LC Classical Mechanics and Relativity 1 Lecture Notes

Year 1 Semester 1
MSci Physics with Particle Physics and Cosmology

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Lectures Index

Tue 30 Sep 2025 12:00

Lecture 1 - Orders of Magnitude and Dimensional Analysis

Given some equation, for example $E = mv^2$, we can decompose it into the basic physical quantities that make it up, for example in terms of Mass, Time and Length. We can denote the dimensions of some quantity by wrapping it in square brackets.

$$E = \frac{1}{2}mv^2$$

$$[E] = ML^2T^{-2}$$

Example:

$$\text{Pressure} \equiv \frac{\text{Force}}{\text{Area}}$$

Suppose we want to test whether pressure and linear momentum flux (amount of linear momentum per unit time, per unit surface) were equivalent quantities, we could do this using dimensional analysis:

$$[P] = \frac{[F]}{[A]}$$

$$[P] = \frac{M \times LT^{-2}}{L^2}$$

$$= M/LT^2$$

And for linear momentum flux ($\Phi(p)$ where lowercase p is momentum):

$$\Phi(p) = \frac{[p]}{[A][\text{time}]}$$

$$= \frac{MLT^{-1}}{L^2T}$$

$$= \frac{M}{LT^2}$$

So yes, they seem to be (at least dimensionally) equivalent.

0.1   Challenging the LHC

We want to use orders of magnitude calculations to challenge the idea that the LHC is the "Big Bang Machine".

The LHC operates on the order of magnitude of approx 10TeV. The age of the universe is approx 13.7Bn Years, or (in orders of magnitude) 10^{10}yrs.

What time was the big bang? The Big Bang started the universe, but we can't really say it happened at $0s$, because that doesn't really make sense. What about 1sec? or 1ms? Well it's clearly less than both of those, so we want to find the smallest possible increment of time "Plank Second" and say it happened after one of them.

Thu 02 Oct 2025 15:00

Lecture 2 - Dimensional Analysis (contd.) and Vectors

0.1 Continuation of Dimensional Analysis

What if, in theory, we could build a system of units entirely from c, the speed of light, G, Newton's constant and h, the Plank Constant?

Cont. from Lec01, we can try to use this to work out the earliest possible cosmic time.

$$h = 6.6 \times 10^{-34} Js$$
$$G = 6.67 \times 10^{-11} Nm^2/kg^2 nvminstall20$$
$$c = 3 \times 10^8 m/s$$

Dimensionally:

$$[h] = \frac{ML^2}{T}$$
$$[G] = \frac{L^3}{T^2 M}$$
$$[c] = \frac{L}{T}$$

We want to use these to build out a time unit, so:

$$[h^u G^v c^z] = T$$
$$\left(\frac{ML^2}{T}\right)^u \left(\frac{L^3}{T^2 M}\right)^v \left(\frac{L}{T}\right)^z = T$$

$$M^{u-v} L^{2u+3v+z} T^{-u-2v-z} = T$$

Solving for:

$$u - v = 0$$

$$2u + 3v + z = 0$$

$$-u - 2v - z = 1$$

Gives us:

$$u = \frac{1}{2} \tag{2.1}$$

$$v = \frac{1}{2} \tag{2.2}$$

$$z = \frac{-5}{2} \tag{2.3}$$

$t_P = \sqrt{\frac{Gh}{c^5}}$ and plugging in the values for G, h, c gives us a value of time, which the earliest possible cosmic time equal to about $10^{-43} s$

0.2 Plank Energy

Doing the same process for energy gives us (this time, the plank energy is the energy at which traditional theories of physics break down):

$$E_p = \frac{hc^{5}}{G}^{0.5} \approx 10^9 J$$

On the other hand, the LHC manages about 10TeV, which is orders of magnitude smaller than this, so the LHC cannot accurate simulate energies of this magnitude.

0.3 More Vectors

Again, vector notation will be $\vec{a}$. We define the x, y, z unit vectors as $\hat{\underline{e_x}}, \hat{\underline{e_y}}, \hat{\underline{e_z}}$.

We can therefore define any vector as:

$$\vec{a} = a_x \hat{\underline{e_x}} + a_y \hat{\underline{e_y}} + a_z \hat{\underline{e_z}}.$$

The length of a vector is again $|\vec{a}|$.

0.4 Vector Multiplication

Given $\vec{a}$ and $\vec{b}$ we can define the dot (scalar) product and the cross (vector) product

$$\vec{a} \cdot \vec{b} = |a||b| \cos\theta$$

Say we want to know the component of a vector along an axis, we can do the following (eg for x):

$$\vec{a} \cdot \hat{\underline{e_x}} = a_x$$

For the vector product, we can define:

$$\vec{a} \times \vec{b} = |a||b| \sin(\theta)\hat{j}$$

As the vector perpendicular to the plane containing a and b. It is in the direction given by the *right hand rule*, where curling four fingers into a fist, and orienting your fist so these fingers sweep from $\vec{a}$ to $\vec{b}$, the new vector will point in the direction of an extended thumb. Theta is the angle between a and b, while j is the unit vector in the direction the new vector will point.

0.5 Solar Energy Example

The world yearly energy usage is about $180,000$TWh, which is about $5 \times 10^{20} J$ total. Is it (theoretically) possible to get this all from solar energy? We can check using an approximate order of magnitude calculation.

The Sun's total luminosity is $L_\odot = 3.8 \times 10^{26}$. This energy is radiated in a spherically symmetric way (we assume). Therefore the energy per time, per unit surface is (using 1AU for distance):

$$\frac{L_\odot}{4\pi \times (1.5 \times 10^6)^2}$$

Which is approximately (using order of magnitude):

$$\frac{3.8 \times 10^{26} W}{10 \times 10^{22} m^2} \approx \frac{1kW}{m^2}$$

This is true in ideal conditions, and real energy supply is lower (due to clouds, atmosphere etc).

