Chapter 3: Conditional Probability and Independence

Conditional Probability

The probability we assign to an event can change if we know that some other event has already occurred.

conditional probability of A given B (has occured)

$$\boxed{P(A|B) = \frac{P(A\cap B)}{P(B)}} \quad (P(B) > 0)$$

Consider two events A and B. You are told that event B has already occurred and asked what is the probability of event A happens (i.e. probability of A given B)

The conditional probability of A given B is

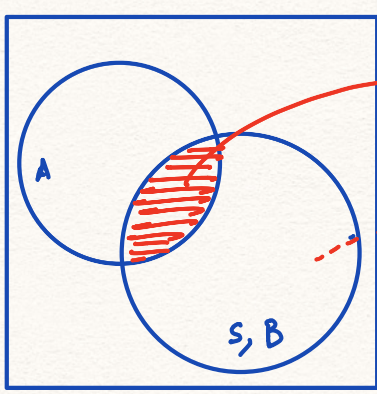
$$\boxed{P(A|B) = \frac{P(A\cap B)}{P(B)}}$$

The conditional probability of B given A is

$$\boxed{P(B|A) = \frac{P(B\cap A)}{P(A)}} \quad \text{provided } P(A) > 0$$

Note: In conditional probability, we can think of "given B" as specifying a new sample space over which we calculate the probability of A.

$P(A|B) \Rightarrow$



$P(A\cap B)$ shaded

Our new sample space is complete B
$\because$ it has already occured

Example 1: A pair of four-sided dice (with sides 1, 2, 3, 4) is rolled and the sum is recorded. Provided that the sum is 3 or 6, find the probability that the sum is at most 4.

a) $A = \{\text{sum is } 3 \text{ or } 6\}$

$B = \{\text{sum is at most } 4\}$

(1,1)	(1,2)	(1,3)	(1,4)
(2,1)	(2,2)	(2,3)	(2,4)
(3,1)	(3,2)	(3,3)	(3,4)
(4,1)	(4,2)	(4,3)	(4,4)

$n(S) = 16$

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{2/16}{5/16} = \frac{2}{5}$$

b) $P(A|B) = ?$

$$= \frac{P(A\cap B)}{P(B)} = \frac{2/16}{6/16} = \frac{1}{3}$$

Properties of Conditional Probability:

1. $0 \le P(A|B) \le 1$ for any A ⟹ important, but then that's every probability
2. $P(B|B) = 1$
3. **a)** If $A_1, \ldots, A_k$ are disjoint, then

$$P\left(\bigcup_{i=1}^{k} A_i \,|B\right) = \sum_{i=1}^{k} P(A_i|B) \qquad \text{(finite additivity)}$$

$$\boxed{P(A_1 \cup A_2 \cup A_3 \cdots A_k|B) = P(A_1|B) + \cdots P(A_k|B)}$$

b) If $A_1, A_2, A_3, \ldots$ are disjoint, then

$$P\left(\bigcup_{i=1}^{\infty} A_i \,|B\right) = \sum_{i=1}^{\infty} P(A_i|B) \qquad \text{(countable additivity)}$$

union leads to cases & additivity

$$\boxed{P(A_1 \cup A_2 \cdots\cdots|B) = P(A_1|B) + P(A_2|B) + \cdots}$$

4. $\boxed{P(A|B) = 1 - P(\bar{A}|B)}$ (as long as condition is B)

↪ so again if easier to find $\bar{A}$, find that & $1 - \ldots$ for $P(A|B)$

The Multiplication Rule:

$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$

Obviously

$P(A) \times P(B|A) \times P(C|A \cap B)$

happens in step 1

Now prob of B if A has happened

Prob of C if both A & B have happened

$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$

$P(A \cap B \cap C) = \cancel{P(A)} \frac{\cancel{P(A \cap B)}}{\cancel{P(A)}} \frac{P(C \cap B \cap C)}{\cancel{P(A \cap B)}}$

Example 2: I give 2 midterms for one of my stat courses. Suppose 25% of the class got A's on the 1st midterm. From previous experience, someone who aced the first midterm has a 60% chance of getting A on the second midterm. What is the probability that a randomly selected student in the class gets A's for both midterms?

