Exam 2

Texas A&M University
AERO-430-500 Numerical Simulation
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December 1, 2021

Due: December 7, 2021

Contents

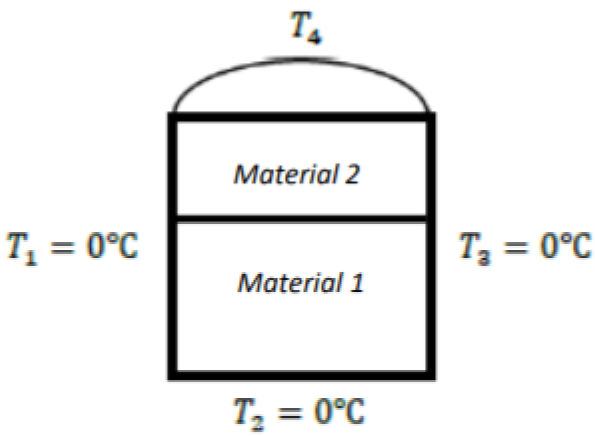
1 Abstract

The purpose of this assignment is to formulate the analytical solution to the problem of heat conduction in a bi-material, 2-dimensional plate. Both analytical and finite difference method (FDM) techniques were used in this analysis. The FDM solution was derived at an accuracy of a second order solution and its convergence was determined at various values of thermal conductivity and number of nodes used. Additionally, the application of a conformal mesh was investigated within this assignment and its performance was compared against the non-conformal mesh case.

2 Analytical Solution

2.1 The Orthotropic Diffusion Equation

A visualization of the problem to be solved within this report is given below:



For this analysis, the following boundary conditions apply:

$$T_1(0, y) = T_2(x, 0) = T_3(1, y) = 0°C$$

$$T_4(x, 1) = 100 * \sin(\pi x)\,°C$$

2.2 Solution Formulation

The analytical solution to this problem can be derived by utilizing the separation of variable solutions scheme and applying boundary conditions. This results in the equations below and will appropriately account for the material interface pictured above.

$$u^1(x, y) = Y^{(1)}(\text{y})\sin(\pi x)$$

$$u^2(x, y) = Y^{(2)}(\text{y})\sin(\pi x)$$

Where a one superscript represents material one, and a two superscript represents material two. Plugging these equations back into the heat equation results in the following:

$$-Y''(y) + K^2\pi^2 Y(y) = 0$$

The material one boundary conditions are as follows:

$$Y^{(1)}(0) = 0 \; and \; Y^{(1)}(\bar{y}) = \bar{Y}$$

While the material two boundary conditions are the following:

$$Y^{(2)}(1) = 100 \; and \; Y^{(1)}(\bar{y}) = \bar{Y}$$

Solving for this partial differential equation for material one yields:

$$Y^1(y) = \bar{Y} * \frac{\sinh(K_1\pi y)}{\sinh\,(K_1\pi\bar{\bar{y}})}$$

Solving this partial differential equation for material two yields:

$$Y^2(y) = \left(100 - \bar{Y}\cosh\left(K_2\pi(1-\bar{y})\right)\right) * \frac{\sinh\left(K_2\pi(y-\bar{y})\right)}{\sinh\left(K_2\pi(1-\bar{y})\right)} + \bar{Y}\cosh(K_2\pi(y-\bar{y}))$$

$\bar{Y}$ can now be solved for by using conservation of heat flux at the material interface:

$$K_{yy_2}\frac{dY^2}{dy}(\bar{y}) = K_{yy_1}\frac{dY^1}{dy}(\bar{y})$$

Plugging in the solution of the partial differential equation for material 2 and rearranging for Y produces the following equation on the left-hand side:

$$K_{yy_2}\frac{d}{dy}\left(100 - \bar{Y}\cosh\left(K_2\pi(1-\bar{y})\right)\right) * \frac{\sinh\left(K_2\pi(y-\bar{y})\right)}{\sinh\left(K_2\pi(1-\bar{y})\right)} + \bar{Y}\cosh\left(K_2\pi(y-\bar{y})\right) =$$

$$K_{yy_2}\left(100 - \bar{Y} * \cosh\left(K_2\pi(1-\bar{y})\right)\right) * K_2\frac{1}{\sinh\,(K_2\pi(1-\bar{y}))}$$

Plugging in the solution of the partial differential equation for material 1 and rearranging for Y produces the following equation on the left-hand side:

$$K_{yy_1}\frac{d}{dy}\left(\bar{Y} * \frac{\sinh K_1\pi y}{\sinh K_1\pi\,\bar{y}}\right) = \bar{Y} * K_1\pi * \frac{\cosh K_1\pi\bar{y}}{\sinh K_1\pi\bar{y}}$$

Setting these two equations equal to each other produces:

$$\bar{Y} = 100 * \frac{1}{\sinh\left(K_2\pi(1-\bar{y})\right)} * \frac{1}{\frac{K_{yy_1}}{K_{yy_2}} * \frac{K_1}{K_2}\coth K_1\pi\bar{y} + \coth\left(K_2\pi(1-\bar{y})\right)}$$

The finial general solution to the heat equation is now:

$$u^1(x,y) = f(x)g(y) = \sinh(\pi x) * \bar{Y} * \frac{\sinh(K_1\pi y)}{\sinh\,(K_1\pi\bar{y})}$$

$$u^2(x,y) = \sinh(\pi x) * \left(100 - \bar{Y} * \cosh\left(K_2\pi(1-\bar{y})\right)) * \frac{\sinh K_2\pi(y-\bar{y}))}{\sinh\,(K_2\pi(1-\bar{y})} + \bar{Y} * \cosh\left(K_2\pi(y-\bar{y})\right)\right)$$

3 Numerical Solution

3.1 Second Order Central Difference Scheme FDM

The second order FDM solution derivation commences with an Ordinary Taylor Series Expansion.

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f''' + \cdots$$
$$f(x) = f(x)$$
$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f''' + \cdots$$

Adapting this equation for a point in 2D space.

$$U_{i-\Delta x,j} = U_{i,j} - \Delta x U'_{i,j} + \frac{\Delta x^2}{2}U''_{i,j} + \cdots$$
$$U_{i,j} = U_{i,j}$$
$$U_{i+\Delta x,j} = U_{i,j} + \Delta x U'_{i,j} + \frac{\Delta x^2}{2}U''_{i,j} + \cdots$$
$$U_{i,j-\Delta y} = U_{i,j} - \Delta y U'_{i,j} + \frac{\Delta y^2}{2}U''_{i,j} + \cdots$$
$$U_{i,j+\Delta y} = U_{i,j} + \Delta y U'_{i,j} + \frac{\Delta y^2}{2}U''_{i,j} + \cdots$$

Subtracting the Taylor series yields the second order approximation of U':

$$U_{i,j-\Delta y} - U_{i,j+\Delta y} = U_{i,j} - \Delta y U'_{i,j} + \frac{\Delta y^2}{2}U''_{i,j} - (U_{i,j} + \Delta y U'_{i,j} + \frac{\Delta y^2}{2}U''_{i,j} + \cdots)$$

Dividing both sides of this equation by $2\Delta y$ produces $\frac{dU}{dy}$:

$$U'_{i,j} \approx \frac{U_{i,j-\Delta y} - U_{i,j+\Delta y}}{2\Delta y}$$

Applying a similar process of adding Taylor Series will result in U'':

$$\frac{d^2U}{dx^2} \approx \frac{U_{i+\Delta x,j} + U_{i-\Delta x,j} - 2U_{i,j}}{\Delta x^2}$$

$$\frac{d^2U}{dy^2} \approx \frac{U_{i,j+\Delta y} + U_{i,j-\Delta y} - 2U_{i,j}}{\Delta y^2}$$

Plugging these equations back into the governing heat equation produces the following equation that can be utilized to produce a stiffness matrix

$$K_{xx} * \frac{\partial^2 u}{\partial x^2}(x,y) + K_{yy}\frac{\partial^2 u}{\partial y^2}(x,y) = 0$$

$$K_{xx} * \frac{U_{i+\Delta x,j} + U_{i-\Delta \mathrm{x,j}} - 2U_{i,j}}{\Delta x^2} + K_{yy}\frac{U_{i,j+\Delta y} + U_{\mathrm{i,j-\Delta y}} - 2U_{i,j}}{\Delta y^2} = 0$$

It is important to note that derived equation above is *only* valid within each individual material and *not* valid for any point along the material interface. The following section will discuss the process needed to follow in order to account for this material interface.

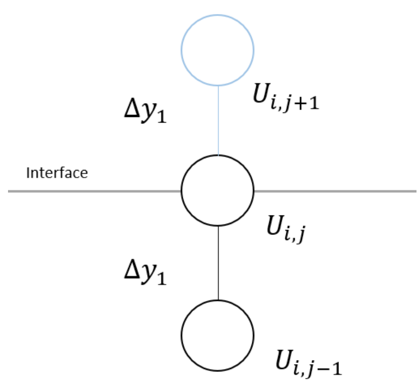
3.2 Interface Equation

The equation at the interface can be generated by utilizing the ghost point method to satisfy the continuity of heat flux across the material interface. First the heat flux from the first and second derivatives must be obtained.

$$K_{yy_1}\frac{\partial u^-}{\partial y} = g_{inter}^-$$

$$K_{yy_2}\frac{\partial u^+}{\partial y} = g_{inter}^+$$

Here, g_{inter} represents the heat flux through the interface. These two equations will ultimately be set equal to each other in order to obtain the interface equation. The derivative terms can now be approximated by utilizing the ghost point method.



Here, the upper point is the ghost point, and the derivative can be approximated by:

$$K_{yy_1}\frac{\partial u^-}{\partial y} = K_{yy_1}\frac{U_{i,j+1} - U_{i-1}}{2\Delta y_1} = g_{inter}^-$$

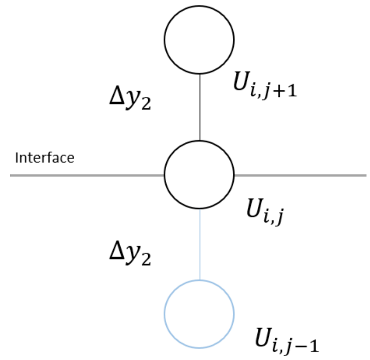
Solving for the ghost point yields:

$$U_{i,j+1} = 2\Delta y_1 g_{inter}^- + U_{i,j-1}$$

Plugging this ghost point into the governing heat equation equation and solving for g_{inter}^{-} produces:

$$g_{inter}^{-} = -\left(\frac{1}{\Delta y_1}\right)U_{i,j-1} + \left(-\frac{1}{2}\frac{K_{xx}}{K_{yy_1}}\frac{\Delta y_1}{\Delta x^2}\right)U_{i-1,j} + \left(\frac{K_{xx}}{K_{yy_1}}\frac{\Delta y_1}{\Delta x^2} + \frac{1}{\Delta y_1}\right)U_{i,j} + \left(-\frac{1}{2}\frac{K_{xx}}{K_{yy_1}}\frac{\Delta y_1}{\Delta x^2}\right)U_{i+1,j}$$

Now the same process must be performed for the other ghost point.



Here, the lower point is the ghost point, and the derivative can be approximated by:

$$K_{yy_2}\frac{\partial u^{+}}{\partial y} = K_{yy_2}\frac{U_{i,j+1} - U_{i-1}}{2\Delta y_2} = g_{inter}^{+}$$

Solving for the ghost point yields:

$$U_{i,j-1} = -2\Delta y_2 g_{inter}^{+} + U_{i,j+1}$$

Plugging this ghost point into the governing heat equation and solving for g_{inter}^{-} produces:

$$g_{inter}^{+} = \left(\frac{1}{\Delta y_1}\right)U_{i,j-1} + \left(\frac{1}{2}\frac{K_{xx}}{K_{yy_2}}\frac{\Delta y_2}{\Delta x^2}\right)U_{i-1,j} - \left(\frac{K_{xx}}{K_{yy_2}}\frac{\Delta y_2}{\Delta x^2} + \frac{1}{\Delta y_2}\right)U_{i,j} + \left(\frac{1}{2}\frac{K_{xx}}{K_{yy_2}}\frac{\Delta y_2}{\Delta x^2}\right)U_{i+1,j} + \left(\frac{1}{\Delta y_2}\right)U_{i,j+1}$$

We can now set the g equations equal to each other and produce the interface equation:

$$\left(\frac{1}{\Delta y_1}\right)U_{i,j-1} + \left(\frac{1}{2}\frac{K_{xx}}{K_{yy_2}}\frac{\Delta y_2}{\Delta x^2}\right)U_{i-1,j} - \left(\frac{1}{\Delta y_1} + \frac{K_{xx}}{K_{yy_1}}\frac{\Delta y_1}{\Delta x^2} + \frac{K_{xx}}{K_{yy_2}}\frac{\Delta y_2}{\Delta x^2} + \frac{1}{\Delta y_2}\right)U_{i,j} + \left(\frac{1}{2}\frac{K_{xx}}{K_{yy_2}}\frac{\Delta y_2}{\Delta x^2}\right)U_{i+1,j} + \left(\frac{1}{\Delta y_2}\right)U_{i,j+1} = 0$$

This equation is valid for all points along the material interface.

3.3 Non-Conformal Mesh

The formulation of the non-conformal mesh can be simply generated with a series of evenly space points between the bounds of the bar.

$$\Delta y = \frac{y_3 - y_1}{N}$$

Here, N is the number of elements the rod is divided into.

3.4 Conformal Mesh

In order to avoid the issue of having a node not located on the interface location, we can define are mesh such that it will always fall on the interface location. Delta x will not have to conform to the interface. The conformal mesh can be derived with the following:

$$\begin{cases} \Delta y_1 = \dfrac{\bar{y} - y_1}{INT\left(\frac{N}{2}\right)}, & y_1 < y < \bar{y}, \\ \Delta y_2 = \dfrac{y_3 - \bar{y}}{INT\left(\frac{N}{2}\right)}, & \bar{y} < x < y_3 \end{cases}$$

4 Heat Flux Through the Top Boundary

4.1 Analytical Solution

The analytical solution to the total heat flux through the top boundary can be calculated using 1/3 Simpson Integration:

$$\dot{q} = \int_0^x -k * thickness * \frac{dT}{dy} * dx$$

Here,

$$\frac{dT}{dy} = \frac{d}{dy}\left(\frac{100\sinh(K\pi y)}{\sinh(K\pi)} * \sin(\pi x)\right)$$

Integrating yields:

$$\dot{q} = -200 * \frac{K}{\tanh\,(K * \pi)}$$

4.2 Numerical Computation

For the numerical computation of the heat flux through the upper boundary, the same equation used for the analytical solution can be used. However, $\frac{dT}{dy}$ must be derived with the following expression:

$$\frac{dT}{dy} \approx \frac{T_{i,jmax-2} - 4 * T_{i,jmax-1} + 3 * T_{i,jmax}}{2 * \Delta x}$$

The heat flux is then derived by applying Simpson integration with this equation.

5 Performance Analysis

5.1 Error

The errors of the numerical solutions are calculated using the equations below:

$$e_h = u(x) - u_h(x)$$

5.2 Percent Error

The percent error of an estimated quantity $Q_{Estimated}$ (calculated using FDM) against its exact values Q_{Exact} is calculated using the equation below:

$$\%Error = \left|\frac{Q_{Exact} - Q_{Estimated}}{Q_{Exact}}\right| \times 100\%$$

5.3 Extrapolation and Convergence

Richardson's Extrapolation was used to extrapolate an approximate of the exact value from a series of approximated values. In general, error is modeled as:

$$Q_{ex} - Q_h = Ch^{\beta}$$

Where Q is the quantity of interest, Q_h is the approximate value at some mesh size h, C is some constant, and β is the convergence rate. In general, it is rare for the exact value to be known, and it is often difficult or impossible to obtain analytical solutions. In this case it is possible to use Richardson's Extrapolation to obtain reasonably accurate approximate value of the exact solution. If we write this equation at another mesh size, say h/2, the two can be divided and the unknown β can be found.

$$Q_{ex} - Q_h = C(h)^{\beta}$$

$$Q_{ex} - Q_{\frac{h}{2}} = C\left(\frac{h}{2}\right)^{\beta}$$

$$\frac{Q_{ex} - Q_h}{Q_{ex} - Q_{\frac{h}{2}}} = \frac{C(h)^{\beta}}{C(\frac{h}{2})^{\beta}}$$

$$log\left(\frac{Q_{ex} - Q_h}{Q_{ex} - Q_{\frac{h}{2}}}\right) = log\left(\frac{C(h)^{\beta}}{C(\frac{h}{2})^{\beta}}\right)$$

$$log(Q_{ex} - Q_h) - log(Q_{ex} - Q_{\frac{h}{2}}) = \beta log(h) - \beta log\left(\frac{h}{2}\right)$$

$$\boxed{\beta = \frac{log(Q_{ex} - Q_h) - log(Q_{ex} - Q_{\frac{h}{2}})}{log(h) - log(\frac{h}{2})}}$$

Again, Richardson's Extrapolation will be used to derive an expression for an extrapolated value. Here we will have to utilize three mesh sizes rather than the previous two.

$$Q_{ex} - Q_h = C(h)^\beta \approx 2^\beta$$

$$Q_{ex} - Q_{\frac{h}{2}} = C\left(\frac{h}{2}\right)^\beta \approx 2^\beta$$

$$Q_{ex} - Q_{\frac{h}{4}} = C\left(\frac{h}{4}\right)^\beta \approx 2^\beta$$

$$\frac{Q_{ex} - Q_h}{Q_{ex} - Q_{\frac{h}{2}}} \approx 2^\beta \approx \frac{Q_{ex} - Q_{\frac{h}{2}}}{Q_{ex} - Q_{\frac{h}{4}}}$$

$$\frac{Q_{extr} - Q_h}{Q_{extr} - Q_{\frac{h}{2}}} = 2^\beta = \frac{Q_{extr} - Q_{\frac{h}{2}}}{Q_{extr} - Q_{\frac{h}{4}}}$$

$$(Q_{extr} - Q_h)(Q_{extr} - Q_{\frac{h}{4}}) = (Q_{extr} - Q_{\frac{h}{2}})^2$$

$$\boxed{Q_{extr} = \frac{Q_{\frac{h}{2}}^2 - Q_h * Q_{\frac{h}{4}}}{2Q_{\frac{h}{2}} - Q_h - Q_{\frac{h}{4}}}}$$

From here Q_{extra} can be substituted to solve for β, which yields:

$$\boxed{\frac{log\left(\frac{Q_{extr} - Q_h}{Q_{extr} - Q_{\frac{h}{2}}}\right)}{log(2)} = \beta = \frac{log\left(\frac{Q_{extr} - Q_{\frac{h}{2}}}{Q_{extr} - Q_{\frac{h}{4}}}\right)}{log(2)}}$$

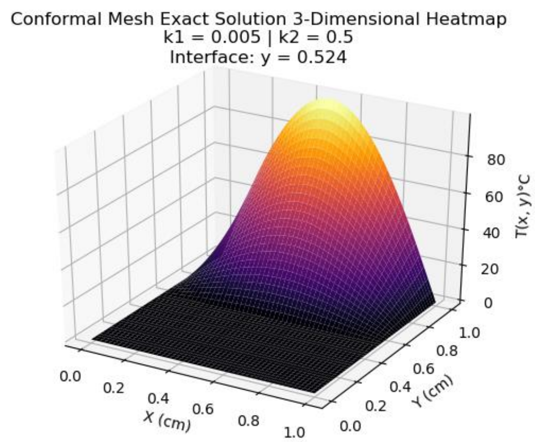
6 Results

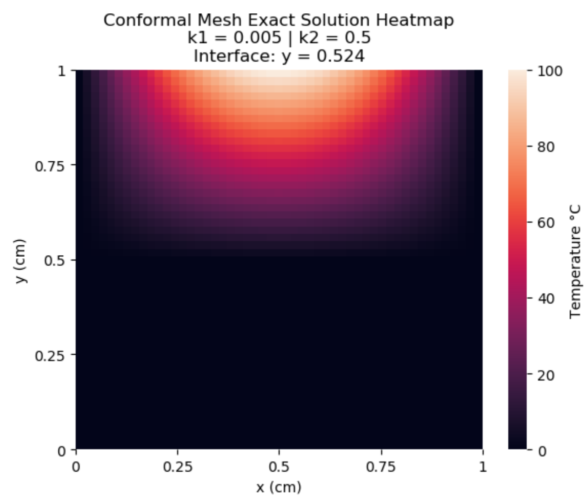
6.1 Conformal Mesh Results

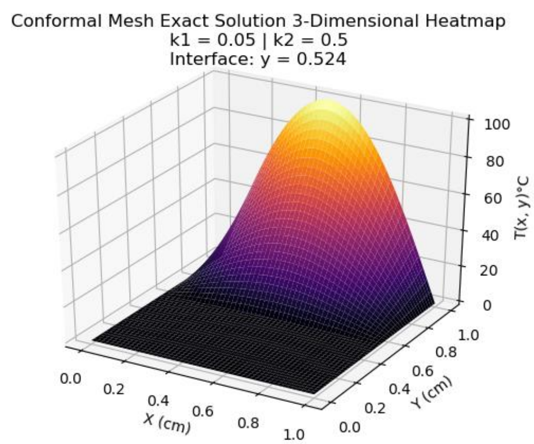
The following section outlines the analytical and second order FDM results with the utilization of a conformal mesh. Additionally, the heat flux and convergence of the approximate and extrapolation solution are provided.

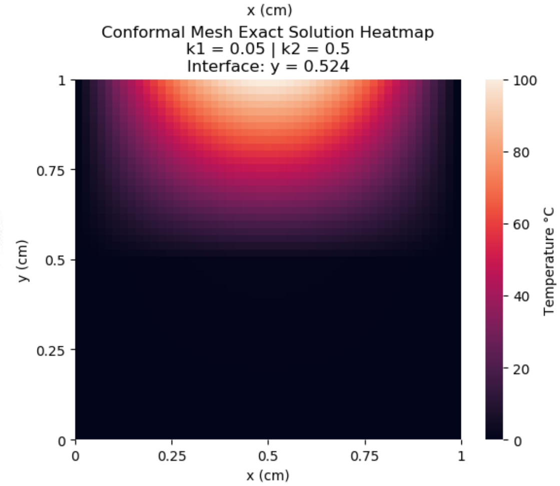
6.1.1 Conformal Temperature Analytical Results

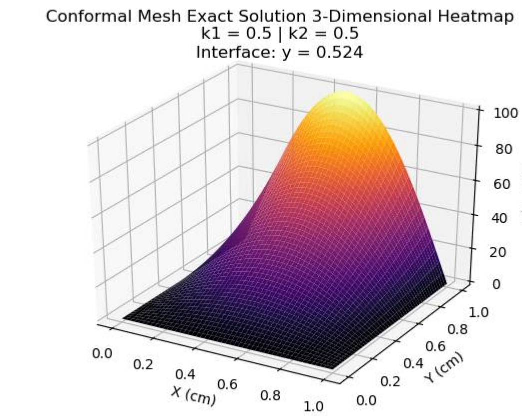
The following section displays the exact solution to the problem with the utilization of an analytical mesh. The results in the following section outline the FDM solution to the conformal mesh and should align with the solutions provide in this section.

