Chapter 2: Probability

Outcome space (sample space)

The collection of all the possible outcomes of a random experiment. They can be finite and infinite

Example 1:

1. Rolling a die

   $S = \{1,2,3,4,5,6\}$

2. Letter grade in this course

   $S = \{A, A-, B+, B, B-, C \ldots F\}$

3. Tossing a coin two times

   $S = \{HH, HT, TH, TT\}$

4. Tossing a coin two times and record the number of tails (T)

   $S = \{0,1,2\}$

5. Tossing a coin and record the number of *tosses* until we get the first tail

   $S = \{1, 2, 3 \ldots\}$ ∵ we atleast need to toss once

6. Tossing a coin and record the number of *H*'s until we get the first T

   $S = \{0, 1, 2, \ldots\}$ ∵ we can get T on first toss too.

7. Brain's reaction time to external stimuli

   $S = \{x : x > 0\} = (0, \infty)$, any real number $(\mathbb{R}^+)$

Events

An event is any subset of the sample space S. Denoted by A, B, C or $A_1, A_2 ---$

S is its own subset too

So, $\boxed{A \subseteq S}$

A occurs if the outcome of the experiment is in A

Example : Roll a die

$A = \{\text{even}\} = \{2,4,6\}$

$P(A) = \frac{3}{6}$

Algebra of sets

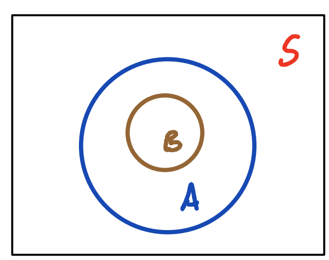
Null/empty set ∅ or event → The event that contains no outcomes

$\boxed{\phi}$

Subset

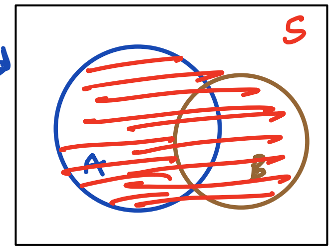
A is a subset of event B if all outcomes of A are in B.

$\boxed{A \subseteq B}$ or $\boxed{A \subset B}$



Union of A & B

$\boxed{A \cup B} \Rightarrow$



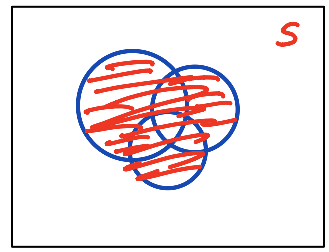
The collection of outcomes that are either in A or in B (or in both)

The union of events $A_1, \dots, A_k$ consists of all outcomes that are *in at least one* of $A_1, \dots, A_k$.

$A_1 \cup A_2 \cup A_3 \cdots \cup A_k$

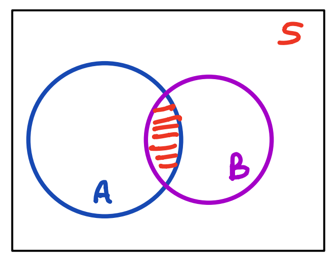
$$\boxed{\bigcup_{i=1}^{k} A_i}$$

Similarly $\boxed{A \cup B \cup C} \Rightarrow$



Intersection of A&B

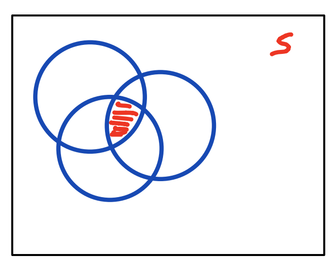
The collection of all outcomes that are in both A and B



The intersection of events $A_1, \dots, A_k$ consists of all outcomes that are *in all* of $A_1, \dots, A_k$.

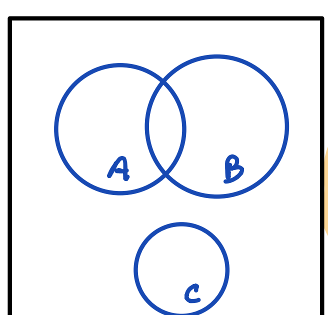
$A_1 \cap A_2 \cap A_3 \cdots \cap A_k$

$\boxed{A \cap B \cap C} \Rightarrow$



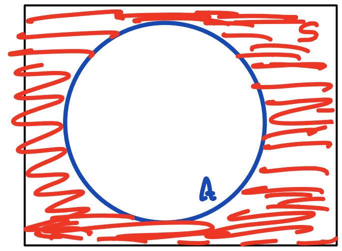
$$\boxed{\bigcap_{i=1}^{K} A_i}$$

If



$\Rightarrow$ In here $A \cap B \cap C = \phi$

Complement of A



The collection of all outcomes that are not in A

$\boxed{A^c}$ or $\boxed{\cancel{\bar{A}}}$ or $\boxed{A'}$

not advised

use this

$$\boxed{\bar{S} = \phi \quad \& \quad \bar{\phi} = S}$$

Example 2: Joe and Zoe take a fitness test in gym. Let A be the event that Joe pass the test and B be the event that Zoe pass the test. Express below in terms of events A and B.

