Chapter 5: Continuous Random Variables

Uniform Distribution

X is a uniform random variable on a finite interval (a, b) if its pdf is

constant pdf

$X \sim Unif(a,b)$ 1) $\underbrace{f(x)}_{\boxed{PDF}} = \begin{cases} \frac{1}{b-a}, & x \in (a,b) \\ 0, & o/w \end{cases}$

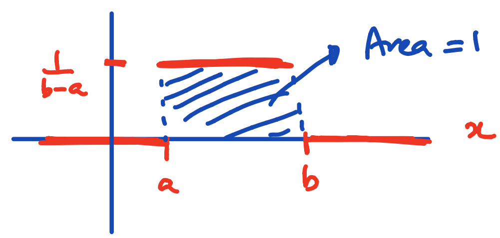
Properties of Uniform Distribution:

2)

Mean: $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = 0 + \int_a^b x \cdot \frac{1}{b-a}\,dx = \frac{a+b}{2}$

Variance: $Var(X) = \frac{(b-a)^2}{12}$

Cdf: $F(x) = P(X \le x) = \int_{-\infty}^{x} f(v)\,dv$



$$F(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a < x < b \\ 1, & x \geq b \end{cases}$$

Selecting any $X \Rightarrow$ prob is the same

Example 2: A point is chosen at random evenly on a line segment of length L. Find the probability that the ratio of the shorter to the longer segment is less than 1/4.

$$X \sim \text{Unif}(0, L) \quad \because X \in (0, L)$$

$$\Rightarrow \boxed{\text{pdf} = \frac{1}{L}}$$

$$P\left(\frac{\text{short}}{\text{longer}} < \frac{1}{4}\right) = P\left(X < \frac{L}{2} \cap \frac{X}{L-X} < \frac{1}{4}\right) + P\left(X \geq \frac{L}{2} \cap \frac{L-X}{X} < \frac{1}{4}\right) \quad \left(\because 4L - 4X < X,\ X > \frac{4L}{5}\right)$$

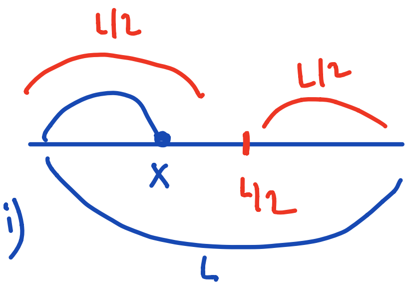
$$= P\left(X < \frac{L}{5}\right) + P\left(X > \frac{4L}{5}\right) \quad \left(\because 4X < L - X,\ 5X < L,\ X < \frac{L}{5}\right)$$

$$= \int_0^{L/5} \frac{1}{L}\,dx + \int_{4/5L}^{L} \frac{1}{L}\,dx = \boxed{0.4}$$

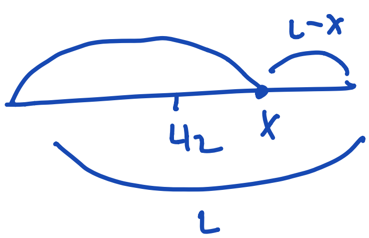
$$= F_X\left(\frac{L}{5}\right) + \left[1 - F_X\left(\frac{4L}{5}\right)\right]$$

$$\left(F_X(L) = 1\right)$$

i)



ii)



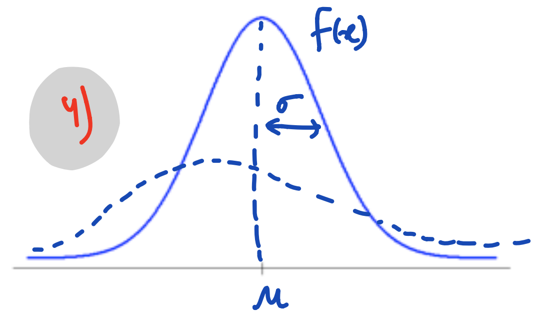
Normal Distribution (Gaussian)

X is a Normal random variable with parameters μ and σ^2, $(-\infty < \mu < \infty,\ \sigma > 0)$, if its pdf is

3) $$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},}\quad -\infty < x < \infty, \ -\infty < \mu < \infty, \ \sigma > 0$$