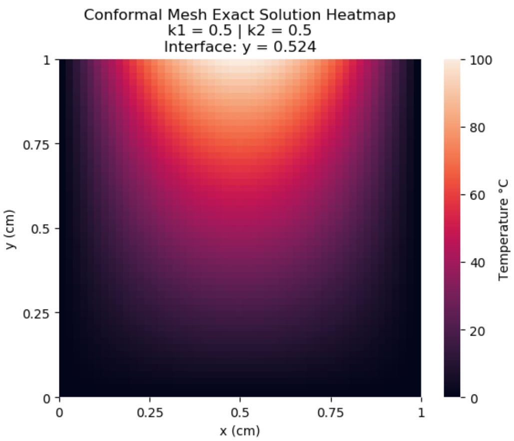


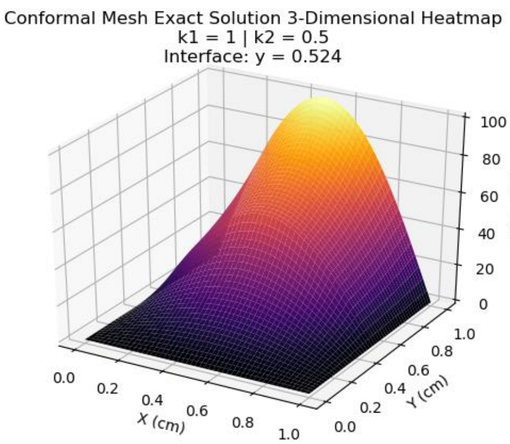


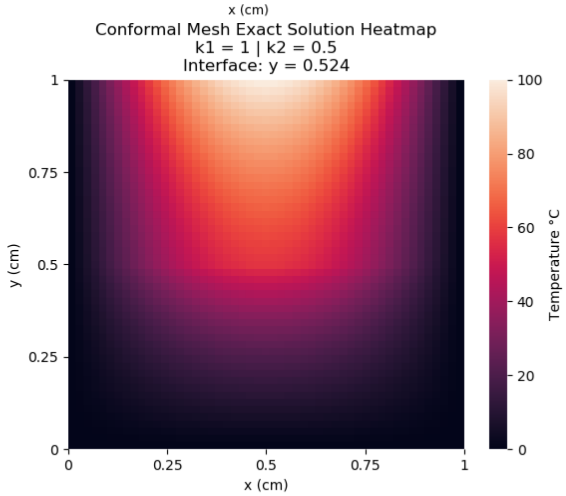


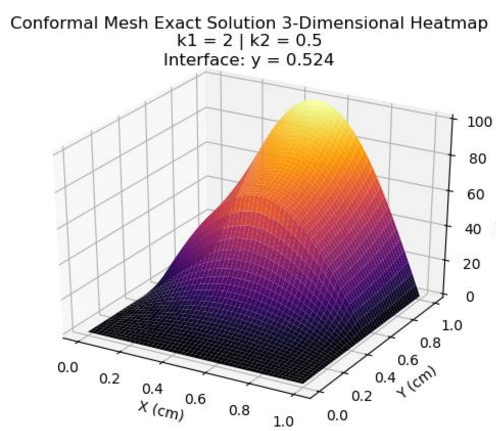


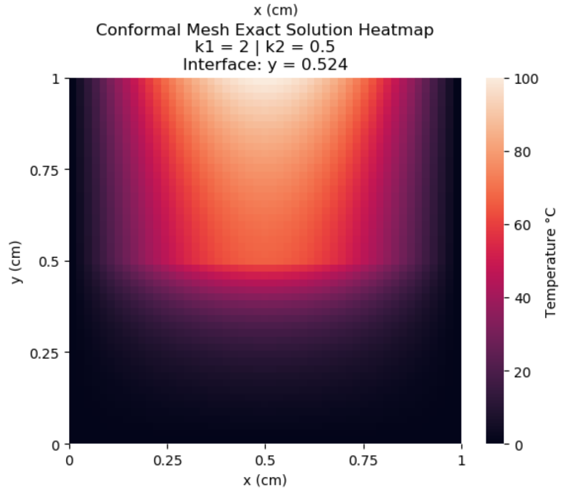

Conformal Mesh Exact Solution 3-Dimensional Heatmap
k1 = 0.5 | k2 = 0.5
Interface: y = 0.524
T(x, y)°C
X (cm)
Y (cm)

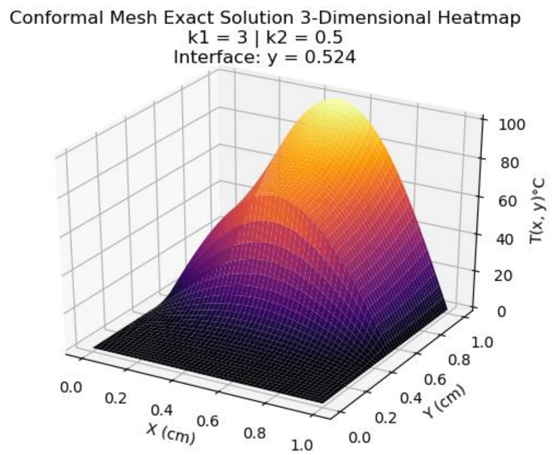

Conformal Mesh Exact Solution Heatmap
k1 = 0.5 | k2 = 0.5
Interface: y = 0.524
y (cm)
x (cm)
Temperature °C

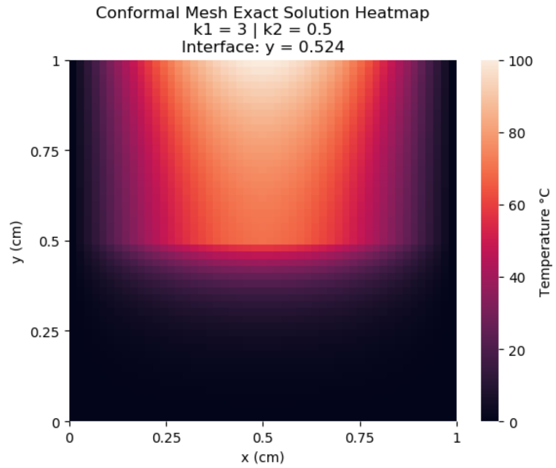

Conformal Mesh Exact Solution 3-Dimensional Heatmap
k1 = 1 | k2 = 0.5
Interface: y = 0.524
T(x, y)°C
X (cm)
Y (cm)


Conformal Mesh Exact Solution Heatmap
k1 = 1 | k2 = 0.5
Interface: y = 0.524
y (cm)
x (cm)
Temperature °C


Conformal Mesh Exact Solution 3-Dimensional Heatmap
k1 = 2 | k2 = 0.5
Interface: y = 0.524
T(x, y)°C
X (cm)
Y (cm)


Conformal Mesh Exact Solution Heatmap
k1 = 2 | k2 = 0.5
Interface: y = 0.524
y (cm)
x (cm)
Temperature °C

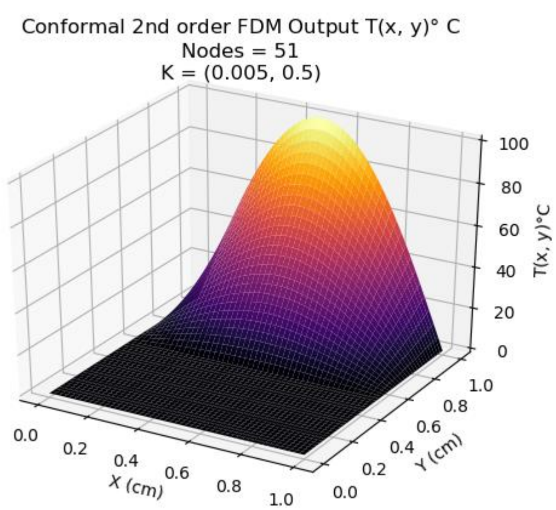


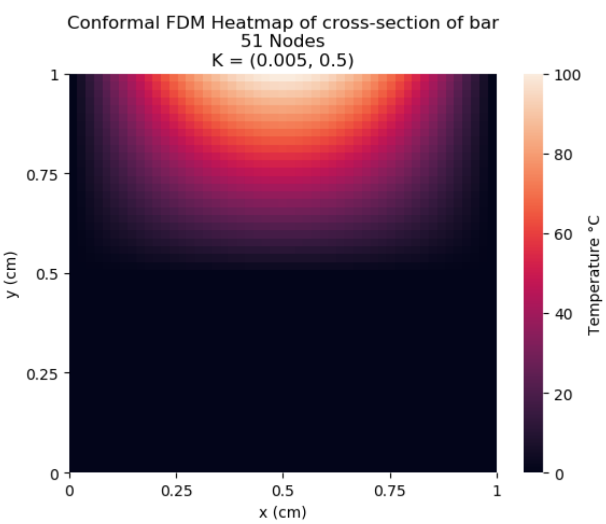


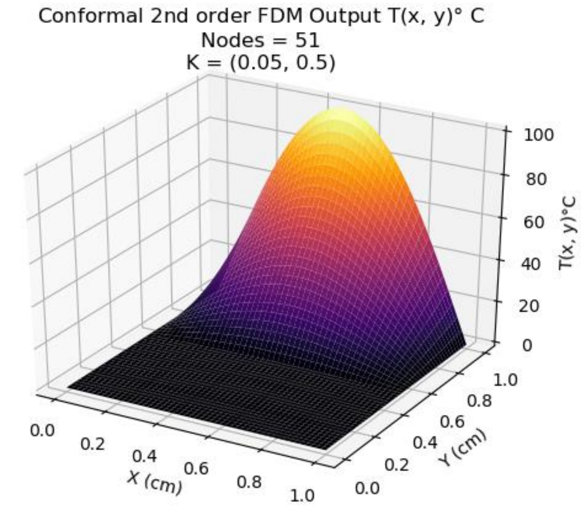
It is interesting to note how the heat transfer differs based upon the relative values of the heat conductivity of the two materials. Additionally, when the two materials have the same thermal conductivity, the solution represents a single material case, as expected.

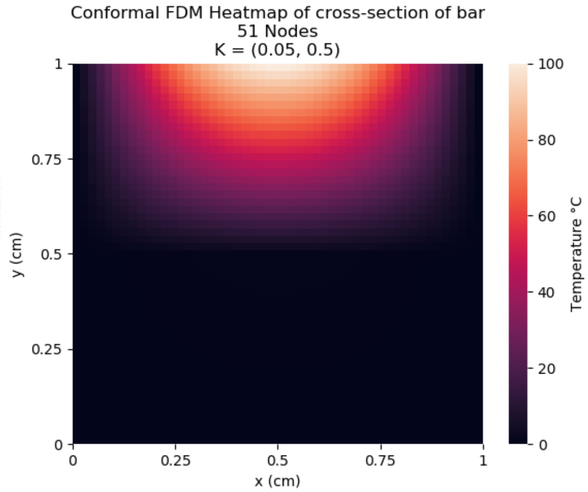
6.1.2 Conformal Temperature FDM Results

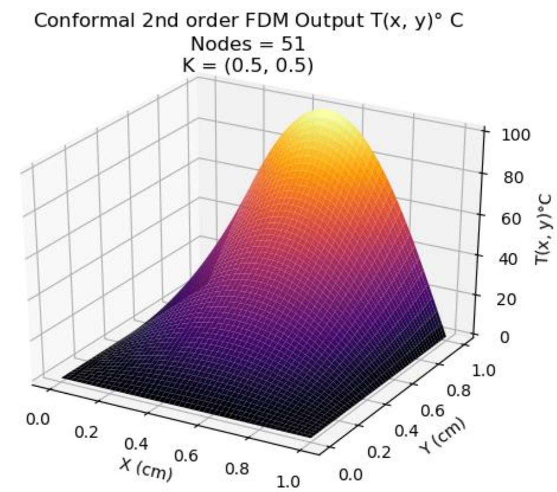
The following section displays the second order FDM solution to the problem with the utilization of an analytical mesh. The results in the previous section outline the exact solution to the conformal mesh and should align with the solutions provide in this section.

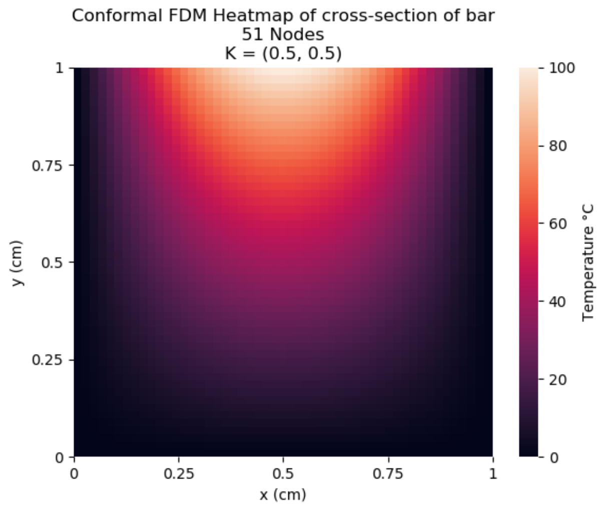


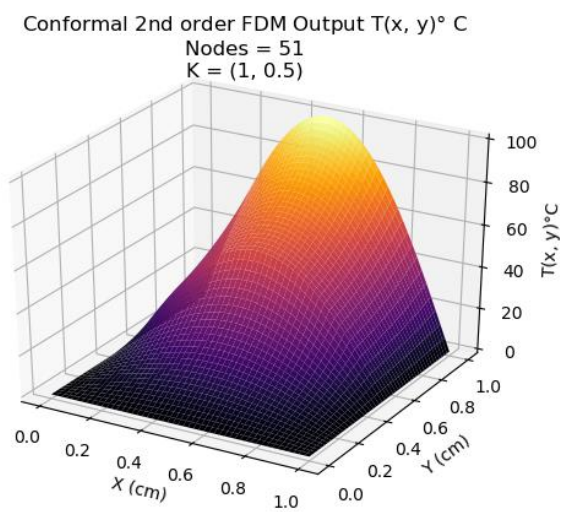


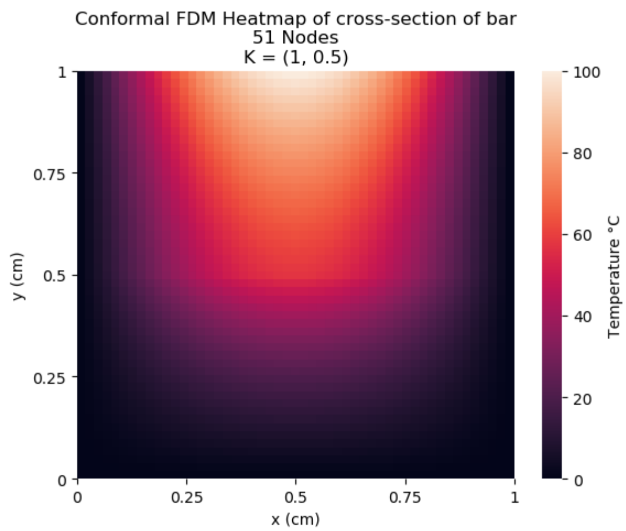

Conformal 2nd order FDM Output T(x, y)° C
Nodes = 51
K = (0.05, 0.5)
X (cm)
Y (cm)
T(x, y)°C

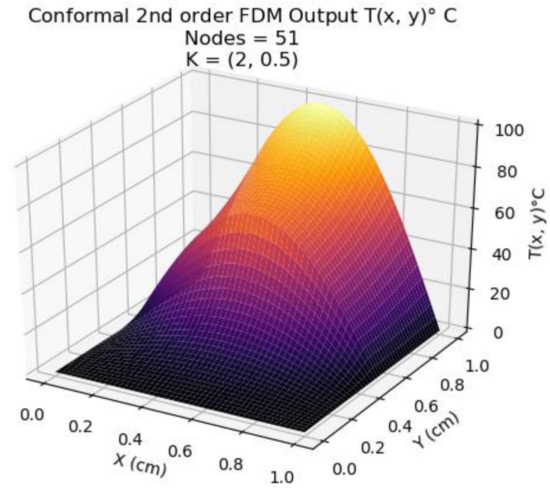

Conformal FDM Heatmap of cross-section of bar
51 Nodes
K = (0.05, 0.5)
x (cm)
y (cm)
Temperature °C

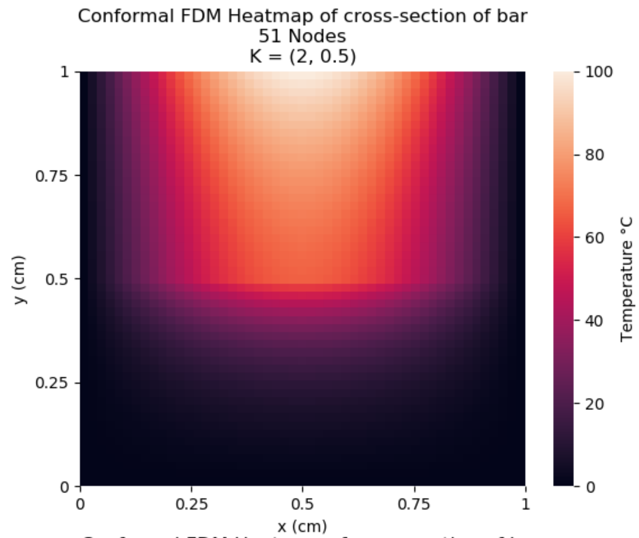

Conformal 2nd order FDM Output T(x, y)° C
Nodes = 51
K = (0.5, 0.5)
X (cm)
Y (cm)
T(x, y)°C

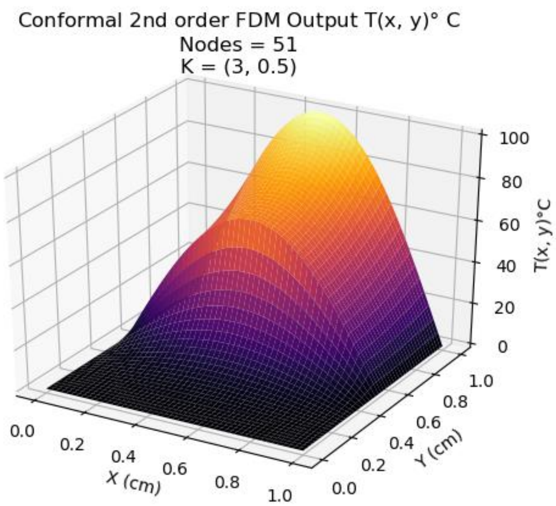

Conformal FDM Heatmap of cross-section of bar
51 Nodes
K = (0.5, 0.5)
x (cm)
y (cm)
Temperature °C

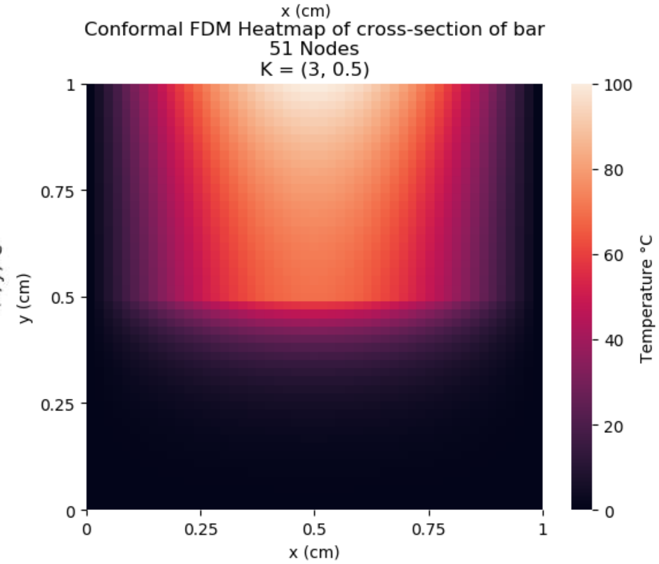

Conformal 2nd order FDM Output T(x, y)° C
Nodes = 51
K = (1, 0.5)
X (cm)
Y (cm)
T(x, y)°C


Conformal FDM Heatmap of cross-section of bar
51 Nodes
K = (1, 0.5)
x (cm)
y (cm)
Temperature °C


Conformal 2nd order FDM Output T(x, y)° C
Nodes = 51
K = (2, 0.5)
T(x, y)°C
X (cm)
Y (cm)

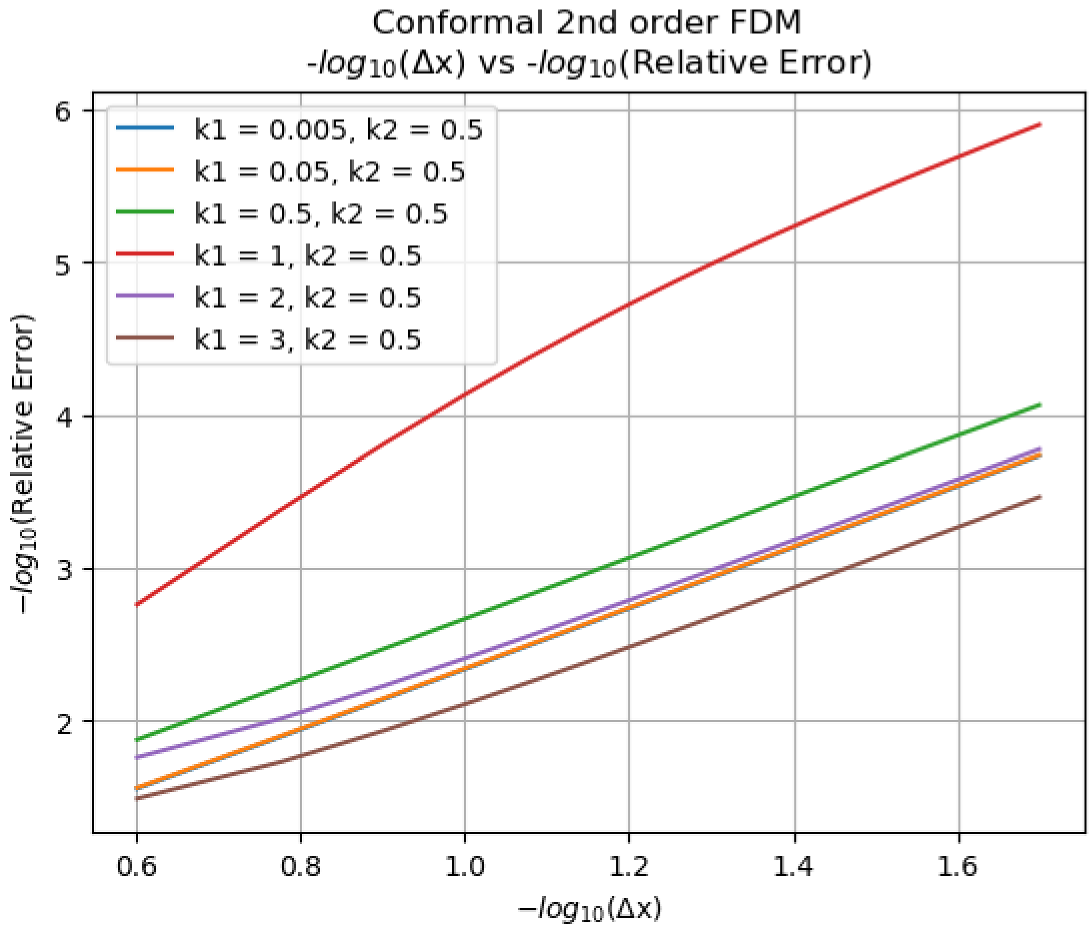

Conformal FDM Heatmap of cross-section of bar
51 Nodes
K = (2, 0.5)
y (cm)
x (cm)
Temperature °C


Conformal 2nd order FDM Output T(x, y)° C
Nodes = 51
K = (3, 0.5)
T(x, y)°C
X (cm)
Y (cm)


Conformal FDM Heatmap of cross-section of bar
51 Nodes
K = (3, 0.5)
y (cm)
x (cm)
Temperature °C

6.1.3 Conformal Temperature FDM Convergence

The following section presents the convergence rates of the temperature of the midpoint of the interface location of the FDM solution. The data is presented for multiple values of thermal conductivity.



