UNIVERSITY OF BIRMINGHAM

School of Physics and Astronomy

Lecture Notes

Year 1 Semester 2

Ash Stewart

MSci Physics with Particle Physics and Cosmology

Lecture Index

LC Classical Mechanics and Relativity 2 . . . . . **2**

LC Electric Circuits **3**
Lecture 3 4

LC Electromagnetism I **5**
Lecture 1: EM1 Intro and Electric Fields 6

LC Introduction to Particle Physics and Cosmology **10**
Lecture 1: The Standard Model of Particle Physics 11
Lecture 2: Luminosity and Particle Signatures 14
Lecture 3: Tracking Systems 19

LC Mathematics for Physicists 1B **20**
Lecture 12: End of Partial Differentiation & Start of ODEs 21

LC Temperature and Matter **24**
Lecture 1 25

LC Classical Mechanics and Relativity 2

LC Electric Circuits

Fri 13 Feb 2026 11:00

Lecture 3

LC Electromagnetism I

Mon 19 Jan 2026 11:00

Lecture 1 - EM1 Intro and Electric Fields

1 Course Intro

Course Materials:

- Background material and derivations etc on PowerPoint.
- Worked examples etc are handwritten on visualiser, these are the bits we really need to know.

Why is EM important?

- Foundations of modern technology and the modern world.
- What gives elements their properties.
- Responsible for life itself.
- Everyday materials are held together by EM forces.
- Optics can only be understood through EM theory.

The course aim is to lay down the foundations, eventually leading us to Maxell's Laws.

1.1 Maxwell's Laws

Maxwell's four equations are:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \wedge \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \wedge \boldsymbol{B} = \mu_0 \left(\boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right)$$

Where:

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$$

Together, these show that the electric and magnetic fields are related and are two aspects of a single force, the electromagnetic force. We don't have to properly understand them yet, but cannot learn them in EMII unless we fundamentally understand E and B fields from this module.

1.2 Course Structure

Part I: Electric Fields

- Charge and Coulomb's Law.
- The electric field.
- Gauss' Law.
- Capacitors.

Part II: Magnetic Fields

- Magnetic Fields
- Charged Particles in B-Fields
- Electromagnetic Induction.
- Magnetic Dipoles

In this lecture:

- Introduction to EM.
- Electric charge.
- Force between charges.
- The concept of the Electric Field (E-Field).

2 Electric Charge

First attributed to Thales circa. 624 - 546 BC. Experiments by Franklin and Coulomb expanded and showed that there was two types of charge, which they called positive and negative.

The "positive electricity" came from rubbing a glass rod with silk, and the negative from rubbing an ebonite (early plastic) rod with fur. They found that like charges repel and opposite charges attract.

We know that the elementary unit charge is the magnitude of charge of an electron/proton and everything else is a multiple of this [1]:

$$e = 1.6 \times 10^{-19} C$$

This has units of the Coulomb.

2.1 Charge Conservation

Electrons and protons are both stable (protons decay with a life greater than 10^{31} years). This means that the total charge of an isolated system is constant and can be conserved.

They have the same magnitude of charge, exactly:

$$|q_p| = |q_e| = e$$

2.2 Electrostatic Force

Like charges repeal and opposite charges attract, along the line of action given by a line drawn between the two charges. The force is proportional to the product of charges so:

$$F \propto q_1 q_2$$

Here, a negative force means attraction and a positive force means repulsion. Newton called this "force at a distance". Like gravity, two charges will exert a force on each other at a distance without any contact.

There must, therefore, be something between them that mediates this force. Later physics gives this as "virtual particles" which isn't a Y1 topic, so classically we say that this medium is the Electric Field.

