LC Mathematics for Physicists 1B Lecture Notes

Year 1 Semester 2

Ash Stewart

MSci Physics with Particle Physics and Cosmology

School of Physics and Astronomy
University of Birmingham

Table of Lectures

Wed 21 Jan 2026 11:00

Lecture 1 - Course Welcome and Introduction to Partial Differentiation

1 Course Welcome

1.1 Recommended books:

- Mathematical Techniques 4e, Jordan & Smith
- Engineering Mathematics 8e, Stroud
- Calculus (Schaum), 6e, Ayres & Mendelson
- Advanced Calculus (Schaum), 6e, Ayres & Mendelson

1.2 Assessment details:

- Maths 1A/1B form a single 20 credit module.
- 80% assessed by a 3 hour exam - Section 1 is 36% with 6 short questions and Section 2 is 64% with 4 long questions.
- 20% assessed by problem sheets.

1.3 Course structure:

1. Partial Differentiation
    - Definition, total differential, chain rule, gradient.
    - Taylor series, stationary points, Lagrange multipliers.
2. Differential Equations
    - Definition, 1st order separable, exact and homogenous.
    - Linear equations: general solution, 1st order and constant coefficients.
3. Integration
    - Definition as area under the curve, fundamental theorem of calculus.
    - Integration by: substitution, parts, partial fractions and tricks.
4. Multiple Integrals
    - Multiple and repeated integrals. Change of order of integration.
    - Change of variables and the Jacobian. Arc length. Solids of revolution.

2 Multivariate Functions

Lots of physics involves functions of more than one variable. A physical quantity defined at every point in space is called a field. We can have both scalar fields and vector fields.

For example, some scalar fields are:

- $V(x, y, z)$: Electrostatic potential. This is often easier to work with compared to the full electric (vector) field.
- $T(x, y, z)$: Temperature.
- $p(x, y, z)$: Pressure.

While some vector fields are:

- $\underline{E}(x, y, z)$: Electric Field.
- $\underline{B}(x, y, z)$: Magnetic Field.
- $\underline{v}(x, y, z)$: Velocity Field (i.e in fluid mechanics).

2.1 Partial Derivatives

Consider a function of two variables. The partial derivative of a function with respect to one variable is the rate of change of a function wrt that variable, while keeping other variables constant. Effectively, we carry out a derivative while treating the other variables as if they were constants.

Suppose we have a function $f(x, y)$. The definition of a partial derivative is:

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f((x_0 + h), y_0) - f(x_0, y_0)}{h}$$
$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{k \to 0} \frac{f(x_0, (y_0 + k)) - f(x_0, y_0)}{k}$$

Just like we denote $\frac{df}{dx}$ as the derivative of a function of a single variable, we denote $\frac{\partial f}{\partial x}$ as the partial derivative of a function of several variables.

Note that this is not delta f and delta y, i.e. not $\frac{\delta f}{\delta x}$

In theory, we'd explicitly notate:

$$\left(\frac{\partial f}{\partial x}\right)_y$$

With the subscript y explicitly stating that y is being kept constant. This is rarely, but sometimes, needed.

Consider $f(x, y, z) = x^2 \sin yz$. We have:

$$\frac{\partial f}{\partial x} = 2x \sin yz$$
$$\frac{\partial f}{\partial y} = x^2 z \cos yz$$
$$\frac{\partial f}{\partial z} = x^2 y \cos yz$$

2.2 Higher Orders

Higher derivatives are defined as they were previously, but they can now be mixed. For example, with $f(x, y) = x^2 \sin y$, we can write:

$$\frac{\partial f}{\partial x} = 2x \sin y \qquad \frac{\partial f}{\partial y} = x^2 \cos y$$
$$\frac{\partial^2 f}{\partial x^2} = 2 \sin y$$

We can also have:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2x \cos y$$

Shorthand notation exists, i.e. $f_{xx} = \frac{\partial^2 f}{\partial x^2}$ or $f_{yx} = \frac{\partial^2 f}{\partial y \partial x}$

For most cases, but not all, mixed derivatives are often independent of the order of partial derivatives, so: $f_{xy} = f_{yx}$

Thu 22 Jan 2026 09:00

Lecture 2 - Partial Differentiation II

1 The Total Differential

In order to generalise the chain rule, we need to define the total differential. Consider the change in a function of two variables, $f(x, y)$ when we move from some point (x, y) to some point $(x + dx, y + dy)$.

