Chapter 5: Continuous Random Variables

Continuous Random Variables

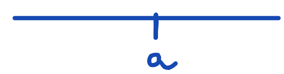
$\boxed{\begin{array}{c}\text{T/F} \\ \text{Regular} \\ \text{MCQ}\end{array}}$ $\Rightarrow$ Exam

RV is continuous if its support is an interval or union of intervals

The cumulative distribution function (cdf)

1) • $F_X(x) = P(X \le x)$

2) X – continuous RV



• $P(X=a) = 0$ for any real number a

3) • $0 \le F(x) \le 1$ for all x

• $F(x)$ is non-decreasing

• $P(a \le X \le b) = F(b) - F(a)$

$\quad\quad ||$

4) • $P(a \le X \le b) = P(a < X < b)$

The probability density function (pdf)

$f(x) \Rightarrow$ probability function

(Pdf density) of a continuous RV X is

5) $f(x) \geq 0$ for all x

&

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

⊛ pmf is for discrete r.v

⊛ Pdf is for cont. r.v

6)

$f(x)$ is a valid pdf iff

i) $f(x) \geq 0$ for all x

ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Support $S_X = \{x : f(x) > 0\}$

- $P(a \leq X \leq b) = F(b) - F(a) = \int_a^b f(x)\,dx$

- $F(a) = P(X \leq a) = P(-\infty < X \leq a) = \int_{-\infty}^{a} f(x)\,dx$

$$\Rightarrow \boxed{f(x) = \frac{dF(x)}{dx} = F'(x)}$$

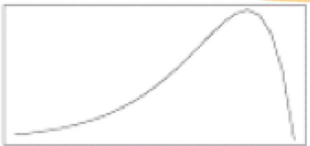
7) Finding pdf from cdf

Density curve describes the shape of the distribution. We can think of the density curve as "smooth" or idealized histogram.

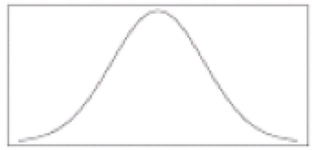
8)

Common shapes of density functions:

Unimodal (1 peak)

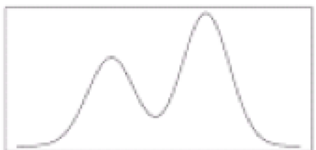


Left-skewed – negatively skewed

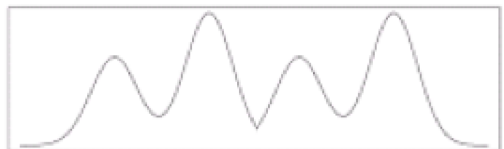


Symmetric

Right-skewed positively skewed



Bimodal (2 peaks)



Multimodal

Example 1: A professor never finishes his lecture before the end of the hour and always finishes his lectures within 2 minutes after the hour. Let X be the "extra" time (in minutes) that elapses between the end of the hour and the end of the lecture, and suppose the pdf of X is

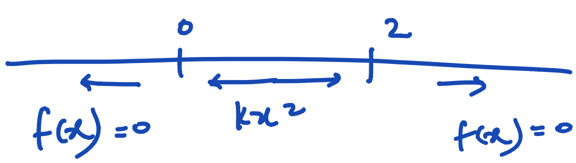
$$f(x) = kx^2, \quad 0 \le x \le 2$$

$\boxed{f(x) \Rightarrow \text{density}}$

a) What is the value of k?
b) What is the probability that the lecture continues beyond the hour for between 60 and 90 seconds?

$$\int_{-\infty}^{0} 0\,dx + \int_{0}^{2} kx^2\,dx + \int_{2}^{\infty} 0\,dx = \int_{0}^{2} kx^2\,dx = k\left[\frac{x^3}{3}\right]_0^2 = \boxed{k \cdot \frac{8}{3}}$$

a)



now, $\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$

& $f(x) \geq = 1$

So, $\int_{-\infty}^{\infty} f(x)\,dx = 1 = \int_{0}^{2} kx^2\,dx = \frac{8k}{3} = 1 \Rightarrow \boxed{k = \frac{3}{8}}$

