LECTURE 4

§4. FINITE SAMPLE SPACES

Definition 3. A sample space Ω is called finite if the number of possible outcomes of the experiment is finite, i. e.

$$\Omega = \{\omega_1, \omega_2, ..., \omega_n\}.$$

In other words, a sample space Ω is defined as finite if it is of finite size which is to say that the random experiment under consideration possesses only a finite number of possible outcomes.

Definition 4. A single–member event is an event that contains exactly one outcome. If an event A has as its only member the outcome ω_i, this fact may be expressed in symbols by writing $A = \{\omega_i\}$. Thus $\{\omega_i\}$ is the event that occurs if and only if the random situation being observed has outcome ω_i.

Let Ω be a finite sample space. Let us take numbers p_i so that $p_i \geq 0$, $i = 1, ..., n$ and $\sum\limits_{i=1}^{n} p_i = 1$. We set

$$P(A) = \sum_{i:\omega_i \in A} p_i. \tag{12}$$

Exercise 3. Prove that the function defined by (12) is a probability.

Hint: For that Axioms 1, 2 and 3 must be checked.

Exercise 4. Show that if P is a probability on $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$ then there exist numbers $p_i \geq 0$, $i = 1, ..., n$ and $\sum\limits_{i=1}^{n} p_i = 1$ such that the relation (12) is satisfied for $p_i = P(\{\omega_i\})$.

Hint: There are 2^n possible events on a sample space of finite size n. Let A be an event. We can represent $A = \{\omega_{i_1}, ..., \omega_{i_k}\}$, $k < \infty$. A probability $P(\cdot)$ defined on Ω can be specified by giving its values $P(\{\omega_i\})$ on the single–member events $\{\omega_i\}$ which correspond to the elements of Ω. Its value $P(A)$ on an event A may then be computed by the formula

$$P(A) = \sum_{j=1}^{k} P(\{\omega_{i_j}\}).$$

Example 12. Let Ω be 2 element sample space, i. e. $\Omega = \{\omega_1, \omega_2\}$. Assume that $p_1 = p$ and $p_2 = 1 - p$, where $0 \leq p \leq 1$. This example describes all experiments which have two outcomes. If our experiment consists of tossing a coin and if we assume that a head is as likely to appear as a tail, then we would have $p = 1/2$; on the other hand, if the coin were biased and a head were twice as likely to appear as a tail, then we would have $p = 2/3$ (see also Examples 2 and 10).

§5. SAMPLE SPACES HAVING EQUALLY LIKELY OUTCOMES

In many probability situations in which finite sample spaces arise it may be assumed that all outcomes are equally likely; that is, all outcomes in Ω have equal probabilities of occurring. More precisely, we define the sample space $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$ as having equally likely outcomes if all the single–member events of Ω have equal probabilities, so that

$$P(\{\omega_1\}) = P(\{\omega_2\}) = ... = P(\{\omega_n\}) = p.$$

Now it follows from Axioms 2 and 3 that

$$1 = P(\Omega) = \sum_{i=1}^{n} P(\{\omega_i\}) = p \cdot n$$

what shows that

$$P(\{\omega_i\}) = p = \frac{1}{n} \qquad \text{for any} \quad i = 1, 2, ..., n.$$

It should be clear that each of the single–member events $\{\omega_i\}$ has probability $1/n$, since there are n such events, each of which has equal probability, and the sum of their probabilities must equal 1, the probability of the certain event.

It follows from Axiom 3 that for any event A

$$P(A) = \sum_{i:\omega_i \in A} P(\{\omega_i\}) = \frac{\text{Number of outcomes in } A}{n}.$$

Therefore the calculation of the probability of an event defined on a sample space with equally likely outcomes can be reduced to the calculation of the size of the event. By (12) probability of A is equal to $1/n$, multiplied by the number of outcomes in A. In other words, *the probability of A is equal to the ratio of the size of A to the size of Ω.*

If, for an event A of finite size, we define by $N(A)$ the size of A (the number of outcomes of A), then the foregoing conclusions can be summed up in a formula

$$P(A) = \frac{N(A)}{N(\Omega)} = \frac{\text{size of } A}{\text{size of } \Omega} = \frac{\text{the number of outcomes favorable to } A}{\text{total number of outcomes}}.$$

This is a precise formulation of the classical definition of the probability of an event, first explicitly formulated by Laplace in 1812. For several centuries, the theory of probability was based on the classical definition. This concept is used today to determine probabilistic data and as a working hypothesis. It is important to note, however, that the significance of the numbers $N(\Omega)$ and $N(A)$ is not always clear.

