LC Optics and Waves

MSci Physics w/ Particle Physics and Cosmology
University of Birmingham

Year 1, Semester 1

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Lectures Index

Thu 02 Oct 2025 13:00

Lecture 2 - Wave Functions

Sine Waves

Mechanical Waves A mechanical wave is a disturbance through a medium. It's formed of a single wave pulse or a periodic wave.

Mechanical Waves have the following properties:

1. Transverse: Where displacement of the medium is perpendicular to the direction of propogation.
2. Longitudinal, displaccement of hte medium is in the same direction as propogation.
3. Propogation depends on the medium the wave moves through (i.e. density, rigidity)
4. The medium does not travel with the wave.
5. Waves have a magnitude and a direction.
6. The disturbance travels with a known exact speed.
7. Waves transport energy but not matter throughout the medium.

Wave Functions

We want to define a wave function in terms of two variables, x and t. In any given moment, if we consider a single point on the wave (i.e. $t = 0$), and wait a short while, the wave will have travelled to some $t = t_1 > 0$.

In order to quantify displacement, we therefore want to specify both the time, and the displacement. This will let us find the wave speed, acceleration and the (new) wave number.

We are also able to talk about the velocity and acceleration of individual particles on the wave.

Wave Function for a Sine Wave

Consider a sine wave. We want to find a wave funtion in the form $y(x, t)$. Consider the particle at $x = 0$.

We can express the wave function at this point as $y(x = 0, t) = A\cos\omega t$. However we want to expand this to any general point. Now consider a point (2) which is one wavelength away. We know the behaviour of particle 1 is mirrored by particle 2 (with a time lag).

Since the string is initially at rest, it takes on period (T) for the propogation of the wave to reach point 2, therefore point 2 is lagging behind the motion of point 1. The wave equation is therefore (if particle two has $x = \lambda$) $y(x = \lambda, t) = A\cos(\omega t - 2\pi)$.

For arbitrary x, $y(x, t) = A\cos(\omega t - \frac{x}{\lambda} \cdot 2\pi)$ to account for this delay. This quantity is called the wave number:

$$\text{Wave Number: } k = \frac{2\pi}{\lambda}$$

So:

$$\begin{aligned} y(x, t) &= A\cos(\omega t - kx) \\ &= A\cos(kx - \omega t) \end{aligned}$$

Note the second step is possible as cos is an even function. k can also be signed to indicate direction: if $k > 0$, the wave travels in the positive x. If $k < 0$, the wave travels in the negative x direction. Again, $\omega = 2\pi f$

Displacement Stuff

Considering a point (starting at equi), the time taken for the particle on the sin wave to reach maximum displacement, minimum displacement and back takes the time period T. The speed of the wave is distance travelled over the time taken. We take the distance to be the wavelength λ, as we know the time by definition this takes is one time period T. Therefore wave speed v is:

$$v = \frac{\lambda}{T} = \lambda f$$

Since $\lambda = \frac{2\pi}{k}$ and $f = \frac{\omega}{2\pi}$ (as ω is defined as $\frac{2\pi}{T}$), we can also write:

$$v = \frac{2\pi}{k} \cdot \frac{\omega}{2\pi} = \frac{\omega}{k}$$

Particle Velocity

We can also determine the velocity of individual particles in the medium. We can use this to determine the acceleration.

We know that

$$y(x,t) = A\cos(kx - \omega t)$$

The vertical velocity v_y is therefore given by:

$$v_y = \frac{dy(x,t)}{dt}$$

Which is unhelpful (as we can't differentiate two variables at once), we can slightly cheat this by looking at purely a certain value of x, and therefore treating x as constant (to get a single variable derivative).

$$v_y = \frac{dy(x,t)}{dt}\bigg|_{x=\text{const.}}$$

However this is notationally yucky, so we therefore use the notation:

$$\frac{\partial y(x,t)}{\partial t}$$

To represent the same thing. Finally (carrying out the partial derivative):

$$v_y(x,t) = \frac{\partial y(x,t)}{\partial t} = \omega A \sin(kx - \omega t)$$

Particle Acceleration

We can work out particle acceleration (transverse acceleration) by differentiating in the same manner again:

$$a_y(x,t) = \frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial y(x,t)}{\partial t}\right) = -\omega^2 A\cos(kx - \omega t) = -\omega^2 y(x,t).$$

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Lecture 3 - Generalised Wavefunctions

Recap: For a sine wave, the wavefunction is:

$$y(x,t) = A\cos(kx - \omega t)$$

Where k is the wave number, $k = \frac{2\pi}{\lambda}$ and omega is $\frac{2\pi}{T} = 2\pi f$

More Wavefunctions

What is the general form of a triangular shaped wave? What about a square stepped wave? TODO: Diagram

We know that each particle (i.e. along a string) copies the motion of its immediate left-hand neighbour (for a particle moving in the positive x), with a time delay proportional to their distance. Every wave is describable $y(x,t)$ wave function, and every mechanical wave relies on a medium (i.e. a piece of string or water, concrete etc) to travel.

Ideally, we want to be able to find a general form of a wave function.

