UNIT 1 : Vector Space

Lecture 3: Preliminaries continued

Dr. P. Venugopal

Professor of Mathematics

School of Science & Humanities

Shiv Nadar University Chennai

SHIV NADAR UNIVERSITY CHENNAI

Contents

Ring

Field

Ring

Definition.

A non-empty set R is said to be a **ring** with two binary operations, $+$ and $\cdot$ and called *addition* and *multiplication*, respectively, if the following properties are satisfied:

1. $(R, +)$ is an abelian group

2. Multiplication is associative

3. The left and right distributive laws hold:

    (i) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

    (ii) $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$

Note.

A ring in which the multiplication operation is commutative is called a *commutative ring*.

SHIV NADAR UNIVERSITY CHENNAI

Examples of ring

1. $(Z;\ +,\ \cdot), (Q;\ +,\ \cdot), (R;\ +,\ \cdot), (C;\ +,\ \cdot)$ are rings.

2. N is NOT a ring for the usual addition and multiplication.

   There is no additive identity element, namely 0 in N

   Also, existence of additive inverse fails, there is no $n\ \in\ N$ for which $1\ +\ n\ =\ 0$

3. Polynomials, with real coefficients, form a commutative ring with identity under the usual addition and multiplication; we denote this by $R[x]$.

Field

Definition. A nonempty set F with at least two elements and two binary operations addition $(+)$ and multiplication $(\cdot)$, is said to be a field, if the following properties are satisfied:

For $a, b, c \ \in F$

For Addition:

1. **Closure**: $a + b \in F$
2. **Associative**:
$$(a + b) + c = a + (b + c)$$
3. **Existence of identity element**: There exists an element $0 \in F$ (called zero element) such that
$$a + 0 = 0 + a = a$$
4. **Existence of inverse**: There exists an element $a^{-1} \in F$ (called additive inverse) such that
$$a + a^{-1} = a^{-1} + a = 0$$
5. **Commutative**: $a + b = b + a$

For Multiplication:(non-zero elements)

1. **Closure**: $a \cdot b \in F$
2. **Associative**:
$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$
3. **Existence of identity element**: There exists an element $1 \in F$ (called unit element) such that
$$a \cdot 1 = 1 \cdot a = a$$
4. **Existence of inverse**: There exists an element $a' \in F$ (called multiplicative inverse) such that
$$a \cdot a' = a' \cdot a = 1$$
5. **Commutative:** $a \cdot b = b \cdot a$

Distributive laws: $a \cdot (b + c) = a \cdot b + a \cdot c$ (left)
$(b + c) \cdot a = b \cdot a + c \cdot a$ (right)

Field – In short

A nonempty set F with atleast two elements and two binary operations addition(+) and multiplication $(\cdot)$, such that

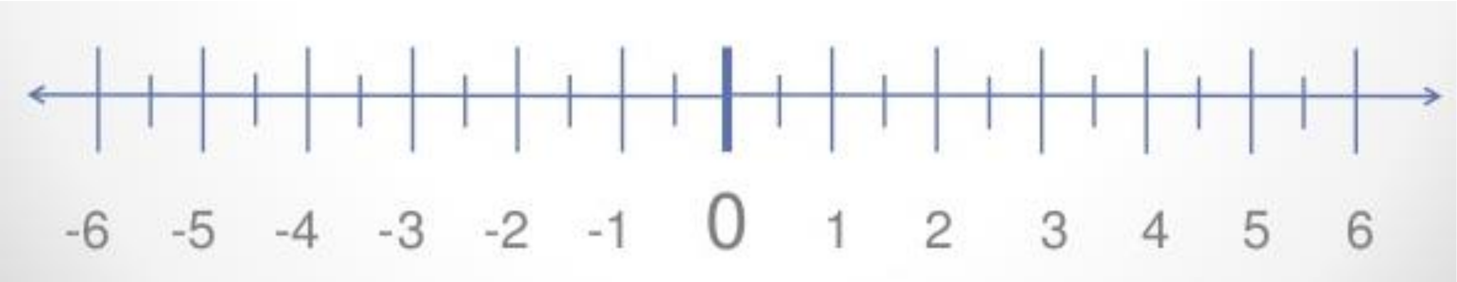
1. $(F, +)$ is an abelian group
2. $(F - \{0\}, \cdot)$ is an abelian group
3. Distributive laws are satisfied

Note: The elements of field are called scalars.

SHIV NADAR UNIVERSITY CHENNAI

Example of Field

The set of real numbers R, with usual addition(+) and usual multiplication (·)



1. $(R, +)$ is an abelian group
2. $(R - \{0\}, \cdot)$ is an abelian group
3. Distributive laws are satisfied

So, the set of real numbers is a field.

Test yourself

1. The rational numbers Q, and the complex numbers C are examples of fields.
2. The set Z of integers is not a field.

Thank you

SHIV NADAR
UNIVERSITY
CHENNAI