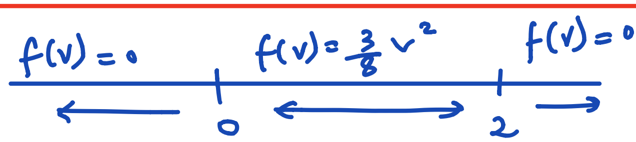
$$\Rightarrow \boxed{f(x) = \begin{cases} \frac{3}{8}x^2, & 0 \le x \le 2 \\ 0, & \text{o/w} \end{cases}}$$

b) $60 \text{ sec} = 1 \text{ minute} \Rightarrow P(1 \le X \le 1.5) \Rightarrow$

$$\int_{1}^{1.5} \frac{3}{8}x^2\,dx = \frac{3}{8}\cdot\left[\frac{x^3}{3}\right]_1^{1.5} = \frac{1.5^3 - 1^3}{8} = \boxed{0.297}$$

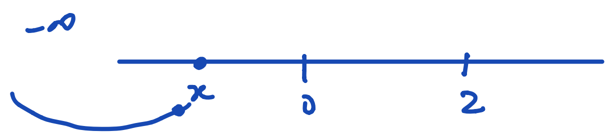
c) cdf of RV X?

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(v)\,dv$$

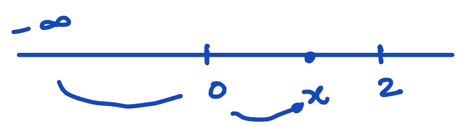


If $x < 0$, $\Rightarrow$ $F(x) = \int_{-\infty}^{x} f(v)\,dv$

$$= \int_{-\infty}^{x} 0\,dv = \boxed{0}$$



If $0 \le x \le 2$, $\Rightarrow$ $F(x) = \int_{-\infty}^{x} f(v)\,dv$

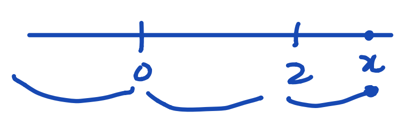


$$= \int_{-\infty}^{0} f(v)\,dv + \int_{0}^{x} f(v)\,dv$$

$$= 0 + \int_{0}^{x} \frac{3}{8}v^2\,dv = \boxed{\frac{x^3}{8}}$$

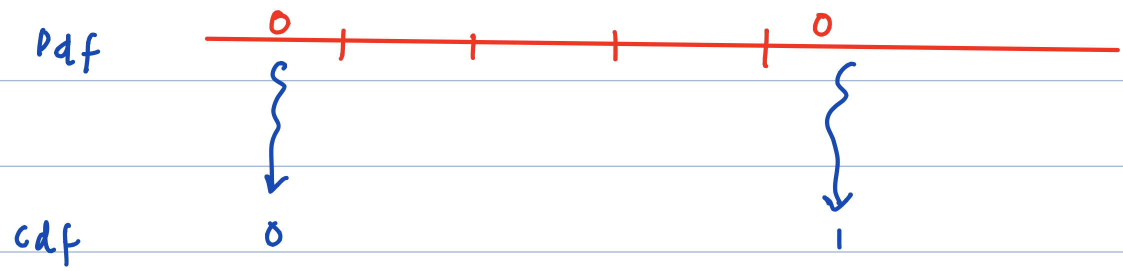
If $x > 2$ $\Rightarrow$ $F(x) = \int_{-\infty}^{x} f(v)\,dv = \int_{-\infty}^{0} f(v)\,dv + \int_{0}^{2} f(v)\,dv + \int_{2}^{x} f(v)\,dv$

$$= 0 + \int_{0}^{2} \frac{3}{8}v^2\,dv + 0 = \left[\frac{3}{8}\frac{v^3}{3}\right]_0^2 = \boxed{1}$$



Therefore, $\Rightarrow F(x) = \begin{cases} 0, & x<0 \\ \frac{x^3}{8}, & 0 \le x \le 2 \\ 1, & x>2 \end{cases}$

⊛ Generalization, (not always true but...)



