LC Classical Mechanics and Relativity 1 Lecture Notes

Year 1 Semester 1

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Table of Lectures

Tue 30 Sep 2025 12:00

Lecture 1 - Orders of Magnitude and Dimensional Analysis

Given some equation, for example $E = mv^2$, we can decompose it into the basic physical quantities that make it up, for example in terms of Mass, Time and Length. We can denote the dimensions of some quantity by wrapping it in square brackets.

$$E = \frac{1}{2}mv^2$$
$$[E] = ML^2T^{-2}$$

Example:

$$\text{Pressure} \equiv \frac{\text{Force}}{\text{Area}}$$

Suppose we want to test whether pressure and linear momentum flux (amount of linear momentum per unit time, per unit surface) were equivalent quantities, we could do this using dimensional analysis:

$$[P] = \frac{[F]}{[A]}$$
$$[P] = \frac{M \times LT^{-2}}{L^2}$$
$$= M/LT^2$$

And for linear momentum flux ($\Phi(p)$ where lowercase p is momentum):

$$\Phi(p) = \frac{[p]}{[A][\text{time}]}$$
$$= \frac{MLT^{-1}}{L^2T}$$
$$= \frac{M}{LT^2}$$

So yes, they seem to be (at least dimensionally) equivalent.

0.1 Challenging the LHC

We want to use orders of magnitude calculations to challenge the idea that the LHC is the "Big Bang Machine".

The LHC operates on the order of magnitude of approx 10TeV. The age of the universe is approx 13.7Bn Years, or (in orders of magnitude) 10^{10}yrs.

What time was the big bang? The Big Bang started the universe, but we can't really say it happened at $0s$, because that doesn't really make sense. What about 1sec? or 1ms? Well it's clearly less than both of those, so we want to find the smallest possible increment of time "Plank Second" and say it happened after one of them.

Thu 02 Oct 2025 15:00

Lecture 2 - Dimensional Analysis (contd.) and Vectors

0.1 Continuation of Dimensional Analysis

What if, in theory, we could build a system of units entirely from c, the speed of light, G, Newton's constant and h, the Plank Constant?

Cont. from Lec01, we can try to use this to work out the earliest possible cosmic time.

$$h = 6.6 \times 10^{-34} Js$$
$$G = 6.67 \times 10^{-11} Nm^2/kg^2 nvminstall20$$
$$c = 3 \times 10^8 m/s$$

Dimensionally:

$$[h] = \frac{ML^2}{T}$$
$$[G] = \frac{L^3}{T^2 M}$$
$$[c] = \frac{L}{T}$$

We want to use these to build out a time unit, so:

$$[h^u G^v c^z] = T$$
$$\left(\frac{ML^2}{T}\right)^u \left(\frac{L^3}{T^2 M}\right)^v \left(\frac{L}{T}\right)^z = T$$

$$M^{u-v} L^{2u+3v+z} T^{-u-2v-z} = T$$

Solving for:

$$u - v = 0$$

$$2u + 3v + z = 0$$

$$-u - 2v - z = 1$$

Gives us:

$$u = \frac{1}{2} \tag{2.1}$$
$$v = \frac{1}{2} \tag{2.2}$$
$$z = \frac{-5}{2} \tag{2.3}$$

$t_p = \sqrt{\frac{Gh}{c^5}}$ and plugging in the values for G, h, c gives us a value of time, which the earliest possible cosmic time equal to about $10^{-43} s$

0.2 Plank Energy

Doing the same process for energy gives us (this time, the plank energy is the energy at which traditional theories of physics break down):

$$E_p = \frac{hc^5}{G}^{0.5} \approx 10^9 J$$

On the other hand, the LHC manages about 10TeV, which is orders of magnitude smaller than this, so the LHC cannot accurate simulate energies of this magnitude.

0.3 More Vectors

Again, vector notation will be $\vec{a}$. We define the x, y, z unit vectors as $\underline{\hat{e_x}}, \underline{\hat{e_y}}, \underline{\hat{e_z}}$.

We can therefore define any vector as:

$$\vec{a} = a_x \underline{\hat{e_x}} + a_y \underline{\hat{e_y}} + a_z \underline{\hat{e_z}}.$$

The length of a vector is again $|\vec{a}|$.

0.4 Vector Multiplication

Given $\vec{a}$ and $\vec{b}$ we can define the dot (scalar) product and the cross (vector) product

$$\vec{a} \cdot \vec{b} = |a||b| \cos\theta$$

Say we want to know the component of a vector along an axis, we can do the following (eg for x):

$$\vec{a} \cdot \underline{\hat{e_x}} = a_x$$

For the vector product, we can define:

$$\vec{a} \times \vec{b} = |a||b| \sin(\theta) \hat{j}$$

As the vector perpendicular to the plane containing a and b. It is in the direction given by the *right hand rule*, where curling four fingers into a fist, and orienting your fist so these fingers sweep from $\vec{a}$ to $\vec{b}$, the new vector will point in the direction of an extended thumb. Theta is the angle between a and b, while j is the unit vector in the direction the new vector will point.

0.5 Solar Energy Example

The world yearly energy usage is about 180,000TWh, which is about $5 \times 10^{20} J$ total. Is it (theoretically) possible to get this all from solar energy? We can check using an approximate order of magnitude calculation.

The Sun's total luminosity is $L_\odot = 3.8 \times 10^{26}$. This energy is radiated in a spherically symmetric way (we assume). Therefore the energy per time, per unit surface is (using 1AU for distance):

$$\frac{L_\odot}{4\pi \times (1.5 \times 10^6)^2}$$

Which is approximately (using order of magnitude):

$$\frac{3.8 \times 10^{26} W}{10 \times 10^{22} m^2} \approx \frac{1kW}{m^2}$$

This is true in ideal conditions, and real energy supply is lower (due to clouds, atmosphere etc).

If we totally covered the earth's surface area ($A_{\text{surface}} \approx \pi R_\odot^2$) which is approximately:

$$A_{\text{surface}} \approx \pi \times (6 \times 10^3 \times 10^3 m)^2 \approx 10^{14} m^2$$

Therefore total energy received is approximately:

$$P = \frac{1kW}{m^2} \times 10^{14} m^2 \approx 10^{17} W$$

And to power the world:

$$E = \frac{5 \times 10^{20} J}{3 \times 10^{7} s} \approx 10^{13} W$$

So, it's theoretically possible, if we could cover enough of the world in solar panels and if we could perfectly capture the sun's energy without losing some to sources such as clouds, atmosphere, areas of the ocean we cannot cover in solar panels etc.

Tue 07 Oct 2025 12:00

Lecture 3 - Kinematics Introduction

For kinematics, we'll treat all objects as points and disregard aspects like rotation/the physical size of the body etc.

Given some point, we can define its position as a function of time $\vec{r}(t)$, and velocity as the derivative wrt time of this:

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

And acceleration:

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

0.1 Position from Unit Vectors

We can define:

$$\vec{r}(t) = r_x(t)\hat{e_x} + r_y(t)\hat{e_y} + r_z(t)\hat{e_z} = \sum_{j=1}^{3} r_j(t)\hat{e}_j$$

So:

$$\begin{aligned} \frac{d\vec{r}}{dt} &= \frac{d}{dt}\left(\sum_{j=1}^{3} r_j\hat{e}_j\right) \\ &= \sum_j \frac{d}{dt}(r_j\hat{e}_j) \\ &= \sum_j \frac{dr_j}{dt}\hat{e}_j \\ \vec{v} &= \sum_j v_j\hat{e}_j \end{aligned}$$

And:

$$\vec{a} = \frac{d\vec{v}}{dt} = \sum_{j=1}^{3} a_j\hat{e}_j$$

Note: Taking the derivative of a vector wrt time is looking at how the variable changes in some infinitesimal time. This can be a change in direction, and/or a change in magnitude. To differentiate a vector we can differentiate it component-wise.

0.2 Cartesian and Polar

Instead of representing a point as x and y components (in 2D), we can instead define it as a distance from the origin r and the angle this distance line forms with the positive x-axis θ.

Therefore (by basic right angle trig) $x = rcos\theta$, $y = rsin\theta$, and hence:

$$\vec{r} = r\cos\theta\hat{e}_x + r\sin\theta\hat{e}_y$$

So:

$$\vec{u(t)} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\cos\theta)\hat{e}_x + \frac{d}{dt}(r\sin\theta)\hat{e}_y$$

$$= \left(\dot{r}\cos\theta + r(-\sin\theta)\dot{\theta}\right)\hat{e}_x + \left(\dot{r}\sin\theta + r(\cos\theta)\dot{\theta}\right)\hat{e}_y$$
$$= \dot{r}\left(\cos\theta\hat{e}_x + \sin\theta\hat{e}_y\right) + r\dot{\theta}\left(-\sin\theta\hat{e}_x + \cos\theta\hat{e}_y\right)$$

0.3 Example

Lets model a particle, in a single dimension moving with constant acceleration (a_0) along a line. What is x(t)?

The introduction of τ here was generally poorly understood by the class at the time. Please see Lec 05 for a more thorough explanation.

$$a = \text{constant} = a_0$$
$$a = a_0 = \frac{dv}{dt}$$

So we can simply integrate to get $v(t)$ and again to get $x(t)$.

What if a is not constant? Consider $a(t) = kt^3$. We begin by redefining $a(t)$ as the following, where tau is a time constant representing one time unit. This could be one second, one year etc.

$$a(t) = \tau^3 k \left(\frac{t}{\tau}\right)^3$$
$$\text{let } a_* \equiv k\tau^3$$
$$a(t) \equiv a_* \left(\frac{t}{\tau}\right)^3$$

$$\int dv = \int a_* \tau \left(\frac{t}{\tau}\right)^3 d\frac{t}{\tau}$$

$$v = \frac{1}{4} a_* \tau \left(\frac{t}{\tau}\right)^4 + v_0$$

I've removed the rest here, as the whole τ thing was confusing in this lecture. Again, please see Lec 05, which re-does this section in a better manner.

Thu 09 Oct 2025 15:00

Lecture 4 - Projectile Motion and Reference Frames

Projectile Motion: The motion of a particle subject to gravitational acceleration, $g \approx 9.81$m/s

1 Projectile Motion

For this to hold, the height of the particle above the ground must be $m << R_e \approxeq 6 \times 10^3$km.

$$x(t) = x_0 + v_0(t - t_0) + \frac{1}{2}a_0(t - t_0)^2$$

Lets begin solely by considering motion in the vertical axis (called z here, for some strange reason). This particle is falling from height hm, to the ground at $h = 0$, with constant acceleration gm/s^2. It has been dropped at time $t = t_0$

At time $t_0, v = 0, z = h$

$$z(t) = h + 0 - \frac{1}{2}gt^2$$
$$\frac{1}{2}gt^2 = h$$
$$\implies t = \sqrt{\frac{2h}{g}}$$

1.1 What about 2D?

Now we can expand our example to (rather than drop the particle from rest) give the particle some initial velocity v_0m/s parallel to the ground. We now want two position functions, $x(t)$ and $z(t)$. As previously calculated:

$$z(t) = h + 0 + \frac{1}{2}(-g)t^2$$

And horizontally:

$$x(t) = 0 + v_0(t) + 0$$

So:

$$\begin{cases} z = h - \frac{1}{2}gt^2 \\ x = v_0 t \end{cases}$$

Rearranging:

$$t = \frac{x}{v_0}$$
$$z = h - \frac{1}{2}g\left(\frac{x}{v_0}\right)^2$$

Since h, g, v_0 are all constants, this is an x^2 parabola.