If we totally covered the earth's surface area ($A_{\text{surface}} \approx \pi R_\odot^2$) which is approximately:

$$A_{\text{surface}} \approx \pi \times (6 \times 10^3 \times 10^3 m)^2 \approx 10^{14} m^2$$

Therefore total energy received is approximately:

$$P = \frac{1kW}{m^2} \times 10^{14}m^2 \approx 10^{17}W$$

And to power the world:

$$E = \frac{5 \times 10^{20}J}{3 \times 10^7 s} \approx 10^{13}W$$

So, it's theoretically possible, if we could cover enough of the world in solar panels and if we could perfectly capture the sun's energy without losing some to sources such as clouds, atmosphere, areas of the ocean we cannot cover in solar panels etc.

Tue 07 Oct 2025 12:00

Lecture 3 - Kinematics Introduction

For kinematics, we'll treat all objects as points and disregard aspects like rotation/the physical size of the body etc.

Given some point, we can define its position as a function of time $\vec{r}(t)$, and velocity as the derivative wrt time of this:

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

And acceleration:

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

0.1 Position from Unit Vectors

We can define:

$$\vec{r}(t) = r_x(t)\hat{e_x} + r_y(t)\hat{e_y} + r_z(t)\hat{e_z} = \sum_{j=1}^{3} r_j(t)\hat{e}_j$$

So:

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\sum_{j=1}^{3} r_j \hat{e}_j\right)$$

$$= \sum_j \frac{d}{dt}(r_j \hat{e}_j)$$

$$= \sum_j \frac{dr_j}{dt}\hat{e}_j$$

$$\vec{v} = \sum_j v_j \hat{e}_j$$

And:

$$\vec{a} = \frac{d\vec{v}}{dt} = \sum_{j=1}^{3} a_j \hat{e}_j$$

Note: Taking the derivative of a vector wrt time is looking at how the variable changes in some infinitesimal time. This can be a change in direction, and/or a change in magnitude. To differentiate a vector we can differentiate it component-wise.

0.2 Cartesian and Polar

Instead of representing a point as x and y components (in 2D), we can instead define it as a distance from the origin r and the angle this distance line forms with the positive x-axis θ.

Therefore (by basic right angle trig) $x = r\cos\theta$, $y = r\sin\theta$, and hence:

$$\vec{r} = r\cos\theta\,\hat{e}_x + r\sin\theta\,\hat{e}_y$$

So:

$$u\vec{(t)} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\cos\theta)\hat{e}_x + \frac{d}{dt}(r\sin\theta)\hat{e}_y$$

$$= \left(\dot{r}\cos\theta + r(-\sin\theta)\dot{\theta}\right)\hat{e}_x + \left(\dot{r}\sin\theta + r(\cos\theta)\dot{\theta}\right)\hat{e}_y$$

$$= \dot{r}\left(\cos\theta\,\hat{e}_x + \sin\theta\,\hat{e}_y\right) + r\dot{\theta}\left(-\sin\theta\,\hat{e}_x + \cos\theta\,\hat{e}_y\right)$$

0.3   Example

Lets model a particle, in a single dimension moving with constant acceleration (a_0) along a line. What is x(t)?

The introduction of τ here was generally poorly understood by the class at the time. Please see Lec 05 for a more thorough explanation.

$$a = \text{constant} = a_0$$

$$a = a_0 = \frac{dv}{dt}$$

So we can simply integrate to get $v(t)$ and again to get $x(t)$.

What if a is not constant? Consider $a(t) = kt^3$. We begin by redefining $a(t)$ as the following, where tau is a time constant representing one time unit. This could be one second, one year etc.

$$a(t) = \tau^3 k \left(\frac{t}{\tau}\right)^3$$

$$\text{let } a_* \equiv k\tau^3$$

$$a(t) \equiv a_* \left(\frac{t}{\tau}\right)^3$$

$$\int dv = \int a_* \tau \left(\frac{t}{\tau}\right)^3 d\frac{t}{\tau}$$

$$v = \frac{1}{4}a_*\tau \left(\frac{t}{\tau}\right)^4 + v_0$$

I've removed the rest here, as the whole τ thing was confusing in this lecture. Again, please see Lec 05, which re-does this section in a better manner.

Thu 09 Oct 2025 15:00

Lecture 4 - Projectile Motion and Relativity Reference Frames

Projectile Motion: The motion of a particle subject to gravitational acceleration, $g \approx 9.81 \text{m/s}$

1   Projectile Motion

For this to hold, the height of the particle above the ground must be $m << R_e \approxeq 6 \times 10^3 \text{km}$.

$$x(t) = x_0 + v_0(t - t_0) + \frac{1}{2}a_0(t - t_0)^2$$

Lets begin solely by considering motion in the vertical axis (called z here, for some strange reason). This particle is falling from height hm, to the ground at $h = 0$, with constant acceleration $g\text{m/s}^2$. It has been dropped at time $t = t_0$

At time t_0, $v = 0, z = h$

$$z(t) = h + 0 - \frac{1}{2}gt^2$$

$$\frac{1}{2}gt^2 = h$$

$$\implies t = \sqrt{\frac{2h}{g}}$$

1.1   What about 2D?

Now we can expand our example to (rather than drop the particle from rest) give the particle some initial velocity v_0m/s parallel to the ground. We now want two position functions, $x(t)$ and $z(t)$. As previously calculated:

$$z(t) = h + 0 + \frac{1}{2}(-g)t^2$$

And horizontally:

$$x(t) = 0 + v_0(t) + 0$$

So:

$$\begin{cases} z = h - \frac{1}{2}gt^2 \\ x = v_0 t \end{cases}$$

Rearranging:

$$t = \frac{x}{v_0}$$

$$z = h - \frac{1}{2}g\left(\frac{x}{v_0}\right)^2$$

Since h, g, v_0 are all constants, this is an x^2 parabola.

