Chapter 6: Several Random Variables

$a_1X_1 + a_2X_2 + \cdots + a_kX_k = \sum_{i=1}^{k} a_iX_i$

→ linear combination of 2 variables $X_1, \ldots X_k$ with coefficients $a_1, \ldots a_k$

Linear Combinations of Independent Normal Random Variables

Proposition: Let $X_1, X_2, \ldots, X_k$ be independent and $X_i \sim N(\mu_i, \sigma_i^2)$, $i = 1,2,\ldots,k$. Then

↳ must be independent and normal

1. $a_1X_1 + a_2X_2 + \cdots + a_kX_k = \sum_{i=1}^{k} a_iX_i \sim$

$$\boxed{\sim N\left(\sum_{i=1}^{k} a_i \mu_i, \sum_{i=1}^{k} a_i^2 \sigma_i^2\right)}$$

2. In particular, the sum $\sum_{i=1}^{k} X_i \sim$ $\boxed{N\left(\sum_{i=1}^{k} \mu_i, \sum_{i=1}^{k} \sigma_i^2\right)}$

linear combination of $X_1, \ldots X_k$ with coefficients $a_1 = 1 \ \ldots \ a_k = 1$ } → all coeff = 1

Example 1: Suppose the useful life (in years) of a refrigerator is normally distributed. The information for the useful life of three most common brands is summarized in the table.

brand	mean	standard deviation
1	10	3
2	9.5	2
3	11	4

The useful lives of refrigerators are independent. Suppose a refrigerator is randomly selected from each brand. Find the probability that the total useful life of the refrigerators from the first two brands will exceed 1.9 times the useful life of the refrigerator from the 3rd brand.

$X_i =$ useful life of refrigerator from brand $i = 1, 2, 3$

$X_1 \sim N(10, 3^2)$ | $X_2 \sim N(9.5, 2^2)$ | $X_3 \sim N(11, 4^2)$

Independent

$$P(X_1 + X_2 > 1.9X_3) = P(\underbrace{X_1 + X_2 - 1.9X_3}_{} > 0)$$

linear comb. of X_1, X_2, X_3 w/ coeff $a_1 = 1, a_2 = 1, a_3 = -1.9$

let $Y = X_1 + X_2 - 1.9X_3 \sim N\left(10 + 9.5 + (-1.9)(11),\ 1^2 \cdot 3^2 + 1^2 \cdot 2^2 + (-1.9)^2 (4)^2\right)$

$$= N(-1.4, 70.76)$$

now, $P(Y > 0) = P\left(Z > \dfrac{0 - (-1.4)}{\sqrt{70.76}}\right) = P(Z > 0.17)$

$= 1 - \Phi(0.17) =$

$1 - 0.5675 = \boxed{0.4325}$

$\Downarrow$

(from table)

Sums of Independent Random Variables (Not Normal) (only independent, not normal)

Proposition:

1. If $X_1, X_2, \dots, X_k$ are independent and $X_i \sim \text{Binom}(n_i, p)$, $i = 1,2,\dots,k$. Then

$$\sum_{i=1}^{k} X_i \sim \text{Binom}\left(\sum_{i=1}^{k} n_i, p\right)$$

2. If $X_1, X_2, \dots, X_k$ are independent and $X_i \sim \text{NegBinom}(r_i, p)$, $i = 1,2,\dots,k$. Then

$$\sum_{i=1}^{k} X_i \sim \text{Neg Binom}\left(\sum_{i=1}^{k} r_i, p\right)$$

3. If $X_1, X_2, \dots, X_k$ are independent and $X_i \sim \text{Pois}(\lambda_i)$, $i = 1,2,\dots,k$. Then

$$\sum_{i=1}^{k} X_i \sim \text{Pois}\left(\sum_{i=1}^{k} \lambda_i\right)$$

4. If $X_1, X_2, \dots, X_k$ are independent and $X_i \sim \text{Gamma}(\alpha_i, \lambda)$, $i = 1,2,\dots,k$. Then

$$\sum_{i=1}^{k} X_i \sim \text{Gamma}\left(\sum_{i=1}^{k} \alpha_i, \lambda\right)$$

Note: Unlike the above, in general, the sum of independent random variables from some class of distributions does not necessarily belong to the same class of distributions!

Example:

$$\left.\begin{array}{l} X_1 \sim \text{Unif}(0,1) \\ X_2 \sim \text{Unif}(0,1) \end{array}\right\} \text{independent}$$

$X_1 + X_2$ is not a uniform RV

$\boxed{\text{Ex 3a, Pg 258}}$