Lecture Notes

MSci Physics w/ Particle Physics and Cosmology
University of Birmingham

Year 1, Semester 1

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Lectures Index

LC Classical Mechanics and Relativity 1

Thu 02 Oct 2025 14:15

Lecture 1 - Orders of Magnitude and Dimensional Analysis

TODO

Thu 02 Oct 2025 15:00

Lecture 2 - Dimensional Analysis (contd.) and Vectors

Continuation of Dimensional Analysis

What if, in theory, we could build a system of units entirely from c, the speed of light, G, Newton's constant and h, the Plank Constant?

Cont. from Lec01, we can try to use this to work out the earliest possible cosmic time.

$$h = 6.6 \times 10^{-34} Js$$
$$G = 6.67 \times 10^{-11} Nm^2/kg^2$$
$$c = 3 \times 10^8 m/s$$

Dimensionally:

$$[h] = \frac{ML^2}{T}$$
$$[G] = \frac{L^3}{T^2 M}$$
$$[c] = \frac{L}{T}$$

We want to use these to build out a time unit, so:

$$[h^u G^v c^z] = T$$
$$\left(\frac{ML^2}{T}\right)^u \left(\frac{L^3}{T^2 M}\right)^v \left(\frac{L}{T}\right)^z = T$$

$$M^{u-v} L^{2u+3v+z} T^{-u-2v-z} = T$$

Solving for:

$$u - v = 0$$

$$2u + 3v + z = 0$$

$$-u - 2v - z = 1$$

Gives us:

$$u = \frac{1}{2} \tag{1}$$
$$v = \frac{1}{2} \tag{2}$$
$$z = \frac{-5}{2} \tag{3}$$

$t_p = \sqrt{\frac{Gh}{c^5}}$ and plugging in the values for G, h, c gives us a value of time, which the earliest possible cosmic time equal to about $10^{-43} s$

Plank Energy

Doing the same process for energy gives us (this time, the plank energy is the energy at which traditional theories of physics break down):

$$E_p = \frac{hc^5}{G}^{0.5} \approx 10^9 J$$

On the other hand, the LHC manages about 10TeV, which is orders of magnitude smaller than this, so the LHC cannot accurate simulate energies of this magnitude.

More Vectors

Again, vector notation will be $\vec{a}$. We define the x, y, z unit vectors as $\underline{\hat{e_x}}, \underline{\hat{e_y}}, \underline{\hat{e_z}}$.

We can therefore define any vector as:

$$\vec{a} = a_x \underline{\hat{e_x}} + a_y \underline{\hat{e_y}} + a_z \underline{\hat{e_z}}.$$

The length of a vector is again $|\vec{a}|$.

Vector Multiplication

Given $\vec{a}$ and $\vec{b}$ we can define the dot (scalar) product and the cross (vector) product

$$\vec{a} \cdot \vec{b} = |a||b| \cos\theta$$

Say we want to know the component of a vector along an axis, we can do the following (eg for x):

$$\vec{a} \cdot \underline{\hat{e_x}} = a_x$$

For the vector product, we can define:

$$\vec{a} \times \vec{b} = |a||b| \sin(\theta) \hat{\jmath}$$

As the vector perpendicular to the plane containing a and b. It is in the direction such that[1]. Theta is the angle between a and b, while j is the unit vector in the direction the new vector will point.

Solar Energy Example

The world yearly energy usage is about 180,000TWh, which is about $5 \times 10^{20} J$ total. Is it (theoretically) possible to get this all from solar energy? We can check using an approximate order of magnitude calculation.

The Sun's total luminosity is $L_\odot = 3.8 \times 10^{26}$. This energy is radiated in a spherically symmetric way (we assume). Therefore the energy per time, per unit surface is (using 1AU for distance):

$$\frac{L_\odot}{4\pi \times (1.5 \times 10^6)^2}$$

Which is approximately (using order of magnitude):

$$\frac{3.8 \times 10^{26} W}{10 \times 10^{22} m^2} \approx \frac{1kW}{m^2}$$

This is true in ideal conditions, and real energy supply is lower (due to clouds, atmosphere etc).

If we totally covered the earth's surface area ($A_{\text{surface}} \approx \pi R_\odot^2$) which is approximately:

$$A_{\text{surface}} \approx \pi \times (6 \times 10^3 \times 10^3 m)^2 \approx 10^{14} m^2$$

Therefore total energy recieved is approximately:

$$P = \frac{1kW}{m^2} \times 10^{14} m^2 \approx 10^{17} W$$

And to power the world:

$$E = \frac{5 \times 10^{20} J}{3 \times 10^7 s} \approx 10^{13} W$$

So, it's theoretically possible, if we could cover enough of the world in solar panels and if we could perfectly capture the sun's energy without losing some to sources such as clouds, atmosphere, areas of the ocean we cannot cover in solar panels etc.

[1] TODO, fix

Thu 04 Sep 2025 12:00

Lecture 3 - Kinematics Introduction

For kinematics, we'll treat all objects as points and disregard aspects like rotation/the physical sie of the body etc.

Given some point, we can define its position as a fucntion of time $\vec{r}(t)$, and velocity as the derivative wrt time of this:

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

And acceleration:

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

Position from Unit Vectors

We can define:

$$\vec{r}(t) = r_x(t)\hat{e_x} + r_y(t)\hat{e_y} + r_z(t)\hat{e_z} = \sum_{j=1}^{3} r_j(t)\hat{e}_j$$

So:

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\sum_{j=1}^{3} r_j\hat{e}_j\right)$$
$$= \sum_j \frac{d}{dt}(r_j\hat{e}_J)$$

Note: Taking the derivative of a vector wrt time is looking at how the variable changes in some infinitestimal time. This can be a change in direction, and/or a change in magnitude. To differentiate a vector we can differentiate it component-wise.

