UNIT - 5 Multivariate calculus

Topics

1. Limit, continuity and differentiability
2. Chain rule
3. Euler's theorem
4. jacobians
5. Extrema
6. Lagrange's method.

basic diff

End. 10

15

10

10

Differentiation

(1) $x^n \rightarrow n x^{n-1}$

(2) $\sin x \rightarrow \cos x$

$\cos x \rightarrow -\sin x$

(3) $\sin^{-1} x \rightarrow 1/\sqrt{1-x^2}$

$\cos^{-1} x \rightarrow -1/\sqrt{1-x^2}$

$\tan^{-1} x \rightarrow \frac{1}{1+x^2}$

(4) $a^x \rightarrow a^x \log a$

(5) $u \cdot v \rightarrow u'v + uv'$

(6) $\frac{u}{v} \rightarrow \frac{v(u') - u(v')}{v^2}$

(7) $\log x \rightarrow \frac{1}{x}$ | $e^x \rightarrow e^x$

Eg

(1) $x^7 \rightarrow 7x^6$; $x^{101} \rightarrow 101 x^{100}$

(4) $5^x \rightarrow 5^x \log 5$

(5) $x \cdot \log x \rightarrow x \cdot \frac{1}{x} + \log x \cdot 1$

$= 1 + \log x$

(6) $\frac{x^2}{e^x} \rightarrow \frac{e^x(2x) - x^2(e^x)}{(e^x)^2}$

Multivariate calculus

* function in one variable with one value

$f(x) = y$

(independent var) → dependent variable.

* function in two variable with one value

$f(x, y) = z$

independent var → dependent

* $f(x, y, z) = w$; fxn in 3 variable with one value

Similarly, we can increase the number of independent variable with one value called as <u>multivariable function</u> with one value.

<u>Ex</u>; $f(x, y, z, k, p, t, r, s \cdots) = \beta$

Partial derivative

Derivative of fxn of two or more variable w.r.t individual variable keeping other variables as constant.

- T/f

$\boxed{f_x = \frac{\partial f}{\partial x}}$ $\boxed{f_y = \frac{\partial f}{\partial y}}$

(1.) $f(x,y) = x^2 + y^2 + 1$

Soln. $\frac{\partial f}{\partial x} = 2x$; $\frac{\partial f}{\partial y} = 2y$

(2.) $f(x,y) = \ln(x/y)$ at $(2,3)$

Soln. $f(x,y) = \ln(x) - \ln(y)$

$\frac{\partial f}{\partial x} = \frac{1}{x}\Big|_{(2,3)} = 1/2$

$\frac{\partial f}{\partial y} = \frac{-1}{y}\Big|_{(2,3)} = -1/3$

(3) Evaluate the 1st order partial derivative of $f(x,y) = x^4 - x^2y^2 + y^4$ at the point $(1,-1)$

Soln. $\frac{\partial f}{\partial x} = 4x^3 - 2y^2x \Rightarrow f_x(1,-1) = 4-2 = 2$

$\frac{\partial f}{\partial y} = -2x^2y + 4y^3 \Rightarrow f_y(1,-1) = -2(1)(-1) + 4(-1)^3$

$= 2 - 4$

$= -2$

Total derivative

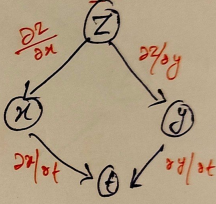
Chain rule (applying)

The total change in the dependent variable due to the change in all the independent variables.

Eg. $z = x^2 + y^2 \qquad x = \frac{t^2-1}{t} \qquad y = \frac{t}{t^2+1} \qquad$ at $t=1$

$$\boxed{\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}}$$

* Chain rule.



$$\frac{\partial z}{\partial x} = 2x \quad ; \quad \frac{\partial z}{\partial y} = 2y \quad ; \quad \frac{\partial x}{\partial t} = \frac{t(2t) - (t^2-1)(1)}{t^2} = \frac{2t^2 - t^2 + 1}{t^2} = \frac{t^2+1}{t^2}$$

$$\frac{\partial y}{\partial t} = \frac{(t^2+1)(1) - (t)(2t)}{(t^2+1)^2} = \frac{t^2+1-2t^2}{(t^2+1)^2} = \frac{1-t^2}{(1+t^2)^2}$$

$$\therefore \frac{\partial z}{\partial t} = 2x \cdot \frac{\partial x}{\partial t} + 2y \frac{\partial y}{\partial t}$$

At $t=1$

$x = \frac{1-1}{1} = 0$

$y = \frac{1}{2}$

$$\left.\frac{\partial x}{\partial t}\right|_{t=1} = \frac{1+1}{1} = 2$$

$$\left.\frac{\partial y}{\partial t}\right|_{t=1} = \frac{1-1}{(1+1)^2} = 0$$

$$\left.\frac{\partial z}{\partial t}\right|_{t=1} = (0 \times 2) + (2 \cdot \tfrac{1}{2} \times 0)$$

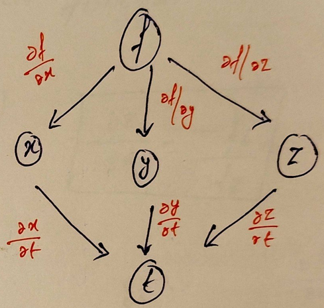
$$= 0$$

$$\therefore \boxed{\left.\frac{\partial z}{\partial t}\right|_{t=1} = 0}$$

Que. find $\frac{df}{dt}$ at $t=0$ $\quad f(x,y,z) = x^3 + xz^2 + y^3 + xyz$ $\quad x = e^t$, $y = \cos t$, $z = t^3$

Sol.

$$\boxed{\frac{df}{dt} = \left(\frac{df}{dx} \times \frac{dx}{dt}\right) + \left(\frac{df}{dy} \times \frac{dy}{dt}\right) + \left(\frac{df}{dz} \times \frac{dz}{dt}\right)}$$



Now

$$\frac{df}{dx} = 3x^2 + z^2 + yz \qquad \frac{dx}{dt} = e^t$$

$$\frac{df}{dy} = 3y^2 + yz \qquad \frac{dy}{dt} = -\sin t$$

$$\frac{df}{dz} = x + xy \qquad \frac{dz}{dt} = 3t^2$$

Now. At $t=0$.

$$\frac{dx}{dt} = e^0 = 1 \qquad x = e^0 = 1$$

$$\frac{dy}{dt} = -\sin 0 = 0 \qquad y = \cos(0) = 1$$

$$\frac{dz}{dt} = 3(0)^2 = 0 \qquad z = 0^3 = 0$$

Now

$$\frac{df}{dt} = (3 \times 1) + (3 \times 0) + (1 \times 0)$$

$$= 3 + 0 + 0$$

$$= 3$$

thus

$$\boxed{\left.\frac{df}{dt}\right|_{t=0} = 3}$$

* Derivative of implicit functions *

A function $f(x,y) = c$ is called an implicit function

if $f(x,y,z) = c$ then

$$\boxed{\frac{\partial y}{\partial x} = \frac{-f_x}{f_y}} \qquad \boxed{\frac{\partial z}{\partial y} = \frac{-f_y}{f_z}} \qquad \boxed{\frac{\partial z}{\partial x} = \frac{-f_x}{f_z}}$$

Or

$$\frac{\partial y}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}} \qquad \frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}$$

Ques Let $x^3 + 3x^2y^2 + y^3 = 1$ then find $\frac{dy}{dx}$.

Soln.

