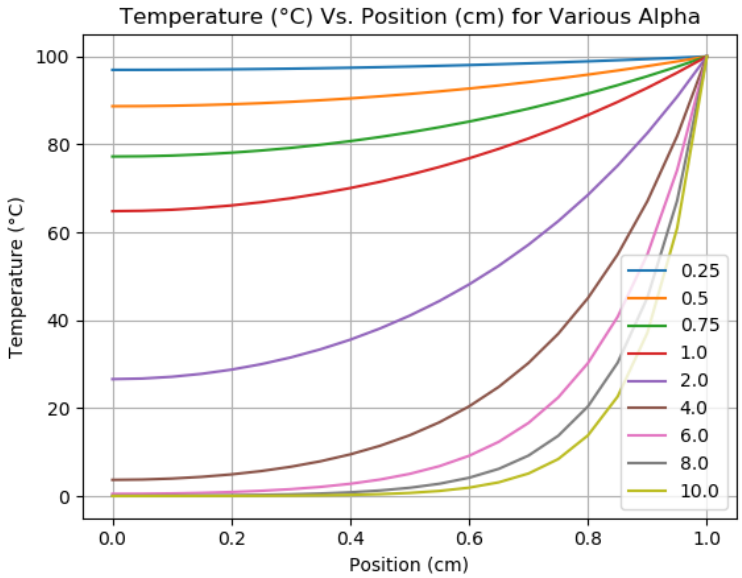
2.7.1 Case 1



Case 1: Zero Ambient Temperature

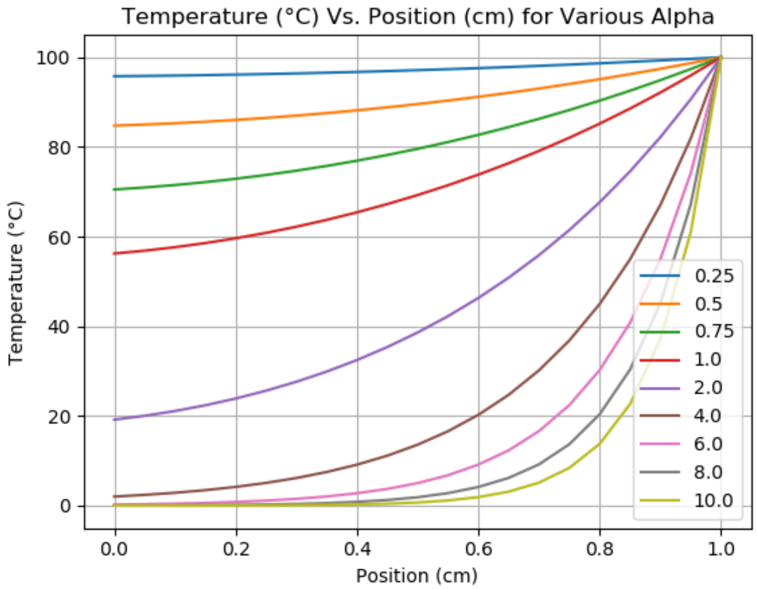
x(cm)	0.25	0.5	0.75	1.0	2.0	4.0	6.0	8.0	10.0
0	0	0	0	0	0	0	0	0	0
0.05	4.94842	4.7981	4.5614	4.2565	2.76303	0.741434	0.153778	0.0288407	0.00517883
0.1	9.89762	9.59919	9.12922	8.52365	5.5537	1.51253	0.321395	0.0622959	0.0116524
0.15	14.8484	14.4063	13.7099	12.8121	8.3999	2.34412	0.517938	0.105719	0.021039
0.2	19.8014	19.2224	18.3098	17.1326	11.3301	3.26948	0.761095	0.166056	0.0356854
0.25	24.7576	24.0505	22.9355	21.4959	14.3736	4.32561	1.07275	0.252963	0.0592531
0.3	29.7176	28.8936	27.5934	25.9129	17.5608	5.55477	1.48096	0.380343	0.0976341
0.35	34.6823	33.7548	32.2902	30.3948	20.9237	7.00613	2.02244	0.568578	0.160424
0.4	39.6524	38.6371	37.0323	34.9526	24.4958	8.73772	2.74595	0.847786	0.263319
0.45	44.6287	43.5436	41.8265	39.5978	28.3128	10.8188	3.7166	1.26264	0.432044
0.5	49.6119	48.4773	46.6796	44.342	32.413	13.3327	5.02174	1.87952	0.708781
0.55	54.6029	53.4412	51.5983	49.1971	36.8373	16.3799	6.77884	2.79712	1.16271
0.6	59.6025	58.4386	56.5895	54.1751	41.63	20.0822	9.14603	4.16225	1.90732
0.65	64.6113	63.4725	61.6604	59.2886	46.8389	24.5879	12.3364	6.19335	3.12876
0.7	69.6303	68.546	66.8179	64.5503	52.5163	30.077	16.637	9.21538	5.13239
0.75	74.6601	73.6624	72.0694	69.9734	58.7188	36.7693	22.4349	13.7119	8.41912
0.8	79.7016	78.8249	77.4222	75.5714	65.5085	44.9323	30.252	20.4023	13.8106
0.85	84.7555	84.0366	82.884	81.3583	72.9533	54.8926	40.7917	30.357	22.6548
0.9	89.8227	89.3008	88.4622	87.3487	81.1277	67.0486	55.0027	45.1689	37.1627
0.95	94.9039	94.6208	94.1649	93.5574	90.1133	81.8866	74.164	67.2078	60.9612
1	100	100	100	100	100	100	100	100	100