Cartestian and Polar

Instead of representing a point as x and y components (in 2D), we can instead define it as a distance from the origin r and the angle this distance line forms with the positive x-axis θ.

Therefore (by basic right angle trig) $x = rcos\theta$, $y = rsin\theta$, and hence:

$$\vec{r} = r\cos\theta\hat{e}_x + r\sin\theta\hat{e}_y$$

So:

$$\vec{u(t)} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\cos\theta)\hat{e}_x + \frac{d}{dt}(r\sin\theta)\hat{e}_y$$

$$= \left(\dot{r}\cos\theta + r(-\sin\theta)\dot{\theta}\right)\hat{e}_x + \left(\dot{r}\sin\theta + r(\cos\theta)\dot{\theta}\right)\hat{e}_y$$
$$= r\left(\cos\theta\hat{e}_x + \sin\theta\hat{e}_y\right) + r\dot{\theta}\left(-\sin\theta\hat{e}_x + \cos\theta\hat{e}_y\right)$$

Example

Lets model a particle, in a single dimension moving with constant acceleration (a_0) along a line. What is x(t)?

Going Backwards

Lets say we have some body with $a(t) = kt^3$. What is the position function $x(t)$?

$$a = \frac{dv}{dt} = kt^3$$

$$v = \int_a^b kt^3 \, dx$$

Thu 09 Oct 2025 15:00

Lecture 4 - Projectile Motion

Projectile Motion: The motion of a particle subject to gravitational acceleration, $g \approx 9.81\text{m/s}$

Projectile Motion

For this to hold, the height of the particle above the ground must be $m << R_e \approxeq 6 \times 10^3\text{km}$.

$$x(t) = x_0 + v_0(t - t_0) + \frac{1}{2}a_0(t - t_0)^2$$

Lets begin soley by considering motion in the vertical axis (called z here, for some strange reason). This particle is falling from height hm, to the ground at $h = 0$, with constant acceleration $g\text{m/s}^2$. It has been dropped at time $t = t_0$

At time t_0, $v = 0, z = h$

$$z(t) = h + 0 - \frac{1}{2}gt^2$$

$$\frac{1}{2}gt^2 = h$$

$$\implies t = \sqrt{\frac{2h}{g}}$$

What about 2D?

Now we can expand our example to (rather than drop the particle from rest) give the particle some initial velocity v_0m/s parallel to the ground. We now want two position functions, $x(t)$ and $z(t)$. As previously calculated:

$$z(t) = h + 0 + \frac{1}{2}(-g)t^2$$

And horizontally:

$$x(t) = 0 + v_0(t) + 0$$

So:

$$\begin{cases} z = h - \frac{1}{2}gt^2 \\ x = v_0 t \end{cases}$$

Rearranging:

$$t = \frac{x}{v_0}$$

$$z = h - \frac{1}{2}g\left(\frac{x}{v_0}\right)^2$$

Since h, g, v_0 are all constants, this is an x^2 parabola.

Interplanetary Example

Lets consider some planet, with $g_{\text{planet}} = 5m/s^2$. You (denoted Y) fall into the atmosphere at some distance h from the ground, and some horizontal distance d from $O(x = 0)$. There is an alien who wants to kill you, by shooting you down. This "gun" can throw pebbles at some constant speed v_0. The only degree of freedom the alien has to target you is change the shooting angle wrt the horizontal, θ. From the alien's persepctive, what is the required θ to hit the incoming spacecraft?

To hit you, there is some time t, when the position of the bullet B, with initial velocity v where B is in the same position as Y

Consider B

$$x_B(t) = v_0 \cos(\theta)t$$

$$z_B(t) = v_0 \sin(\theta)t - \frac{1}{2}g_p t^2$$

Consider Y

$$x_Y(t) = d$$

$$z_Y(t) = h - \frac{1}{2}g_p t^2$$

We want to find a θ where $x_B = x_Y$ and $z_B = z_Y$ at the same t:

$$v_0 \cos(\theta)t = d \tag{4}$$

$$v_0 \sin(\theta)t - \frac{1}{2}g_p t^2 = h - \frac{1}{2}g_p t^2 \tag{5}$$

From 2:

$$v_0 \sin(\theta)t = h$$

$$\implies t = \frac{h}{v_0 \sin\theta}$$

And subsituting:

$$v_0 \cos(\theta)\left(\frac{h}{v_0 \sin\theta}\right) = d$$

$$\frac{\cos(\theta)h}{\sin(\theta)} = d$$

$$\frac{\cos\theta}{\sin\theta} = \frac{d}{h}$$

$$\tan\theta = \frac{h}{d}$$

Since we have the value of θ in terms of two constants, yes, the alien can always hit the spaceship provided it correctly selects the angle corresponding to the value of these two constants. This means that the required angle does not depend on velocity, in this example.

Frames of Reference

"Observer" represents a frame of reference.

LC Introduction to Probability and Statistics

Wed 01 Oct 2025 12:00

Lecture 1 - Introdution and Descriptive Statistics

Course Welcome

- First half of the semester: Statistics
- Second hald the of semester: Probability
- All slides and notes on Canvas.

Why Descriptive Statistics? If we want to share an interesting bit of data, sharing the whole data is going to be confusing. Instead, we can share a small number of stats which describe and summarise the data.

Sample Statistics

One of the most simple is the number of samples (N), and the sample mean:

$$\text{Sample Mean: } \bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

We can also calculate the sample standard deviation as the average of mean squared error across the points in the sample:

$$\text{Sample STDev: } s_n^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2$$

We can also use median or mode as measures of central tendancy. The mode is a poor estimator however (as it massively depends on how binning is done, for a continuous measurement), while the median is more resistant to outliers.