$X \sim N(\mu, \sigma^2)$

The shape of the normal pdf curve

4) 

- bell shaped
- symmetric around μ
- unimodel

$$\boxed{\begin{aligned} E(X) &= \mu \\ Var(X) &= \sigma^2 \\ X &\sim N(\mu, \sigma^2) \end{aligned}}$$

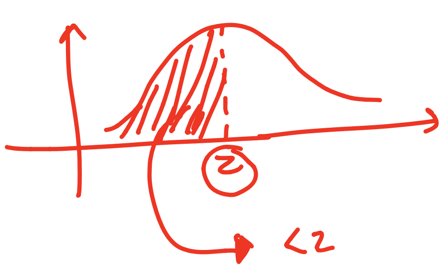
CDF ⇒ $\boxed{F(c) = P(X \le c) = \int_{-\infty}^{c} f(x)\,dx = \text{No closed form}}$

⊛ Some important notes ⇒ $\mu \in \mathbb{R}$ and defines the center of distribution

σ is the standard deviation of the normal dist.

σ must be +

Suppose we want some probability $P(c < X < d)$



from the z table, see the value — given below

i) of z (from the first row & column)

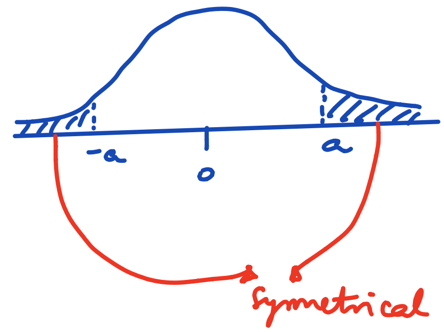
ii) see what it corresponds to ⇒ $\int < z$ is given by that

$\boxed{\phi(a) = P(Z \le a)}$

6) **Special case:** Standard Normal $\mu = 0, \ \sigma = 1$ $\Rightarrow$ $\boxed{\begin{array}{l} E(X) = \mu = 0 \\ Var(X) = \sigma^2 = 1 \end{array}}$

$\boxed{\frac{X-\mu}{\sigma} \leftarrow Z \sim N(0,1)}$

Cdf of a standard normal $\Phi(a) = P(Z \le a)$



5)

- positive a Use table
- negative a

$\phi(-a) = P(Z \le -a) = 1 - \phi(a)$

$P(a < Z < b) = \phi(b) - \phi(a)$

⊛ Any normal RV can be represented by standard Normal RV as,

$X = \mu + \sigma Z$

$Z \rightleftarrows X$

$Z = \frac{X-\mu}{\sigma}$

(Standardization)

$E(X) = E(\mu + \sigma Z) = \mu + \sigma E(Z) = \mu + 0 = \mu$

$Var(X) = Var(\mu + \sigma Z) = \sigma^2 Var(Z) = \sigma^2$

How does this help with X?

- If $X \sim N(\mu, \sigma^2)$, then $\frac{(X-\mu)}{\sigma} \sim Z$
- If $Z \sim N(0,1)$, then $\mu + \sigma Z \sim X$

$\boxed{Z = \frac{X-\mu}{\sigma}}$

$\boxed{\text{If } X \sim N(\mu, \sigma^2), \text{ then } P(c < X < d) = \ ???}$

integer part and first digit after .

second digit after .

TABLE 5.1: AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF X

X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

NOTE: *<u>When the precise z-value is not in the table</u>*, you may round it to two decimal places and read that value from the table.

⊛ See table for -ve z values too

This is incomplete

6) Conversion into Z is a must

Example 3: Birth weight of baby elephants in an elephant orphanage has a normal distribution with mean 234 lbs and standard deviation 34 lbs.