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{3x^2 + 3y^2(2x)}{3x^2(2y) + 3y^2} = -\frac{3x^2 + 6xy^2}{6x^2y + 3y^2}$$

$$\Rightarrow -\left(\frac{x^2 + 2xy^2}{2x^2y + y^2}\right) \text{ Ans}$$

Extrema

Extrema

$\xleftrightarrow{CA}$ Extd.

- **Higher order derivatives.**

1. $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = f_{xx}$

2. $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial(y)}\left(\dfrac{\partial f}{\partial y}\right) = f_{yy}$

3. $\dfrac{\partial^2 f}{\partial y\,\partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = f_{yx}$

4. $\dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = f_{xy}$

How to find critical points.

find f_x & f_y

then put $f_x = 0$ & $f_y = 0$

$r = f_{xx}$

$t = f_{yy}$

$s \Rightarrow f_{xy}$

Que. $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

$f_x \to y - 2x - 2$ $\Rightarrow f_x = 0$ & $f_y = 0$

$f_y \Rightarrow x - 2y - 2$ $\Rightarrow y - 2x - 2 = 0$ & $x - 2y - 2 = 0$ $\Big\}$ $(x, y = -2, -2)$

$\therefore$ C. points $\Rightarrow (-2, -2)$

⊛ How to find maxima / minima at critical points (a, b)

Soln. Calculate f_{xx}, f_{yy}, f_{xy}

$f_{xx} \downarrow r$ $\quad$ $f_{yy} \downarrow t$ $\quad$ $f_{xy} \downarrow s$

Now. calculate $\boxed{rt - s^2}$

$\boxed{rt - s^2}$

- $\boxed{rt - s^2 \neq 0}$
  - $\boxed{rt - s^2 > 0}$
    - $\boxed{r > 0}$ $\downarrow$ (Local minima)
    - $\boxed{r < 0}$ $\downarrow$ (local maxima)
  - $\boxed{rt - s^2 < 0}$ $\downarrow$ (Saddle point) (No maxima or minima)
- $rt - s^2 = 0$ $\downarrow$ (No result) (test is inconclusive)

Ques. Test the function for relative maxima or minima

$$f(x,y) = xy + \frac{9}{x} + \frac{3}{y}$$

Sol^n.

$f_x = y - \frac{9}{x^2}$ $\qquad f_y = x - \frac{3}{y^2}$

$\boxed{r = f_{xx} = \frac{18}{x^3}}$ $\qquad \boxed{t = f_{yy} = \frac{6}{y^3}}$ $\qquad \boxed{s = f_{xy} = 1}$

now for finding critical points.

for maxima & minima. $f_x = 0$ & $f_y = 0$ $\Rightarrow y - \frac{9}{x^2} = 0$ & $x - \frac{3}{y^2} = 0$

$\Rightarrow y = \frac{9}{x^2}$ $\qquad \Rightarrow x - \frac{3 \times x^4}{81} = 0$

$\Rightarrow 27x - x^4 = 0$

$\Rightarrow x(27 - x^3) = 0$

$\Rightarrow \boxed{x = 0 \ \& \ x = 3}$

$\Rightarrow y = \frac{9}{0} = \text{N.D}$

$y = \frac{9}{9} = 1$

$\Rightarrow \boxed{C.P \Rightarrow (3,1)}$

$rt - s^2 = \frac{18}{x^3} \cdot \frac{6}{y^3} - 1$

$= \frac{18}{27} \cdot 6 - 1$

$\Rightarrow 4 - 1 = 3$

$r = \frac{18}{x^3} = \frac{18}{27}$

$= 2/3$

• $rt - s^2 > 0$

$r > 0$

$\Rightarrow$ relative minima.

thus; (3 1) is the point of relative minima.

$f_{min}(3,1)$

$\Rightarrow 3 \cdot 1 + \frac{9}{3} + \frac{3}{1}$

$\Rightarrow 3 + 3 + 3$

$\Rightarrow 9$

Ques. find relative maxima and minima values of the fun

$$f(x) = 2(x^2-y^2) - x^4 + y^4$$

Solⁿ

$f = 2x^2 - 2y^2 - x^4 + y^4$

$f_x \rightarrow 4x - 4x^3 \Rightarrow f_x = 0 \Rightarrow 4x(1-x^2) \Rightarrow x = 0, 1, -1$

$f_y \rightarrow -4y + 4y^3$ & $f_y = 0$ $\Rightarrow -4y(1-y^2) \Rightarrow y = 0, 1, -1$

$\because$ Critical points are (0 0) (0 1) (0 -1) (1 0) (1 1) (1 -1) (-1 0) (-1 1) (-1 -1)

$r = f_{xx} \rightarrow 4 - 12x^2$ & $t = f_{yy} = -4 + 12y^2$ $s \rightarrow f_{xy} \rightarrow 0$

Critical points (x,y)	r	t	$rt - s^2$
(0 0)	4	-4	-16
(0 1)	4	8	32
(0 -1)	4	8	32
(1 0)	-8	-4	32
(1 1)	-8	8	-64
(1 -1)	-8	8	-64
(-1 0)	-8	-4	32
(-1 1)	-8	8	-64
(-1 -1)	-8	8	-64

* Saddle points $\rightarrow \boxed{rt - s^2 < 0}$

$\rightarrow$ (0 0) (1 1) (1 -1) (-1 1) & (-1 -1) are saddle points.

* Local minima. ($rt - s^2 > 0$ & $r > 0$)

$\hookrightarrow$ (0 1) & (0 -1) are points of local minima.

Minimum value $\rightarrow 2(0-1) - 0 + 1 = -1$

* Local maxima. $rt - s^2 > 0$ & $r < 0$

(1, 0) & (-1, 0) are the points of local maxima

Max. value $\rightarrow 1$

END TERM

Ques. $z = \cos y + x \sin y$ value of $\frac{\partial^2 z}{\partial x \partial y}$

~~a.)~~ $\sin y$ ~~b.)~~ $\cos y$ c.) $-\sin y$ d.) $-\cos y$

Soln.

$$\frac{\partial z}{\partial y} \to \underbrace{-\sin y + x\cos y}_{\partial x}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \to \cos y$$

Ques. $f(x,y) = x^3 + y^3 - 2x^2y^2$ $f_{yy}(1,1) = ?$

~~a.)~~ 1 ~~b.)~~ 2 c.) −2 d.) 0

Soln.

$f_y \to 3y^2 - 4x^2y$

$f_{yy} \to 6y - 4x^2 \Big|_{(1,1)} \to 6 - 4 = 2$

Ques. The function $f(x,y) = y^2 - x^2$ has

a.) minimum at (0,0)

b.) minimum at (1,1)

~~c.)~~ Neither minimum nor maximum at (0,0)

d.) maximum at (1,1)

Soln.

$f_x \to -2x$ $f_y \to 2y$

$r = f_{xx} \to -2$ $t \to f_{yy} \to 2$

$s \to f_{xy} \to 0$

$rt - s^2 = -2 \times 2 = -4 < 0$

$\therefore \boxed{rt - s^2 < 0} \Rightarrow$ saddle point

$\Rightarrow$ Neither minimum nor maximum.

Que. $p=0 \quad q=0 \quad rt-s^2>0 \quad r<0 \quad f(x,y) \rightarrow$

~~a.)~~ minimum ~~b.)~~ maximum c.) saddle point d.) NOT

r. max.