Thu 02 Oct 2025 09:00

Lecture 2 - Population Statistics

Accuracy and Precision

We usually take measurements to determine some kind of true value. Usually, we can't actually know what this true value is, but if we could there are two bits of terminology that is particularly important:

Accuracy: Accuracy is the 'closeness' between our value and the 'true' value.

Precision: Precision is the 'closeness' between our measurements, i.e. how spread out are our various measurements.

Error

Random Error: is uncertainty related to the fact that our measurements are only a finite sample, so is not going to be immediately representative of the true value. The smaller this error, the more precise the measurement is.

Systematic Error: is related to some kind of issue with the measurement or the equipment. This shifts all values, and negatively affects accuracy (but leaves precision unchanged)

Taking many repeat measurements decreases the effects of random error, but the effects of systematic error are much harder to combat.

Ideally, we want to be both precise and accurate, however accuracy is arguably more important. This is because a value which is precise, but not accurate may lead to false conclusions around the inaccurate value.

Wed 08 Oct 2025 12:02

Lecture 3 - Error Propogation and Combinations of Variables

Office Hours: 11:00 to 13:00 Thursdays, Physics West Rm 122

Types of Error

Broadly two types of error: Statistical/Random Error (resulting from low precision) and Systematic Error (from Low Accuracy).

Random error widens the distribution, while systematic error shifts the whole distribution up or down, meaning no matter how many repeats you take and how precise you think you are, the value is still nonsense as all datapoints have been equally shifted (i.e. by a poor experimental setup).

For example, you are trying to measure the length of an object using a ruler that has been unknowingly stretched. You cannot get a true value no matter the number of repeats or degree of precision.

Accuracy vs Precision

High accuracy is preferable to high precision - having high precision but low accuracy can lead to false conclusions (as an incorrect value appears confidently correct). Accuracy is more difficult to improve - precision can be improved by gathering more data, while higher accuracy can only be improved by a better experimental design.

Error Propogation

If we take a distribution, and add a constant value to all points, the distribution is shifted up/down without changing the variance.

$$\langle x + k \rangle = \langle x \rangle + k$$

$$Var(x + k) = Var(x)$$

If we multiply by a constant value, the mean is multiplied by this value, but the distribution becomes stretched and the variance grows:

$$\langle xk \rangle = k\langle x \rangle$$

$$Var(kx) = k^2 Var(x)$$

Or taking the natural log:

$$\langle \ln x \rangle \approx ln\langle x \rangle$$

$$Var(\ln x) \approx \frac{Var(x)}{x^2}$$

As this is a non-linear operator, these become good approximations rather than strict rules of equivalence.

And another example:

$$\langle e^x \rangle \approx e^{\langle x \rangle}$$

$$Var(e^x) \approx (e^x)^2 Var(x)$$

Note here, even though our underlying distribution is Normal and symmetric, the new distribution after e^x is neither, and these are an even worse approximation than before.

Combining Operators

We can apply some linear transformation $mx + c$, we can chain these rules together by doing the multiplicative transformation m first, then the linear scale c.

$$\langle mx + c \rangle = m\langle x \rangle + c$$

$$Var(mx + c) = m^2 Var(x)$$

Multiple Variables

What if we have mutliple distributed variables we want to add?

$$\langle A + B \rangle = \langle A \rangle + \langle B \rangle$$

$$Var(A + B) = Var(A) + Var(B)$$

And multiplying them (again this are now approximations)?

$$\langle AB \rangle \approx \langle A \rangle \langle B \rangle$$

$$Var(AB) \approx \langle B \rangle^2 Var(A) + \langle A \rangle^2 Var(B)$$

$$\frac{Var(AB)}{\langle AB \rangle^2} \approx \frac{Var(A)}{\langle A \rangle^2} + \frac{Var(B)}{\langle B \rangle^2}$$

Or division?

$$\left\langle \frac{A}{B} \right\rangle \approx \frac{\langle A \rangle}{\langle B \rangle}$$

$$Var\left(\frac{A}{B}\right) = \frac{Var(A)}{\langle B \rangle^2} + \frac{Var(B)}{\langle A \rangle^2}$$

One Rule to Rule Them All

$$Var(f) \approx \left(\frac{\partial f}{\partial A}\Big|_{A=\langle A \rangle, B=\langle B \rangle}\right)^2 Var(A) + \left(\frac{\partial f}{\partial A}\Big|_{A=\langle A \rangle, B=\langle B \rangle}\right)^2 Var(B)$$

Thu 09 Oct 2025 09:00

Lecture 4 - Covariance and Correlation

Office Hours: Thursday 11am - 1pm, Physics West Rm 222 (b.becsy@bham.ac.uk)

Previous, when looking at two or more variables for error propogation/combinations etc, we assumed that they were independant of one another. Today we look at how to handle multiple variables which may be correlated.

Covariance

Covariance is a measure that indicates how much two variables fluctuate together:

$$\text{Cov}(x, y) = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})$$

Covariance matrices represent all combinations of covariance (noting Cov(x, y) = Cov(y, x) and Cov(x, y) = Var(x))

$$\Sigma = \begin{pmatrix} Cov(x, x) & Cov(x, y) \\ Cov(y, x) & Cov(y, y) \end{pmatrix}$$

We can then define correlation:

$$Corr(x, y) = \frac{Cov(x, y)}{\sqrt{Var(x)Var(y)}} = \frac{\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{N}(x_i - \bar{x})^2 \sum_{i=1}^{N}(y_i - \bar{y})^2}}$$

