June 19

Numerical Analysis

2012

The aim of this assignment is to compare and analyze the behavior of different numerical methods studied in class for solving Simultaneous Linear Equations (Gaussian elimination with and without pivoting, LU Decomposition, the Jacobi and Gauss-Seidel Methods)

Solving linear system of equations

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1- Problem statement

The aim of this assignment is to compare and analyze the behavior of different numerical methods studied in class for solving Simultaneous Linear Equations (Gaussian elimination with and without pivoting, LU Decomposition, the Jacobi and Gauss-Seidel Methods)

Required

- Implementation of the above methods
- Analysis of performance for each method
- Analysis of solved examples using each method

2- Pseudo Code

a- Forward elimination pseudo code

```
Input: augmented matrix A, boolean pivoting
Output: vector x

For k = 1 ... m:
    if pivoting = true
        //Find pivot for column k
        r_max := k
        for r := k+1 …. m do
            if abs(A[r,c]) > abs(A[r_max,c]) then
                r_max := r
            end if
        end for
        //swap rows
        swap rows(k, r_max)
    end if

    if A[k, k] = 0
        error "Matrix is singular!"

    //Do for all rows below pivot:
    for r = k + 1 ... m:
        //Do for all remaining elements in current row:
        for c = k ... n:
        A[r, c] :=
            A[r, c] - A[k, c] * (A[r, k] / A[k, k])
end for
```

b- back substitution pseudo code

```
Input : coefficient  matrix A , right hand side vector b
Output : solution vector x

nrow := size(A)

x[nrow] := b[nrow] / A[nrow, nrow]

for i = nrow - 1  to  1
```

$$x[i] := \frac{b[i] - \sum_{k=i+1}^{nrow} x[k]*A[i,k]}{A[i,i]}$$

```
end for
```

c- forward substitution pseudo code

```
Input : coefficient matrix A , right hand side vector b
Output : solution vector x

nrow := size(A)

for i = 1  to  nrow
```

$$x[i] := \frac{b[i] - \sum_{k=1}^{i+1} x[k]*A[i,k]}{A[i,i]}$$

```
end for
```

d- LUdecomposition pseudo code

```
Input : matrix A
Output : lower matrix L , upper matrix U

//initialize L and U to zeros:
for r = 1…m
    for c = 1….n
        U[r, c] = 0
        if r = c
            L[r, c] = 1
        else
            L[r, c] = 0
    end for
end for
for k = 1 ... m
    //Find pivot for column k
    r_max := k
    for r := k+1 …. m do
        if abs(A[r,c]) > abs(A[r_max,c]) then
            r_max := r
        end if
    end for
    swap rows(k, r_max)
    if A[k, k] = 0
        error "Matrix is singular!"
    //Do for all rows below pivot:
    for r = k + 1 ... m:
        //set the U matrix
        L[r,k] := A[r, k] / A[k, k]
        //Do for all remaining elements in current row
        for c = k ... n:
            U[r, c] :=
                 A[r, c] - A[k, c] * (A[r, k] / A[k, k])
```

e- LU solve pseudo code

```
Input : matrix A
output : solution vector x

L,U = LUdecomposition(A)
Z  = forward substitution(L, C)
X  = backward substitution(U, Z)
```

f- Gauss Seidel pseudo code

```
Input: matrix A , int MaxIteration , int maxError
Output: solution vector x

//Rearranging matrix A to be diagonally dominant:
Diagonal = rearrange(A)

If(diagonal = false)
    Error "not diagonally dominant"

While( itr < MaxIteration)
    For i = 1  to  nrow
        err = 0
```

$$newX[i] := \frac{b[i] - \sum_{k=1}^{i+1} x[k]*A[i,k]}{A[i,i]}$$

```
        err = max(err , (newX[i]-x[i]) / x[i])
        newX[i] = x[i]
    end for
    if(error < maxError)
        return
    end if
end while
```

g- Jacobi pseudo code

```
Input: matrix A , int MaxIteration , int maxError
Output: solution vector x

//Rearranging matrix A to be diagonally dominant:
Diagonal = rearrange(A)
If(diagonal = false)
    Error "not diagonally dominant"

While( itr < MaxIteration)
    For i = 1  to  nrow
        err = 0
```

$$newX[i] := \frac{b[i] - \sum_{k=1}^{i+1} x[k]*A[i,k]}{A[i,i]}$$

```
        err = max(err , (newX[i]-x[i]) / x[i])
    end for

    x[1:n]  = newX[1:n]

    if(error < maxError)
        return
    end if
end while
```