1.2 Interplanetary Example

Lets consider some planet, with $g_{\text{planet}} = 5m/s^2$. You (denoted Y) fall into the atmosphere at some distance h from the ground, and some horizontal distance d from $O(x = 0)$. There is an alien who wants to kill you, by shooting you down. This "gun" can throw pebbles at some constant speed v_0. The only degree of freedom the alien has to target you is change the shooting angle wrt the horizontal, θ. From the alien's perspective, what is the required θ to hit the incoming spacecraft?

To hit you, there is some time t, when the position of the bullet B, with initial velocity v where B is in the same position as Y

Consider B

$$x_B(t) = v_0 \cos(\theta)t$$

$$z_B(t) = v_0 \sin(\theta)t - \frac{1}{2}g_p t^2$$

Consider Y

$$x_Y(t) = d$$

$$z_Y(t) = h - \frac{1}{2}g_p t^2$$

We want to find a θ where $x_B = x_Y$ and $z_B = z_Y$ at the same t:

$$v_0 \cos(\theta)t = d \tag{4.1}$$

$$v_0 \sin(\theta)t - \frac{1}{2}g_p t^2 = h - \frac{1}{2}g_p t^2 \tag{4.2}$$

From 2:

$$v_0 \sin(\theta)t = h$$

$$\implies t = \frac{h}{v_0 \sin \theta}$$

And substituting:

$$v_0 \cos(\theta)\left(\frac{h}{v_0 \sin \theta}\right) = d$$

$$\frac{\cos(\theta)h}{\sin(\theta)} = d$$

$$\frac{\cos \theta}{\sin \theta} = \frac{d}{h}$$

$$\tan \theta = \frac{h}{d}$$

Since we have the value of θ in terms of two constants, yes, the alien can always hit the spaceship provided it correctly selects the angle corresponding to the value of these two constants (excluding cases where the particle is too far to the left to possibly be hit regardless of angle). This means that the required angle does not depend on velocity, in this example.

2 Frames of Reference

"Observer" represents a frame of reference. The way that one person sees the world (in terms of relative positions and velocities) is different to how another person may see the world. We observe the same core physics, but need to do coordinate translations to go from one reference frame to another.

Say we have two reference frames, A and B. We can represent the translation from A to B as a vector, denoted $\vec{r}$. Some vector $\vec{b_B}$ in B's frame of reference is therefore equal to:

$$\vec{b_B} = \vec{r} + \vec{b_A}$$

Assume that the frames are moving with a constant uniform velocity u with respect to reach other:

$$\frac{d}{dt}(\vec{b_B}) = \frac{d}{dt}(\vec{r}) + \frac{d}{dt}(\vec{b_A})$$

$$\vec{v_b} = \frac{d\vec{r}}{dt} + \vec{v}_r = u + \vec{v}_r$$

This is known as the "Galilean Transformation".

Tue 14 Oct 2025 12:00

Lecture 5 - End of Kinematics and Special Relativity I

In this lecture:

- Vecchio clarifying things he'd been asked from Kinematics.

- The start of Special Relativity.

1   Use of Tau

This caused quite a bit of confusion for people in Lec 03. We have a particle subject to constant acceleration $\boldsymbol{a} = a_0$.

The displacement of a particle at time t is given by:

$$x(t) = x_0 + v_0(t - t_0) + \frac{1}{2}a_0(t - t_0)^2$$

This is only true for a constant acceleration. More generally, we have:

$$a(t) = \frac{dv}{dt}$$

So we can integrate twice to get $x(t)$. Consider the example where $a(t) = kt^3$. This is non-constant acceleration. We assume that $t_0 = 0$ to simplify things a little. We further assume that $v(t = t_0) = v(t = 0) = 0$ and $x(t = t_0) = x(t = 0) = 0$.

We want to determine $x(t)$.

$$\frac{dv}{dt} = kt^3 \implies dv = kt^3 dt$$

$$v - v_0 = \left.\frac{kt^4}{4}\right|_{t_0}^{t}$$

Since we have $v_0 = t_0 = 0$, we have:

$$v(t) = \frac{k}{4}t^4$$

And integrating again:

$$dx = v dt$$

$$x - x_0 = \left.\frac{k}{4}\frac{t^5}{5}\right|_0^t = \frac{k}{20}t^5$$

Again, $x_0 = 0$ so we finally get:

$$x(t) = \frac{k}{20}t^5$$

Note we have simplified by assuming the initial conditions are all 0, hence we can disregard v_0 etc. If we didn't have this, we'd have to include them in the integration all the way down.

The goal of using τ is to make the problem clearer and easier to understand. Going back to $a(t) = kt^3$, we can tell by dimensions that k must have units of an acceleration divided by a time cubed. This is a messy constant with dimensions then of $[k] = L/T^5$. It is therefore difficult to see what an increase in one time unit actually causes $a(t)$ to do.

We can pick a constant timescale called τ. Tau can be whatever we like, one hour, one millisecond, fifteen years etc etc. We rewrite:

$$a(t) = kt^3 = k\frac{t^2}{\tau^3}\tau^3 = k\tau^3\left(\frac{t}{\tau}\right)^3$$

We now have a new constant with units of acceleration, $k\tau^3$ which we call a_*.

$$a(t) = a_*\left(\frac{t}{\tau}\right)^3$$

This lets us think about the problem a little more clearly, as we know that after one τ has passed, the object will have acceleration a_*. After two τs of time have passed, the object will have acceleration $2^3 a_* = 8a_*$ etc. The acceleration now nicely scales in a cubic manner.

Reintegrating with τ gives:

$$dv = a\,dt$$

$$v - v_0 = a_*\tau\left(\frac{t}{\tau}\right)^4\frac{1}{4}$$

In our case for a particle starting at rest:

$$v(t) = \frac{1}{4}a_*\tau\left(\frac{t}{\tau}\right)^4$$

And for x:

$$x - x_0 = \frac{1}{4}a_*\tau^2\frac{1}{5}\left(\frac{t}{\tau}\right)^5$$

$$x = \frac{1}{20}a_*\tau^2\left(\frac{t}{\tau}\right)^5$$

The benefit of τ for v and x is a bit less stark, but it's still somewhat present. For constant acceleration, we can write either:

$$x(t) = \frac{1}{2}a_0 t^2$$

$$x(t) = \frac{1}{2}a_0\tau^2\left(\frac{t}{\tau}\right)^2$$

The distance travelled over some time τ is $\frac{1}{2}a_0\tau^2$. Note that we can compare this to the derived result for non-constant acceleration, so using τ gives us a more comfortable and familiar form even in the non-constant scenario.