The partial derivative only tells us what happens when we change one variable, but we're changing two here. The total differential sums these two in order to get the full total change.

$$df = f(x + dx, y + dy) - f(x, y)$$

We have to look at the change in a single variable at a time, so we split it into two pieces where only x changes in the first, and only y changes in the second.

$$df = \underbrace{[f(x + dx, y + dy) - f(x, y + dy)]}_{\text{isolates change in } x} + \underbrace{[f(x, y + dy) - f(x, y)]}_{\text{isolates change in } y}$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

More generally for a function of $f(x_1, x_1, \dots, x_n)$:

$$df = \frac{\partial f}{x_1}dx_1 + \frac{\partial f}{x_2}dx_2 + \cdots + \frac{\partial f}{x_n}dx_n = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}dx_i$$

The total change in the function $f(x_1, x_1, \dots, x_n)$ is the sum of partial changes due to changing a single variable.

2 The Chain Rule

Recall that if $y = y(x)$, $x = x(t)$, then:

$$dy = \frac{dy}{dx}dx = \frac{dy}{dx}\frac{dx}{dt}dt \implies \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Now, if $f = f(x, y)$ and $x = x(t)$, $y = y(t)$, we can adjust the chain rule to say:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$= \frac{\partial f}{\partial x}\frac{dx}{dt}dt + \frac{\partial f}{\partial y}\frac{dy}{dt}dt$$

$$\implies \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example

$$f(x, y) = x^2 + y^2, \qquad x(t) = t^2, \qquad y(t) = t^3$$

Hence:

$$f(t) = t^4 + t^5 \implies \frac{df}{dt} = 4t^3 + 6t^5$$

By rewriting in terms of one variable:

$$\frac{\partial f}{\partial x} = 2x = 2t^2 \qquad \frac{\partial f}{\partial y} = 2y = 2t^3$$

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 3t^2$$

And instead using the new chain rule:

$$\begin{aligned} &\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= (2t^2 + 2t) + (2t^3)(3t^2) \\ &= 4t^3 + 6t^5 \end{aligned}$$

Hence the new chain rule works!

2.1 Polar Coordinates

Suppose our x and y are now functions of two different variables themselves, so:

$$f = f(x, y) \qquad x = x(r, \theta) \qquad y = y(r, \theta)$$

From r, θ we want to calculate x, y and then from x, y we want to calculate f.

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta$$

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta$$

Hence:

$$df = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial r}\,dr + \frac{\partial x}{\partial \theta}\,d\theta\right) + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial r}\,dr + \frac{\partial y}{\partial \theta}\,d\theta\right)$$

$$df = \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}\right)dr + \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}\right)d\theta$$

We also know (if we substitute x, y into the original function to get a function of r, θ):

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta$$

We can read off the final partial derivatives:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial x}{\partial r}, \qquad \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

As expected!

2.2 Generalising

Suppose we have two functions which map $\mathbb{R}^m \to \mathbb{R}^p$, and $\mathbb{R}^p \to \mathbb{R}^n$, respectively:

$$(x_1, x_2, \ldots, x_m) \to (y_1, y_2, \ldots, y_p) \to (z_1, z_2, \ldots, z_n)$$

Then we have:

$$dz_i = \sum_{k=1}^{p} \frac{\partial z_i}{\partial y_k}dy_k$$

$$dy_k = \sum_{l=1}^{m} \frac{\partial y_k}{\partial x_l}dx_l$$

And substituting:

$$dz_i = \sum_{k=1}^{p} \sum_{l=1}^{m} \frac{\partial z_i}{\partial y_k} \frac{\partial y_k}{\partial x_l} dx_l$$

And (as the two sums are independent), we can pull out the inner sum:

$$= \sum_{l=1}^{m} \left[\sum_{k=1}^{p} \frac{\partial z_i}{\partial y_k} \frac{\partial y_k}{\partial x_l} \right] dx_l$$

$$= \sum_{l=1}^{m} \frac{\partial z_i}{\partial x_l} dx_l$$

The partial derivatives $\partial z_i / \partial x_j$ are therefore given by:

$$\frac{\partial z_i}{\partial x_j} = \sum_{k=1}^{p} \frac{\partial z_i}{\partial y_k} \frac{\partial y_k}{\partial x_j}$$