3- Analysis

All the analysis done in double precision
R = real solution , C = calculated solution

a- Ex1

X1	X2	X3	B
0	-7	0	7
-3	2.099	6	3.901
5	-1	5	6

R
0
-1
1

Method	X		Error = $\frac{\lvert R-C \rvert}{R}$	Time ms
Gaussian Elimination	X1	Can't be solved		
	X2			
	X3			
Pivoting	X1	0.0	0%	0.029434
	X2	-1.00000000000002	≈ 0%	
	X3	1.0	0%	
LU decomposition	X1	0.0	0%	0.878948
	X2	-1.00000000000002	≈ 0%	
	X3	1.0	0%	
Jacobi	X1	∞		0.028981
	X2	∞		
	X3	-∞		
	after 3 iterations , X0 = [1 , 0 , 0]T , accuracy = 0.01			
Gauss-seidel	X1	∞		0.017208
	X2	∞		
	X3	NaN		
	After 2 iterations , X0 = [1 , 0 , 0]T , accuracy = 0.01			

LU inverse	-0.052365079	-0.1111111112	0.1333333336
	-0.1428571428	0.0	0.0
	0.02379365079	0.11111111112	0.0666666667
	executed in 0.878948 ms		

b- Ex2

X1	X2	X3	B
3	7	13	76
1	5	3	28
12	3	-5	1

R
1
3
4

Method	X		Error = $\frac{\lvert R-C \rvert}{R}$	Time ms
Gaussian Elimination	X1	1.0	0%	0.017208
	X2	2.9999999999999996	$1.3 * 10^{-9}$ %	
	X3	4.0	0%	
Pivoting	X1	0.999999999999994	$6 * 10^{-9}$ %	0.029434
	X2	3.000000000000013	$4.3 * 10^{-9}$ %	
	X3	3.9999999999999996	$1 * 10^{-8}$ %	
LU decomposition	X1	1.0	0%	0.187473
	X2	3.0	0%	
	X3	4.0	0%	
Jacobi	X1	3.101564760026298	210.156%	0.037132
	X2	6.977100591715977	132.57%	
	X3	-4.588934911242604	214.72%	
	after 5 iterations , X0 = [1 , 0 , 0]T , accuracy = 0.3			
Gauss-seidel	X1	2.6460442691211923	164.6%	0.026717
	X2	7.82787201402586	160.929%	
	X3	-4.748710531195739	218.7%	
	after 4 iterations , X0 = [1 , 0 , 0]T , accuracy = 0.3			

LU inverse	0.06115107913	-0.13309352517	0.07913669064
	-0.07374100719	0.307553956834	-0.00719424460
	0.102517985611	-0.13489208633	-0.01438848920
	executed in 0.187473 ms		

c- Ex3

X1	X2	X3	B
12	3	-5	1
1	5	3	28
3	7	13	76

R
1
3
4

Method	X		Error = $\frac{\lvert R-C \rvert}{R}$	Time ms
Gaussian Elimination	X1	0.999999999999994	$6 * 10^{-9}$ %	0.016754
	X2	3.000000000000013	$4.3 * 10^{-9}$ %	
	X3	3.999999999999999	$2.5 * 10^{-10}$ %	
Pivoting	X1	0.999999999999994	$6 * 10^{-9}$ %	0.016755
	X2	3.000000000000013	$4.3 * 10^{-9}$ %	
	X3	3.999999999999996	$1 * 10^{-7}$ %	
LU decomposition	X1	1.0	0%	0.16936
	X2	3.0	0%	
	X3	4.0	0%	
Jacobi	X1	1.010834976988823	1.083%	0.039397
	X2	2.8069822485207103	6.433%	
	X3	3.7172781065088754	7.06%	
	after 5 iterations , X0 = [1 , 0 , 0]T , accuracy = 0.3			
Gauss-seidel	X1	0.9318101976530846	6.81%	0.032604
	X2	3.0356901022789216	1.189%	
	X3	3.9965183608529458	0.087%	
	after 5 iterations , X0 = [1 , 0 , 0]T , accuracy = 0.3			