2.3 Electric Field

A charge produces a field around it. Another charge also interacts with this field, and this interaction is what causes a force:

$$\underline{F} = \underline{E} q$$

Where F is the force exerted on a test charge of charge q by a charge Q producing a field E. The magnitude of the electric field has units NC^{-1} (force per units charge).

$$|\underline{E}| \propto Q \qquad |\underline{F}| \propto Qq$$

[1] While quarks have fractional charge, we don't get free quarks

Consider a point charge with a spherical electric field spreading out around it. As the distance from the charge increases, the surface area increases as $4\pi r^2$. Therefore the magnitude of the electric field must decrease with $4\pi r^2$

Therefore:

$$|\underline{E}| \propto \frac{Q}{4\pi r^2}$$

We need a (inverse) constant of proportionality. This depends on the medium, but for a vacuum we call it the permittivity of free space ϵ_0:

$$\epsilon_0 = 8.854 \times 10^{-12} C^2 m^{-2} N^{-1}$$

Hence:

$$\boxed{E = |\underline{E}| = \frac{Q}{4\pi\epsilon_0 r^2}}$$

2.4 Direction of the E-Field

Force is a vector, so the E-field must be too. Consider an E-field from a charge Q at distance r. We give E components E_r, E_θ, E_ϕ, where, since it's a sphere ϕ and θ represent the unit vectors in the two possible tangential directions.

If there was a component in E_θ this would be clockwise from one perspective, but anticlockwise from another (walking behind it). This is not possible, as the field must behave in the same manner from all viewpoints. Therefore:

$$E_\theta = E_\phi = 0$$

$$\underline{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{t}$$

2.5 Force between two charges

Consider two charges q_1 and q_2. The force on q_2 due to the e-field from q_1 is:

$$\underline{F_1} = \underline{E_1} q_2$$

This is equal to the force on q_1 due to the e-field from q_2, given by:

$$\underline{F_2} = \underline{E_2} q_1$$

So the force between two charges is:

$$\underline{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{r}_{12}$$

2.6 Force between many charges

If we have more than two positive charges, we use the "principle of superposition". Effectively, you consider each pair of charges at the same time and vector sum of the forces together. I.e. if we have three points and we care about the net force on one, we take the vector sum of the two vectors from that point to the others.

In general:

$$\begin{aligned}\underline{F} &= \underline{F_1} + \underline{F_2} + \underline{F_3} + \cdots \\ &= q\sum_i \frac{q_i}{4\pi\epsilon_0 r^2}\hat{r_j}\end{aligned}$$

Where r_j is the distance between q_i and q, with unit vector $\hat{r_j}$ between them.

Since $\underline{F} = q\underline{E}$, the electric field at a test charge q must be:

$$\underline{E} = \sum_i \frac{q_i}{4\pi\epsilon_0 r_i^2}\hat{r_j}$$

2.7 Example

Say we have a square of side length a. Clockwise, these corners have charge Q, q, $-2Q$, $3Q$.

What is the net force exerted on the q charge?

LC Introduction to Particle Physics and Cosmology

Thu 22 Jan 2026 16:00

Lecture 1 - The Standard Model of Particle Physics

1 Course Introduction

Course Structure

- Particle Physics: 6 lectures in weeks 1 to 6.
    1. Introduction and the standard model.
    2. Experimental measurements.
    3. Interactions with matter
    4. Tracking detectors I
    5. Tracking detectors II
    6. Calorimeters and Particle Identification.
- Cosmology: 4 lectures in weeks 7 to 11.

Course Aims

- Overview of current methods in Particle Physics experiments.
- An emphasis placed on the questions and challenges.

For example, the LHC has already been programmed with experiments all the way up to 2041. Therefore, any detectors we design today only become relevant in over a decade, which makes good detector design decisions incredibly important. This course will equip us to understand what drives those design choices.

The course is assessed by a single one hour long exam, weighted half particle physics and half cosmology.

Recommended Texts

- Detectors for particle radiation (2nd edition), K. Kleinknecht (1998)
- Particle Physics, Martin and Shaw.
- High Energy Physics, D. H. Perkins (2nd through 4th editions)
- Feynman Lectures.
- Modern Particle Physics, M. Thomson.

2 Matter Particles

Fermions all have quantum spin 1/2. Spin is a purely inherent quantum property (like mass or charge) and has no classical representation, but is analogous to angular momentum. They are subject to Fermi-Dirac statistics, which means that no identical fermion in a system of fermions can have the same quantum number as any other. Fermions are divided into two types, quarks and leptons.