Fri 23 Jan 2026 12:00

Lecture 3 - Partial Differentiation III

Fri 13 Feb 2026 12:00

Lecture 12 - End of Partial Differentiation & Start of ODEs

Recap of lecture 11:

- The tangent plane to a surface $f(x, y, z) = 0$ at (x_0, y_0, z_0) is given by:

$$\left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0) + \left(\frac{\partial f}{\partial z}\right)_0 (z - z_0) = 0$$

  Such that $\underline{\nabla} f(x_0, y_0, z_0)$ is the normal vector to the plane

- The parametric representation of a curve $\underline{r}(t)$ has:
  - Unit tangent: $\hat{\underline{T}} = \frac{dr}{dt} / \left|\frac{dr}{dt}\right|$
  - Arc length $s(t)$: $\frac{ds}{dt} = \left|\frac{dr}{dt}\right| \rightarrow \hat{\underline{T}} = \frac{dr}{ds}$.
  - Unit normal and curvature:

1 Orthonormal Triads

We can create an *orthonormal triad* by introducing a new normal vector called the unit binormal, $\hat{\underline{B}} = \hat{\underline{T}} \times \hat{\underline{N}}$. Since $\hat{\underline{N}} \times \hat{\underline{N}}$, differentiating wrt s gives:

$$\hat{\underline{N}} \cdot \frac{d\hat{\underline{N}}}{ds} = 0$$

TODO

We have:

$$\frac{d\hat{\underline{T}}}{ds} = \kappa \hat{\underline{N}}$$

$$\frac{d\hat{\underline{N}}}{ds} = -\kappa \hat{\underline{T}} + \tau \hat{\underline{B}}$$

Hence:

$$\begin{aligned}
\frac{d\hat{\underline{B}}}{ds} &= \frac{d}{ds}\left(\underline{T} \hat{\times} \hat{\underline{N}}\right) \\
&= \frac{d\hat{\underline{T}}}{ds} \times \hat{\underline{N}} + \hat{\underline{T}} \times \frac{d\hat{\underline{N}}}{ds} \\
&= \kappa \hat{\underline{N}} \times \hat{\underline{N}} + \hat{\underline{T}} \times \left(-\kappa \hat{\underline{T}} + \tau \hat{\underline{B}}\right) \\
&= \tau \hat{\underline{T}} \times \hat{\underline{B}} = \tau \hat{\underline{T}} \times \left(\hat{\underline{T}} \times \hat{\underline{N}}\right) \\
&= \tau \left[\left(\hat{\underline{T}} \cdot \hat{\underline{N}}\right) \hat{\underline{T}} - \left(\hat{\underline{T}} \cdot \hat{\underline{T}}\right) \hat{\underline{N}}\right] \\
&= \tau \hat{\underline{N}}
\end{aligned}$$

This gives the Frenet-Serret Formulae:

This concludes partial differentiation! :D

2 Ordinary Differential Equations

A differential equation is any equation that involves derivatives. We care, because most laws of physics manifest themselves in the form of differential equations. For example:

$$\text{Newton's Second Law:} \qquad \underline{F} = m\frac{d^2 r}{dt^2}$$

$$\text{3D Time-Independent Schrödinger Eqn:} \qquad -\frac{\hbar}{2m}\left(\frac{\partial^2\psi}{dx^2} + \frac{\partial^2\psi}{dy^2} + \frac{\partial^2\psi}{dz^2}\right) + V(x,y,z)\psi = E\psi$$

$$\begin{aligned}\text{3D Wave Eqn:} \qquad & \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \\ & = \nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}\end{aligned}$$

$$\text{Gauss' Law:} \qquad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0}$$

$$\text{Navier-Stokes Eqn:} \qquad \rho\left(\frac{\partial \underline{v}}{\partial t} + \left(\underline{v}\cdot\underline{\nabla}\right)\underline{v}\right) = -\underline{\nabla}p + \rho\underline{g} + \mu\nabla^2\underline{v}$$

In this course, we will only solve DEs of a single variable, i.e. Ordinary Differential Equations (ODEs). We don't look at Partial DEs of multiple variables yet.

