Chapter 1: Combinatorial Analysis

"The best thing about being a statistician is that you get to play in everyone's backyard"
John Tukey

Why Statistics and Probability?
~ variability!

To make sense of this, we need appropriate probabilistic mathematical models!

In many fields of study statistics and probability are necessary to answer research questions!

- *in medicine*: Compare the effectiveness of a new drug with one already in the market.
- *in astronomy*: Objects under study are not easy to reach. So, they observe some external characteristics and predict underlying properties.
- *in politics*: How likely do you think that Donald Trump will be reelected?
- *in sports*: Sports teams use statistics to prepare for upcoming opponents. They also use statistics to predict the chances of certain countries wining next world championships.
- *in finance*: Modeling and prediction of stock prices.

Probability

Random experiments

Example:

- Rolling a die
- Letter grade in this course
- Tossing a coin two times

Outcome space (sample space)

An event

Equally Likely Outcomes

$$\boxed{P(A) = \frac{\#\text{ of outcomes in } A}{\text{total }\#\text{ of outcomes in } S}}$$

S is sample space

A is event

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Example 1: Suppose we roll a die twice and want to find the probability that the sum of two rolls is 8.

$A = \{\text{sum of 2 rolls is } 8\}$ $\quad n(A) = 5$

$P(A) = ?$

(1)

$S = \{(1,1), (1,2), (1,3) \; --- \; (1,6)$ $\quad n(S) = 36$

$(6,1) \; (6,2) \; (6,3) \; --- \; (6,6)\}$

$$\boxed{\underset{\text{sum}=8}{P(A)} = \frac{5}{36}}$$

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(2) Sample Space S (using Basic Principle of Counting)

1st roll $n_1 = 6$ $\quad$ 2nd roll $n_2 = 6$ $\Rightarrow$ 36 total

Event A

1st roll $n_1 = 5$ (2 to 6) $\quad$ 2nd role $n_2 = 1$ $\Rightarrow$ 5

(for each values say 2 there is only 6 | for 3 it's 5 | for 4 it's 4 and so on)

Counting Techniques

(1) **Basic Principle of Counting**

Suppose the experiment consists of 2 steps. If step 1 has n_1 possible outcomes or step 1 is performed in n_1 ways and for each outcomes of step 1, step 2 has n_2 possible outcomes.

Then the entire experiment has $n_1 \times n_2$ possible outcomes / performed in $n_1 \times n_2$ ways

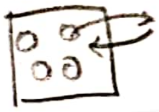
Generalized Principle of Counting

If an experiment can be performed in r steps, and step i has n_i possible outcomes, $i = 1, \ldots, r$, regardless of the outcomes of the previous steps, then the entire experiment has $n_1 n_2 \ldots n_r$ possible outcomes.

You have to pay attention:

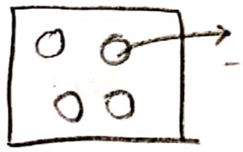
- Selection is done with or without replacement

Selection with replacement:



→ Pool remains same each time

Selection without replacement:



Each time pool gets smaller

- Order matters or not

Order matters (ordered selection):

$AB \neq BA$

Permutation → ${}^nP_r = \frac{n!}{(n-r)!}$

Order doesn't matter (unordered selection):

$AB = BA$

Combination → ${}^nC_r = \frac{n!}{r!(n-r)!}$

Example 2:

a) How many different letter arrangements can be made from the letters BOARD, if you are allowed to repeat the letters?

$\underline{5}\ \underline{5}\ \underline{5}\ \underline{5}\ \underline{5} \Rightarrow \boxed{5^5 \text{ ways}}$

order matters

b) How many different letter arrangements can be made from the letters BOARD, if you are not allowed to repeat the letters.

$\underline{1}\ \underline{2}\ \underline{3}\ \underline{4}\ \underline{5} \Rightarrow \boxed{5! \text{ ways}}$

Order matters

Example 3:

a) There are 5 finalists named Bill, Jack, Rob, Sam, and Ted. We select the 1st, 2nd, and 3rd place winners. In how many way can this be done?