2.7.2 Case 2



x(cm)	0.25	0.5	0.75	1.0	2.0	4.0	6.0	8.0	10.0
0	96.9544	88.6824	77.2411	64.8106	26.6016	3.68627	0.506914	0.0707016	0.0100484
0.05	96.962	88.7101	77.2954	64.8916	26.7346	3.75999	0.529725	0.0763577	0.0113045
0.1	96.9847	88.7933	77.4584	65.1348	27.1349	3.98412	0.600212	0.0942311	0.0153866
0.15	97.0226	88.9319	77.7304	65.5409	27.8066	4.36761	0.724717	0.127181	0.0233154
0.2	97.0756	89.1262	78.1116	66.1108	28.7564	4.9258	0.914447	0.180481	0.0370731
0.25	97.1438	89.3761	78.6027	66.846	29.9937	5.68103	1.18648	0.262657	0.0600991
0.3	97.2272	89.6819	79.2043	67.7484	31.531	6.66349	1.56529	0.386858	0.0981498
0.35	97.3258	90.0438	79.9173	68.8201	33.3836	7.9125	2.08498	0.572957	0.160738
0.4	97.4396	90.4619	80.7427	70.0638	35.57	9.47801	2.79232	0.850729	0.263511
0.45	97.5686	90.9366	81.6816	71.4827	38.1121	11.4226	3.75096	1.26462	0.432161
0.5	97.7128	91.4681	82.7354	73.0803	41.0353	13.8242	5.0472	1.88085	0.708852
0.55	97.8724	92.0567	83.9056	74.8606	44.3689	16.7787	6.79768	2.79801	1.16276
0.6	98.0472	92.7029	85.1937	76.8281	48.1462	20.4043	9.15995	4.16285	1.90735
0.65	98.2373	93.4071	86.6016	78.9876	52.405	24.8461	12.3466	6.19375	3.12878
0.7	98.4428	94.1696	88.1314	81.3446	57.1877	30.2818	16.6445	9.21565	5.1324
0.75	98.6637	94.991	89.785	83.905	62.5424	36.9287	22.4403	13.7121	8.41913
0.8	98.8999	95.8717	91.5649	86.6751	68.5225	45.0528	30.2558	20.4024	13.8106
0.85	99.1517	96.8124	93.4736	89.6619	75.1878	54.979	40.7944	30.3571	22.6548
0.9	99.4189	97.8136	95.5137	92.8729	82.605	67.1044	55.0044	45.169	37.1627
0.95	99.7017	98.8759	97.6882	96.3161	90.8483	81.9139	74.1648	67.2079	60.9612
1	100	100	100	100	100	100	100	100	100

