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Course No: EE 3122
Sessional on Numerical Methods & Statistics

Experiment No. 2
Name of the Experiment: Solution of Equations

Objectives:

[1] To determine the solution of algebraic and transcendental equations
[2] To determine the solution of equation by Gaussian elimination / Gaussian elimination with Row pivoting method
[3] To find the LU decomposition of the matrix
[4] To find the inverse of a matrix using LU decomposition
[5] To determine the solution of equation by Gauss-Seidel method

NB: Study the materials before writing code

Theory/Introduction: How is a set of equations solved numerically?

One of the most popular techniques for solving simultaneous linear equations is the Gaussian elimination method. The approach is designed to solve a general set of n equations and n unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$. \qquad\qquad .$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

Gaussian elimination consists of two steps

1. Forward Elimination of Unknowns: In this step, the unknown is eliminated in each equation starting with the first equation. This way, the equations are *reduced* to one equation and one unknown in each equation.
2. Back Substitution: In this step, starting from the last equation, each of the unknowns is found.

Forward Elimination of Unknowns:

In the first step of forward elimination, the first unknown, x_1 is eliminated from all rows below the first row. The first equation is selected as the pivot equation to eliminate x_1. So, to eliminate x_1 in the second equation, one divides the first equation by a_{11} (hence called the pivot element) and then multiplies it by a_{21}. This is the same as multiplying the first equation by a_{21}/a_{11} to give

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

Now, this equation can be subtracted from the second equation to give

$$\left(a_{22}-\frac{a_{21}}{a_{11}}a_{12}\right)x_2+...+\left(a_{2n}-\frac{a_{21}}{a_{11}}a_{1n}\right)x_n=b_2-\frac{a_{21}}{a_{11}}b_1$$

or

$$a'_{22}x_2+...+a'_{2n}x_n=b'_2$$

where

$$\begin{aligned}a'_{22}&=\quad a_{22}-\frac{a_{21}}{a_{11}}a_{12}\\ &\vdots\\ a'_{2n}&=\quad a_{2n}-\frac{a_{21}}{a_{11}}a_{1n}\end{aligned}$$

This procedure of eliminating x_1, is now repeated for the third equation to the n^{th} equation to reduce the set of equations as

$$a_{11}x_1+a_{12}x_2+a_{13}x_3+...+a_{1n}x_n=b_1$$

$$a'_{22}x_2+a'_{23}x_3+...+a'_{2n}x_n=b'_2$$

$$a'_{32}x_2+a'_{33}x_3+...+a'_{3n}x_n=b'_3$$

$$.\qquad\qquad\qquad.\qquad\qquad\qquad.$$

$$a'_{n2}x_2+a'_{n3}x_3+...+a'_{nn}x_n=b'_n$$

This is the end of the first step of forward elimination. Now for the second step of forward elimination, we start with the second equation as the pivot equation and a'_{22} as the pivot element. So, to eliminate x_2 in the third equation, one divides the second equation by a'_{22} (the pivot element) and then multiply it by a'_{32}. This is the same as multiplying the second equation by a'_{32}/a'_{22} and subtracting it from the third equation. This makes the coefficient of x_2 zero in the third equation. The same procedure is now repeated for the fourth equation till the n^{th} equation to give

$$a_{11}x_1+a_{12}x_2+a_{13}x_3+...+a_{1n}x_n=b_1$$

$$a'_{22}x_2+a'_{23}x_3+...+a'_{2n}x_n=b'_2$$

$$a''_{33}x_3+...+a''_{3n}x_n=b''_3$$

$$.\qquad\qquad.$$

$$a''_{n3}x_3+...+a''_{nn}x_n=b''_n$$

The next steps of forward elimination are conducted by using the third equation as a pivot equation and so on. That is, there will be a total of $n-1$ steps of forward elimination. At the end of $n-1$ steps of forward elimination, we get a set of equations that look like

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + ... + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + ... + a''_{3n}x_n = b''_3$$

$$. \qquad .$$

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

Back Substitution:

Now the equations are solved starting from the last equation as it has only one unknown.

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Then the second last equation, that is the $(n-1)^{\text{th}}$ equation, has two unknowns: x_n and x_{n-1}, but x_n is already known. This reduces the $(n-1)^{\text{th}}$ equation also to one unknown. Back substitution hence can be represented for all equations by the formula

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)}x_j}{a_{ii}^{(i-1)}} \qquad \text{for } i = n-1, n-2, \ldots, 1$$

and

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Example 1: The upward velocity of a rocket is given at three different times in Table 1.