Que. $f(x,y) = \sin xy + x^2 \log y \qquad f_{yx}$ at $(0, \pi/2)$

a.) 33
b.) 0
c.) 3
~~d.)~~ 1

Soln.

$$f_x \rightarrow \cos(xy)\cdot y + \log y \cdot 2x$$

$$= y\cos(xy) + 2x\log y$$

$$f_{yx} \rightarrow \left[\begin{matrix} y(-\sin(xy))\cdot x \\ + \cos(xy) \end{matrix}\right] + \left[\frac{2x}{y}\right]\Bigg|_{(0,\pi/2)} = \left[\pi/2\,(-)\cdot 0 + \cos(0) + 0\right]$$

$$\rightarrow \cos(0) \rightarrow \underline{1}$$

Ques. NO. of critical points for $f(x,y) = 4x - x^4 - 4y^3$

~~A.)~~ 1 B.) 2 C.) 3 d.) 4

Soln. $f_x \rightarrow 4 - 4x^3 \quad | \quad f_y \rightarrow -12y^2$

$\Rightarrow 4(1-x^3) = 0 \qquad -12y^2 = 0$

$\rightarrow 1 - x^3 = 0 \qquad \Rightarrow \boxed{y=0}$

$\Rightarrow \boxed{x=1}$

$\therefore \rightarrow (1,0)$ is the only 1 C.P

Q26 → nature of (1,1) for $f(x,y) = 4 + x^3 + y^3 - 3xy$

a.) Relative minima
b.) Relative maxima
c.) Saddle point
d.) Not

Sol^n

$f_x \rightarrow 3x^2 - 3y$ $\quad$ $f_y \rightarrow 3y^2 - 3x$

$f_{xx} \rightarrow 6x$ $\quad$ $f_{yy} \rightarrow 6y$ $\quad$ $s \Rightarrow f_{xy} \rightarrow -3$

$r \rightarrow$ $\quad$ $t \nearrow$

$rt - s^2 = (6x)(6y) - (-3)^2$

$\Rightarrow 36xy - 9$

$\Rightarrow 36 - 9 = 27 > 0$

$\because \boxed{rt - s^2 > 0}$

$\&$ $r \rightarrow 6x$
$\rightarrow 6$ $\quad$ $\boxed{r > 0}$

$\boxed{\text{R. Min.}}$

Ques. Critical point of $f(x,y) = 2x^2 + 2xy + 2y^2 - 6x$.

A.) (1,2) $\quad$ B.) (0,1) $\quad$ C.) (−2,3) $\quad$ d.) (2,−1)

Soln.

$f_x \rightarrow 4x + 2y - 6$ $\quad$ $f_y \rightarrow 2x + 4y$

$\Rightarrow f_x \rightarrow 0 \rightarrow 4x + 2y - 6 = 0$ $\;\&\;$ $2x + 4y = 0$

$\rightarrow 2x + y - 3 = 0$ $\;\&\;$ $x + 2y = 0 \Rightarrow \boxed{x = -2y}$

$\rightarrow -4y + y = 3$

$\rightarrow -3y = 3$ $\quad$ $\boxed{y = -1}$

$x = (-2)(-1)$
$= 2$

$\rightarrow (x, y)$
$\downarrow$
$(2, -1)$

END TERM

Ques. If $z = f(x, y)$ & $x = r\cos\theta$, $y = r\sin\theta$ then $\frac{\partial z}{\partial r}$

(a) $\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$ (b) $\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta$ c.) $\frac{\partial f}{\partial x}\cos\theta - \frac{\partial f}{\partial y}\sin\theta$ d.) $\frac{\partial f}{\partial x}\sin\theta - \frac{\partial f}{\partial y}\cos\theta$

Sol^n

$z = f$

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial r} = \left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r}\right) + \left(\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}\right)$$

$$\frac{\partial x}{\partial r} = \cos\theta \; ; \; \frac{\partial y}{\partial r} = \sin\theta \quad \Rightarrow \boxed{\frac{\partial f}{\partial x}\cdot\cos\theta + \frac{\partial f}{\partial y}\cdot\sin\theta}$$

(a).

Ques. $x^y + y^x = c$ find value of $\frac{dy}{dx}$ at (1,1) a.) 0 b.) 1 c.) -1 d.) -2

Sol^n. $a^x \to a^x \log a$

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = -\left[\frac{yx^{y-1} + y^x\log y}{x^y\log x + xy^{x-1}}\right]$$

$\boxed{\log 1 = 0}$

$$\text{at } (1,1) \to -\left[\frac{1+0}{0+1}\right] = -1$$

Que. If $f(x,y) = 0$ then $\frac{dy}{dx} = ??$

a) $\frac{\frac{\partial y}{\partial f}}{\frac{\partial x}{\partial f}}$ (b) $-\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}}$ (c) $-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$ d.) $\frac{\partial y}{\partial x}\cdot\frac{\partial f}{\partial y}$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

(c)

Ques $U = y^x$ then $\frac{du}{dx}$ a.) xy^{x-1} (b) 0 (c) $y^x \log y$ (d) NOT

$a^x \to a^x \log a$. $\to \frac{du}{dx} \Rightarrow y^x \log y$. (c.)

Que. $x = r\cos\theta$ $y = r\sin\theta$ $\frac{\partial r}{\partial x}$ a.) $\sec\theta$ b.) $\sin\theta$ c.) $\cos\theta$ d.) $\mathrm{cosec}\,\theta$.

Sol^n. $r = x/\cos\theta$

$$\frac{dr}{dx} = \frac{1}{\cos\theta} = \sec\theta$$

Que. $U = \frac{x^2+y^2+xy}{x+y}$ then $\frac{x\,du}{dx} + \frac{y\,du}{dy} = ?$ a.) 1 b.) 0 c.) U d.) 2U

Sol^n. $\frac{du}{dx} = \frac{(x+y)[2x+y] - [x^2+y^2+xy](1)}{(x+y)^2}$ $\quad \frac{du}{dy} \Rightarrow \frac{(x+y)[2y+x] - (x^2+y^2+xy](1)}{(x+y)^2}$

$\to$ A/q. $x\left[\frac{(x+y)(2x+y) - (x^2+y^2+xy)}{(x+y)^2}\right] + y\left[\frac{(x+y)(2y+x) - (x^2+y^2+xy)}{(x+y)^2}\right]$

$\to \frac{-(x^2+y^2+xy)[x+y] + (x+y)[2x^2+xy+2y^2+xy]}{(x+y)^2} \Rightarrow \frac{(x+y)[-x^2-y^2-xy+2x^2+2y^2+2xy]}{(x+y)^2}$

$\to \left[\frac{x^2+y^2+xy}{x+y}\right] = U$ A

Ques. $U = x^2 + y^2$ then $\frac{\partial u}{\partial x} = ??$ a.) 0 b.) 2 c.) $2x + 2y$ d.) $2x$.

Soln $\frac{du}{dx} \to 2x$.

Ques. $u = f(x/y)$ then. ~~a)~~ $x\frac{du}{dx} - y\frac{du}{dy} = 0$ ~~b)~~ $x\frac{du}{dx} + y\frac{du}{dy} = 0$ c) $x\frac{du}{dx} + y\frac{du}{dy} = u$

d.) $x\frac{du}{dx} + y\frac{du}{dy} = 1$.