This is bounded between -1 (x = -y), y (x = y) and zero for no correlation. We can again put this in a matrix, noting it is symmetrical:

$$\begin{pmatrix} 1 & Corr(x, y) \\ Corr(y, x) & 1 \end{pmatrix}$$

Variable Combinations

Now, with correlated variables, we can say:

$$\langle x + y \rangle = \langle x \rangle + \langle y \rangle$$

$$Var(x, y) = Var(x) + Var(y) + 2Cov(x, y)$$

And (noting the mean slightly increases with correlated variables):

$$\langle xy \rangle = \langle x \rangle \langle y \rangle + Cov(x, y)$$

And the one formula to rule them all

$$Var(f) \approx \frac{\partial f}{\partial A}^2 Var(A) + \frac{\partial f}{\partial B}^2 Var(B) + 2\frac{\partial f}{\partial A}\frac{\partial f}{\partial B}Cov(A, B)$$

LC Mathematics for Physicists 1A

Thu 02 Oct 2025 15:53

Lecture 1

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Lecture 2

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Lecture 3

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Lecture 4

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Lecture 5

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Lecture 6

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Lecture 7

Mon 13 Oct 2025 12:00

Lecture 8 - More Planes

Recap

Given the origin O, a point on the plane O' and a vector $\vec{a}$ between them, we can take two vectors $\vec{b}$ and $\vec{c}$ from this point (which are not parallel). Using some combination of these two vectors, we can reach any point on the plane:

$$\vec{r}(s,t) = \vec{a} + s\vec{b} + t\vec{c}$$

This is the parametric equation of a plane, and is very robust. We can describe a flat plane in any dimensional space using this.

We can also define the scalar equation of a plane. Given these same two vectors, we can define a normal vector $\vec{\hat{n}}$ which is perpendicular to any vector that sits within the plane. We can construct this by using the cross product:

$$\vec{n} = \vec{b} \times \vec{c}$$

Given some generic point P:

$$\vec{OP} = \vec{a} + \vec{O'P}$$

And:

$$\underline{r}(s,t) = \underline{a} + s\underline{b} + t\underline{c}$$

We have:

$$(\underline{b} \times \underline{c}) \cdot \underline{r} = (\underline{b} \times \underline{c}) \cdot \underline{a} + s(\underline{b} \times \underline{c}) \cdot \underline{b} + t(\underline{b} \times \underline{c}) \cdot \underline{c}$$

Which (as a vector dotted with itself is 0) simplifies to (using $\underline{b} \times \underline{c} = \underline{n}$):

$$\underline{n} \cdot (\underline{r} - \underline{a}) = 0$$

LC Optics and Waves

Wed 01 Oct 2025 11:00

Lecture 1 - Intro to Waves and SHM Recap

Course Objectives

- Have a sound understanding of basic wave properties
- Have a basic understanding of interference effects, inc diffraction
- Be able to use simple geometric optics and understand the fundamentals of optical instruments.

Recommended Textbooks

1. University Physics, Young and Freedman (Ch 15, 16 for Waves, Ch 33-36 for Optics)
2. Physics for Scientists and Engineers (Ch 20, 21 for Waves, Ch 22-24 for Optics)
3. 5e, Tipler and Mosca, (Ch 15, 16 for Waves, 31-33 for Optics)
4. Fundamentals of Optics, Jenkins and White
5. Optics, Hecht and Zajac

What is a wave? Waves occur when a system is disturbed from equilibrium and the disturbance can travel from one region to another region. Waves carry energy, but do not move mass. The course aim is to derive basic equations for describing waves, and learn their physical properties.

Periodic Motion

Waves are very linked to periodic motion. Therefore we recap periodic motion first.

It has these characterstics:

- A period, T (the time for one cycle)
- A frequency, f, the number of cycles per unit time ($f = \frac{1}{T}$)
- An amplitude, A, the maximum displacement from equilibrium.

Periodic motion continues due to the restoring force. When an object is displaced from equilibrium, the restoring force acts back towards the equi point. The object reaches equi with a non-zero speed, so the motion continues past the equi point and continues forever.

Energy

Periodic motion is an exchange between potential and kinetic energy, with no energy loss. Energy is conserved.

Simple Harmonic Motion

If the restoring force is directly proportional to the displacement $F = -kx$, then the periodic motion becomes Simple Harmonic Motion and the object is called a harmonic oscillator.

In a single dimension, displacement is given by:

$$x = A\cos(\omega t + \phi)$$

Where $\omega = 2\pi f$ is the angular velocity, and ϕ is the phase angle. In cases like this, where the phase angle is 90 deg we can simplify to $x = -A\sin(\omega t)$

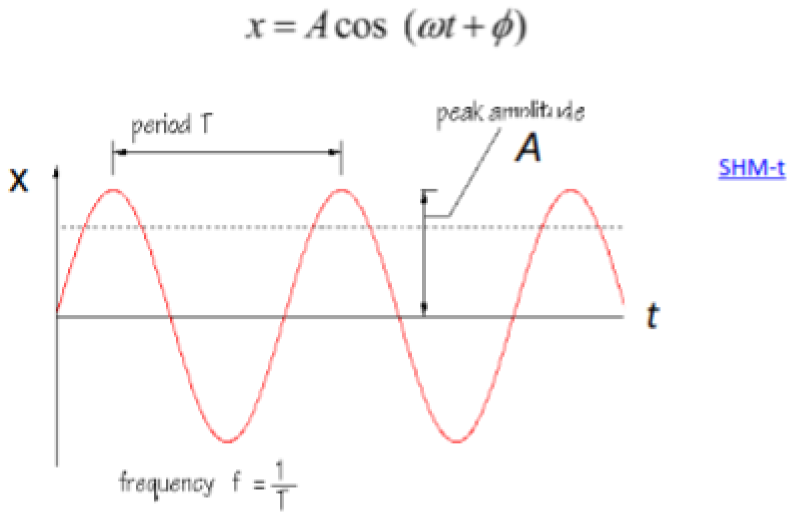


Figure 1: A Phase Angle of 90

More SHM Equations

Velocity

$$v_x = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

Acceleration

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega t + \phi)$$

Both properties are signed to indicate direction, as they are both vectors.