$M_1 = \{$student gets A in midterm 1$\}$

$M_2 = \{$student gets A in midterm 2$\}$

$P(M_1 \cap M_2) = ?$

$P(M_1) = 1/4$

$P(M_2|M_1) = \frac{6}{10} = \frac{3}{5} = \frac{P(M_1 \cap M_2)}{P(M_1)} \Rightarrow \frac{3}{5} \times \frac{1}{4} = P(M_1 \cap M_2)$

$= \boxed{\frac{3}{20}}$

Example: Say we collected data from 500 PSU students on whether they have football season tickets or not, also whether they live on campus or off campus. A student is randomly selected. Suppose the student lives on campus. What is the probability he has the tickets?

	On campus	Off campus	total
Tickets	125	25	150
No tickets	200	150	350
total	325	175	500

$A = \{$lives on campus$\}$; $B = \{$has tickets$\}$

$P(B|A) = ?$ $\quad \frac{P(A \cap B)}{P(A)} = \frac{\frac{125}{500}}{\frac{325}{500}} = \frac{125}{325}$

Bayes' Theorem & Law of Total Probability

Little bit of History: Rev. Thomas Bayes (1701-1761) gave us a way to find the conditional probability $P(A|B)$, when the "reverse" conditional probability $P(B|A)$ is known, $P(A), P(B) > 0$

Example 3: A pharmaceutical company has developed a new in-home pregnancy test. This test isn't perfect, but the company expects it to give more accurate results compared to the tests that are already in the market. The test gives positive results for 99% of those who are really pregnant and gives false positives 2% of the times. Suppose in a test trial they ask a group of women to take the test in which 20% of them are really pregnant. We randomly sample a woman in that group.

a) What is the probability that her test is positive?

b) Suppose her test is positive. What is the chance that she is really pregnant?

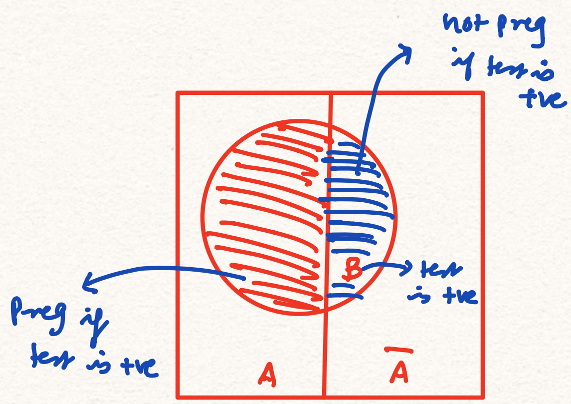
↳ our condition

$A = \{\text{woman is pregnant}\}$; $B = \{\text{test taken is positive}\}$

Given, $P(B|A) = 0.99$; $P(B|\bar{A}) = 0.02$; $P(A) = 0.20$

$\frac{P(B\cap A)}{P(A)} = 0.99$	$\frac{P(B\cap \bar{A})}{P(\bar{A})} = 0.02$	$P(A) = 0.2$

$\Downarrow$



a) $P(B) = P(B\cap A) + P(B\cap \bar{A})$

$= P(A)\cdot P(B|A) + P(\bar{A}) \times P(B|\bar{A})$

$= 0.2 \times 0.99 + 0.8 \times 0.02$

$= 0.198 + 0.016$

$= \boxed{0.214}$

b) $P(A|B) = ?$ $\quad P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)\cdot P(B|A)}{P(B)} = \frac{0.99 \times 0.214}{0.214}$