1.2 Interplanetary Example

Lets consider some planet, with $g_{\text{planet}} = 5m/s^2$. You (denoted Y) fall into the atmosphere at some distance h from the ground, and some horizontal distance d from $O(x = 0)$. There is an alien who wants to kill you, by shooting you down. This "gun" can throw pebbles at some constant speed v_0. The only degree of freedom the alien has to target you is change the shooting angle wrt the horizontal, θ. From the alien's perspective, what is the required θ to hit the incoming spacecraft?

To hit you, there is some time t, when the position of the bullet B, with initial velocity v where B is in the same position as Y

Consider B

$$x_B(t) = v_0 \cos(\theta)t$$

$$z_B(t) = v_0 \sin(\theta)t - \frac{1}{2}g_p t^2$$

Consider Y

$$x_Y(t) = d$$

$$z_Y(t) = h - \frac{1}{2}g_p t^2$$

We want to find a θ where $x_B = x_Y$ and $z_B = z_Y$ at the same t:

$$v_0 \cos(\theta)t = d \tag{4.1}$$

$$v_0 \sin(\theta)t - \frac{1}{2}g_p t^2 = h - \frac{1}{2}g_p t^2 \tag{4.2}$$

From 2:

$$v_0 \sin(\theta)t = h$$

$$\implies t = \frac{h}{v_0 \sin\theta}$$

And substituting:

$$v_0 \cos(\theta)\left(\frac{h}{v_0 \sin\theta}\right) = d$$

$$\frac{\cos(\theta)h}{\sin(\theta)} = d$$

$$\frac{\cos\theta}{\sin\theta} = \frac{d}{h}$$

$$\tan\theta = \frac{h}{d}$$

Since we have the value of θ in terms of two constants, yes, the alien can always hit the spaceship provided it correctly selects the angle corresponding to the value of these two constants (excluding cases where the particle is too far to the left to possibly be hit regardless of angle). This means that the required angle does not depend on velocity, in this example.

2 Frames of Reference

"Observer" represents a frame of reference. The way that one person sees the world (in terms of relative positions and velocities) is different to how another person may see the world. We observe the same core physics, but need to do coordinate translations to go from one reference frame to another.

Say we have two reference frames, A and B. We can represent the translation from A to B as a vector, denoted r. Some vector b_B in B's frame of reference is therefore equal to:

$$b_B = r + b_A$$

Assume that the frames are moving with a constant uniform velocity u with respect to reach other:

$$\frac{d}{dt}(b_B) = \frac{d}{dt}(r) + \frac{d}{dt}(b_A)$$

$$v_b = \frac{dr}{dt} + v_r = u + v_r$$

This is known as the “Galilean Transformation”.

Tue 14 Oct 2025 12:00

Lecture 5 - The Tau Incident and Special Relativity I

In this lecture:

- Vecchio clarifying things he'd been asked from Kinematics.
- The start of Special Relativity.

1 Use of Tau

This caused quite a bit of confusion for people in Lec 03. We have a particle subject to constant acceleration $\boldsymbol{a} = a_0$.

The displacement of a particle at time t is given by:

$$x(t) = x_0 + v_0(t - t_0) + \frac{1}{2}a_0(t - t_0)^2$$

This is only true for a constant acceleration. More generally, we have:

$$a(t) = \frac{dv}{dt}$$

So we can integrate twice to get $x(t)$. Consider the example where $a(t) = kt^3$. This is non-constant acceleration. We assume that $t_0 = 0$ to simplify things a little. We further assume that $v(t = t_0) = v(t = 0) = 0$ and $x(t = t_0) = x(t = 0) = 0$.

We want to determine $x(t)$.

$$\frac{dv}{dt} = kt^3 \implies dv = kt^3dt$$

$$v - v_0 = \left.\frac{kt^4}{4}\right|_{t_0}^{t}$$

Since we have $v_0 = t_0 = 0$, we have:

$$v(t) = \frac{k}{4}t^4$$

And integrating again:

$$dx = vdt$$

$$x - x_0 = \left.\frac{k}{4}\frac{t^5}{5}\right|_0^t = \frac{k}{20}t^5$$

Again, $x_0 = 0$ so we finally get:

$$x(t) = \frac{k}{20}t^5$$

Note we have simplified by assuming the initial conditions are all 0, hence we can disregard v_0 etc. If we didn't have this, we'd have to include them in the integration all the way down.

The goal of using τ is to make the problem clearer and easier to understand. Going back to $a(t) = kt^3$, we can tell by dimensions that k must have units of an acceleration divided by a time cubed. This is a messy constant with dimensions then of $[k] = L/T^5$. It is therefore difficult to see what an increase in one time unit actually causes $a(t)$ to do.

We can pick a constant timescale called τ. Tau can be whatever we like, one hour, one millisecond, fifteen years etc etc. We rewrite:

$$a(t) = kt^3 = k\frac{t^2}{\tau^3}\tau^3 = k\tau^3\left(\frac{t}{\tau}\right)^3$$

We now have a new constant with units of acceleration, $k\tau^3$ which we call a_*.

$$a(t) = a_*\left(\frac{t}{\tau}\right)^3$$

This lets us think about the problem a little more clearly, as we know that after one τ has passed, the object will have acceleration a_*. After two τs of time have passed, the object will have acceleration $2^3a_* = 8a_*$ etc. The acceleration now nicely scales in a cubic manner.

Reintegrating with τ gives:

$$dv = adt$$

$$v - v_0 = a_*\tau\left(\frac{t}{\tau}\right)^4\frac{1}{4}$$

In our case for a particle starting at rest:

$$v(t) = \frac{1}{4}a_*\tau\left(\frac{t}{\tau}\right)^4$$

And for x:

$$x - x_0 = \frac{1}{4}a_*\tau^2\frac{1}{5}\left(\frac{t}{\tau}\right)^5$$

$$x = \frac{1}{20}a_*\tau^2\left(\frac{t}{\tau}\right)^5$$

The benefit of τ for v and x is a bit less stark, but it's still somewhat present. For constant acceleration, we can write either:

$$x(t) = \frac{1}{2}a_0t^2$$

$$x(t) = \frac{1}{2}a_0\tau^2\left(\frac{t}{\tau}\right)^2$$

The distance travelled over some time τ is $\frac{1}{2}a_0\tau^2$. Note that we can compare this to the derived result for non-constant acceleration, so using τ gives us a more comfortable and familiar form even in the non-constant scenario.

2 Normalisation

This is where we described a particle's position not in terms of unit vectors $\hat{e}_x, \hat{e}_y, \hat{e}_z$ and instead using polar form $\hat{e}_\theta$ and $\hat{e}_r$.

Consider a particle in circular motion (i.e. a child on a merry-go-round). Lets say the child wants to accelerate their motion, we want to keep the distance from the origin constant (or the child would fly off!) while increasing the speed around the circle. Doing this with the former notation would change the coefficients all three unit vectors, while using the latter notation allows us to express it as only a single constant multiplied by the unit vector changing unit vector.

3 Special Relativity

We will cover:

- The Lorentz Factor γ.
- Time Dilation
- Length Contraction

Special relativity is about how two different observers observe the kinematics of objects. For special relativity to hold, these observers cannot be accelerating. They must move with constant velocity with respect to each other. For an observer, we describe an event with four coordinates: (x, y, z, t), where t is time. For a moving observer, it will see the same event, but at a different set of coordinates (x', y', z', t').

We note that the speed of light c must be constant and independent of any observer. If two observers measure c in a vacuum, they will both determine the same value regardless of motion. This breaks the standard rules of kinematics that we've seen so far, and it means that time and space are both relative - i.e. one second for one observer may be different to one second for another.

We have these assumptions:

- Two inertial observers will observe the same physics.
- Two inertial observers will observe the same speed of light.

Everything in special relativity scales with the 'Lorentz Factor' in some form, given by:

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{u}{c}\right)^2}}$$

Gamma is always larger than 1, as $u < c$. We have two key results which we will derive later:

- Moving clocks run slow - a moving observer experiences time slower relative to a static observer.
- Moving objects shrink - a static observer will observe that a moving object has shrunk relative to what the moving observer measures about itself.

Thu 16 Oct 2025 15:00

Lecture 6 - Special Relativity II

1 Special Relativity

We consider theoretical observers that are unaccelerated with respect to each other. Either both observers are at rest, or moving with respect to each other a a constant speed. For ease in CMR1, we only consider motion in one dimension.

We also say that:

- The First Law of Dynamics (Newton's First Law) still holds true, so an object at rest will remain at rest, and an object in constant motion will remain in constant motion, unless an external force acts upon it.
- The distance between two points is constant (relative to an observer).
- We can synchronise clocks between two observers, and they will tick at the same rate.
- We only deal with Euclidean geometry.

We have two key postulates:

1. The laws of physics are the same for every inertial observer.
2. The speed of light in a vacuum is constant for every inertial observer. It is independent of any motion of the source or the observer. Even if a source travelling at $0.5c$ shines a laser facing forward, that light will still travel at c, and not $1.5c$.

1.1 Lorentz Factor

We have some stationary observer A, and a second observer B which is moving at velocity um/s relative to A. The Lorentz Factor γ is defined as:

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{u}{c}\right)^2}}$$

We may also see it written as:

$$\gamma = \frac{1}{\sqrt{1 - (u)^2}}$$

Which holds only if u is already measured in units of the speed of light. For this course, we use the first definition. This is also known as the "Relativistic Factor". Note that it is dimensionless and is a positive number $\gamma > 1$, as $u < c$.

Taking Limits: We take limits of γ to see it's behaviour as u changes relative to the speed of light.

If $u \ll c$:

$$\frac{u}{c} \ll 1$$

We use ϵ to denote a very small value. Let $\epsilon \equiv u/c$.

$$\gamma = \frac{1}{\sqrt{1 - \epsilon^2}}$$

We expand this using a Taylor Series:

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0) + \cdots$$
$$\gamma = 1 - (-\frac{1}{2}\epsilon^2) + \cdots$$

If u is very fast, say $u = 30$km/s, then:

$$\epsilon = \frac{3 \times 10 \times 10^3}{3 \times 10^8}$$
$$\epsilon = 10^{-4}$$

Hence:

$$\gamma = 1 + \frac{1}{2}10^{-8} + \cdots$$

So even for speeds which are classically extremely fast, $\gamma \approx 1$ and we therefore do not encounter relativistic effects in classical mechanics.

