UNIT 1 : Vector Space

Lecture 2: Preliminaries continued

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Definition

A nonempty set S together with a binary operation $*$ defined on it, is said to be a semigroup if the following properties are satisfied

1. $a * b \in G,\ \forall a, b \ \in S$ (Closure)

2. $(a * b) * c = a * (b * c), \forall a, b, c \ \in S$ (Associative)

Note:

It is denoted by $(S,*)$.

Example 1:

Let $A = \{1,3,5,\cdots\}$, the set of all odd positive integers and the binary operation $*$ is usual multiplication. Is $(A,*)$is a semigroup?

Solution.

Closure property: Multiplication of two positive odd integers is again a positive odd integer. Therefore, the closure property is satisfied.

Associative property: Clearly $(a * b) * c = a * (b * c), \forall a, b, c \in A$

$\therefore$ $(A,*)$ is a semigroup.

To discuss:

Is $(A,*)$, where the binary operation $*$ is usual addition a semigroup?

Test yourself:

Check which of the algebraic systems are semigroups.

1. $(N, +)$

2. $(\mathrm{R}, -)$

3. $(\mathrm{Z}, .)$ here dot denotes usual multiplication.

Answer:

(1) and (3) are semigroups.

(2) is NOT a semigroup as associative property is not satisfied.

Monoid

Definition.

A nonempty set M is said to be a group with respect to a binary operation $*$ if it satisfies the following properties

1. $a * b \in M$, $\forall a, b \in M$ (Closure)

2. $(\mathrm{a} * \mathrm{b}) * \mathrm{c} = \mathrm{a} * (\mathrm{b} * \mathrm{c}), \forall\, \mathrm{a}, \mathrm{b}, \mathrm{c} \in M$ (Associative)

3. For every $a \in M$, there exists an element $e \in M$ (called identity) such that

   $a * e = e * a = a$ (Existence of identity)

Note:

It is denoted by $(M, *)$.

Group

Definition.

A nonempty set G is said to be a group with respect to a binary operation $*$ if it satisfies the following properties

1. $a * b \in G, \ \forall a, b \in G$ (Closure)

2. $(a * b) * c = a * (b * c), \forall a, b, c \in G$ (Associative)

3. For every $a \in G$, there exists an element $e \in G$ (called identity) such that $a * e = e * a = a$

   (Existence of identity)

4. For every $a \in G$, there exists an element $a^{-1} \in G$ (called inverse) such that

   $a * a^{-1} = a^{-1} * a = e$ (Existence of inverse)

Note:

It is denoted by $(G, *)$.

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Examples of group

Example -1

Set of integers Z, with usual addition (+)

The sum of two integers results in an integer. So, the closure property is satisfied.

Three integers can be added in any order. So, the associate property is satisfied.

The integer zero (0) serves as the identity element.

$$(a + 0 = 0 + a = a)$$

For every integer a, the additive inverse is $-a$.

$$(a + (-a) = (-a) + a = 0)$$

So, the set of integers is a group with usual addition.

Example -2

Set of real numbers R, with usual addition (+)

The sum of two real numbers results in a real number. So, the closure property is satisfied.

Three real numbers can be added in any order. So associate property is satisfied.

The integer zero (0) serves as the identity element.

For every real number a, the additive inverse is $-a$.

So, the set of real numbers with addition is a group.

To Discuss

Example 3:

Set of real numbers R, with usual multiplication ($\times$)

Why only non-zero reals?

Example -4

Set of non-zero real numbers R ($R^* = R - \{0\}$), with usual multiplication ($\times$)

The product of two real numbers results in a real number. So, the closure property is satisfied.

Three real numbers can be multiplied in any order. So associate property is satisfied.

The integer one (1) serves as the identity element.

$$(a \times 1 = 1 \times a = a)$$

For every real number a, the multiplicative inverse is $\frac{1}{a}$.

$$(a \times \frac{1}{a} = \frac{1}{a} \times a = 1)$$

So, the set of non-zero real numbers along with multiplication is a group.

To Discuss

Example - 5

Set of natural numbers N, with usual addition (+)

Closure and associative property are satisfied.

But identity element (0) does not exist, since 0 is not a natural number.

Example - 6

$N' = N \cup \{0\} = \{0,1,2,3,\cdots\}$, with usual addition (+).

Closure and associative property are satisfied.

Why this is not a group?

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Abelian group

Definition.

A group $(G, *)$ is said to be an Abelian group or commutative group if $a * b = b * a$, for every $a, b \in G$.

Examples.

1. (Z; +), (Q; +), (R; +), (C; +) are abelian groups
2. Set of non-zero real numbers R, with usual multiplication (×)
3. $(Q - \{0\}, \cdot)$, $(R - \{0\}, \cdot)$, $(C - \{0\}, \cdot)$ are abelian groups.

Example: Not Abelian Group

The set of square matrices of order n, with matrix multiplication $(M_{n\times n}, \times)$

Closure and associate property are satisfied.

The identity element is $I_{n\times n} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$, the identity matrix.

The inverse for any matrix A, is the usual matrix inverse, A^{-1}.

Matrix multiplication is not commutative, i.e.. $AB \neq BA$.

So, not an Abelian group.

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Thank you