- **a)** Zoe passes the test but Joe fails
- **b)** At least one of them pass the test
- **c)** Exactly one of them pass

A = { Joe passes the test } ; B = { Zoe passes the test }

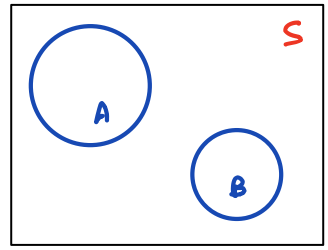
a) $B \cap \bar{A}$

b) $B \cup A$

c) $(B \cup A) \cap (\overline{A \cap B})$ | $(A \cap B') \cup (A' \cap B)$

Mutually exclusive or disjoint

A and B are mutually exclusive if $\boxed{A \cap B = \phi}$

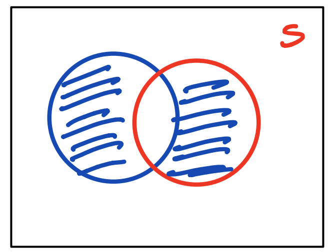


Events $\underbrace{A_1, \dots, A_k}_{\text{Partitions of } S}$ are mutually exclusive or disjoint $\boxed{\text{if } A_i \cap A_j = \emptyset \text{ for every } i \neq j.}$

Partition

Events $A_1, \dots, A_k$ form a partition (of the sample space) if they are

- mutually exclusive (disjoint)
- exhaustive



- A and $\bar{A}$ are always disjoint
- But $A \cup \bar{A} = S$ always so A & $\bar{A}$ form partitions

Commutative Laws:

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Associative Laws:

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

Distributive Laws:

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

De Morgan's Laws:

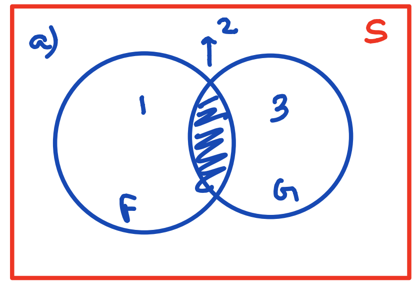
$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$
$$\& \quad \overline{A \cup B} = \bar{A} \cap \bar{B}$$

Example 3:

In a class, 30% of students speak French, 35% speak German, and 50% speak only one of the two languages. Let F denote F = {students who speak French} and G = {students who speak German}. What are the Venn diagram, set notation, and the percentage for:

- **a)** students who speak both French and German
- **b)** students who speak at least one of the languages
- **c)** students who do not speak French
- **d)** students who speak German, but not French



a) Students speak both F & G ⇒ $\boxed{F\cap G}$

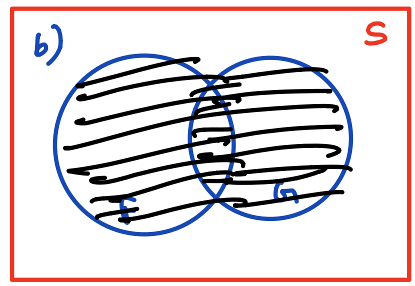
$$n(A\cup B) = n(A) + n(B) - n(A\cap B)$$

$$\Rightarrow n(A\cap B) = \underbrace{30}_{(1+2)} + \underbrace{35}_{(2+3)} - \underbrace{(1+2+3)}_{50}$$