Soln. $\frac{du}{dx} = f'\left(\frac{x}{y}\right)\cdot\left(\frac{1}{y}\right)$

$\frac{du}{dy} \Rightarrow f'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right)$

Option check a.) $x \cdot f'\left(\frac{x}{y}\right)\cdot\frac{1}{y} - y \cdot f'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right)$

$\Rightarrow \frac{x}{y} f'\left(\frac{x}{y}\right) + \frac{x}{y} f'(x/y) \neq 0$

b. $x \cdot f'\left(\frac{x}{y}\right)\cdot\left(\frac{1}{y}\right) + y \cdot f'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right) = \frac{x}{y} f'\left(\frac{x}{y}\right) - \frac{x}{y} f'\left(\frac{x}{y}\right) = 0$

Ques. $f = x^2 + y^2$ $x = r + 3s$ $y = 2r - s$ then $\frac{\partial f}{\partial r} = ??$

a.) $4x + 2y$ b.) $2x + y$ c.) $2x + 4y$ d.) $x + 4y$

Soln $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{dx}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = (2x \cdot 1) + (2y \cdot 2)$

$= 2x + 4y$

Q.9. $U = x^2 \tan^{-1}(y/x)$ then $x\frac{du}{dx} + y\frac{du}{dy}$ at $x=y=1$ a.) $\pi/4$ ~~b.)~~ $\pi/2$
c.) $-\pi/4$ d.) π

Sol^n. $\frac{du}{dx} \to x^2 \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{-y}{x^2}\right) + \tan^{-1}\left(\frac{y}{x}\right) \cdot 2x \Big|_{(1,1)} \to -\frac{1}{2} + 2\tan^{-1}(1)$

$\left(\tan^{-1}(1) = \pi/4\right)$

$\frac{du}{dy} \to x^2 \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{1}{x}\right) \Big|_{(1,1)} \to \left(\frac{1}{2}\right)$ A/q $\to 1 \cdot 1 \cdot \left(\cancel{-\frac{1}{2}} + 2\tan^{-1}(1)\right) + 1 \cdot \cancel{\frac{1}{2}}$

$\Rightarrow 2\tan^{-1}(1) \Rightarrow 2 \times \frac{\pi}{4} = \pi/2$

Q. $f(x,y) = x^3 + y^2$ $x = \log t + e^t$ $y = t^2 + \frac{1}{t}$ $\frac{df}{dt}$ at $t=1$

~~(A)~~ 0 ~~(B)~~ $e^2 - e + 5$ (C) 4 ✓(d) $3e^2(1+e)+4$

Sol^n. $\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt} \Rightarrow \Big|_{t=1} \to 3e^2(1+e) + 4 \cdot 1$

$\Rightarrow \boxed{3e^2(1+e)+4}$

(d)

$\frac{df}{dx} = 3x^2$; $\frac{df}{dy} = 2y$

$\frac{dx}{dt} \to \frac{1}{t} + e^t$; $\frac{dy}{dt} = 2t - \frac{1}{t^2}$

$\boxed{\text{at } t=1}$ $\boxed{x=e}$ $\boxed{y=2}$ $\Rightarrow \frac{df}{dx} = 3e^2$; $\frac{df}{dy} = 4$; $\frac{dx}{dt} \Rightarrow 1+e$; $\frac{dy}{dt} = 1$

Ques. $\underbrace{x^2z + x^3y + xy^3z}=6$ $\frac{\partial y}{\partial z}=??$

(a) $-\frac{(2xy+yz^2)}{x^3+3xyz^2}$ ✗

(b) $-\frac{(x+y^3)}{x^2+3y^2z}$ ✓

(c) $-\frac{(x^2+3yz^2)}{(2xy+yz^2)}$

(d) $-\frac{(x^2+3xyz)}{x^2+yz^3}$

Sol^n $\frac{\partial y}{\partial z} = -\frac{fz}{fy} = -\frac{x^2+xy^3}{x^3+3y^2xz}$

$\Rightarrow \frac{-x(x^2+y^3)}{x(x^2+3y^2z)} = \frac{-(x^2+y^3)}{x^2+3y^2z}$

Ques. $z=f(u,v)$ $u=xy$ $v=3x-2y$ then $\frac{\partial z}{\partial y}=??$

(a) $x\frac{dz}{du} - 2\frac{dz}{dv}$ ✓

(B) $y\frac{dz}{du} + 3\frac{dz}{dv}$ ✗

(c) $x\frac{du}{dz} - 2\frac{dv}{dx}$ ✗

(d) $y\frac{du}{dz} + z\frac{dv}{dz}$ ✗

Sol^n. $\frac{dz}{dy} \Rightarrow \frac{dz}{du}\times\frac{du}{dy} + \frac{dz}{dv}\times\frac{dv}{dy}$

$\Rightarrow \frac{du}{dy}\to x$ $\Rightarrow \frac{dz}{du}\times(x) + \frac{dz}{dv}\times(-2)$

$\frac{dv}{dy}\to -2$

$\longrightarrow x\frac{dz}{du} - 2\frac{dz}{dv}$ (a)

Ques $z = e^{\left(\frac{x^2+y^2}{x+y}\right)}$ $x\frac{dz}{\partial x} + y\frac{dz}{dy}$

A.) 0 ~~c)~~ $z \ln z$
B.) $z^2 \ln z$ d) z.

Solⁿ ⟹ taking log both sides..

$$\log z = \log e^{\left(\frac{x^2+y^2}{x+y}\right)} = \frac{x^2+y^2}{x+y} \log e$$

$\log a^x$ ↳ $x \log a$

$$\to \frac{dz}{dx} \Rightarrow \frac{1}{z}\frac{dz}{dx} = \log e \left[\frac{(x+y)(2x) - (x^2+y^2)}{(x+y)^2}\right]$$

$$\to \frac{dz}{dy} \to \frac{1}{z}\frac{dz}{dy} = \log e \left[\frac{(x+y)(2y) - (x^2+y^2)}{(x+y)^2}\right]$$

$$\frac{x \cdot z \log e}{(x+y)^2}\left[(x+y)(2x) - (x^2+y^2)\right] + \frac{y \cdot z \cdot \log e}{(x+y)^2}\left[(x+y)(2y) - (x^2+y^2)\right]$$

$$\Rightarrow \frac{z \log e\left[2x^2(x+y) - x(x^2+y^2) + (x+y)(2y^2) - (x^2+y^2)\cdot y\right]}{(x+y)^2}$$

$$\Rightarrow \frac{z \log e}{(x+y)^{\cancel{2}}}\left[2\cancel{(x+y)}\left[x^2+y^2\right] - (x^2+y^2)\cancel{[x+y]}\right] \Rightarrow \frac{z \log e}{x+y}\left[x^2+y^2\right]$$

$$\Rightarrow z \log e \cdot \frac{\log z}{\log e} = z \log z$$

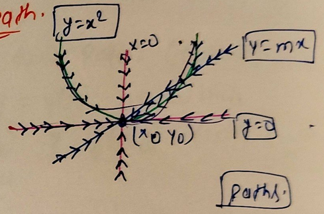
A

Limit AND Continuity

Limit

- In case of single variable if LHL & RHL exists then we say limit exist.
- In case of multivariable if limit exists from all possible paths and equal then limit exists.

• eg → path.



Note.

1. If $\lim_{(x,y)\to(x_0 y_0)} f(x,y) = L$ exists, then it is unique.
2. The limit is path independent
3. if the limit depends on path, then limit does not exist.

Type 1

- we can find limit by putting the value of x & y
  (only when fxn value is defined)

eg → $\lim_{(x,y)\to(2,1)} (3x+4y)$

So^n → $3(2)+4(1)$

$= 6+4$

$= 10$

(*) Type (2)

indeterminant form $\left(\frac{0}{0}\right)\left(\frac{\infty}{\infty}\right)$ ..... or not defined value.