$\boxed{= 0.925}$ or $\boxed{92.5\%}$

Let $A_1, A_2, \ldots, A_k$ be a partition of the sample space, such that $A_1, A_2, \ldots$ are mutually exclusive & exhaustive

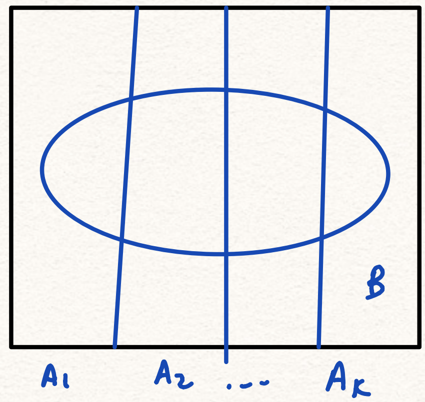
$(A_1 \cap A_2 \ldots A_k = \phi)$ $(A_1 \cup A_2 \cup \ldots A_k = S)$

$\boxed{P(B) > 0}$

Multiply to get total prob

Know: $P(A_i)$ $\quad i = 1,2,\ldots,k$
$P(B|A_i)$ $\quad i = 1,2,\ldots,k$ → keep in mind

(Either given or find)



Law of Total Probability

$$P(B) = P(A_1 \cap B) + \ldots + P(A_k \cap B)$$

$$\Rightarrow P(A_1)P(B|A_1) + \ldots + P(A_k)P(B|A_k)$$

$$\boxed{P(B) = \sum_{i=1}^{k} P(A_i)P(B|A_i)}$$

Bayes' Theorem

$P(A_j|B) =$

j = which element of partition we are looking at $= 1, 2, \ldots k$

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(A_j)\,P(B|A_j)}{\sum_{i=1}^{k} P(A_i)\,P(B|A_i)}$$

$$\boxed{\Rightarrow P(A_j|B) = \frac{P(A_j)P(B|A_j)}{\sum_{i=1}^{k} P(A_i)P(B|A_i)}}$$

what is the prob that A_j happened if B happened

$P(A_i) \Rightarrow$ Prior probability then B has happened

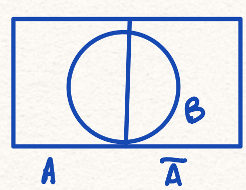
$\Rightarrow P(A_i|B)$ is needed $\because$ B happened after

$\Downarrow$

Posterior probability

Example 3

$k = 2$; $A_1 = A$;
$A_2 = \bar{A}$ & B



Need to know,
i) $P(A)$ & $P(\bar{A})$
ii) $P(B|A)$ & $P(B|\bar{A})$

Example 4: A quality control manager is responsible for checking for defects in the latest iPhones produced by Apple Inc. For simplicity, suppose there are three factories where iPhones are manufactured. The QC manager knows that 36% of phones are produced in the first factory, 34% in the second, and 30% in the third. Also, of the phones produced in the first factory, 1.5% are defective, of those in the second factory 1% is defective, and of those in the third factory 2% are defective.

a) What is the probability that a randomly selected phone is found to be defective?

b) If a randomly selected phone is found to be defective, what is the probability it was manufactured in the third factory?

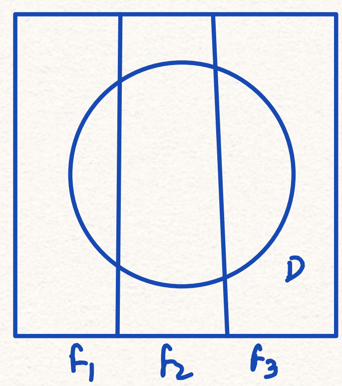
a) $F_1 = \{\text{produced in factory 1}\}$; $F_2 = \{\text{produced in factory 3}\}$; $F_3 = \{\text{produced in factory 3}\}$

