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Course No: EE 3122
Sessional on Numerical Methods & Statistics

Experiment No. 4

Name of the Experiment: Numerical differentiation and integration

Objectives:

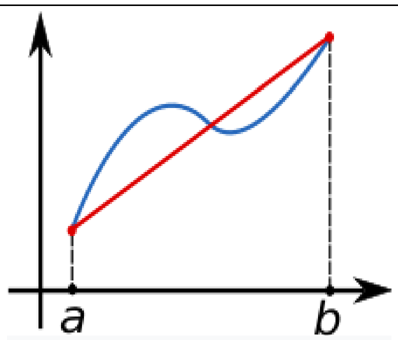
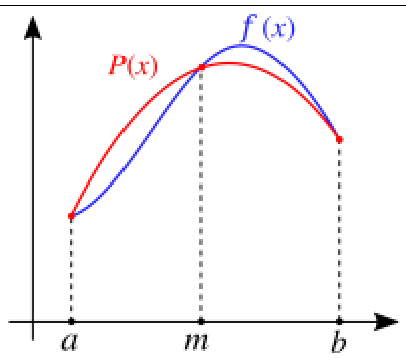
[1] To perform numerical integration using Trapezoidal rule
[2] To perform numerical integration using Simpson's 1/3 and Simpson's 3/8 rule
[3] To perform differentiation by Newton's interpolation formula
[4] To perform differentiation by forward/ backward/ central divided difference formula

Theory/Introduction:

What is integration?

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus.

Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral.
Here, we will discuss the trapezoidal rule / Simpson's rule of approximating integrals of the form

$I = \int_a^b f(x)dx$ where $f(x)$ is called the integrand, $a =$ lower limit of integration $b =$ upper limit of integration	 The function $f(x)$ (in blue) is approximated by a linear function (in red).	 Simpson's rule can be derived by approximating the integrand $f(x)$ (in blue) by the quadratic interpolant $P(x)$ (in red).

For reference, you can see any numerical textbook

Trapezoidal rule:

The integral of the single-segment trapezoidal rule is given by

$$\int_a^b f(x)dx \approx (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

To dividing $[a,b]$ into n equal segments and applying the trapezoidal rule over each segment, the sum of the results obtained for each segment is the approximate value of the integral. Divide $(b-a)$ into n equal segments as shown in Figure 3. Then the width of each segment is

$$h=\frac{b-a}{n}$$

The integral I can be broken into h integrals as

$$I=\int_a^b f(x)dx=\int_a^{a+h} f(x)dx+\int_{a+h}^{a+2h} f(x)dx+\ldots+\int_{a+(n-2)h}^{a+(n-1)h} f(x)dx+\int_{a+(n-1)h}^{b} f(x)dx$$

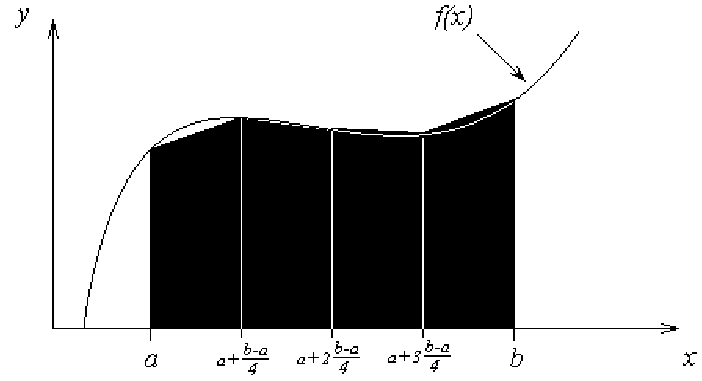


Figure 3 Multiple ($n=4$) segment trapezoidal rule

Applying trapezoidal rule on each segment gives the integral of the function-

$$\int_a^b f(x)dx=\frac{h}{2}\left[f(a)+2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\}+f(b)\right]=\frac{b-a}{2n}\left[f(a)+2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\}+f(b)\right]$$

Error:

The true error for a single segment Trapezoidal rule is given by

$$E_t=-\frac{(b-a)^3}{12}f''(\zeta),\quad a<\zeta<b \quad \text{where } \zeta \text{ is some point in } [a,b].$$

The total error in the multiple-segment trapezoidal rule is

$$E_t=-\frac{(b-a)^3}{12n^2}\frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$$

Simpson's rule:

For Simpson 1/3 rule, the interval $[a,b]$ is broken into 2 segments, the segment width

$$h = \frac{b-a}{2}$$

The Simpson's 1/3 rule is given by

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] = \frac{h}{3}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

Since the above form has 1/3 in its formula, it is called Simpson's 1/3 rule.