If u/c is 'large', i.e. $u/c \to 1$:

Now, $1 - u/c$ is small, so we define $\epsilon \equiv 1 - \frac{u}{c} \ll 1$ instead.

$$\frac{u}{c} = 1 - \epsilon$$

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{u}{c}\right)^2}} = \frac{1}{\sqrt{\left(1 - \frac{u}{c}\right)\left(1 + \frac{u}{c}\right)}}$$
$$\gamma = \frac{1}{\sqrt{(\epsilon)(1 + (1 - \epsilon))}}$$
$$\gamma = \frac{1}{\sqrt{2\epsilon - \epsilon^2}}$$

Since $\epsilon \ll 1$, we say that the ϵ^2 term is small enough to disregard, so we have:

$$\gamma = \frac{1}{\sqrt{2}}\epsilon^{-1/2}$$

Again since ϵ is very small, $\epsilon^{1/2}$ tends to infinity, so as $\gamma \propto \epsilon^{-1/2}$, $\gamma \to \infty$. For non-relativistic objects, we therefore treat $\gamma = 1$, but a curve of γ against u/c has an asymptote at $u/c = 1$, hence γ rapidly increases unbounded as $u \to c$.

2 Einstein's Thought Experiment

We want to design a clock. We do so by creating a perfect cylinder, with a perfectly reflective top and bottom.

We place a light source (laser) at the bottom, and we shine this laser up towards the top of the cylinder. The photons travel to the top, hit the ceiling, which is perfectly reflective, so travels back down.

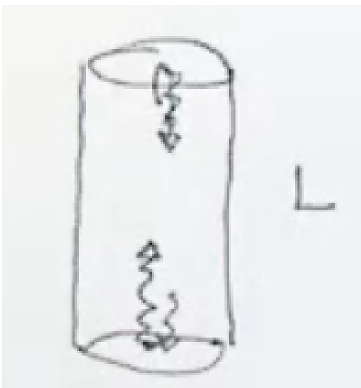

Figure 6.1

We also have a perfect clock, which measures the round-trip time for the photon to go up, hit the ceiling and hit the bottom again. The cylinder is L distance units high, so the total photon path is $2L$ for the round trip. The time taken is therefore:

$$\Delta t = \frac{2L}{c}$$

We add two observers, (B) who is fixed to the top of the cylinder (and is travelling with it). We have some other observer (A) who has designed the problem to place the whole cylinder on a moving trolley, moving in 1D with speed u. Observer A is standing stationary on the ground as the trolley speeds past them.

Observer (B) while moving sees a photon emitted at the bottom, travel up and reflect back down, with no issues.

However, Observer (A) sees the whole setup moving. It sees a photon emitted and travel up, and while it travels up the trolley has moved some distance. The trolley (and photon) have moved some distance when the photon strikes the top and reflects. As the photon travels back down, the trolley (and photon) have moved some distance again.

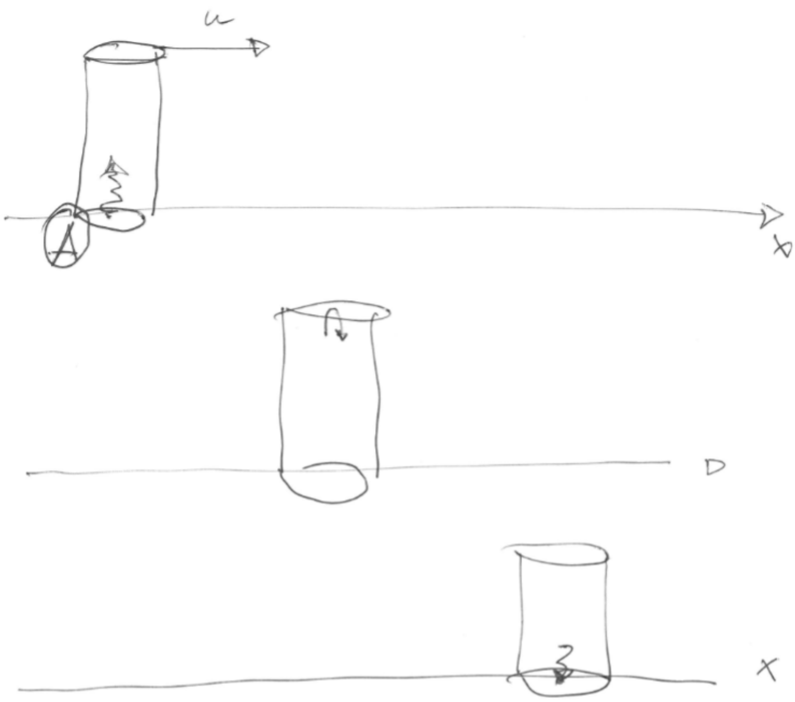

Figure 6.2: The trolley at point of photon emission, point of reflection, and point of detection. Note the photon has moved with the trolley.

We note that the height of the cylinder is not affected by the motion of the trolley, as the motion is perpendicular to this length. We are given this as fact. The only lengths which may be affected are the lengths with components in the direction of motion.

From the perspective of (B), the photon has taken the standard and simple up-down path, in some time Δt. However, from (A)'s perspective, the photon has taken a much longer path which includes horizontal motion:

Figure 6.3

In Observer (A)'s reference frame:

The two (equal) lengths that form the bases of the two right-angled triangles have length $u\frac{\Delta t}{2}$, and the two hypotenuses have length $c\frac{\Delta t}{2}$.

We therefore can simply use Pythagoras to get:

$$\left(u\frac{\Delta t}{2}\right)^2 + L^2 = \left(c\frac{\Delta t}{2}\right)^2$$

In Observer (B)'s reference frame:

$$\Delta t = \frac{2L}{c}$$

We donate the clocks held by (A) to give measurements:

$$\Delta t_B = 2\frac{L_B}{c_B}$$

and for (A):

$$\left(u\frac{\Delta t_A}{2}\right)^2 + L_A^2 = \left(c_A\frac{\Delta t_A}{2}\right)^2$$

We know that the speed of light is identical in every reference frame, so $c_A = c_B = c$. We have been told that L is unaffected, since it is perpendicular to the direction of motion, so $L_A = L_B = L$.

Hence:

$$\Delta t_B = \frac{2L}{c}$$

Which we can rearrange and substitute to get:

$$\left(u\frac{\Delta t_A}{2}\right)^2 + \left(\frac{c\Delta t_B}{2}\right)^2 = \left(c\frac{\Delta t_A}{2}\right)^2$$

$$(c\Delta t_B)^2 = (c\Delta t_A)^2 - (u\Delta t_A)^2$$

For this to be true, $t_B \neq t_A$, so the two observers can no longer agree on the time the photon took. This gives us time dilation, where moving clocks (i.e. the clock used by (A)) run slower, and record a longer time between two events compared to a stationary observer.

Tue 21 Oct 2025 12:00

Lecture 7 - Special Relativity III

1 Time Dilation

We concluded the previous lecture with:

$$(c\Delta t_B)^2 = (c\Delta t_A)^2 - (u\Delta t_A)^2$$

We rearrange to get:

$$c^2\Delta t_B^2 = \left(c^2 - u^2\right)\Delta t_A^2$$
$$\Delta t_B^2 = \left(1 - \left(\frac{u}{c}\right)^2\right)\Delta t_A^2$$
$$\Delta t_B = \sqrt{1 - \left(\frac{u}{c}\right)^2}\Delta t_A$$

Hence:

$$\Delta t_A = \frac{1}{\sqrt{1 - \left(\frac{u}{c}\right)^2}}\Delta t_B$$
$$\boxed{\Delta t_A = \gamma\Delta t_B}$$

This tells us that the time recorded for the photon to travel by the two observers is different. The moving clock runs slower, so the stationary observer measures a longer duration than the moving clock records. For non-relativistic speeds, $\gamma \approx 1$ so the difference is negligible; however, at larger speeds the disparity becomes much larger and grows without bounds. This makes physical sense, as for faster speeds, the trolley will have travelled a larger horizontal distance, therefore (A) will measure a longer path, and hence require a larger time.

Generally, we have:

$$\Delta T = \gamma\Delta t_0$$

Where t_0 is the "proper time" and is defined as the time interval taken between two events that take place in the same frame, by an observer in that frame.

2 Length Contraction

We have the same identical setup, except the setup is now horizontally on the trolley. Observer (B) is again attached to the cylinder, with photons again bouncing along the length of the cylinder, just with the left/right instead of top/bottom surfaces. The cylinder is still moving on a trolley, and observer (A) is still stationary.

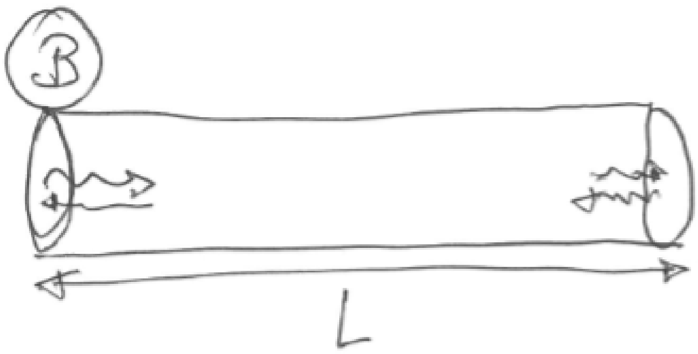

Figure 7.1

We define Δt as the interval between emission and detection, again with a subscript to denote who is making the measurement.

$$\Delta t_B = \frac{2L_B}{c}$$

In (B)'s frame of reference:

$$c\Delta t_B = 2L_B$$

In (A)'s frame of reference:

The cylinder has moved to the right as the photon travels, this adds some extra length that the photon must travel.

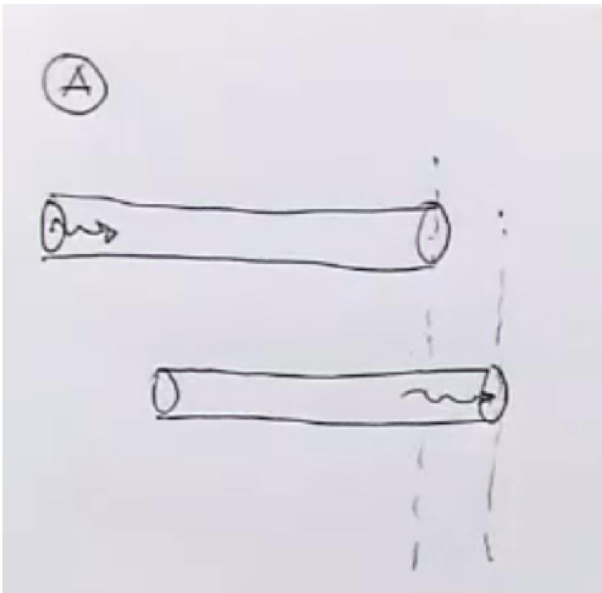

Figure 7.2

This extra length (between the two dotted lines) is $u\Delta t_1$ where Δt_1 is the time for the photon to hit the right wall. The photon now hits the wall and bounces back, while the cylinder is still moving to the right. The cylinder will move $u\Delta t_2$, where Δt_2 is the time taken for the photon to travel back and hit the left wall.