Conformal Convergence of 2nd Order FDM Temperature Solution (k1 = 0.005, k2 = 0.5):

Num. Elements	dx	Exact Midpoint Temp	Approx. Midpoint Temp	Percent Error	Beta
4	0.25	0.0100267	0.010304	2.76642	n/a
6	0.166667	0.0100267	0.0101532	1.26234	1.9350369047947347
8	0.125	0.0100267	0.0100985	0.716642	1.967947832578277
10	0.1	0.0100267	0.0100728	0.460613	1.980871447427557
12	0.0833333	0.0100267	0.0100588	0.320613	1.9872818350476524
14	0.0714286	0.0100267	0.0100503	0.235882	1.9909295672149674
16	0.0625	0.0100267	0.0100448	0.180761	1.9932037703047714
18	0.0555556	0.0100267	0.010041	0.142912	1.9947173784814527
20	0.05	0.0100267	0.0100383	0.115811	1.9957757890098342
22	0.0454545	0.0100267	0.0100363	0.0957427	1.99654511976195
24	0.0416667	0.0100267	0.0100347	0.0804706	1.9971214885995068
26	0.0384615	0.0100267	0.0100335	0.0685801	1.997564756355294
28	0.0357143	0.0100267	0.0100326	0.059142	1.9979131325831565
30	0.0333333	0.0100267	0.0100318	0.0515256	1.9981920073470831
32	0.03125	0.0100267	0.0100312	0.0452908	1.9984168588702598
34	0.0294118	0.0100267	0.0100307	0.0401226	1.9986038214481765
36	0.0277778	0.0100267	0.0100303	0.0357909	1.9987600660409732
38	0.0263158	0.0100267	0.0100299	0.0321245	1.9988893318493146
40	0.025	0.0100267	0.0100296	0.0289939	1.9990022434946713
42	0.0238095	0.0100267	0.0100293	0.0262995	1.9990958724235852
44	0.0227273	0.0100267	0.0100291	0.0239638	1.999177083304061
46	0.0217391	0.0100267	0.0100289	0.0219261	1.9992506205830418
48	0.0208333	0.0100267	0.0100287	0.0201375	1.9993126312827603
50	0.02	0.0100267	0.0100285	0.0185592	1.9993694253335448

Conformal Convergence of 2nd Order FDM Temperature Solution (k1 = 0.05, k2 = 0.5):

Num. Elements	dx	Exact Midpoint Temp	Approx. Midpoint Temp	Percent Error	Beta
4	0.25	0.987782	1.01479	2.73418	n/a
6	0.166667	0.987782	1.00011	1.24757	1.9351378661514578
8	0.125	0.987782	0.994778	0.708254	1.9679832891574638
10	0.1	0.987782	0.992278	0.45522	1.9808892952961064
12	0.0833333	0.987782	0.990912	0.316858	1.9872926369238102
14	0.0714286	0.987782	0.990085	0.233119	1.9909368267339715
16	0.0625	0.987782	0.989546	0.178644	1.9932089547620766
18	0.0555556	0.987782	0.989177	0.141238	1.9947213634583187
20	0.05	0.987782	0.988912	0.114454	1.9957789288237733
22	0.0454545	0.987782	0.988716	0.0946212	1.996547492169034
24	0.0416667	0.987782	0.988567	0.079528	1.9971235929151667
26	0.0384615	0.987782	0.988451	0.0677767	1.9975665638106161
28	0.0357143	0.987782	0.988359	0.0584491	1.997914497529045
30	0.0333333	0.987782	0.988285	0.050922	1.9981927707181026
32	0.03125	0.987782	0.988224	0.0447603	1.9984188252433368
34	0.0294118	0.987782	0.988173	0.0396526	1.9986049495798721
36	0.0277778	0.987782	0.988131	0.0353716	1.998760041804606
38	0.0263158	0.987782	0.988095	0.0317482	1.9988906130599142
40	0.025	0.987782	0.988065	0.0286542	1.9990016228423466
42	0.0238095	0.987782	0.988039	0.0259914	1.9990967105797577
44	0.0227273	0.987782	0.988016	0.0236831	1.9991788684234064
46	0.0217391	0.987782	0.987996	0.0216692	1.9992502940541264
48	0.0208333	0.987782	0.987978	0.0199016	1.9993127944435323
50	0.02	0.987782	0.987963	0.0183418	1.9993677685028186

Conformal Convergence of 2nd Order FDM Temperature Solution (k1 = 0.5, k2 = 0.5):

Num. Elements	dx	Exact Midpoint Temp	Approx. Midpoint Temp	Percent Error	Beta
4	0.25	39.9071	40.4361	1.32572	n/a
6	0.166667	39.9071	40.1447	0.595588	1.9734416682007774
8	0.125	39.9071	40.0413	0.336289	1.9868369553483343
10	0.1	39.9071	39.9931	0.215603	1.992131784397899
12	0.0833333	39.9071	39.9669	0.149867	1.9947646377560748
14	0.0714286	39.9071	39.951	0.11017	1.996264635694922
16	0.0625	39.9071	39.9407	0.0843806	1.9972004504922818
18	0.0555556	39.9071	39.9337	0.0666881	1.9978236006389438
20	0.05	39.9071	39.9286	0.0540273	1.9982594580523614
22	0.0454545	39.9071	39.9249	0.0446567	1.9985762646337797
24	0.0416667	39.9071	39.922	0.0375279	1.9988137689853336
26	0.0384615	39.9071	39.9198	0.031979	1.9989964071371185
28	0.0357143	39.9071	39.9181	0.0275755	1.9991398722785456
30	0.0333333	39.9071	39.9167	0.0240226	1.999254622096276
32	0.03125	39.9071	39.9155	0.0211145	1.9993478393100705
34	0.0294118	39.9071	39.9145	0.0187041	1.9994246008255874
36	0.0277778	39.9071	39.9137	0.0166841	1.9994885553251875
38	0.0263158	39.9071	39.913	0.0149745	1.9995424122688028
40	0.025	39.9071	39.9125	0.0135148	1.9995881849122643
42	0.0238095	39.9071	39.912	0.0122585	1.9996274247146846
44	0.0227273	39.9071	39.9115	0.0111696	1.9996612883997489
46	0.0217391	39.9071	39.9111	0.0102196	1.9996907693792072
48	0.0208333	39.9071	39.9108	0.0093858	1.9997165314167644
50	0.02	39.9071	39.9105	0.00865005	1.9997392057546235

Conformal Convergence of 2nd Order FDM Temperature Solution (k1 = 1, k2 = 0.5):

Num. Elements	dx	Exact Midpoint Temp	Approx. Midpoint Temp	Percent Error	Beta
4	0.25	57.6326	57.7324	0.173188	n/a
6	0.166667	57.6326	57.6564	0.0412577	3.5380113070520194
8	0.125	57.6326	57.6415	0.0153982	3.425953250242716
10	0.1	57.6326	57.6369	0.00744345	3.257627390453988
12	0.0833333	57.6326	57.635	0.00424044	3.0861200309706556
14	0.0714286	57.6326	57.6342	0.0026992	2.930341064083107
16	0.0625	57.6326	57.6337	0.00185822	2.7958640607780447
18	0.0555556	57.6326	57.6334	0.0013548	2.68256631356813
20	0.05	57.6326	57.6332	0.00103145	2.5881303125645037
22	0.0454545	57.6326	57.6331	0.000812024	2.5096622343603587
24	0.0416667	57.6326	57.633	0.000656446	2.4443825515423154
26	0.0384615	57.6326	57.6329	0.000542153	2.389873189955925
28	0.0357143	57.6326	57.6329	0.000455698	2.344125300536319
30	0.0333333	57.6326	57.6328	0.000388684	2.3055083987376177
32	0.03125	57.6326	57.6328	0.000335657	2.272712620590276
34	0.0294118	57.6326	57.6328	0.000292951	2.2446905073152994
36	0.0277778	57.6326	57.6327	0.000258031	2.2206031024565784
38	0.0263158	57.6326	57.6327	0.000229096	2.19977865618121
40	0.025	57.6326	57.6327	0.000204842	2.18167518054226
42	0.0238095	57.6326	57.6327	0.0001843	2.165854158627069
44	0.0227273	57.6326	57.6327	0.000166743	2.151957715034113
46	0.0217391	57.6326	57.6327	0.000151615	2.1396973178163865
48	0.0208333	57.6326	57.6327	0.000138482	2.128828364797785
50	0.02	57.6326	57.6327	0.000127006	2.119154774056971

Conformal Convergence of 2nd Order FDM Temperature Solution (k1 = 2, k2 = 0.5):

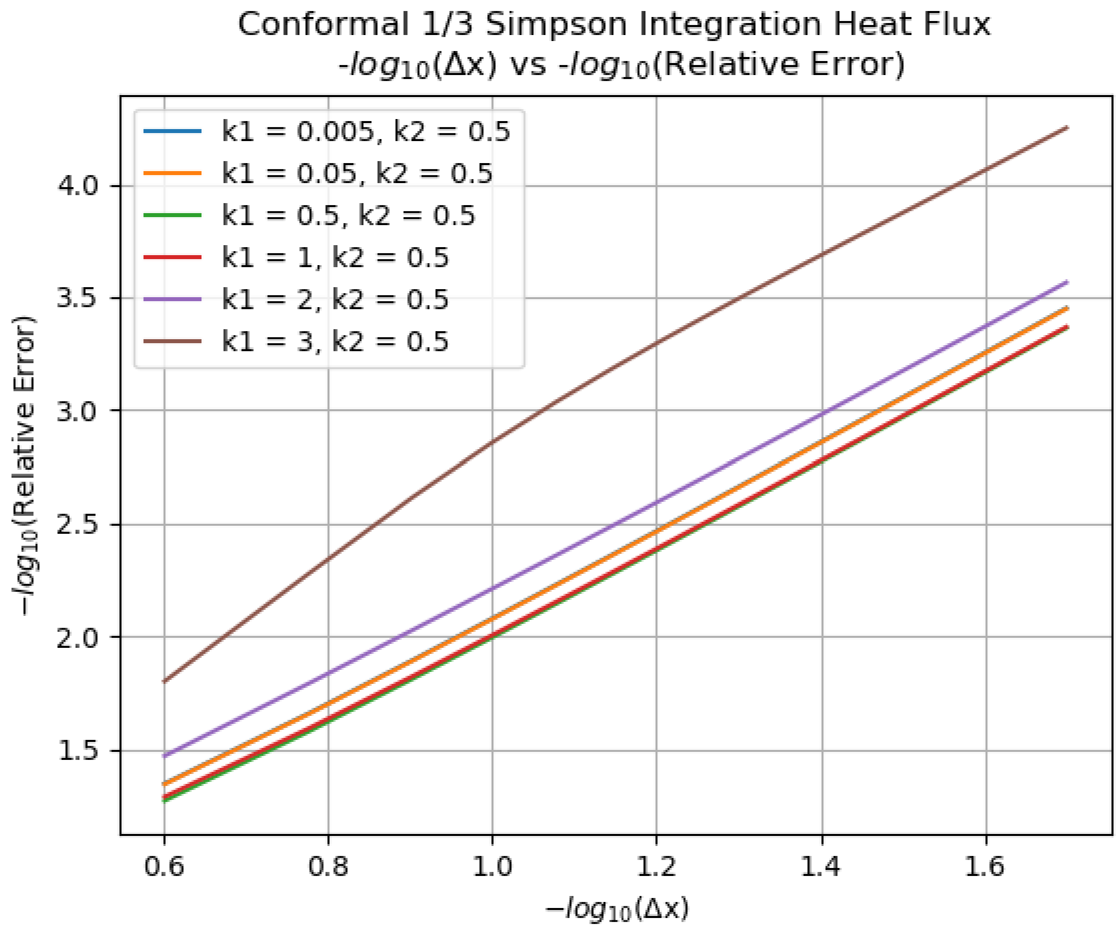
Num. Elements	dx	Exact Midpoint Temp	Approx. Midpoint Temp	Percent Error	Beta
4	0.25	66.7147	65.5601	1.73068	n/a
6	0.166667	66.7147	66.0734	0.961253	1.450264414867067
8	0.125	66.7147	66.3226	0.587707	1.7102489530515539
10	0.1	66.7147	66.4536	0.39143	1.8213495502982078
12	0.0833333	66.7147	66.5293	0.277885	1.8790945042987286
14	0.0714286	66.7147	66.5767	0.206922	1.9128547712363744
16	0.0625	66.7147	66.6081	0.159821	1.934259831063974
18	0.0555556	66.7147	66.6299	0.127044	1.9486664130859208
20	0.05	66.7147	66.6457	0.103353	1.958818013440129
22	0.0454545	66.7147	66.6575	0.0856913	1.966236375594274
24	0.0416667	66.7147	66.6665	0.0721812	1.9718199933261493
26	0.0384615	66.7147	66.6736	0.0616212	1.9761267870998962
28	0.0357143	66.7147	66.6792	0.0532133	1.9795178972319547
30	0.0333333	66.7147	66.6837	0.0464115	1.9822353771349859
32	0.03125	66.7147	66.6875	0.0408323	1.9844463520039368
34	0.0294118	66.7147	66.6905	0.0361999	1.986269195570343
36	0.0277778	66.7147	66.6931	0.032312	1.9877896502609482
38	0.0263158	66.7147	66.6953	0.0290174	1.9890710287651303
40	0.025	66.7147	66.6972	0.0262014	1.9901609225846806
42	0.0238095	66.7147	66.6988	0.0237758	1.9910956422998722
44	0.0227273	66.7147	66.7002	0.0216716	1.9919033037371625
46	0.0217391	66.7147	66.7015	0.0198346	1.9926058976105243
48	0.0208333	66.7147	66.7025	0.0182214	1.9932209119044593
50	0.02	66.7147	66.7035	0.0167971	1.993762288350547

Conformal Convergence of 2nd Order FDM Temperature Solution (k1 = 3, k2 = 0.5):

Num. Elements	dx	Exact Midpoint Temp	Approx. Midpoint Temp	Percent Error	Beta
4	0.25	69.9292	67.6796	3.21693	n/a
6	0.166667	69.9292	68.6383	1.84599	1.3698125695191223
8	0.125	69.9292	69.1222	1.15408	1.6327454317058698
10	0.1	69.9292	69.3844	0.779009	1.7613601850256917
12	0.0833333	69.9292	69.5392	0.557648	1.8335444669936751
14	0.0714286	69.9292	69.6372	0.417494	1.8777596643483339
16	0.0625	69.9292	69.7029	0.323653	1.9066457792844544
18	0.0555556	69.9292	69.7488	0.25795	1.926484717911627
20	0.05	69.9292	69.7822	0.21025	1.9406647048442545
22	0.0454545	69.9292	69.8071	0.174571	1.9511348960328232
24	0.0416667	69.9292	69.8263	0.147212	1.9590770850574246
26	0.0384615	69.9292	69.8412	0.125784	1.9652398206856247
28	0.0357143	69.9292	69.8532	0.108697	1.970115062559069
30	0.0333333	69.9292	69.8629	0.0948572	1.9740364963833228
32	0.03125	69.9292	69.8708	0.0834931	1.9772367093000318
34	0.0294118	69.9292	69.8774	0.0740496	1.9798817141590401
36	0.0277778	69.9292	69.883	0.066118	1.9820925124834163
38	0.0263158	69.9292	69.8877	0.0593929	1.9839589341242245
40	0.025	69.9292	69.8917	0.0536418	1.9855487876371554
42	0.0238095	69.9292	69.8952	0.0486858	1.9869140102514506
44	0.0227273	69.9292	69.8982	0.044385	1.9880949305825621
46	0.0217391	69.9292	69.9008	0.040629	1.9891232129712533
48	0.0208333	69.9292	69.9031	0.0373296	1.9900240433333463
50	0.02	69.9292	69.9051	0.0344158	1.9908176023363782

6.1.4 Conformal Heat Flux FDM Results

The following section presents the convergence rates of the heat flux of the upper boundary of the bar with a conformal mesh FDM solution. The data is presented for multiple values of thermal conductivity.



Conformal Convergence of Heat Flux (k1 = 0.005, k2 = 0.5):

Num. Elements	dx	Exact Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-157.679	-150.62	4.47655	n/a
6	0.166667	-157.679	-154.241	2.18051	1.7739985109212075
8	0.125	-157.679	-155.668	1.27534	1.8643671264394837
10	0.1	-157.679	-156.364	0.833884	1.9040416260465258
12	0.0833333	-157.679	-156.753	0.586934	1.9261653630975284
14	0.0714286	-157.679	-156.993	0.435211	1.9401870791433726
16	0.0625	-157.679	-157.15	0.335449	1.9498281863257774
18	0.0555556	-157.679	-157.259	0.266396	1.9568445673818269
20	0.05	-157.679	-157.337	0.216643	1.9621694502824134
22	0.0454545	-157.679	-157.396	0.179619	1.9663430843466478
24	0.0416667	-157.679	-157.44	0.151328	1.9696991303270706
26	0.0384615	-157.679	-157.475	0.129227	1.9724544119386263
28	0.0357143	-157.679	-157.503	0.111634	1.9747557382456467
30	0.0333333	-157.679	-157.525	0.0974021	1.9767059447066797
32	0.03125	-157.679	-157.544	0.0857268	1.9783791450214931
34	0.0294118	-157.679	-157.559	0.0760309	1.9798300700564904
36	0.0277778	-157.679	-157.572	0.067891	1.9810999868175572
38	0.0263158	-157.679	-157.583	0.0609912	1.9822205923065
40	0.025	-157.679	-157.592	0.055092	1.9832166181486954
42	0.0238095	-157.679	-157.6	0.0500088	1.9841076439711107
44	0.0227273	-157.679	-157.607	0.0455979	1.9849093606712775
46	0.0217391	-157.679	-157.613	0.0417457	1.985634506012005
48	0.0208333	-157.679	-157.618	0.0383617	1.9862934942031105
50	0.02	-157.679	-157.623	0.0353731	1.9868949540191196

Conformal Convergence of Heat Flux (k1 = 0.05, k2 = 0.5):

Num. Elements	dx	Exact Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-156.487	-149.443	4.50138	n/a
6	0.166667	-156.487	-153.055	2.19294	1.7736215732512968
8	0.125	-156.487	-154.479	1.2827	1.8641171932834744
10	0.1	-156.487	-155.174	0.838732	1.9038632473756567
12	0.0833333	-156.487	-155.563	0.590361	1.9260289357816505
14	0.0714286	-156.487	-155.802	0.43776	1.940077497753537
16	0.0625	-156.487	-155.959	0.337417	1.9497370229918245
18	0.0555556	-156.487	-156.067	0.267962	1.95676672749194
20	0.05	-156.487	-156.146	0.217918	1.9621016506465745
22	0.0454545	-156.487	-156.204	0.180677	1.9662830987145352
24	0.0416667	-156.487	-156.249	0.15222	1.9696453859962537
26	0.0384615	-156.487	-156.283	0.129989	1.972405759976969
28	0.0357143	-156.487	-156.311	0.112293	1.9747113156921963
30	0.0333333	-156.487	-156.333	0.0979771	1.9766650876697567
32	0.03125	-156.487	-156.352	0.0862332	1.978341332778602
34	0.0294118	-156.487	-156.367	0.0764801	1.9797948867565194
36	0.0277778	-156.487	-156.38	0.0682922	1.981067095540224
38	0.0263158	-156.487	-156.391	0.0613518	1.9821897163830344
40	0.025	-156.487	-156.4	0.0554178	1.9831875274814907
42	0.0238095	-156.487	-156.408	0.0503046	1.9840801460081356
44	0.0227273	-156.487	-156.415	0.0458676	1.9848832912061642
46	0.0217391	-156.487	-156.421	0.0419927	1.9856097253285978
48	0.0208333	-156.487	-156.426	0.0385888	1.9862698824272826
50	0.02	-156.487	-156.431	0.0355825	1.9868724057322413

Conformal Convergence of Heat Flux (k1 = 0.5, k2 = 0.5):

Num. Elements	dx	Exact Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-109.033	-103.225	5.3265	n/a
6	0.166667	-109.033	-106.173	2.62273	1.7473222217138373
8	0.125	-109.033	-107.352	1.54223	1.8457469988306605
10	0.1	-109.033	-107.93	1.01155	1.8900327434091084
12	0.0833333	-109.033	-108.255	0.713433	1.9149991156575679
14	0.0714286	-109.033	-108.456	0.529766	1.9309248103433372
16	0.0625	-109.033	-108.587	0.40876	1.9419235303281768
18	0.0555556	-109.033	-108.679	0.32488	1.949954356951327
20	0.05	-109.033	-108.745	0.264374	1.9560649752712709
22	0.0454545	-109.033	-108.794	0.219307	1.960864637738539
24	0.0416667	-109.033	-108.832	0.184845	1.9647309419616277
26	0.0384615	-109.033	-108.861	0.157906	1.9679099613025608
28	0.0357143	-109.033	-108.884	0.136451	1.9705687048715579
30	0.0333333	-109.033	-108.903	0.119087	1.9728243958091647
32	0.03125	-109.033	-108.919	0.104837	1.9747616663445422
34	0.0294118	-109.033	-108.932	0.0929989	1.9764431134388445
36	0.0277778	-109.033	-108.943	0.0830575	1.9779159991096902
38	0.0263158	-109.033	-108.952	0.0746285	1.979216673485578
40	0.025	-109.033	-108.96	0.0674201	1.9803735279364167
42	0.0238095	-109.033	-108.966	0.0612075	1.981409066655953
44	0.0227273	-109.033	-108.972	0.0558155	1.9823413337802616
46	0.0217391	-109.033	-108.977	0.0511056	1.983185004998966
48	0.0208333	-109.033	-108.982	0.0469676	1.9839520684707816
50	0.02	-109.033	-108.986	0.0433125	1.9846524829543963

Conformal Convergence of Heat Flux (k1 = 1, k2 = 0.5):

Num. Elements	dx	Exact Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-87.4207	-82.9475	5.11689	n/a
6	0.166667	-87.4207	-85.2007	2.53947	1.7278731772128582
8	0.125	-87.4207	-86.1087	1.50078	1.8282881068649488
10	0.1	-87.4207	-86.5573	0.987692	1.874902018828227
12	0.0833333	-87.4207	-86.8103	0.698285	1.9018291343661862
14	0.0714286	-87.4207	-86.9666	0.519445	1.9193290743193159
16	0.0625	-87.4207	-87.0699	0.401349	1.9315928600216563
18	0.0555556	-87.4207	-87.1416	0.31934	1.9406529596023208
20	0.05	-87.4207	-87.1933	0.260097	1.9476133101255255
22	0.0454545	-87.4207	-87.232	0.215919	1.9531244411955784
24	0.0416667	-87.4207	-87.2615	0.182102	1.957594153198126
26	0.0384615	-87.4207	-87.2847	0.155646	1.961290842635925
28	0.0357143	-87.4207	-87.3031	0.134559	1.9643982515705631
30	0.0333333	-87.4207	-87.318	0.117483	1.9670463371767575
32	0.03125	-87.4207	-87.3303	0.103461	1.9693295640155024
34	0.0294118	-87.4207	-87.3405	0.0918066	1.9713182283304598
36	0.0277778	-87.4207	-87.349	0.0820154	1.9730656952128087
38	0.0263158	-87.4207	-87.3563	0.0737105	1.9746132153148208
40	0.025	-87.4207	-87.3625	0.0666057	1.9759931514310298
42	0.0238095	-87.4207	-87.3679	0.0604805	1.977231260766493
44	0.0227273	-87.4207	-87.3725	0.0551628	1.9783482756446635
46	0.0217391	-87.4207	-87.3766	0.0505166	1.9793611188513534
48	0.0208333	-87.4207	-87.3801	0.0464335	1.9802836537253623
50	0.02	-87.4207	-87.3833	0.0428261	1.9811274348857717