2   Normalisation

This is where we described a particle's position not in terms of unit vectors $\hat{e}_x, \hat{e}_y, \hat{e}_z$ and instead using polar form $\hat{e}_\theta$ and $\hat{e}_r$.

Consider a particle in circular motion (i.e. a child on a merry-go-round). Lets say the child wants to accelerate their motion, we want to keep the distance from the origin constant (or the child would fly off!) while increasing the speed around the circle. Doing this with the former notation would change the coefficients all three unit vectors, while using the latter notation allows us to express it as only a single constant multiplied by the unit vector changing unit vector.

3   Special Relativity

We will cover:

- The Lorentz Factor γ.

- Time Dilation

- Length Contraction

Special relativity is about how two different observers observe the kinematics of objects. For special relativity to hold, these observers cannot be accelerating. They must move with constant velocity with respect to each other. For an observer, we describe an event with four coordinates: (x, y, z, t), where t is time. For a moving observer, it will see the same event, but at a different set of coordinates (x', y', z', t').

We note that the speed of light c must be constant and independent of any observer. If two observers measure c in a vacuum, they will both determine the same value regardless of motion. This breaks the standard rules of kinematics that we've seen so far, and it means that time and space are both relative - i.e. one second for one observer may be different to one second for another.

We have these assumptions:

- Two inertial observers will observe the same physics.

- Two inertial observers will observe the same speed of light.

Everything in special relativity scales with the 'Lorentz Factor' in some form, given by:

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{u}{c}\right)^2}}$$

Gamma is always larger than 1, as $u < c$. We have two key results which we will derive later:

- Moving clocks run slow - a moving observer experiences time slower relative to a static observer.

- Moving objects shrink - a static observer will observe that a moving object has shrunk relative to what the moving observer measures about itself.

Thu 16 Oct 2025 15:00

Lecture 6 - Special Relativity II

1 Special Relativity

We consider theoretical observers that are unaccelerated with respect to each other. Either both observers are at rest, or moving with respect to each other a a constant speed. For ease in CMR1, we only consider motion in one dimension.

We also say that:

- The First Law of Dynamics (Newton's First Law) still holds true, so an object at rest will remain at rest, and an object in constant motion will remain in constant motion, unless an external force acts upon it.

- The distance between two points is constant (relative to an observer).

- We can synchronise clocks between two observers, and they will tick at the same rate.

- We only deal with Euclidean geometry.

We have two key postulates:

1. The laws of physics are the same for every inertial observer.

2. The speed of light in a vacuum is constant for every inertial observer. It is independent of any motion of the source or the observer. Even if a source travelling at $0.5c$ shines a laser facing forward, that light will still travel at c, and not $1.5c$.

1.1 Lorentz Factor

We have some stationary observer A, and a second observer B which is moving at velocity um/s relative to A. The Lorentz Factor γ is defined as:

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{u}{c}\right)^2}}$$

We may also see it written as:

$$\gamma = \frac{1}{\sqrt{1 - (u)^2}}$$

Which holds only if u is already measured in units of the speed of light. For this course, we use the first definition. This is also known as the "Relativistic Factor". Note that it is dimensionless and is a positive number $\gamma > 1$, as $u < c$.

Taking Limits: We take limits of γ to see it's behaviour as u changes relative to the speed of light.

If $u \ll c$:

$$\frac{u}{c} \ll 1$$

We use ϵ to denote a very small value. Let $\epsilon \equiv u/c$.

$$\gamma = \frac{1}{\sqrt{1-\epsilon^2}}$$

We expand this using a Taylor Series:

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0) + \cdots$$

$$\gamma = 1 - (-\frac{1}{2}\epsilon^2) + \cdots$$

If u is very fast, say $u = 30$km/s, then:

$$\epsilon = \frac{3 \times 10 \times 10^3}{3 \times 10^8}$$
$$\epsilon = 10^{-4}$$

Hence:

$$\gamma = 1 + \frac{1}{2}10^{-8} + \cdots$$

So even for speeds which are classically extremely fast, $\gamma \approx 1$ and we therefore do not encounter relativistic effects in classical mechanics.

If u/c is 'large', i.e. $u/c \to 1$:

Now, $1 - u/c$ is small, so we define $\epsilon \equiv 1 - \frac{u}{c} \ll 1$ instead.

$$\frac{u}{c} = 1 - \epsilon$$

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{u}{c}\right)^2}} = \frac{1}{\sqrt{\left(1 - \frac{u}{c}\right)\left(1 + \frac{u}{c}\right)}}$$
$$\gamma = \frac{1}{\sqrt{(\epsilon)(1 + (1 - \epsilon))}}$$
$$\gamma = \frac{1}{\sqrt{2\epsilon - \epsilon^2}}$$

Since $\epsilon \ll 1$, we say that the ϵ^2 term is small enough to disregard, so we have:

$$\gamma = \frac{1}{\sqrt{2}}\epsilon^{-1/2}$$

Again since ϵ is very small, $\epsilon^{1/2}$ tends to infinity, so as $\gamma \propto \epsilon^{-1/2}$, $\gamma \to \infty$. For non-relativistic objects, we therefore treat $\gamma = 1$, but a curve of γ against u/c has an asymptote at $u/c = 1$, hence γ rapidly increases unbounded as $u \to c$.

2 Einstein's Thought Experiment

We want to design a clock. We do so by creating a perfect cylinder, with a perfectly reflective top and bottom.