The time taken between emission and detection Δt is given by:

$$\Delta t = \Delta t_1 + \Delta t_2$$

For the first part of the trip, the distance is $L_A + u\Delta t_1$, so:

$$c\Delta t_1 = L_A + u\Delta t_1$$

$$\Delta t_1 = \frac{L_A}{c-u}$$

For the second part of the trip, the distance is less as the cylinder "catches up" with the photon as it moves, giving us a distance of $L_A - u\Delta t_2$. This gives us:

$$c\Delta t_2 = L_A - u\Delta t_2$$

$$\Delta t_2 = \frac{L_A}{c+u}$$

Hence the round-trip time is:

$$\begin{aligned}
\Delta t &= \frac{L_A}{c+u} + \frac{L_A}{c-u} \\
&= L_A\left(\frac{c+u+c-u}{(c-u)(c+u)}\right) \\
&= L_A\left(\frac{2c}{(c-u)(c+u)}\right) \\
&= 2cL_A\left(\frac{1}{c\left(1-\frac{u}{c}\right)c\left(1+\frac{u}{c}\right)}\right) \\
&= \frac{2L_A}{c}\frac{1}{1-\frac{u^2}{c^2}}
\end{aligned}$$

So:

$$\Delta t_A = \frac{2L_A}{c}\left(\frac{1}{1-\frac{u^2}{c^2}}\right)$$

And we know that:

$$\Delta t_B = \frac{2L_B}{c}$$

The fact A and B don't agree is fine, we can apply the time dilation formula:

$$\Delta t_A = \gamma\Delta t_B$$

$$\frac{2L_A}{c}\left(\frac{1}{1-\frac{u^2}{c^2}}\right) = \frac{1}{\sqrt{1-\left(\frac{u^2}{c^2}\right)}}\frac{2L_B}{c}$$

$$L_A\left(\frac{1}{1-\frac{u^2}{c^2}}\right) = \frac{1}{\sqrt{1-\left(\frac{u^2}{c^2}\right)}}L_B$$

$$L_A\left(\frac{\sqrt{1}}{\sqrt{1-\frac{u^2}{c^2}}}\right)^2 = \frac{1}{\sqrt{1-\left(\frac{u^2}{c^2}\right)}}L_B$$

$$L_A\left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\right) = L_B$$

$$L_A = \frac{L_B}{\gamma}$$

This tells us that measurements of lengths for a stationary vs moving observer also do not agree. Just like time runs slower for a moving object, a moving object will be measured to be smaller by a stationary observer. Effectively, moving lengths shrink. This is called length contraction.

Generally, we have:

$$\boxed{\Delta L = \frac{\Delta L_0}{\gamma}}$$

Where ΔL_0 is the "proper length" of an object, i.e. the length of an object measured by an observer at rest relative to it.

3 Example

We have someone on a spaceship, coming back to earth. The spaceship is travelling rightwards, directly towards the earth in 1D.

The spaceship has an astronaut, and we consider one person on the earth in Mission Control. MC spots the spaceship at distance $l = 3,000$km and is at speed u corresponding to $\gamma = 10$ (so $u \approx c$). The earth is stationary.

When the spaceship is at this distance l, the spaceship sends earth a distress signal saying $\Delta t_r = 10^{-3}$s, where Δt_r is how long the spaceship's oxygen supply lasts. We assume the astronaut cannot hold their breath and dies immediately if the oxygen supply runs out before the ship makes it back to Earth. Will the astronaut survive?

MC's child, who has no knowledge of relativity, calculates the travel time:

$$\Delta t = \frac{l}{u} \approx \frac{l}{c} = \frac{3 \times 10^3 \times 10^3}{3 \times 10^8} = 10^{-2} s.$$

The child compares this to the oxygen supply ($10^{-3}s$) and concludes the astronaut dies. This isn't quite accurate however, as we need to treat the oxygen time relativistically. To MC, the clock on the spaceship runs slow (Time Dilation):

$$\Delta t_{MC} = \gamma \Delta t_0$$
$$\Delta t_{MC} = 10 \times 10^{-3}\text{s} = 10^{-2} s$$

Comparing the relativistic oxygen duration ($10^{-2}s$) to the travel time ($10^{-2}s$), we see that the astronaut (just barely!!) survives.

We solved this in the Earth frame, where the distance l is a proper length. We could alternatively solve this in the Astronaut's frame, where the distance to Earth is length contracted to $L = l/\gamma = 300$km. In that frame, the travel time is $10^{-3}s$, which matches the proper time of the oxygen supply, so gives the same result.

Thu 23 Oct 2025 15:00

Lecture 8 - Special Relativity IV and Intro to Dynamics

1 The Relativistic Doppler Effect

For an emitted frequency f_0, emitted by an object moving with velocity (in 1D) u relative to an observer, the received frequency f is given by:

$$f = \sqrt{\frac{1 + u/c}{1 - u/c}} f_0$$

For a relativistic speed u. Note that we cannot use the standard Doppler formula for relativistic speeds. Also note the lack of $\pm$, as we encode this into u. If the object moves towards the observer, u is positive, and if the object is moving away u is negative.

If u is non-relativistic, we assume that $u/c \ll 1$:

$$f = \left(1 + \frac{u}{c}\right)^{1/2} \left(1 - \frac{u}{c}\right)^{-1/2} f_0$$

And Taylor Series expanding:

$$= \left(1 + \frac{1}{2}\frac{u}{c} + \cdots\right)\left(1 - \frac{-1}{2}\frac{u}{c} + \cdots\right) f_0$$

Since u/c is small, we ignore any quadratic, cubic etc terms of u/c, as these are very small.

$$= \left(1 + \frac{1}{2}\frac{u}{c} + \frac{1}{2}\frac{u}{c}\right) f_0$$

Hence:

$$f \approx \left(1 + \frac{u}{c}\right) f_0$$

Since we assume $u/c \ll 1$:

$$f = f_0 + \frac{u}{c} f_0$$

$$f - f_0 = \Delta f = \frac{u}{c} f_0$$

$$\frac{\Delta f}{f_0} = \frac{u}{c} \qquad \text{plus higher order terms we ignore}$$

This is the classical result that we're familiar with, for non-relativistic speeds.

2 Lorentz Transformation

Say we have a reference frame s' which is moving along the x-direction relative to a static reference frame s.

An event in the s frame has coordinates (x, y, z, t) and the same event in the s' frame has coordinates (x', y', z', t'). We have the following transformations:

$$t' = \gamma\left(t - \frac{u}{c^2}x\right)$$

$$x' = \gamma\,(x - ut)$$

Noting that $y = y'$, and $z = z'$ as these are orthogonal to the direction of motion. We also have:

$$u'_x = \frac{u_x - u}{1 - \frac{u}{c^2}u_x}$$

Please note that CMR1 does not include derivations of these equations (which collectively form the Lorentz Transformations), however for understanding's sake I'll include them here regardless.

2.1 Derivations

We want a transformation in time and space between a stationary frame S and the moving frame S'. We have three postulates to do this:

- **Linearity**: The transformation must be linear, i.e. a straight line in S must map to a straight line in S'.
- **Standard rule for c**: c is invariant and has the same velocity in all frames, regardless of motion.
- **Inverse symmetry**: The inverse transformation ($S' \to S$) is the same, but with $u \to -u$, as in S''s reference frame, it is static with S moving with speed $-u$.

Deriving x': We derive the transformation in position using length contraction:

Imagine a ruler at rest in the moving frame S'. It has one end on the origin O' and the right end at some coordinate x'. The ruler therefore has proper length $L_0 = x'$.

From the stationary frame S:

- The origin O' has moved distance ut after some time t.
- The ruler is moving, so has been length contracted to the observer in S. The ruler now appears to have length x'/γ.

The total coordinate as seen by S is therefore:

$$x = ut + \frac{x'}{\gamma}$$

And rearranging gives:

$$x' = \gamma(x - ut)$$

Deriving t': We derive the transformation for time using the third postulate above:

If the transformation from $S \to S'$ is:

$$x' = \gamma(x - ut)$$

Then the transformation from $S' \to S$ is:

$$x = \gamma(x' + ut')$$

As the scenario is the same: S' is moving with speed u relative to S, if we instead consider S''s reference frame then it is static, and S is moving in the opposite direction with the same magnitude of velocity, so the transformation must be the same with $u \to -u$.

Substituting the position transformation for x' into this:

$$x = \gamma[\gamma(x - ut) + ut']$$

$$\frac{x}{\gamma} = \gamma x - \gamma ut + ut'$$

Rearranging for t':

$$t' = \gamma t - \frac{x}{u}\left(\gamma - \frac{1}{\gamma}\right)$$

Using the identity $\gamma - 1/\gamma = \beta^2\gamma = \frac{u^2}{c^2}\gamma$:

$$t' = \gamma t - \frac{x}{u}\left(\gamma\frac{u^2}{c^2}\right)$$

$$t' = \gamma\left(t - \frac{u}{c^2}x\right)$$

Which gives us the time transformation.

Deiving u'_x: We find velocity in the moving frame as dx'/dt'.

The velocity in the moving frame S' is defined as $u'_x = \frac{dx'}{dt'}$. Taking these derivatives

$$dx' = \gamma(dx - udt) \quad \text{and} \quad dt' = \gamma\left(dt - \frac{u}{c^2}dx\right)$$

Substituting these into the definition of velocity:

$$u'_x = \frac{\gamma(dx - udt)}{\gamma\left(dt - \frac{u}{c^2}dx\right)} = \frac{dx - udt}{dt - \frac{u}{c^2}dx}$$

Dividing through by dt:

$$u'_x = \frac{\frac{dx}{dt} - u}{1 - \frac{u}{c^2}\frac{dx}{dt}}$$

And using $u_x = \frac{dx}{dt}$:

$$u'_x = \frac{u_x - u}{1 - \frac{uu_x}{c^2}}$$

3 Dynamics

Kinematics is effectively looking at objects in motion. Dynamics is effectively "why" they move (Newton's laws, static friction etc).

3.1 Newton's Laws

We have three:

1. If there is no resultant force acting upon an object, there are two possibilities:
   - The object was initially at rest, and stays at rest.
   - The object moves at a constant speed with no acceleration.
2. $F = ma$. Force and acceleration are proportional with a constant of proportionality m, the "inertial mass". We call it inertial mass because this is theoretically distinct from gravitational mass, however all experiments give them as having them same value.
3. The Reaction Principle. If a body A is producing a force F on a body B, then B acts back upon A with a force of the same magnitude but the opposite direction "Every action has an equal and opposite reaction".

3.2 Superposition Principle

If we have some body with N forces acting upon it, the final resultant force that acts upon an object is a vector sum of these forces:

$$F = F_1 + F_2 + \cdots + F_n = \sum_{i=1}^{N} F_i = ma$$

3.3 Example

Consider an object of mass m hanging from the ceiling with an "ideal string". This means that string is inextensible and is massless. The body is initially at rest.

Two forces act upon this body:

- The weight force due to local gravitational acceleration: $w = mg$.
- The force produced by the string (tension).

Since the body is at rest, the resultant force must be zero and:

$$T - mg = 0$$

$$T = mg$$

3.4 Example II

Consider a body on a horizontal surface. We laterally pull the object with force F. We have these forces:

- Again a weight force: $w = mg$.
- The normal force produced by the table acting back upon the body iaw Newton's Third Law.
- A frictional force acting in opposition to the direction of motion, F_{fric}

The frictional force is proportional to the Normal force with a coefficient depending on the materials used:

$$F_{\text{fric}} = \mu N$$

When moving an object there are two stages:

- Attempting to take an object from stationary to actually moving.
- Continuing the motion of the object once it's moving (this is easier).