Conformal Convergence of Heat Flux (k1 = 2, k2 = 0.5):

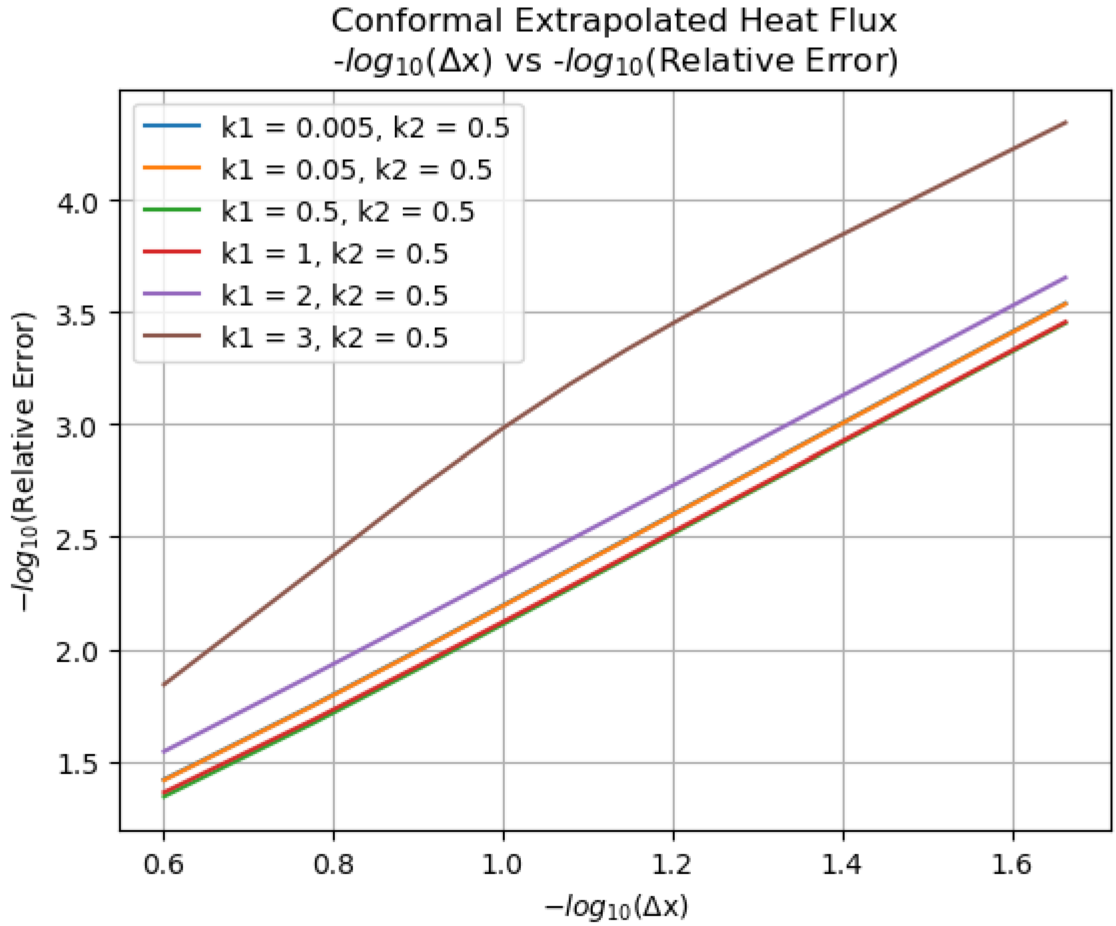
Num. Elements	dx	Exact Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-76.3471	-73.7704	3.37493	n/a
6	0.166667	-76.3471	-75.12	1.60717	1.829749942406464
8	0.125	-76.3471	-75.6309	0.938099	1.8714201328026712
10	0.1	-76.3471	-75.8771	0.615553	1.8881786876772009
12	0.0833333	-76.3471	-76.0147	0.435309	1.9002967442505547
14	0.0714286	-76.3471	-76.0995	0.324274	1.9102565019945619
16	0.0625	-76.3471	-76.1555	0.250984	1.9186308198135262
18	0.0555556	-76.3471	-76.1943	0.200051	1.9257263728450202
20	0.05	-76.3471	-76.2225	0.16321	1.9317794762219445
22	0.0454545	-76.3471	-76.2435	0.135697	1.9369815489567248
24	0.0416667	-76.3471	-76.2596	0.114605	1.9414865133385055
26	0.0384615	-76.3471	-76.2722	0.0980794	1.9454172163088304
28	0.0357143	-76.3471	-76.2823	0.0848895	1.9488715306305422
30	0.0333333	-76.3471	-76.2904	0.0741939	1.951927651080448
32	0.03125	-76.3471	-76.2971	0.0654006	1.9546483743665688
34	0.0294118	-76.3471	-76.3027	0.0580836	1.9570844826779372
36	0.0277778	-76.3471	-76.3074	0.0519299	1.9592773188400772
38	0.0263158	-76.3471	-76.3114	0.0467052	1.961260834179197
40	0.025	-76.3471	-76.3148	0.0422313	1.9630630692750954
42	0.0238095	-76.3471	-76.3178	0.0383711	1.9647073906882824
44	0.0227273	-76.3471	-76.3203	0.0350171	1.966213352307955
46	0.0217391	-76.3471	-76.3226	0.0320845	1.9675975353045134
48	0.0208333	-76.3471	-76.3245	0.0295055	1.968873901240977
50	0.02	-76.3471	-76.3263	0.0272256	1.9700544784576988

Conformal Convergence of Heat Flux (k1 = 3, k2 = 0.5):

Num. Elements	dx	Exact Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-72.4277	-71.2855	1.57704	n/a
6	0.166667	-72.4277	-72.0482	0.523985	2.7174719126849607
8	0.125	-72.4277	-72.2519	0.242717	2.6750626453723507
10	0.1	-72.4277	-72.3268	0.139353	2.4866786890715034
12	0.0833333	-72.4277	-72.3614	0.0915538	2.3040727198842705
14	0.0714286	-72.4277	-72.3802	0.065574	2.1650725117604748
16	0.0625	-72.4277	-72.3917	0.0497515	2.067975339978185
18	0.0555556	-72.4277	-72.3992	0.0392989	2.002350011627101
20	0.05	-72.4277	-72.4045	0.0319714	1.9585563254243734
22	0.0454545	-72.4277	-72.4084	0.0266007	1.9295233477022737
24	0.0416667	-72.4277	-72.4114	0.0225269	1.9104290805606279
26	0.0384615	-72.4277	-72.4137	0.0193518	1.8980584633542992
28	0.0357143	-72.4277	-72.4155	0.0168222	1.8902724271033116
30	0.0333333	-72.4277	-72.417	0.0147701	1.885638257079178
32	0.03125	-72.4277	-72.4182	0.0130798	1.8831862601916975
34	0.0294118	-72.4277	-72.4192	0.0116692	1.8822525530331782
36	0.0277778	-72.4277	-72.4201	0.0104789	1.88237722828635
38	0.0263158	-72.4277	-72.4208	0.00946443	1.8832384426590334
40	0.025	-72.4277	-72.4215	0.00859235	1.884608575303026
42	0.0238095	-72.4277	-72.422	0.00783686	1.8863254928329187
44	0.0227273	-72.4277	-72.4225	0.00717782	1.888272658769305
46	0.0217391	-72.4277	-72.4229	0.00659932	1.8903664521814905
48	0.0208333	-72.4277	-72.4233	0.00608861	1.8925457708642441
50	0.02	-72.4277	-72.4236	0.00563542	1.894766877566099

6.1.5 Conformal Heat Flux Richardson Extrapolation

The following section presents the convergence rates of the extrapolated heat flux of the upper boundary of the bar with a conformal mesh FDM solution. The data is presented for multiple values of thermal conductivity.



```
Conformal Convergence of Extrapolated Heat Flux (k1 = 0.005, k2 = 0.5):
```

Num. Elements	dx	Extrapolated Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-156.597	-150.62	3.81648	2.29569
6	0.166667	-157.027	-154.241	1.77424	2.49595
8	0.125	-157.248	-155.668	1.00469	2.60324
10	0.1	-157.374	-156.364	0.641652	2.67181
12	0.0833333	-157.452	-156.753	0.443668	2.71981
14	0.0714286	-157.504	-156.993	0.324437	2.75541
16	0.0625	-157.54	-157.15	0.247293	2.78292
18	0.0555556	-157.565	-157.259	0.194601	2.80482
20	0.05	-157.585	-157.337	0.157054	2.82269
22	0.0454545	-157.6	-157.396	0.129376	2.83755
24	0.0416667	-157.611	-157.44	0.108397	2.85011
26	0.0384615	-157.62	-157.475	0.0921226	2.86085
28	0.0357143	-157.628	-157.503	0.0792476	2.87016
30	0.0333333	-157.634	-157.525	0.0688888	2.8783
32	0.03125	-157.639	-157.544	0.060432	2.88547
34	0.0294118	-157.643	-157.559	0.0534393	2.89185
36	0.0277778	-157.647	-157.572	0.0475917	2.89755
38	0.0263158	-157.65	-157.583	0.0426525	2.90268
40	0.025	-157.653	-157.592	0.0384432	2.90732
42	0.0238095	-157.655	-157.6	0.0348268	2.91154
44	0.0227273	-157.657	-157.607	0.0316971	2.91539
46	0.0217391	-157.659	-157.613	0.0289706	2.91892

```
Conformal Convergence of Extrapolated Heat Flux (k1 = 0.05, k2 = 0.5):
```

Num. Elements	dx	Extrapolated Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-155.407	-149.443	3.83773	2.2952
6	0.166667	-155.836	-153.055	1.78438	2.49561
8	0.125	-156.056	-154.479	1.0105	2.60299
10	0.1	-156.182	-155.174	0.645387	2.67162
12	0.0833333	-156.26	-155.563	0.446261	2.71965
14	0.0714286	-156.312	-155.802	0.326339	2.75528
16	0.0625	-156.348	-155.959	0.248745	2.7828
18	0.0555556	-156.373	-156.067	0.195745	2.80472
20	0.05	-156.393	-156.146	0.157979	2.8226
22	0.0454545	-156.408	-156.204	0.130138	2.83747
24	0.0416667	-156.419	-156.249	0.109036	2.85003
26	0.0384615	-156.428	-156.283	0.0926662	2.86079
28	0.0357143	-156.436	-156.311	0.0797154	2.8701
30	0.0333333	-156.442	-156.333	0.0692956	2.87824
32	0.03125	-156.447	-156.352	0.060789	2.88542
34	0.0294118	-156.451	-156.367	0.0537551	2.8918
36	0.0277778	-156.455	-156.38	0.047873	2.89751
38	0.0263158	-156.458	-156.391	0.0429048	2.90264
40	0.025	-156.461	-156.4	0.0386706	2.90728
42	0.0238095	-156.463	-156.408	0.0350328	2.9115
44	0.0227273	-156.465	-156.415	0.0318846	2.91535
46	0.0217391	-156.467	-156.421	0.029142	2.91888

Conformal Convergence of Extrapolated Heat Flux (k1 = 0.5, k2 = 0.5):

Num. Elements	dx	Extrapolated Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-108.136	-103.225	4.54087	2.26214
6	0.166667	-108.489	-106.173	2.13405	2.47154
8	0.125	-108.672	-107.352	1.21494	2.58441
10	0.1	-108.777	-107.93	0.778367	2.65654
12	0.0833333	-108.842	-108.255	0.539296	2.70698
14	0.0714286	-108.886	-108.456	0.39493	2.74436
16	0.0625	-108.916	-108.587	0.301342	2.77321
18	0.0555556	-108.937	-108.679	0.237325	2.79617
20	0.05	-108.954	-108.745	0.191658	2.81489
22	0.0454545	-108.966	-108.794	0.157964	2.83045
24	0.0416667	-108.976	-108.832	0.132406	2.84358
26	0.0384615	-108.984	-108.861	0.112568	2.85483
28	0.0357143	-108.99	-108.884	0.0968655	2.86456
30	0.0333333	-108.995	-108.903	0.0842263	2.87306
32	0.03125	-108.999	-108.919	0.0739039	2.88056
34	0.0294118	-109.003	-108.932	0.0653657	2.88722
36	0.0277778	-109.006	-108.943	0.0582237	2.89318
38	0.0263158	-109.009	-108.952	0.0521896	2.89854
40	0.025	-109.011	-108.96	0.0470459	2.90339
42	0.0238095	-109.013	-108.966	0.0426258	2.90779
44	0.0227273	-109.015	-108.972	0.0387999	2.91181
46	0.0217391	-109.016	-108.977	0.0354663	2.91549

Conformal Convergence of Extrapolated Heat Flux (k1 = 1, k2 = 0.5):

Num. Elements	dx	Extrapolated Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-86.7217	-82.9475	4.35204	2.24147
6	0.166667	-86.9952	-85.2007	2.06271	2.45149
8	0.125	-87.1376	-86.1087	1.18077	2.56615
10	0.1	-87.2195	-86.5573	0.759261	2.6401
12	0.0833333	-87.2706	-86.8103	0.527436	2.69215
14	0.0714286	-87.3045	-86.9666	0.386995	2.73089
16	0.0625	-87.3281	-87.0699	0.295728	2.7609
18	0.0555556	-87.3452	-87.1416	0.233179	2.78485
20	0.05	-87.358	-87.1933	0.18849	2.80442
22	0.0454545	-87.3678	-87.232	0.155475	2.82071
24	0.0416667	-87.3755	-87.2615	0.130407	2.83449
26	0.0384615	-87.3816	-87.2847	0.110931	2.8463
28	0.0357143	-87.3865	-87.3031	0.0955028	2.85653
30	0.0333333	-87.3906	-87.318	0.0830764	2.86548
32	0.03125	-87.394	-87.3303	0.0729219	2.87338
34	0.0294118	-87.3969	-87.3405	0.0645183	2.8804
36	0.0277778	-87.3993	-87.349	0.0574856	2.88668
38	0.0263158	-87.4013	-87.3563	0.0515414	2.89234
40	0.025	-87.4031	-87.3625	0.0464725	2.89745
42	0.0238095	-87.4047	-87.3679	0.0421153	2.90211
44	0.0227273	-87.406	-87.3725	0.0383426	2.90635
46	0.0217391	-87.4072	-87.3766	0.0350545	2.91025

```
Conformal Convergence of Extrapolated Heat Flux (k1 = 2, k2 = 0.5):
```

Num. Elements	dx	Extrapolated Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-75.9419	-73.7704	2.85945	2.39621
6	0.166667	-76.1063	-75.12	1.29593	2.53628
8	0.125	-76.189	-75.6309	0.732609	2.6079
10	0.1	-76.2355	-75.8771	0.470104	2.65721
12	0.0833333	-76.2641	-76.0147	0.326983	2.69495
14	0.0714286	-76.2829	-76.0995	0.240464	2.7252
16	0.0625	-76.296	-76.1555	0.184207	2.75006
18	0.0555556	-76.3054	-76.1943	0.145587	2.77088
20	0.05	-76.3125	-76.2225	0.117938	2.78856
22	0.0454545	-76.3179	-76.2435	0.0974682	2.80376
24	0.0416667	-76.3221	-76.2596	0.0818933	2.81695
26	0.0384615	-76.3254	-76.2722	0.0697695	2.8285
28	0.0357143	-76.3282	-76.2823	0.0601485	2.8387
30	0.0333333	-76.3304	-76.2904	0.0523865	2.84777
32	0.03125	-76.3323	-76.2971	0.0460341	2.85589
34	0.0294118	-76.3339	-76.3027	0.0407696	2.86319
36	0.0277778	-76.3352	-76.3074	0.0363584	2.86979
38	0.0263158	-76.3363	-76.3114	0.0326256	2.87579
40	0.025	-76.3373	-76.3148	0.0294391	2.88127
42	0.0238095	-76.3382	-76.3178	0.0266972	2.88628
44	0.0227273	-76.3389	-76.3203	0.024321	2.89089
46	0.0217391	-76.3396	-76.3226	0.0222483	2.89515

```
Conformal Convergence of Extrapolated Heat Flux (k1 = 3, k2 = 0.5):
```

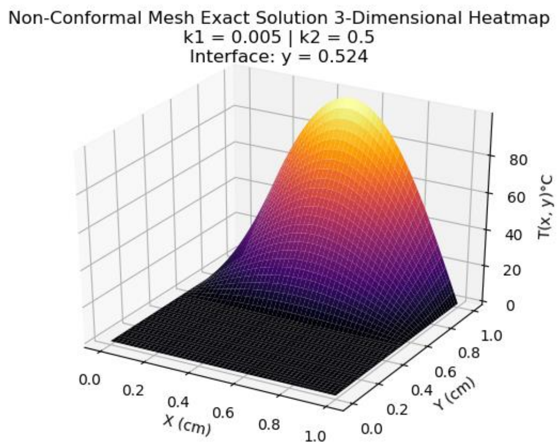
Num. Elements	dx	Extrapolated Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-72.3261	-71.2855	1.43884	3.25586
6	0.166667	-72.3703	-72.0482	0.44504	3.47972
8	0.125	-72.3912	-72.2519	0.192378	3.4563
10	0.1	-72.4026	-72.3268	0.104749	3.34401
12	0.0833333	-72.4095	-72.3614	0.0664668	3.21687
14	0.0714286	-72.414	-72.3802	0.0466295	3.10482
16	0.0625	-72.417	-72.3917	0.0349664	3.01581
18	0.0555556	-72.4191	-72.3992	0.0274418	2.94873
20	0.05	-72.4207	-72.4045	0.0222442	2.89966
22	0.0454545	-72.4218	-72.4084	0.0184679	2.86447
24	0.0416667	-72.4227	-72.4114	0.0156178	2.83969
26	0.0384615	-72.4234	-72.4137	0.0134026	2.8226
28	0.0357143	-72.4239	-72.4155	0.0116405	2.81114
30	0.0333333	-72.4244	-72.417	0.0102123	2.80382
32	0.03125	-72.4248	-72.4182	0.00903648	2.79954
34	0.0294118	-72.4251	-72.4192	0.00805571	2.79747
36	0.0277778	-72.4253	-72.4201	0.00722833	2.79704
38	0.0263158	-72.4256	-72.4208	0.00652349	2.7978
40	0.025	-72.4258	-72.4215	0.00591783	2.79942
42	0.0238095	-72.4259	-72.422	0.00539338	2.80166
44	0.0227273	-72.4261	-72.4225	0.00493612	2.80435
46	0.0217391	-72.4262	-72.4229	0.00453491	2.80737

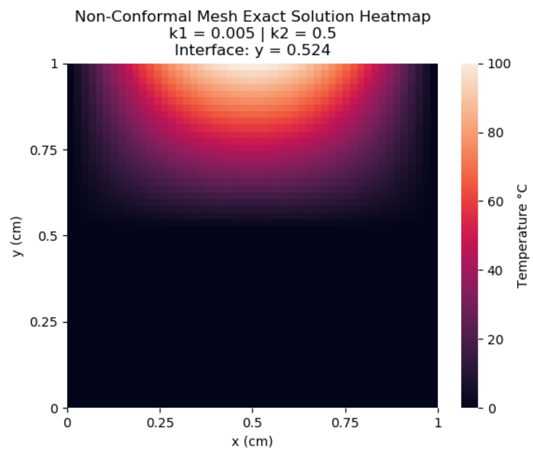
6.2 Non-Conformal Mesh Results

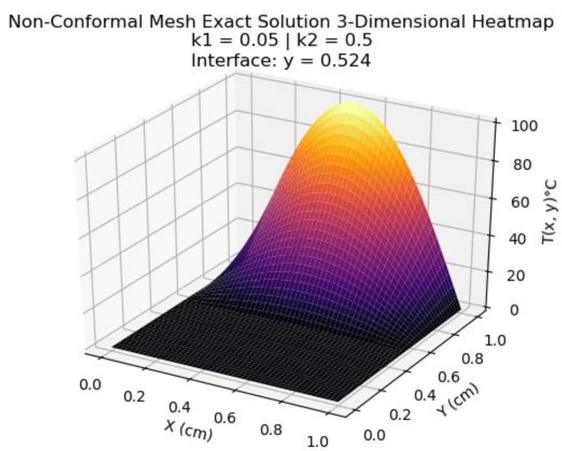
The following section outlines the analytical and second order FDM results with the utilization of a non-conformal mesh. Additionally, the heat flux and convergence of the approximate and extrapolation solution are provided.

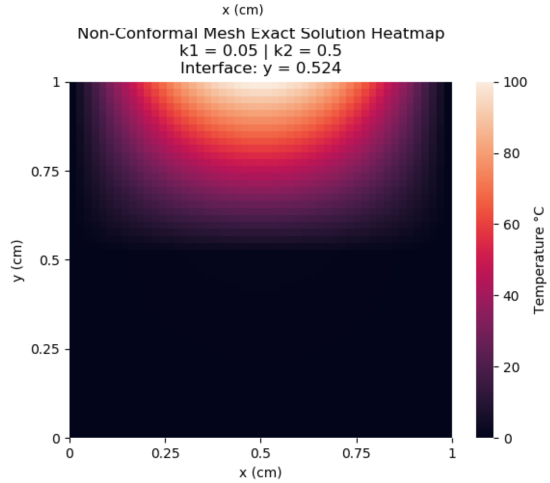
6.2.1 Non-Conformal Temperature Analytical Results

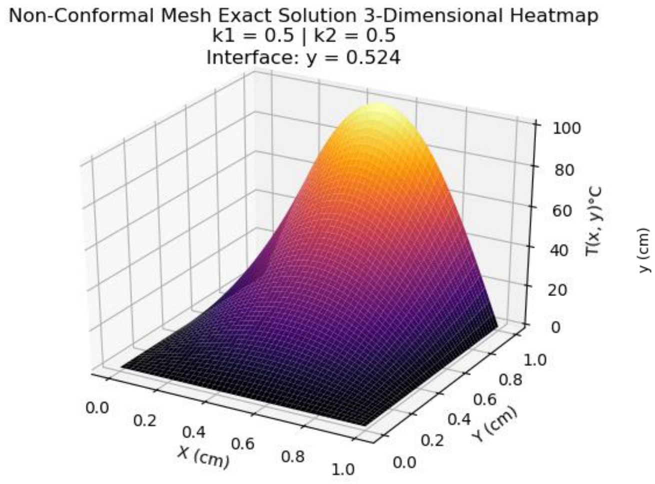
The following section displays the exact solution to the problem with the utilization of an analytical mesh. The results in the following section outline the FDM solution to the conformal mesh and should align with the solutions provide in this section. However, it will be seen that because this solution utilizes a non-conformal mesh, that the results will differ between the analytical and approximated solutions.