$D = \{\text{phone is defective}\}$; $\bar{D} = \{ \dots \}$

$P(F_1) = 0.36$; $P(F_2) = 0.34$; $P(F_3) = 0.30$

$P(D|F_1) = 0.015$; $P(D|F_2) = 0.01$; $P(D|F_3) = 0.02$ $\Rightarrow$ know

$$P(D) = P(F_1)\,P(D|F_1) + P(F_2)\,P(D|F_2) + P(F_3)\,P(D|F_3)$$
$$= \boxed{0.0148}$$



b) $$P(F_3|D) = ? = \frac{P(F_3)\,P(D|F_3)}{P(D)} = \frac{0.3 \times 0.02}{0.0148} = \boxed{0.405}$$

A_i

$B|A_i$ $\Rightarrow$ know

	F_1	F_2	F_3
D	$P(D\cap F_1) = P(D|F_1)P(F_1) = 0.36 \times 0.015$	$P(D\cap F_2) = P(D|F_2) \times P(F_2)$	$P(D\cap F_3) = P(F_3) \times P(D|F_3)$
$\bar{D}$	$P(\bar{D}\cap F_1) = P(\bar{D}|F_1)P(F_1) = (1-P(D|F_1)).P(F_1)$	$P(\bar{D}\cap F_2) = \dots$	$P(\bar{D}\cap F_3) = \dots$

$F_1 <^{D\,=}_{\bar{D}\,=}$

$F_2 <^{D\,=}_{\bar{D}\,=}$ ....

$F_3 <^{D\,=}_{\bar{D}\,=}$

Independent Events

Two events are said to be independent if occurrence of one event has no effect on the chance of occurrence of the other event.

i) $P(A \cap B) = P(A) \cdot P(B|A) = P(A) \cdot P(B)$ ($\because$ B & A are independent)

Example 5: Suppose family has three children. Also assume that girls and boys are equally likely and the gender of a child has nothing to do with the gender of the next child.

- **a)** Let A be the event 1st child is a girl, and B be the event 2nd child is a girl. Are A and B independent?
- **b)** Let C = {at least one boy and a one girl} and D = {gender of the 2nd child is same as the gender of the gender of the first child}. Are two events independent?

check i), ii) & iii) to see if the events are independent, any would work

a) A = { 1st child is a girl } ; B = { 2nd child is a girl }

$\underbrace{P(A \cap B)}_{\left(\frac{1}{4} = \frac{2}{8}\right)} = P(A) \cdot P(B) = \left(\frac{1}{2} = \frac{4}{8}\right) \times \left(\frac{1}{2} = \frac{4}{8}\right) = \frac{1}{4}$

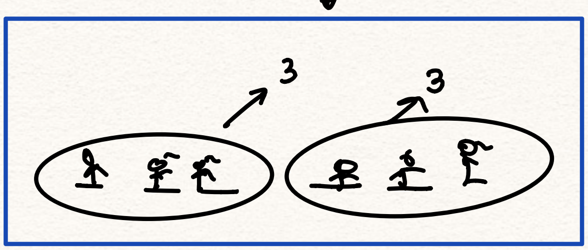
$2 \times 2 \times 2 = 8$ possibilities

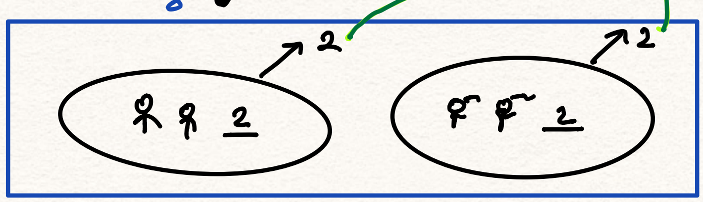
$\boxed{G \; G \; 2}$

b) C = { at least a boy and a girl } ; D = { gender of 2nd child is same as 1st child }