2.1 Quarks

There are three generations of quarks:

First Generation

- Up Quark (u), mass of ≈ 0.001GeV
- Down Quark (d), mass of ≈ 0.001GeV

Second Generation

- Charm Quark (c), mass of ≈ 1.3GeV
- Strange Quark (s) mass of ≈ 4.3GeV

Third Generation

- Top Quark (t), mass of ≈ 175GeV
- Bottom Quark (b), mass of ≈ 4.3GeV

"up-type quarks", u, c, t have electromagnetic charge $+2/3$ and "down-type quarks", d, s, b, have electromagnetic charge $-1/3$ (charges in units of e). Quarks do not ever exist alone in isolation.

2.2 Leptons

There are three generations of leptons, given by the electron e^-, muon, μ^- and tau τ^-. These all have charge of -1 (in units of e). These have masses (in MeV) of approximately $0.5, 105, 1800$.

These also have associated neutrinos, the electron neutrino, the mu neutrino and the tau neutrino, ν_e, ν_μ, ν_τ. These are not massless, but have a tiny mass many orders of magnitude smaller than their corresponding non-neutrino counterparts. These are neutrally charged.

Leptons are not subject to the strong interaction.

2.3 Hadrons

Quarks do not exist in isolation, but form bound states subject to the strong force called hadrons. There are two types of hadrons - baryons and mesons.

Baryons are formed from three quarks, which may be the same or different $q_1 q_2 q_2$.

Mesons are formed from a quark-antiquark pair, $q_1 \bar{q}_2$. Examples of baryons include the proton and the neutrino, given by uud and udd.

3 Forces

As far as we know, there are four fundamental forces:

- Gravity
- Electromagnetism
- Strong
- Weak

For considering particle interactions, we disregard gravity as it becomes incredibly weak for small masses. Creating a complete theory that incorporates all four is an open question in physics. It's okay to neglect it, but it is unsatisfying.

We consider these forces as arising by the exchange (between two particles subject to a force between them) of particles called bosons. These have spin-1, so are called "gauge bosons". These are subject to Bose-Einstein statistics, which does not impose the same restriction as Fermi-Dirac for quantum numbers in a system.

For EM: The exchange particle is a photon, γ. This is represented on a Feynman diagram as a wiggly line. They are massless and couple to electric charge.

For Weak: The exchange particle is a $W^{\pm}$ or Z^0 boson. This is represented by a wiggly line or a dotted straight line. They are not massless, and have masses of approx. 80 and 90GeV respectively.

For Strong: The exchange particle is a gluon g. This is represented by a series of curls on a Feynman diagram. They are massless. They couple to "colour charge" which is just another quantum number analogous to electric charge. Just like electric charge has values $\pm$, colour charge has values we denote r, g, b

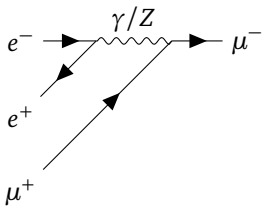
Quarks are subject to the strong, electromagnetic and weak interactions

Leptons are not subject to the strong interaction, but the e, μ, τ are subject to EM (ν is not as it is neutrally charged), and all are subject to the weak interaction. This makes neutrinos very difficult to detect as they are only affected by the weak interaction.

4 Feynman Diagrams

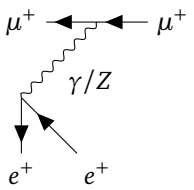
Feynman diagrams are space (y-axis), time (x-axis) diagrams to show allowed interactions between particles.

Consider a simple example of electron-positron annihilation. They travel towards each other, meeting and annihilating into either a photon or a Z boson. This is called a 'time-like exchange'. The boson then decays and we see pair production of two muons (one μ^- muon and one μ^+ antimuon).



Arrows in Feynman diagrams convey "fermion flow". This means that for a matter particle, the arrows aligns to the time axis. For antimatter particles, they antialign. Some conservation laws (i.e. charge) apply at the vertex level, while others only apply across whole processes.