We place a light source (laser) at the bottom, and we shine this laser up towards the top of the cylinder. The photons travel to the top, hit the ceiling, which is perfectly reflective, so travels back down.
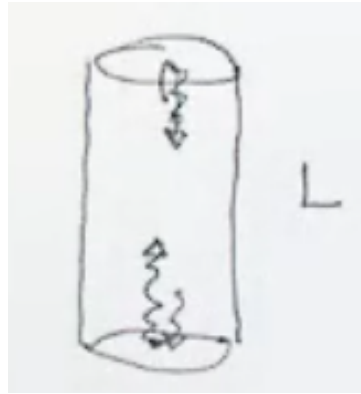
Figure 6.1

We also have a perfect clock, which measures the round-trip time for the photon to go up, hit the ceiling and hit the bottom again. The cylinder is L distance units high, so the total photon path is $2L$ for the round trip. The time taken is therefore:

$$\Delta t = \frac{2L}{c}$$

We add two observers, (B) who is fixed to the top of the cylinder (and is travelling with it). We have some other observer (A) who has designed the problem to place the whole cylinder on a moving trolley, moving in 1D with speed u. Observer A is standing stationary on the ground as the trolley speeds past them.

Observer (B) while moving sees a photon emitted at the bottom, travel up and reflect back down, with no issues.

However, Observer (A) sees the whole setup moving. It sees a photon emitted and travel up, and while it travels up the trolley has moved some distance. The trolley (and photon) have moved some distance when the photon strikes the top and reflects. As the photon travels back down, the trolley (and photon) have moved some distance again.
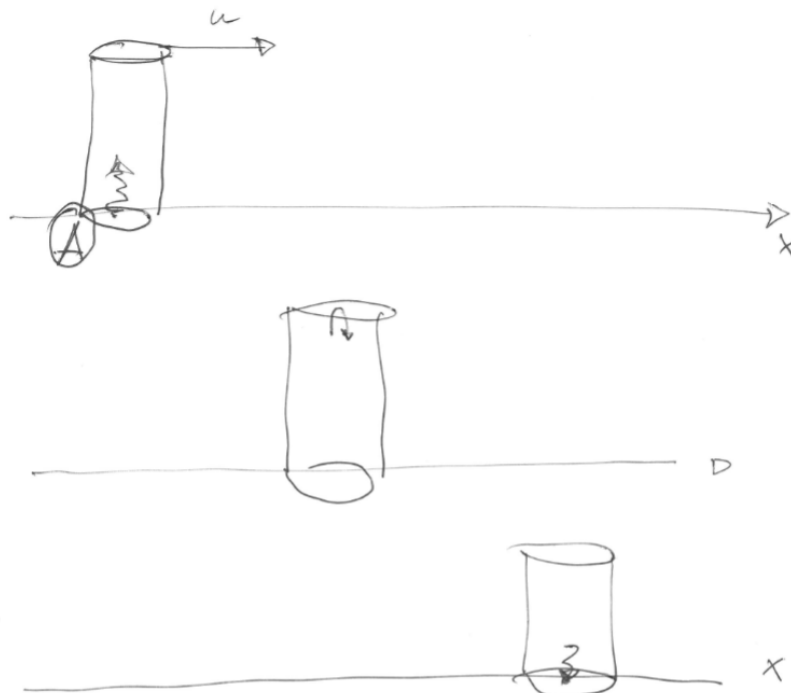

Figure 6.2: The trolley at point of photon emission, point of reflection, and point of detection. Note the photon has moved with the trolley.

We note that the height of the cylinder is not affected by the motion of the trolley, as the motion is perpendicular to this length. We are given this as fact. The only lengths which may be affected are the lengths with components in the direction of motion.

From the perspective of (B), the photon has taken the standard and simple up-down path, in some time Δt. However, from (A)'s perspective, the photon has taken a much longer path which includes horizontal motion:

Figure 6.3

In Observer (A)'s reference frame:

The two (equal) lengths that form the bases of the two right-angled triangles have length $u\frac{\Delta t}{2}$, and the two hypotenuses have length $c\frac{\Delta t}{2}$.

We therefore can simply use Pythagoras to get:

$$\left(u\frac{\Delta t}{2}\right)^2 + L^2 = \left(c\frac{\Delta t}{2}\right)^2$$

In Observer (B)'s reference frame:

$$\Delta t = \frac{2L}{c}$$

We donate the clocks held by (A) to give measurements:

$$\Delta t_B = 2\frac{L_B}{c_B}$$

and for (A):

$$\left(u\frac{\Delta t_A}{2}\right)^2 + L_A^2 = \left(c_A\frac{\Delta t_A}{2}\right)^2$$

We know that the speed of light is identical in every reference frame, so $c_A = c_B = c$. We have been told that L is unaffected, since it is perpendicular to the direction of motion, so $L_A = L_B = L$.

Hence:

$$\Delta t_B = \frac{2L}{c}$$

Which we can rearrange and substitute to get:

$$\left(u\frac{\Delta t_A}{2}\right)^2 + \left(\frac{c\Delta t_B}{2}\right)^2 = \left(c\frac{\Delta t_A}{2}\right)^2$$

$$(c\Delta t_B)^2 = (c\Delta t_A)^2 - (u\Delta t_A)^2$$

For this to be true, $t_B \neq t_A$, so the two observers can no longer agree on the time the photon took. This gives us time dilation, where moving clocks (i.e. the clock used by (A)) run slower, and record a longer time between two events compared to a stationary observer.

