LECTURE 3

PROPERTIES OF PROBABILITY

Example 10. In Example 2, if we assume that a head is equally likely to appear as a tail, then we would have

$$P(\{\omega_1\}) = P(\{\omega_2\}) = \frac{1}{2}.$$

On the other hand, if we had a biased coin and felt that a head was twice as likely to appear as a tail, then we would have

$$P(\{\omega_1\}) = \frac{2}{3} \qquad P(\{\omega_2\}) = \frac{1}{3}.$$

In Example 3, if we supposed that all six outcomes were equally likely to appear, then we would have

$$P(\{\omega_1\}) = P(\{\omega_2\}) = ... = P(\{\omega_6\}) = \frac{1}{6}.$$

From Axiom 3 follows that the probability of getting an even number would equal

$$P(\{\omega_2, \omega_4, \omega_6\}) = P(\{\omega_2\}) + P(\{\omega_4\}) + P(\{\omega_6\}) = \frac{1}{2}.$$

These axioms will now be used to prove the simplest properties concerning probabilities.

Property 1. $P(\emptyset) = 0.$

That is, the impossible event has probability 0 of occurring.

The proof of Property 1 you can find in Appendix-1 of the present lecture.

It should also be noted that it follows that for any finite number of mutually exclusive events $A_1, ..., A_n$

$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P(A_k). \tag{3}$$

In particular, for any two mutually exclusive events A and B

$$P(A \cup B) = P(A) + P(B). \tag{4}$$

The proof of (3) you can find in Appendix-1 of the present lecture.

Therefore Axiom 3 is valid both for finite number of events (see (3) and (4)) and for countable number of events.

Property 2. For any event A

$$P(\overline{A}) = 1 - P(A).$$

Proof of Property 2: We first note that A and $\overline{A}$ are always mutually exclusive and since $A \cup \overline{A} = \Omega$ we have by Axiom 3 that

$$P(\Omega) = P(A \cup \overline{A}) = P(A) + P(\overline{A}) \quad \text{and by Axiom 2} \quad P(A) + P(\overline{A}) = 1.$$

The proof is complete.

As a special case we find that $P(\emptyset) = 1 - P(\Omega) = 0$, since the impossible event is the complement of Ω.

Property 3. For any two events A and B

$$P(B \setminus A) = P(B) - P(A \cap B). \tag{5}$$

Proof: The events $A \cap B$ and $B \cap \overline{A}$ are mutually exclusive, and their union is B. Therefore, by Axiom 3, $P(B) = P(A \cap B) + P(B \cap \overline{A})$, from which (5) follows immediately because $B \setminus A = B \cap \overline{A}$.

Property 4. If $A \subset B$ then

$$P(B \setminus A) = P(B) - P(A).$$

Proof: Property 4 is a corollary of Property 3.

Property 5. If $A \subset B$ then $P(A) \leq P(B)$, that is probability P is nondecreasing function.

Proof of Property 5: As $P(B \setminus A) \geq 0$, then Property 5 implies from Property 4.

Property 6. For any event A

$$P(A) \leq 1.$$

Property 6 immediately follows from both Property 5 where we substitute $B = \Omega$ and from Axiom 2 (i. e. any event A is a subevent of the certain event).

Therefore Axiom 1 and Property 6 state that the probability that the outcome of the experiment is contained in A is some number between 0 and 1, i. e.

$$0 \le P(A) \le 1.$$

Property 7. For any two events A and B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B). \tag{6}$$

The proof of Property 7 you can find in Appendix-2 of the present lecture.

Example 11. A card is selected at random from a deck of 52 playing cards. We will win if the card is either a club or a king. What is the probability that we will win?