LU inverse	0.07913669064	-0.13309352517	0.06115107913
	-0.0071942446	0.307553956834	-0.0737410071
	-0.01438848920	-0.13489208633	0.1025179856
	executed in 0.16936 ms		

d- Ex4

X1	X2	X3	B
1	1	1	3
2	3	4	9
1	7	1	9

R
1
1
1

Method	X		Error = $\frac{\|R-C\|}{R}$	Time ms
Gaussian Elimination	X1	1.0	0%	0.016755
	X2	1.0	0%	
	X3	1.0	0%	
Pivoting	X1	1.0	0%	0.029434
	X2	1.0	0%	
	X3	1.0	0%	
LU decomposition	X1	1.0	0%	0.128152
	X2	1.0	0%	
	X3	1.0	0%	
Jacobi	X1	-∞		1.945823
	X2	-∞		
	X3	-∞		
	after 561 iterations , X0 = [1 , 0 , 0]T , accuracy = 0.01			
Gauss-seidel	X1	NaN		0.704155
	X2	NaN		
	X3	NaN		
	after 391 iterations , X0 = [1 , 0 , 0]T , accuracy = 0.01			

LU inverse	2.08333333335	-0.5	-0.083333331
	-0.1666666669	0.0	0.166666666
	-0.9166666667	0.5	-0.083333334
	executed in 0.128152 ms		

e- Ex5

X1	X2	X3	B
5	2	3	3
2	5	3	10
2	3	5	11

R
$-\frac{61}{60}$ = -1.0167
$\frac{79}{60}$ = 1.3167
$\frac{109}{60}$ = 1.8167

Method	X		Error = $\frac{\lvert R-C \rvert}{R}$	Time ms
Gaussian Elimination	X1	-1.016666666666667	0%	0.010868
	X2	1.3166666666666669	0%	
	X3	1.8166666666666669	0%	
Pivoting	X1	-1.016666666666667	0%	0.020378
	X2	1.3166666666666669	0%	
	X3	1.8166666666666669	0%	
LU decomposition	X1	-1.016666666666668	0%	0.129963
	X2	1.316666666666669	0%	
	X3	1.8166666666666667	0%	
Jacobi	X1	Out of memory		
	X2	Out of memory		
	X3	Out of memory		
	after … iterations , X0 = [1 , 0 , 0]T , accuracy = 0.01			
Gauss-seidel	X1	-1.0183261229154303	0.16%	0.009962
	X2	1.3292976988264855	0.959%	
	X3	1.8097518298702808	0.38%	
	after 7 iterations , X0 = [1 , 0 , 0]T , accuracy = 0.01			

LU inverse	0.26666666667	-0.016666666663	-0.15
	-0.066666666667	0.31666666665	-0.15
	-0.066666666667	-0.18333333338	0.350000003
	executed in 0.129963 ms		