We now consider a space-like exchange where the exchange is aligned with the vertical (space) axis. An antimuon scatters off a position like such:



5 Luminosity

We can determine the rate of interactions with the following:

$$W = \mathscr{L}\sigma$$

Where W (s^{-1}) is interaction rate, $\mathscr{L}$ ($cm^{-2}s^{-1}$) represents the luminosity (an attribute of the accelerator being used), and σ (cm^{-2}) is the cross section, representing the underlying physics of the interaction.

These are investigated in greater detail in Lecture 03.

Mon 02 Feb 2026 12:00

Lecture 2 - Luminosity and Particle Signatures

1 Cross Sections

Consider a proton-proton interaction, producing some unknown particle X:

$$pp \to ppX$$

We have said that the rate of interaction is given by:

$$W = \mathscr{L}\sigma$$

Where $\mathscr{L}$ in $\text{cm}^{-2}\text{s}^{-1}$ is (coarsely) a parameter of the accelerator, describing its ability to produce collisions, and σ in cm^{-2} is a measure of interaction probability. Even though the particles are point-like, we treat them as having an effective area, and the magnitude of that area dictates how likely an interaction is to take place.

In this interaction, we have two protons (modelled as solid balls) passing immediately next to each other (one travelling clockwise and one counter-clockwise) around the accelerator. Assuming they pass immediately next to each other, and we model them as having radius 10^{-15}m, we have a a separation between the centres of each proton as $2 \times 10^{-15}\text{m}$, therefore a cross section of:

$$\pi\left(2 \times 10^{-28}\text{m}^2\right) \sim 0.12 \times 10^{-28}\text{m}^2$$

To move this to a less annoying length scale, we define a new unit, the barn:

$$1\text{barn} \equiv 10^{-28}\text{m}^2 = 10^{-24}cm^2$$

In reality, this model may approximate a cross section, but it's not accurate. In reality, there's a much wider variation:

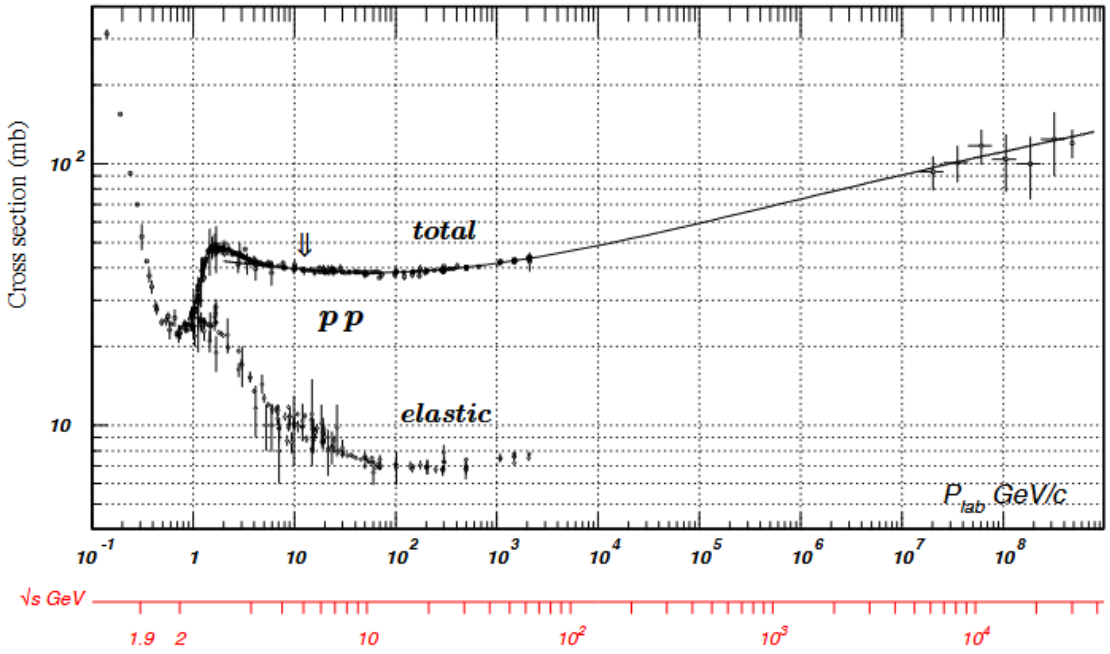


Figure 4.1

Here we have two x-axes running in parallel - the black axis is the momentum in a lab frame (as it hits some fixed target) while the red axis is the corresponding "centre of mass energy". How do we relate these two?