$X \sim N(\mu, \sigma^2) \Updownarrow Z \sim N(0,1)$

- **a)** What is the probability that a baby elephant weighs less than 250 lbs?
- **b)** What is the probability that a baby elephant weighs less than 216 lbs?
- **c)** What proportion of baby elephants weigh more than 260 lbs?
- **d)** What proportion of baby elephants weigh between 220 lbs and 280 lbs?
- e) $P(X>375)$

$$X = \text{weight} \sim N(\mu = 234, \sigma^2 = 34^2)$$

a) $P(X<250) = P\left(\frac{X-234}{34} < \frac{250-234}{34}\right) = P(Z<0.47)$

$= \Phi(0.47) = \boxed{0.6808}$

b) $P(X<216) = P\left(\frac{X-234}{34} < \frac{216-234}{34}\right)$

$= P(Z<-0.53) = \Phi(-0.53) = 1 - \Phi(0.53)$

$= 1 - 0.7019 = \boxed{0.2981}$

c) $P(X>260) = P\left(\frac{X-234}{34} > \frac{260-234}{34}\right)$

$P(Z>0.76) = 1 - \Phi(0.76) =$

$1 - 0.7764 = \boxed{0.2236}$

d) $P(220 < X < 280) = P\left(\frac{220-234}{34} < \frac{X-234}{34} < \frac{280-234}{34}\right)$

$\Rightarrow P(-0.41 < Z < 1.35) = \Phi(1.35) - \Phi(-0.41) = \Phi(1.35) - (1 - \Phi(0.41))$

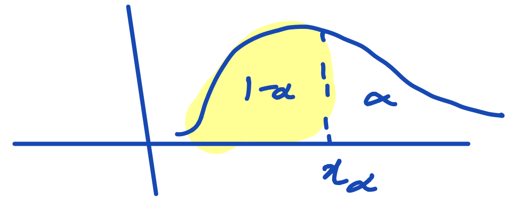
$= \Phi(1.35) - (1 - (\Phi(0.41))) = \boxed{0.5743}$

e) $P(X>345) = P\left(\frac{X-234}{34} > \frac{375-234}{34}\right) = P(Z>4.15) =$

$1 - \Phi(4.15) = 1 - 1 = \boxed{0}$

100(1−α) th percentile

Finding percentiles using Standard Normal Table

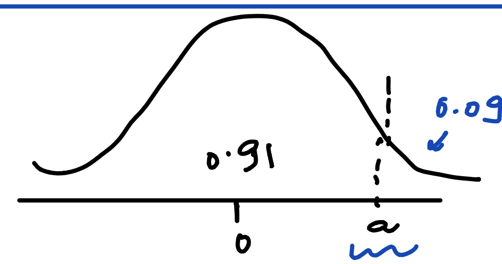


$F_X(x_\alpha) = 1-\alpha$

Q. $Z \sim N(0,1)$

z value

find a such that $P(Z \leq a) = 0.91$



a is 91st percentile of Z $\Rightarrow$ $\boxed{a = z_{0.09}}$

$$\boxed{x_\alpha = \mu + \sigma z_\alpha}$$

Example 4: Grades in Prof. Jones' course are normally distributed with mean 70 and variance 280. He wants to give 10% of the class an A. What cutoff score should he use to determine who gets A?

$$\boxed{X = \text{grade} \sim N(\mu = 70, \sigma^2 = 280)}$$

$$\boxed{z = \frac{X-\mu}{\sigma}}$$

90th percentile of X $\Rightarrow$ $x_{0.10} = 70 + (1.28)(\sqrt{280})$

$= \boxed{91.42}$

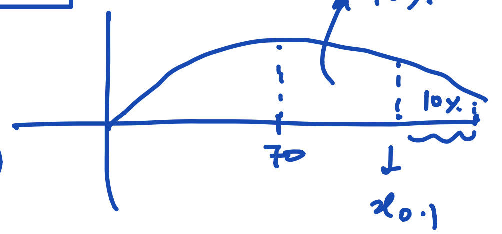
↳ This is z value

But we need x, not z so this formula

$z_{0.10}$: $\Phi(z_{0.10}) = 0.90$ ($\because$ sum before $x_{0.1}$)

$\downarrow$

$P(Z \leq z_{0.10}) = 0.90$



$\boxed{z_{0.1} = 1.28}$ $\Rightarrow$ from table

Example 3 (cont.): Birth weight of baby elephants in an elephant orphanage has a normal distribution with mean 234 lbs and standard deviation 34 lbs. What weight value is exceeded by 82% of baby elephants?