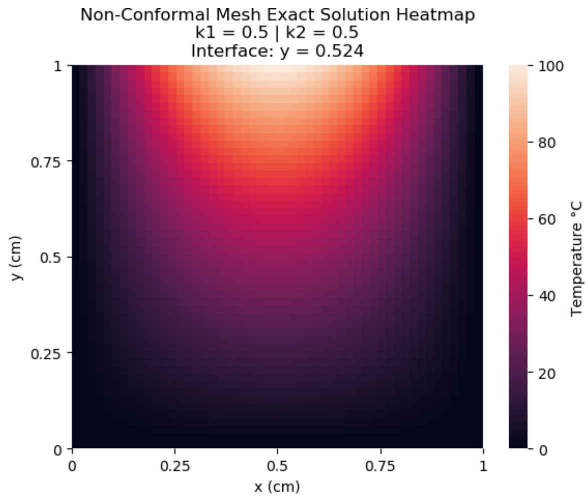


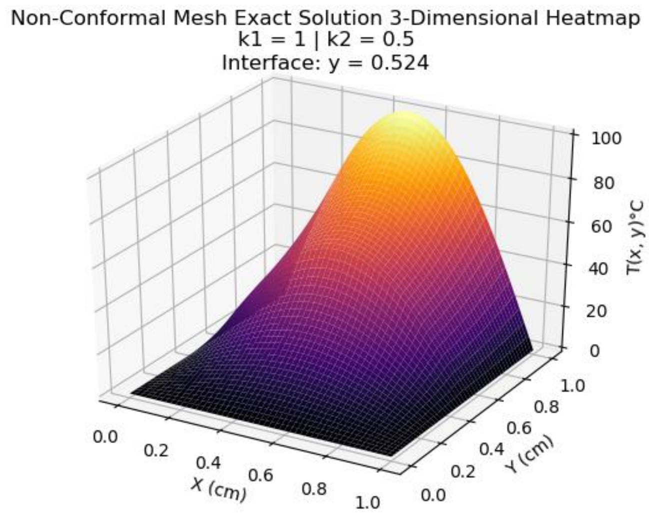


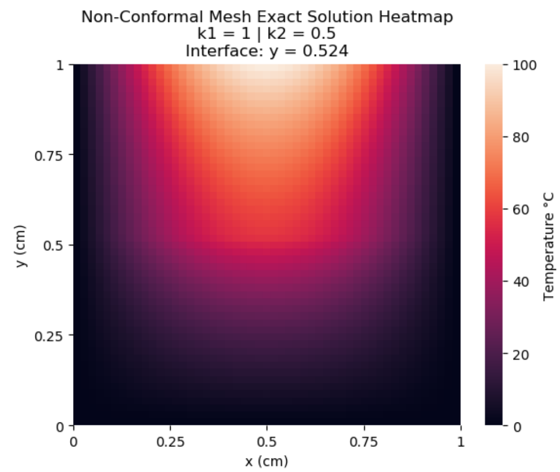


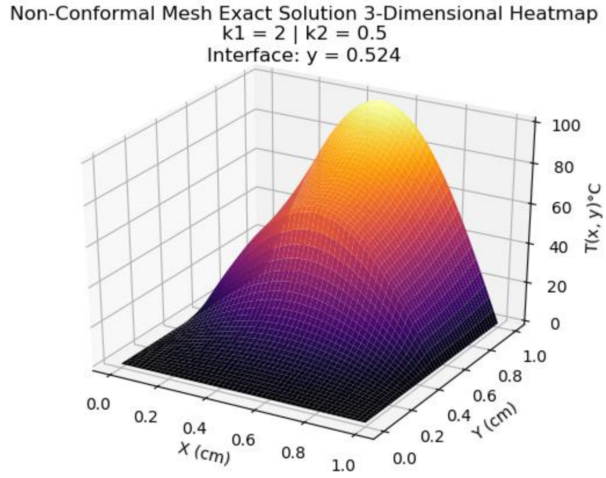


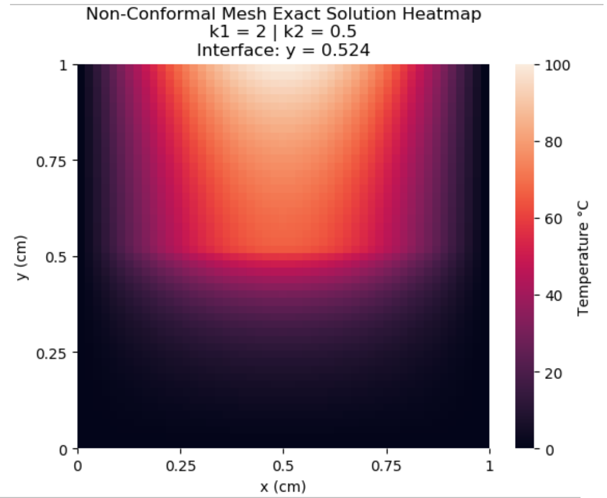

Non-Conformal Mesh Exact Solution 3-Dimensional Heatmap
k1 = 0.5 | k2 = 0.5
Interface: y = 0.524
T(x, y)°C
X (cm)
Y (cm)

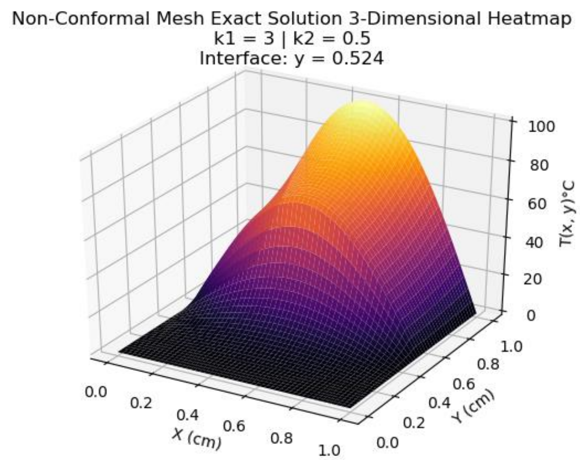

Non-Conformal Mesh Exact Solution Heatmap
k1 = 0.5 | k2 = 0.5
Interface: y = 0.524
y (cm)
x (cm)
Temperature °C

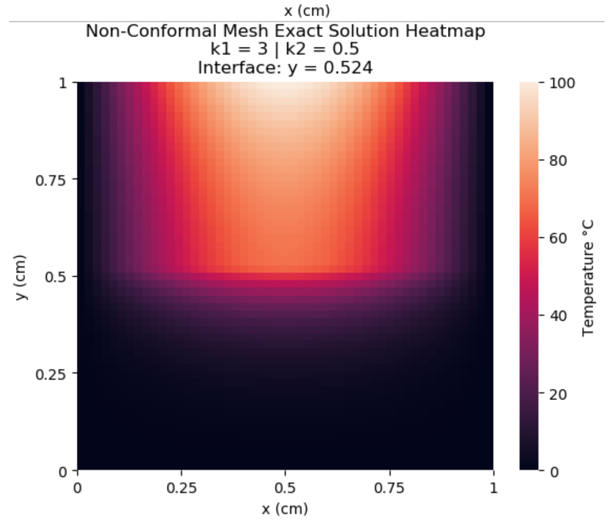

Non-Conformal Mesh Exact Solution 3-Dimensional Heatmap
k1 = 1 | k2 = 0.5
Interface: y = 0.524
T(x, y)°C
X (cm)
Y (cm)


Non-Conformal Mesh Exact Solution Heatmap
k1 = 1 | k2 = 0.5
Interface: y = 0.524
y (cm)
x (cm)
Temperature °C



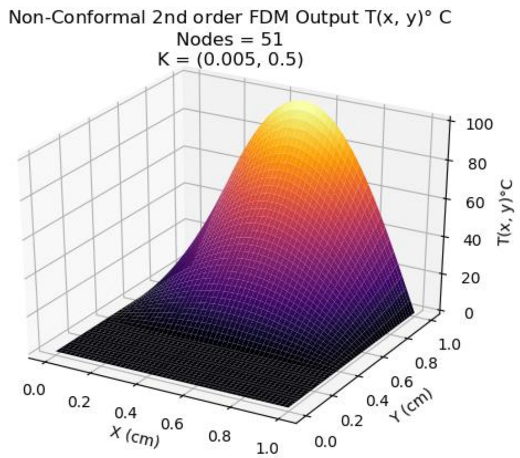


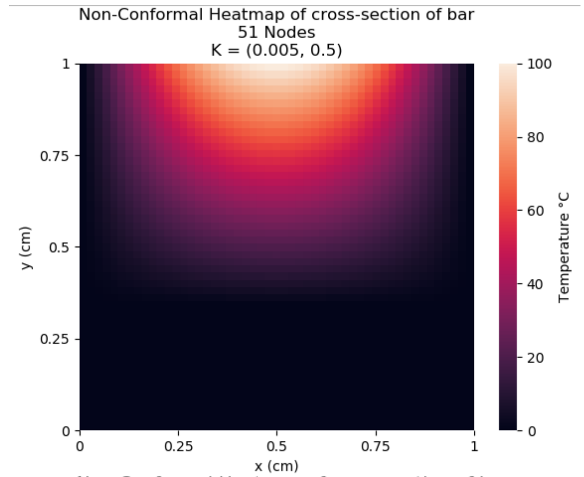


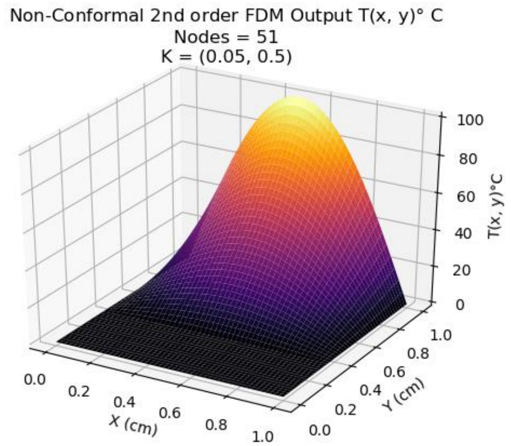


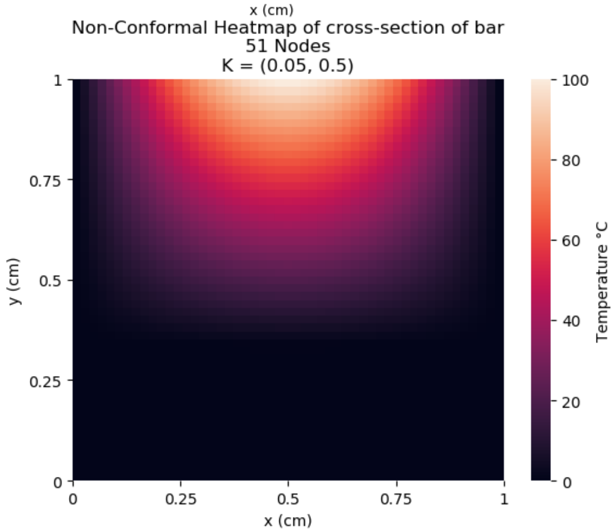
6.2.2 Non-Conformal Temperature FDM Results

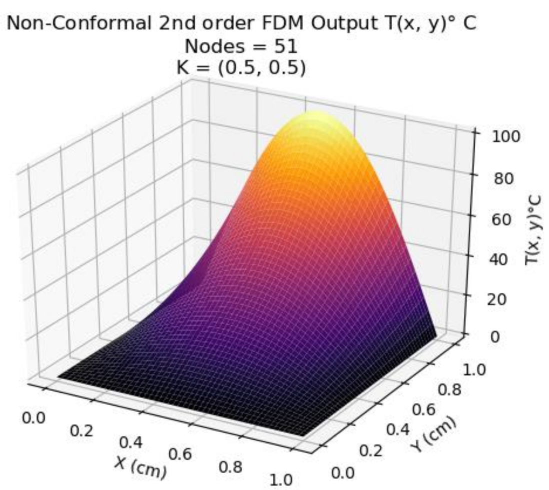
The following section displays the second order FDM solution to the problem with the utilization of an analytical mesh. The results in the previous section outline the exact solution to the conformal mesh and should align with the solutions provide in this section.

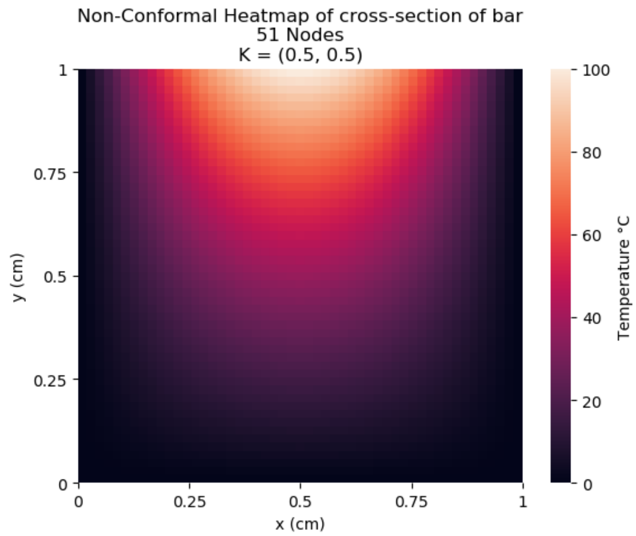

Non-Conformal 2nd order FDM Output T(x, y)° C
Nodes = 51
K = (0.005, 0.5)
X (cm)
Y (cm)
T(x, y)°C

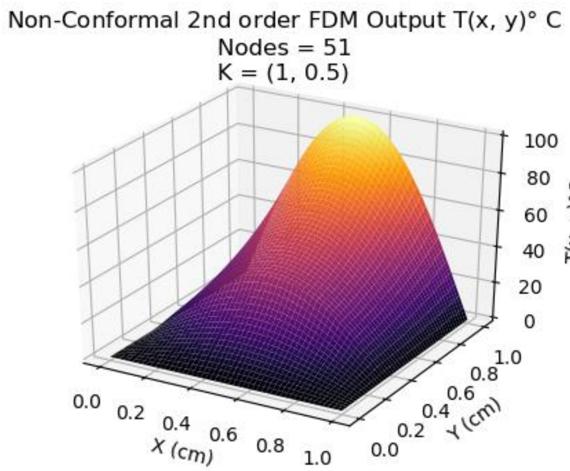

Non-Conformal Heatmap of cross-section of bar
51 Nodes
K = (0.005, 0.5)
x (cm)
y (cm)
Temperature °C

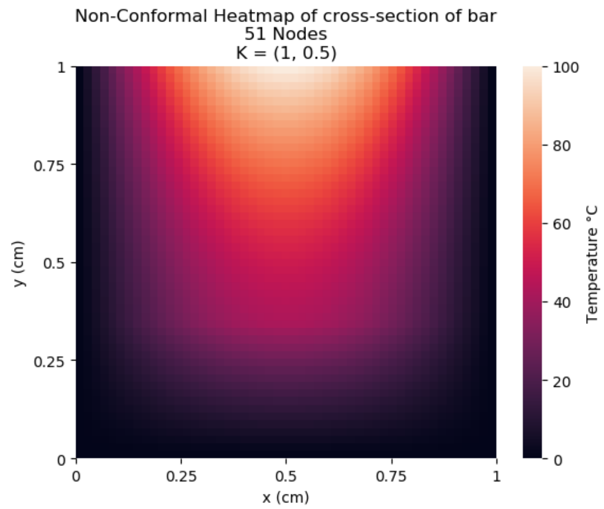

Non-Conformal 2nd order FDM Output T(x, y)° C
Nodes = 51
K = (0.05, 0.5)
X (cm)
Y (cm)
T(x, y)°C

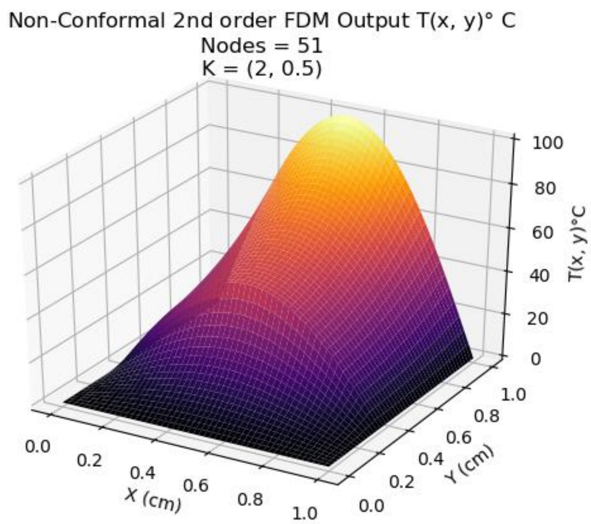

Non-Conformal Heatmap of cross-section of bar
51 Nodes
K = (0.05, 0.5)
x (cm)
y (cm)
Temperature °C

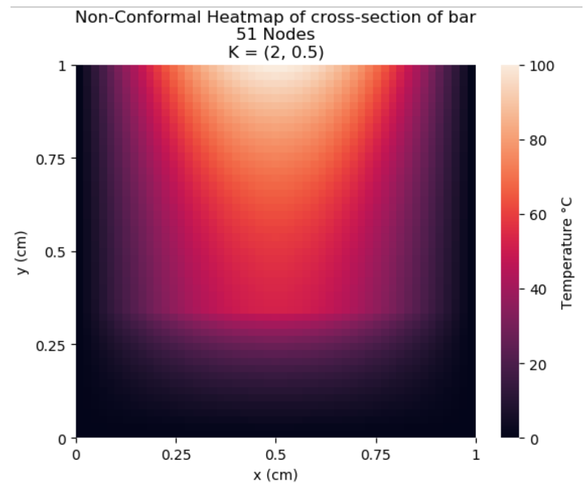

Non-Conformal 2nd order FDM Output T(x, y)° C
Nodes = 51
K = (0.5, 0.5)
X (cm)
Y (cm)
T(x, y)°C

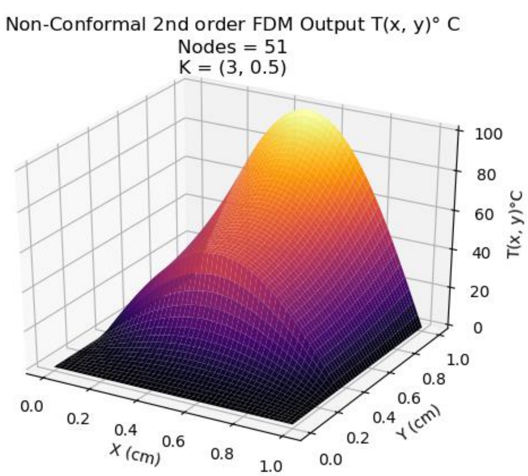

Non-Conformal Heatmap of cross-section of bar
51 Nodes
K = (0.5, 0.5)
x (cm)
y (cm)
Temperature °C

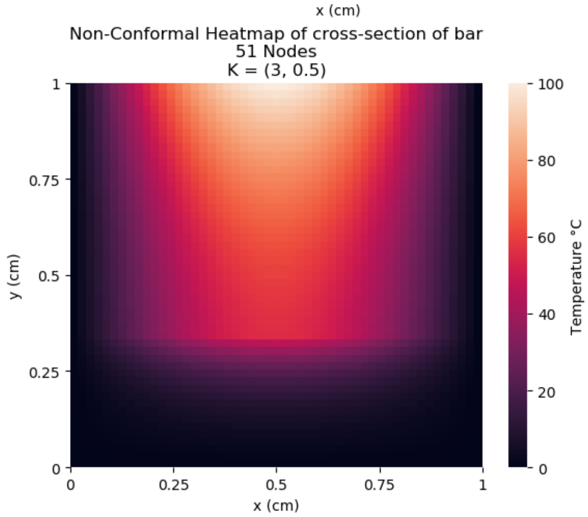

Non-Conformal 2nd order FDM Output T(x, y)° C
Nodes = 51
K = (1, 0.5)
X (cm)
Y (cm)
T(x, y)°C


Non-Conformal Heatmap of cross-section of bar
51 Nodes
K = (1, 0.5)
y (cm)
x (cm)
Temperature °C


Non-Conformal 2nd order FDM Output T(x, y)° C
Nodes = 51
K = (2, 0.5)
X (cm)
Y (cm)
T(x, y)°C

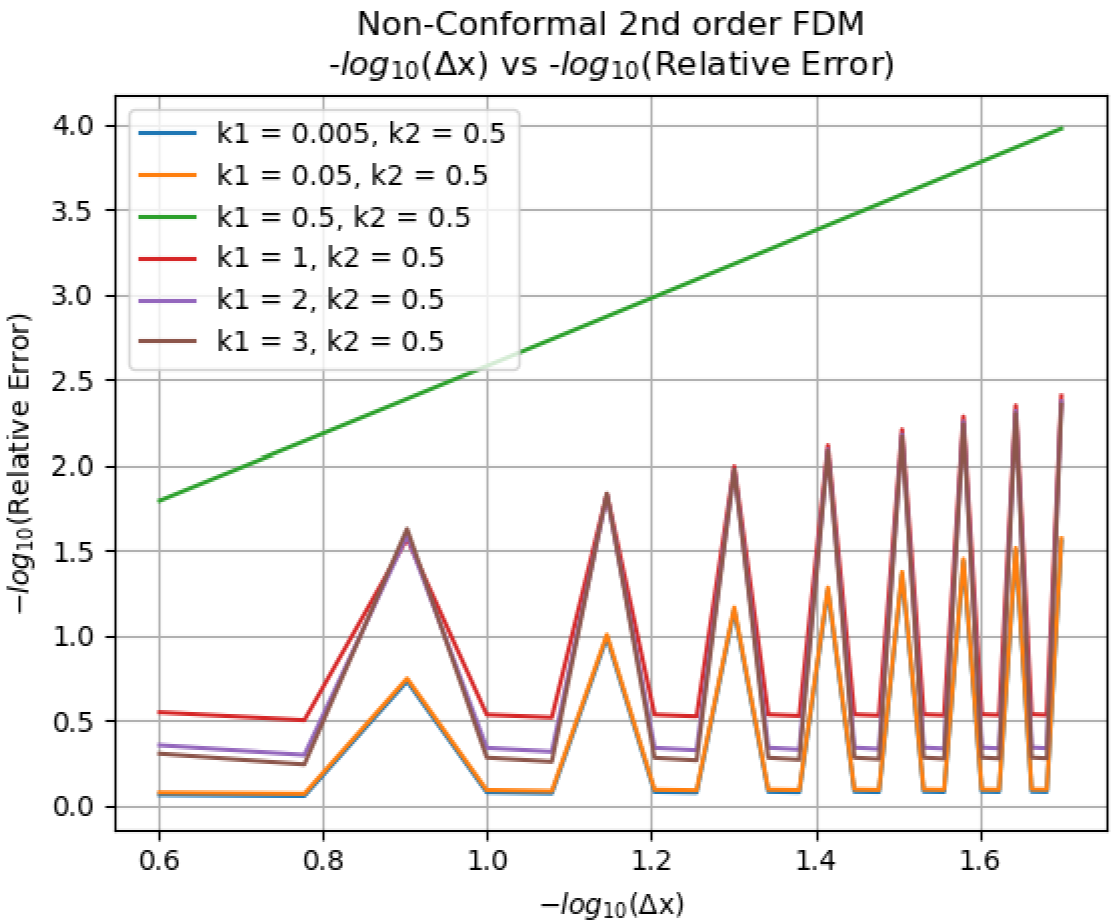

Non-Conformal Heatmap of cross-section of bar
51 Nodes
K = (2, 0.5)
y (cm)
x (cm)
Temperature °C


Non-Conformal 2nd order FDM Output T(x, y)° C
Nodes = 51
K = (3, 0.5)
X (cm)
Y (cm)
T(x, y)°C


Non-Conformal Heatmap of cross-section of bar
51 Nodes
K = (3, 0.5)
y (cm)
x (cm)
Temperature °C

6.2.3 Non-Conformal Temperature FDM Convergence

The following section presents the convergence rates of the temperature of the midpoint of the interface location of the FDM solution for non-conformal mesh. The data is presented for multiple values of thermal conductivity.