Table 1 Velocity vs. time data.

Time, t (s)	Velocity, v (m/s)
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as

$$v(t) = a_1t^2 + a_2t + a_3, \qquad 5 \le t \le 12$$

The coefficients a_1, a_2, and a_3 for the above expression are given by

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Find the values of $a_1, a_2,$ and a_3 using the Naïve Gauss elimination method. Find the velocity at $t = 6, 7.5, 9, 11$ seconds.

Solution

Forward Elimination of Unknowns

Since there are three equations, there will be two steps of forward elimination of unknowns.

First step

Divide Row 1 by 25 and then multiply it by 64, that is, multiply Row 1 by $64/25 = 2.56$.

$([25 \quad 5 \quad 1] \qquad [106.8]) \times 2.56$ gives Row 1 as

$[64 \quad 12.8 \quad 2.56] \qquad [273.408]$

Subtract the result from Row 2

$$\begin{array}{rrrrr} & [64 & 8 & 1] & [177.2] \\ - & [64 & 12.8 & 2.56] & [273.408] \\ \hline & 0 & -4.8 & -1.56 & -96.208 \end{array}$$

to get the resulting equations as

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 279.2 \end{bmatrix}$$

Divide Row 1 by 25 and then multiply it by 144, that is, multiply Row 1 by $144/25 = 5.76$.

$([25 \quad 5 \quad 1] \qquad [106.8]) \times 5.76$ gives Row 1 as

$[144 \quad 28.8 \quad 5.76] \qquad [615.168]$

Subtract the result from Row 3

$$\begin{array}{rrrrr} & [144 & 12 & 1] & [279.2] \\ - & [144 & 28.8 & 5.76] & [615.168] \\ \hline & 0 & -16.8 & -4.76 & -335.968 \end{array}$$

to get the resulting equations as

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ -335.968 \end{bmatrix}$$

Second step

We now divide Row 2 by −4.8 and then multiply by −16.8, that is, multiply Row 2 by $-16.8/-4.8 = 3.5$.

$$\left(\begin{bmatrix}0 & -4.8 & -1.56\end{bmatrix} \quad \begin{bmatrix}-96.208\end{bmatrix}\right) \times 3.5$$ gives Row 2 as

$$\begin{bmatrix}0 & -16.8 & -5.46\end{bmatrix} \quad \begin{bmatrix}-336.728\end{bmatrix}$$

Subtract the result from Row 3

$$\begin{array}{rrrrr} & [0 & -16.8 & -4.76] & [-335.968] \\ - & [0 & -16.8 & -5.46] & [-336.728] \\ \hline & 0 & 0 & 0.7 & 0.76 \end{array}$$

to get the resulting equations as

$$\begin{bmatrix}25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7\end{bmatrix}\begin{bmatrix}a_1 \\ a_2 \\ a_3\end{bmatrix} = \begin{bmatrix}106.8 \\ -96.208 \\ 0.76\end{bmatrix}$$

Back substitution

From the third equation

$$0.7a_3 = 0.76$$

$$a_3 = \frac{0.76}{0.7} \qquad = 1.08571$$

Substituting the value of a_3 in the second equation,

$$-4.8a_2 - 1.56a_3 = -96.208$$

$$a_2 = \frac{-96.208 + 1.56a_3}{-4.8} \qquad = \frac{-96.208 + 1.56 \times 1.08571}{-4.8} \qquad = 19.6905$$

Substituting the value of a_2 and a_3 in the first equation,

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25} \qquad = \frac{106.8 - 5 \times 19.6905 - 1.08571}{25} \qquad = 0.290472$$

Hence the solution vector is

$$\begin{bmatrix}a_1 \\ a_2 \\ a_3\end{bmatrix} = \begin{bmatrix}0.290472 \\ 19.6905 \\ 1.08571\end{bmatrix}$$

The polynomial that passes through the three data points is then

$$v(t) = a_1t^2 + a_2t + a_3$$

$$= 0.290472t^2 + 19.6905t + 1.08571, \quad 5 \le t \le 12$$

Since we want to find the velocity at $t = 6, 7.5, 9$ and 11 seconds, we could simply substitute each value of t in $v(t) = 0.290472t^2 + 19.6905t + 1.08571$ and find the corresponding velocity. For example, at $t = 6$

$$v(6) = 0.290472(6)^2 + 19.6905(6) + 1.08571$$
$$= 129.686 \text{ m/s}$$

However we could also find all the needed values of velocity at t = 6, 7.5, 9, 11 seconds using matrix multiplication.