§6. GEOMETRIC PROBABILITIES

Let Ω be the same as in Example 8 i. e. we choose a point at random in the bounded subset D of n–dimensional Euclidean space $\mathbb{R}^n$. Therefore $\Omega = D$ and events are subsets of D. We suppose that volume of D does not equal 0, $V(D) \neq 0$ (V stands for volume). We define $P(A)$ by the following formula

$$P(A) = \frac{V(A)}{V(\Omega)}. \tag{13}$$

In particular, when $n = 2$

$$P(A) = \frac{S(A)}{S(\Omega)}, \tag{14}$$

in which S stands for area.

Exercise 5. Prove that P which is defined by formula (13) is a probability.

Remark 2. You know from school that $0 + 0 = 0$, i. e. the sum of finite number of zeros is equal to zero. Later we learned that the sum of countable number of zeros is zero, i. e. $\sum_{k=0}^{\infty} 0 = 0$ (see Lemma 2 in Appendix-3 of the present lecture). But for the sum of uncountable number of events it is not true. In order to be convinced let us consider the following example.

Example 13. Let $\Omega = [0, 1] \times [0, 1]$.

$$P(A) = \frac{S(A)}{S(\Omega)} = S(A) \qquad (S(\Omega) = 1).$$

Denote by

$$A_y = \{(x,y):\quad y \quad \text{is fixed}\}.$$

$$P(A_y) = 0 \quad \text{for any} \quad y \in [0,1] \quad \text{and} \quad \Omega = \bigcup_{y\in[0,1]} A_y, \quad P(\Omega) = 1.$$

APPENDIX-3:

§7. PROBABILITY AS A CONTINUOUS SET FUNCTION

A sequence $\{A_n,\ n \geq 1\}$ is called an increasing sequence if

$$A_1 \subset A_2 \subset ... \subset A_n \subset A_{n+1} \subset ...$$

and decreasing sequence if

$$A_1 \supset A_2 \supset ... \supset A_n \supset A_{n+1} \supset ....$$

Definition 5. **We will say that a sequence $\{A_n\}$ monotone increasing tends to the event A and will denote it by**

$$\lim_{n\to\infty} \uparrow A_n = A \qquad \text{or} \quad A_n \uparrow A$$

if the following two conditions are satisfied:

i) $\{A_n\}$ **is an increasing sequence of events (i. e.** $A_n \subset A_{n+1}$, **for any** $n = 1, 2, ...$**)**

ii) $A = \bigcup\limits_{n=1}^{\infty} A_n$.

If we have an increasing sequence then the limit is naturally determined as the union of all events A_n.

Definition 6. **We will say that a sequence $\{A_n\}$ monotone decreasing tends to the event A and will denote it by**

$$\lim_{n\to\infty} \downarrow A_n = A \qquad \text{or} \quad A_n \downarrow A$$

if the following two conditions are satisfied:

i) $\{A_n\}$ **is an decreasing sequence of events (i. e.** $A_n \supset A_{n+1}$, **for any natural** n**)**

ii) $A = \bigcap\limits_{n=1}^{\infty} A_n$.

If we have a decreasing sequence then the limit is naturally determined as the intersection of all events A_n.

Therefore, there exists a limit for any monotone sequence $\{A_n\}$.