↳ 82% are heavier than that weight

$X = \text{weight} \sim N(\mu = 234, \sigma^2 = 34^2)$

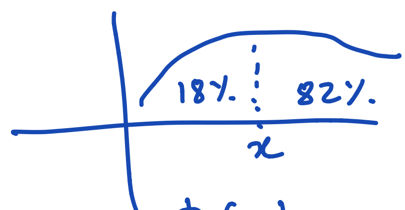
Since $0.18 < 0.5$

$\Rightarrow \phi(z) = 0.18 = 1 - \phi(-z)$ 18th percentile of X

$= \phi(-z) = 1 - 0.18 = 0.82$

$\Rightarrow -z = 0.92$

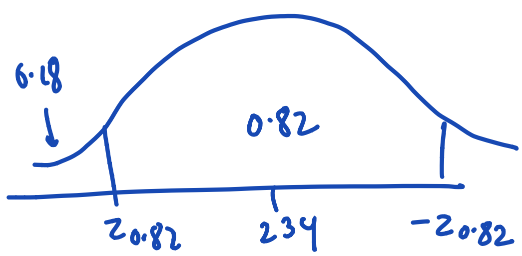
$\Rightarrow \boxed{z = -0.92}$



$\phi(x) = \underline{0.18}$

$x_{0.82} = 234 + 34 \cdot z_{0.82}$

$\boxed{= 202.87 \text{ lbs}}$



$-z_{0.82} = z_{0.18}$

$\Rightarrow -z_{\alpha} = z_{1-\alpha}$

$\Rightarrow \Phi(-z_{0.82}) = 0.82$

$\Rightarrow -z_{0.82} = 0.92$

$\Rightarrow z_{0.82} = -0.92$

Extension of Poisson, we see the event until first occurrence

Exponential, Gamma and Chi-square Distributions

Exponential Distribution

Example 5: Suppose customers arrive to a shop according to a Poisson process with rate 20 per hour. What can we say about the number of customers arriving during an hour? What can we say about the waiting time till the first customer arrives?

event happening at a rate.

In this context,

- the number of customers arriving during certain time period

  → discrete

  → Poisson

- the waiting time till the first/next customer arrives

  7)

  → continuous

  → exponential

Suppose occurrences of events happen according to a Poisson process with rate λ.

Let X be the waiting time until the first/next occurrence

$$\boxed{X \sim Exp(\lambda), \quad \lambda > 0}\;;\; \boxed{\text{1st next event}}$$

Properties of Exponential Distribution:

Pdf: $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

8)

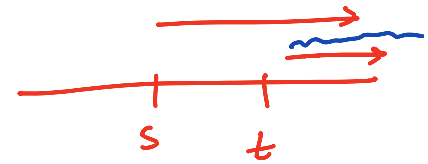
Cdf: $F(x) = \int_{-\infty}^{x} f(v)\,dv = F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

Mean and variance: $E(X) = \frac{1}{\lambda}$ $\quad Var(X) = \frac{1}{\lambda^2}$

$SD(X) = \sigma_\lambda = \frac{1}{\lambda}$

Memoryless property:

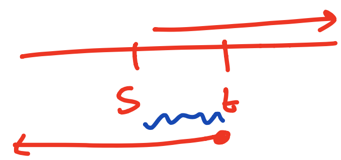
$X \sim Exp(\lambda) \Rightarrow$ for any $s, t > 0$



9)

$$P(X > s+t \mid X > s) = P(X > t)$$

$$P(X \leq s+t \mid X > s) = P(X \leq t)$$



Example 6: The number of miles that a particular car can run before its battery wears out is exponentially distributed with an average of 10,000 miles. The owner needs to take a 5,000 mile trip. Find the probability that he will be able to complete the trip without replacing the battery.

at least 5000 mile

X = lifetime of battery (in miles)

$X \sim Exp(\lambda)$

$P(X \geq 5000)$ $\quad E(X) = 10000 = \frac{1}{\lambda} \Rightarrow \boxed{\lambda = \frac{1}{10000}}$

- $P(X \geq 5000) = \int_{5000}^{\infty} \frac{1}{10000} e^{-\frac{1}{10000}x}\,dx = e^{-\frac{1}{2}} = \boxed{0.607}$

OR

- $P(X \geq 5000) = 1 - P(X < 5000) = 1 - F_X(5000) = 1 - \left(1 - e^{-\frac{5000}{10000}}\right)$

$= e^{-\frac{1}{2}} = \boxed{0.607}$

Example 5 (cont.): Customers arrive to a shop according to a Poisson process with rate 20 per hour. Let X be the waiting time for the arrival of the next customer (in minutes).

a) Find the probability that we have to wait no more than 5 minutes for the next customer.
b) If no one entered the shop in the last 3 minutes, what it the probability that at least one customer will enter in the next 2 minutes?
c) What is the mean time until the next customer arrives?
d) What is the median time until the next customer arrives?