$P(C \cap D) \stackrel{?}{=} P(C) \cdot P(D)$

$\Rightarrow P(C) = \frac{6}{8} = 1 - \frac{2}{8}$; $P(D) = \frac{4}{8}$; $P(C \cap D) = ? = \frac{2}{8}$





This becomes 1 for $C \cap D$, so $\boxed{2}$

Since $P(C \cap D) \neq P(C) \cdot P(D)$

C & D are not independent

NOTES:

ii) ⊛ $P(A|B) = P(A)$

iii) ⊛ $P(B|A) = P(B)$ if A & B are independent events

∵ A will happen no matter B happens or not

⊛ If A & B are independent then,

A and B are NOT mutually exclusive & vice versa

i) A & B' are independent
ii) A' & B are independent
iii) A' & B' are independent

Independence of more than two events

Independence of more than two events can be extended in a similar way, but you need to be cautious!!!!

Events A, B and C are mutually independent

i) $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$	ii) $P(A \cap B) = P(A) \cdot P(B)$	iii) $P(B \cap C) = P(B) P(C)$

iv) $P(A \cap C) = P(A) \cdot P(C)$

So in order for A, B, C to be mutually independent all these 4 eqns have to be true, not just one!!!

Example 6: An urn contains four balls numbered 1,2,3 and 4. One ball is to be drawn at random from the urn. Let events A, B and C be defined by $\underbrace{A=\{1,2\}}_{\text{means either 1 or 2}}$, B={1,3} and C={1,4}. Are A, B and C mutually independent?

$$\underbrace{P(A\cap B)}_{\frac{1}{4}} \stackrel{?}{=} P(A)\cdot P(B) = \left(\frac{1}{4}+\frac{1}{4}\right)\cdot\left(\frac{1}{4}+\frac{1}{4}\right) = \frac{1}{4}$$

$\because A\cap B=\{1\} \Rightarrow$ A & B are independent

show this for all resp. 2 pairs & then for $P(A\cap B\cap C)$.

$$P(A\cap B\cap C) \stackrel{?}{=} P(A)\cdot P(B)\cdot P(C)$$

$$\Rightarrow \frac{1}{4} \neq \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$$

$\because$ $\Rightarrow$ A, B & C are not mutually independent

Example 7: Suppose a company has 3 quality control managers and each one of them inspect the items in the company's production line independently. Probability of detecting an item as defective is 0.99, 0.98 and 0.96 for the three managers. What is the probability that of all three of them identifying an item as defective?

a) $P(A\cap B\cap C) = 0.99\times0.98\times0.96$ $\because$ independent

b) Probability that at least one of them identifies defect

i) $\underbrace{P(A\cup B\cup C)} = P(A)+P(B)+\cdots$ apply this ✓

ii) $1-P(\bar{A}\cap\bar{B}\cap\bar{C}) = 1-\left(P(\bar{A})\cdot P(\bar{B})\cdot P(\bar{C})\right) = 1-(\cdots)$ ✓

$\because$ complements DON'T change independence, so we can apply this!

Example: An oil exploration company has projects in Asia and Europe. Let A be the event that the Asian project is successful, and B be the event that the European project is successful. Suppose A and B are independent, and $P(A) = 0.4$ and $P(B) = 0.7$.

Q: A company has projects in Asia & Europe $P(A) = 0.4$; $P(B) = 0.7$

$A = \{$ Asian project is successfull $\}$; $B = \{$ European project is successfull $\}$

a) If Asian project is not successfull, what's the probability that the European one is also not successfull?

$$P(\bar{B} | \bar{A}) = P(\bar{B}) = 0.3$$

b) what is probability that at least one of the projects are successfull?

$$P(A \cup B) = 1 - P(\bar{A} \cap \bar{B}) = 1 - (0.6 \times 0.3) = 1 - 0.18 = 0.82$$

(c) If at least one of the two projects is successful, what is the probability that only the Asian project is successful?