$\Rightarrow$ We have a simple consideration to show that limit does not exists.
We say, $\lim_{x,y \to (a,b)}$ does not exists if there exist ($\exists$) two paths (curves) $y = \phi(x)$ & $y = \psi(x)$ on which $(x,y) \to (a,b)$ & $f(x,y)$ tends two diff. values.

for Ex.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{x+y}{x-y}$$

sol^n $\rightarrow$ $\frac{0+0}{0-0} = \frac{0}{0}$

now take a curve $\boxed{y = mx}$

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{x-y} = \lim_{(x,y)\to(0,0)} \frac{x+mx}{x-mx} = \lim_{(x,y)\to(0,0)} \frac{x(1+m)}{x(1-m)} = \boxed{\frac{1+m}{1-m}}$$

** value of function depends on m

for $m = 2$ function value is -3

for $m = 3$ fxn value is -2

*** $\Rightarrow$ D/f value exists for different values of m, so limit does not exists.

Ques. $\lim_{(x,y)\to(1,-1)} \frac{x-1}{x+y}$

Sol^n → $\frac{0}{0}$ put $y = -1/x$

$\lim_{(x,y)\to(1,-1)} \left(\frac{x-1}{x+y}\right)$ $\frac{x-1}{x-\frac{1}{x}} = \frac{(x-1)x}{x^2-1} \Rightarrow \frac{\cancel{(x-1)}\,x}{\cancel{(x-1)}(x+1)}$

$\Rightarrow \frac{x}{x+1} = \frac{1}{2}$ —(1)

put $y = \frac{-1}{x^2}$

$\lim_{(x,y)\to(1,-1)} \frac{x-1}{x+y} = \frac{x-1}{x-\frac{1}{x^2}} \Rightarrow \frac{(x-1)(x^2)}{(x^3-1)} = \frac{x^2(x-1)}{(x-1)(x^2+1+x)} = \frac{x^2}{x^2+1+x}$

$= \frac{1}{1+1+1} = 1/3$ —(2)

from 1st & 2nd

we get diff values of given fxn.

∴ limit doesn't exists.

Ques. ① $\lim_{(x,y)\to(0,0)} \frac{3x-2y}{2x-3y}$

Sol^n

$\frac{0}{0} \Rightarrow$ let $y = mx$.

$\Rightarrow \frac{3x-2mx}{2x-3mx} \Rightarrow \frac{x(3-2m)}{x(2-3m)}$

$\Rightarrow \frac{3-2m}{2-3m}$

$\Rightarrow$ value of fxn depends on m; so, limit does not exists.

② $f(x,y) = \begin{cases} \frac{x^2y^2}{x^4+y^4} & ; (x,y) \neq (0,0) \\ & (x,y) = (0,0) \end{cases}$

Sol^n $\frac{0}{0}$ $\boxed{y = mx}$

$\lim_{(x,y\to 0,0)} \frac{x^2 \cdot m^2x^2}{x^4+m^4x^4} \Rightarrow \frac{\cancel{x^4}\, m^2}{\cancel{x^4}(1+m^4)}$

$= \frac{m^2}{1+m^4}$

value of fxn depends on m

$\therefore$ Limit does not exists

Q3. $f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & x,y \neq (0,0) \\ & x,y = (0,0) \end{cases}$

Sol^n. $\boxed{y = mx}$ $\Rightarrow \frac{x^2 \cdot mx}{x^4+m^2x^2} = \frac{mx}{x^2+m^2}$

$\boxed{x \to 0}$ $\Rightarrow \frac{0}{0+m^2} = 0$

now take $\boxed{y = mx^2}$

$\frac{x^2 \cdot mx^2}{x^4+m^2x^4} \Rightarrow \frac{x^4 m}{x^4(1+m^2)}$

$= \frac{m}{1+m^2}$

$\Rightarrow$ Limit does not exists.

(Because both curve give diff value)

CA PYQ

Ques. Discuss the limit of the fxn at the point (00) $f(x,y) = \frac{(x-y)^2}{x^2+y^2}$

Sol^n

$$\lim_{(x,y)\to(0,0)} f(x,y) \quad \frac{(x-y)^2}{x^2+y^2} = \frac{0-0}{0+0} = \frac{0}{0}$$

(indeterminant form).

So we take a curve $\boxed{y = mx}$

now.

$$\lim_{(x,y)\to(00)} f(x,y) \quad \frac{(x-y)^2}{x^2+y^2} = \frac{(x-mx)^2}{x^2+m^2x^2} = \frac{x^2(1-m)^2}{x^2(1+m^2)}$$

$$= \boxed{\frac{(1-m)^2}{1+m^2}}$$

∴ Since value of given function depends on m, so Limit does not exists.

for example

$m = 0 \Rightarrow$ fxn value $= 1$.

& if $m = 2 \Rightarrow$ fxn value $= \frac{(-1)^2}{1+4} = \frac{1}{5}$] → diff

Ques. find the limit of the given fxn at the points (0 0)

(CR)

$$f(x,y) = \frac{x^4 y^2}{(x^4+y^2)^2}$$

Sol^n

$$\lim_{(x,y)\to(0,0)} \frac{x^4 y^2}{(x^4+y^2)^2} = \frac{0}{0} \quad \text{(indeterminant form)}$$

$\Rightarrow$ $\boxed{\text{let } y = mx^2}$

$$\therefore \lim_{(x,y)\to(0\,0)} \frac{x^4 y^2}{(x^4+y^2)^2} = \frac{x^4 \cdot m^2 x^4}{(x^4 + m^2 x^4)^2} = \frac{x^8\, m^2}{x^8[1+m^2]^2} \quad \boxed{\frac{m^2}{(1+m^2)^2}}$$

$\therefore$ fn value depend on m

$\therefore$ limit does not exists.

Basics - ($11^{th}/12^{th}$)

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$

2. $\lim_{x \to 0} \frac{\tan x}{x} = 1$

* L Lopital rule.

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

Eg. $\lim_{x \to 0} \frac{\sin x}{x} \Rightarrow \lim_{x \to 0} \frac{\cos x}{1}$

$\Rightarrow \frac{\cos(0)}{1} = 1$

continuity

In case of one variable fxn $y = f(x)$

for continuity $\underbrace{\lim_{x \to a} f(x)}_{\text{Limit of fxn}} = \underbrace{f(a)}_{\text{value of fxn.}}$

In case of two variable fxn $f(x,y) = z$ for continuity

$\underbrace{\lim_{(x,y) \to (a,b)} f(x,y)}_{\text{Limit of fxn}} = \underbrace{f(a,b)}_{\text{value of fxn.}}$

* If limit not exists

$\Rightarrow$ fxn is discontinuous

i.e not continuous.

END TERM

Ques. $\lim_{x \to 2} (x-2) \sec\left(\frac{\pi x}{4}\right)$ a.) $\pi/2$ b.) 0 c.) $-\frac{4}{\pi}$ d.) -1

Soln. $\frac{x-2}{\cos\left(\frac{\pi x}{4}\right)} \longrightarrow \frac{1}{-\sin\left(\frac{\pi x}{4}\right) \cdot \frac{\pi}{4}}$ $\boxed{x=2}$ $\Rightarrow \frac{-1 \quad 4}{\sin\left(\frac{\pi}{2}\right) \cdot \pi} \Rightarrow \boxed{-4/\pi}$

Ques. $\lim_{x \to 0} (\operatorname{cosec} x)^{\tan x} = ??$ a.) π B.) 0 c.) 1 d.) e

Soln $a^b = e^{b \log(a)}$

$\Rightarrow e^{\tan x \cdot \log(\operatorname{cosec} x)}$ $x=0 \Rightarrow e^0 \Rightarrow 1$ Ans.