In particular, $A_n \downarrow \emptyset$, if $\{A_n, \quad n \geq 1\}$ is a decreasing sequence and

$$\bigcap_{n=1}^{\infty} A_n = \emptyset.$$

Definition 7. **A function P defined on the set of events is said to be continuous at $\emptyset$ if for any sequence $\{A_n\}$ of events such as $A_n \downarrow \emptyset$ implies**

$$P(A_n) \downarrow 0.$$

Theorem 1. *Any probability P is continuous at $\emptyset$.*

Property 11. **If $A_n \downarrow A$ then $P(A_n) \downarrow P(A)$.**

Property 12. **If $A_n \uparrow A$ then $P(A_n) \uparrow P(A)$.**

Properties 11 and 12 state that any probability is continuous under monotone increasing or decreasing sequences of events.

Proof of Theorem 1: **Let $A_n \downarrow \emptyset$, i. e. $A_{n+1} \subset A_n$ for any n and**

$$\bigcap_{n=1}^{\infty} A_n = \emptyset. \tag{15}$$

We have to prove that $P(A_n) \downarrow \emptyset$.
It is not difficult to verify that for any decreasing sequence $\{A_n\}$ we have

$$A_n = \bigcup_{k=n}^{\infty} (A_k \setminus A_{k+1}) \bigcup \left(\bigcap_{n=1}^{\infty} A_n \right).$$

We must prove that if an outcome belongs to the left–hand side of the identity then the outcome belongs to the right–hand side and vice versa (the proof of this assertion is left to the reader).

It follows from (15) that

$$A_n = \bigcup_{k=n}^{\infty} (A_k \setminus A_{k+1})$$

and $\{A_k \setminus A_{k+1}\}$ are mutually exclusive. Thus

$$P(A_n) = \sum_{k=n}^{\infty} P(A_k \setminus A_{k+1}). \tag{16}$$

Let us write this equation for $n = 1$, we have

$$P(A_1) = \sum_{k=1}^{\infty} P(A_k \setminus A_{k+1}).$$

Since by Axiom 1 and Property 6 $0 \le P(A) \le 1$ we come to conclusion that the series

$$\sum_{k=1}^{\infty} P(A_k \setminus A_{k+1})$$

is convergent. As we know that the remainder $\sum_{k=N}^{\infty} a_k$ of the convergent series $\sum_{k=1}^{\infty} a_k$ tends to zero as $N \to \infty$, therefore by (16) $P(A_n) \downarrow 0$.

Proof of Property 11: **$A_n \downarrow A$ if and only if $(A_n \setminus A) \downarrow \emptyset$. Therefore by Theorem 1 $P(A_n \setminus A) \downarrow \emptyset$. By Property 4 we get $P(A_n) - P(A) \downarrow \emptyset$. The proof is complete.**

Proof of Property 12: **$A_n \uparrow A$ if and only if $(A \setminus A_n) \downarrow \emptyset$. Therefore by Theorem 1 $P(A \setminus A_n) \downarrow \emptyset$. By Property 4 we get $P(A) - P(A_n) \downarrow \emptyset$. Therefore $P(A_n) \uparrow P(A)$. The proof is complete.**

Remark 3. **We have supposed that $P(A)$ is defined for all events A of the sample space. Actually, when the sample space is an uncountably infinite set, $P(A)$ is defined for only the so–called measurable events. However, this restriction need not concern us as all events of any practical interest are measurable.**

Exercise 6. **Give counterexamples to the following assertions:**

i) if $P(A) = 0$ then $A = \emptyset$,
ii) if $P(A) = 1$ then $A = \Omega$.

Lemma 1. **If each event of finite or infinite sequence $A_1, A_2, ..., A_n, ...$ has probability equal to one (i. e. $P(A_k) = 1$ for any $k = 1, 2, ...$) then**

$$P\left(\bigcap_{k=1}^{\infty} A_k\right) = 1.$$

Lemma 2. **If each event of finite or infinite sequence $A_1, A_2, ..., A_n, ...$ has probability equal to zero (i. e. $P(A_k) = 0$ for any $k = 1, 2, ...$) then**

$$P\left(\bigcup_{k=1}^{\infty} A_k\right) = 0.$$

Exercise 7. **Prove Lemmas 1 and 2.**