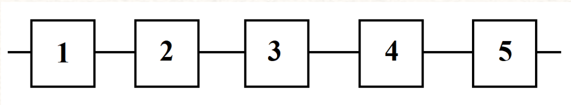
SOLUTION:

We need $P(A \cap B'|A \cup B) = \frac{P((A \cap B') \cap (A \cup B))}{P(A \cup B)}$. We found $P(A \cup B) = 0.82$ in class. Since $(A \cap B') \cap (A \cup B) = A \cap B'$, and from independence of A and B', $P(A \cap B') = P(A)P(B')$, we have $P(A \cap B'|A \cup B) = \frac{0.4 * (1 - 0.7)}{0.82} = 0.146$.

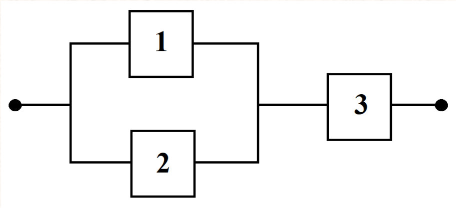
Chapter 3 Additional Practice

NOTE: For better learning, make a serious effort to solve these examples on your own before looking at the solutions on next pages!

Example 1: Consider a system of 5 components connected in series (as in the picture below). The system fails if one component fails. Each component fails independently with probability 0.1. What is the probability that the system fails?



Example 2: Consider the system consisting of 3 components connected as in the picture below. The system functions if there is path from left to right. Each component operates independently with probabilities 0.9, 0.95, 0.85, respectively. What is the probability that the system functions?



Example 3: (Problem #3.14, Ross, 10th ed) Suppose that an ordinary deck of 52 cards (which contains 4 aces) is randomly divided into 4 hands of 13 cards each. We are interested in determining the probability that each hand has an ace.
Hint: Let E_i be the event that the ith hand has exactly one ace. Then the probability of interest is $P(E_1 \cap E_2 \cap E_3 \cap E_4)$. Use the multiplication rule to determine the probability.

Example 4: (Modified Example 3n, Ross, 10th ed) A bin contains 3 types of flashlights. Suppose that 60 percent of the flashlights in the bin are type 1, 10 percent are type 2, and the remaining flashlights are type 3. The probability that a type 2 flashlight will last than 100 hours is one third of the probability that a type 1 flashlight will last more than 100 hours, and is four times the probability type 3 flashlight will last more than 100 hours. A flashlight is randomly selected from the bin. Suppose the selected flashlight lasted more than 100 hours. What is the probability it was type 1?

SOLUTIONS:

1. Denote $C_i = \{\text{the } i\text{th component functions}\}$, $i = 1, \ldots, 5$. Then we are given that $P(C_i') = 0.1$, $i = 1, \ldots, 5$, and $C_1', \ldots, C_5'$ are independent. Note that the latter implies that $C_1, \ldots, C_5$ are independent too.

The system fails if at least one component fails, i.e. if any one component fails, or any two, or any three, or any four, or all five fail. It's easier to deal with the complement then, for which the system functions if all five components function. In terms of probabilities, $P(\text{system fails}) = P(C_1' \cup \ldots \cup C_5') = 1 - P(C_1 \cap \ldots \cap C_5)$ and then using independence of $C_1, \ldots, C_5$ (see above), $P(\text{system fails}) = 1 - P(C_1) \ldots P(C_5) = 1 - (1 - 0.1)^5 = 0.4095$

2. Denote $C_i = \{\text{the } i\text{th component functions}\}$, $i = 1, 2, 3$. Then $P(C_1) = 0.9$, $P(C_2) = 0.95$, $P(C_3) = 0.85$, and C_1, C_2, C_3 are independent.