⊛ Now, $P(1 \le X \le 1.5)$? $\boxed{\text{using cdf}}$

$$\Rightarrow F(1.5) - F(1) = \frac{1.5^3}{8} - \frac{1^3}{8} = \boxed{0.297}$$

$\because \frac{x^3}{8}$

$\boxed{\text{Outside support } f(x)=0}$

9) If she asks to show the CDF $\Rightarrow$ Proceed in the most general way starting $\int_{-\infty}^{x} f(x)\,dx$ & accounting for all x.

Continuous $\Rightarrow \int$, Discrete $= \Sigma$

Parameters of Continuous Distribution

Let X be a continuous random variable with the pdf $f(x)$.

+, −, 0 possible

The expected value of X $\qquad E(X) = \mu_X = \int_{-\infty}^{\infty} x \cdot f(x)\,dx \Leftrightarrow$ (weighted average)

10)

The variance of X $\qquad Var(X) = \sigma_X^2 = E((X-\mu_X)^2) = \int_{-\infty}^{\infty} (x-\mu_x)^2 f(x)\,dx$

≥0 always

$$= E(X^2) - \mu_X^2 = \left(\int_{-\infty}^{\infty} x^2 f(x)\,dx\right) - \mu_x^2$$

11)

≥0 always

The standard deviation $\qquad \sigma_X = \sqrt{Var(X)}$

12)

Expectation of a function $h(X)$

13)

1. In general, $E(h(X)) = \int_{-\infty}^{\infty} h(x) \cdot f(x)\,dx$, where $f(x)$ is density fn

2. If $h(X)$ is a linear function, i.e. $h(X) = aX + b$, then

Expectation

$\boxed{E(aX+b) = a\,E(X) + b}$

Variance

$\boxed{Var(aX+b) = a^2 Var(X) = \sigma_{aX+b}^2}$

SD

$\boxed{\sigma_{aX+b} = |a| \cdot Var(X)}$

E(X) ⇒ 1st moment of X

3. If $r = 1, 2, 3 \ldots$ the r^{th} moment of $X \Rightarrow$ $\boxed{E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx}$

4. r^{th} moment about constant b $\Rightarrow$ $\boxed{E[(X-b)^r] = \int_{-\infty}^{\infty} (x-b)^r f(x)\,dx}$

If $b = \mu$ & $r = 2$ $\Rightarrow E[(X-\mu)^2] = 2^{nd}$ moment about μ

Example 1 (cont.): A professor never finishes his lecture before the end of the hour and always finishes his lectures within 2 minutes after the hour. The pdf of X, the "extra" time (in minutes) that elapses between the end of the hour and the end of the lecture, is $f(x) = \frac{3}{8}x^2,\ 0 \le x \le 2$.

a) What is the mean of X?

b) What are the variance and standard deviation of X?

X = extra time

a) $$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^2 x \cdot \frac{3}{8}x^2\,dx = \frac{3}{8}\left[\frac{x^4}{4}\right]_0^2 = \boxed{1.5}$$

b) $$Var(X) = E(X^2) - (E(X))^2$$

$$\Rightarrow E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_0^2 x^2 \cdot \frac{3}{8}x^2\,dx = \boxed{2.4}$$

$$\Rightarrow (E(X))^2 = (1.5)^2 \Rightarrow Var(X) = 2.40 - 2.25 = \boxed{0.15}$$

$$\underline{SD} = \sqrt{0.15} = \boxed{0.387}$$

c) Mean & Variance of RV $Y = 2X+5$?

d) Mean & Variance of RV $W = X^{1/5}$?

Also do Taylor & Geometric Expansions

Percentiles

for cont.

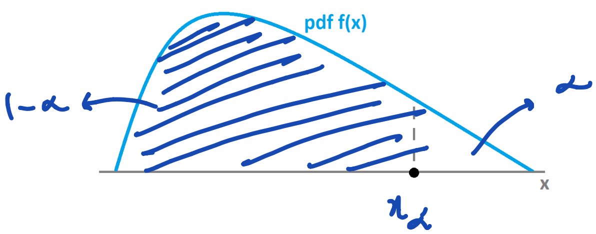
Let X be a continuous random variable with the pdf $f(x)$ and cdf $F(x)$. Let $0 < \alpha < 1$.