f- Ex6

X1	X2	X3	B
6	2	3	5
2	5	3	10
2	3	5	11

R
$-\frac{61}{76}$ = -0.8
$\frac{24}{19}$ = 1.26
$\frac{67}{38}$ = 1.76

Method	X		Error = $\frac{\lvert R-C\rvert}{R}$	Time ms
Gaussian Elimination	X1	-0.8026315789473685	0%	0.016755
	X2	1.2631578947368423	0%	
	X3	1.763157894736842	0%	
Pivoting	X1	-0.8026315789473685	0%	0.020831
	X2	1.2631578947368423	0%	
	X3	1.763157894736842	0%	
LU decomposition	X1	-0.8026315789473685	0%	0.169812
	X2	1.2631578947368423	0%	
	X3	1.763157894736842	0%	
Jacobi	X1	-0.6896163396177828	14%	0.025812
	X2	1.3920821814269844	10.2%	
	X3	1.8920819971987175	7.3%	
	after 30 iterations , X0 = [1 , 0 , 0]T , accuracy = 0.2			
Gauss-seidel	X1	-0.802033777777778	0.07%	0.004528
	X2	1.3402183111111111	6.1%	
	X3	1.7166825244444446	2.6%	
	after 5 iterations , X0 = [1 , 0 , 0]T , accuracy = 0.2			

LU inverse	0.21052631578	-0.0131578947	-0.1184210526
	-0.0526315789	0.31578947368	-0.1578947368
	-0.0526315789	-0.1842105263	0.34210526315
	executed in 0.169812 ms		

4- Conclusions from analysis

1- From ex1
   a. Partial pivoting avoids division by zero

2- From ex2 to ex6
   a. Partial pivoting decreases round off error
   b. Partial pivoting solution nearly equal to forward elimination solution
   c. Forward elimination executes faster than partial pivoting

3- From ex1 to ex6
   a. LU solves all the matrices
   b. All matrices as notice have AA-1 = 1 property and we can get A-1 by using LU decomposition as shown in the examples
   c. LU is slower than forward elimination and partial pivoting but more accurate than both

4- From ex1, ex2, ex4
   a. Gauss and Jacobi failed to converge because the matrices are not strictly diagonal matrix – SDD –
      - SDD for Matrix A means this :

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \,!= i}}^{n} |a_{ij}| \quad \& \quad |a_{ii}| > \sum_{\substack{j=1 \\ j \,!= i}}^{n} |a_{ij}| \ \ for\ at\ least\ one$$

5- From ex3
   a. The matrix is in SDD format so Gauss and Jacobi converge to true solution with reasonable error as

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \,!= i}}^{n} |a_{ij}| \quad for\ all\ i$$

6- From ex5

   a. When the matrix is DD only

      - Jacobi overflows the memory and terminates with no solution
      - Gauss converge with reasonable error
      - Diagonal Dominant matrix DD means that