Non-Conformal Convergence of 2nd Order FDM Temperature Solution (k1 = 0.005, k2 = 0.5):

Num. Elements	dx	Exact Midpoint Temp	Approx. Midpoint Temp	Percent Error	Beta
4	0.25	0.0100267	39.463	393481	n/a
6	0.166667	0.0100267	39.7538	396380	-0.018108028620092056
8	0.125	0.0100267	17.2689	172130	2.8994678784490433
10	0.1	0.0100267	38.8591	387457	-3.636027062551235
12	0.0833333	0.0100267	39.0963	389823	-0.033386047844605644
14	0.0714286	0.0100267	19.0282	189676	4.67318456041473
16	0.0625	0.0100267	38.7572	386441	-5.329540514351172
18	0.0555556	0.0100267	38.9203	388068	-0.03568045067742433
20	0.05	0.0100267	19.6986	196363	6.4655958287614315
22	0.0454545	0.0100267	38.718	386050	-7.092679132712544
24	0.0416667	0.0100267	38.8408	387275	-0.036398779417108194
26	0.0384615	0.0100267	20.0524	199891	8.262625833857104
28	0.0357143	0.0100267	38.6977	385847	-8.874491034339096
30	0.0333333	0.0100267	38.7957	386826	-0.03670289734176572
32	0.03125	0.0100267	20.2711	202071	10.06148027139487
34	0.0294118	0.0100267	38.6853	385724	-10.664010324658095
36	0.0277778	0.0100267	38.7669	386538	-0.03685500508105875
38	0.0263158	0.0100267	20.4196	203553	11.861238560139821
40	0.025	0.0100267	38.677	385642	-12.45745520261114
42	0.0238095	0.0100267	38.7468	386338	-0.03693952454526849
44	0.0227273	0.0100267	20.5271	204625	13.661509813305589
46	0.0217391	0.0100267	38.6711	385583	-14.253171127686374
48	0.0208333	0.0100267	38.732	386190	-0.03698996941047025
50	0.02	0.0100267	20.6085	205436	15.462099992042145

Non-Conformal Convergence of 2nd Order FDM Temperature Solution (k1 = 0.05, k2 = 0.5):

Num. Elements	dx	Exact Midpoint Temp	Approx. Midpoint Temp	Percent Error	Beta
4	0.25	0.987782	39.4509	3893.89	n/a
6	0.166667	0.987782	39.7347	3922.62	-0.018131259536570427
8	0.125	0.987782	17.654	1687.24	2.9326435535016744
10	0.1	0.987782	38.8482	3832.88	-3.6771125878160293
12	0.0833333	0.987782	39.0825	3856.59	-0.033828199346550826
14	0.0714286	0.987782	19.3669	1860.65	4.728221016287532
16	0.0625	0.987782	38.7468	3822.61	-5.39205265142786
18	0.0555556	0.987782	38.9081	3838.94	-0.03618810762244568
20	0.05	0.987782	20.0206	1926.82	6.54252253023117
22	0.0454545	0.987782	38.7079	3818.67	-7.176878178129469
24	0.0416667	0.987782	38.8293	3830.96	-0.03692795845646141
26	0.0384615	0.987782	20.3657	1961.76	8.361438875510714
28	0.0357143	0.987782	38.6877	3816.62	-8.980460269689312
30	0.0333333	0.987782	38.7847	3826.44	-0.03724168016229734
32	0.03125	0.987782	20.5791	1983.36	10.182175748327037
34	0.0294118	0.987782	38.6754	3815.38	-10.791787075962421
36	0.0277778	0.987782	38.7561	3823.55	-0.03739887239669269
38	0.0263158	0.987782	20.724	1998.04	12.003813799574893
40	0.025	0.987782	38.6672	3814.55	-12.60705943293874
42	0.0238095	0.987782	38.7362	3821.53	-0.03748639644402314
44	0.0227273	0.987782	20.829	2008.66	13.825963008038334
46	0.0217391	0.987782	38.6614	3813.96	-14.424614768826567
48	0.0208333	0.987782	38.7216	3820.06	-0.03753875858049988
50	0.02	0.987782	20.9084	2016.71	15.648429885600352

Non-Conformal Convergence of 2nd Order FDM Temperature Solution (k1 = 0.5, k2 = 0.5):

Num. Elements	dx	Exact Midpoint Temp	Approx. Midpoint Temp	Percent Error	Beta
4	0.25	39.9071	38.3548	3.88961	n/a
6	0.166667	39.9071	38.0212	4.72559	-0.48014895792168155
8	0.125	39.9071	37.9021	5.02417	-0.2129725328454187
10	0.1	39.9071	37.8465	5.16345	-0.12254446361727475
12	0.0833333	39.9071	37.8162	5.2394	-0.08008828809295723
14	0.0714286	39.9071	37.7979	5.28529	-0.056576154276279926
16	0.0625	39.9071	37.786	5.31512	-0.04214387480319011
18	0.0555556	39.9071	37.7778	5.33559	-0.032631350133096614
20	0.05	39.9071	37.7719	5.35024	-0.02602347211420298
22	0.0454545	39.9071	37.7676	5.36108	-0.02124371977244625
24	0.0416667	39.9071	37.7643	5.36933	-0.017673126016489276
26	0.0384615	39.9071	37.7618	5.37575	-0.014934751693683499
28	0.0357143	39.9071	37.7597	5.38085	-0.012788169365051255
30	0.0333333	39.9071	37.7581	5.38496	-0.011074056719519054
32	0.03125	39.9071	37.7567	5.38833	-0.009683397159396685
34	0.0294118	39.9071	37.7556	5.39112	-0.008539529095322285
36	0.0277778	39.9071	37.7547	5.39346	-0.007587256456885635
38	0.0263158	39.9071	37.7539	5.39544	-0.006786011821719698
40	0.025	39.9071	37.7532	5.39713	-0.006105434133644412
42	0.0238095	39.9071	37.7526	5.39858	-0.005522432671943414
44	0.0227273	39.9071	37.7521	5.39984	-0.005019191666106657
46	0.0217391	39.9071	37.7517	5.40094	-0.004581786781155846
48	0.0208333	39.9071	37.7513	5.40191	-0.0041992068410800065
50	0.02	39.9071	37.751	5.40276	-0.003862651717987331

Non-Conformal Convergence of 2nd Order FDM Temperature Solution (k1 = 1, k2 = 0.5):

Num. Elements	dx	Exact Midpoint Temp	Approx. Midpoint Temp	Percent Error	Beta
4	0.25	57.6326	36.0083	37.521	n/a
6	0.166667	57.6326	34.412	40.2908	-0.17565751837645452
8	0.125	57.6326	51.5066	10.6293	4.631871315787926
10	0.1	57.6326	35.5523	38.3122	-5.745864067083826
12	0.0833333	57.6326	34.9839	39.2984	-0.1393944148783765
14	0.0714286	57.6326	50.8974	11.6864	7.867370316809721
16	0.0625	57.6326	35.5586	38.3013	-8.889743889265699
18	0.0555556	57.6326	35.2036	38.9173	-0.13546063548599435
20	0.05	57.6326	50.6671	12.086	11.09891952452337
22	0.0454545	57.6326	35.5777	38.2682	-12.092808680367789
24	0.0416667	57.6326	35.318	38.7187	-0.1345152514013994
26	0.0384615	57.6326	50.5463	12.2957	14.330777878997996
28	0.0357143	57.6326	35.5931	38.2414	-15.311042731967731
30	0.0333333	57.6326	35.388	38.5972	-0.1342267089201566
32	0.03125	57.6326	50.4719	12.4247	17.563009033093373
34	0.0294118	57.6326	35.6048	38.2211	-18.53534185020467
36	0.0277778	57.6326	35.4353	38.5152	-0.13414053890997352
38	0.0263158	57.6326	50.4215	12.5121	20.795501818349393
40	0.025	57.6326	35.6138	38.2054	-21.762660255952895
42	0.0238095	57.6326	35.4693	38.4563	-0.13412807374314845
44	0.0227273	57.6326	50.3852	12.5753	24.028173156243355
46	0.0217391	57.6326	35.6209	38.1931	-24.991695718517455
48	0.0208333	57.6326	35.4949	38.4118	-0.13414449451430302
50	0.02	57.6326	50.3577	12.623	27.26096923289077

Non-Conformal Convergence of 2nd Order FDM Temperature Solution (k1 = 2, k2 = 0.5):

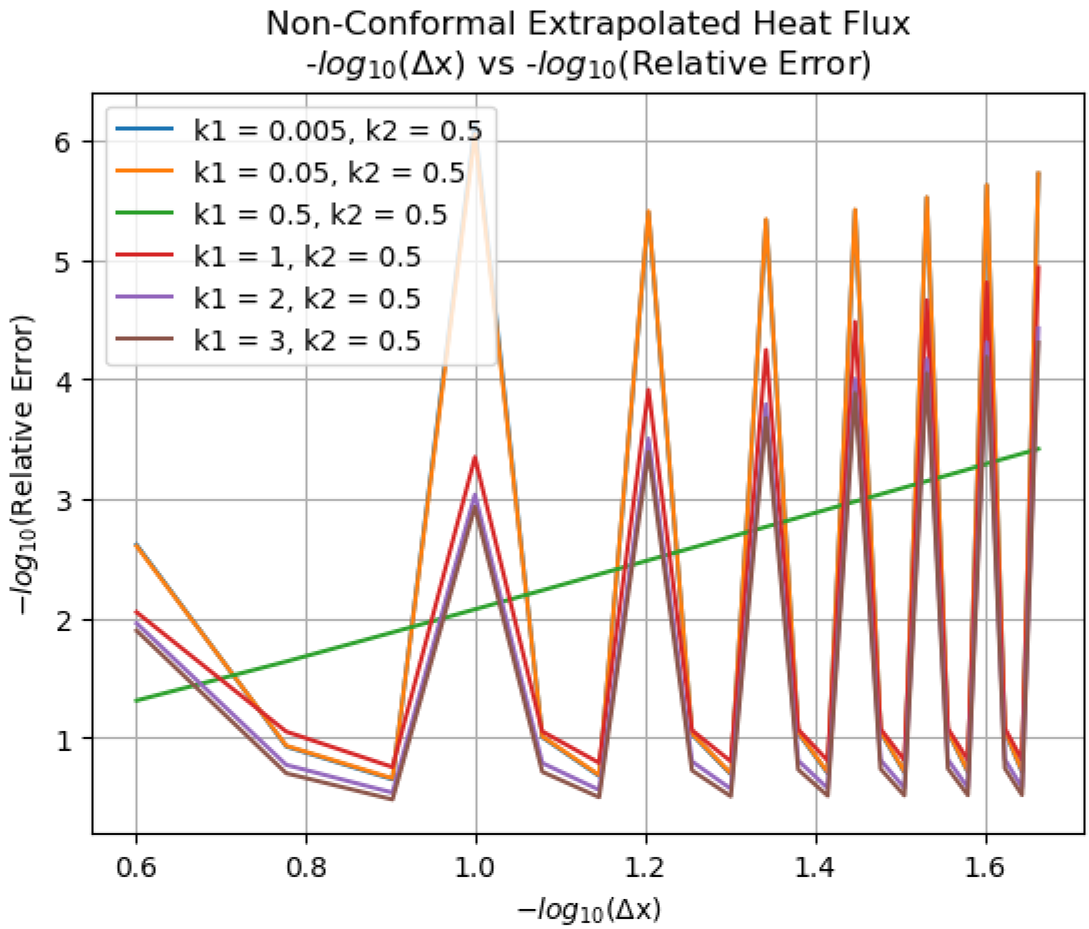
Num. Elements	dx	Exact Midpoint Temp	Approx. Midpoint Temp	Percent Error	Beta
4	0.25	66.7147	32.1232	51.8499	n/a
6	0.166667	66.7147	28.5266	57.2409	-0.24395285082101129
8	0.125	66.7147	58.8459	11.7947	5.49083998452543
10	0.1	66.7147	31.1764	53.269	-6.756649659929615
12	0.0833333	66.7147	29.7993	55.3333	-0.20852869108310473
14	0.0714286	66.7147	58.2245	12.7261	9.534299391821744
16	0.0625	66.7147	31.194	53.2427	-10.718124880745481
18	0.0555556	66.7147	30.3206	54.5519	-0.20622825778527756
20	0.05	66.7147	57.9707	13.1066	13.53485185735492
22	0.0454545	66.7147	31.2417	53.1712	-14.69311704239465
24	0.0416667	66.7147	30.5984	54.1355	-0.20656653231432986
26	0.0384615	66.7147	57.8326	13.3136	17.524441816464886
28	0.0357143	66.7147	31.2804	53.1132	-18.670589774851855
30	0.0333333	66.7147	30.7702	53.8779	-0.20719636059561672
32	0.03125	66.7147	57.7457	13.4438	21.50971363455338
34	0.0294118	66.7147	31.3098	53.0691	-22.648865233030225
36	0.0277778	66.7147	30.8868	53.7031	-0.2077817512908561
38	0.0263158	66.7147	57.6861	13.5332	25.492863523469765
40	0.025	66.7147	31.3325	53.0351	-26.627482127687752
42	0.0238095	66.7147	30.9711	53.5768	-0.20827693989198012
44	0.0227273	66.7147	57.6426	13.5983	29.47481921745782
46	0.0217391	66.7147	31.3503	53.0084	-30.606270233605496
48	0.0208333	66.7147	31.0348	53.4813	-0.20868890997835304
50	0.02	66.7147	57.6095	13.6479	33.45603806150971

Non-Conformal Convergence of 2nd Order FDM Temperature Solution (k1 = 3, k2 = 0.5):

Num. Elements	dx	Exact Midpoint Temp	Approx. Midpoint Temp	Percent Error	Beta
4	0.25	69.9292	30.1926	56.824	n/a
6	0.166667	69.9292	25.5419	63.4746	-0.27296922914604815
8	0.125	69.9292	61.0264	12.7311	5.584603412246739
10	0.1	69.9292	28.5799	59.1302	-6.8820863538633805
12	0.0833333	69.9292	26.8017	61.6731	-0.23094542151746927
14	0.0714286	69.9292	60.4927	13.4944	9.857738065041037
16	0.0625	69.9292	28.5047	59.2378	-11.0782019904655
18	0.0555556	69.9292	27.3852	60.8386	-0.22639859925610153
20	0.05	69.9292	60.2535	13.8364	14.055783028049902
22	0.0454545	69.9292	28.5328	59.1976	-15.251054277560549
24	0.0416667	69.9292	27.7104	60.3737	-0.2260885437714823
26	0.0384615	69.9292	60.1173	14.0312	18.231115332855445
28	0.0357143	69.9292	28.5675	59.1479	-19.414318533742737
30	0.0333333	69.9292	27.916	60.0796	-0.22652044529359236
32	0.03125	69.9292	60.0293	14.157	22.396867805870396
34	0.0294118	69.9292	28.5974	59.1052	-23.57308589584442
36	0.0277778	69.9292	28.0575	59.8773	-0.22705840712879083
38	0.0263158	69.9292	59.9679	14.2448	26.557733153135928
40	0.025	69.9292	28.6218	59.0702	-27.72942492536276
42	0.0238095	69.9292	28.1607	59.7297	-0.22756188232928562
44	0.0227273	69.9292	59.9226	14.3096	30.715781738071144
46	0.0217391	69.9292	28.6418	59.0418	-31.884312584894285
48	0.0208333	69.9292	28.2392	59.6175	-0.2280032275905304
50	0.02	69.9292	59.8878	14.3594	34.87206322060208

6.2.4 Non-Conformal Heat Flux Richardson Extrapolation

The following section presents the convergence rates of the extrapolated heat flux of the upper boundary of the bar with a conformal mesh FDM solution. The data is presented for multiple values of thermal conductivity.



It is worth noting that when the mesh falls upon the interface location, the convergence of the FDM greatly improves. Additionally, the convergence of a single material case follows a second order convergence, as expected.

Non-Conformal Convergence of Extrapolated Heat Flux (k1 = 0.005, k2 = 0.5):

Num. Elements	dx	Extrapolated Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-101.293	-101.539	0.242801	-5.8992
6	0.166667	-118.051	-103.983	11.917	0.369869
8	0.125	-106.722	-130.699	22.4662	28.3329
10	0.1	-106.679	-106.679	7.52789e-05	-34.489
12	0.0833333	-118.509	-106.722	9.9456	0.201989
14	0.0714286	-107.335	-129.934	21.0542	40.695
16	0.0625	-107.434	-107.434	0.000403565	-46.004
18	0.0555556	-118.499	-107.336	9.42037	0.148946
20	0.05	-107.576	-129.488	20.3691	56.0459
22	0.0454545	-107.68	-107.681	0.000465635	-61.3
24	0.0416667	-118.448	-107.576	9.17838	0.125498
26	0.0384615	-107.698	-129.211	19.9753	73.2156
28	0.0357143	-107.792	-107.793	0.000384883	-78.5808
30	0.0333333	-118.4	-107.698	9.03856	0.112986
32	0.03125	-107.77	-129.024	19.7218	91.3682
34	0.0294118	-107.853	-107.853	0.000302683	-96.8667
36	0.0277778	-118.36	-107.77	8.9472	0.105468
38	0.0263158	-107.816	-128.89	19.5456	110.219
40	0.025	-107.89	-107.89	0.000238631	-115.847
42	0.0238095	-118.327	-107.817	8.8827	0.100566
44	0.0227273	-107.849	-128.789	19.4163	129.618
46	0.0217391	-107.914	-107.914	0.000190921	-135.366

Non-Conformal Convergence of Extrapolated Heat Flux (k1 = 0.05, k2 = 0.5):

Num. Elements	dx	Extrapolated Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-101.3	-101.553	0.24921	-5.84818
6	0.166667	-117.84	-104.004	11.7416	0.375368
8	0.125	-106.738	-130.26	22.0374	27.9057
10	0.1	-106.692	-106.692	8.92631e-05	-33.9737
12	0.0833333	-118.322	-106.738	9.78969	0.204476
14	0.0714286	-107.349	-129.545	20.6764	40.7289
16	0.0625	-107.446	-107.446	0.000392601	-46.0463
18	0.0555556	-118.32	-107.35	9.2718	0.150514
20	0.05	-107.589	-129.118	20.0104	56.023
22	0.0454545	-107.692	-107.692	0.000459328	-61.2776
24	0.0416667	-118.274	-107.59	9.03362	0.126672
26	0.0384615	-107.711	-128.851	19.6267	73.1617
28	0.0357143	-107.804	-107.804	0.000381129	-78.5251
30	0.0333333	-118.229	-107.711	8.89612	0.113953
32	0.03125	-107.782	-128.67	19.3793	91.2895
34	0.0294118	-107.864	-107.865	0.000300238	-96.785
36	0.0277778	-118.191	-107.782	8.80632	0.106313
38	0.0263158	-107.828	-128.539	19.2072	110.117
40	0.025	-107.901	-107.901	0.000236919	-115.742
42	0.0238095	-118.159	-107.829	8.74294	0.101333
44	0.0227273	-107.861	-128.441	19.0809	129.496
46	0.0217391	-107.925	-107.926	0.000189657	-135.24

Non-Conformal Convergence of Extrapolated Heat Flux (k1 = 0.5, k2 = 0.5):

Num. Elements	dx	Extrapolated Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-108.061	-102.761	4.90486	2.25675
6	0.166667	-108.443	-105.938	2.30983	2.46585
8	0.125	-108.641	-107.211	1.31681	2.57934
10	0.1	-108.755	-107.837	0.844392	2.65209
12	0.0833333	-108.826	-108.189	0.585411	2.70304
14	0.0714286	-108.873	-108.406	0.428899	2.74083
16	0.0625	-108.905	-108.549	0.327377	2.77002
18	0.0555556	-108.929	-108.648	0.257902	2.79326
20	0.05	-108.947	-108.72	0.208322	2.81222
22	0.0454545	-108.96	-108.773	0.17173	2.82798
24	0.0416667	-108.971	-108.814	0.143968	2.84129
26	0.0384615	-108.979	-108.846	0.122414	2.85268
28	0.0357143	-108.986	-108.871	0.10535	2.86255
30	0.0333333	-108.992	-108.892	0.0916128	2.87117
32	0.03125	-108.996	-108.909	0.0803922	2.87878
34	0.0294118	-109	-108.923	0.0711099	2.88553
36	0.0277778	-109.004	-108.935	0.0633446	2.89157
38	0.0263158	-109.007	-108.945	0.0567832	2.89701
40	0.025	-109.009	-108.953	0.0511896	2.90192
42	0.0238095	-109.011	-108.961	0.0463825	2.90639
44	0.0227273	-109.013	-108.967	0.0422214	2.91047
46	0.0217391	-109.015	-108.973	0.0385954	2.9142

Non-Conformal Convergence of Extrapolated Heat Flux (k1 = 1, k2 = 0.5):

Num. Elements	dx	Extrapolated Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-106.295	-105.346	0.892	-3.36932
6	0.166667	-100.978	-110.012	8.94603	-0.0839945
8	0.125	-111.486	-91.7235	17.7267	13.2499
10	0.1	-110.509	-110.459	0.0451324	-16.0127
12	0.0833333	-102.324	-111.433	8.90155	0.118621
14	0.0714286	-111.619	-93.3811	16.3398	26.7338
16	0.0625	-111.12	-111.106	0.0123185	-30.2979
18	0.0555556	-102.674	-111.605	8.69835	0.170161
20	0.05	-111.633	-93.9018	15.8837	41.4013
22	0.0454545	-111.297	-111.29	0.0056984	-45.4118
24	0.0416667	-102.819	-111.627	8.56605	0.189662
26	0.0384615	-111.622	-94.1443	15.6579	56.851
28	0.0357143	-111.367	-111.363	0.00331509	-61.1645
30	0.0333333	-102.894	-111.618	8.4784	0.198614
32	0.03125	-111.606	-94.2816	15.5226	72.9101
34	0.0294118	-111.4	-111.397	0.00218306	-77.4522
36	0.0277778	-102.939	-111.603	8.41703	0.203214
38	0.0263158	-111.59	-94.3691	15.4322	89.4749
40	0.025	-111.417	-111.415	0.00155268	-94.2006
42	0.0238095	-102.968	-111.588	8.37193	0.205746
44	0.0227273	-111.576	-94.4294	15.3673	106.474
46	0.0217391	-111.426	-111.425	0.00116383	-111.353

Non-Conformal Convergence of Extrapolated Heat Flux (k1 = 2, k2 = 0.5):