We want to know what the maximum mass of the particle we can generate is. In the lab frame, this requires us to take the momentum of the incoming and generated particle into account. We then have to take the final momentum of the system into account to conserve momentum, as the whole system must continue moving in the direction of motion for conservation.

Translating into the frame of reference given by the system centre of mass gives us a system where the two masses can be thought of as approaching each other with equal and opposite momenta. Since the total momentum is zero, the objects (incoming particle and the target) can theoretically hit each other and come to a complete stop. Because the system does not have to keep moving after the collision, all of the energy in this frame is free to be converted into the mass of a new particle.

While this may not be accurate in practice, it gives us a hard upper maximum for the possible energy available for production.

We can show relativistically that the energy in the centre of mass frame (labelled $\sqrt{s}$, the energy available for particle production) is:

$$E_{\text{com}} = \sqrt{s} \propto \sqrt{p_{\text{lab}}}$$

This can be seen in the final values of the x-axis, which (both starting approx. 1) are 10^8 and 10^4. This tells us that we reach diminishing returns with a fixed target collider - increasing energies by 8 orders of magnitude only increases the energy available for particle production by 4 orders. This is an inherent inefficiency of a fixed target collider.

This is much cheaper as its easier to align, we just fire a beam at a block of (for example) lead. It also makes it quite easy to change the target material. Changing materials in a colliding-beam collider (where two beams are fired in opposite directions, one clockwise and one anticlockwise, and collide with each other), i.e. to fire lead nuclei requires an extensive recalibration process.

In a dual-beam collider, only a very small proportion of the accelerated material from each beam actually interact with each other. In a fixed target collider, the target is much more dense, so we see a higher rate of interactions.

In summary, the advantages of a fixed target collider are:

1. Easier to collide.
2. Easier to change the target.
3. Very high density

2 Luminosity

2.1 Fixed Target Case

Consider a fixed target collider. We want to build an expression for the luminosity of this setup.

We have some incoming flux of particles (per second per unit area), J, incident on the block of material with density ρ, thickness t and mass of one nucleus m_A. The beam, modelled by a cylinder, illuminates some circular portion of the block, with area A.

Consider our interaction rate W. This is given by (where V is the volume of a cylinder from the illuminated beam, of thickness equal to the target):

$$W = \underbrace{JA}_{\substack{\text{incident particles} \\ \text{per unit time}}} \times \underbrace{\frac{\rho}{m_A} V}_{\substack{\text{total number of} \\ \text{target particles}}} \times \underbrace{\frac{\sigma}{A}}_{\substack{\text{probablity of} \\ \text{interaction}}}$$

$$W = JA \times \frac{\rho}{m_A} At \times \frac{\sigma}{A}$$

$$W = J \times \frac{\rho}{m_A} At \times \sigma$$

$$W = \frac{J\rho At}{m_A} \sigma$$

Comparing to $W = \mathscr{L}\sigma$ gives the luminosity as:

$$\mathscr{L} = \frac{J\rho At}{m_A}$$

2.2 Colliding Beam Case

In a colliding beam case, the derivation is more (and too) complex. It is equal to:

$$\mathscr{L} = \frac{f_{\text{rep}} n_b N_1 N_2}{4\pi \sigma_x \sigma_y}$$

Where:

- f_{rep} is the repetition frequency, i.e the rate of the beam passing the collision point.
- n_b is the number of bunches in the beam.
  - A beam can be thought of as as string of pearls, rather than a single discrete constant beam - i.e. clusters of particles "bunches", followed by empty space between them.
- N_1, N_2 are the number of particles per bunch for each beam
- σ_x, σ_y are the dimensions of the beam in the x- and y-direction, not a cross section as previously.