We therefore have multiple coefficients of friction. Here we consider the coefficient of static friction μ_s and the coefficient of kinetic friction μ_k. There is also the coefficient of rolling friction μ_r seen in labs. Generally, $\mu_s > \mu_k$.

Tue 28 Oct 2025 12:00

Lecture 9 - Dynamics Exercises

1 Exercise I

We have a (very heavy) book on a table. We pull that book at some angle θ from the horizontal with constant force F. The book has $m = 10$kg and there is some coefficient of **kinetic** friction μ.

What is the minimum F so that the book moves with constant velocity?

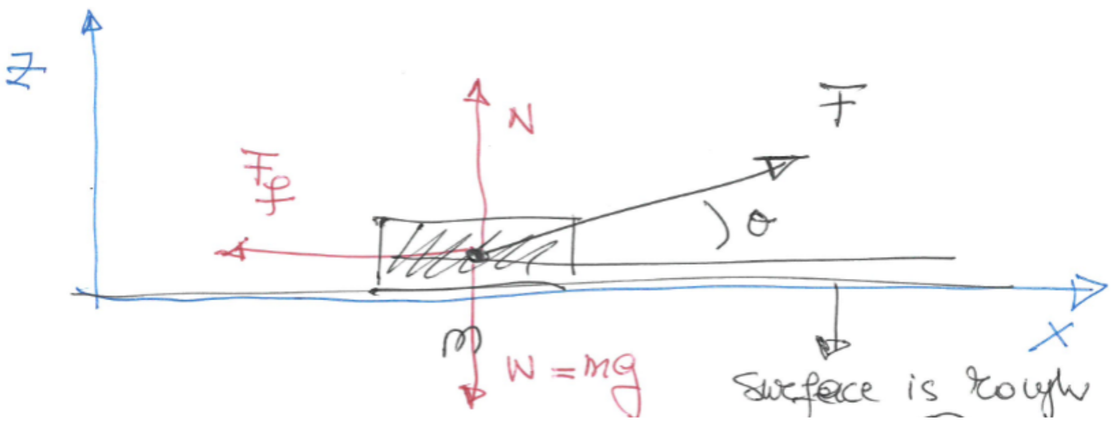

Figure 9.1

We have the following forces acting on the object.

Vertical (in z):

- The Normal reaction force N acting perpendicular to the table (directly up).
- The weight force $w = mg$ acting perpendicular to the table (directly down).
- A vertical component of F given my $F \sin\theta$.

Horizontal (in x):

- The frictional force $-\mu N$.
- A horizontal component of the pulling force F given by $F \cos\theta$.

The velocity is constant, so $dv/dt = a$. Hence $v = \int a\,dt$. For v to be constant, a must be zero. Hence there is no resultant force. Consider the total forces with $F = ma$:

$$\text{For } T_x : -\mu N + F\cos\theta = ma_x = 0$$

$$F\cos\theta - \mu N = 0$$

$$\text{For } T_z : N - mg + F\sin\theta = 0$$

Therefore solving for N:

$$\mu N = F\cos\theta \implies N = \frac{F\cos\theta}{\mu}$$

And substituting into z:

$$\frac{F\cos\theta}{\mu} - mg + F\sin\theta = 0$$
$$\implies F(\cos\theta + \mu\sin\theta) - \mu mg = 0$$
$$\implies F = \frac{\mu mg}{\cos\theta + \mu\sin\theta}$$

We can now ask “what is the right choice of θ to minimise the required F”?. We take the derivative of F wrt θ so that we can solve for a minima.

$$\frac{dF(\theta)}{d\theta} = 0$$

This will give us stationary points, and we can classify the stat points to ensure we find a minima. Alternatively, we can save a little bit of faff and do this quicker by recognising that the numerator is a constant. The minimum of F is therefore the maximum of $\cos\theta + \mu\sin\theta$.

$$\frac{d}{d\theta}(\cos\theta + \mu\sin\theta) = 0$$
$$-\sin\theta + \mu\cos\theta = 0$$
$$\mu = \frac{\sin\theta}{\cos\theta} = \tan\theta \implies \theta = \arctan\mu$$

So the optimum angle depends on the coefficient of friction.

2 Exercise II

We have an inclined plane at angle θ. There is a pulley at the top of plane and one block with mass m_1 on the plane, connected by an ideal string over the pulley to another block. This block has mass m_2 and is hanging off the plane. The plane is frictionless.

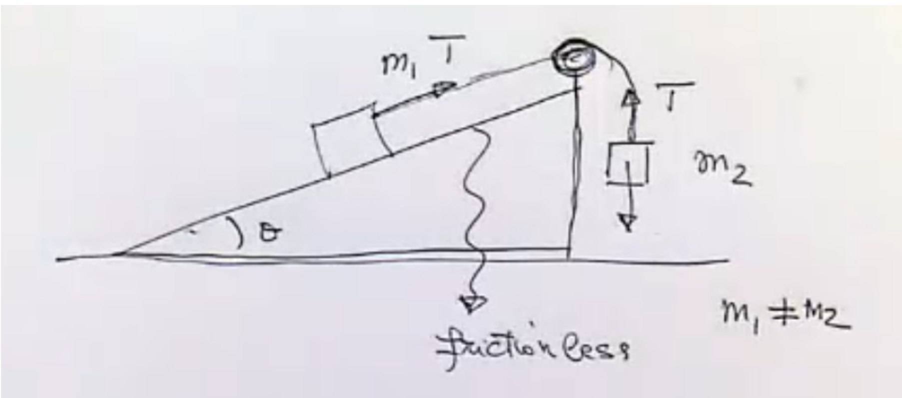

Figure 9.2

What is θ such that the two masses move together with constant speed?

Both objects will move together, either up or down (depending on which mass is greater).

For m_2:

- The weight $w = m_2 g$ acting downwards.
- The tension force T acting upwards.

For m_2:

- The normal force acting perpendicular to the plane.
- The tension force T acting in the direction of motion.

- The weight force $w = m_1 g$ acting immediately down.

We ignore the normal force and the component of the weight acting perpendicular to the plane, as these are not in the direction of motion (and we have no frictional force that relies on them).

Therefore (taking the vertical axis positive down):

$$m_2 g - T = 0$$

And (taking the horizontal axis positive right parallel to the direction of motion, and the vertical axis now positive up perpendicular to the direction of motion):

$$T - m_1 g \sin\theta = 0 \implies T = m_1 g \sin\theta$$

Substituting the second into the first:

$$m_2 g - m_1 g \sin\theta = 0$$
$$m_2 - m_1 \sin\theta = 0$$
$$\sin\theta = \frac{m_2}{m_1} \implies \theta = \arcsin\frac{m_2}{m_1}$$

Thu 30 Oct 2025 15:00

Lecture 10 - Terminal Velocity

1 Connected Bodies

Consider N blocks on a frictionless surface, all connected by an ideal string:

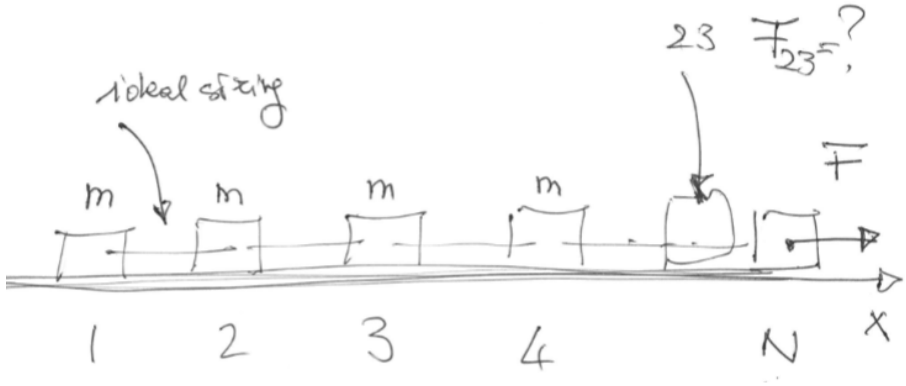

Figure 10.1

We pull the final block, with some magnitude of force F. All bricks are identical and have the same mass m.

What is the net force on brick number i, where i is arbitrary (and i ≤ N). For example, what is the net force on brick 23, *denoted* F_{23}

The total acceleration of the entire system considered together is:

$$a = \frac{F}{M} \qquad M = Nm \text{ (total system mass)}$$

And for our choice of brick:

$$F_{23} = m_{23}a_{23}$$

As all the strings are ideal, the acceleration is equal for every brick, and is equal to the total system acceleration, hence:

$$F_{23} = m\frac{F}{M} = m\frac{F}{Nm} = \frac{F}{N}$$

2 Air Drag and Terminal Velocity

We have some body moving with velocity v through some medium (air, water etc). This medium has a frictional force (the 'drag force') which acts upon the object in opposition to the direction of motion.

For sufficiently low velocities, this force is proportional to speed. However, as velocity increases this no longer applies, and may increase with (for example v^2). In a simple case of proportionality:

$$F_d = kv$$

$$F_d = -kv\underline{\hat{v}}$$

2.1 Terminal Velocity

Consider jumping [1] off a very tall tower. The only two forces that act upon you are:

- $F_w = mg$ acting downwards.
- F_d acting in the opposite direction to motion (upwards)

At some point, we have:

$$F_d = F_w$$

At this time, there is no resultant force, no acceleration and therefore you move at a constant velocity. This velocity, which we will denote v_* is the "terminal velocity" such that $F = a = 0$

$$F_d + F_w = 0$$

$$-kv_* + mg = 0 \implies kv_* = mg \implies v_* = \frac{mg}{k}$$

We want to investigate the behaviour of v as it approaches terminal velocity, so want to build expressions for $v(t)$ and $z(t)$, where z is the vertical axis.

$$\begin{aligned} F &= ma \\ mg - kv &= ma \\ mg - kv &= m\frac{dv}{dt} \\ \frac{mg - kv}{mg - kv} &= \frac{m}{mg - kv}\frac{dv}{dt} \\ 1 &= \frac{m}{mg - kv}\frac{dv}{dt} \\ dt &= \frac{m}{mg - kv}dv \end{aligned}$$

And integrating:

$$\int_{t_0}^{t} dt = \int_{v_0}^{v} \frac{m}{mg - kv}\, dv$$

Since our jump starts from rest at t_0, we have $t_0 = v_0 = 0$

$$\begin{aligned} \int_0^t dt &= \int_0^v \frac{m}{mg - kv}\, dv \\ t &= \frac{m}{mg}\int_0^v \frac{1}{1 - \frac{k}{mg}v}\, dv \\ &= \frac{1}{g}\int_0^v \frac{1}{1 - \left(\frac{v}{v_*}\right)}\, dv \end{aligned}$$

We then stop integrating wrt v alone and make a substitution such that we are integrating wrt the fraction of terminal velocity achieved:

$$\begin{aligned} t &= \frac{v_*}{g}\int_0^{v/v_*} \frac{1}{1 - \frac{v}{v_*}}\, d\left(\frac{v}{v_*}\right) \\ t &= \frac{v_*}{g}\left[-\log(1 - x)\right]_0^{v/v_*} \\ t &= \frac{v_*}{g}\left[-\log\left(1 - \frac{v}{v_*} + \log(1)\right)\right] \\ t &= -\frac{v_*}{g}\log\left(1 - \frac{v}{v_*}\right) \end{aligned}$$

[1] with a parachute!

$$-g\frac{t}{v_*} = \log\left(1 - \frac{v}{v_*}\right)$$

Let $\tau \equiv v_*/g$:

$$-\frac{t}{\tau} = \log\left(1 - \frac{v}{v_*}\right)$$

$$e^{-t/\tau} = 1 - \frac{v}{v_*}$$

Hence:

$$v(t) = v_*\left(1 - e^{-t/\tau}\right)$$

Tue 04 Nov 2025 12:00

Lecture 11 - Uniform Circular Motion and Work Done I

1 Uniform Circular Motion

Consider a circle of radius r. A particle sits on this circle, travelling around it with velocity v and a force F acting from the particle towards the centre of the circle.