Thu 02 Oct 2025 13:00

Lecture 2 - Wave Functions

Sine Waves

Mechanical Waves A mechanical wave is a disturbance through a medium. It's formed of a single wave pulse or a periodic wave.

Mechanical Waves have the following properties:

1. Transverse: Where displacement of the medium is perpendicular to the direction of propogation.
2. Longitudinal, displaccement of hte medium is in the same direction as propogation.
3. Propogation depends on the medium the wave moves through (i.e. density, rigidity)
4. The medium does not travel with the wave.
5. Waves have a magnitude and a direction.
6. The disturbance travels with a known exact speed.
7. Waves transport energy but not matter throughout the medium.

Wave Functions

We want to define a wave function in terms of two variables, x and t. In any given moment, if we consider a single point on the wave (i.e. $t = 0$), and wait a short while, the wave will have travelled to some $t = t_1 > 0$.

In order to quantify displacement, we therefore want to specify both the time, and the displacement. This will let us find the wave speed, acceleration and the (new) wave number.

We are also able to talk about the velocity and acceleration of individual particles on the wave.

Wave Function for a Sine Wave

Consider a sine wave. We want to find a wave funtion in the form $y(x, t)$. Consider the particle at $x = 0$.

We can express the wave function at this point as $y(x = 0, t) = A \cos \omega t$. However we want to expand this to any general point. Now consider a point (2) which is one wavelength away. We know the behaviour of particle 1 is mirrored by particle 2 (with a time lag).

Since the string is initially at rest, it takes on period (T) for the propogation of the wave to reach point 2, therefore point 2 is lagging behind the motion of point 1. The wave equation is therefore (if particle two has $x = \lambda$) $y(x = \lambda, t) = A \cos(\omega t - 2\pi)$.

For arbitrary x, $y(x, t) = A \cos(\omega t - \frac{x}{\lambda} \cdot 2\pi)$ to account for this delay. This quantity is called the wave number:

$$\text{Wave Number: } k = \frac{2\pi}{\lambda}$$

So:

$$\begin{aligned} y(x, t) &= A \cos(\omega t - kx) \\ &= A \cos(kx - \omega t) \end{aligned}$$

Note the second step is possible as cos is an even function. k can also be signed to indicate direction: if $k > 0$, the wave travels in the positive x. If $k < 0$, the wave travels in the negative x direction. Again, $\omega = 2\pi f$

Displacement Stuff

Considering a point (starting at equi), the time taken for the particle on the sin wave to reach maximum displacement, minimum displacement and back takes the time period T. The speed of the wave is distance travelled over the time taken. We take the distance to be the wavelength λ, as we know the time by definition this takes is one time period T. Therefore wave speed v is:

$$v = \frac{\lambda}{T} = \lambda f$$

Since $\lambda = \frac{2\pi}{k}$ and $f = \frac{\omega}{2\pi}$ (as ω is defined as $\frac{2\pi}{T}$), we can also write:

$$v = \frac{2\pi}{k} \cdot \frac{\omega}{2\pi} = \frac{\omega}{k}$$

Particle Velocity

We can also determine the velocity of individual particles in the medium. We can use this to determine the acceleration.

We know that

$$y(x,t) = A\cos(kx - \omega t)$$

The vertical velocity v_y is therefore given by:

$$v_y = \frac{dy(x,t)}{dt}$$

Which is unhelpful (as we can't differentiate two variables at once), we can slightly cheat this by looking at purely a certain value of x, and therefore treating x as constant (to get a single variable derivative).

$$v_y = \frac{dy(x,t)}{dt}\bigg|_{x=\text{const.}}$$

However this is notationally yucky, so we therefore use the notation:

$$\frac{\partial y(x,t)}{\partial t}$$

To represent the same thing. Finally (carrying out the partial derivative):

$$v_y(x,t) = \frac{\partial y(x,t)}{\partial t} = \omega A \sin(kx - \omega t)$$

Particle Acceleration

We can work out particle acceleration (transverse acceleration) by differentiating in the same manner again:

$$a_y(x,t) = \frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial y(x,t)}{\partial t}\right) = -\omega^2 A\cos(kx - \omega t) = -\omega^2 y(x,t).$$

Wed 08 Oct 2025 11:00

Lecture 3 - Generalised Wavefunctions

Recap: For a sine wave, the wavefunction is:

$$y(x,t) = A\cos(kx - \omega t)$$

Where k is the wave number, $k = \frac{2\pi}{\lambda}$ and omega is $\frac{2\pi}{T} = 2\pi f$

More Wavefunctions

What is the general form of a triangular shaped wave? What about a square stepped wave? TODO: Diagram

We know that each particle (i.e. along a string) copies the motion of its immediate left-hand neighbour (for a particle moving in the positive x), with a time delay proportional to their distance. Every wave is describable $y(x,t)$ wave function, and every mechanical wave relies on a medium (i.e. a piece of string or water, concrete etc) to travel.

Ideally, we want to be able to find a general form of a wave function.

In a single fixed instant of time, the moving wave pulse is stationary, so purely a function of $y = f(x)$. We want a wave function where we can input any value of t, so we need a moving frame of reference. We define this frame of reference as O' (for the origin) and x', y' axes. This frame of reference moves with the wave pulse and at the same speed, therefore y' is a function of x' only, independant of speed.