Solution: Denote by A the event that the card is clubs and by B that it is a king. The desired probability is equal to $P(A \cup B)$. It follows from Property 7 that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

As

$$P(A) = \frac{1}{4}, \quad P(B) = \frac{4}{52} \quad \text{and} \quad P(A \cap B) = \frac{1}{52}$$

we obtain

$$P(A \cup B) = \frac{1}{4} + \frac{4}{52} - \frac{1}{52} = \frac{4}{13}.$$

Property 8 (Inclusion–Exclusion Principle). For any events A_1, A_2, ..., A_n we have

$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P(A_k) - \sum_{k<j} P(A_k \cap A_j) + \sum_{k<j<i} P(A_k \cap A_j \cap A_i) - ... + (-1)^{n-1} P\left(\bigcap_{k=1}^{n} A_k\right). \tag{7}$$

In words, formula (7) states that the probability of the union of n events equals the sum of the probabilities of these events taken one at a time minus the sum of the probabilities of these events taken two at a time plus the sum of the probabilities of these events taken three at a time, and so on.

Exercise 2. Prove Property 8.

Hint: We note that (6) is a special case of (7) when $n = 2$. For finishing the proof we have to apply the method of mathematical induction[1].

Property 9. For any two events A and B the inequality

$$P(A \cup B) \leq P(A) + P(B) \tag{8}$$

holds.

The proof follows from (6).

Property 10 (Boole's inequality). For any sequence of events A_1, ..., A_n, ...

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n). \tag{9}$$

The proof of Boole's inequality you can find in Appendix-2.

Properties 9 and 10 state that for any events the probability that at least one of these events occurs is less than or equal to the sum of their respective probabilities.

Definition 2. A pair (Ω, P) is called a ***probability space***, where Ω is a sample space on which a probability P (satisfying Axioms 1, 2 and 3) has been defined.

[1]The principle of mathematical induction states that a proposition $p(n)$ which depends on an integer n is true for $n = 1, 2, ...$ if one shows that (i) it is true for $n = 1$ and (ii) it satisfies the implication: $p(n)$ implies $p(n+1)$.

APPENDIX-1:

Proofs of (3) and Property 1

Proof of Property 1: **If we consider a sequence of events $A_1, A_2, ...$, where $A_1 = \Omega$, $A_k = \emptyset$ for $k > 1$ then, as the events are mutually exclusive and as $\Omega = \bigcup_{n=1}^{\infty} A_n$, we have from Axiom 3 that**

$$P(\Omega) = \sum_{n=1}^{\infty} P(A_n) = P(\Omega) + \sum_{n=2}^{\infty} P(A_n)$$

and by Axiom 2 $P(\Omega) = 1$ we obtain

$$\sum_{n=2}^{\infty} P(\emptyset) = 0$$

implying that

$$P(\emptyset) = 0.$$

Proof of (3): **It follows from Axiom 3 by defining A_i to be the impossible event for all values of i greater than n. Indeed,**

$$P\left(\bigcup_{i=1}^{n} A_i \bigcup \left[\bigcup_{i=n+1}^{\infty} \emptyset\right]\right) = \sum_{i=1}^{n} P(A_i) + \sum_{i=n+1}^{\infty} P(\emptyset).$$

As $P(\emptyset) = 0$ we obtain (3).

APPENDIX-2:
Proofs of Properties 7 and 10

Proof of Property 7: **It is not difficult to prove the following identity:**

$$A \cup B = A \cup (B \cap \overline{A}),$$

where A and $B \cap \overline{A}$ are mutually exclusive.
By Axiom 3 we get

$$P(A \cup B) = P(A) + P(B \cap \overline{A}). \tag{10}$$

Because $B \cap \overline{A} = B \setminus A$, by Property 3 we obtain

$$P(B \cap \overline{A}) = P(B) - P(A \cap B). \tag{11}$$

Substituting (11) into (10) we have (6). The Property 7 is proved.

Proof of Property 10: **To prove (9) we represent $\bigcup\limits_{n=1}^{\infty} A_n$ in the form**

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \left(A_n \cap B_n\right),$$

where

$$B_n = \overline{\bigcup_{k=1}^{n-1} A_k} \quad (B_1 = \Omega,\ B_2 = \overline{A_1},\ B_3 = \overline{A_1 \cup A_2} \quad ... \quad).$$

Therefore

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = P\left(\bigcup_{n=1}^{\infty} (A_n \cap B_n)\right).$$

It is not difficult to verify that the events $\{A_n \cap B_n\}$ are mutually exclusive. Hence we have

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n \cap B_n).$$

Since $P(A_n \cap B_n) \leq P(A_n)$ (by Property 5) the proof is complete.