$$v(t) = \begin{bmatrix} 0.290472 & 19.6905 & 1.08571 \end{bmatrix} \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}$$

So if we want to find $v(6), v(7.5), v(9), v(11),$ it is given by

$$\begin{bmatrix} v(6) \, v(7.5) & v(9) \, v(11) \end{bmatrix} = \begin{bmatrix} 0.290472 & 19.6905 & 1.08571 \end{bmatrix} \begin{bmatrix} 6^2 & 7.5^2 & 9^2 & 11^2 \\ 6 & 7.5 & 9 & 11 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.290472 & 19.6905 & 1.08571 \end{bmatrix} \begin{bmatrix} 36 & 56.25 & 81 & 121 \\ 6 & 7.5 & 9 & 11 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 129.686 & 165.104 & 201.828 & 252.828 \end{bmatrix}$$

$$v(6) = 129.686 \text{ m/s}$$

$$v(7.5) = 165.104 \text{ m/s}$$

$$v(9) = 201.828 \text{ m/s}$$

$$v(11) = 252.828 \text{ m/s}$$

Example 2

Use Naïve Gauss elimination to solve

$$20x_1 + 15x_2 + 10x_3 = 45$$

$$-3x_1 - 2.249x_2 + 7x_3 = 1.751$$

$$5x_1 + x_2 + 3x_3 = 9$$

Use six significant digits with chopping in your calculations.

Solution

Working in the matrix form

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Forward Elimination of Unknowns

First step

Divide Row 1 by 20 and then multiply it by −3, that is, multiply Row 1 by $-3/20 = -0.15$.

$([20 \quad 15 \quad 10] \quad [45]) \times -0.15$ gives Row 1 as

$[-3 \quad -2.25 \quad -1.5] \qquad [-6.75]$

Subtract the result from Row 2

$$\begin{array}{rrrrr} & [-3 & -2.249 & 7] & [1.751] \\ - & [-3 & -2.25 & -1.5] & [-6.75] \\ \hline & 0 & 0.001 & 8.5 & 8.501 \end{array}$$

to get the resulting equations as

$$\begin{bmatrix} 20 & 15 & 10 \\ 0 & 0.001 & 8.5 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 8.501 \\ 9 \end{bmatrix}$$

Divide Row 1 by 20 and then multiply it by 5, that is, multiply Row 1 by $5/20 = 0.25$

$([20 \quad 15 \quad 10] \quad [45]) \times 0.25$ gives Row 1 as

$[5 \quad 3.75 \quad 2.5] \qquad [11.25]$

Subtract the result from Row 3

$$\begin{array}{rrrrr} & [5 & 1 & 3] & [9] \\ - & [5 & 3.75 & 2.5] & [11.25] \\ \hline & 0 & -2.75 & 0.5 & -2.25 \end{array}$$

to get the resulting equations as

$$\begin{bmatrix} 20 & 15 & 10 \\ 0 & 0.001 & 8.5 \\ 0 & -2.75 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 8.501 \\ -2.25 \end{bmatrix}$$

Second step

Now for the second step of forward elimination, we will use Row 2 as the pivot equation and eliminate Row 3: Column 2.

Divide Row 2 by 0.001 and then multiply it by −2.75, that is, multiply Row 2 by $-2.75/0.001 = -2750$.

$([0 \quad 0.001 \quad 8.5] \quad [8.501]) \times -2750$ gives Row 2 as

$[0 \quad -2.75 \quad -23375] \qquad [-23377.75]$

Rewriting within 6 significant digits with chopping

$[0 \quad -2.75 \quad -23375] \qquad [-23377.7]$

Subtract the result from Row 3

$$\begin{array}{rcccc} & [0 & -2.75 & 0.5] & [-2.25] \\ - & [0 & -2.75 & -23375] & [-23377.7] \\ \hline & 0 & 0 & 23375.5 & 23375.45 \end{array}$$

Rewriting within 6 significant digits with chopping

$$[0 \quad 0 \quad 23375.5] \qquad [-23375.4]$$

to get the resulting equations as

$$\begin{bmatrix} 20 & 15 & 10 \\ 0 & 0.001 & 8.5 \\ 0 & 0 & 23375.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 8.501 \\ 23375.4 \end{bmatrix}$$

This is the end of the forward elimination steps.