New System

x is the distance from the origin O to the relevant point, while x' is the distance from O'.

$$x = x' + vt$$

$$x' = x - vt$$

$$y' = f(x') = f(x - vt)$$

However, as the wave is moving purely in one direction (along x), $y = y'$, so:

$$y = f(x - vt)$$

Back to Basics

Going back to:

$$\begin{aligned} y(x,t) &= A\cos(kx - \omega t) \\ &= A\cos\left[k\left(x - \frac{\omega}{k}t\right)\right] \\ &= A\cos\left[k\left(x - vt\right)\right] \end{aligned}$$

(Note, this is true for a wave in the positive x, for a wave moving in the negative x this would be x + vt)

Equivalent Rrepresentations

There are some equivalent representations for a sine wave:

$$y(x,t) = A\cos\left(2\pi f\frac{x}{v} - \omega t\right) = A\cos\left(2\pi\left[\frac{x}{\lambda} - \frac{t}{T}\right]\right) = TODO$$

More Differentiation

We've already looked at differentiating with respect to t, but what about x? This would give us the slope of the string at that point:

$$\frac{\partial y(x,t)}{\partial x}$$

And the curvature of the string:

$$\frac{\partial^2 y(x,t)}{\partial x^2} = -k^2 A\cos(kx - \omega t) = -k^2 y(x,t)$$

Note the similarities here with the equation for transverse acceleration:

$$(1)\frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 y(x,t)$$

$$(2)\frac{\partial^2 y(x,t)}{\partial x^2} = -k^2 y(x,t)$$

Dividing (1) by (2):

$$\frac{\partial^2 y(x,t)/\partial t^2}{\partial^2 y(x,t)/\partial x^2} = \frac{\omega^2}{k^2} = v^2$$

Therefore:

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

We call this the 'wave equation' as every wave function y(x, t) must satisfy it, regardless of whether or not it is periodic or its direction of travel. If y(x, t) does not satisfy this, it is not a wave function.

An Example

$$y(x,t) = \frac{x^3 - vt^2}{e^t}$$

TODO, the example in our own time

Wave Equation for a String

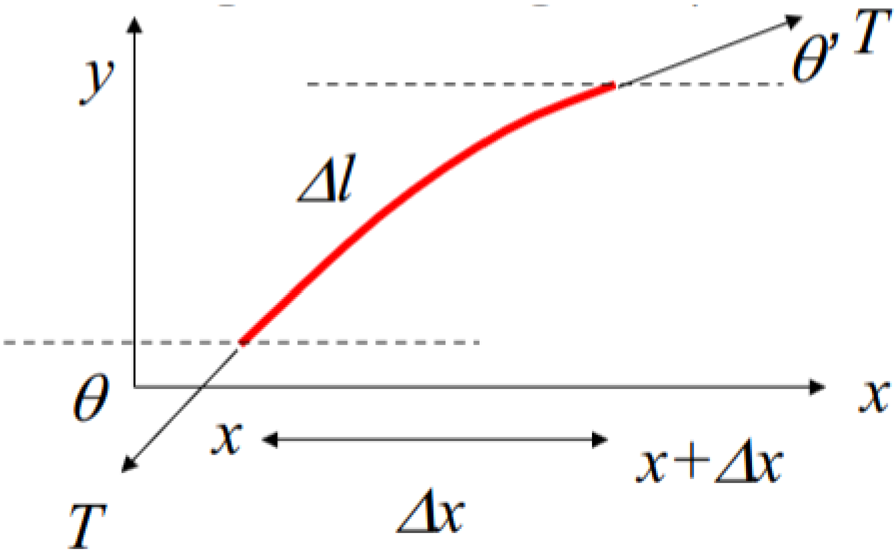


Figure 2: A snapshot of the wave.

Lets say we have some string, suspended horizontally under tension. We generate a single wave pulse and allow it to propogate down the string.

We assume the string is 1D, under tension T (constant throughout) and has mass per unit length μ.

Consider a small segment of string of length ΔL from x to $x + \Delta x$. This string makes angle θ with the horizontal at the bottom of the string, and angle θ' with the horizontal at the top of the string.

Net force (transverse in y) is:

$$F_y = T \sin\theta' - T \sin\theta$$

And using the small angle approx $\sin\theta \approx \tan\theta = \frac{dy}{dx}$:

$$F_y = T \left(\frac{dy}{dx}\bigg|_{x+\Delta x} - \frac{dy}{dx}\bigg|_{x} \right)$$

Using differentiation by first principles:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

We can say:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \lim_{\Delta x \to 0} \frac{\frac{dy}{dx}\Big|_{x+\Delta x} - \frac{dy}{dx}\Big|_{x}}{\Delta x}$$

Therefore:

$$F_y = T \frac{d^2y}{dx} \Delta x$$

The mass of this section of string is $\mu \Delta x$, and considering the acceleration in the y direction we can plug into $F = ma$ to get:

$$F = ma$$

$$T \frac{d^2y}{dx^2} \Delta x = \mu \Delta x \frac{d^2y}{dt^2}$$

$$\frac{d^2y}{dx^2} = \frac{\mu}{T} \frac{d^2y}{dt^2}$$

The LHS is the rate of change of the string's gradient, which as mentioned is the curvature of the string. The RHS includes the transverse acceleration of the string, therefore acceleration is proportional to curvature. To evaluate this at a fixed time/position we should write:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

And comparing to the wave equation we get:

$$\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

$$\frac{1}{v^2} = \frac{\mu}{T}$$

$$v = \sqrt{\frac{T}{\mu}}$$

So a wave travels faster under a higher tension with a lower mass per unit length.

Thu 09 Oct 2025 13:00

Lecture 4 - Waves at Boundaries

Recap

We previously derived:

$$v = \sqrt{\frac{T}{\mu}}$$

And we also have:

$$v = \frac{\omega}{k}$$

The former is useful explictly for a wave travelling over a string, while the latter is applicable to the movement of any wave. On a string, a higher tension yields a higher restoring force and therefore a higher speed, while a higher mass per unit length gives a higher mass for some arbitrary length of string, therefore a lower acceleration and lower speed.