2.7.3 Case 3



Case 3: Newton's Cooling

x(cm)	0.25	0.5	0.75	1.0	2.0	4.0	6.0	8.0	10.0
0	95.7814	84.7645	70.5193	56.2371	19.1788	2.03504	0.22629	0.0262398	0.00315708
0.05	95.8488	85.0029	70.9659	56.8704	20.0456	2.40786	0.321603	0.0464759	0.00710342
0.1	95.9311	85.2945	71.5122	57.646	21.113	2.87699	0.445861	0.0741482	0.0128256
0.15	96.0284	85.6394	72.1591	58.5656	22.3914	3.46121	0.610246	0.113684	0.0217542
0.2	96.1407	86.0379	72.9074	59.6317	23.8938	4.18387	0.829553	0.17141	0.0361214
0.25	96.2681	86.4901	73.7583	60.8469	25.6351	5.07388	1.12352	0.256561	0.0595189
0.3	96.4105	86.9963	74.713	62.2141	27.6328	6.16685	1.5186	0.382761	0.0977961
0.35	96.5679	87.557	75.7726	63.737	29.9068	7.5065	2.05036	0.570204	0.160522
0.4	96.7404	88.1723	76.9389	65.4191	32.4799	9.14641	2.76665	0.848879	0.263379
0.45	96.9281	88.8428	78.2133	67.2648	35.3778	11.1522	3.73194	1.26337	0.432081
0.5	97.1309	89.5688	79.5977	69.2787	38.6294	13.604	5.0331	1.88001	0.708803
0.55	97.3489	90.3507	81.0941	71.4657	42.2673	16.6	6.78725	2.79745	1.16273
0.6	97.5821	91.1892	82.7045	73.8314	46.3279	20.26	9.15224	4.16248	1.90733
0.65	97.8305	92.0846	84.4312	76.3817	50.8518	24.7304	12.3409	6.1935	3.12877
0.7	98.0942	93.0376	86.2766	79.123	55.8842	30.1901	16.6403	9.21548	5.13239
0.75	98.3732	94.0487	88.2433	82.062	61.4755	36.8573	22.4373	13.7119	8.41912
0.8	98.6677	95.1186	90.3342	85.2063	67.6815	44.9988	30.2537	20.4023	13.8106
0.85	98.9775	96.248	92.5521	88.5635	74.5643	54.9403	40.7929	30.3571	22.6548
0.9	99.3028	97.4375	94.9001	92.1421	82.1928	67.0794	55.0035	45.1689	37.1627
0.95	99.6436	98.6879	97.3816	95.9511	90.6432	81.9017	74.1643	67.2078	60.9612
1	100	100	100	100	100	100	100	100	100

2.7.4 Case 1 Convergence

In this section, convergence plots are presented at $\alpha = 6$ for gradually smaller dx values. The table shows the percent error, approximate heat transfer and exact heat transfer as a function of dx. It can be seen from this table that the precent error is minimized to below 1% for dx < 0.025.

dx	q_dot_exact	q_dot	% error
0.05	-9.42489	-9.16509	2.75656
0.033	-9.42489	-9.30131	1.3112
0.025	-9.42489	-9.35299	0.762916
0.02	-9.42489	-9.37793	0.498248
0.017	-9.42489	-9.39184	0.350697
0.014	-9.42489	-9.40038	0.260142
0.013	-9.42489	-9.40599	0.20061
0.011	-9.42489	-9.40987	0.159395
0.01	-9.42489	-9.41267	0.129689
0.009	-9.42489	-9.41476	0.107574

2.7.5 Case 2 Convergence

In this section, convergence plots are presented at $\alpha = 6$ for gradually smaller dx values. The table shows the percent error, approximate heat transfer and exact heat transfer as a function of dx. It can be seen from this table that the precent error is minimized to below 1% for dx < 0.025.

dx	q_dot_exact	q_dot	% error
0.05	-9.42466	-9.16486	2.75665
0.033	-9.42466	-9.30108	1.31124
0.025	-9.42466	-9.35276	0.762942
0.02	-9.42466	-9.3777	0.498265
0.017	-9.42466	-9.39161	0.350708
0.014	-9.42466	-9.40014	0.26015
0.013	-9.42466	-9.40575	0.200616
0.011	-9.42466	-9.40964	0.159401
0.01	-9.42466	-9.41244	0.129693
0.009	-9.42466	-9.41452	0.107577

2.7.6 Case 3 Convergence

In this section, convergence plots are presented at $\alpha = 6$ for gradually smaller dx values. The table shows the percent error, approximate heat transfer and exact heat transfer as a function of dx. It can be seen from this table that the precent error is minimized to below 1% for dx < 0.025.

dx	q_dot_exact	q_dot	% error
0.05	-9.42487	-9.16499	2.75746
0.033	-9.42487	-9.30121	1.3121
0.025	-9.42487	-9.35288	0.763825
0.02	-9.42487	-9.37783	0.499157
0.017	-9.42487	-9.39174	0.351605
0.014	-9.42487	-9.40027	0.261051
0.013	-9.42487	-9.40588	0.201518
0.011	-9.42487	-9.40977	0.160304
0.01	-9.42487	-9.41257	0.130598
0.009	-9.42487	-9.41465	0.108482