Ques. $\lim_{(x,y) \to (0,0)} \frac{3x - 2y}{x + 5y} = ??$ A.) 3 B.) $\frac{-2}{5}$ C.) $\frac{2}{5}$ d.) does not exists.

0/0

Soln. $\boxed{y = mx}$ $\frac{3x - 2mx}{x + 5mx} \Rightarrow \frac{x(3-2m)}{x(1+5m)} = \frac{3-2m}{1+5m}$

$\therefore$ fxn value depend on m

$\Rightarrow$ limit not exist.

Ques. value of ~~limit~~ $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$ a.) 0 b.) 1 c.) 2 ~~d.)~~ does not exist.

Sol^n Already done →

Ques. ** $\lim_{(x,y)\to(0,1)} \left[\tan^{-1}\left(\frac{y}{x}\right)\right]$ a.) −1 b.) 0 c.) 2 ~~d.)~~ Limit not exists

Sol^n → $\tan^{-1}\left(\frac{1}{0}\right) \Rightarrow \tan^{-1}(\infty) = \pi/2$

let $\tan^{-1}(\infty) = \alpha$

$\tan\alpha = \infty = \tan(\pi/2)$

$\Rightarrow \boxed{\alpha = \pi/2}$

$\pi/2$ (1)

Ques. $\lim_{(x,y)\to(0,0)} (x+y)\sin\left(\frac{1}{x+y}\right)$ is a.) Limit not exists ~~b.)~~ 0 c.) 1 d.) −1

Sol^n. $\sin(x) \to [-1, 1]$

$(x+y)\sin\left(\frac{1}{x+y}\right)$

$(0,0) \to (-1,1) = 0$

Ques. $\lim_{(x,y)\to(0,-1)} \frac{1}{x^2+y}$ a.) 2 b.) 0 c.) limit does not exist ~~d.)~~ −1

Sol → $\frac{1}{0-1} = -1$

Ques. value of ~~limit~~ $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ a.) 0 b.) 1 c.) 2 ~~d.)~~ does not exists.

Sol^n Already done →

Ques. **
$\lim_{(x,y)\to(0,1)} \left[\tan^{-1}\left(\frac{y}{x}\right)\right]$ a.) −1 b.) 0 c.) 2 ~~d.)~~ Limit not exists.

Sol^n
$\to \tan^{-1}\left(\frac{1}{0}\right) \Rightarrow \tan^{-1}(\infty) = \pi/2$

let $\tan^{-1}(\infty) = \alpha$
$\tan\alpha = \infty = \tan(\pi/2)$
$\Rightarrow \boxed{\alpha = \pi/2}$

$\pi/2$ 1

Ques. $\lim_{(x,y)\to(0,0)} (x+y)\sin\left(\frac{1}{x+y}\right)$ is a.) Limit not exists ~~b.)~~ 0 c.) 1 d.) −1

Sol^n.
$\sin(x) \to [-1, 1]$

$(x+y)\,\underbrace{\sin\left(\frac{1}{x+y}\right)}_{(-1,1)}$

$(0+0) \to (-1,1) = 0$

Ques. $\lim_{(x,y)\to(0,-1)} \frac{1}{x^2+y}$ a.) 2 b.) 0 c.) limit does not exist ~~d.)~~ −1

Sol → $\frac{1}{0-1} = -1$

Ques. $\lim_{(x,y) \to (0,0)} \dfrac{\sec(y)\sin(x)}{x}$ ~~a.)~~ 1 b.) 0 c.) -1 d.) Infinity

Soln. $\dfrac{\sin x}{x} = 1 \Rightarrow \dfrac{\sec y \sin x}{x} = \sec y = \dfrac{1}{\cos y} \Rightarrow \dfrac{1}{\cos(0)} = \dfrac{1}{1} = 1$

$\Rightarrow 1$

continuous Questions

Q. find the value of α $f(x,y) = \begin{cases} \dfrac{\sec y}{x \operatorname{cosec} 2x} & x \neq 0 \\ \alpha & x = 0 \end{cases}$ continuous at $(0,0)$

~~a.)~~ 2 (b) 0 (c) 1/3 (d) 4.5

Soln $\Rightarrow \lim_{(x,y) \to (0,0)} \dfrac{\sec y}{x \operatorname{cosec} 2x}$

$\Rightarrow \dfrac{\sin 2x}{x \times \cos y} \times \dfrac{2}{2} \Rightarrow \dfrac{\sin 2x}{2x} \cdot \dfrac{2}{\cos y} \Rightarrow \dfrac{2}{\cos(0)} = \dfrac{2}{1} = 2$

$\dfrac{\sin 2x}{2x} \Rightarrow 1$, $y = 0$

$\Rightarrow$ fxn value = Limit value

$\Rightarrow \boxed{\alpha = 2}$

Ques. $f(x) = \begin{cases} y \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$

~~a.)~~ $\lim_{(x,y) \to (0,0)} f(x)$ does not exists.

~~c.)~~ continuous at $(0,0)$

~~b.)~~ not continuous at $(0,0)$

~~d.)~~ $f(0,0) \neq 0$

Ques: Show that the following fxn is continuous at the point (0,0)

$$f(x,y) = \begin{cases} \dfrac{2x^4+3y^4}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0\,0) \end{cases}$$

Sol^n.

① $f(0,0) = 0$ — ①

② $\lim\limits_{(x,y)\to(0\,0)} \dfrac{2x^4+3y^4}{x^2+y^2}$ $\boxed{y = mx}$ $\dfrac{2x^4+3m^4x^4}{x^2+m^2x^2}$

$\Rightarrow \dfrac{x^4(2+3m^4)}{x^2(1+m^2)} \Rightarrow \dfrac{x^2(2+3m^4)}{1+m^2}$

$\boxed{x \to 0}$

$= 0$

from ① & ②
∴ value of fxn = value of limit

therefore $f(x,y)$ is continuous at (0,0).

Ques. $f(x,y) = \begin{cases} \dfrac{2xy^2}{x^3+3y^3} & (xy) \neq (00) \\ 0 & (xy) = (00) \end{cases}$ check continuity at (00)

Soln $\boxed{f(x,y) = 0}$

$\boxed{y = mx}$ $\dfrac{2x \cdot m^2x^2}{x^3+3m^3x^3} = \dfrac{2x^3m^2}{x^3(1+3m^3)} \Rightarrow \dfrac{2m^2}{1+3m^3} \Rightarrow$ Limit not exists

$\Rightarrow$ not continuous at (00)

Q11 $f(x,y) = \dfrac{x^2+y^2}{x-y}$ is continuous for

(a.) $x \neq y$

(b.) only for $x > y$

(c.) only for $x < y$

(d.) $x = y$

Soln,

99.99%

5

15

12/19

Jacobian (j).

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

Note.

