L E C T U R E 1

§1. INTRODUCTION

Today we begin to study probability theory which is a branch of mathematics. Every person has some intuitive interpretation of the meaning of the concept of probability. We may ask such questions as, "what is the probability that it will rain tomorrow", "what is the probability that the waiting time for a bus is less than 10 minutes", and so on. When speaking about probability, the usual picture formed in one's mind is that of dice throwing, card playing, or any other games of chance.

The birth of Probability Theory is considered to be in the XVII century and is connected with the combinatorial problems of gambling games. It is difficult to consider gambling games a serious business. But the problems arising from them couldn't be solved within the framework of the mathematical models that existed at that time. For understanding probability models which we will consider in this course, we will often return to different examples from gambling games.

In the beginning of the last century from natural sciences more serious problems have been arisen. These problems led to the development of a vast part of mathematics which is probability theory. This field of knowledge up to the present is in a state of intensive development. Today, probability theory finds applications in a large and growing list of areas. It forms the basis of the Mendelian theory of heredity, and hence has played a major part in the development of the science of genetics. Modern theories in physics concerning atomic particles make use of probability models. The spread of an infectious disease through a population is studied in epidemiology, a branch of probability theory. Queuing theory uses probability models to investigate customer waiting times under the provision of various levels and types of service.

Any realistic model of real-world phenomenon must take into account the possibility of randomness. Uncertainties are unavoidable in the design and planning of engineering systems. Therefore, the tools of engineering analysis should include methods and concepts for evaluating the significance of uncertainty on system performance and design. In this regard, the principles of probability offer the mathematical basis for modeling uncertainty and the analysis of its effects on engineering design.

The idea of attaching a number to a set or to an object is familiar to everybody. We may talk about the length of a segment, the area of a triangle, the volume of a ball, or the mass of a physical body, its temperature and so on. All these facts can be expressed in numbers. Probability is also expressed in terms of numbers attached to events. The way of doing this is very much analogous to that of length, area and volume.

Probability theory is one of the most beautiful branches of mathematics. The problems that it can address and the answers that it provides are often strikingly structured and beautiful. At the same time, probability theory is one the most applicable branches of mathematics. It is used as the primary tool for analyzing statistical methodologies.

As in the cases of all parts of mathematics, probability theory is constructed by means of an axiomatic method. In 1933, A. N. Kolmogorov provided an axiomatic basis for probability theory, and it is now the universally accepted model. Nowadays the following approach is accepted in probability theory.

§2. SAMPLE SPACE AND EVENTS

Below we introduce the concept of the probability of an event and then show how these probabilities can be computed in certain situations. As a preliminary, however, we need the concept of the *sample space* and *events* of an experiment. To each experiment corresponds some nonempty set Ω. This set consists of all possible outcomes of the experiment. This Ω is called the *set of all possible outcomes* or *sample space* and a generic element of Ω is denoted by ω. We generally denote a sample space by the capital Greek letter Ω. Let's consider some examples.

Example 1. If the outcome of an experiment consists of the determination of the sex of a newborn child, then

$$\Omega = \{\omega_1, \omega_2\},$$

where the outcome ω_1 means that the child is a girl and ω_2 that it is a boy.

Example 2. If an experiment consists of tossing a coin on a smooth surface then Ω is the same as in Example 1, that is

$$\Omega = \{\omega_1, \omega_2\},$$

where ω_1 corresponds to a head and ω_2 corresponds to a tail.

Example 3. If we toss a symmetrical (or a fair) die on a smooth surface then

$$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\},$$

where ω_i corresponds to the outcome when on the opened lateral face appear i points, $i = 1, 2, ..., 6$.

Example 4. Now an experiment consists of tossing a coin twice. In this case we take as the set of possible outcomes (i.e. as the sample space) the 4–element set

$$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\},$$

where the outcomes are the following
$\omega_1 = hh$ is the outcome: heads on the first toss and heads on the second;
$\omega_2 = ht$ is the outcome: heads on the first toss and tails on the second;
$\omega_3 = th$ is the outcome: tails on the first toss and heads on the second;
$\omega_4 = tt$ is the outcome: tails on the first toss and tails on the second.

Example 5. Now two symmetrical dice are thrown on a smooth surface then

$$\Omega = \{\omega_1 = (1, 1), \quad \omega_2 = (1, 2), \quad \omega_3 = (2, 1), ..., \quad \omega_{36} = (6, 6)\},$$

where (i, j) is the outcome that the first die equals i and the second die equals j, i. e. Ω is a 36 element set.

Example 6. A card is selected at random from a deck of 52 playing cards, then Ω is 52 element set.

Remark 1. It should be pointed out that the sample space of an experiment can be defined in more than one way. Observers with different conceptions of what could possibly be observed will arrive at different sample spaces. For instance, in Example 2, the sample space Ω might consist of three elements, if we desired to include the possibility that the coin might stand on its edge or rim. Then

$$\Omega = \{\omega_1, \omega_2, \omega_3\},$$

where ω_1, ω_2 are as in Example 2 and the outcome ω_3 represents the possibility of the coin standing on its rim.

There is yet a fourth possibility; the coin might be lost by rolling away when it lands. The sample space Ω would then be

$$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$$

in which the outcome ω_4 denotes the possibility of loss.

In these examples Ω is finite. However, there exist experiments for which the number of elements of Ω is infinite.