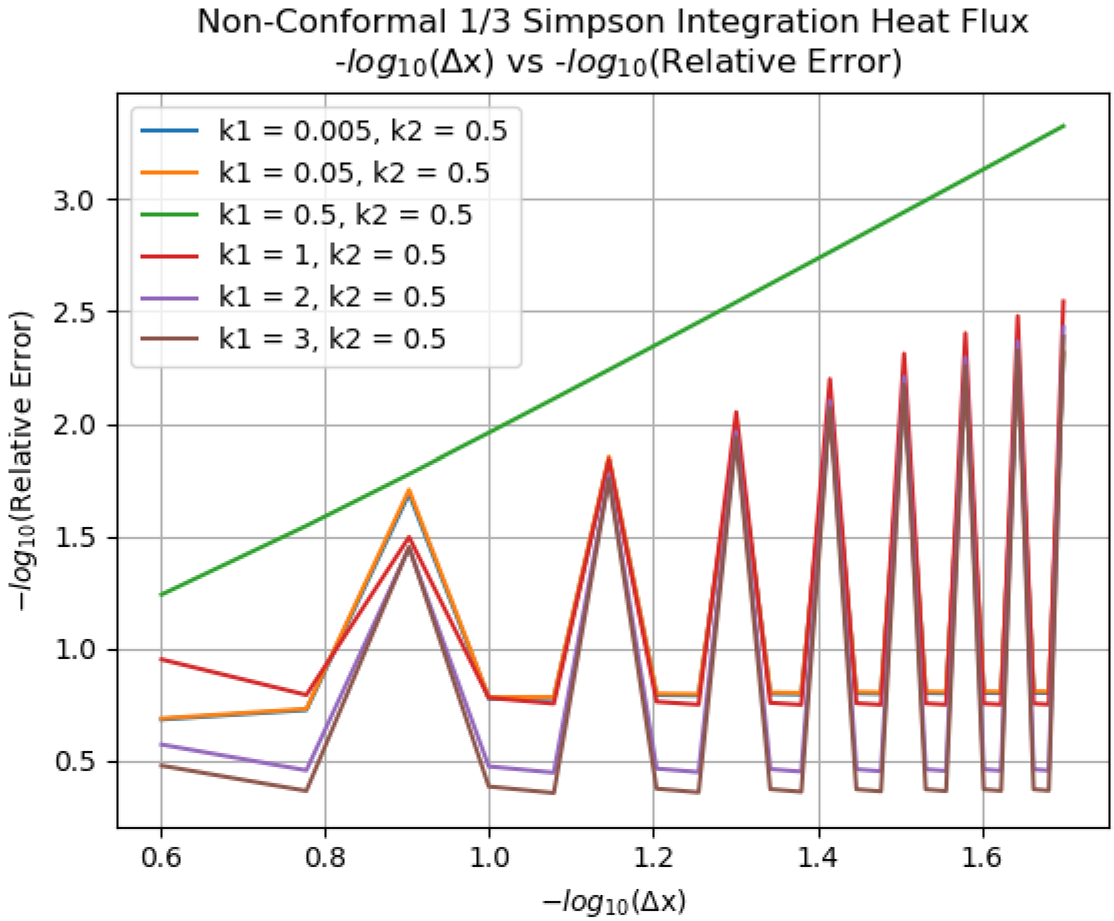
Num. Elements	dx	Extrapolated Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-110.852	-109.628	1.10471	-3.83631
6	0.166667	-99.7074	-116.654	16.9962	0.126958
8	0.125	-117.493	-83.3687	29.0435	12.6407
10	0.1	-115.567	-115.46	0.0921625	-15.5225
12	0.0833333	-100.854	-117.372	16.3774	0.256412
14	0.0714286	-117.254	-84.9771	27.5272	25.0534
16	0.0625	-116.153	-116.116	0.0316144	-28.556
18	0.0555556	-101.123	-117.214	15.9119	0.285418
20	0.05	-117.073	-85.5092	26.961	38.577
22	0.0454545	-116.293	-116.275	0.016252	-42.4499
24	0.0416667	-101.225	-117.053	15.6369	0.29479
26	0.0384615	-116.942	-85.7653	26.66	52.8936
28	0.0357143	-116.335	-116.323	0.00999594	-57.0306
30	0.0333333	-101.273	-116.93	15.4604	0.298242
32	0.03125	-116.845	-85.914	26.4716	67.8378
34	0.0294118	-116.346	-116.339	0.00680181	-72.181
36	0.0277778	-101.299	-116.836	15.3386	0.299482
38	0.0263158	-116.77	-86.0104	26.3421	83.3031
40	0.025	-116.347	-116.341	0.00494016	-87.8156
42	0.0238095	-101.314	-116.764	15.2497	0.29979
44	0.0227273	-116.711	-86.0779	26.2473	99.2137
46	0.0217391	-116.344	-116.339	0.00375612	-103.87

Non-Conformal Convergence of Extrapolated Heat Flux (k1 = 3, k2 = 0.5):

Num. Elements	dx	Extrapolated Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-113.197	-111.755	1.2737	-3.8345
6	0.166667	-100.048	-120.023	19.9655	0.144604
8	0.125	-120.966	-80.8865	33.1329	12.3661
10	0.1	-118.566	-118.428	0.116822	-15.2632
12	0.0833333	-101.181	-120.805	19.3947	0.276215
14	0.0714286	-120.639	-82.3756	31.7174	24.5838
16	0.0625	-119.252	-119.203	0.0409818	-28.0694
18	0.0555556	-101.432	-120.585	18.8835	0.306251
20	0.05	-120.401	-82.8859	31.1583	37.9011
22	0.0454545	-119.414	-119.388	0.0212636	-41.7445
24	0.0416667	-101.52	-120.373	18.5708	0.315991
26	0.0384615	-120.229	-83.138	30.8506	52.008
28	0.0357143	-119.459	-119.444	0.0131436	-56.1095
30	0.0333333	-101.56	-120.213	18.3669	0.319582
32	0.03125	-120.103	-83.287	30.6536	66.7416
34	0.0294118	-119.47	-119.459	0.00897125	-71.0456
36	0.0277778	-101.579	-120.092	18.2247	0.320873
38	0.0263158	-120.006	-83.385	30.5162	81.9957
40	0.025	-119.468	-119.46	0.00652939	-86.4669
42	0.0238095	-101.59	-119.998	18.1205	0.321197
44	0.0227273	-119.93	-83.4541	30.4145	97.695
46	0.0217391	-119.462	-119.456	0.00497185	-102.309

6.2.5 Non-Conformal Heat Flux FDM Results

The following section presents the convergence rates of the heat flux of the upper boundary of the bar with a conformal mesh FDM solution. The data is presented for multiple values of thermal conductivity.



Non-Conformal Convergence of Heat Flux (k1 = 0.005, k2 = 0.5):

Num. Elements	dx	Exact Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-157.679	-101.539	35.6037	n/a
6	0.166667	-157.679	-103.983	34.0542	0.10974389019763087
8	0.125	-157.679	-130.699	17.1108	2.3923792592090165
10	0.1	-157.679	-106.679	32.3439	-2.8534013521892723
12	0.0833333	-157.679	-106.722	32.3166	0.00464091626684979
14	0.0714286	-157.679	-129.934	17.5958	3.9436641790746307
16	0.0625	-157.679	-107.434	31.8653	-4.44730881673358
18	0.0555556	-157.679	-107.336	31.9276	-0.016580525542476098
20	0.05	-157.679	-129.488	17.8786	5.503625685872631
22	0.0454545	-157.679	-107.681	31.7088	-6.011842848820052
24	0.0416667	-157.679	-107.576	31.775	-0.023973288538207255
26	0.0384615	-157.679	-129.211	18.0544	7.062375976822411
28	0.0357143	-157.679	-107.793	31.6379	-7.569583487559346
30	0.0333333	-157.679	-107.698	31.6977	-0.02736342113808088
32	0.03125	-157.679	-129.024	18.173	8.619753189893478
34	0.0294118	-157.679	-107.853	31.5994	-9.125027036381873
36	0.0277778	-157.679	-107.77	31.6522	-0.02918844294164602
38	0.0263158	-157.679	-128.89	18.2581	10.176150120861173
40	0.025	-157.679	-107.89	31.576	-10.679507045342119
42	0.0238095	-157.679	-107.817	31.6227	-0.03027976437956255
44	0.0227273	-157.679	-128.789	18.3221	11.731868668401603
46	0.0217391	-157.679	-107.914	31.5606	-12.233522992102774
48	0.0208333	-157.679	-107.849	31.6023	-0.030982638703466705
50	0.02	-157.679	-128.71	18.3719	13.287108482482331

Non-Conformal Convergence of Heat Flux (k1 = 0.05, k2 = 0.5):

Num. Elements	dx	Exact Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-156.487	-101.553	35.1046	n/a
6	0.166667	-156.487	-104.004	33.5381	0.11259040949984875
8	0.125	-156.487	-130.26	16.7594	2.4114133259587724
10	0.1	-156.487	-106.692	31.8206	-2.8732763116456757
12	0.0833333	-156.487	-106.738	31.7908	0.005130654747747032
14	0.0714286	-156.487	-129.545	17.2163	3.9787149296835373
16	0.0625	-156.487	-107.446	31.3386	-4.485794871254254
18	0.0555556	-156.487	-107.35	31.4	-0.016610101328706967
20	0.05	-156.487	-129.118	17.4894	5.5543900419639085
22	0.0454545	-156.487	-107.692	31.1811	-6.066707571029194
24	0.0416667	-156.487	-107.59	31.2468	-0.02418747136712911
26	0.0384615	-156.487	-128.851	17.6603	7.128657135120421
28	0.0357143	-156.487	-107.804	31.1098	-7.640217527845308
30	0.0333333	-156.487	-107.711	31.1692	-0.027663121479843282
32	0.03125	-156.487	-128.67	17.7761	8.701442958804638
34	0.0294118	-156.487	-107.865	31.0711	-9.21118783127193
36	0.0277778	-156.487	-107.782	31.1236	-0.02953450800313579
38	0.0263158	-156.487	-128.539	17.8593	10.273184968627492
40	0.025	-156.487	-107.901	31.0475	-10.781075688892145
42	0.0238095	-156.487	-107.829	31.094	-0.030653717677806433
44	0.0227273	-156.487	-128.441	17.9219	11.844208569842696
46	0.0217391	-156.487	-107.926	31.0321	-12.350432973479075
48	0.0208333	-156.487	-107.861	31.0735	-0.03137464877862372
50	0.02	-156.487	-128.365	17.9707	13.414726723450443

Non-Conformal Convergence of Heat Flux (k1 = 0.5, k2 = 0.5):

Num. Elements	dx	Exact Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-109.033	-102.761	5.75292	n/a
6	0.166667	-109.033	-105.938	2.8387	1.74210104445759
8	0.125	-109.033	-107.211	1.67154	1.8409159937269834
10	0.1	-109.033	-107.837	1.09736	1.885951837982921
12	0.0833333	-109.033	-108.189	0.77445	1.9115229219484502
14	0.0714286	-109.033	-108.406	0.575342	1.9279126055127038
16	0.0625	-109.033	-108.549	0.444083	1.9392716829908854
18	0.0555556	-109.033	-108.648	0.353053	1.9475883360767416
20	0.05	-109.033	-108.72	0.287364	1.95393038750077
22	0.0454545	-109.033	-108.773	0.238423	1.9589209053435634
24	0.0416667	-109.033	-108.814	0.200988	1.9629471216554142
26	0.0384615	-109.033	-108.846	0.171719	1.9662619758079976
28	0.0357143	-109.033	-108.871	0.148404	1.9690374813394
30	0.0333333	-109.033	-108.892	0.129532	1.9713945833705349
32	0.03125	-109.033	-108.909	0.114042	1.973420735939064
34	0.0294118	-109.033	-108.923	0.101172	1.9751807065750682
36	0.0277778	-109.033	-108.935	0.0903631	1.9767234653406567
38	0.0263158	-109.033	-108.945	0.0811976	1.9780867085284215
40	0.025	-109.033	-108.953	0.0733587	1.9792999105880595
42	0.0238095	-109.033	-108.961	0.0666022	1.9803864592400418
44	0.0227273	-109.033	-108.967	0.0607377	1.9813651220417796
46	0.0217391	-109.033	-108.973	0.0556149	1.9822511665032863
48	0.0208333	-109.033	-108.977	0.0511137	1.9830570912060053
50	0.02	-109.033	-108.982	0.0471375	1.9837932601970953

Non-Conformal Convergence of Heat Flux (k1 = 1, k2 = 0.5):

Num. Elements	dx	Exact Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-87.4207	-105.346	20.5051	n/a
6	0.166667	-87.4207	-110.012	25.8415	-0.5704698075035343
8	0.125	-87.4207	-91.7235	4.92197	5.764255144487407
10	0.1	-87.4207	-110.459	26.3531	-7.51928054348285
12	0.0833333	-87.4207	-111.433	27.4674	-0.22714749064306106
14	0.0714286	-87.4207	-93.3811	6.81799	9.039444561752658
16	0.0625	-87.4207	-111.106	27.0933	-10.332560446110353
18	0.0555556	-87.4207	-111.605	27.6643	-0.1770690877507941
20	0.05	-87.4207	-93.9018	7.41369	12.498169855142221
22	0.0454545	-87.4207	-111.29	27.3044	-13.6787225204441
24	0.0416667	-87.4207	-111.627	27.689	-0.160727258540144
26	0.0384615	-87.4207	-94.1443	7.69101	16.003736949180784
28	0.0357143	-87.4207	-111.363	27.3877	-17.1377152730654
30	0.0333333	-87.4207	-111.618	27.6792	-0.1534616532894218
32	0.03125	-87.4207	-94.2816	7.84812	19.529536642053753
34	0.0294118	-87.4207	-111.397	27.4266	-20.639144624207283
36	0.0277778	-87.4207	-111.603	27.6622	-0.1496314439611663
38	0.0263158	-87.4207	-94.3691	7.94823	23.066061775386192
40	0.025	-87.4207	-111.415	27.4468	-24.16105304100201
42	0.0238095	-87.4207	-111.588	27.6449	-0.14737978975300486
44	0.0227273	-87.4207	-94.4294	8.0172	26.608986854876058
46	0.0217391	-87.4207	-111.425	27.4579	-27.69438780098057
48	0.0208333	-87.4207	-111.574	27.629	-0.14595160098919183
50	0.02	-87.4207	-94.4733	8.06743	30.156045595802354

Non-Conformal Convergence of Heat Flux (k1 = 2, k2 = 0.5):

Num. Elements	dx	Exact Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-76.3471	-109.628	43.5915	n/a
6	0.166667	-76.3471	-116.654	52.7942	-0.47239619226129226
8	0.125	-76.3471	-83.3687	9.19704	6.074482817934771
10	0.1	-76.3471	-115.46	51.2306	-7.696640359992516
12	0.0833333	-76.3471	-117.372	53.7344	-0.2617076701796136
14	0.0714286	-76.3471	-84.9771	11.3037	10.112959693818734
16	0.0625	-76.3471	-116.116	52.09	-11.44179697158415
18	0.0555556	-76.3471	-117.214	53.5275	-0.23112889615521318
20	0.05	-76.3471	-85.5092	12.0006	14.19166026780569
22	0.0454545	-76.3471	-116.275	52.2973	-15.444204400746615
24	0.0416667	-76.3471	-117.053	53.317	-0.22192549569053174
26	0.0384615	-76.3471	-85.7653	12.3361	18.286793367238893
28	0.0357143	-76.3471	-116.323	52.3613	-19.507183443718592
30	0.0333333	-76.3471	-116.93	53.1556	-0.21821410679702388
32	0.03125	-76.3471	-85.914	12.5308	22.390233805868128
34	0.0294118	-76.3471	-116.339	52.3811	-23.59363979255478
36	0.0277778	-76.3471	-116.836	53.0332	-0.21646591314065092
38	0.0263158	-76.3471	-86.0104	12.6572	26.49842675985496
40	0.025	-76.3471	-116.341	52.3847	-27.691583464910206
42	0.0238095	-76.3471	-116.764	52.9386	-0.21556446588700917
44	0.0227273	-76.3471	-86.0779	12.7454	30.609584724568016
46	0.0217391	-76.3471	-116.339	52.3819	-31.7959862094263
48	0.0208333	-76.3471	-116.707	52.8636	-0.215075229834338
50	0.02	-76.3471	-86.1275	12.8105	34.72271427077798

Non-Conformal Convergence of Heat Flux (k1 = 3, k2 = 0.5):

Num. Elements	dx	Exact Heat Flux	Approx. Heat Loss	Percent Error	Beta
4	0.25	-72.4277	-111.755	54.2992	n/a
6	0.166667	-72.4277	-120.023	65.7136	-0.47055829776757463
8	0.125	-72.4277	-80.8865	11.6789	6.0049582480957335
10	0.1	-72.4277	-118.428	63.5118	-7.589015058209802
12	0.0833333	-72.4277	-120.805	66.7943	-0.2763918037630972
14	0.0714286	-72.4277	-82.3756	13.735	10.260571711149286
16	0.0625	-72.4277	-119.203	64.5826	-11.59278075705659
18	0.0555556	-72.4277	-120.585	66.4907	-0.24721427382907438
20	0.05	-72.4277	-82.8859	14.4395	14.493966399370127
22	0.0454545	-72.4277	-119.388	64.838	-15.758246193982476
24	0.0416667	-72.4277	-120.373	66.1981	-0.23859851726575898
26	0.0384615	-72.4277	-83.138	14.7876	18.72579132921086
28	0.0357143	-72.4277	-119.444	64.9141	-19.961091746258763
30	0.0333333	-72.4277	-120.213	65.9763	-0.23525165608273216
32	0.03125	-72.4277	-83.287	14.9933	22.958272502606622
34	0.0294118	-72.4277	-119.459	64.9356	-24.17816764488O325
36	0.0277778	-72.4277	-120.092	65.809	-0.23375793398949854
38	0.0263158	-72.4277	-83.385	15.1286	27.191555959621827
40	0.025	-72.4277	-119.46	64.9373	-28.40211594654029
42	0.0238095	-72.4277	-119.998	65.6799	-0.23304506475291686
44	0.0227273	-72.4277	-83.4541	15.2241	31.425502272265874
46	0.0217391	-72.4277	-119.456	64.9316	-32.62989088788212
48	0.0208333	-72.4277	-119.924	65.5779	-0.23270124617224755
50	0.02	-72.4277	-83.5055	15.295	35.65996208079114

7 Discussion

This report investigated the application of the FDM to solve a 2-dimensional boundary condition problem with a material interface. Furthermore, this report investigated the effect that varying the number of nodes and the value of the thermal conductivity had on the resulting temperature distribution, heat transfer through the upper boundary, the convergence, and accuracy of the results. Finally, the report observed the differences in results, accuracy, and convergence in the utilization of a conformal and non-conformal mesh.

For the conformal mesh case, it was observed that the temperature distribution within the bar varied drastically as the value of K increased. This result is expected as highly values of K represent a greater ability of the material to resist a change in temperature. In addition, the convergence of the results consistently approximated to a value of 2. This result is also expected as the FDM solution was derived to a second order.

In addition, the relative thermal conductivity had an interesting effect on the distribution of the temperature within the bar. For cases when material two behaved as in insulator, the temperature distribution within material one was near zero. Whereas for cases when material two behaved as a conductor as compared to material one, much more heat was able to flow into the bar.

For the non-conformal case, the accuracy and convergence of the solution was found to be unreliable and sporadic. For cases when the non-conformal mesh aligned with the material interface location, the convergence and accuracy of the solution followed a slope of 2, corresponding to the second order derivation of the FDM solution. However, for all other cases the solution was found to be unable to converge. For cases where the two materials possessed the same thermal conductivity, the accuracy and convergence of the solution improved significantly. This resulted in a convergence rate of two for the non-conformal case.

8 References

The following reports and codes were used to assist in the creation of this report and the code utilized within.