Back substitution

We can now solve the above equations by back substitution. From the third equation,

$$23375.5x_3 = 23375.4$$

$$x_3 = \frac{23375.4}{23375.5} \qquad = 0.999995$$

Substituting the value of x_3 in the second equation

$$0.001x_2 + 8.5x_3 = 8.501$$

$$x_2 = \frac{8.501 - 8.5x_3}{0.001} \qquad = \frac{8.501 - 8.5 \times 0.999995}{0.001} \qquad = \frac{8.501 - 8.49995}{0.001} \qquad = \frac{0.00105}{0.001} \qquad = 1.05$$

Substituting the value of x_3 and x_2 in the first equation,

$$20x_1 + 15x_2 + 10x_3 = 45$$

$$x_1 = \frac{45 - 15x_2 - 10x_3}{20} \qquad = \frac{45 - 15 \times 1.05 - 10 \times 0.999995}{20}$$

$$= \frac{45 - 15.75 - 9.99995}{20}$$

$$= \frac{29.25 - 9.99995}{20}$$

$$= \frac{19.2500}{20}$$

$$= 0.9625$$

Hence the solution is

$$[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad = \begin{bmatrix} 0.9625 \\ 1.05 \\ 0.999995 \end{bmatrix}$$

Compare this with the exact solution of

$$[X]=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \quad =\begin{bmatrix}1\\1\\1\end{bmatrix}$$

LU Decomposition

We already studied two numerical methods of finding the solution to simultaneous linear equations – Naïve Gauss elimination and Gaussian elimination with partial pivoting. Then, why do we need to learn another method? To appreciate why LU decomposition could be a better choice than the Gauss elimination techniques in some cases, let us discuss first what LU decomposition is about.
For a nonsingular matrix $[A]$ on which one can successfully conduct the Naïve Gauss elimination forward elimination steps, one can always write it as

$$[A]=[L][U]$$

Where, $[L]$= Lower triangular matrix, $[U]$ = Upper triangular matrix

Then if one is solving a set of equations

$$[A][X]=[C],$$

Then, $[L][U][X]=[C]$ as $([A]=[L][U])$

Multiplying both sides by $[L]^{-1}$,

$$[L]^{-1}[L][U][X]=[L]^{-1}[C]$$

$$[I][U][X]=[L]^{-1}[C] \text{ as } ([L]^{-1}[L]=[I])$$

$$[U][X]=[L]^{-1}[C] \text{ as } ([I][U]=[U])$$

Let

$$[L]^{-1}[C]=[Z]$$

Then, $[L][Z]=[C]$ (1)

And, $[U][X]=[Z]$ (2)

So we can solve Equation (1) first for $[Z]$ by using forward substitution and then use Equation (2) to calculate the solution vector $[X]$ by back substitution.

How do I decompose a non-singular matrix $[A]$, that is, how do I find $[A]=[L][\mathrm{U}]$?

If forward elimination steps of the Naïve Gauss elimination methods can be applied on a nonsingular matrix, then $[A]$ can be decomposed into LU as

$$[A]=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & \dots & 0 \\ \ell_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \ell_{n1} & \ell_{n2} & \cdots & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

The elements of the $[U]$ matrix are exactly the same as the coefficient matrix one obtains at the end of the forward elimination steps in Naïve Gauss elimination.

The lower triangular matrix $[L]$ has 1 in its diagonal entries. The non-zero elements on the non-diagonal elements in $[L]$ are multipliers that made the corresponding entries zero in the upper triangular matrix $[U]$ during forward elimination.

Let us look at this using the same example as used in Naïve Gaussian elimination.