This velocity is tangential to the circle, so is perpendicular to the force.

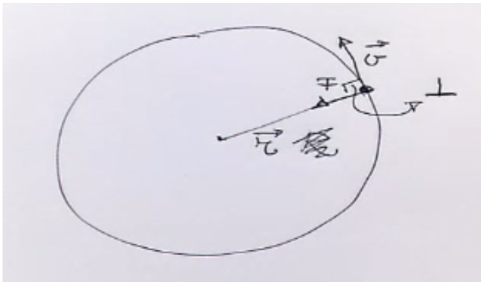

Figure 11.1

The particle travels in a circular path with constant radius and constant speed. Consider the particle after some infinitesimal angle change $d\theta$. The direction vector of the particle (relative to the centre of the circle) goes from r_1 to r_2, which have the same magnitude but different directions.

There is some tiny change dx in horizontal position. This is given by:

$$dr = rd\theta$$

The particle goes from v_1 to v_2 (again, same magnitude with slightly different direction). We therefore have:

$$dv = vd\theta$$

1.1 Determining Acceleration

The magnitude of the infinitesimal displacement $|dr|$ is related to r and $d\theta$ by:

$$|dr| = r\,d\theta \implies d\theta = \frac{|dr|}{r}$$

The velocity vectors rotate through the same angle $d\theta$ so we also have:

$$|dv| = v\,d\theta$$

$$|dv| = v\left(\frac{|dr|}{r}\right) = \frac{v}{r}|dr|$$

$$a = \frac{|dv|}{dt} = \frac{v}{r}\frac{|dr|}{dt}$$

Using $\frac{|dr|}{dt} = v$:

$$a = \frac{v}{r}(v) = \frac{v^2}{r}$$

1.2 Angular Frequency

We define a new (constant) quantity called "angular frequency", ω:

$$\omega \equiv \frac{d\theta}{dt} = \dot{\theta}$$

The particle has time period to complete a whole rotation ($\theta = 2\pi$), T. Therefore:

$$T = \frac{2\pi}{\omega} \qquad \omega = \frac{2\pi}{T}$$

Using:

$$\frac{dr}{d\theta} = r\frac{d\theta}{dt}$$

We have:

$$v = \omega r$$

Hence:

$$a = \frac{\omega^2 r^2}{r} = \omega^2 r$$

2 Unit Vectors

We define the position vector using the radial unit vector $\hat{e}_r$:

$$r = r\hat{e}_r$$

To differentiate this, we first define the unit vectors in Cartesian coordinates to determine their time derivatives:

$$\hat{e}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \quad \text{and} \quad \hat{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$$

Differentiating $\hat{e}_r$ wrt time:

$$\dot{\hat{e}}_r = \frac{d}{dt}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) = (-\sin\theta\cdot\dot{\theta})\hat{\imath} + (\cos\theta\cdot\dot{\theta})\hat{\jmath} = \dot{\theta}\hat{e}_\theta$$

Differentiating $\hat{e}_\theta$ wrt time:

$$\dot{\hat{e}}_\theta = \frac{d}{dt}(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) = (-\cos\theta\cdot\dot{\theta})\hat{\imath} - (\sin\theta\cdot\dot{\theta})\hat{\jmath} = -\dot{\theta}\hat{e}_r$$

2.1 Velocity

Using the product rule on $r = r\hat{e}_r$:

$$v = \frac{d}{dt}(r\hat{e}_r) = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r$$

Substituting $\dot{\hat{e}}_r = \dot{\theta}\hat{e}_\theta$:

$$v = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

2.2 Acceleration

Using the product rule on v:

$$a = \frac{d}{dt}(\dot{r}\hat{e}_r) + \frac{d}{dt}(r\dot{\theta}\hat{e}_\theta)$$

Expanding individual terms:

$$\frac{d}{dt}(\dot{r}\hat{e}_r) = \ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r = \ddot{r}\hat{e}_r + \dot{r}(\dot{\theta}\hat{e}_\theta)$$

$$\frac{d}{dt}(r\dot{\theta}\hat{e}_\theta) = (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta = (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta + r\dot{\theta}(-\dot{\theta}\hat{e}_r)$$

Grouping the $\hat{e}_r$ and $\hat{e}_\theta$ components:

$$a = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

For UCM, radius is constant and $\dot{\theta} = \omega$:

$$a = (r\omega^2)\hat{e}_r + \hat{e}_\theta = -r\omega^2\hat{e}_r$$

Using $v = r\omega$:

$$a = -\frac{v^2}{r}\hat{e}_r$$
$$a = \frac{-v^2}{r}$$

3 Work Done by a Force

We have some force F that causes the particle to move by a infinitesimal displacement ds. We define the infinitesimal work done by this force as:

$$dW = F \cdot ds$$

If the force acts at some angle θ to the direction of displacement, we have:

$$dW \equiv |F||ds| \cos\theta = F \times ds \times \cos\theta$$

This has dimensions of:

$$[dW] = \frac{ML}{T^2}L = \frac{ML^2}{T^2} = M\left(\frac{L}{T}\right)^2$$

We notice this has the same dimensions as mv^2 which therefore gives us dimensions of energy.

Say a force now displaces the object along a (not necessarily straight, so not necessarily constant F) path from point (1) to point (2). The work done by this motion is given by the integral:

$$W = \int_{(1)}^{(2)} F \cdot ds$$

3.1 Example

We again have some force with magnitude F, producing an infinitesimal displacement ds. Say the two are in the same direction as each other:

$$dW = F \cdot ds = Fds$$

And if they are orthogonal:

$$dW = F \cdot ds$$
$$= Fds \cos\frac{\pi}{2}$$

And has $\cos\pi/2 = 0$, the the force does no work in this case (cannot cause the displacement).

Finally, if the force and the displacement are in opposite directions to each other:

$$dW = F \cdot ds = Fds\cos\pi = -Fds$$

So signs are important to uphold conservation of energy.

3.2 Example II

You, at point (1) want to travel to your friend along a beach at point (2) immediately below you. There is some frictional force opposing your motion (F_0).

What is the work against friction to go from (1) to (2), assuming you walk in a straight line.

$$W = \int_{(1)}^{(2)} F \cdot ds$$

$$= -F_0 \int_{(1)}^{(2)} ds$$

$$-F_0((2) - (1)) = -F_0 L$$

Say you are a little drunk. You take the path (walking three sides of a square instead of a straight line downwards):

- lm at an angle of 90° from the correct direction to your friend.
- You realise you've gone wrong, and walk Lm downwards.
- You walk back lm at angle 90° to finally meet them.

This gives us:

$$W = -F_0 l + -F_0 L - F_0 l = -F_0(2l + L)$$

So we need to have our force and our path well defined along the whole route to be able to accurately determine work, as different paths will yield different works.

Thu 06 Nov 2025 15:00

Lecture 12 - Work Done II

1 Work Done by Gravity

Consider a falling object with no forces other than gravity. The object falls down the vertical axis from z_1 to z_2 so $z_2 < z_1$.

$$F = -mg\hat{e}_z$$
$$ds = dz = \hat{e}_z dz$$

Hence work over an infinitesimal change in height is:

$$dW = F \cdot ds$$

$$dW = -mg\hat{e}_z \cdot \hat{e}_z dz$$
$$dW = -mgdz$$

And across the whole fall between the two points:

$$W = \int_{z_1}^{z_2} -mgdz$$

$$-mg \int_{z_1}^{z_2} dz = -mg\,[z]_{z_1}^{z_2} = -mg(z_2 - z_1)$$

If we let the change of height be the magnitude $h = |z_2 - z_1|$. Since $z_2 - z_1 < 0$ (falling), this term is equal to $-h$. This gives:

$$W = -mg(-h)$$
$$W = mgh$$

So if work is being done by gravity (the object is going down and $z_2 < z_1$), work is positive. In the opposite case (i.e. an object being raised up), work is being done against gravity and $W < 0$.

2 Work and Kinetic Energy

Let's revisit the general formula:

$$dW = F \cdot ds$$
$$ma \cdot ds$$
$$= m\frac{dv}{dt} \cdot ds$$

For a small change in time, we can write $ds = vdt$, hence:

$$= m\frac{dv}{dt} \cdot vdt$$

Let's consider this:

$$v \cdot v = v^2$$
$$\frac{d}{dt}(v \cdot v) = \frac{d}{dt} \cdot v + v \cdot \frac{dv}{dt}$$
$$= 2\frac{dv}{dt} \cdot v$$

Hence:

$$2\frac{dv}{dt} \cdot v = \frac{d}{dt}(v \cdot v) = \frac{d}{dt}v^2$$

So:

$$dW = \frac{m}{2}\frac{d}{dt}(v^2)\,dt$$

And since mass is constant:

$$dW = \frac{d}{dt}\left(\frac{m}{2}v^2\right)dt$$
$$= d(\frac{1}{2}mv^2) = dE_k$$

Hence work done is equal to the change in kinetic energy. This explains why work can be both positive or negative, depending on the resultant change in kinetic energy.

3 Work Done on a Curved Path

Consider a frictionless skateboard ramp, made of a quarter circle with radius R:

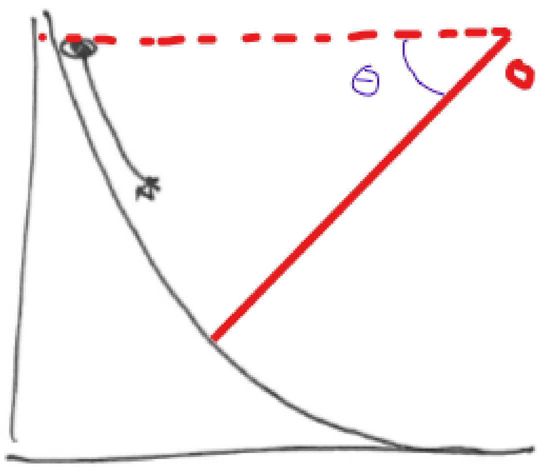

Figure 12.1

A person of mass m rolls down the ramp from point (1) at the top to point (2) at the bottom. The only force on the object is due to gravity.

We will again work out the work done over an infinitesimal displacement and integrate to get the full work done over the path.