3 Examples of Detectors

We have some interaction point producing a spray of particles, and surround this with a series of different detector layers. Each produced particle will trigger a different subset of these layers, defining a unique signature we can use to identify produced particles.

Broadly, in some generic detector, we have:

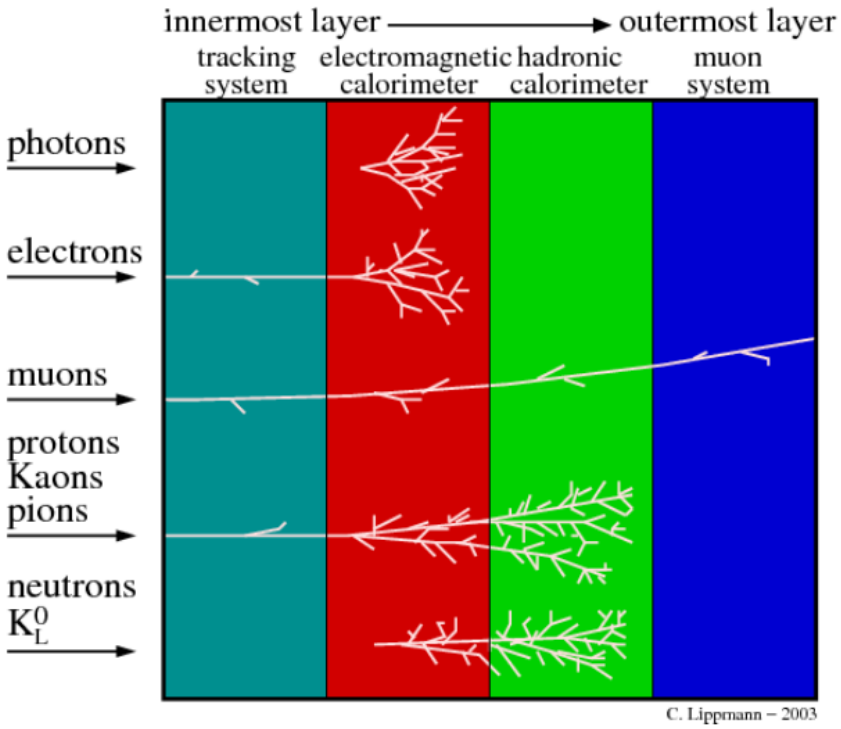


Figure 4.2

The tracking system is non-destructive. Formed of layers of silicon, a charged particle with ionise small portions of each layer. This can be turned into a signal. Neutral hadrons and photons will pass straight through, but charged particles will leave a deposit of charge and pass through unaffected.

We then move to destructive layers. Particles leave tree-like structures as they pass through these layers and create a shower of particles.

3.1 Decays

Most decays take place over a very low time scale, $< 10^{-8}$s and produce a final state made up of some subset of the following: $\gamma, \pi^+, \pi^-, \kappa^+, \kappa^-, p, n, \pi^0, e^+, e^-, \mu^+, \mu^-$.

So, in order to detect some exotic particle, we assume that it will either persist long enough to be detected itself, or decay into some subset of these known particles which will reach out detector.

Consider a parent particle A, for example B^0 $(\bar{b}d)$ decaying into two child particles B, C, given by a "J Psi" $(c\bar{c})$, J/ψ and a "K Short", K_s^0 $(d\bar{s})$:

$$B \to J/\psi \quad K_s^0$$
$$\bar{b}d \to c\bar{c} \quad d\bar{s}$$

This decay has a lifetime of 10^{-12}s, via the weak interaction due to the change in quark flavour. The J Psi decays into e^+e^- or $\mu^+\mu^-$ via the EM interaction very rapidly in 10^{-21}s (its lifetime is governed by the strong interaction, which it may also use to decay via, even though we detect it via the EM decay path). The K Short decays into $\pi^+\pi^-$ or $\pi^0\pi^0$ again via the weak interaction with lifetime in 10^{-10}s.