$\lambda = \frac{1}{3}$ per min

X = time until next customer (in min)

$$X \sim Exp\left(\tfrac{1}{3}\right)$$

a) $$P(X<5) = \int_0^5 \frac{1}{3} e^{-\frac{1}{3}x}\,dx = 1 - e^{-5/3} = \boxed{0.811}$$

$$\boxed{= F_X(5) = 1 - e^{-\frac{1}{3}\cdot 5}}$$

b) $$P(X \le 3+2 \mid X > 3) = P(X \le 2) \text{ (Memoryless property)}$$

$$= F_X(2) = 1 - e^{-\frac{1}{3}\cdot 2} = \boxed{0.487}$$

c) $$E(X) = \frac{1}{\lambda} = \frac{1}{1/3} = 3 \text{ min}$$

d) $$q_2 \qquad \int_0^{q_2} \frac{1}{3} e^{-\frac{x}{3}}\,dx = 0.5$$

$$F_X(q_2) = 0.5 = 1 - e^{-\frac{1}{3}q_2}$$

$$\Rightarrow e^{-\frac{q_3}{3}} = 1 - 0.5 = \boxed{0.5}$$

$$\boxed{\Rightarrow q_3 = 2.08 \text{ min}}$$

→ Again we now see the event until α th occurrence

Gamma Distribution

What can we say about the waiting time for the arrival of the first two customers?

$\alpha = 2$

Suppose occurrences of events happen according to a Poisson process with rate λ.
Let X be the waiting time until the α^{th} occurrence

X is Gamma RV with parameters α and λ → $\boxed{X \sim Gamma(\alpha, \lambda)}$

10) The pdf of X is $f(x) = \dfrac{\lambda^\alpha}{\Gamma(\alpha)} \cdot x^{\alpha-1} \cdot e^{-\lambda x}$, $\boxed{x \geq 0} \Rightarrow S_x$

Some properties of Gamma function $\Gamma(\alpha) = \int_0^\infty t^{(\alpha-1)} e^{-t} dt$

11)
- If $\alpha > 1$, then $\Gamma(\alpha) = (\alpha - 1) \cdot \Gamma(\alpha - 1)$
- If $\alpha = n$ positive integer
- $\Gamma(n) = (n-1)!$ ⊛ $\boxed{\Gamma(\tfrac{1}{2}) = \sqrt{\pi}}$ → substitute

- $\Gamma(10) = 9\,\Gamma(9) = 9 \cdot 8 \cdot \Gamma(8) = \cdots = 9!$
- $\Gamma(10.5) = 9.5\,\Gamma(9.5) = (9.5) \cdot (8.5)\,\Gamma(8.5) = \cdots = (9.5).(8.5).(1.5).\,0.5\,\Gamma(0.5) = (9.5).(8.5)\ldots(1.5)(0.5)\sqrt{\pi}$

Properties of Gamma Distribution:

12)

Cdf: in general no closed-form expression

Mean and variance: $E(X) = \dfrac{\alpha}{\lambda}$ $Var(X) = \dfrac{\alpha}{\lambda^2}$

A special case: when $\alpha = 1$

$\boxed{X \sim Gamma(\alpha, \lambda)}$

$X \sim Gamma(\alpha = 1, \lambda) \sim Exp(\lambda)$

Example 5 (cont.): Customers arrive in a shop according to a Poisson process at a mean rate of 20 per hour.

a) What is the probability that we will have to wait more than 5 minutes for the arrival of the first two customers?

b) What is the expected time till the next 4 customers arrive?