1. If $J = 0$, then variables are functionally related i.e dependent on each other.

2. If $j \neq 0$ then variables are independent of any relation.

Ques. If $u = e^{2x} \sin 3y$ $\quad v = e^{2x} \cos 3y$ $\quad$ find $\frac{\partial(u, v)}{\partial(x, y)}$

Sol^n
$$j = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2e^{2x} \sin 3y & 3e^{2x} \cos 3y \\ 2e^{2x} \cos 3y & -3e^{2x} \sin 3y \end{vmatrix}$$

$$\frac{\partial u}{\partial x} = \sin 3y \; e^{2x} \cdot 2$$

$$\frac{\partial u}{\partial y} = e^{2x} \cos 3y \cdot 3$$

$$\frac{\partial v}{\partial x} = \cos 3y \; e^{2x} \cdot 2$$

$$\frac{\partial v}{\partial y} = e^{2x} (-\sin 3y) \cdot 3$$

$$\Rightarrow -6e^{4x} (\sin^2 3y) - 6e^{4x} \cos^2 3y$$

$$\Rightarrow -6e^{4x} [\sin^2 3y + \cos^2 3y]$$

$$\Rightarrow -6e^{4x} \qquad \therefore \boxed{j = -6e^{4x}}$$

Ans

Ques. Check whether the variables are functionally related or not

$u = x + 3z \qquad v = x - y - z \qquad w = y^2 + 16z^2 + 8yz.$

Soln

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$u_x = 1$

$v_x = 1$

$w_x = 0$

$u_y \to 0$

$v_y \to -1$

$w_y \to 2y + 8z$

$u_z = 3$

$v_z = -1$

$w_z = 32z + 8y$

$$\Rightarrow \begin{vmatrix} 1 & 0 & 3 \\ 1 & -1 & -1 \\ 0 & 2y+8z & 32z+8y \end{vmatrix}$$

$\Rightarrow 1[-32z - 8y + 2y + 8z] - 0 + 3[2y + 8z]$

$\Rightarrow -24z - 6y + 6y + 24z$

$= 0 + 0$

$= 0$

$\therefore$ Since Jacobian $(J) = 0 \Rightarrow$ variables are functionally related.

Euler's Theorem for homogeneous function

Homogeneous function

A fxn $f(x,y)$ is said to be homogeneous of degree n in x & y iff

2 var $\rightarrow$ $\boxed{f(\lambda x, \lambda y) = \lambda^n f(x,y)}$

3 var $\rightarrow$ $\boxed{f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x,y,z)}$

Matlab $\hookrightarrow$ ($x, y, z \rightarrow \lambda x, \lambda y, \lambda z$ put karna h)

Q.1 $x^2 + xy \rightarrow f(x)$

Soln.
$\Rightarrow (\lambda x)^2 + (\lambda x)(\lambda y)$
$\Rightarrow \lambda^2 x^2 + \lambda^2 xy$
$\Rightarrow \lambda^2 (x^2 + xy)$
$\Rightarrow \boxed{\lambda^2 \cdot f(x)}$

$\Rightarrow$ degree $\rightarrow 2$.

$\rightarrow$ hom

2. $\tan^{-1}(y/x) \rightarrow f(x)$

$\rightarrow \tan^{-1}\left(\frac{\lambda y}{\lambda x}\right)$
$\Rightarrow 1 \tan^{-1}\left(\frac{y}{x}\right)$
$\Rightarrow \boxed{\lambda^0 \cdot f(x)}$

degree $\rightarrow 0$

hom

3. $\frac{1}{x+y}$ $f(x)$

$\rightarrow \frac{1}{\lambda(x) + \lambda(y)}$
$\rightarrow \frac{1}{\lambda(x+y)}$
$\Rightarrow \boxed{\lambda^{-1} f(x)}$

$\boxed{n = -1}$

$\boxed{\text{degree} = -1}$

hom

4. $\frac{x^3 + y}{x^2 + y}$ $\boxed{f(x)}$

$\rightarrow \frac{\lambda^3 x^3 + \lambda y}{\lambda^2 x^2 + \lambda y}$
$\Rightarrow \frac{\lambda(\lambda^2 x^3 + y)}{\lambda(\lambda x^2 + y)}$
$\Rightarrow \frac{\lambda^2 x^3 + y}{\lambda x^2 + y}$

$\Rightarrow$ not homogeneous. $\neq f(x)$

Q: Find term. which of the following is homogeneous.

~~a)~~ $\frac{x^3 - xy^2}{x-1}$ ~~b)~~ $\sin\left(\frac{x^5}{x^2+y^2}\right)$ ✓ c) $\tan\left(\frac{x^2-y^2}{x^2+y^2}\right)$ ~~d)~~ $\frac{x^2-y}{y^2-xy}$

Sol^n.

(a) $f(x) = \frac{x^3 - xy^2}{x-1}$

(1) put $x \to \lambda x$, $y \to \lambda y$

$\Rightarrow \frac{\lambda^3 x^3 - \lambda(x)\cdot\lambda^2 y^2}{\lambda x - 1}$

$\Rightarrow \frac{\lambda^3(x^3 - xy^2)}{\lambda x - 1}$

$\neq f(x)$

$\Rightarrow$ not homo.

(b) $f(x) \nearrow$

$\Downarrow$

$\sin\left(\frac{(\lambda x)^5}{\lambda^2 x^2 + \lambda^2 y^2}\right)$

$\sin\left(\frac{\lambda^5 x^5}{\lambda^2(x^2+y^2)}\right)$

$\sin\left(\frac{\lambda^3 x^5}{x^2+y^2}\right)$

$\neq f(x)$

$\Downarrow$

not homog.

(c) $f(x) \nearrow$

$\Downarrow$

$\tan\left[\frac{(\lambda x)^2 - (\lambda y)^2}{(\lambda x)^2 + (\lambda y)^2}\right]$

$\tan\left(\frac{\lambda^2(x^2-y^2)}{\lambda^2(x^2+y^2)}\right)$

$\tan\left(\frac{x^2-y^2}{x^2+y^2}\right) = f(x)$

$\therefore \Rightarrow$ (c) $\to$ homo.

(d) $\frac{x^2-y}{y^2-xy} \Rightarrow \frac{\lambda^2 x^2 - \lambda y}{\lambda^2 y^2 - \lambda^2(xy)}$

$\Rightarrow \frac{\lambda(\lambda x^2 - y)}{\lambda^2(y^2 - xy)} \Rightarrow \frac{\lambda x^2 - y}{\lambda(y^2 - xy)}$

$\neq f(x)$

Q. End term $x = u(1+v)$ $y = v(1+u)$ $\frac{\partial(x,y)}{\partial(u,v)}$

a.) $2uv$ (b) $1-u-v$ (c) $1+u+v$ (d.) 1

$\Rightarrow \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

Sol^n

$x_u = 1+v$

$x_v = u$

$\boxed{x = u + uv}$ $\boxed{y = v + uv}$

$y_u = v$

$y_v = 1+u$

$\Rightarrow \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$

$\Rightarrow 1+u+v+\cancel{uv} \cancel{-uv}$

$\Rightarrow \boxed{1+u+v}$

(c)

End term

Ques. $x = 3u - v^2$, $y = 5u + v^2$

$\frac{\partial(x,y)}{\partial(u,v)} = ?$

a.) 0 (b) $2v$ (c) $16v$ d. 2.

$\Rightarrow \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

Sol^n $x_u = 3$ $y_u = 5$

$x_v = -2v$ $y_v = 2v$

$\Rightarrow \begin{vmatrix} 3 & -2v \\ 5 & 2v \end{vmatrix}$

$\Rightarrow 6v + 10v$

$= 16v$

Euler's theorem.