The range of a particle is given by:

$$\text{Range} = \beta\gamma c\tau$$

Where τ is a time scale (lifetime), c is the speed of light, γ is the Lorentz Factor and β scales the range based on the speed actually being travelled. We also have:

$$E = \gamma m$$

$$p = \beta\gamma m$$

And familiarly:

$$E^2 - p^2 = m^2$$

If the B^0 has energy 20GeV and mass 5GeV, we have $\gamma = 4$ and this gives a range of $\approx$ 1mm. This is so small we will never observe it directly. The K_s^0 however has range $\approx$ 30cm, so is detectable.

Crucially:

- SI lifetimes: $\sim 10^{-21} - 10^{-24}$s
- EM lifetimes: $\sim 10^{-16} - 10^{-20}$s
- WI lifetimes: $\sim 10^{-12}$s

Thu 05 Feb 2026 16:00

Lecture 3 - Tracking Systems

LC Mathematics for Physicists 1B

Fri 13 Feb 2026 12:00

Lecture 12 - End of Partial Differentiation & Start of ODEs

Recap of lecture 11:

- The tangent plane to a surface $f(x,y,z)=0$ at (x_0,y_0,z_0) is given by:

$$\left(\frac{\partial f}{\partial x}\right)_0 (x-x_0)+\left(\frac{\partial f}{\partial y}\right)_0 (y-y_0)+\left(\frac{\partial f}{\partial z}\right)_0 (z-z_0)=0$$

  Such that $\underline{\nabla} f(x_0,y_0,z_0)$ is the normal vector to the plane
- The parametric representation of a curve $\underline{r}(t)$ has:
  - Unit tangent: $\hat{\underline{T}}=\frac{d\underline{r}}{dt}/\left|\frac{d\underline{r}}{dt}\right|$
  - Arc length $s(t)$: $\frac{ds}{dt}=\left|\frac{d\underline{r}}{dt}\right| \rightarrow \hat{\underline{T}}=\frac{d\underline{r}}{ds}$.
  - Unit normal and curvature:

1 Orthonormal Triads

We can create an *orthonormal triad* by introducing a new normal vector called the unit binormal, $\hat{\underline{B}}=\hat{\underline{T}}\times\hat{\underline{N}}$.

Since $\hat{\underline{N}}\times\hat{\underline{N}}$, differentiating wrt s gives:

$$\hat{\underline{N}}\cdot\frac{d\hat{\underline{N}}}{ds}=0$$

TODO

We have:

$$\frac{d\hat{\underline{T}}}{ds}=\kappa\hat{\underline{N}}$$

$$\frac{d\hat{\underline{N}}}{ds}=-\kappa\hat{\underline{T}}+\tau\hat{\underline{B}}$$

Hence:

$$\frac{d\hat{\underline{B}}}{ds}=\frac{d}{ds}\left(\underline{T}\hat{\times}\hat{\underline{N}}\right)$$

$$=\frac{d\hat{\underline{T}}}{ds}\times\hat{\underline{N}}+\hat{\underline{T}}\times\frac{d\hat{\underline{N}}}{ds}$$

$$=\kappa\hat{\underline{N}}\times\hat{\underline{N}}+\hat{\underline{T}}\times\left(-\kappa\hat{\underline{T}}+\tau\hat{\underline{B}}\right)$$

$$=\tau\hat{\underline{T}}\times\hat{\underline{B}}=\tau\hat{\underline{T}}\times\left(\hat{\underline{T}}\times\hat{\underline{N}}\right)$$

$$=\tau\left[\left(\hat{\underline{T}}\cdot\hat{\underline{N}}\right)\hat{\underline{T}}-\left(\hat{\underline{T}}\cdot\hat{\underline{T}}\right)\hat{\underline{N}}\right]$$

$$=\tau\hat{\underline{N}}$$

This gives the Frenet-Serret Formulae:

This concludes partial differentiation! :D

2 Ordinary Differential Equations

A differential equation is any equation that involves derivatives. We care, because most laws of physics manifest themselves in the form of differential equations. For example:

$$\text{Newton's Second Law:} \qquad \underline{F} = m\frac{d^2\underline{r}}{dt^2}$$

$$\text{3D Time-Independent Schrödinger Eqn:} \qquad -\frac{\hbar}{2m}\left(\frac{\partial^2\psi}{dx^2}+\frac{\partial^2\psi}{dy^2}+\frac{\partial^2\psi}{dz^2}\right)+V(x,y,z)\psi = E\psi$$

$$\text{3D Wave Eqn:} \qquad \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

$$= \nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

$$\text{Gauss' Law:} \qquad \frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0}$$

$$\text{Navier-Stokes Eqn:} \qquad \rho\left(\frac{\partial \underline{v}}{\partial t}+\left(\underline{v}\cdot\underline{\nabla}\right)\underline{v}\right) = -\underline{\nabla}p+\rho\underline{g}+\mu\nabla^2\underline{v}$$

In this course, we will only solve DEs of a single variable, i.e. Ordinary Differential Equations (ODEs). We don't look at Partial DEs of multiple variables yet.

In order to think about solving these, we need to classify them. Most DEs aren't soluble in closed form with elementary functions and need to be solved numerically. Here, we only consider nice soluble functions, but this is a vast minority in reality. We want to identify classes of DEs we can reasonable solve with a method for each.

We can generally solve linear equations by breaking them into small chunks and solving them individually, for example.

3 Types of DEs

3.1 Partial vs. Ordinary

In the examples above, only the first was an ODE, and the rest PDEs. Ordinary Differential Equations (ODEs) involve only a single variable.

Consider a vector $\underline{r}(t) = (x(t), y(t), z(t))$. t is called the independent variable, with x, y, z being dependant variables. While we have 3 dependant variables, we only have one independent variable (so only one thing to differentiate wrt), so this would end up being ordinary.

PDEs involve equations of two or more variables and hence involve partial derivatives.

3.2 Order

The order of a DE is given by the order of the highest derivative involved, so Newton's 2nd Law is a second order DE, as the highest order derivative is a second derivative.

3.3 Degree

The degree of a DE is a less important measure than the others. It is given by the highest power of the highest order derivative. For example, Newton's 2nd is a first order, while an equation containing a^3 would be third degree (and second order, as a is a second derivative).

Ideally, we want this to be 1 for ease of solving, and higher degrees are rare but they do exist. For example, from Lagrangian Mechanics we have:

$$\frac{1}{2m}\left[\left(\frac{\partial s}{\partial x}\right)^2+\left(\frac{\partial s}{\partial y}\right)^2+\left(\frac{\partial s}{\partial z}\right)^2\right]+V(x,y,z)=\frac{ds}{dt}$$

3.4 Homogenous and Inhomogeneous

A homogenous DE is a DE that does not have any terms of only the independent variable(s), while an inhomogeneous DE does.

For example, Newton's 2nd is homogenous as there is no term that involves t alone. This would be inhomogeneous:

$$\frac{\partial^2 x}{\partial t^2}=t+x$$

While this would be homogenous:

$$\frac{\partial^2 x}{\partial t^2}=tx$$

As t is a coefficient and not a pure term in its own right.

3.5 Linear and Non-Linear

A DE is linear if the dependant variable(s) and all of its/their derivatives occur purely as linear functions. For example:

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$$

This is linear, as the dependant variable y never has a coefficient greater than 1.

$$\frac{dy}{dx}+xy=0$$

Is also linear, while this is not:

$$\frac{dy}{dx}+xy^2=0$$

This is also non-linear (as shown by the Taylor Expansion of sine):

$$\frac{d^2\theta}{dt^2}=-\frac{g}{l}\sin\theta$$

3.6 Examples

$$(1)\quad \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=u^2$$

Homogenous first-order second-degree non-linear PDE.

$$(2)\quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=x^2+y^2+z^2$$

Inhomogeneous second-order first-degree linear PDE.

$$(3)\quad \frac{\partial y}{\partial x}+y^2=x$$

Inhomogeneous first-order second-degree non-linear ODE.

LC Temperature and Matter

Tue 20 Jan 2026 12:00

Lecture 1