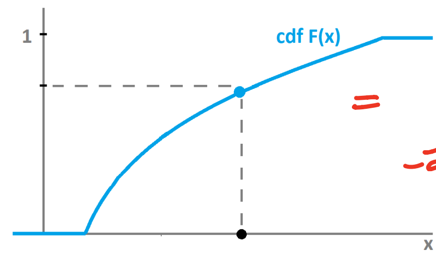
14) The $100(1-\alpha)$th percentile of X is x_α such that

15)
- $\int_{-\infty}^{x_\alpha} f(x)dx = 1-\alpha$
- $F(x_\alpha) = 1-\alpha$

So, $0<\alpha<1$, and $100(1-\alpha)^{th}$ percentile of X is $\ni$

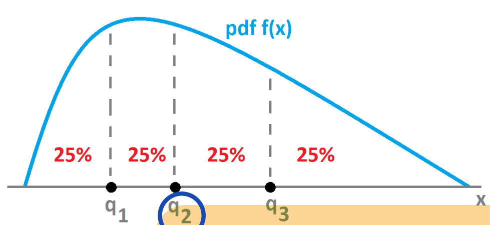
$x_\alpha = F_X(x_\alpha) = P(X \le x_\alpha) = 1-\alpha = \int_{-\infty}^{x_\alpha} f(x)dx = 1-\alpha$



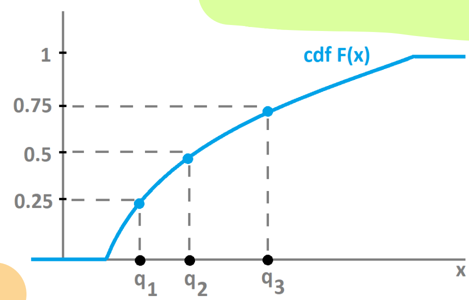


name = bottom prob ∋ to left
α = top prob ∋ whatever to right

The quartiles:



kinda center of distribution



- Lower quartile = 1sr quartile = 25 th percentile
  To find solve:

$q_1 = x_\alpha = x_{0.75}$ ∵ to right of q_1 is 75%
& $\int_{-\infty}^{q_1} f(x)dx = F(q_1) = 0.25$

- Median = 2nd quartile = 50 th percentile
  To find solve: $= \tilde{\mu}$

$q_2 = x_{0.5} = \tilde{\mu}$
& $F(q_2) = 0.5 = \int_{-\infty}^{q_2} f(x)dx = 0.5$

- Upper quartile = 3rd quartile = 75 th percentile
  To find solve:

$q_3 = x_{0.25}$
& $F(q_3) = 0.75 = \int_{-\infty}^{q_3} f(x)dx = 0.75$

Interquartile Range (IQR)

IQR = $q_3 - q_1$ = A kind of measure of a spread

Example 1 (cont.): Find the median and the interquartile range of the "extra" time that elapses between the end of the hour and the end of the lecture.

$X =$ "extra" time ; $f(x) = \frac{3}{8}x^2 \quad 0 \le x \le 2$

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^3}{8}, & 0 \le x \le 2 \\ 1, & x > 2 \end{cases}$$

16) Very necessary for quartile questions

cdf approach

q_2: $F(q_2) = 0.5 \Rightarrow \because 0.5 \Rightarrow 0 \le q_2 \le 2$ & $F(q_2) = \frac{q_2^3}{8} = 0.5$

$q_2 = 4^{1/3} = \boxed{1.59} = \tilde{\mu}$

Density approach

So, $\int_{-\infty}^{q_2} f(x)\,dx = 0.5 = \int_0^{q_2} \frac{3}{8}x^2\,dx = 0.5 \Rightarrow$ .... solve & find q_2

cdf approach

q_1: $F(q_1) = 0.25$

$\Rightarrow 0 \le q_1 \le 2 \Rightarrow \frac{q_1^3}{8} = 0.25 \Rightarrow q_1 = 2^{1/3} = \boxed{1.26}$

