L E C T U R E 9

§12. INDEPENDENT TRIALS.

It is sometimes the case that the probability experiment under consideration consists of performing a sequence of subexperiments. For instance, if the experiment consists of tossing a coin n times, we may think of each toss as being a subexperiment. In many cases it is reasonable to assume that the outcomes of any group of the subexperiments have no effect on the probabilities of the outcomes of the other subexperiments. If such is the case, we say that the subexperiments are independent.

If each subexperiment is identical — that is, if each subexperiment has the same sample space and the same probability function on its events — then the subexperiments are called *Trials*.

Many problems in Probability Theory involve independent repeated trials of an experiment whose outcomes have been classified in two categories, called "successes" (the event A) and "failures" (the event $\overline{A}$). The probability of the event A is usually denoted by p $(P(A) = p)$ and therefore $P(\overline{A}) = 1 - p$ where $0 \leq p \leq 1$.
Such an experiment is called a Bernoulli trial.
Consider now n independent repeated Bernoulli trials, in which the word "repeated" is meant to indicate that the probabilities of success $(P(A) = p)$ and failure $(P(\overline{A}) = 1 - p)$ remain the same throughout the trials. The sample space Ω of n independent repeated Bernoulli trials contains 2^n outcomes.
Therefore, in the problems we study in this section, we shall always make the following assumptions:

1. *There are only two possible outcomes for each trial* (arbitrarily called "success" and "failure", without inferring that a success is necessary desirable).

2. *The probability of a success is the same for each trial.*

3. *There are n trials, where n is a constant.*

4. *The n trials are independent.*

If the assumptions cannot be met, the theory we shall develop here does not apply. Frequently, the only fact about outcome of a succession of n Bernoulli trials in which we are interested is the *number of successes*.

We now compute the probability that the number of successes will be k, for any integer k from $0, 1, 2, ..., n$. The event "k successes in n trials" can happen in as many ways as k letters A may be distributed among n places; this is the same as the number of subsets of size k that may be formed from a set containing n members. Consequently, there are $\binom{n}{k}$ outcomes containing exactly k successes and $n-k$ failures (in which $k = 0, 1, ..., n$) is given by

$$P_n(k) = \binom{n}{k} p^k (1-p)^{n-k}. \tag{25}$$

The law expressed by (25) is called the *binomial law* because of the role the quantities in (25) play in the binomial theorem, which states that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

for any real a and b. Taking $a = p$ and $b = 1 - p$, it follows immediately that

$$1 = (p + (1-p))^n = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} P_n(k).$$

Letting $a = b = 1$, one finds that the sum of all binomial coefficients is 2^n:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Letting $b = 1$ and $a = -1$, we obtain

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

However, (25) represents the solution to a probability problem that does not involve equally likely outcomes.

Let us compute the probability that at least one success occurs in n trials;

In order to determine the probability of at least 1 success in n trials, it is easiest to compute first the probability of the complementary event, that of no successes in n trials. We have

$$P(\text{at least 1 success}) = 1 - P_n(0) = 1 - (1-p)^n.$$

Example 30. Four fair coins are flipped. If the outcomes are assumed independent, what is the probability that two heads and two tails are obtained?

Solution: We have four repeated trials $n = 4$ with parameters $p = 0.5$ and $k = 2$. Hence, by (25) we get

$$P_4(2) = \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = \frac{3}{8}.$$

Example 31. A fair die is rolled five times. We shall find the probability that "six" will show twice.

Solution: In the single roll of a die $A = \{\text{six}\}$ is an event with probability $1/6$. Setting $n = 5,\ k = 2,\ p = P(A) = 1/6$ in (25), we obtain

$$P_5(2) = \binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 = \frac{625}{3,888} = 0.160751.$$

Example 32. A pair of fair dice is rolled four times. We shall find the probability that "seven" will not show at all.

Solution: The event $A = \{$ the sum of the dice equals $7\}$ consists of the six favorable outcomes

$$\{(3,4),(4,3),(5,2),(2,5),(6,1),(1,6)\}.$$

Therefore, $P(A) = p = 1/6$. With $n = 4$ and $k = 0$, (25) yields

$$P_4(0) = \left(\frac{5}{6}\right)^4.$$

Example 33. We place at random n points in the interval $(0,T)$. What is the probability that k of these points are in the interval (t_1,t_2), $t_1 > 0$, $t_2 < T$?

Solution: This problem can be considered as a problem in repeated trials. The experiment is the placing of a single point in the interval $(0,T)$. In this experiment, $A = \{$ the point is in the interval $(t_1,t_2)\}$ is an event with probability

$$p = P(A) = \frac{t_2 - t_1}{T}.$$

The event $\{A \text{ occurs k times }\}$ means that k of the n points are in the interval (t_1, t_2). Hence, by (25)

$$P_n(k) = \binom{n}{k} \left(\frac{t_2 - t_1}{T}\right)^k \left(1 - \frac{t_2 - t_1}{T}\right)^{n-k}.$$

Exercise 9 (The behavior of the binomial probabilities). Show, that as k goes from 0 to n, probabilities $P_n(k)$ increase monotonically, then decrease monotonically, reaching their largest value when k satisfying the inequalities

$$n \cdot p - (1 - p) \leq k \leq n \cdot p + p.$$