In order to think about solving these, we need to classify them. Most DEs aren't soluble in closed form with elementary functions and need to be solved numerically. Here, we only consider nice soluble functions, but this is a vast minority in reality. We want to identify classes of DEs we can reasonable solve with a method for each.

We can generally solve linear equations by breaking them into small chunks and solving them individually, for example.

3 Types of DEs

3.1 Partial vs. Ordinary

In the examples above, only the first was an ODE, and the rest PDEs. Ordinary Differential Equations (ODEs) involve only a single variable.

Consider a vector $\underline{r}(t) = (x(t), y(t), z(t))$. t is called the independent variable, with x, y, z being dependant variables. While we have 3 dependant variables, we only have one independent variable (so only one thing to differentiate wrt), so this would end up being ordinary.

PDEs involve equations of two or more variables and hence involve partial derivatives.

3.2 Order

The order of a DE is given by the order of the highest derivative involved, so Newton's 2nd Law is a second order DE, as the highest order derivative is a second derivative.

3.3 Degree

The degree of a DE is a less important measure than the others. It is given by the highest power of the highest order derivative. For example, Newton's 2nd is a first degree, while an equation containing a^3 would be third degree (and second order, as a is a second derivative).

Ideally, we want this to be 1 for ease of solving, and higher degrees are rare but they do exist. For example, from Lagrangian Mechanics we have:

$$\frac{1}{2m}\left[\left(\frac{\partial s}{\partial x}\right)^2 + \left(\frac{\partial s}{\partial y}\right)^2 + \left(\frac{\partial s}{\partial z}\right)^2\right] + V(x,y,z) = \frac{ds}{dt}$$

3.4 Homogenous and Inhomogeneous

A homogenous DE is a DE that does not have any terms of only the independent variable(s), while an inhomogeneous DE does.

For example, Newton's 2nd is homogenous as there is no term that involves t alone. This would be inhomogeneous:

$$\frac{\partial^2 x}{\partial t^2} = t + x$$

While this would be homogenous:

$$\frac{\partial^2 x}{\partial t^2} = tx$$

As t is a coefficient and not a pure term in its own right.

3.5 Linear and Non-Linear

A DE is linear if the dependant variable(s) and all of its/their derivatives occur purely as linear functions. For example:

$$\left(1 - x^2\right)\frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

This is linear, as the dependant variable y never has a power greater than 1.

$$\frac{dy}{dx} + xy = 0$$

Is also linear, while this is not:

$$\frac{dy}{dx} + xy^2 = 0$$

This is also non-linear (as shown by the Taylor Expansion of sine):

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

3.6 Examples

$$(1) \quad \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = u^2$$

Homogenous first-order second-degree non-linear PDE.

$$(2) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = x^2 + y^2 + z^2$$

Inhomogeneous second-order first-degree linear PDE.

$$(3) \quad \frac{\partial y}{\partial x} + y^2 = x$$

Inhomogeneous first-order second-degree non-linear ODE.

Wed 18 Feb 2026 11:00

Lecture 13 - ODEs II

Summary of Lecture 12:

- Frenet-Serret Equations for a curve $\underline{r}(s)$, where $\hat{T} = d\underline{r}/ds$.
- Classification of differential equations:
  - Partial vs Ordinary
  - Order
  - Degree
  - Homogeneous vs Inhomogeneous
  - Linear vs Non-Linear

1 Equations Soluble By Direct Integration

Given some:

$$\frac{dy}{dx} = f(x) \implies y(x) = \int^x f(x')dx' + c$$

We can generalise this to some repeated derivative:

$$\frac{d^n y}{dx^n} = f(x)$$

Instead of having one undetermined constant here, we now have n. As each round of integration picks up a factor of the integration subject, these constants will form an nth degree polynomial.

1.1 Example: Particle Falling

$$\frac{d^2 z}{dt^2} = -g$$
$$\implies \frac{dz}{dt} = -gt + v_0$$
$$z = -\frac{1}{2}gt^2 + v_0 t + z_0$$

Here we consider our unknown constants as boundary conditions, i.e. the initial velocity and initial height.