Multiple segment Simpson's rule: Just like in multiple-segment trapezoidal rule, one can subdivide the interval $[a,b]$ into n segments or strips and apply Simpson's 1/3 rule repeatedly over every two segments. Note that n needs to be even. Divide interval $[a,b]$ into n equal segments, so that the segment width is given by

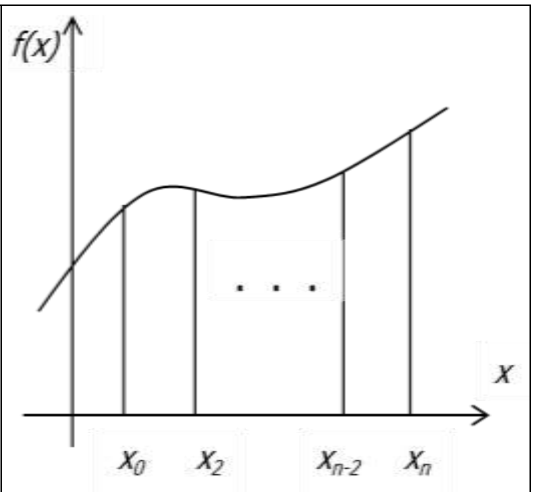
$$h = \frac{b-a}{n}.$$

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots..$$

$$\ldots. + \int_{x_{n-4}}^{x_{n-2}} f(x)dx + \int_{x_{n-2}}^{x_n} f(x)dx$$

$$\int_a^b f(x)dx = \frac{h}{3}[f(x_0)+4\{f(x_1)+f(x_3)+\ldots+f(x_{n-1})\}+\ldots]$$

$$\ldots+2\{f(x_2)+f(x_4)+\ldots+f(x_{n-2})\}+f(x_n)\}]$$



Therefore Simpson's 1/3 rule over the entire interval is given by-

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(x_0)+4\sum_{\substack{i=1\\ i=odd}}^{n-1} f(x_i)+2\sum_{\substack{i=2\\ i=even}}^{n-2} f(x_i)+f(x_n)\right] = \int_a^b f(x)dx = \frac{b-a}{3n}\left[f(a)+4\sum_{\substack{i=1\\ i=odd}}^{n-1} f(x_i)+2\sum_{\substack{i=2\\ i=even}}^{n-2} f(x_i)+f(b)\right]$$

Error: the total error in Multiple Segment Simpson's 1/3rd Rule is

$$E_t = -\frac{(b-a)^5}{90n^4}\,\frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n} = -\frac{h^5}{90} f^{(4)}(\varsigma_i)\text{, where } \xi \in [\text{a, b}]$$

Matlab/C/C++ Function Helpline

```
function I = Trap (f, a, b, n)
%%find integral using multiple-segment trapezoidal rule%%
%% f=function
%% a= lowest limit
%% b= highest limit
h = (b-a)/n;
S = feval(f, a);
for i=1: n-1
     x(i)= a + h*i
     S = S + 2 * feval (f, x(i))
end
S=S + feval (f, b);
I=h*S/2
```

Algorithms for evaluating Simpson's rule

```
function I = Simp (f, a, b, n)
%%find integral using multiple-segment Simpson's rule%%
%% n must be even %%
Evaluate h
Evaluate function for smallest value a
Find sum for odd functions
Find sum for even functions
Evaluate function for the highest value b
Evaluate integral
```

Performance:

Problem1: Use 4-segment Simpson's 1/3rd Rule to approximate the distance covered by a rocket from t= 8 to t=30 as given by

$$x = \int_{8}^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

a) Use *n*=2, 4, …, 20 segment Simpson's 1/3rd Rule and Trapezoidal rule to find the approximate value of x.

b) Plot segment vs. absolute error, where the error is given by

$$\left| \in_t \right| = \left| \frac{ExactIntegral - ApproxmiateIntegral}{ExactIntegral} \right| \times 100\%$$

Problem2: Using Simpson's rule, compute the integral $\int_a^b f(x)dx$. Also get an error estimate of the computed integral.

x	a = 0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	b=5.5
y	0.493	0.946	1.325	1.605	1.778	1.849	1.833	1.925	2.056	2.312	2.215

Problem3: *Using Trapezoidal rule compute the integral* $\int_0^1 e^{x^2} dx$ *where the values of* $y = e^{x^2}$ *is given table below:*

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	1.00000	1.01005	1.04081	1.09417	1.17351	1.28402	1.43332	1.63231	1.89648	2.2479	2.71828