We consider some infinitesimal ds. Instead of talking about a Cartesian dx and dz, we'll think about it in terms of arc lengths. Consider the angle θ made between the upper horizontal and the radius connecting to the person. At point (1), $\theta = 0$, and at (2) $\theta = \pi/2$.

As the person moves an infinitesimal distance, there is an infinitesimal change in angle, $d\theta$ such that the change in displacement is $ds = Rd\theta$ and therefore $ds = Rd\theta\hat{e}_\theta$.

To determine work we need to find the component of gravity acting in the direction of motion. We can disregard the component of gravity acting perpendicular to the direction of motion as this does not contribute to work done. The component of gravity along $\hat{e}_\theta$ is $F\cos\theta$.

At the top of the ramp, $\cos\theta = 1$ (and $\hat{e}_\theta$ acts straight downwards) so gravity acts entirely in the direction of motion, while at the bottom of the ramp as $\cos\theta = 1$ there is no component of gravity acting along $\hat{e}_\theta$.

Hence:

$$dW = F \cdot \hat{e}_\theta R d\theta$$
$$dW = mg\cos\theta R d\theta$$

$$W = \int_0^{\pi/2} mg\cos\theta R\, d\theta$$
$$= mgR \int_0^{\pi/2} \cos\theta\, d\theta$$
$$W = mgR$$

Note that this is the same as if the object had simply fallen from point (1) immediately down, without following the curved path.

4 Conservative Forces

There is a "special class of forces", including gravity such that the work does not depend on the path taken. It depends only on the start and end position.

Formally, the work along a closed loop is zero, so going from some point A to some point B back to A is zero, regardless of path.

We can define a quantity for conservative forces that depends purely on position, called "potential energy", denoted $u(x)$.

Tue 11 Nov 2025 12:00

Lecture 13 - Work III and Potential Energy

1 Conservative Forces

For a conservative force:

$$W_{(1)\to(2)} \text{ is independent of the path taken between the points}$$

Equivalently, the work on a closed loop:

$$W_{(1)\to(1)} = \oint F \cdot ds = 0$$

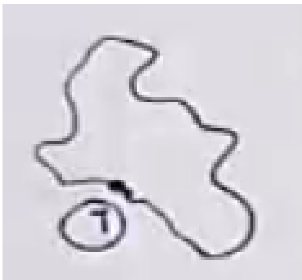

Figure 13.1: A closed loop.

We can also (for any force, not just conservative forces) express the work done as a change in kinetic energy between the two points:

$$W_{(1)\to(2)} = E_{k(2)} - E_{k(1)}$$

For conservative forces, we can define the (scalar) potential energy, a quantity depending solely on position. Notation differs, we will use U, but Φ is also common.

This is defined such that the **force is the negative gradient of the potential** and has units of energy:

$$\boxed{F = -\nabla U}$$

2 Local Gravitational Potential

Consider a particle falling from z_1 to z_2 (where $z_1 > z_1$). The particle is affected only by the gravitational force $F = mg\hat{e}_z$.

We have shown that:

$$\begin{aligned} W_{(1)\to(2)} &= -mg(z_2 - z_1) \\ &= mgz_1 - mgz_2 \end{aligned}$$

Since this is computed as the difference between two points (the work done to bring a particle to each point) it follows that potential energy is:

$$U(z) = mgz + \text{const.}$$

Where the constant depends on what height we consider to be $z = 0$. We formally define work done (for conservative forces) as the negative difference in potential energy at those two points. Therefore:

$$W_{(1)\to(2)} = -mg(z_2 - z_1) = mgz_1 - mgz_2 = \boxed{-(U_2 - U_1)}$$

2.1 Connecting Potential and Force

As this is a 1D problem where force, displacement are both in the downwards z direction:

$$dW = F \cdot dz = Fdz = -dU$$

So:

$$F = -\frac{dU}{dz}$$

$$F = -\frac{dU}{dz}\hat{e}_z$$

Recall we have a potential given by:

$$U(z) = mgz + \xi$$

Where ξ ("xi") is our constant. So:

$$F = -\frac{d}{dz}\left(U(z)\right)\hat{e}_z$$

$$= -\frac{d}{dz}\left(mgz + \xi\right)\hat{e}_z$$

$$= -mg\hat{e}_z$$

So the constant becomes irrelevant and disappears once we move into force or work.

3 Generalising

3.1 Multiple Dimensions

We have $F = -\nabla U$. In Cartesian $\mathbb{R}^3$ space, this is given by the operator:

$$\nabla \equiv \frac{\partial}{\partial x}\hat{e}_x + \frac{\partial}{\partial y}\hat{e}_y + \frac{\partial}{\partial z}\hat{e}_z$$

I.e. applying to some function f:

$$\nabla f = \frac{\partial f}{\partial x}\hat{e}_x + \frac{\partial f}{\partial y}\hat{e}_y + \frac{\partial f}{\partial z}\hat{e}_z$$

For our falling particle example, gravity acts purely downwards, so $U(x), U(y)$ are zero, and their derivatives are zero.

3.2 Conservation of Energy

As a reminder, we again have:

$$W_{(1)\to(2)} = E_{k2} - E_{k1} = -(U_2 - U_1)$$

We can rearrange to get:

$$E_{k2} + U_2 = E_{k1} + U_1 = \text{constant!}$$

We have just discovered conservation of energy. For a conservative force, the sum of kinetic and potential energy in an isolated system is constant.

3.3 Multiple Forces

Say we have N forces, all of which are conservative, given by:

$$F = F_1 + F_2 + \cdots + F_N$$

Work between two points a and b is given by:

$$W_{a\to b} = \int_a^b F\,ds = \int_a^b \sum_k F_k\,ds$$

What if we have a force which has two components, one conservative and one not?

$$F = F_{\text{conservative}} + F_{\text{non-conservative}} = F_c + F_n$$

$$W_{a\to b} = W_{a\to b}^{\text{conservative}} + W_{a\to b}^{\text{non-conservative}}$$

We have an expression in terms of potential energy for the work done conservatively, and we know that work done is a change in kinetic energy, therefore:

$$-U_b + U_a + W_{a\to b}^{\text{non-conservative}} = K_{k_b} - E_{k_a}$$

We can rearrange to get:

$$\boxed{E_{k_a} + U_a + W_{a\to b}^{\text{non-conservative}} = K_{k_b} + U_2}$$

This makes sense, as doing work (or having work done) adds or removes work from the system. This therefore isn't an isolated system, so we get the previous conservation of energy equation plus any energy introduced from outside the system into it, or extracted from the system.

Thu 13 Nov 2025 15:00

Lecture 14 - Conservation of Energy

1 Work Done Example I

Do not read this lecture without also taking Section 1 of Lec 15 into account.

We have a straight portion of track with length D which becomes a quarter circle ramp with radius R. We kick a box of mass m, providing initial horizontal velocity v_0.

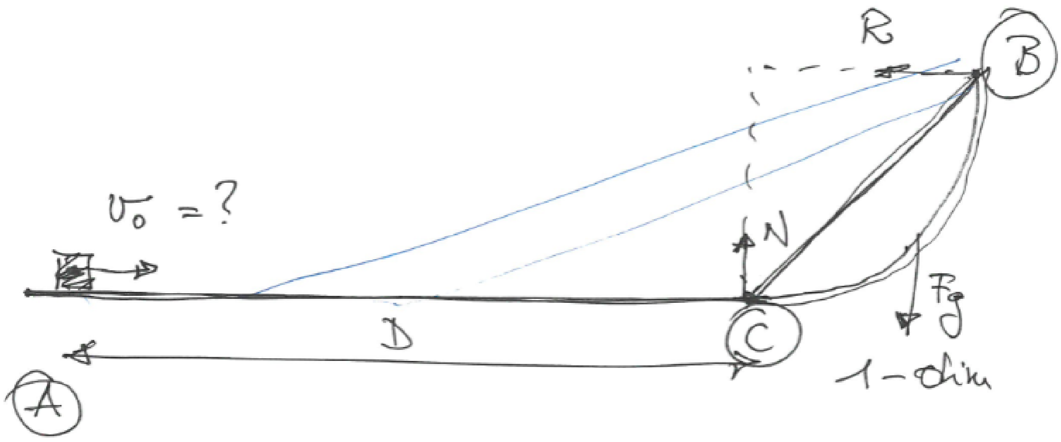

Figure 14.1

1.1 Assuming No Resistive Friction/Air Resistance

The only forces on the block are gravity and the normal contact force. *What v_0 must we impart on the block for it to (perfectly) reach the top of the ramp at point (B)?*.

Since there's no resistive forces along the horizontal section, the length of it is irrelevant. We can solve for the curved section using solely energy conservation:

$$\frac{1}{2}mv_0^2 = mgR + \xi$$

We take the height of points (A) and (C) as the $U = 0$ baseline, so $\xi = 0$.

$$\frac{1}{2}v_0^2 = gR$$
$$v_0 = \sqrt{2gR}$$

This was much easier compared to solving it using forces. Where possible, it's often extremely powerful to use conservation of energy in cases where we only care about the final and initial state of the object.

1.2 Assuming Friction

Assume there is some friction with $\mu_{\text{kinetic}} = \mu$, we therefore have to account for work done by this:

$$E_A + W_{A\to B} = E_B$$

Where E is a general energy term encompassing both kinetic and potential.

$$W_{A\to B} = \int_A^B F_{\text{fric}}\, ds$$

Breaking up the integral into the horizontal section and the ramp:

$$W_{A\to B} = \int_A^C F_f \cdot ds + \int_C^B F_f \cdot ds$$

$$= -\mu N \int_{(A)}^{(C)} \hat{e}_s \cdot ds + \int_C^B F_f \cdot ds$$

$$= -\mu N D + \int_C^B F_f \cdot ds$$

Where (as no other forces are acting horizontally), $N = mg$:

$$= -\mu mgD + \int_C^B F_f \cdot ds$$

The weight force acting down has two components, we care about the projection of the weight force along the direction of motion, $mg\cos\theta$. Where θ is the angle that joins the vertical and a radius to the box's position, going from 0 at (C) to $90° = \pi/2$ at (B).

$$= -\mu mgD + (-\mu mg)\int_{(C)}^{(B)} \cos\theta ds$$

$$= -\mu mgD - \mu mg\int_{(C)}^{(B)} \cos\theta R d\theta$$

$$= -\mu mgD - \mu mgR\int_0^{\pi/2} \cos\theta d\theta$$

$$= -\mu mgD - \mu mgR\left[\sin\theta\right]_0^{\pi/2}$$

$$= -\mu mgD - \mu mgR$$

$$= -\mu mg\,(D + R)$$

Note these **important things**:

- The work done is negative, as it is work done against the motion by friction.
- We assume that we move infinitesimally slowly, therefore we disregard centripetal force. Yes this is slightly (very) contradictory to the premise of the problem, as high velocity is required to climb but it's an approximation. This is looked at next lecture.
- This is exactly the same as if the object simply carried along a straight stretch of road of length R after the horizontal stretch - the fact the ramp is a curve changes nothing.

Putting it all together, we therefore finally have:

$$E_A + W_{A\to B} = E_B$$

$$E_A - \mu mg\,(D + R) = E_B$$

$$\frac{1}{2}mv_0^2 - \mu mg\,(D + R) = mgR$$

As the object starts at our $U = 0$ baseline so has no potential energy, and we provide perfectly enough initial velocity such that the object reaches (B) with zero kinetic energy.