- Antonio Diaz's Code and Report
- Valentina Musu's Report

9 Appendix A: Code

```
"""
AERO 430
Exam 2 Code

Andrew Hollister
127008398
"""

import tabulate as tbl
import matplotlib.pyplot as plt
from mpl_toolkits import mplot3d
from matplotlib import cm
import seaborn as sns
import numpy as np
import math
import time

"""
Data Class Structure
"""


class DataStructure:
    def __init__(self):
        self.data = []

    def add_data(self, nodes, k, x_vals, y_vals, temp_mesh):
        self.data.append((nodes, k, x_vals, y_vals, temp_mesh))

    def return_data(self, nodes, k):
        x_vals = next(item[2] for item in self.data if item[0] == nodes and item[1] == k)
        y_vals = next(item[3] for item in self.data if item[0] == nodes and item[1] == k)
        temp_mesh = next(item[4] for item in self.data if item[0] == nodes and item[1] == k)
        return x_vals, y_vals, temp_mesh


"""
Analytical Solution
"""


def exact_temp(x, y, k1, k2, interface):

    # Heat flow constant across y for material 1 and 2
    kyy1 = kxx/k1**2
    kyy2 = kxx/k2**2

    # Y bar function
    y_bar = 100/(np.sinh(k2*np.pi*(1-interface))*((kyy1*k1*(1/np.tanh(interface*np.pi*k1)))
                                              / (kyy2*k2)+(1/np.tanh(k2*np.pi*(1-interface)))))

    # Material 1 Exact Solution
    temp1 = y_bar*np.sin(np.pi*x)*np.sinh(k1*np.pi*y)/np.sinh(interface*np.pi*k1)

    # Material 2 Exact Solution
    temp2 = np.sin(np.pi*x)*((100-y_bar*np.cosh(k2*np.pi*(1-interface)))*np.sinh(k2*np.pi*(y-interface))
                           / np.sinh(k2*np.pi*(1-interface))+y_bar*np.cosh(k2*np.pi*(y-interface)))

    temp = np.where(y < interface, temp1, temp2)

    return temp


"""
2nd Order FDM Solution
"""


def bc_temp_func(x):
    return -100*np.sin(np.pi*x)  # deg c


def approx_temp(x, y, interface, k1, k2):

    # Heat Conduction Variables
```

```
    kyy1 = kxx/k1**2
    kyy2 = kxx/k2**2

    # Sizing mesh
    n_nodes = len(x)
    dim = n_nodes**2

    # Creating Meshes for reference
    y_mesh = np.meshgrid(x, y)[1]

    # Defining dx and dy
    dx = x[1]-x[0]
    dy1 = y[1]-y[0]
    dy2 = y[-1]-y[-2]

    # Building A matrix
    a = np.zeros([dim, dim])

    count = 0
    mesh_row = 0
    for row_i, row in enumerate(a):

        # Need to figure out a way to properly figure out the y value of the previous and next node
        if count == n_nodes:
            mesh_row += 1
            count = 0
        count += 1

        # Heat Conduction Variable
        if np.reshape(y_mesh, dim)[row_i] < interface:  # Material Behind Interface

            # Adding Diagonal Terms
            a[row_i][row_i] = 2*(k1**2*dy1**2/dx**2+1)

            # Adding Off-Diagonal Terms
            if row_i - 1 < 0:
                pass
            else:
                a[row_i][row_i - 1] = -k1**2*dy1**2/dx**2
            try:
                a[row_i][row_i + 1] = -k1**2*dy1**2/dx**2
            except IndexError:
                pass
            try:
                a[row_i][row_i + n_nodes] = -1
            except IndexError:
                pass
            if row_i - n_nodes < 0:
                pass
            else:
                a[row_i][row_i - n_nodes] = -1

        elif np.reshape(y_mesh, dim)[row_i] > interface:  # Material Ahead of Interface

            # Adding Diagonal Terms
            a[row_i][row_i] = 2*(k2**2*dy2**2/dx**2+1)

            # Adding Off-Diagonal Terms
            if row_i - 1 < 0:
                pass
            else:
                a[row_i][row_i - 1] = -k2**2*dy2**2/dx**2
            try:
                a[row_i][row_i + 1] = -k2**2*dy2**2/dx**2
            except IndexError:
                pass
            try:
                a[row_i][row_i + n_nodes] = -1
            except IndexError:
                pass
            if row_i - n_nodes < 0:
                pass
            else:
                a[row_i][row_i - n_nodes] = -1

        elif np.reshape(y_mesh, dim)[row_i] == interface:  # Material Along Interface

            # Adding Diagonal Terms
            a[row_i][row_i] = (kyy1/dy1+kxx*dy1/dx**2+kxx*dy2/dx**2+kyy2/dy2)
```

```
            # Adding Off-Diagonal Terms
            a[row_i][row_i - 1] = -(0.5*kxx*dy1/dx**2+0.5*kxx*dy2/dx**2)
            a[row_i][row_i + 1] = -(0.5*kxx*dy1/dx**2+0.5*kxx*dy2/dx**2)
            a[row_i][row_i - n_nodes] = -kyy1/dy1
            a[row_i][row_i + n_nodes] = -kyy2/dy2

    # Removing Extraneous Terms
    for i in range(dim):
        if i % n_nodes == 0 and i != 0:
            a[i-1][i] = 0
            a[i][i-1] = 0

    # Generating B Matrix
    b = np.zeros([dim, 1])
    dxl = length/(n_nodes+1)
    for i in range(n_nodes):
        xl = dxl*(i+1)
        b[-i-1][0] = -bc_temp_func(xl)

    # Solving Matrix
    temps = np.reshape(np.linalg.solve(a, b), [n_nodes, n_nodes])
    temps = np.flip(temps)
    temps = np.row_stack((-bc_temp_func(np.linspace(0, length, n_nodes+2))[1:-1], temps))
    temps = np.row_stack((temps, np.zeros(n_nodes)))
    temps = np.column_stack((temps, np.zeros(n_nodes+2)))
    temps = np.column_stack((np.zeros(n_nodes+2), temps))

    return np.flip(temps)


"""
Heat Flux Functions
"""


# Exact Heat Transfer Function
def exact_heat_transfer(k1, k2, interface):

    # Heat flow constant across y for material 1 and 2
    kyy1 = kxx / k1 ** 2
    kyy2 = kxx / k2 ** 2

    # Y bar function
    y_bar = 100 / (np.sinh(k2 * np.pi * (1 - interface)) * ((kyy1 * k1 * (1 / np.tanh(interface * np.pi * k1)))
                                                            / (kyy2 * k2) + (
                                                                        1 / np.tanh(k2 * np.pi * (1 -
interface)))))

    # Heat FLux Through Top Boundary
    q_ex = ((100-y_bar*np.cosh(k2*np.pi*(1-interface))) * np.cosh(k2*np.pi*(length-interface))
            / np.sinh(k2*np.pi*(length-interface)) + y_bar*np.sinh(k2*np.pi*(length-interface)))

    return -q_ex


# Approximate Heat Flux Function
def approx_heat_transfer(lower, upper, nodes, fdm_temp, k, y):
    dtdy = []
    dy = y[-1]-y[-2]
    for i in range(len(fdm_temp)):
        col = fdm_temp[:, i]
        dtdy.append((col[-3]-4*col[-2]+3*col[-1])/(2*dy))

    summation = 0
    for i in range(nodes + 1):
        summand = dtdy[i]
        if (i != 0) and (i != nodes):
            summand *= (2 + (2 * (i % 2)))
        summation += summand

    val = ((upper - lower) / (3 * nodes)) * summation
    return -2*k*val


"""
Richardson Extrapolation Function
"""
```

```python
def q_rich_extr(q, q2, q4):
    q_extr = (q2**2 - q*q4)/(2*q2-q-q4)
    perc_err = abs(abs(q_extr-q)/q_extr)*100
    return q_extr, perc_err


"""
Convergence Rate Function
"""


def get_beta(exact, approx, approx_2, h, h_2):
    a = np.log(abs(exact - approx))
    b = np.log(abs(exact - approx_2))
    c = np.log(h) - np.log(h_2)
    return -(a - b) / c


"""
Main Function
"""


def main():

    # Material (1, 2) Heat Conductivity
    k_list = [(0.005, 0.5),
              (0.05, 0.5),
              (0.5, 0.5),
              (1, 0.5),
              (2, 0.5),
              (3, 0.5)]
    global kxx
    kxx = 1  # Heat Conductivity of both materials in x flow

    # Interface location list
    inter_list = [np.pi/6]

    # Defining mesh sizes to loop through
    node_list = range(3, 51, 2)

    # Bar properties
    global length
    length = 1

    # Data Structures
    nonc_fdm_data = DataStructure()
    conf_fdm_data = DataStructure()

    """
    Data Generation
    """

    for inter in inter_list:
        for k in k_list:
            k1, k2 = k
            for n in node_list:
                n_nodes = n

                # Non-Conformal Mesh
                x_vals = np.linspace(0, length, n_nodes)
                y_vals = np.linspace(0, length, n_nodes)

                # Conformal Mesh
                y_vals_a = np.linspace(inter, length, 1+math.trunc(n_nodes / 2))
                y_vals_b = np.linspace(0, inter, math.ceil(n_nodes / 2))[:-1]
                y_vals_c = np.append(y_vals_b, y_vals_a)

                # Generating Data
                conf_sol = approx_temp(x_vals, y_vals_c, inter, k1, k2)
                nonc_sol = approx_temp(x_vals, y_vals, 1/3, k1, k2)

                # Resizing Meshes
                x_vals = np.linspace(0, length, n_nodes+2)
                y_vals = np.linspace(0, length, n_nodes+2)
                y_vals_a = np.linspace(inter, length, 1+math.trunc((n_nodes + 2) / 2))
                y_vals_b = np.linspace(0, inter, math.ceil((n_nodes + 2) / 2))[:-1]
                y_vals_c = np.append(y_vals_b, y_vals_a)
```

```
            # Appending Data to Data Structures
            conf_fdm_data.add_data(n_nodes+2, k, x_vals, y_vals_c, conf_sol)
            nonc_fdm_data.add_data(n_nodes+2, k, x_vals, y_vals, nonc_sol)

    """
    Conformal Mesh FDM Heat Flux and Error Calculations
    """

    # Loop Lists
    re_log_err = []
    re_dx_err = []
    hf_log_err = []
    hf_dx_err = []
    tc_log_err = []
    tc_dx_err = []

    for inter in inter_list:
        for k in k_list:

            k1, k2 = k

            heat_flux_data = []
            temp_error_data = []
            for n in node_list:
                n_nodes = n
                x_vals, y_vals, z = conf_fdm_data.return_data(n_nodes + 2, k)

                # Heat Flux Calculations and Error Calculations
                q_exact = exact_heat_transfer(k1, k2, inter)
                q_approx = approx_heat_transfer(0, 1, n_nodes + 1, z, k2, y_vals)
                error_q = abs((q_exact - q_approx) / q_exact * 100)
                heat_flux_data.append([n_nodes + 1, length / (n_nodes + 1), q_exact, q_approx, error_q])

                # Temperature Error Calculations
                inter_index = int(len(z) / 2)
                midpoint = length / (n_nodes + 1) * inter_index
                approx_t = z[inter_index][inter_index]
                exact_t = exact_temp(midpoint, inter, k1, k2, inter)
                error_t = abs(exact_t - approx_t) / exact_t * 100
                temp_error_data.append([n_nodes + 1, length / (n_nodes + 1), exact_t, approx_t, error_t])

            # Richardson Extrapolation and Convergence
            heat_flux_rc_data = []
            for i in range(len(heat_flux_data) - 2):
                q_extr, perc_err = q_rich_extr(heat_flux_data[i][3], heat_flux_data[i + 1][3],
                                               heat_flux_data[i + 2][3])
                beta = get_beta(q_extr, heat_flux_data[i][3], heat_flux_data[i + 1][3],
                                heat_flux_data[i][0], heat_flux_data[i + 1][0])
                heat_flux_rc_data.append([heat_flux_data[i][0], heat_flux_data[i][1], q_extr,
heat_flux_data[i][3],
                                          perc_err, beta])
            print(f'\nConformal Convergence of Extrapolated Heat Flux (k1 = {k1}, k2 = {k2}):')
            print(tbl.tabulate(heat_flux_rc_data,
                               headers=['Num. Elements', 'dx', 'Extrapolated Heat Flux', 'Approx. Heat Loss',
                                        'Percent Error', 'Beta']))

            re_log_err.append([-np.log10(item[4] / 100) for item in heat_flux_rc_data])
            re_dx_err.append([-np.log10(item[1]) for item in heat_flux_rc_data])

            # Heat Flux Convergence
            heat_flux_data[0].append('n/a')
            for i in range(len(heat_flux_data) - 1):
                heat_flux_data[i + 1].append(
                    get_beta(heat_flux_data[i][2], heat_flux_data[i][3], heat_flux_data[i + 1][3],
                             heat_flux_data[i][0], heat_flux_data[i + 1][0]))
            print(f'\nConformal Convergence of Heat Flux (k1 = {k1}, k2 = {k2}):')
            print(tbl.tabulate(heat_flux_data, headers=['Num. Elements', 'dx', 'Exact Heat Flux', 'Approx. Heat
Loss',
                                                        'Percent Error', 'Beta']))
            hf_log_err.append([-np.log10(item[4] / 100) for item in heat_flux_data])
            hf_dx_err.append([-np.log10(item[1]) for item in heat_flux_data])

            # Temperature Convergence
            temp_error_data[0].append('n/a')
            for i in range(len(temp_error_data) - 1):
                temp_error_data[i + 1].append(
                    get_beta(temp_error_data[i][2], temp_error_data[i][3], temp_error_data[i + 1][3],
                             temp_error_data[i][0], temp_error_data[i + 1][0]))
```

```
            print(f'\nConformal Convergence of 2nd Order FDM Temperature Solution (k1 = {k1}, k2 = {k2}):')
            print(tbl.tabulate(temp_error_data,
                               headers=['Num. Elements', 'dx', 'Exact Midpoint Temp', 'Approx. Midpoint Temp',
                                        'Percent Error', 'Beta']))
            tc_log_err.append([-np.log10(item[4] / 100) for item in temp_error_data])
            tc_dx_err.append([-np.log10(item[1]) for item in temp_error_data])

        # Plotting Convergence Graphs
        for i in range(len(k_list)):
            plt.plot(re_dx_err[i], re_log_err[i], label=f'k1 = {k_list[i][0]}, k2 = {k_list[i][1]}')
        plt.title('Conformal Extrapolated Heat Flux\n' + r'-$log_{10}$($\Delta$x) vs -$log_{10}$(Relative
Error)')
        plt.xlabel(r'$-log_{10}$($\Delta$x)')
        plt.ylabel(r'$-log_{10}$(Relative Error)')
        plt.grid()
        plt.legend()
        plt.show()

        for i in range(len(k_list)):
            plt.plot(hf_dx_err[i], hf_log_err[i], label=f'k1 = {k_list[i][0]}, k2 = {k_list[i][1]}')
        plt.title('Conformal 1/3 Simpson Integration Heat Flux\n '
                  + r'-$log_{10}$($\Delta$x) vs -$log_{10}$(Relative Error)')
        plt.xlabel(r'$-log_{10}$($\Delta$x)')
        plt.ylabel(r'$-log_{10}$(Relative Error)')
        plt.grid()
        plt.legend()
        plt.show()

        for i in range(len(k_list)):
            plt.plot(tc_dx_err[i], tc_log_err[i], label=f'k1 = {k_list[i][0]}, k2 = {k_list[i][1]}')
        plt.title('Conformal 2nd order FDM\n' + r'-$log_{10}$($\Delta$x) vs -$log_{10}$(Relative Error)')
        plt.xlabel(r'$-log_{10}$($\Delta$x)')
        plt.ylabel(r'$-log_{10}$(Relative Error)')
        plt.grid()
        plt.legend()
        plt.show()

    """
    Non-Conformal Mesh FDM Heat Flux and Error Calculations
    """

    # Loop Lists
    re_log_err = []
    re_dx_err = []
    hf_log_err = []
    hf_dx_err = []
    tc_log_err = []
    tc_dx_err = []

    for inter in inter_list:
        for k in k_list:

            k1, k2 = k

            heat_flux_data = []
            temp_error_data = []
            for n in node_list:
                n_nodes = n
                x_vals, y_vals, z = nonc_fdm_data.return_data(n_nodes + 2, k)

                # Heat Flux Calculations and Error Calculations
                q_exact = exact_heat_transfer(k1, k2, inter)
                q_approx = approx_heat_transfer(0, 1, n_nodes + 1, z, k2, y_vals)
                error_q = abs((q_exact - q_approx) / q_exact * 100)
                heat_flux_data.append([n_nodes + 1, length / (n_nodes + 1), q_exact, q_approx, error_q])

                # Temperature Error Calculations
                inter_index = int(len(z) / 2)
                midpoint = length / (n_nodes + 1) * inter_index
                approx_t = z[inter_index][inter_index]
                exact_t = exact_temp(midpoint, inter, k1, k2, inter)
                error_t = abs(exact_t - approx_t) / exact_t * 100
                temp_error_data.append([n_nodes + 1, length / (n_nodes + 1), exact_t, approx_t, error_t])

            # Richardson Extrapolation and Convergence
            heat_flux_rc_data = []
            for i in range(len(heat_flux_data) - 2):
                q_extr, perc_err = q_rich_extr(heat_flux_data[i][3], heat_flux_data[i + 1][3],
                                               heat_flux_data[i + 2][3])
```

```
                beta = get_beta(q_extr, heat_flux_data[i][3], heat_flux_data[i + 1][3],
                                heat_flux_data[i][0], heat_flux_data[i + 1][0])
                heat_flux_rc_data.append([heat_flux_data[i][0], heat_flux_data[i][1], q_extr,
heat_flux_data[i][3],
                                          perc_err, beta])
            print(f'\nConformal Convergence of Extrapolated Heat Flux (k1 = {k1}, k2 = {k2}):')
            print(tbl.tabulate(heat_flux_rc_data,
                               headers=['Num. Elements', 'dx', 'Extrapolated Heat Flux', 'Approx. Heat Loss',
                                        'Percent Error', 'Beta']))

            re_log_err.append([-np.log10(item[4] / 100) for item in heat_flux_rc_data])
            re_dx_err.append([-np.log10(item[1]) for item in heat_flux_rc_data])

            # Heat Flux Convergence
            heat_flux_data[0].append('n/a')
            for i in range(len(heat_flux_data) - 1):
                heat_flux_data[i + 1].append(
                    get_beta(heat_flux_data[i][2], heat_flux_data[i][3], heat_flux_data[i + 1][3],
                             heat_flux_data[i][0], heat_flux_data[i + 1][0]))
            print(f'\nNon-Conformal Convergence of Heat Flux (k1 = {k1}, k2 = {k2}):')
            print(tbl.tabulate(heat_flux_data, headers=['Num. Elements', 'dx', 'Exact Heat Flux', 'Approx. Heat
Loss',
                                                        'Percent Error', 'Beta']))
            hf_log_err.append([-np.log10(item[4] / 100) for item in heat_flux_data])
            hf_dx_err.append([-np.log10(item[1]) for item in heat_flux_data])

            # Temperature Convergence
            temp_error_data[0].append('n/a')
            for i in range(len(temp_error_data) - 1):
                temp_error_data[i + 1].append(
                    get_beta(temp_error_data[i][2], temp_error_data[i][3], temp_error_data[i + 1][3],
                             temp_error_data[i][0], temp_error_data[i + 1][0]))
            print(f'\nNon-Conformal Convergence of 2nd Order FDM Temperature Solution (k1 = {k1}, k2 = {k2}):')
            print(tbl.tabulate(temp_error_data,
                               headers=['Num. Elements', 'dx', 'Exact Midpoint Temp', 'Approx. Midpoint Temp',
                                        'Percent Error', 'Beta']))
            tc_log_err.append([-np.log10(item[4] / 100) for item in temp_error_data])
            tc_dx_err.append([-np.log10(item[1]) for item in temp_error_data])

            # Plotting Convergence Graphs
        for i in range(len(k_list)):
            plt.plot(re_dx_err[i], re_log_err[i], label=f'k1 = {k_list[i][0]}, k2 = {k_list[i][1]}')
        plt.title('Non-Conformal Extrapolated Heat Flux\n' + r'-$log_{10}$($\Delta$x) vs -$log_{10}$(Relative
Error)')
        plt.xlabel(r'$-log_{10}$($\Delta$x)')
        plt.ylabel(r'$-log_{10}$(Relative Error)')
        plt.grid()
        plt.legend()
        plt.show()

        for i in range(len(k_list)):
            plt.plot(hf_dx_err[i], hf_log_err[i], label=f'k1 = {k_list[i][0]}, k2 = {k_list[i][1]}')
        plt.title('Non-Conformal 1/3 Simpson Integration Heat Flux\n '
                  + r'-$log_{10}$($\Delta$x) vs -$log_{10}$(Relative Error)')
        plt.xlabel(r'$-log_{10}$($\Delta$x)')
        plt.ylabel(r'$-log_{10}$(Relative Error)')
        plt.grid()
        plt.legend()
        plt.show()

        for i in range(len(k_list)):
            plt.plot(tc_dx_err[i], tc_log_err[i], label=f'k1 = {k_list[i][0]}, k2 = {k_list[i][1]}')
        plt.title('Non-Conformal 2nd order FDM\n' + r'-$log_{10}$($\Delta$x) vs -$log_{10}$(Relative Error)')
        plt.xlabel(r'$-log_{10}$($\Delta$x)')
        plt.ylabel(r'$-log_{10}$(Relative Error)')
        plt.grid()
        plt.legend()
        plt.show()

    for inter in inter_list:
        for k in k_list:
            k1, k2 = k
            for n_nodes in range(3, 53, 10):

                """
                Conformal Mesh Related Plotting
                """

                # Extracting values
```

```python
x_vals, y_vals, z = conf_fdm_data.return_data(n_nodes+2, k)
x, y = np.meshgrid(x_vals, y_vals)
x1, y1, = np.meshgrid(x_vals, y_vals)
z1 = exact_temp(x1, y1, k1, k2, inter)

"""
Exact Solution Plotting
"""

# 3-Dimensional Heat Maps
ax = plt.axes(projection='3d')
ax.plot_surface(x1, y1, z1, rstride=1, cstride=1,
                cmap='inferno', edgecolor='none')
plt.title(f'Conformal Mesh Exact Solution 3-Dimensional Heatmap\nk1 = {k1} |'
          f' k2 = {k2}\nInterface: y = {round(inter, 3)}')
ax.set_xlabel('X (cm)')
ax.set_ylabel('Y (cm)')
ax.set_zlabel(r'T(x, y)$\degree$C')
plt.show()

# 2-Dimensional Heat Maps
ax = sns.heatmap(z1, xticklabels=False, yticklabels=False,
                 cbar_kws={'label': u'Temperature \N{DEGREE SIGN}C'})
plt.xticks((n_nodes + 2) * np.array([0, 0.25, 0.5, 0.75, 1]), labels=[0, 0.25, 0.5, 0.75, 1])
ax.set_xlabel('x (cm)')
plt.yticks((n_nodes + 2) * np.array([0, 0.25, 0.5, 0.75, 1]), labels=[0, 0.25, 0.5, 0.75, 1])
ax.set_ylabel('y (cm)')
ax.set_title(f'Conformal Mesh Exact Solution Heatmap\nk1 = {k1} '
             f'| k2 = {k2}\nInterface: y = {round(inter, 3)}')
plt.gca().invert_yaxis()
plt.show()

"""
Conformal Mesh FDM Plotting
"""

# 3-Dimensional Heat Maps
ax = plt.axes(projection='3d')
ax.plot_surface(x, y, z, rstride=1, cstride=1,
                cmap='inferno', edgecolor='none')
plt.title(
    r'Conformal 2nd order FDM Output T(x, y)$\degree$ '
    r'C' + '\nNodes = ' + str(n_nodes+2) + '\nK = ' + str(k))
ax.set_xlabel('X (cm)')
ax.set_ylabel('Y (cm)')
ax.set_zlabel(r'T(x, y)$\degree$C')
plt.show()

# 2-Dimensional Heat Maps
ax = sns.heatmap(z, xticklabels=False, yticklabels=False,
                 cbar_kws={'label': u'Temperature \N{DEGREE SIGN}C'})
plt.xticks((n_nodes + 2) * np.array([0, 0.25, 0.5, 0.75, 1]), labels=[0, 0.25, 0.5, 0.75, 1])
ax.set_xlabel('x (cm)')
plt.yticks((n_nodes + 2) * np.array([0, 0.25, 0.5, 0.75, 1]), labels=[0, 0.25, 0.5, 0.75, 1])
ax.set_ylabel('y (cm)')
ax.set_title('Conformal FDM Heatmap of cross-section'
             ' of bar\n' + str(n_nodes + 2) + ' Nodes' + '\nK = ' + str(k))
plt.gca().invert_yaxis()
plt.show()

"""
Non-Conformal Mesh Related Plotting
"""

# Extracting values
x_vals, y_vals, z = nonc_fdm_data.return_data(n_nodes + 2, k)
x, y = np.meshgrid(x_vals, y_vals)
x1, y1, = np.meshgrid(x_vals, y_vals)
z1 = exact_temp(x1, y1, k1, k2, inter)

"""
Exact Solution Plotting
"""

# 3-Dimensional Heat Maps
ax = plt.axes(projection='3d')
ax.plot_surface(x1, y1, z1, rstride=1, cstride=1,
                cmap='inferno', edgecolor='none')
plt.title(f'Non-Conformal Mesh Exact Solution 3-Dimensional Heatmap\nk1 = {k1} |'
```

```
                          f' k2 = {k2}\nInterface: y = {round(inter, 3)}')
                ax.set_xlabel('X (cm)')
                ax.set_ylabel('Y (cm)')
                ax.set_zlabel(r'T(x, y)$\degree$C')
                plt.show()

                # 2-Dimensional Heat Maps
                ax = sns.heatmap(z1, xticklabels=False, yticklabels=False,
                                 cbar_kws={'label': u'Temperature \N{DEGREE SIGN}C'})
                plt.xticks((n_nodes + 2) * np.array([0, 0.25, 0.5, 0.75, 1]), labels=[0, 0.25, 0.5, 0.75, 1])
                ax.set_xlabel('x (cm)')
                plt.yticks((n_nodes + 2) * np.array([0, 0.25, 0.5, 0.75, 1]), labels=[0, 0.25, 0.5, 0.75, 1])
                ax.set_ylabel('y (cm)')
                ax.set_title(f'Non-Conformal Mesh Exact Solution Heatmap\nk1 = {k1}'
                             f' | k2 = {k2}\nInterface: y = {round(inter, 3)}')
                plt.gca().invert_yaxis()
                plt.show()

                """
                Non-Conformal Mesh FDM Plotting
                """

                # 3-Dimensional Heat Maps
                ax = plt.axes(projection='3d')
                ax.plot_surface(x, y, z, rstride=1, cstride=1,
                                cmap='inferno', edgecolor='none')
                plt.title(r'Non-Conformal 2nd order FDM Output T(x, y)$\degree$ C'
                          r'' + '\nNodes = ' + str(n_nodes + 2) + '\nK = ' + str(k))
                ax.set_xlabel('X (cm)')
                ax.set_ylabel('Y (cm)')
                ax.set_zlabel(r'T(x, y)$\degree$C')
                plt.show()

                # 2-Dimensional Heat Maps
                ax = sns.heatmap(z, xticklabels=False, yticklabels=False,
                                 cbar_kws={'label': u'Temperature \N{DEGREE SIGN}C'})
                plt.xticks((n_nodes + 2) * np.array([0, 0.25, 0.5, 0.75, 1]), labels=[0, 0.25, 0.5, 0.75, 1])
                ax.set_xlabel('x (cm)')
                plt.yticks((n_nodes + 2) * np.array([0, 0.25, 0.5, 0.75, 1]), labels=[0, 0.25, 0.5, 0.75, 1])
                ax.set_ylabel('y (cm)')
                ax.set_title('Non-Conformal Heatmap of cross-section'
                             ' of bar\n' + str(n_nodes + 2) + ' Nodes' + '\nK = ' + str(k))
                plt.gca().invert_yaxis()
                plt.show()


if __name__ == "__main__":
    print('\nNote: This program may take a minute to generate all data...')
    main()
```