$$|a_{ii}| = \sum_{\substack{j=1 \\ j \,!= i}}^{n} |a_{ij}| \quad for\ all\ i$$

7- From ex6

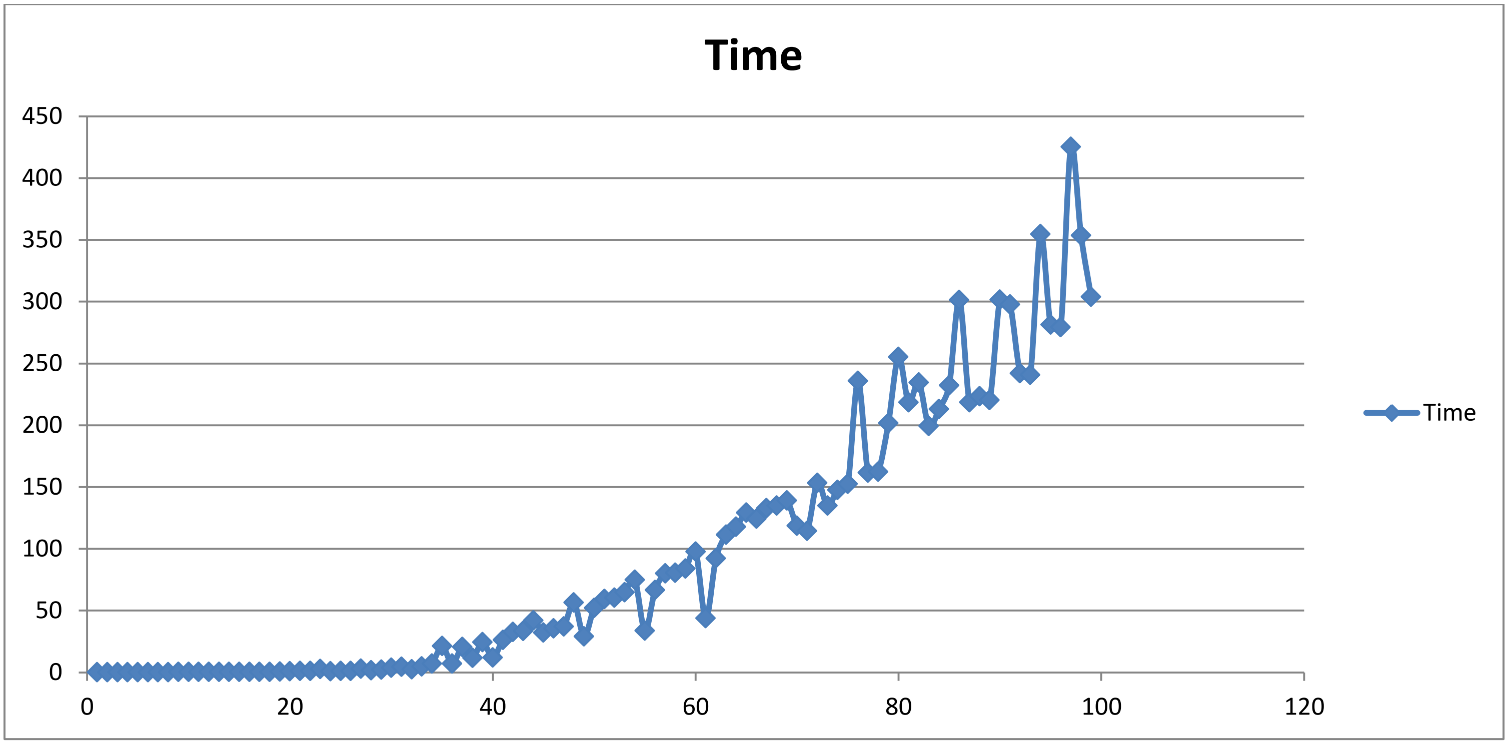
   a. When the matrix SDD Gauss and Jacobi converges to true solution with reasonable error

8- So for Gauss and Jacobi when matrix SDD then Gauss and Jacobi tends to converge but if the matric DD then Gauss and Jacobi probably converge or not while if matrix not SDD then Gauss and Jacobi doesn’t converge at all