2 Separable Equations

These are equations in the form:

$$\frac{dy}{dx} = \frac{f(x)}{g(x)}$$

$$g(y)dy = f(x)dx \implies \int g(y)dy = \int f(x)dx + c$$

2.1 Example I: Falling Particle with Air Resistance

$$\frac{dv}{dt} = -g - kv$$
$$\int \frac{dv}{g+kv} = -\int dt$$
$$\implies \frac{1}{k}\ln(g+kv) = -t + c$$
$$\implies k + gv = Ae^{-kt}$$
$$\implies v(t) = -\frac{g}{k} + \left(v_0 + \frac{g}{k}\right)e^{-kt}$$

2.2 Example II

$$\frac{dy}{dx} - x^2y^2 = x^2$$
$$\frac{dy}{dx} = x^2 + x^2y^2 = x^2(y^2+1)$$
$$\implies \frac{1}{y^2+1}dy = x^2dx$$
$$\implies \arctan y = \frac{1}{3}x^3 + c$$
$$\implies y = \tan\left(\frac{1}{3}x^3 + c\right)$$

2.3 Example III

$$\frac{dy}{dx} = -\frac{x}{y}$$
$$ydy = -xdx$$
$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + c$$
$$y^2 + x^2 = 2c$$

This is the equation for a circle.

3 Exact Equations

Suppose a function $y(x)$ is implicitly defined such that $f(x,y) = c$. It follows that the total differential:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

As $f(x,y)$ is a constant for all values of x, y, the constant differentiates to zero.

We can set:

$$M(x,y) = \frac{\partial f}{\partial x} \qquad N(x,y) = \frac{\partial f}{\partial y}$$

To try to solve this equation:

$$M(x,y)dx + N(x,y)dy = 0$$

If we can do this (i.e. if the function can be decomposed into gradient components M and N), the equation is called "exact". For this to be true, we need:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

If this relationship holds, we have an exact equation and can integrate M and N to get $f(x,y)$ to solve the equation in the form $f(x,y) = c$

3.1 Example

$$\frac{dy}{dx} = -\frac{2x+y}{x+2y}$$
$$(2x+y)dx + (x+2y)dy = 0$$

So:

$$M(x,y) = 2x+y \qquad N(x,y) = x+2y$$

And:

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 1$$

So yes, it is an exact function. It follows that:

$$\frac{\partial f}{\partial x} = 2x+y \implies f(x,y) = x^2 + xy + c(y)$$
$$\frac{\partial f}{\partial y} = x+2y \implies f(x,y) = xy + y^2 + d(x)$$
$$\implies f(x,y) = x^2 + xy + y^2 = c$$

Thu 19 Feb 2026 09:00

Lecture 14

Recap of last lecture:

- Soluble by direct integration:

$$\frac{dy}{dx} = f(x) \implies y(x) = \int f(x)dx + c$$

- Separable equations:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \implies \int f(x)dx = \int g(y)dy$$

- Exact equations:

  Given in the form (or re-arrangeable to the form):

$$M(x,y)dx + N(x,y)dy = 0$$

  These are soluble if:

$$M(x,y) = \frac{df}{dx} \qquad N(x,y) = \frac{df}{dy}$$

  With $f(x,y) = c$.

  The condition for an equation being exact is:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

1 "Homogeneous" Equations

Here, the word homogenous has a different meaning to classification of differentiating equations. If we have some function that satisfied this:

$$g(\lambda x, \lambda y) = \lambda^p g(x,y)$$

We call it a homogeneous function of order p.

"Homogeneous" differential equations in this case are functions of the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Here, we make the substitution $v(x) = y(x)/x$. We can write:

$$y(x) = v(x)x$$

And substitute back in. We use the product rule where:

$$\frac{dy}{dx} = x\frac{dv}{dx} + v = f(v)$$

$$x\frac{dv}{dx} = f(v) - v$$

This is now separable, so:

$$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x}$$

1.1 Example

$$2xydy - (x^2 + y^2)dx = 0$$

This is not exact, so we put it into a standard form and substitute:

$$2xydy = (x^2 + y^2)dx$$

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy} = \frac{(y/x)^2 + 1}{2(y/x)}$$

Let $u(x) = y/x$:

Fri 20 Feb 2026 11:56

Lecture 15 - Differential Equations III

Summary of last lecture:

- "Homogeneous" Equations
$$\frac{dy}{dx} = f\left(\frac{x}{y}\right) \to x\frac{dv}{dx} = f(v) - v \text{ where } y(x) = xv(x)$$
- Linear Equations
$$\sum_{k=0}^{n} a_k(x)\frac{d^k y}{dx^k} = f(x) \to y(x) = \sum_{k=1}^{n} \alpha_k y_k(x) + y_{PI}(x)$$
The solution is a sum of the complementary function (the general solution of the homogenous equation) and the particular integral (one solution of the inhomogeneous equation).
- First Order Linear Equations
$$\frac{dy}{dx} = P(x)y = Q(x) \to \frac{d}{dx}\left[y(x)e^{\int P(x)dx}\right] = Q(x)e^{\int P(x)dx}$$

1 Examples

1.1 Example I

$$(1-x^2)\frac{dy}{dx} - xy = 1$$

Rewriting in the standard form:

$$\frac{dy}{dx} - \frac{x}{1-x^2}y = \frac{1}{1-x^2}$$

Hence:

$$P(x) = \frac{-x}{1-x^2}$$
$$I(x) = \exp\left(-\int \frac{xdx}{1-x^2}\right) = \exp\left(\frac{1}{2}\ln(1-x^2)\right) = \sqrt{1-x^2}$$

Multiplying through by the integrating factor:

$$\sqrt{1-x^2}\frac{dy}{dx} - \frac{x}{\sqrt{1-x^2}}y = \frac{1}{\sqrt{1-x^2}}$$

The L.H.S is now the derivative of a product:

$$\frac{d}{dx}\left(y\sqrt{1-x^2}\right) = \frac{1}{\sqrt{1-x^2}}$$

And integrating both sides:

$$y\sqrt{1-x^2} = \arcsin x + c$$
$$\implies \boxed{y = \frac{c}{\sqrt{1-x^2}} + \frac{\arcsin x}{\sqrt{1-x^2}}}$$

1.2 Example II

2 Linear ODEs with Constant Coefficients

In the most general form, we have:

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = f(x)$$

Firstly, we solve the homogenous equation:

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots a_1 \frac{dy}{dx} + a_0 y = 0$$

Let $y(x) = e^{\lambda x}$:

$$e^{\lambda x} \left(a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 \right) = 0$$

This reduces to finding the zeroes of the corresponding nth order polynomial. If this polynomial has n distinct zeroes, then the complimentary function is given by:

$$y_{CF}(x) = \alpha_1 e^{\lambda_1 x} + \alpha_2 e^{\lambda_2 x} + \cdots + \alpha_n e^{\lambda_n x}$$

If these roots contain a repeated root, this will reduce the number of unique solutions by one. If we have any complex solutions, they wil come in complex conjugate pairs:

$$a^{(\pm ib)x} = e^{ax} \left(\cos bx \pm i \sin bx \right)$$

This therefore has independent real solutions:

$$e^{ax} \cos bx$$

$$e^{ax} \sin bx$$

3 Equidimensional Equations

These have coefficients which do depend on x, but where the coefficients are functions of x such that the nth derivative has a coefficient of $a_n x^n$

The general form of a homogeneous equidimensional equation is:

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = f(x)$$

This has a solution in the general form $y(x) = x^\lambda$. When we differentiate k times with respect to x we lose k powers of x, as each differentiation decreases the power. We need to restore this therefore with a x^k prefactor.

4 Mass on a Spring (SHM)

A mass, m on a spring is displaced from its equilibrium position by some distance x. There is a restoring force given by $F = -kx$.

$$F = ma \implies F = m \frac{d^2}{dt}$$

$$m \frac{d^2 x}{dt^2} + kx = 0$$

$$\frac{d^2 x}{dt^2} + \omega^2 X = 0 \qquad \text{where } \omega^2 = \frac{k}{m}$$

We have:

$$x(t) = e^{\lambda t}$$

$$(\lambda^2 + \omega^2)e^{\lambda t} = 0$$
$$\lambda^2 + \omega^2 = 0$$

Hence:

$$\lambda = e^{\pm i\omega t}$$