2 Hooke's Law

Consider a 1D x-axis. We are given that there is a force characterised by the potential:

$$U(x) = \frac{1}{2}kx^2 \qquad \text{where } k \text{ is a constant.}$$

As this is a one dimensional problem:

$$\begin{aligned} F &= -\frac{dU}{dx}\hat{e}_x \\ &= -\frac{d}{dx}\frac{1}{2}kx^2\hat{e}_x \\ &= -\frac{1}{2}k(2x)\hat{e}_x \\ &= -kx\hat{e}_x \end{aligned}$$

So:

$$F = -kx$$

We have encountered this before at A-Level, it's the restoring force of a spring.

The spring is characterised by some spring constant k, and we assume that the spring is ideal and massless. The position $x = 0$ is the relaxed (unstretched and uncompressed) portion of the spring. The restoring force acts back towards the $x = 0$ equilibrium point.

$F = -kx$ is known as "Hooke's Law". We will see applications of it later.

3 Linear Momentum

We now pivot from Conservation of Energy to Conservation of Linear Momentum.

If we have particle with mass m and velocity v, we define linear momentum as:

$$\phi = mv$$

We can therefore use $F = ma$ to rewrite force as:

$$F = \frac{d\phi}{dt}$$

We also define "impulse" as the change of linear momentum over a time:

$$\begin{aligned} J &= \int_{t_1}^{t_2} F\,dt \\ &= \int_{t_1}^{t_2} \frac{d\phi}{dt}\,dt \\ &= \phi\big|_{t_2} - \phi\big|_{t_1} \end{aligned}$$

Tue 18 Nov 2025 12:00

Lecture 15 - Conservation of Linear Momentum

1 Curved Path Exercise Last Lecture

Last lecture, we had a setup like this:

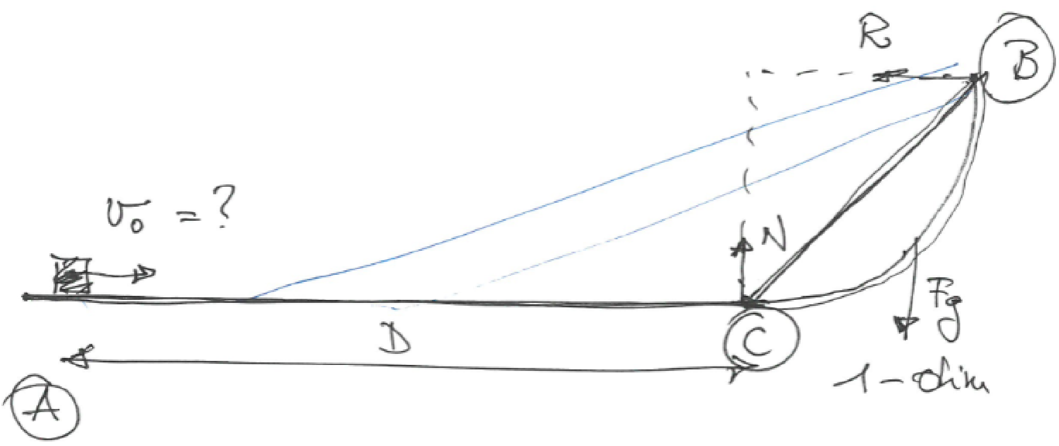

Figure 15.1

An issue arose then in how we treat the mass over the curved section. We assume today that the mass travels from $C \to B$ and we've already solved the part from $A \to B$.

At the base of the ramp, the object has some velocity v_0. Previously, we neglected centripetal force.

Since we neglected centripetal acceleration, last lecture we got to a solution which is only valid if we consider the particle to be in static equilibrium with no centripetal acceleration at every single point, i.e when v is infinitesimal. However, a reasonable velocity is required for the object to get up the ramp, so we have a physically inconsistent solution.

We have a centripetal force in the same direction as the normal force:

$$F_c = N - mg\cos\theta = mr\omega^2 = mr\dot{\theta}^2$$

This is going to change as the object climbs, as the relative magnitude of relevant forces changes.

In the tangential direction, we have two forces, and a tangential acceleration given by circular motion as $mr\ddot{\theta}$:

$$-(mg\sin\theta + F_{\text{fric}}) = m|a|$$

$$-(mg\sin\theta + F_{\text{fric}}) = mr\ddot{\theta}$$

And using $F_{\text{fric}} = \mu N$ and $N = mr\dot{\theta}^2 + mg\cos\theta$:

$$-(mg\sin\theta + \mu(mr\dot{\theta}^2 + mg\cos\theta)) = mr\ddot{\theta}$$

We haven't yet done the integration technique to actually solve this, but this is the correct form at least of the equation now we aren't neglecting centripetal acceleration!

2 Conservation of Linear Momentum

If there is no external force acting on a system in equilibrium:

$$F = 0 \implies \frac{d\phi}{dt} = 0 \implies \phi = \text{const.}$$

So linear momentum does not change for a system in equilibrium unless a force is externally supplied. Linear momentum, like energy, must be conserved.

2.1 Collisions

For this section, we assume $u \ll c$ and special relativity can be disregarded. We consider two *main* types of ideal collision although a collision in real life is not ideal, so can be in-between.

- Totally inelastic collisions.
  1. The two bodies stick together after the collision and behave together as one.
  2. Linear momentum is conserved.
  3. Kinetic energy *of the two bodies* is not conserved, as some is lost to friction/sound/etc.
- Totally elastic collisions.
  1. The two objects perfectly rebound off each other with no loss of energy (in real life some energy is dissipated, but this is an ideal)
  2. Linear momentum is conserved.
  3. Kinetic energy of the two bodies is conserved.

For simplicity, we consider motion in 1D, with two masses m_1 and m_2. Before the hit, they have velocities u_1 and u_2. After the collision, they have velocities v_1 and v_2.

We assume the collision is totally inelastic. Both particles are travelling in the same direction, but m_1 (on the left) is travelling faster than m_2 on the right, so catches up and collides from behind.

After the collision, they travel together with mass $m = m_1 + m_2$ and velocity $v = v_1 = v_2$.

The linear momentum in the system before is:

$$m_1 u_1 + m_2 u_2$$

And after:

$$(m_1 + m_2)v$$

So conserving:

$$\begin{aligned} m_1 u_1 + m_2 u_2 &= (m_1 + m_2)v \\ \implies v &= \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} \end{aligned}$$

Lets assume a situation (i.e. bowling ball hitting a marble) where $m_1 \gg m_2$. The ball has some velocity u_1, and the marble is stationary $u_2 = 0$.

Therefore:

$$\begin{aligned} v &= \frac{m_1 u_1}{m_1 + m_2} \\ &= \frac{m_1}{m_1 + m_2} u_1 \\ &= \frac{1}{1 + \frac{m_2}{m_1}} \end{aligned}$$

Since m_1 is much larger than m_2, we can express this as:

$$\approx \left(1 - \frac{m_2}{m_1}\right) u_1$$

We now assume that the collision is totally elastic, so we can conserve kinetic energy and we can conserve linear momentum.

$$
\begin{aligned}
m_1 u_1 + m_2 u_2 &= m_1 v_1 + m_2 v_2 \\
\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 &= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \\
\implies m_1 u_1^2 + m_2 u_2^2 &= m_1 v_1^2 + m_2 v_2^2
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
m_1(u_1^2 - v_1^2) &= m_2(v_2^2 - u_2^2) \\
m_1(u_1 + v_1)(u_1 - v_1) &= m_2(v_2 - u_2)(v_2 + u_2) \qquad (1)
\end{aligned}
$$

And as the masses don't change, conservation of momentum gives us that the sum of the velocities must be the same before and after. Rearranging the momentum equation:

$$
m_1(u_1 - v_1) = m_2(v_2 - u_2) \qquad (2)
$$

Dividing equation (1) by equation (2):

$$
\begin{aligned}
\frac{m_1(u_1 + v_1)(u_1 - v_1)}{m_1(u_1 - v_1)} &= \frac{m_2(v_2 - u_2)(v_2 + u_2)}{m_2(v_2 - u_2)} \\
u_1 + v_1 &= v_2 + u_2
\end{aligned}
$$

And the standard result:

$$
v_2 - v_1 = -(u_1 - u_2)
$$

3 Multiple Particles

With many particles, we can say:

$$
\sum_{k=1}^{N} p_k^{\text{(before)}} = \sum_{k=1}^{N} p_k^{\text{(after)}}
$$

And if the collision is totally elastic (so energy can be conserved):

$$
\sum_k E_k^{\text{(before)}} = \sum_k E_k^{\text{(after)}}
$$

$$
E = \frac{1}{2} m v^2 = \frac{1}{2} \frac{m^2 v^2}{m} = \frac{1}{2m} p^2
$$

Thu 20 Nov 2025 15:00

Lecture 16

Tue 25 Nov 2025 12:00

Lecture 17 - Gravitation

1 Gravitation

Consider two particles, with masses m_1 and m_2 at distance $\vec{r}$. We know there is an attractive force between the two masses:

$$\vec{F} = -G\frac{m_1 m_2}{r^2}\hat{e}_r \tag{17.1}$$

$$G = 6.67 \times 10^{-11} m^3/kg/s^2$$

1.1 Key Properties

- "Long Range" force. A force will exist between any two masses anywhere in the universe, regardless of distance and cannot be cancelled. Negligible at large distances, as force quickly tends to zero as distance increases - but never zero.
- Weak.
- $\vec{F} \propto \frac{1}{r^2}$

1.2 Mass Caveats

While we don't practically make a distinction between them, m in gravity refers to 'gravitational mass', m_g, while mass in Newton's Second Law $\vec{F} = m\vec{a}$ is 'inertial mass', m_i.

Einstein's Equivalence Principle says that they're equal, i.e:

$$\frac{dm_g}{dm_i} = 1$$

Why 'they could be different but they fundamentally are not' is important is beyond me, but it's in the lecture so here it goes...

2 Freefall

Consider an object of mass m at height h from the ground. It has force:

$$F = mg$$

And potential energy:

$$\Delta U(h) = mgh$$

However this first expression looks quite different to our definition of force.

2.1 Derivation

Gravitational force is conservative, and there is a potential energy at all points in the gravitational field, associated with this force.

Give two bodies, M and m. m is moved away from M from r_1 to r_2 along the radial direction (preserving angle, i.e. moving only in $\hat{e}_r$).

$$\vec{F} = -G\frac{Mm}{r^2}\hat{e}_r$$

This is the negative gradient of the potential (as potential is area under a force curve):

$$\vec{F} = -\nabla U = -\frac{dU}{dr}\hat{e}_r$$

$$\vec{F}dr = -dU\hat{e}_r$$

The work done from r_1 to r_2 is:

$$w_{1\to 2} = \int_{r_1}^{r_2} \vec{F} \cdot \hat{e}_r \, dx$$

Thu 27 Nov 2025 15:00

Lecture 18

Tue 02 Dec 2025 12:00

Lecture 19

Thu 04 Dec 2025 15:00

Lecture 20