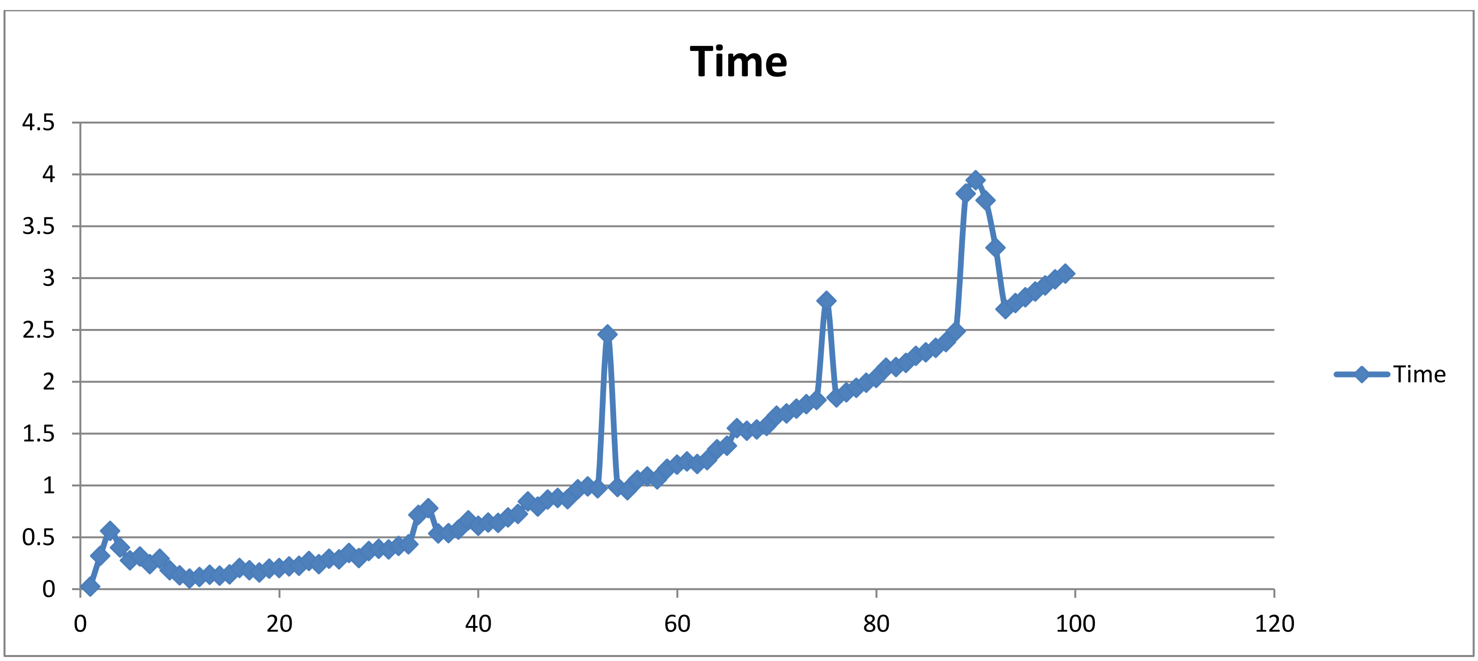
5- Each method analysis

No. of Equations Vs Time (ms)