Tue 21 Oct 2025 12:00

Lecture 7 - Special Relativity III

1   Time Dilation

We concluded the previous lecture with:

$$(c\Delta t_B)^2 = (c\Delta t_A)^2 - (u\Delta t_A)^2$$

We rearrange to get:

$$c^2\Delta t_B^2 = \left(c^2 - u^2\right)\Delta t_A^2$$

$$\Delta t_B^2 = \left(1 - \left(\frac{u}{c}\right)^2\right)\Delta t_A^2$$

$$\Delta t_B = \sqrt{1 - \left(\frac{u}{c}\right)^2}\,\Delta t_A$$

Hence:

$$\Delta t_A = \frac{1}{\sqrt{1 - \left(\frac{u}{c}\right)^2}}\Delta t_B$$

$$\boxed{\Delta t_A = \gamma \Delta t_B}$$

This tells us that the time recorded for the photon to travel by the two observers is different. The moving clock runs slower, so the stationary observer measures a longer duration than the moving clock records. For non-relativistic speeds, $\gamma \approx 1$ so the difference is negligible; however, at larger speeds the disparity becomes much larger and grows without bounds. This makes physical sense, as for faster speeds, the trolley will have travelled a larger horizontal distance, therefore (A) will measure a longer path, and hence require a larger time.

Generally, we have:

$$\Delta T = \gamma \Delta t_0$$

Where t_0 is the "proper time" and is defined as the time interval taken between two events that take place in the same frame, by an observer in that frame.

2   Length Contraction

We have the same identical setup, except the setup is now horizontally on the trolley. Observer (B) is again attached to the cylinder, with photons again bouncing along the length of the cylinder, just with the left/right instead of top/bottom surfaces. The cylinder is still moving on a trolley, and observer (A) is still stationary.

Figure 7.1

We define Δt as the interval between emission and detection, again with a subscript to denote who is making the measurement.

$$\Delta t_B = \frac{2L_B}{c}$$

In (B)'s frame of reference:

$$c\Delta t_B = 2L_B$$

In (A)'s frame of reference:

The cylinder has moved to the right as the photon travels, this adds some extra length that the photon must travel.
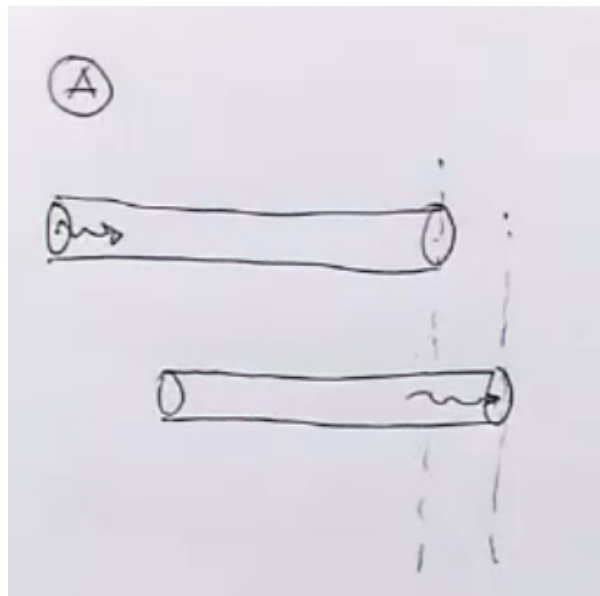

Figure 7.2

This extra length (between the two dotted lines) is $u\Delta t_1$ where Δt_1 is the time for the photon to hit the right wall. The photon now hits the wall and bounces back, while the cylinder is still moving to the right. The cylinder will move $u\Delta t_2$, where Δt_2 is the time taken for the photon to travel back and hit the left wall.

The time taken between emission and detection Δt is given by:

$$\Delta t = \Delta t_1 + \Delta t_2$$

For the first part of the trip, the distance is $L_A + u\Delta t_1$, so:

$$c\Delta t_1 = L_A + u\Delta t_1$$

$$\Delta t_1 = \frac{L_A}{c - u}$$

For the second part of the trip, the distance is less as the cylinder "catches up" with the photon as it moves, giving us a distance of $L_A - u\Delta t_2$. This gives us:

$$c\Delta t_2 = L_A - u\Delta t_2$$

$$\Delta t_2 = \frac{L_A}{c + u}$$

Hence the round-trip time is:

$$\Delta t = \frac{L_A}{c + u} + \frac{L_A}{c - u}$$

$$= L_A \left(\frac{c + u + c - u}{(c - u)(c + u)} \right)$$

$$= L_A \left(\frac{2c}{(c - u)(c + u)} \right)$$

$$= 2cL_A \left(\frac{1}{c\left(1 - \frac{u}{c}\right) c\left(1 + \frac{u}{c}\right)} \right)$$

$$= \frac{2L_A}{c} \frac{1}{1 - \frac{u^2}{c^2}}$$

So:

$$\Delta t_A = \frac{2L_A}{c} \left(\frac{1}{1 - \frac{u^2}{c^2}} \right)$$

And we know that:

$$\Delta t_B = \frac{2L_B}{c}$$

The fact A and B don't agree is fine, we can apply the time dilation formula:

$$\Delta t_A = \gamma \Delta t_B$$

$$\frac{2L_A}{c} \left(\frac{1}{1 - \frac{u^2}{c^2}} \right) = \frac{1}{\sqrt{1 - \left(\frac{u^2}{c^2}\right)}} \frac{2L_B}{c}$$

$$L_A \left(\frac{1}{1 - \frac{u^2}{c^2}} \right) = \frac{1}{\sqrt{1 - \left(\frac{u^2}{c^2}\right)}} L_B$$

$$L_A \left(\frac{\sqrt{1}}{\sqrt{1 - \frac{u^2}{c^2}}} \right)^2 = \frac{1}{\sqrt{1 - \left(\frac{u^2}{c^2}\right)}} L_B$$

$$L_A \left(\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \right) = L_B$$

$$L_A = \frac{L_B}{\gamma}$$

This tells us that measurements of lengths for a stationary vs moving observer also do not agree. Just like time runs slower for a moving object, a moving object will be measured to be smaller by a stationary observer. Effectively, moving lengths shrink. This is called length contraction.

Generally, we have:

$$\boxed{\Delta L = \frac{\Delta L_0}{\gamma}}$$

Where ΔL_0 is the "proper length" of an object, i.e. the length of an object measured by an observer at rest relative to it.

