LECTURE 6

Example 19. Two balls are drawn from a bowl containing six balls, of which four are white and two are red. Find the probability that

i) both balls will be white;

ii) both balls will be the same color;

iii) at least one of the balls will be white.

Solution: **I.** Let us first consider that the balls are drawn without replacement.

i) Denote by A_1 the event that both balls will be white. By classical definition of probability

$$P(A_1) = \frac{\text{size of } A_1}{\text{size of } \Omega} = \frac{\binom{4}{2}}{\binom{6}{2}} = \frac{2}{5}.$$

ii) Denote by A_2 the event that both balls will be red, and A is the event that both balls will be the same color. We have

$$P(A) = P(A_1 \cup A_2) = P(A_1) + P(A_2) = \frac{2}{5} + \frac{1}{15} = \frac{7}{15}.$$

iii) Let B be the event that at least one of the balls will be white.

$$P(B) = 1 - P(\overline{B}) = 1 - P(A_2) = 1 - \frac{1}{15} = \frac{14}{15}.$$

II. Let us consider that the balls are drawn with replacement.

$$\text{i) } P(A_1) = \frac{4^2}{6^2} = \frac{4}{9}; \quad \text{ii) } P(A) = P(A_1 \cup A_2) = \frac{4}{9} + \frac{2^2}{6^2} = \frac{5}{9};$$

$$iii)\, P(B) = 1 - P(A_2) = 1 - \frac{1}{9} = \frac{8}{9}.$$

Example 20. An ordinary deck of 52 playing cards is randomly divided into 4 piles of 13 card each. Compute the probability that each pile has exactly 1 ace.

Solution: Let us use the classical definition of probability. Denote by A the event that each pile has exactly 1 ace.

$$P(A) = \frac{\text{size of}\, A}{\text{size of}\, \Omega}.$$

Ω consists of

$$\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13} = \frac{52!}{(13!)^4}$$

outcomes.

Let us count the number of outcomes in A. The first pile which contains only one ace may be chosen by $4 \cdot \binom{48}{12}$ different ways, the second pile by $3 \cdot \binom{36}{12}$ different ways, the third pile by $2 \cdot \binom{24}{12}$ and the fourth only 1. Therefore

$$P(A) = \frac{4 \cdot \binom{48}{12} \cdot 3 \cdot \binom{36}{12} \cdot 2 \cdot \binom{24}{12}}{\frac{52!}{(13!)^4}} = \frac{39 \cdot 26 \cdot 13}{51 \cdot 50 \cdot 49} = 0.105.$$

Example 21. Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen, matters.

Solution: The number of ways to choose six cookies is the number of 6–combinations of a set with four elements. From Lemma 4 this equals $\binom{4+6-1}{6} = \binom{9}{6}$. Since

$$\binom{9}{6} = \binom{9}{3} = 84,$$

there are 84 different ways to choose the six cookies.

Lemma 4 can also be used to find the number of solutions of certain linear equations where the variables are integers subject to constraints. The outcome of the experiment of distributing the k balls into n urns can be described by a vector $(x_1, x_2, ..., x_n)$, where x_i denotes the number of balls that are distributed into the ith urn. Hence the problem reduces to finding the number of distinct nonnegative integer–valued vectors $(x_1, x_2, ..., x_n)$ such that

$$x_1 + x_2 + ... + x_n = k.$$

Hence from Lemma 4, we obtain the following statement:

There are

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

distinct nonnegative integer–valued vectors $(x_1, x_2, ..., x_n)$ *satisfying* $x_1+x_2+...+x_n = k$, $x_i \geq 0$, $i = 1, ..., n$.

This illustrated by the following example.

Example 22. How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2 and x_3 are nonnegative integers?

Solution: To count the number of solutions, we note that a solution corresponds to a way of selecting 11 items from a set with three elements, so that x_1 items of type one, x_2 items of type two, and x_3 items of type three are chosen. Hence, the number of solutions is equal to the number of 11–combinations with repetition allowed from a set with three elements. From Lemma 4 it follows that there are

$$\binom{3+11-1}{11} = \binom{13}{11} = \binom{13}{2} = 78$$

solutions.

To obtain the number of positive solutions, note that the number of positive solutions of

$$x_1 + x_2 + ... + x_n = k$$

is the same as the number of nonnegative solutions of

$$y_1 + y_2 + ... + y_n = k - n$$

(seen by letting $y_i = x_i - 1$). Hence from Lemma 4, we obtain the following statement:

There are

$$\binom{k-1}{n-1}$$

distinct positive integer–valued vectors $(x_1, x_2, ..., x_n)$ *satisfying* $x_1 + x_2 + ... + x_n = k$*,* $x_i > 0$*,* $i = 1, ..., n$.

Therefore the number of positive solutions in Example 22 is

$$\binom{10}{2} = 45.$$

In general, the number of all solutions of the equation

$$x_1 + x_2 + ... + x_n = k$$

that satisfies the additional conditions

$$x_1 \geq l_1 \quad x_2 \geq l_2 \quad ... \quad x_n \geq l_n,$$

where $l_i \geq 0$, $i = 1, 2, ..., n$ is the same as the number of nonnegative solutions of

$$y_1 + y_2 + ... + y_n = k - \sum_{i=1}^{n} l_i$$

(seen by letting $y_i = x_i - l_i$). Hence from Lemma 4, we obtain the following statement:

There are

$$\binom{n + k - \sum\limits_{i=1}^{n} l_i - 1}{n - 1}$$

distinct integer–valued vectors $(x_1, x_2, ..., x_n)$ *satisfying* $x_1 + x_2 + ... + x_n = k$*,* $x_i \geq l_i$*,* $i = 1, ..., n$*, if* $k \geq \sum\limits_{i=1}^{n} l_i$.

Remark 5. The sample is said to have been chosen *at random* and is called a *random sample* if all possible ordered sequences are equally probable. We showed that, under random sampling without replacement, the $\binom{n}{k}$ possible unordered samples are equally likely. However, for random sampling with replacement unordered samples are not equally probable. In general, the probability of an unordered sample depends upon how many repeated elements it contains and how often they appear.

For example, suppose that three digits are chosen at random with replacement from $0, 1, 2, ..., 9$. There are $10^3 = 1,000$ equally probable ordered outcomes 000, 001, ...999. The three digits $0, 1, 2$ can appear in $3!$ different arrangements, and so the unordered outcome

$\{0, 1, 2\}$ has probability 0.006. However the three digits $0, 0, 1$ can be arranged in only 3 ways, and the three digits $0, 0, 0$ can be arranged in only 1 way. Hence the unordered outcomes $\{0, 0, 1\}$ and $\{0, 0, 0\}$ have probabilities 0.003 and 0.001 respectively.

In general, suppose that the ith member of the population occurs r_i times in the sample $(i = 1, 2, ..., k)$ where $\sum_{i=1}^{k} r_i = n$. The number of arrangements or permutations of the n elements in the sample is

$$\frac{n!}{r_1!\, r_2! \, ...r_k!} \tag{17}$$