Example 1

Find the LU decomposition of the matrix

$$[A]=\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Solution

$$[A]=[L][U], \qquad =\begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

The $[U]$ matrix is the same as found at the end of the forward elimination of Naïve Gauss elimination method, that is

$$[U]=\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

To find ℓ_{21} and ℓ_{31}, find the multiplier that was used to make the a_{21} and a_{31} elements zero in the first step of forward elimination of the Naïve Gauss elimination method. It was

$$\ell_{21}=\frac{64}{25} \quad =2.56, \quad \ell_{31}=\frac{144}{25} \quad =5.76$$

To find ℓ_{32}, what multiplier was used to make a_{32} element zero? Remember a_{32} element was made zero in the second step of forward elimination. The $[A]$ matrix at the beginning of the second step of forward elimination was

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

So, $\ell_{32}=\dfrac{-16.8}{-4.8} \qquad =3.5$

Hence

$$[L]=\begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}$$

Confirm $[L][U]=[A]$.

$$[L][U]=\begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 0 & 0 & 0.7\end{bmatrix}=\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 144 & 12 & 1\end{bmatrix}$$

Example 2

Use the LU decomposition method to solve the following simultaneous linear equations.

$$\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 144 & 12 & 1\end{bmatrix}\begin{bmatrix}a_1\\ a_2\\ a_3\end{bmatrix}=\begin{bmatrix}106.8\\ 177.2\\ 279.2\end{bmatrix}$$

Solution

Recall that

$$[A][X]=[C]$$

and if, $[A]=[L][U]$

then first solving, $[L][Z]=[C]$

and then, $[U][X]=[Z]$

gives the solution vector $[X]$.

Now in the previous example, we showed

$$[A]=[L][U]\quad=\begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 0 & 0 & 0.7\end{bmatrix}$$

First solve

$$[L][Z]=[C]$$

$$\begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}=\begin{bmatrix}106.8\\ 177.2\\ 279.2\end{bmatrix}$$

to give

$$z_1=106.8$$

$$2.56z_1+z_2=177.2$$

$$5.76z_1+3.5z_2+z_3=279.2$$

Forward substitution starting from the first equation gives

$$z_1 = 106.8$$

$$z_2 = 177.2 - 2.56 z_1 \qquad = 177.2 - 2.56 \times 106.8 \qquad = -96.208$$

$$z_3 = 279.2 - 5.76 z_1 - 3.5 z_2 \qquad = 279.2 - 5.76 \times 106.8 - 3.5 \times (-96.208) \qquad = 0.76$$

Hence

$$[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \qquad = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

This matrix is same as the right hand side obtained at the end of the forward elimination steps of Naïve Gauss elimination method. Is this a coincidence?

Now solve

$$[U][X] = [Z]$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$-4.8a_2 - 1.56a_3 = -96.208$$

$$0.7a_3 = 0.76$$

From the third equation

$$0.7a_3 = 0.76$$

$$a_3 = \frac{0.76}{0.7} \qquad = 1.0857$$

Substituting the value of a_3 in the second equation,

$$-4.8a_2 - 1.56a_3 = -96.208$$

$$a_2 = \frac{-96.208 + 1.56a_3}{-4.8} \qquad = \frac{-96.208 + 1.56 \times 1.0857}{-4.8} \qquad = 19.691$$

Substituting the value of a_2 and a_3 in the first equation,

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25} \qquad = \frac{106.8 - 5 \times 19.691 - 1.0857}{25} \qquad = 0.29048$$

Hence the solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.29048 \\ 19.691 \\ 1.0857 \end{bmatrix}$$

How do I find the inverse of a square matrix using LU decomposition?

A matrix $[B]$ is the inverse of $[A]$ if

$$[A][B] = [I] = [B][A].$$

How can we use LU decomposition to find the inverse of the matrix? Assume the first column of $[B]$ (the inverse of $[A]$) is

$$[b_{11}\, b_{12} \ldots \ldots\, b_{n1}]^{\mathrm{T}}$$

Then from the above definition of an inverse and the definition of matrix multiplication

$$[A]\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Similarly the second column of $[B]$ is given by

$$[A]\begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Similarly, all columns of $[B]$ can be found by solving n different sets of equations with the column of the right hand side being the n columns of the identity matrix.

Example 3

Use LU decomposition to find the inverse of

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Solution

Knowing that

$$[A] = [L][U] \quad = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

We can solve for the first column of $[B] = [A]^{-1}$ by solving for

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

First solve, $[L][Z] = [C]$,

that is

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

to give

$$z_1 = 1$$

$$2.56z_1 + z_2 = 0$$

$$5.76z_1 + 3.5z_2 + z_3 = 0$$

Forward substitution starting from the first equation gives

$$z_1 = 1$$

$$z_2 = 0 - 2.56z_1 \qquad = 0 - 2.56(1) \qquad = -2.56$$

$$z_3 = 0 - 5.76z_1 - 3.5z_2 \qquad = 0 - 5.76(1) - 3.5(-2.56) \qquad = 3.2$$

Hence

$$[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \qquad = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