In a single fixed instant of time, the moving wave pulse is stationary, so purely a function of $y = f(x)$. We want a wave function where we can input any value of t, so we need a moving frame of reference. We define this frame of reference as O' (for the origin) and x', y' axes. This frame of reference moves with the wave pulse and at the same speed, therefore y' is a function of x' only, independant of speed.

New System

x is the distance from the origin O to the relevant point, while x' is the distance from O'.

$$x = x' + vt$$

$$x' = x - vt$$

$$y' = f(x') = f(x - vt)$$

However, as the wave is moving purely in one direction (along x), $y = y'$, so:

$$y = f(x - vt)$$

Back to Basics

Going back to:

$$\begin{aligned} y(x,t) &= A\cos(kx - \omega t) \\ &= A\cos\left[k\left(x - \frac{\omega}{k}t\right)\right] \\ &= A\cos\left[k\left(x - vt\right)\right] \end{aligned}$$

(Note, this is true for a wave in the positive x, for a wave moving in the negative x this would be x + vt)

Equivalent Rrepresentations

There are some equivalent representations for a sine wave:

$$y(x,t) = A\cos\left(2\pi f\frac{x}{v} - \omega t\right) = A\cos\left(2\pi\left[\frac{x}{\lambda} - \frac{t}{T}\right]\right) = TODO$$

More Differentiation

We've already looked at differentiating with respect to t, but what about x? This would give us the slope of the string at that point:

$$\frac{\partial y(x,t)}{\partial x}$$

And the curvature of the string:

$$\frac{\partial^2 y(x,t)}{\partial x^2} = -k^2 A\cos(kx - \omega t) = -k^2 y(x,t)$$

Note the similarities here with the equation for transverse acceleration:

$$(1)\frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 y(x,t)$$

$$(2)\frac{\partial^2 y(x,t)}{\partial x^2} = -k^2 y(x,t)$$

Dividing (1) by (2):

$$\frac{\partial^2 y(x,t)/\partial t^2}{\partial^2 y(x,t)/\partial x^2} = \frac{\omega^2}{k^2} = v^2$$

Therefore:

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

We call this the 'wave equation' as every wave function y(x, t) must satisfy it, regardless of whether or not it is periodic or its direction of travel. If y(x, t) does not satisfy this, it is not a wave function.

An Example

$$y(x,t) = \frac{x^3 - vt^2}{e^t}$$

TODO, the example in our own time

Wave Equation for a String

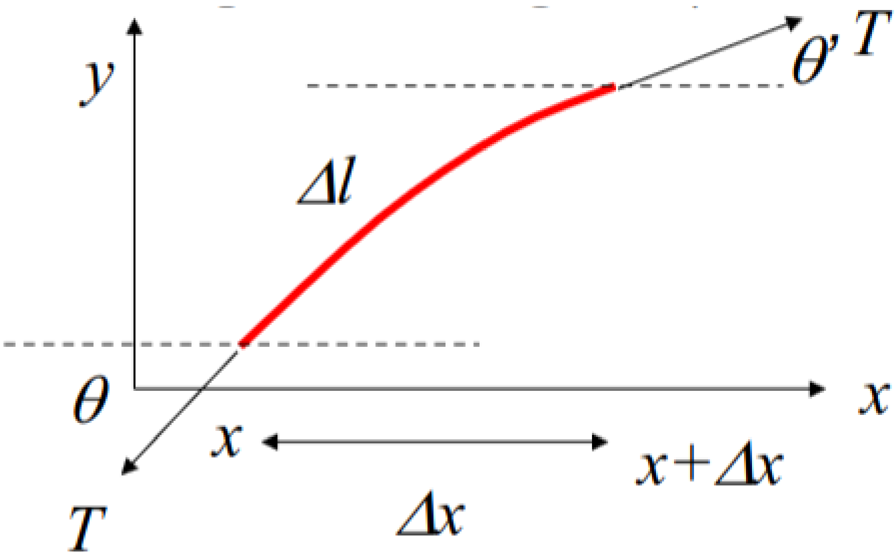

Figure 1: A snapshot of the wave.

Lets say we have some string, suspended horizontally under tension. We generate a single wave pulse and allow it to propogate down the string.

We assume the string is 1D, under tension T (constant throughout) and has mass per unit length μ.

Consider a small segment of string of length ΔL from x to $x + \Delta x$. This string makes angle θ with the horizontal at the bottom of the string, and angle θ' with the horizontal at the top of the string.

Net force is:

$$F_y = T \sin\theta' - T \sin\theta$$

And using the small angle approx $\sin\theta \approx \tan\theta = \frac{dy}{dx}$:

$$F_y = T \left(\frac{dy}{dx}\bigg|_{x+\Delta x} - \frac{dy}{dx}\bigg|_{x} \right)$$

Using differentiation by first principles:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

We can say:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \lim_{\Delta x \to 0} \frac{\frac{dy}{dx}\Big|_{x+\Delta x} - \frac{dy}{dx}\Big|_{x}}{\Delta x}$$

Therefore:

$$F_y = T \frac{d^2y}{dx} \Delta x$$