LECTURE 2

In these examples Ω is finite. However, there exist experiments for which the number of elements of Ω is infinite.

Example 7. Assume we toss a coin until a tail opens. Then

$$\Omega = \{\omega_1, \omega_2, ..., \omega_n, ...\},$$

where ω_i corresponds to the outcome for which a coin rolls exactly i times, i. e. we will have to toss the coin i times until a tail opens.

Example 8. If we choose a point at random in the bounded subset D of n–dimensional Euclidean space $\mathbb{R}^n$ then

$$\Omega = D.$$

In Example 7 Ω is a countable set and in Example 8 Ω is a continuum set.
In general, a sample space is said to be *discrete* if it has finitely many or a countable infinity of elements. If the points (outcomes) of a sample space constitute a continuum, for example, all the points on a line, all the points on a line segment, or all the points in a plane, the sample space is said to be *continuous*.

Definition 1. A subset A of the sample space is called an *event*.
If the outcome of the experiment is contained in A then we say that A has occurred. The notation $\omega \in A (\omega \notin A)$ will mean that ω is (is not) an element of A.

Some examples of events are the following.
In Example 1 if $A = \{\omega_2\}$ then A is the event that the child is a boy.
In Example 2 if $A = \{\omega_1\}$, then A is the event that a head appears on the flip of the coin.
In Example 3 if $A = \{\omega_2, \omega_4, \omega_6\}$ then A is the event when appears even face.
In Example 4 if $A = \{\omega_1, \omega_2, \omega_3\}$ then A is the event that at least one head appears.
In Example 7 if $A = \{\omega_1, \omega_2, ..., \omega_7\}$ then A is the event that the number of the throwing of a coin does not exceed 7.

Example 9. Consider an experiment that consists of counting the number of traffic accidents at a given intersection during a specified time interval. The sample space is the set

$$\Omega = \{0, 1, 2, 3, ...\}.$$

The statement "the number of accidents is less than or equal to seven" describes the event $\{0, 1, ..., 7\}$. The event $A = \{5, 6, 7, ...\}$ occurs if and only if the number of accidents is greater than or equal to 5.

In particular, the sample space Ω is a subset of itself and is thus an event. We call the sample space Ω the *certain event*, since by the method of construction of Ω it will always occur.

For any two events A and B of a sample space Ω we define the new event

$$A \cup B$$

called the *union* of the events A and B, consisting of all outcomes which belong to at least one, either A or B.

Similarly, for any two events A and B,

$$A \cap B$$

is called the *intersection* of A and B and consists of all outcomes which belong to both A and B.

For instance, in Example 1 if $A = \{\omega_1\}, B = \{\omega_2\}$, then $A \cup B = \Omega$, that is $A \cup B$ would be the certain event Ω, and $A \cap B$ does not contain any outcomes and hence cannot occur. To give such an event a name we introduce the notion of the *impossible event* and denote it by $\emptyset$. Thus $\emptyset$ means the event consists of no outcome.

If $A \cap B = \emptyset$ implying that A and B cannot both occur, then A and B are said to be *mutually exclusive*.

For any event A we define the event $\overline{A}$ referred to as the *complement* of A and consists of all outcomes in Ω that are not in A.

In Example 3 if $A = \{\omega_2, \omega_4, \omega_6\}$ then $\overline{A} = \{\omega_1, \omega_3, \omega_5\}$ is the event when appears odd face.

In Example 4 the events $A = \{\omega_1\}$ and $B = \{\omega_4\}$ are mutually exclusive.

For any two events A and B, if all of the outcomes in A are also in B, then we say that A is a subevent of B and write $A \subset B$. If $A \subset B$ and $B \subset A$ then we say that A and B are equal and we write $A = B$.

We can define unions and intersections of more than two events. The *union* of the events $A_1, A_2, ..., A_n$ denoted by

$$\bigcup_{k=1}^{n} A_k$$

is defined to be the event consisting of all outcomes that are in A_i for at least one $i = 1, ..., n$. Similarly, the *intersection* of the events A_i denoted by

$$\bigcap_{k=1}^{n} A_k$$

is defined to be the event consisting of those outcomes that are in all of the events A_i, $i = 1, 2, ..., n$. In other words, the union of the A_i occurs when at least one of the events A_i occurs; while the intersection occurs when all of the events A_i occurs.

Similarly we can define

$$\bigcup_{n=1}^{\infty} A_n \qquad \text{and} \qquad \bigcap_{n=1}^{\infty} A_n$$

for any sequence of events $\{A_n\}_{n=1}^{\infty}$.

The following useful relations between the three basic operations (unions, intersections and complements of events) are known as De Morgan's law.

$$\overline{\bigcup_i A_i} = \bigcap_i \overline{A_i}, \qquad \overline{\bigcap_i A_i} = \bigcup_i \overline{A_i} \tag{1}$$

for any family of events $\{A_i\}$.

In particular, we have

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \quad \text{and} \quad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

Exercise 1. Prove De Morgan's law.

Hint: Relations stating the equality of two sets is proved by the following way: we take a point belonging to the left–hand side of the equality and show that the point belongs to the right–hand side of the equality and vice versa. Proof of such type of assertions I will always omit.

§3. AXIOMS OF PROBABILITY

Let us consider an experiment and let Ω be the sample space of the experiment. A *probability* on Ω is a real function P which is defined on events of Ω and satisfies the following three axioms.

Axiom 1. $P(A) \geq 0$, for any event A.

Axiom 2. $P(\Omega) = 1$.

Axiom 3. For any sequence of mutually exclusive events A_1, A_2, ... (that is, events for which $A_i \cap A_j = \emptyset$ if $i \neq j$)

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n). \tag{2}$$

We call $P(A)$ the probability of event A.

Thus, Axiom 1 states that the probability that outcome of the experiment is contained in A is some nonnegative number. Axiom 2 states that the probability of the certain event is always equal to one. Axiom 3 states that for any sequence of mutually exclusive events $\{A_n\}_{n=1}^{\infty}$ the probability that at least one of these events occurs is equal to the sum of their respective probabilities.