Now solve

$$[U][X] = [Z]$$

that is

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

$$25b_{11} + 5b_{21} + b_{31} = 1$$

$$-4.8b_{21} - 1.56b_{31} = -2.56$$

$$0.7b_{31} = 3.2$$

Backward substitution starting from the third equation gives

$$b_{31} = \frac{3.2}{0.7} \qquad = 4.571$$

$$b_{21} = \frac{-2.56 + 1.56b_{31}}{-4.8} \qquad = \frac{-2.56 + 1.56(4.571)}{-4.8} \qquad = -0.9524$$

$$b_{11} = \frac{1 - 5b_{21} - b_{31}}{25} \qquad = \frac{1 - 5(-0.9524) - 4.571}{25} \qquad = 0.04762$$

Hence the first column of the inverse of $[A]$ is

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

Similarly by solving

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ gives } \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

and solving

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ gives } \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

Hence

$$[A]^{-1} = \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$

Can you confirm the following for the above example?

$$[A][A]^{-1} = [I] = [A]^{-1}[A]$$

Gauss-Seidel Method

Why do we need another method to solve a set of simultaneous linear equations?

In certain cases, such as when a system of equations is large, iterative methods of solving equations are more advantageous. Elimination methods, such as Gaussian elimination, are prone to large round-off errors for a large set of equations. Iterative methods, such as the Gauss-Seidel method, give the user control of the round-off error. Also, if the physics of the problem are well known, initial guesses needed in iterative methods can be made more judiciously leading to faster convergence.

What is the algorithm for the Gauss-Seidel method? Given a general set of n equations and n unknowns, we have

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = c_2$$

$$. \qquad\qquad .$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = c_n$$

If the diagonal elements are non-zero, each equation is rewritten for the corresponding unknown, that is, the first equation is rewritten with x_1 on the left hand side, the second equation is rewritten with x_2 on the left hand side and so on as follows

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}}$$

$$\vdots$$

$$\vdots$$

$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$

These equations can be rewritten in a summation form as

$$x_1 = \frac{c_1 - \sum_{\substack{j=1 \\ j\neq 1}}^{n} a_{1j}x_j}{a_{11}}$$

$$x_2 = \frac{c_2 - \sum_{\substack{j=1 \\ j\neq 2}}^{n} a_{2j}x_j}{a_{22}}$$

.

$$x_{n-1} = \frac{c_{n-1} - \sum_{\substack{j=1 \\ j\neq n-1}}^{n} a_{n-1,j}x_j}{a_{n-1,n-1}}$$

$$x_n = \frac{c_n - \sum_{\substack{j=1 \\ j\neq n}}^{n} a_{nj}x_j}{a_{nn}}$$

Hence for any row i,

$$x_i = \frac{c_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}x_j}{a_{ii}}, i = 1,2,\ldots,n.$$

Now to find x_i's, one assumes an initial guess for the x_i's and then uses the rewritten equations to calculate the new estimates. Remember, one always uses the most recent estimates to calculate the next estimates, x_i. At the end of each iteration, one calculates the absolute relative approximate error for each x_i as

$$\left|\in_a\right|_i = \left|\frac{x_i^{\text{new}} - x_i^{\text{old}}}{x_i^{\text{new}}}\right| \times 100$$

where x_i^{new} is the recently obtained value of x_i, and x_i^{old} is the previous value of x_i.

When the absolute relative approximate error for each x_i is less than the pre-specified tolerance, the iterations are stopped.

Example 1

The upward velocity of a rocket is given at three different times in the following table

Table 1 Velocity vs. time data.

Time, t (s)	Velocity, v (m/s)
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as

$$v(t) = a_1 t^2 + a_2 t + a_3, \qquad 5 \le t \le 12$$

Find the values of a_1, a_2, and a_3 using the Gauss-Seidel method. Assume an initial guess of the solution as

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$ and conduct two iterations.