The system operates if at least one in the group of components 1 and 2 operates AND component 3 operates, i.e. $P(\text{system operates}) = P((C_1 \cup C_2) \cap C_3)$. Since C_3 is independent from C_1 and C_2 (and thus from $C_1 \cup C_2$), $P(\text{system operates}) = P(C_1 \cup C_2)P(C_3) = [P(C_1) + P(C_2) - P(C_1 \cap C_2)]\, P(C_3)$. By independence of C_1 and C_2, $P(\text{system operates}) = [P(C_1) + P(C_2) - P(C_1)P(C_2)]\, P(C_3) = [0.9 + 0.95 - 0.9 * 0.95] * 0.85 = 0.846$

3. Following the hint, let $E_i = \{\text{the } i\text{th hand has exactly one ace}\}$, $i = 1, 2, 3, 4$. Note events $\{\text{each hand has an ace}\}$ and $E_1 \cap E_2 \cap E_3 \cap E_4 = \{\text{each hand has exactly one ace}\}$ are equivalent. , $P(E_1 \cap E_2 \cap E_3 \cap E_4) = P(E_1)P(E_2|E_1)P(E_3|E_1 \cap E_2)P(E_4|E_1 \cap E_2 \cap E_3)$ by the multiplication rule.

Let's consider the process step-by-step:

- for the 1st hand we select (without replacement, unordered) 13 cards so that 1 card comes from the 4 aces, 12 cards from 48 non-aces $\Rightarrow P(E_1) = \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}}$ ways.
- once the 1st hand is selected, for the 2nd hand we select 13 cards from 39 cards left so that 1 card comes from the 3 aces left and 12 cards from 36 non-aces left $\Rightarrow P(E_2|E_1) = \frac{\binom{3}{1}\binom{36}{12}}{\binom{39}{13}}$ ways.
- once the first 2 hands are selected, for the 3rd hand we select 13 cards from the 26 cards left so that 1 card comes from the 2 aces left and 12 cards from 24 non-aces left $\Rightarrow P(E_3|E_1 \cap E_2) = \frac{\binom{2}{1}\binom{24}{12}}{\binom{26}{13}}$ ways.
- the remaining 13 cards have 1 ace and 12 non-aces, so the 4th hand is determined uniquely, once the first 3 hands are selected, or $P(E_4|E_1 \cap E_2 \cap E_3) = \frac{\binom{1}{1}\binom{12}{12}}{\binom{13}{13}}$ ways.

Then $P(E_1 \cap E_2 \cap E_3 \cap E_4) = \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}} * \frac{\binom{3}{1}\binom{36}{12}}{\binom{39}{13}} * \frac{\binom{2}{1}\binom{24}{12}}{\binom{26}{13}} * 1 = 0.1055$

4. Let $T_i = \{\text{flashlight is type } i\}$, $i = 1, 2, 3$, and $L = \{\text{flashlight lasts } > 100 \text{ hours}\}$. Note that T_1, T_2, T_3 form a partition. Since $P(T_1) = 0.6$ and $P(T_2) = 0.1$, $P(T_3) = 0.3$. We are given $P(L|T_2) = \frac{1}{3}P(L|T_1)$ and $P(L|T_2) = 4P(L|T_3)$. We need to find $P(T_1|L)$.

Denote $P(L|T_3) = a$, then $P(L|T_2) = 4a$ and $P(L|T_1) = 3P(L|T_2) = 12a$.

By Bayes' Theorem, $P(T_1|L) = \frac{P(L|T_1)P(T_1)}{P(L|T_1)P(T_1)+P(L|T_2)P(T_2)+P(L|T_3)P(T_3)} = \frac{12a*0.6}{12a*0.6+4a*0.1+a*0.3}$. We can cancel out a's, and then $P(T_1|L) = \frac{12*0.6}{12*0.6+4*0.1+0.3} = 0.9114$

NOTE: From the information provided in the problem, we cannot find $P(L|T_1)$, $P(L|T_2)$, and $P(L|T_3)$ exactly, but the information about the relationships between these probabilities is sufficient to find $P(T_1|L)$.