L E C T U R E 8

§11. TOTAL PROBABILITY AND BAYES' FORMULAE

Sometimes the probability of an event A cannot be determined directly. However, its occurrence is always accompanied by the occurrence of other events B_i, $i \geq 1$ such that the probability of A will depend on which of the events B_i has occurred. In such a case the probability of A will be an expected probability (that is, the average probability weighted by those of B_i). Such problems require the *Theorem of Total Probability.*

Let A be a subevent of $\bigcup\limits_{n\geq 1} B_n$ (i. e. $A \subset \bigcup\limits_{n\geq 1} B_n$), $\{B_n\}$ be mutually exclusive and $P(B_n) \neq 0$ for any n. Then

$$P(A) = \sum_{n=1}^{\infty} P(B_n)\, P(A\,/B_n\,). \tag{22}$$

We will call formula (22) by *formula of Complete or Total Probability.*

The proof of Total probability formula we can find in Appendix-8 of the present lecture.

Example 28. Urn I contains 6 white and 4 black balls. Urn II contains 5 white and 2 black balls. From urn I one ball is transferred to urn II. Then 2 balls are drawn without replacement from urn II. What is the probability that the two balls are white?

Solution: Denote by B_1 the event that from urn I a white ball is transferred to urn II, and by B_2 the event that from urn I is a black ball transferred to urn II. Denote by A the event that from urn II are selected two white balls.

By formula (22)

$$P(A) = P(B_1)\, P(A\,/B_1\,) + P(B_2)\, P(A\,/B_2\,).$$

Since

$$P(B_1) = \frac{3}{5}, \quad P(B_2) = \frac{2}{5}, \quad P(A\,/B_1\,) = \frac{15}{28}, \quad P(A\,/B_2\,) = \frac{5}{14}$$

we obtain

$$P(A) = \frac{13}{28}.$$

Thus, for any event A one may express the unconditional probability $P(A)$ of A in terms of the conditional probabilities $P(A\,/B_1\,), ..., P(A\,/B_n\,)...$ and the unconditional probabilities

$P(B_1), ..., P(B_n)....$

There is an interesting consequence of the formula of complete probability. Suppose that all conditions of the previous formula are satisfied. Then the following formula

$$P(B_i\,/A) = \frac{P(B_i) \cdot P(A\,/B_i\,)}{\sum\limits_{n=1}^{\infty} P(B_n)\,P(A\,/B_n\,)} \qquad i = 1, 2, ... \tag{23}$$

is known as *Bayes' formula*.

If we think of the events B_n as being possible "hypotheses" about some subject matter, then Bayes' formula may be interpreted as showing us how opinions about these hypotheses held before the experiment [that is, the $P(B_n)$] should be modified by the evidence of the experiment.

Let us prove (23). Applying (20) to the events A and B_i, we have

$$P(B_i) \cdot P(A\,/B_i\,) = P(A) \cdot P(B_i\,/A\,).$$

Therefore we obtain the desired probability

$$P(B_i\,/A\,) = \frac{P(B_i) \cdot P(A\,/B_i\,)}{P(A)}. \tag{24}$$

Using the total probability formula, (24) becomes (23).

Example 29. Suppose that there are n balls in an urn. Every ball can be either white or black. The number of white and black balls in the urn is unknown. Let our aim be to find out that number. Determine the following hypotheses:

Denote by B_i the event that the urn consists of exactly i white balls (from n balls), $i = 0, 1, 2, ..., n$.

As we do not have any additional information about the contents of the urn, therefore all hypotheses are equally likely, that is

$$P(B_i) = \frac{1}{n+1}, \quad \text{for any} \quad i = 0, 1, ..., n.$$

Suppose that we have selected a ball and it is white (the event A). It is obvious that we have to modify the probabilities of B_i. For example

$$P(B_0\,/A\,) = 0.$$

What are the remained probabilities equal to? Let us use the Bayes' formula.

$$P(B_i\,/A) = \frac{P(B_i) \cdot P(A\,/B_i\,)}{\sum\limits_{k=0}^{n} P(B_k)\, P(A\,/B_k\,)} \qquad i = 0, 1, ...n.$$

Since

$$P(B_i) = \frac{1}{n+1}, \qquad P(A\,/B_i\,) = \frac{i}{n}$$

we get

$$P(B_i\,/A) = \frac{\frac{i}{n}}{\sum\limits_{k=0}^{n} \frac{k}{n}} = \frac{2i}{n(n+1)}, \qquad i = 0, 1, ...n.$$

In particular, for $n = 3$ we obtain

$$P(B_0\,/A) = 0, \quad P(B_1\,/A) = \frac{1}{6}, \quad P(B_2\,/A) = \frac{1}{3}, \quad P(B_3\,/A) = \frac{1}{2}.$$

APPENDIX-8.

Proof of TOTAL PROBABILITY formula.

Since A is a subevent of the union of B_n, then

$$A = \bigcup_{n=1}^{\infty} (A \cap B_n),$$

where $\{A \cap B_n\}$ are mutually exclusive because $\{B_n\}$ are mutually exclusive. Therefore by Axiom 3

$$P(A) = \sum_{n=1}^{\infty} P(A \cap B_n)$$

and by formula (20) $P(A \cap B_n) = P(B_n) \cdot P(A\,/B_n\,)$. The proof is complete.