3   Example

We have someone on a spaceship, coming back to earth. The spaceship is travelling rightwards, directly towards the earth in 1D.

The spaceship has an astronaut, and we consider one person on the earth in Mission Control. MC spots the spaceship at distance $l = 3,000$km and is at speed u corresponding to $\gamma = 10$ (so $u \approx c$). The earth is stationary.

When the spaceship is at this distance l, the spaceship sends earth a distress signal saying $\Delta t_r = 10^{-3}$s, where Δt_r is how long the spaceship's oxygen supply lasts. We assume the astronaut cannot hold their breath and dies immediately if the oxygen supply runs out before the ship makes it back to Earth. Will the astronaut survive?

MC's child, who has no knowledge of relativity, calculates the travel time:

$$\Delta t = \frac{l}{u} \approx \frac{l}{c} = \frac{3 \times 10^3 \times 10^3}{3 \times 10^8} = 10^{-2} s.$$

The child compares this to the oxygen supply ($10^{-3}s$) and concludes the astronaut dies. This isn't quite accurate however, as we need to treat the oxygen time relativistically. To MC, the clock on the spaceship runs slow (Time Dilation):

$$\Delta t_{MC} = \gamma \Delta t_0$$
$$\Delta t_{MC} = 10 \times 10^{-3}\text{s} = 10^{-2} s$$

Comparing the relativistic oxygen duration ($10^{-2}s$) to the travel time ($10^{-2}s$), we see that the astronaut (just barely!!) survives.

We solved this in the Earth frame, where the distance l is a proper length. We could alternatively solve this in the Astronaut's frame, where the distance to Earth is length contracted to $L = l/\gamma = 300$km. In that frame, the travel time is $10^{-3}s$, which matches the proper time of the oxygen supply, so gives the same result.

Thu 23 Oct 2025 15:00

Lecture 8 - Special Relativity IV and Intro to Dynamics

1   The Relativistic Doppler Effect

For an emitted frequency f_0, emitted by an object moving with velocity (in 1D) u relative to an observer, the received frequency f is given by:

$$f = \sqrt{\frac{1 + u/c}{1 - u/c}} f_0$$

For a relativistic speed u. Note that we cannot use the standard Doppler formula for relativistic speeds. Also note the lack of $\pm$, as we encode this into u. If the object moves towards the observer, u is positive, and if the object is moving away u is negative.

If u is non-relativistic, we assume that $u/c \ll 1$:

$$f = \left(1 + \frac{u}{c}\right)^{1/2} \left(1 - \frac{u}{c}\right)^{-1/2} f_0$$

And Taylor Series expanding:

$$= \left(1 + \frac{1}{2}\frac{u}{c} + \cdots\right)\left(1 - \frac{-1}{2}\frac{u}{c} + \cdots\right) f_0$$

Since u/c is small, we ignore any quadratic, cubic etc terms of u/c, as these are very small.

$$= \left(1 + \frac{1}{2}\frac{u}{c} + \frac{1}{2}\frac{u}{c}\right) f_0$$

Hence:

$$f \approx \left(1 + \frac{u}{c}\right) f_0$$

Since we assume $u/c \ll 1$:

$$f = f_0 + \frac{u}{c} f_0$$

$$f - f_0 = \Delta f = \frac{u}{c} f_0$$

$$\frac{\Delta f}{f_0} = \frac{u}{c} \qquad \text{plus higher order terms we ignore}$$

This is the classical result that we're familiar with, for non-relativistic speeds.

2   Lorentz Transformation

Say we have a reference frame s' which is moving along the x-direction relative to a static reference frame s.

An event in the s frame has coordinates (x, y, z, t) and the same event in the s' frame has coordinates (x', y', z', t'). We have the following transformations:

$$t' = \gamma\left(t - \frac{u}{c^2}x\right)$$

$$x' = \gamma (x - ut)$$

Noting that $y = y'$, and $z = z'$ as these are orthogonal to the direction of motion. We also have:

$$u'_x = \frac{u_x - u}{1 - \frac{u}{c^2} u_x}$$

Please note that CMR1 does not include derivations of these equations (which collectively form the Lorentz Transformations), however for understanding's sake I'll include them here regardless.

2.1   Derivations

We want a transformation in time and space between a stationary frame S and the moving frame S'. We have three postulates to do this:

- **Linearity**: The transformation must be linear, i.e. a straight line in S must map to a straight line in S'.

- **Standard rule for** c: c is invariant and has the same velocity in all frames, regardless of motion.

- **Inverse symmetry**: The inverse transformation $(S' \to S)$ is the same, but with $u \to -u$, as in S''s reference frame, it is static with S moving with speed $-u$.

Deriving x':   We derive the transformation in position using length contraction:

Imagine a ruler at rest in the moving frame S'. It has one end on the origin O' and the right end at some coordinate x'. The ruler therefore has proper length $L_0 = x'$.

From the stationary frame S:

- The origin O' has moved distance ut after some time t.

- The ruler is moving, so has been length contracted to the observer in S. The ruler now appears to have length x'/γ.

The total coordinate as seen by S is therefore:

$$x = ut + \frac{x'}{\gamma}$$

And rearranging gives:

$$x' = \gamma(x - ut)$$

Deriving t':   We derive the transformation for time using the third postulate above:

If the transformation from $S \to S'$ is:
$$x' = \gamma(x - ut)$$

Then the transformation from $S' \to S$ is:
$$x = \gamma(x' + ut')$$

As the scenario is the same: S' is moving with speed u relative to S, if we instead consider S''s reference frame then it is static, and S is moving in the opposite direction with the same magnitude of velocity, so the transformation must be the same with $u \to -u$.

Substituting the position transformation for x' into this:

$$x = \gamma[\gamma(x - ut) + ut']$$

$$\frac{x}{\gamma} = \gamma x - \gamma ut + ut'$$

Rearranging for t':

$$t' = \gamma t - \frac{x}{u}\left(\gamma - \frac{1}{\gamma}\right)$$

Using the identity $\gamma - 1/\gamma = \beta^2 \gamma = \frac{u^2}{c^2}\gamma$:

$$t' = \gamma t - \frac{x}{u}\left(\gamma \frac{u^2}{c^2}\right)$$

$$t' = \gamma\left(t - \frac{u}{c^2}x\right)$$

Which gives us the time transformation.