Gaussian Elimination

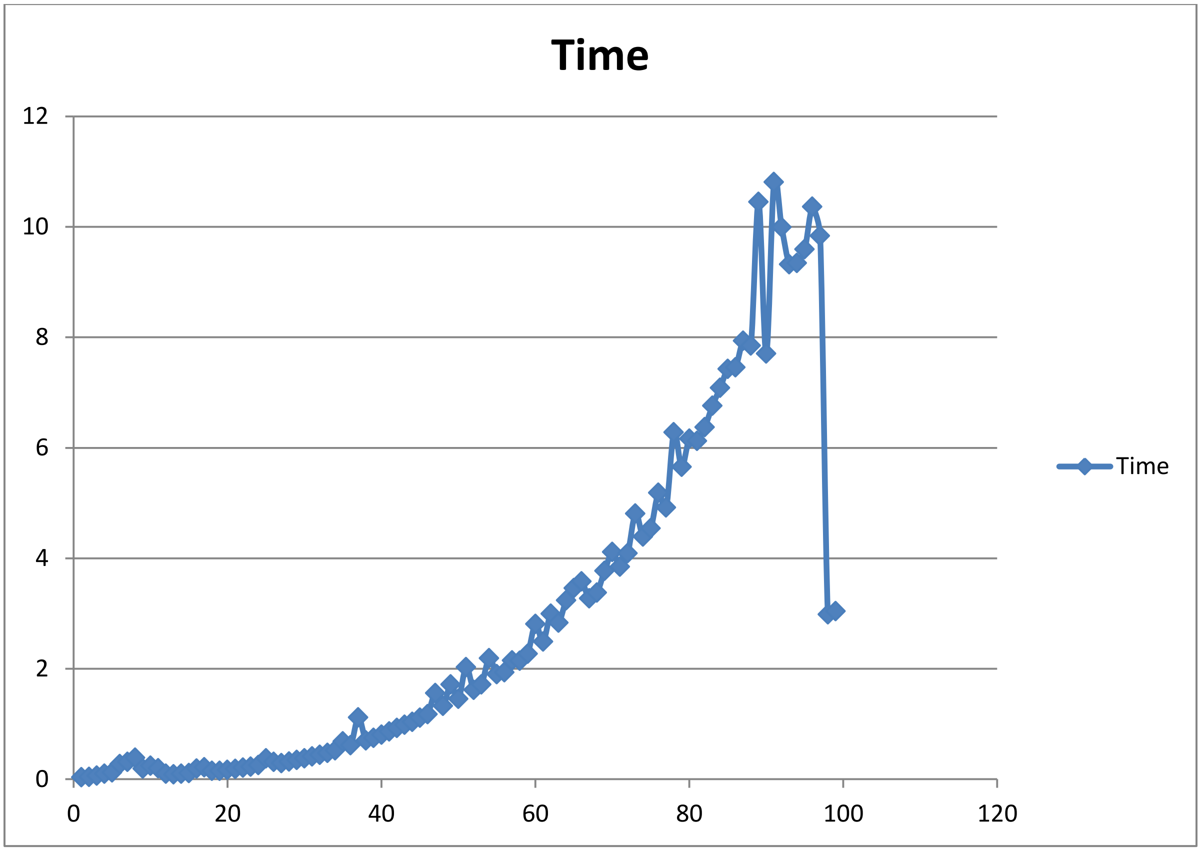


Jacobi and Gauss Seidel



LU Decomposition

No. of Equations Vs Time (ms) and number of B's is the same as the number of equations



Comment:

Gaussian Elimination takes more time and to solve higher order of equations takes much time in comparison to other methods. Other methods have a linear order and Gaussian Elimination has a quadratic and that's the same as the result that was concluded in theoretical analysis.

6- Data structure used

1- Arrays

Array gives a better performance and time.

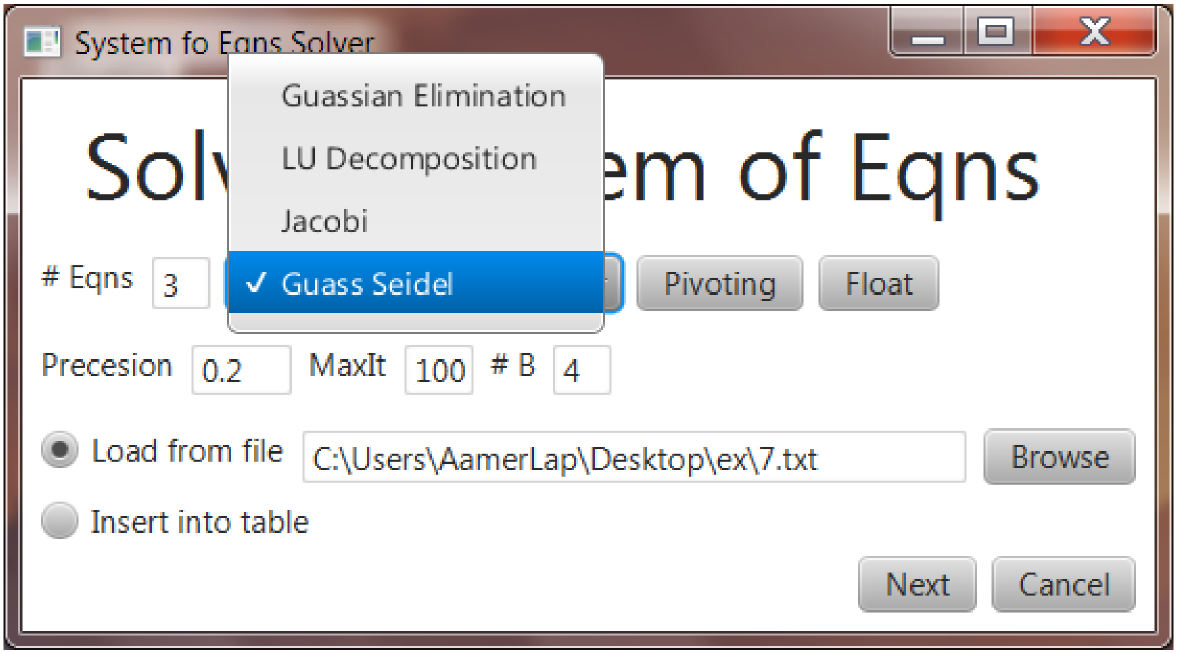
2- ArrayList

ArrayList gives a bit better performance and time and it’s extendable.