If $f(x,y)$ is a homogeneous fxn of degree n in x & y then

① $x\frac{df}{dx} + y\frac{df}{dy} = nf$ & ② $x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}$

$= n(n-1)f.$

Eg. $f(x,y) = \frac{x^4+y^4}{x^2+y^2}$

find degree

① → $\frac{\lambda^4 x^4 + \lambda^4 y^4}{\lambda^2 x^2 + \lambda^2 y^2} = \frac{\lambda^4(x^4+y^4)}{\lambda^2(x^2+y^2)}$

$\Rightarrow \boxed{\lambda^2 f(x,y)}$

$\boxed{\text{degree} = 2}$

$$\boxed{x\frac{df}{dx} + \frac{y\,df}{dy} = 2\cdot\frac{x^4+y^4}{x^2+y^2}}$$

② $u = \sqrt{y^2-x^2}\cdot\sin^{-1}\left(\frac{x}{y}\right)$ find $x\frac{du}{dx} + y\frac{du}{dy}$

$f=u$

$n=1$

Sol^n. ① → $\Rightarrow \sqrt{(\lambda y)^2-(\lambda x)^2}\cdot\sin^{-1}\left(\frac{\lambda x}{\lambda y}\right)$

$\Rightarrow \sqrt{\lambda^2 y^2-\lambda^2 x^2}\cdot\sin^{-1}\left(\frac{x}{y}\right)$

$\Rightarrow \lambda\sqrt{y^2-x^2}\cdot\sin^{-1}(x/y) \Rightarrow \lambda^1\cdot u$

$\boxed{\text{degree} = 1}$ $\Rightarrow$ nf $\Rightarrow nu$ $\Rightarrow u$

$x\frac{du}{dx} + y\frac{du}{dy} \Rightarrow \sqrt{y^2-x^2}\,\sin^{-1}(x/y)$

Lagrange's method

(*) $f = \_\_\_$ $\quad g = \_\_\_$

$F = \boxed{f + \lambda g}$

$F_x = 0$ $\quad F_y = 0$ $\quad F_z = 0$

find x, y, z. (from g)

put in f

get extrema.

Ques. find extrema value of xyz, when $x + y + z = a$, $a > 0$

Sol^n Let $f(x,y,z) = xyz$ & $g(x,y,z) = x + y + z - a$

By lagrange's method.

$\boxed{F = f + \lambda g} \Rightarrow \boxed{F = xyz + \lambda(x + y + z - a)}$

$F_x = 0 \Rightarrow yz + \lambda \quad \Rightarrow \lambda = -yz \quad -(1)$

$F_y = 0 \Rightarrow xz + \lambda \quad \Rightarrow \lambda = -xz \quad -(2)$

$F_z = 0 \Rightarrow xy + \lambda \quad \Rightarrow \lambda = -xy \quad -(3)$

from (1) & (2) $\quad$ & from (2) & (3)

$-yz = -xz \quad\quad -xz = -xy$

$\Rightarrow \boxed{x = y} \quad\quad \Rightarrow \boxed{y = z}$

$\Rightarrow \boxed{x = y = z}$

Since, $x + y + z = a$

$\Rightarrow 3x = a$

$\boxed{x = a/3} \Rightarrow \boxed{x = y = z = a/3}$

$\therefore f_{ext} = xyz = \frac{a}{3} \cdot \frac{a}{3} \cdot \frac{a}{3}$

$= a^3/27$ Ans.

Que. find the extreme value of $x^3+8y^3+64z^3$ when $xyz=1$

Soln Let $f = x^3+8y^3+64z^3$ & $g = xyz-1$

By Lagrange's method.

$$F = f + \lambda g \Rightarrow F = x^3+8y^3+64z^3 + \lambda(xyz-1)$$

$\because F_x \to 3x^2 + yz\lambda \Rightarrow F_x = 0 \Rightarrow \lambda = \frac{-3x^2}{yz}$ —(1)

$F_y \to 24y^2 + xz\lambda \Rightarrow F_y = 0 \Rightarrow \lambda = \frac{-24y^2}{xz}$ —(2)

$F_z \to 192z^2 + xy\lambda \Rightarrow F_z = 0 \Rightarrow \lambda = \frac{-192z^2}{xy}$ —(3)

from (1) & (2)

$$\frac{-3x^2}{yz} = \frac{-24y^2}{xz}$$

$\Rightarrow x^3 = 8y^3 \Rightarrow \boxed{x=2y}$

& from (2) & (3)

$$\frac{-24y^2}{xz} = -\frac{192z^2}{xy}$$

$\Rightarrow \boxed{y^3 = 8z^3}$

$\Rightarrow \boxed{y = 2z}$

$\Rightarrow \boxed{x = 2y = 4z}$

Now: $\because xyz = 1$

$\Rightarrow 4z \cdot 2z \cdot z = 1$

$\Rightarrow \boxed{z = 1/2} \Rightarrow \boxed{x=2,\ y=1}$

$\because F_{ext} = x^3+8y^3+64z^3$

$= 8 + 8 + 64/8$

$= 8 + 8 + 8$

$= 24$ Ans.

Que: find the extrema values of $f(x,y,z) = 2x + 3y + z$ such that $x^2 + y^2 = 5$ and $x + z = 1$

Soln.

$$(2x + 3y + z) + \lambda_1(x^2 + y^2 - 5) + \lambda_2(x + z - 1)$$

$F_x \Rightarrow 2 + 2\lambda_1 x + \lambda_2$

$F_y = 3 + 2\lambda_1 y$

$F_z = 1 + \lambda_2$

$\boxed{\lambda_2 = -1}$

$F_x = 0 \Rightarrow 2 + 2\lambda_1 x + \lambda_2 \Rightarrow 2 + 2\lambda_1 x - 1 = 0 \Rightarrow \boxed{\lambda_1 = \frac{-1}{2x}}$

$F_y = 0 \Rightarrow 3 + 2\lambda_1 y = 0 \Rightarrow \boxed{\lambda_1 = \frac{-3}{2y}}$

$\Rightarrow \frac{-1}{2x} = \frac{-3}{2y} \Rightarrow \boxed{y = 3x}$

Since, $x^2 + y^2 = 5 \Rightarrow x^2 + 9x^2 = 5 \Rightarrow 10x^2 = 5 \Rightarrow \boxed{x = \pm\frac{1}{\sqrt{2}}}$

If $x = \frac{1}{\sqrt{2}} \Rightarrow y = \frac{3}{\sqrt{2}} \Rightarrow z = 1 - 1/\sqrt{2}$

& $x = \frac{-1}{\sqrt{2}} \Rightarrow y = \frac{-3}{\sqrt{2}} \Rightarrow z = 1 + 1/\sqrt{2}$

$\therefore F_{ext} = \frac{2}{\sqrt{2}} + \frac{9}{\sqrt{2}} + 1 - \frac{1}{\sqrt{2}} = 1 + \frac{10}{\sqrt{2}} \Rightarrow 1 + 5\sqrt{2}$

$F_{ext} = \frac{-2}{\sqrt{2}} - \frac{9}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} = 1 - 5\sqrt{2} \Rightarrow$

when $(x,y,z) = \left(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 1 - \frac{1}{\sqrt{2}}\right)$

$\boxed{F_{ext} = 1 + 5\sqrt{2}}$

when $(x,y,z) = \left(\frac{-1}{\sqrt{2}}, \frac{-3}{\sqrt{2}}, 1 + \frac{1}{\sqrt{2}}\right)$

$\boxed{F_{ext} = 1 - 5\sqrt{2}.}$