Order matters

1st place → 5

2nd place → 4

3rd place → 3

$\Rightarrow \boxed{5 \times 4 \times 3 = 60 \text{ ways}}$

b) Now suppose that we select just 3 winners from this group of 5. In how many way can this be done?

order doesn't matter

$${}^5C_3 = \frac{5!}{3!(2!)} = \frac{\cancel{5!}\ \cancel{4}^{2} \times 5}{\cancel{3!}(\cancel{2!})} = 10 \text{ ways}$$

so one group of 3 is arranged in 6 ways so $\boxed{\frac{60}{6} = 10}$

Summary Table: The number of ways to choose r objects from a set of n different objects.

Permutation ^nP_r

w/o replacement

Combination ^nC_r

	Without replacement	With replacement
Order matters	(Select diff items everytime) $\underset{1st}{n}\ \underset{2nd}{n-1}\ \underset{3rd}{n-2}\ \cdots\ \underset{rth}{n-(r-1)}$ $\underbrace{n\cdot(n-1)\cdot(n-2)\cdots(n-r-1)}$ $= \boxed{{}^nP_r = \frac{n!}{(n-r)!} = nPr}$ $= n\cdot(n-r-1)!$ Example 3(a) Permutations case	(Select same item more than once) $\underset{1st}{n}\ \underset{2nd}{n}\ \underset{3rd}{n}\ \cdots\ \underset{rth}{n}$ $\boxed{n^r}$ Example 2(a) Here, r can be greater than n
Order doesn't matter	Example 3(b) $\boxed{{}^nC_r = \frac{n!}{(n-r)!\,r!} = nCr}$ $\binom{n}{r}$ means n choose r Binomial expressions Combinations case	$\boxed{{}^{n+r-1}C_r}$

Let's get to know the notations and definitions in the above table:

Binomial Theorem

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{r} a^r b^{n-r}$$

$$\boxed{T_n = {}^nC_r \, a^{n-r} b^r}$$

Example 4: Suppose there are 20 faculty members in the statistics department and we want to form a committee of 4 faculty members. How many different ways to form the committee?

$$^{20}C_4 = \frac{20!}{16!\,4!} = \frac{17\times \cancel{18}\times 19\times \cancel{20}}{\cancel{4}\times\cancel{6}} = 51\times 95$$

without replacement
unordered

n = 20
r = 4

Example 5: Same problem, but now there are four executive positions in the committee: President, Vice President, Secretary and Treasurer. How many ways?

without replacement
ordered ∵ A B C D → P VP S T
is diff than
A B C D → VP P S T

$$\underset{P}{20}\times\underset{VP}{19}\times\underset{S}{18}\times\underset{T}{17} = 116280$$

or

$$^{20}P_4 = \frac{20!}{16!}$$

Example 6: How many 3 digit telephone area codes can you make from numbers {1,2,….9}

with replacement
ordered ∵
192
is diff than 912

$$\underset{9}{\_\_} \times \underset{9}{\_\_} \times \underset{9}{\_\_} \Rightarrow \boxed{729 \text{ ways}}$$

Example 7: If a six-sided die is rolled 3 times, how many possible unordered outcomes are there?
→ Think of this as a selection problem

with replacement
unordered

n = 6
r = 3

__ __ __ = $\boxed{56 \text{ ways}}$

$$^{6+3-1}C_3 = {}^{8}C_3 = \frac{\boxed{4\times5}\times6\times\boxed{7\times8}}{}$$
$$20\times56\times6 = \frac{\dots}{5!\,8}$$
$$\Rightarrow \frac{8!\,\cancel{5}\times 7\times 8}{\cancel{5!}\,3!}$$

Multinomial Coefficients

Example: How many ways are there to arrange the letters in word MISSISSIPPI? 11 letters

M → 1
I → 4
S → 4
P → 2

$\boxed{M}\ \boxed{I\ I\ I\ I}\ \boxed{P\ P}\ \boxed{S\ S\ S\ S}$

$\boxed{\dfrac{11!}{4!\,2!\,4!}}$ Answer $= 34650$

Multinomial Coefficients

n objects → category 1 has n_1 obj
→ Category 2 has n_2 obj such that $n_1 + n_2 + \cdots + n_r = n$.
⋮
category r has n_r obj

$$\boxed{\text{So, } \binom{n}{n_1, n_2 .. n_r} = {}^{n}C_{n_1, n_2 .. n_r} = \frac{n!}{(n_1!\, n_2!\, n_3! \cdots n_r!)\,(0!)}}$$ → Above example

$$\boxed{(x_1 + x_2 + \cdots + x_r)^n = \sum_{n_1, n_2, .. n_r} \binom{n}{n_1, n_2 .. n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}}$$

Binomial coeff are a special case of multinomial coeff with $r = 2$.