Solution

The polynomial is going through three data points $(t_1, v_1), (t_2, v_2)$, and (t_3, v_3) where from the above table

$$t_1 = 5, \quad v_1 = 106.8$$

$$t_2 = 8, \quad v_2 = 177.2$$

$$t_3 = 12, \quad v_3 = 279.2$$

Requiring that $v(t) = a_1 t^2 + a_2 t + a_3$ passes through the three data points gives

$$v(t_1) = v_1 = a_1 t_1^2 + a_2 t_1 + a_3$$

$$v(t_2) = v_2 = a_1 t_2^2 + a_2 t_2 + a_3$$

$$v(t_3) = v_3 = a_1 t_3^2 + a_2 t_3 + a_3$$

Substituting the data $(t_1, v_1), (t_2, v_2)$, and (t_3, v_3) gives

$$a_1(5^2) + a_2(5) + a_3 = 106.8$$

$$a_1(8^2) + a_2(8) + a_3 = 177.2$$

$$a_1(12^2) + a_2(12) + a_3 = 279.2$$

or

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$64a_1 + 8a_2 + a_3 = 177.2$$

$$144a_1 + 12a_2 + a_3 = 279.2$$

The coefficients $a_1, a_2, \text{and } a_3$ for the above expression are given by

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Rewriting the equations gives

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25}$$

$$a_2 = \frac{177.2 - 64a_1 - a_3}{8}$$

$$a_3 = \frac{279.2 - 144a_1 - 12a_2}{1}$$

Iteration #1

Given the initial guess of the solution vector as

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

we get

$$a_1 = \frac{106.8 - 5(2) - (5)}{25} \qquad = 3.6720$$

$$a_2 = \frac{177.2 - 64(3.6720) - (5)}{8} \qquad = -7.8150$$

$$a_3 = \frac{279.2 - 144(3.6720) - 12(-7.8510)}{1} \qquad = -155.36$$

The absolute relative approximate error for each x_i then is

$$|\epsilon_a|_1 = \left|\frac{3.6720 - 1}{3.6720}\right| \times 100 \qquad = 72.76\%$$

$$|\epsilon_a|_2 = \left|\frac{-7.8510 - 2}{-7.8510}\right| \times 100 \qquad = 125.47\%$$

$$|\epsilon_a|_3 = \left|\frac{-155.36 - 5}{-155.36}\right| \times 100 \qquad = 103.22\%$$

At the end of the first iteration, the estimate of the solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3.6720 \\ -7.8510 \\ -155.36 \end{bmatrix}$$

and the maximum absolute relative approximate error is 125.47%.

Iteration #2

The estimate of the solution vector at the end of Iteration #1 is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3.6720 \\ -7.8510 \\ -155.36 \end{bmatrix}$$

Now we get

$$a_1 = \frac{106.8 - 5(-7.8510) - (-155.36)}{25} \qquad = 12.056$$

$$a_2 = \frac{177.2 - 64(12.056) - (-155.36)}{8} \qquad = -54.882$$

$$a_3 = \frac{279.2 - 144(12.056) - 12(-54.882)}{1} \qquad = -798.34$$

The absolute relative approximate error for each x_i then is

$$\left|\in_a\right|_1 = \left|\frac{12.056 - 3.6720}{12.056}\right| \times 100 \qquad = 69.543\%$$

$$\left|\in_a\right|_2 = \left|\frac{-54.882 - (-7.8510)}{-54.882}\right| \times 100 \qquad = 85.695\%$$

$$\left|\in_a\right|_3 = \left|\frac{-798.34 - (-155.36)}{-798.34}\right| \times 100 \qquad = 80.540\%$$

At the end of the second iteration the estimate of the solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 12.056 \\ -54.882 \\ -798.54 \end{bmatrix}$$

and the maximum absolute relative approximate error is 85.695%.

Conducting more iterations gives the following values for the solution vector and the corresponding absolute relative approximate errors.

Iteration	a_1	$\left\|\in_a\right\|_1 \%$	a_2	$\left\|\in_a\right\|_2 \%$	a_3	$\left\|\in_a\right\|_3 \%$
1	3.6720	72.767	−7.8510	125.47	−155.36	103.22
2	12.056	69.543	−54.882	85.695	−798.34	80.540
3	47.182	74.447	−255.51	78.521	−3448.9	76.852
4	193.33	75.595	−1093.4	76.632	−14440	76.116
5	800.53	75.850	−4577.2	76.112	−60072	75.963
6	3322.6	75.906	−19049	75.972	− 249580	75.931

As seen in the above table, the solution estimates are not converging to the true solution of

$a_1 = 0.29048$, $a_2 = 19.690$, and $a_3 = 1.0857$

The above system of equations does not seem to converge. Why?