Deiving u_x':   We find velocity in the moving frame as dx'/dt'.

The velocity in the moving frame S' is defined as $u_x' = \frac{dx'}{dt'}$. Taking these derivatives

$$dx' = \gamma(dx - udt) \quad \text{and} \quad dt' = \gamma\left(dt - \frac{u}{c^2}dx\right)$$

Substituting these into the definition of velocity:

$$u_x' = \frac{\gamma(dx - udt)}{\gamma\left(dt - \frac{u}{c^2}dx\right)} = \frac{dx - udt}{dt - \frac{u}{c^2}dx}$$

Dividing through by dt:

$$u_x' = \frac{\frac{dx}{dt} - u}{1 - \frac{u}{c^2}\frac{dx}{dt}}$$

And using $u_x = \frac{dx}{dt}$:

$$u_x' = \frac{u_x - u}{1 - \frac{uu_x}{c^2}}$$

3   Dynamics

Kinematics is effectively looking at objects in motion. Dynamics is effectively "why" they move (Newton's laws, static friction etc).

3.1   Newton's Laws

We have three:

1. If there is no resultant force acting upon an object, there are two possibilities:

   - The object was initially at rest, and stays at rest.
   - The object moves at a constant speed with no acceleration.

2. $\vec{F} = m\vec{a}$. Force and acceleration are proportional with a constant of proportionality m, the "inertial mass". We call it inertial mass because this is theoretically distinct from gravitational mass, however all experiments give them as having them same value.

3. The Reaction Principle. If a body A is producing a force $\vec{F}$ on a body B, then B acts back upon A with a force of the same magnitude but the opposite direction "Every action has an equal and opposite reaction".

3.2   Superposition Principle

If we have some body with N forces acting upon it, the final resultant force that acts upon an object is a vector sum of these forces:

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n = \sum_{i=1}^{N}\vec{F}_i = m\vec{a}$$

3.3   Example

Consider an object of mass m hanging from the ceiling with an "ideal string". This means that string is inextensible and is massless. The body is initially at rest.

Two forces act upon this body:

- The weight force due to local gravitational acceleration: $w = mg$.

- The force produced by the string (tension).

Since the body is at rest, the resultant force must be zero and:

$$T - mg = 0$$

$$T = mg$$

3.4   Example II

Consider a body on a horizontal surface. We laterally pull the object with force F. We have these forces:

- Again a weight force: $w = mg$.

- The normal force produced by the table acting back upon the body iaw Newton's Third Law.

- A frictional force acting in opposition to the direction of motion, F_{fric}

The frictional force is proportional to the Normal force with a coefficient depending on the materials used:

$$F_{\text{fric}} = \mu N$$

When moving an object there are two stages:

- Attempting to take an object from stationary to actually moving.

- Continuing the motion of the object once it's moving (this is easier).

We therefore have multiple coefficients of friction. Here we consider the coefficient of static friction μ_s and the coefficient of kinetic friction μ_k. There is also the coefficient of rolling friction μ_r seen in labs. Generally, $\mu_s > \mu_k$.

Tue 28 Oct 2025 12:00

Lecture 9

Thu 30 Oct 2025 15:00

Lecture 10

Lecture 11

Thu 06 Nov 2025 15:00

Lecture 12

Tue 11 Nov 2025 12:00

Lecture 13

Thu 13 Nov 2025 15:00

Lecture 14

Tue 18 Nov 2025 12:00

Lecture 15

Thu 20 Nov 2025 15:00

Lecture 16

Lecture 17 - Gravitation

1   Gravitation

Consider two particles, with masses m_1 and m_2 at distance $\vec{r}$. We know there is an attractive force between the two masses:

$$\vec{F} = -G\frac{m_1 m_2}{r^2}\hat{e}_r \tag{17.1}$$

$$G = 6.67 \times 10^{-11} m^3/kg/s^2$$

1.1   Key Properties

- "Long Range" force. A force will exist between any two masses anywhere in the universe, regardless of distance and cannot be cancelled. Negligible at large distances, as force quickly tends to zero as distance increases - but never zero.

- Weak.

- $\vec{F} \propto \frac{1}{r^2}$

1.2   Mass Caveats

While we don't practically make a distinction between them, m in gravity refers to 'gravitational mass', m_g, while mass in Newton's Second Law $\vec{F} = m\vec{a}$ is 'inertial mass', m_i.

Einstein's Equivalence Principle says that they're equal, i.e:

$$\frac{dm_g}{dm_i} = 1$$

Why 'they could be different but they fundamentally are not' is important is beyond me.

2   Freefall

Consider an object of mass m at height h from the ground. It has force:

$$F = mg$$

And potential energy:

$$\Delta U(h) = mgh$$

However this first expression looks quite different to our definition of force.

2.1   Derivation

Gravitational force is conservative, and there is a potential energy at all points in the gravitational field, associated with this force.

Give two bodies, M and m. m is moved away from M from r_1 to r_2 along the radial direction (preserving angle, i.e. moving only in $\hat{e}_r$).

$$\vec{F} = -G\frac{Mm}{r^2}\hat{e}_r$$

This is the negative gradient of the potential (as potential is area under a force curve):

$$\vec{F} = -\nabla U = -\frac{dU}{dr}\hat{e}_r$$

$$\vec{F}dr = -dU\hat{e}_r$$

The work done from r_1 to r_2 is:

$$w_{1\to2} = \int_{r_1}^{r_2}\vec{F}\cdot\hat{e}_r\,dx$$

Thu 27 Nov 2025 15:00

Lecture 18

Tue 02 Dec 2025 12:00

Lecture 19

Thu 04 Dec 2025 15:00

Lecture 20