But we know $1+3 = 50$, $2 = n(A\cap B)$

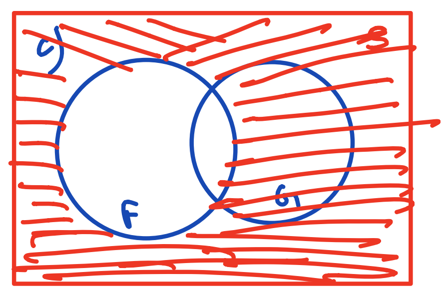
$$\Rightarrow 2n(A\cap B) = 65 - (50) = 15$$

$$\Rightarrow \boxed{n(A\cap B) = 7.5\%}$$



b) $\boxed{F\cup G}$

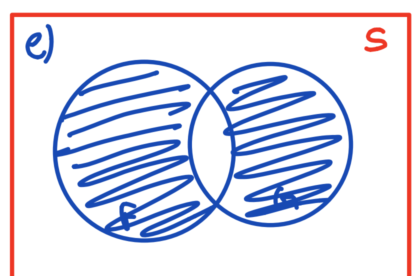
$$\boxed{n(A\cup B) = 65 - 7.5 = 57.5\%}$$



c) $\boxed{\bar{F}} = n(S) - n(F)$

$= 70\%$

d) $\boxed{G\cap \bar{F}}$



$n(G) = 35\%$

Now, $n(G) - n(G\cap F) = 35 - 7.5 = \boxed{27.5\%}$

e) 50% speak only 1 of the 2 languages ⇒ $\boxed{(\bar{F}\cap G) \cup (\bar{G}\cap F)}$

f) Are F & G mutually exclusive = ? ⇒ $\boxed{F\cap G \neq \phi \Rightarrow \text{Not mutually exclusive}}$

g) Example of a partition ⇒ $\boxed{(F\cup G) \text{ and } (F\cup G)'}$; $\boxed{F,\ G\cap \bar{F},\ (F\cup G)'}$

Probability interpretation in long run

Example: Suppose you roll a fair die and the event of interest is A = {observe "2"}. The probability of A is 1/6. What does it mean?

Suppose, N = # of times experiment is performed

(rolling a die)

N_A = # of times A happened

$$\boxed{\text{Relative frequency of A or, probability of A in long run} = \lim_{N \to \infty} \frac{N_A}{N} = \frac{1}{6}}$$

Probability Properties

Probability of an event A in the sample space S

Probability is a set of function with values in $[0,1] \Rightarrow \boxed{P(A)}$

3 axioms of probability:

1. $0 \le P(A) \le 1$ for any A
2. P(S) = 1
3. a) If $A_1, \dots, A_k$ are disjoint, then

$$P\left(\cup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} P(A_i) \quad \text{(finite additivity)}$$

$\ni A_i \cap A_j = \phi_{i \neq j}$

$\boxed{P(A_1 \cup A_2 \dots A_k) = P(A_1) + P(A_2) + \dots}$

b) If $A_1, A_2, A_3, \dots$ are disjoint, then

$$P\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \quad \text{(countable additivity)}$$

A probability model is a way of assigning probabilities to the sample space *S*.

Example 4: Roll a six-sided die. Which of the probability assignments below are valid and which are not? Why?

	"1"	"2"	"3"	"4"	"5"	"6"
Model 1	1	1	1/2	1/2	1	1/2
Model 2	1/6	1/6	1/6	1/6	1/6	1/6
Model 3	1/8	1/8	1/8	1/8	1/8	1/8
Model 4	1/3	1/6	1/6	1/6	0	1/6
Model 5	1/12	1/6	1/6	1/4	1/6	1/6

because of this, Model 4 is not practical ∵ it is no longer 6 sided, it is 5 sided

Concept to be used is $\boxed{P(S) = 1}$ should be valid

P(Model 1) & P(Model (3) don't add up to 1 ⇒ not valid models

$\boxed{\therefore \text{ Model 2 \& Model 4 are valid models}}$

$\boxed{\therefore \text{ Model 2 is the practical model out of all}}$ ⇒ Case of fair die

$\boxed{\text{Model 5 is also practical \& is a model}}$ ⇒ Case of non fair die

More Properties:

4. For any event A

$$\boxed{P(A) = 1 - P(\bar{A})}$$ for any A

Proof:

A, A^c disjoint

$\Rightarrow P(A \cup \bar{A}) = P(A) + P(A^c)$ (Axiom 3a)

$\Rightarrow P(S) = 1 \quad \because \boxed{A \cup \bar{A} = S}$

$\Rightarrow$ so if $P(\bar{A})$ is easier to find $\Rightarrow$ find it & for $P(A) \Rightarrow$ 1 − ans

5. Probability of null set is zero

$$\boxed{P(\phi) = 0}$$

Proof: $P(\phi) = \underbrace{1 - P(\phi')}_{\text{Prop 4}} = 1 - P(S) \underbrace{=}_{\text{axiom 2}} 0$