3- LinkedList

LinkedList gives less performance but it’s extendable and we used to save things and not to operate on its content.

7- User Guide



How to run?!!

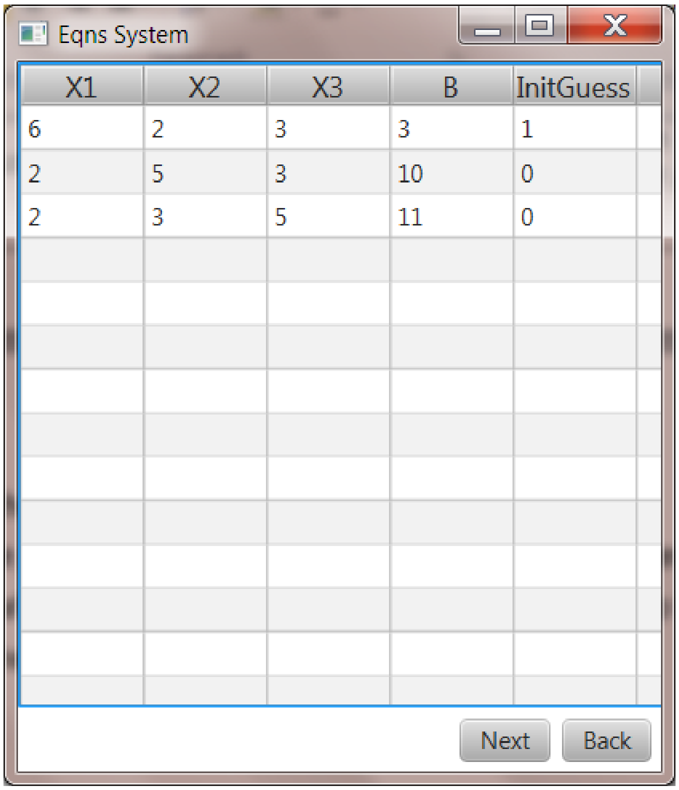
1- Set the # Eqns to the number of equations you have in the system
2- Choose the appropriate method to solve with
3- Set the rest of the fields according to this table

Method	Appropriate fields to set
Gaussian Elimination	**Pivoting :** use pivoting or not **Precession field** : ignore **MaxIt field** : ignore **#B** : ignore
LU Decomposition	**Pivoting** : ignore **Precession field** : ignore **MaxIt field** : ignore **#B** : you can set number of solutions
Jacobi/Gauss-Seidel	**Pivoting** : ignore **Precession field** : set precession **MaxIt field :** set max number of iterations **#B** : ignore

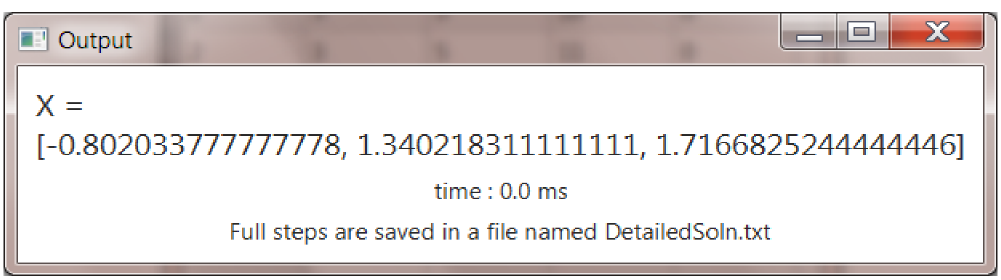
4- Choose either to Load from file or Insert into table

5- Choose the precision **double is default** can be overridden with **float**

6- Click **next** after setting everything and it will take you to the tabulated data like the figure below



7- Click **next** again to see the solution in a window like the figure below



8- Keep in mind that for Jacobi or Gauss-Seidel there is output file to show the iterations performed