$$\boxed{\lambda = \frac{1}{3} \text{ per min}}$$

y = time until 2nd customer (min)

$$\boxed{y \sim \text{Gamma}\left(\alpha = 2,\ \lambda = \frac{1}{3}\right)}$$

a) $$P(y>5) = \int_5^\infty \frac{(1/3)^2}{\Gamma(2)} \cdot y^{2-1} \cdot e^{-\frac{y}{3}}\, dy = \frac{1}{9 \cdot 1!} \int_5^\infty y e^{-y/3}\, dy$$

$$= \frac{1}{9}(-3) \int_5^\infty y\, de^{-y/3} = -\frac{1}{3}\left[y e^{-y/3} \Big|_5^\infty - \int_5^\infty e^{-y/3}\, dy \right]$$

$$= \frac{1}{3}\left[0 - 5e^{-5/3} - (-3)e^{-y/3} \Big|_5^\infty \right]$$

$$= \frac{5}{3} e^{-5/3} - \left(0 - e^{-5/3}\right)$$

$$= \frac{5}{3} e^{-5/3} + e^{-5/3} = e^{-5/3}\left(\frac{8}{3}\right) = \boxed{0.504}$$

b) W = time until the 4th customer

$$W \sim \text{Gamma}\left(\alpha = 4,\ \lambda = \frac{1}{3}\right)$$

- $\alpha \rightarrow$ may or may not be integers
- α & λ have to be +

$$\boxed{E(W) = \frac{\alpha}{\lambda} = \frac{4}{1/3} = 12 \text{ min}}$$

Chi-Square Distribution

X is Gamma RV, with $\alpha = \frac{r}{2}$, where r is a positive integer and $\lambda = \frac{1}{2}$, then X is a chi-square with r degrees of freedom

13) $\boxed{X \sim \chi^2_r}$ or $\boxed{X \sim \chi^2(r)}$

$\alpha = \frac{r}{2} \Rightarrow r$ DOF
$\lambda = \frac{1}{2}$

⊛ special case of Gamma distribution, so properties of Gamma dist. holds

$$\boxed{E(X) = \frac{r/2}{1/2} = r}$$

$$\boxed{Var(X) = \frac{r/2}{(1/2)^2} = 2r}$$

⊛ **Property**

⊛ Sum of squared independent normal RV

If $Z_1, Z_2 .. Z_k$ independent $N(0,1)$ then,

14) $$Z_1^2 + Z_2^2 + Z_3^2 + \dots Z_k^2 \sim \chi_k^2 \text{ or } \chi^2(k)$$

$$\boxed{\text{If } k=1, \quad Z \sim N(0,1) \Rightarrow Z^2 \sim \chi_1^2 \text{ or } \chi^2(1)}$$

Trick to Avoid Integration

Suppose $X \sim \text{Gamma}(\alpha, \lambda)$

$$E(X) = \int_0^\infty x \cdot \underbrace{\frac{\lambda^\alpha}{\Gamma(\alpha)}}_{\text{constant}} \cdot x^{\alpha-1} \cdot e^{-\lambda x}\, dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty \underbrace{x^{(\alpha+1)-1} e^{-\lambda x}}_{\text{upto a constant pdf of Gamma distribution } (\alpha+1, \lambda)}\, dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} \int_0^\infty \overbrace{\frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)} x^{(\alpha+1)-1} e^{-\lambda x}}^{\text{Pdf of Gamma } (\alpha+1, \lambda)}\, dx$$

constant $\times$ & $\div$

entire support

→ full density integrated over full support = 1

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} \cdot 1 = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \cdot \frac{1}{\lambda}$$

$$= \frac{\alpha\,\Gamma(\alpha)}{\Gamma(\alpha)\cdot\lambda} = \boxed{\frac{\alpha}{\lambda}}$$

Generalization ⇒

$$\int_{\text{entire support}} \text{density (of any kind)} = 1$$

15)

Distribution of a Function of a Random Variable

Let X be a continuous random variable with the pdf $f_X(x)$ and the cdf $F_X(x)$.

Let $Y = g(X)$.

We are interested in the distribution of Y. $\boxed{\text{pdf of } Y}$ & $\boxed{\text{cdf of } Y}$?