Density approach

$\int_0^{q_1} \frac{3}{8}x^2\,dx = 0.25 \Rightarrow$ ---- solve for q_1

cdf approach

q_3: $F(q_3) = 0.75 = \frac{q_3^3}{8} \Rightarrow q_3 = \boxed{1.82}$

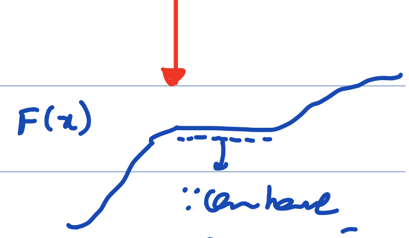
Density approach

$\int_0^{q_3} \frac{3}{8}x^2\,dx = 0.75 \Rightarrow$ ... find q_3

$IQR = q_3 - q_1$
$= 1.82 - 1.26$
$= \boxed{0.56}$

It is imp to write all the values of $F(x)$ depending on values of x, $\because$ will be used later.

⊛ cdf is non decreasing but continuous but not necessarily strictly increasing



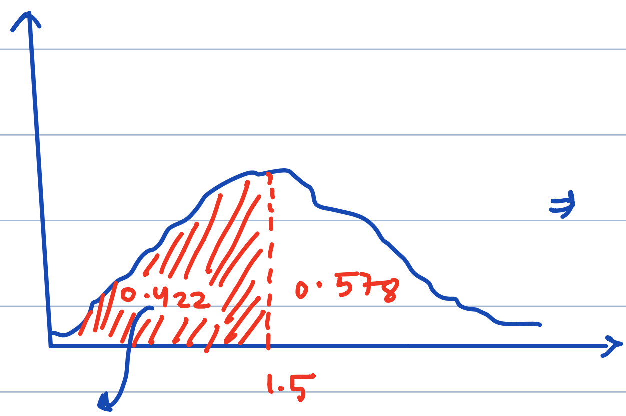
$\because$ can have flat regions as well

b) What percentile of X does the mean 1.5 represent?

$F(1.5) = \frac{1.5^3}{8}$ $\quad (\because 0 \le 1.5 \le 2) \quad \Rightarrow \quad F(1.5) = \boxed{0.422}$

or, $\int_0^{1.5} \frac{3}{8} x^2 \, dx = \ldots\ldots$ find $F(1.5)$ on solving this $\int_a^b$

So,



$\Rightarrow$ 1.5 is the 42.2th percentile of X

name (left)

So prob until 1.5 is 0.422 = cdf until 1.5

$\boxed{\Rightarrow \quad 1.5 = x_{0.578}}$

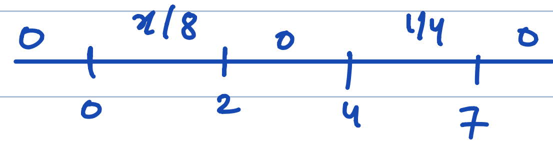
α (right)

Piece-wise pdf problem

Suppose X has pdf $f(x) = \begin{cases} \frac{1}{8}x, & \boxed{0 < x < 2} \\ \frac{1}{4}, & \boxed{4 \le x \le 7} \\ 0, & \text{otherwise} \end{cases}$

a) $E(X) = ?$

So, $S_X = (0,2) \cup [4,7]$



$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = 0 + \int_0^2 x\, f(x)\,dx + 0 + \int_4^7 x\, f(x)\,dx + 0$$

$$= \int_0^2 x \cdot \frac{x}{8}\,dx + \int_4^7 x \cdot \frac{1}{4}\,dx$$

$$= \int_0^2 \frac{x^2}{8}\,dx + \left.\frac{x^2}{8}\right]_4^7$$

$$= \left[\frac{x^3}{24}\right]_0^7 + \left(\frac{49}{8} - \frac{4}{8}\right) = \frac{343}{24} - \frac{45}{8}$$

$$= \boxed{4.46}$$

b) Find the CDF of X?

$$F(x) = \begin{cases} \overset{\text{Always}}{\boxed{0}} & , x \le 0 \\ & , 0 < x < 2 \\ & , 2 \le x < 4 \\ & , 4 \le x \le 7 \\ \underset{\text{Always}}{\boxed{1}} & , x > 7 \end{cases}$$

// See canvas for solutions

// the equality signs don't matter ∵ continuous