6. If events A and B are such that $A \subseteq B$, then

$$\boxed{P(A) \leq P(B)}$$

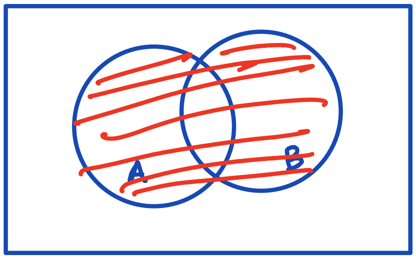
7. If A and B are disjoint, then

$$\boxed{P(\underbrace{A \cap B}_{\phi}) = 0}$$

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B)}$$

8. For any two events A and B $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Illustration



$\Rightarrow P(A \cup B)$

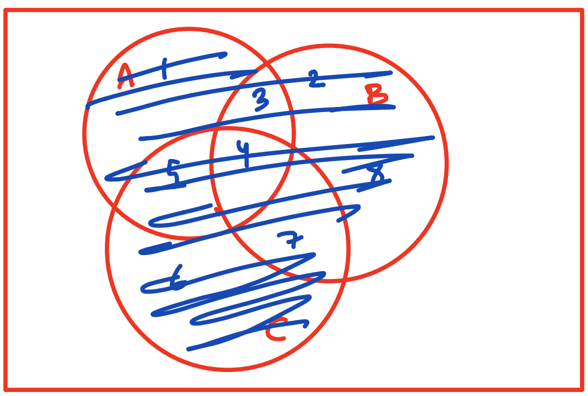
Venn Diagrams are never a prove

9. For three events A, B, and C,

$$\underbrace{P(A \cup B \cup C)}_{1+2+3+4+5+6+7+8} = \underbrace{P(A)}_{(1+3+5+4)} + \underbrace{P(B)}_{(2+3+4+7+8)} + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

. . - - - .

Because until now it wasn't included so we add.



10. Inclusion-Exclusion Formula

$$P(A_1 \cup A_2 \cup \ldots \cup A_n)$$
$$= \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots$$
$$+ (-1)^{n+1} P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

$$\boxed{\therefore \text{ General Term } = \sum_{i=1}^{n} P(A_i) + (-1)^{n+1} P(A_1 \cap A_2 \cdots A_n)}$$

$\Downarrow$

Proof by induction

Example 5: The probability that a company will open a branch office in Toronto is 0.7, that it will open one in Mexico City is 0.4, and that it will open in at least one of the cities is 0.8. Find the probability that they will open an office in:

- **a)** neither of the cities
- **b)** both cities
- **c)** exactly one of the cities

$$P(T) = 0.7$$
$$P(M) = 0.4$$
$$P(T \cup M) = 0.8$$

a) $\boxed{P(\overline{T \cup M}) = 1 - 0.8 = 0.2}$

b) $P(T \cap M) = ?$ $\Rightarrow P(T \cup M) = P(T) + P(M) - P(T \cap M)$

$\Rightarrow 0.8 = 0.7 + 0.4 - x$

$\Rightarrow x = 1.1 - 0.8 = 0.3$

$\Rightarrow \boxed{P(T \cap M) = 0.3}$

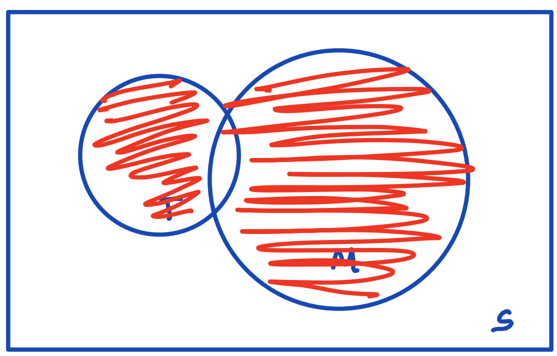
c) $P((\bar{T} \cap M) \cup (T \cap \bar{M})) \Rightarrow$

$(0.7 - 0.3) + (0.4 - 0.3)$

$= 0.4 + 0.1 = \boxed{0.5}$

$\Downarrow$

Same as $\boxed{P((T \cup M) \cap (T \cap M)')}$



Example ⇒ An urn contains 5 red, 6 blue and 8 green balls. A set of 3 balls is chosen at random without replacement

a) Probability they are all of different colors?

$$\boxed{P(E) = \frac{{}^{5}C_1 \times {}^{6}C_1 \times {}^{8}C_1}{{}^{19}C_3}}$$

b) Probability they are of the same color?

$$\boxed{P(E) = \frac{{}^{5}C_3}{{}^{19}C_3} + \frac{{}^{6}C_3}{{}^{19}C_3} + \frac{{}^{8}C_3}{{}^{19}C_3}}$$

⇓

3 cases