Example: Suppose random variable X is uniformly distributed on interval $(0,2)$. Consider $Y = \ln(X^3)$. Find the pdf and the cdf of Y.

support of RV X

S_X

Recall: $f_X(x) = \begin{cases} 1/2 & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$ $\quad F_X(x) = \begin{cases} 0 & x \le 0 \\ x/2 & 0 < x < 2 \\ 1 & x \ge 2 \end{cases}$

$\frac{x-0}{2-0} = \frac{x}{2}$

CDF approach:

replace X by equivalent Y in CDF & find PDF thereafter

$F_Y(y) = P(Y \le y) =$

Capital ⇒ RV | Small ⇒ a particular number

⇓

matters in cdf approach

$$y = \ln(x^3) = 3\ln(x) = g(x)$$

$$\Rightarrow \ln(x) = \frac{y}{3}, \quad x = e^{y/3} = h(y)$$

CDF approach

$$F_Y(y) = P(Y \le y) = P(3\ln X \le y) = P\left(\ln X \le \frac{y}{3}\right)$$

$$= P(X \le e^{y/3}) = F_X(e^{y/3}) =$$

This goes away

$$= \begin{cases} 0, & e^{y/3} \le 0 \\ \frac{e^{y/3}}{2}, & 0 < e^{y/3} < 2 \\ 1, & e^{y/3} \ge 2 \end{cases}$$

$$\Rightarrow F_Y(y) = \begin{cases} \frac{1}{2} e^{y/3}, & -\infty < \frac{y}{3} < \ln 2 \Leftrightarrow -\infty < y < 3\ln 2 \Leftrightarrow y < 3\ln 2 \\ 1, & y/3 \ge \ln 2 \Leftrightarrow y \ge 3\ln 2 \end{cases}$$

$f_Y(y) = F_Y'(y) = \begin{cases} \frac{1}{6}e^{y/3}, & y < 3\ln 2 \\ 0, & y \geq 3\ln 2 \end{cases}$

This doesn't remain $\frac{1}{2}$ as in X $\because$ $\int$ pdf has to give cdf mandatorily

PDF approach (change of variable):

$y = g(x) \qquad \Rightarrow \qquad x = h(y)$

find f_Y using

$f_Y(y) = f_X(h(y)) * |h'(y)|$ and determine the support S_Y

and then cdf using $\int$

$y = 3\ln x = g(x)$ | $x = e^{y/3} = h(y)$

$h'(y) = \frac{1}{3}e^{y/3}$

$f_Y(y) = f_X(e^{y/3}) \cdot \left|\frac{1}{3}e^{y/3}\right| = \frac{1}{2} \cdot \frac{1}{3}e^{y/3} = \frac{1}{6}e^{y/3}$

$\frac{1}{2}$, $0 < e^{y/3} < 2$; 0 o/w

since $f_X(x) = \begin{cases} \frac{1}{2}, & 0 < x < 2 \\ 0, & o/w \end{cases}$

Support

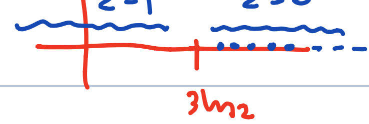
$\boxed{0 < e^{y/3} < 2} \Rightarrow -\infty < y/3 < \ln 2 \Rightarrow -\infty < y < 3\ln 2$

$= f_Y(y) = \begin{cases} \frac{1}{6}e^{y/3} & y < 3\ln 2 \\ 0 & y \geq 3\ln 2 \end{cases}$

$= F_Y(y) = \int_{-\infty}^{y} f_Y(v)\,dv$

$1/6\, e^{v/3}$ | 0 ; $3\ln 2$

$= F_Y(y) = \begin{cases} \int_{-\infty}^{y} \frac{1}{6}e^{v/3}dv & y < 3\ln 2 \\ 1, & y \geq 3\ln 2 \end{cases}$



$$\boxed{= \; f_Y(y) = \begin{cases} \frac{1}{2} e^{y/3} & , \; y < 3\ln 2 \\ 1 & , \; y \geq 3\ln 2 \end{cases}}$$

As for any point out of support ⇒ CDF = 1

⊛ Choose approach based on what's given, if cdf of X is given then cdf approach else if density of X is given, density approach is useful