Well, a pitfall of most iterative methods is that they may or may not converge. However, the solution to a certain classes of systems of simultaneous equations does always converge using the Gauss-Seidel method. This class of system of equations is where the coefficient matrix $[A]$ in $[A][X]=[C]$ is diagonally dominant, that is

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \text{ for all } i$$

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \text{ for at least one } i$$

If a system of equations has a coefficient matrix that is not diagonally dominant, it may or may not converge. Fortunately, many physical systems that result in simultaneous linear equations have a diagonally dominant coefficient matrix, which then assures convergence for iterative methods such as the Gauss-Seidel method of solving simultaneous linear equations.

Example 2

Find the solution to the following system of equations using the Gauss-Seidel method.

$$12x_1 + 3x_2 - 5x_3 = 1$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$3x_1 + 7x_2 + 13x_3 = 76$$

Use

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

as the initial guess and conduct two iterations.

Solution

The coefficient matrix

$$[A] = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

is diagonally dominant as

$$|a_{11}| = |12| = 12 \geq |a_{12}| + |a_{13}| = |3| + |-5| = 8$$

$$|a_{22}| = |5| = 5 \geq |a_{21}| + |a_{23}| = |1| + |3| = 4$$

$$|a_{33}| = |13| = 13 \geq |a_{31}| + |a_{32}| = |3| + |7| = 10$$

and the inequality is strictly greater than for at least one row. Hence, the solution should converge using the Gauss-Seidel method.

Rewriting the equations, we get

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}$$

Assuming an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Iteration #1

$$x_1 = \frac{1 - 3(0) + 5(1)}{12} \quad = 0.50000$$

$$x_2 = \frac{28 - (0.50000) - 3(1)}{5} \quad = 4.9000$$

$$x_3 = \frac{76 - 3(0.50000) - 7(4.9000)}{13} \quad = 3.0923$$

The absolute relative approximate error at the end of the first iteration is

$$|\in_a|_1 = \left|\frac{0.50000 - 1}{0.50000}\right| \times 100 \quad = 100.00\%$$

$$|\in_a|_2 = \left|\frac{4.9000 - 0}{4.9000}\right| \times 100 \quad = 100.00\%$$

$$|\in_a|_3 = \left|\frac{3.0923 - 1}{3.0923}\right| \times 100 \quad = 67.662\%$$

The maximum absolute relative approximate error is 100.00%

Iteration #2

$$x_1 = \frac{1 - 3(4.9000) + 5(3.0923)}{12} \quad = 0.14679$$

$$x_2 = \frac{28 - (0.14679) - 3(3.0923)}{5} \quad = 3.7153$$

$$x_3 = \frac{76 - 3(0.14679) - 7(3.7153)}{13} \quad = 3.8118$$

At the end of second iteration, the absolute relative approximate error is

$$\left|\in_a\right|_1 = \left|\frac{0.14679 - 0.50000}{0.14679}\right| \times 100 \quad = 240.61\%$$

$$\left|\in_a\right|_2 = \left|\frac{3.7153 - 4.9000}{3.7153}\right| \times 100 \quad = 31.889\%$$

$$\left|\in_a\right|_3 = \left|\frac{3.8118 - 3.0923}{3.8118}\right| \times 100 \quad = 18.874\%$$

The maximum absolute relative approximate error is 240.61%. This is greater than the value of 100.00% we obtained in the first iteration. Is the solution diverging? No, as you conduct more iterations, the solution converges as follows.

Iteration	x_1	$\left\|\in_a\right\|_1\%$	x_2	$\left\|\in_a\right\|_2\%$	x_3	$\left\|\in_a\right\|_3\%$
1	0.50000	100.00	4.9000	100.00	3.0923	67.662
2	0.14679	240.61	3.7153	31.889	3.8118	18.874
3	0.74275	80.236	3.1644	17.408	3.9708	4.0064
4	0.94675	21.546	3.0281	4.4996	3.9971	0.65772
5	0.99177	4.5391	3.0034	0.82499	4.0001	0.074383
6	0.99919	0.74307	3.0001	0.10856	4.0001	0.00101

This is close to the exact solution vector of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Experiment:

1. Study the materials
2. Solve simultaneous linear equations by using Gauss elimination, LU decomposition, and Gauss-Seidel method (For any number of equations).

Report:

i. Perform the manual calculation of the supplied problem (in the class) by using different methods as well as perform by using Matlab/C/C++