$\binom{n}{k}$ but we instead of only selecting k, we partition into k & n-k

Select k from n w/out rept & unordered.

Now we only want to select cat 1 of k & not cat 2 of n-k

$$\therefore \underbrace{\binom{n}{k}}_{\text{Multinomial}} = \underbrace{\binom{n}{k,\ n-k}}_{\text{Binomial}}$$

Example 8: How many different letter arrangement can be made from the letters ACCESS ?

A − 1
E − 1
C − 2
S − 2

A E $\boxed{C\ C}$ $\boxed{S\ S}$

$\frac{6!}{2!2!} = 180$ ways

Example 9: Prof. Jones has 10 students in his physics class. The possible grades in the course are A, B, C, F.

a) How many possible grade assignments are there? selection prob again

b) Suppose Prof. Jones curves the grades. He wants to have 20% As, 30% Bs, 30% Cs, and 20% Fs. How many possible ways can he assign grades to his 10 students?

a) with replacement
ordered : ABCF ≠ BACF

$\underline{4}\ \underline{4}\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \underline{4} = 4^{10}$

b) $\boxed{A\ A}$ $\boxed{B\ B\ B}$ $\boxed{C\ C\ C}$ $\boxed{F\ F}$

$\frac{10!}{2!2!3!3!} = \binom{10}{2,3,3,2} = 25200$

order again matters

Q. A person has 8 friends of whom 5 will be invited to a party.

a) How many ways can you do this?

without replacement & unordered $\Rightarrow$ $\boxed{{}^{8}C_{5} = \frac{6 \times 7 \times 8}{3!} = 56}$

b) How many choices are there if 2 of the friends are feuding & can't attend together?

<u>Case i)</u> <u>Invite only one of them</u>

$\boxed{{}^{6}C_{4} \times {}^{2}C_{1}}$ & not $\boxed{{}^{7}C_{5}}$

<u>Case ii)</u> <u>Invite neither</u>

But there are two ways to choose whom not to invite

Also, $\boxed{{}^{6}C_{5}}$

Therefore, final answer $\boxed{{}^{6}C_{4} \times {}^{2}C_{1} + {}^{6}C_{5}}$

The Number of Positive / Non-negative Integer Solutions of Equations

Number of Positive Integer Solutions of Equations

$$x_1 + x_2 + \cdots + x_r = n, \qquad x_i - \text{integer} > 0, \quad i = 1, \dots, r$$

only Positive

$$\boxed{x_1 + x_2 + x_3 + \cdots x_r = n}$$

$x_i > 0$ integers ; $i = 1, 2, \dots r$

$\boxed{\binom{n-1}{r-1}} \Rightarrow$ # + integer solutions of

See the reason why the formula is this

Positive and 0

Number of Non-negative Integer Solutions of Equations

$$x_1 + x_2 + \cdots + x_r = n, \qquad x_i - \text{integer} \geq 0, \quad i = 1, \dots, r$$

Now, $y_i = x_i + i$; $i = 1, 2, \dots r$

$y_i > 0$ integers

$$\boxed{y_1 + y_2 + y_3 + \cdots y_r = n + r}$$

$\boxed{\binom{n+r-1}{r-1}} \Rightarrow$ # of $\boxed{+ \text{ or } 0}$ integer solutions of

Example 10: There are 8 teachers and 4 schools.

a) How many ways are there to distribute teachers among schools if every school has to get at least one teacher?

b) How many ways are there to distribute teachers among schools without restrictions?

at least one in each is not there

a) x_i = # of teachers sent to school i, $i = 1, 2, 3, 4$

And, $x_1 + x_2 + x_3 + x_4 = 8$ has to be true

So, number of ways to distribute this is $\binom{8-1}{4-1} = {}^7C_3 = \boxed{35}$

b) So x_i can also be 0 in this case,

$x_i \geq 0$ integers $\Rightarrow$ Ways $= \binom{8+4-1}{4-1} = {}^{11}C_3 = \boxed{165}$