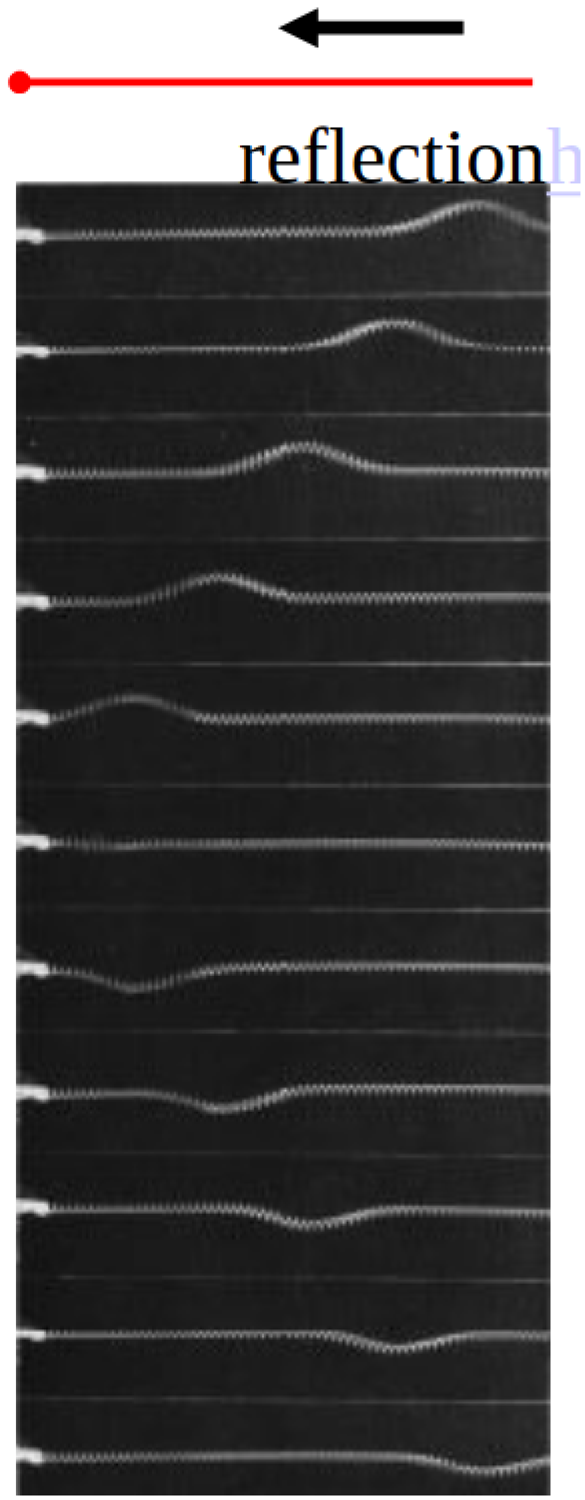
A Quick Interlude

The speed of a mechanical wave has the general form:

$$v = \sqrt{\frac{\text{Restoring force returning to equilibrium}}{\text{Inertia resisting return to equilibrium}}}$$

Reflection

When a wave hits a fixed boundary, it is reflected and inverted. Lets consider a case where a string is fixed on the LHS and is reflected back:


reflection

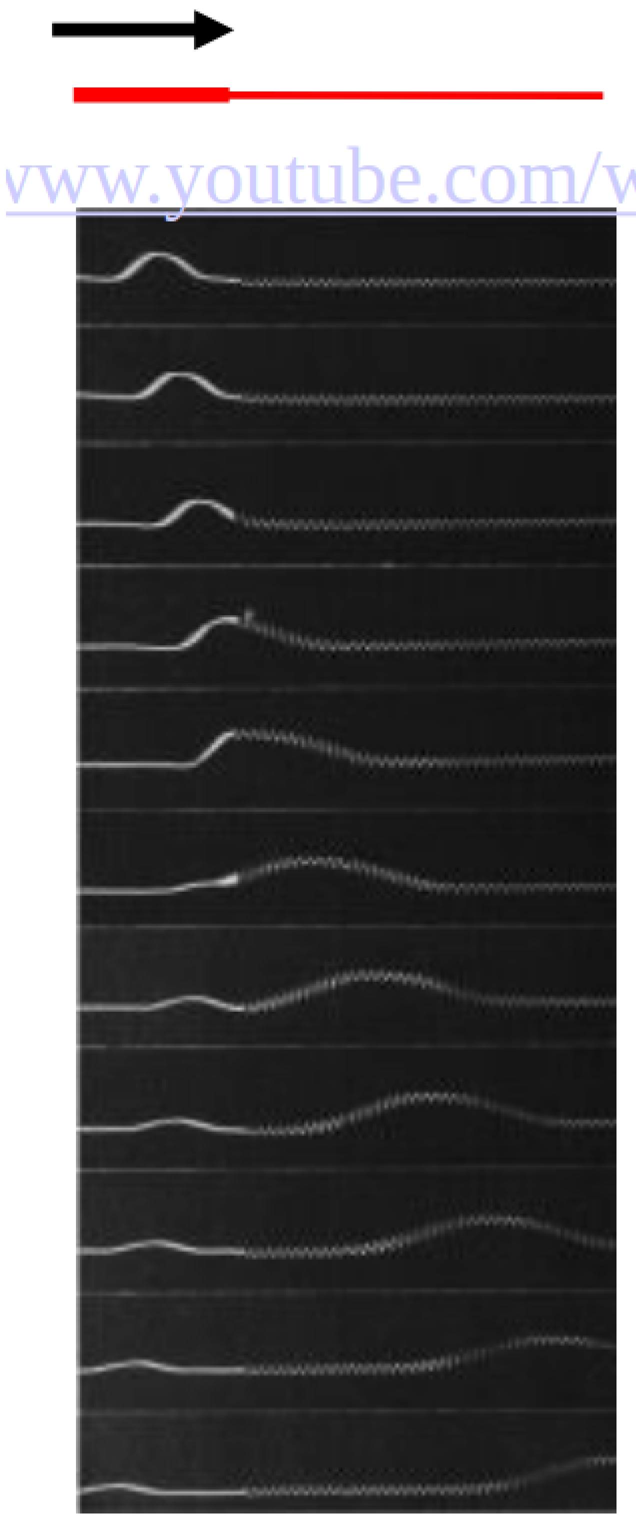


Figure 3: Or with two strings, going from thick to thin:

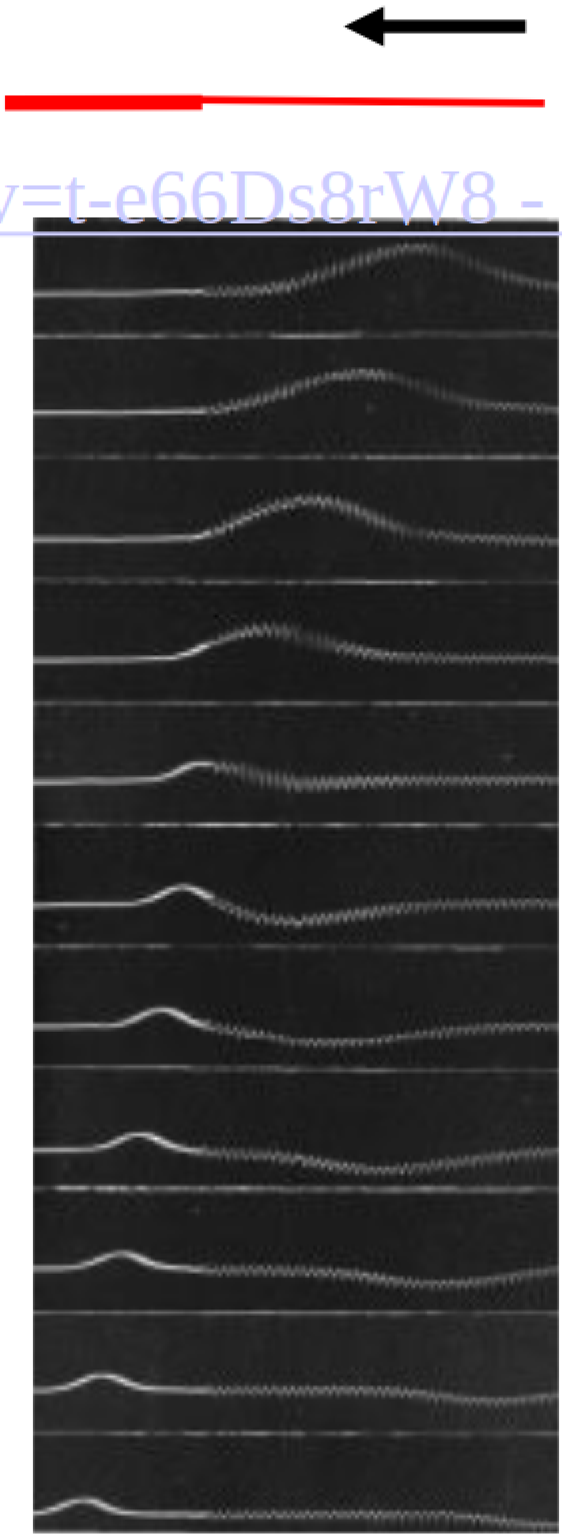


Figure 4: Or from thin to thick.

Waves Interacting at Boundaries

Say we have two pieces of string connected to each other, one thin string with mass per unit length μ_1 and a thicker string with μ_2 (both under the same tension, T). If a wave pulse is passed along from the thin string to the thicker string, whath happens at the point of connection, P?

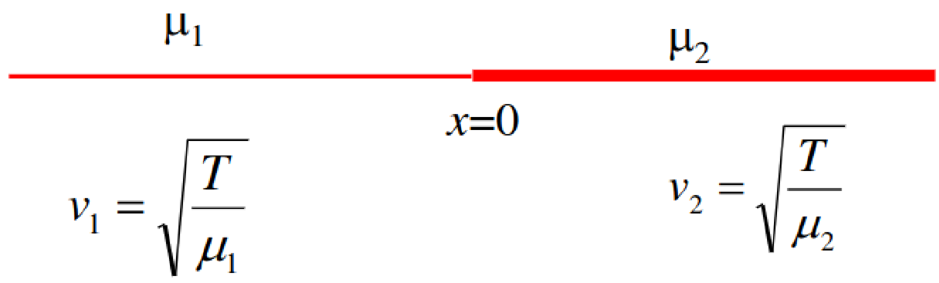


Figure 5: The two connected strings.

Consider a travelling wave coming from the left:

$$y_1 = A\cos(k_1 x - \omega_1 t)$$

$$y_2 = B\cos(k_2 x - \omega_2 t)$$

At the point of connection, we'll define this as $x = 0$. Here (as the string is not broken, so it must be connected):

$$y_1 = y_2 \tag{6}$$

Also, force is finite, therefore curvature must be finite (as force is proportional to curvature). Therefore we cannot have any discontinuities in curvature.

$$\frac{\partial y_1}{\partial x_1} = \frac{\partial y_2}{\partial x_2} \tag{7}$$

Disregarding non-linear effects, so assuming that the frequency with which the waves travel down the string is the same for both parts of the string, $\omega_1 = \omega_2 = \omega$.

From (1) and noting $x = 0$:

$$A\cos(-\omega t) = B\cos(\omega t) \tag{8}$$

And from (2) with the same note:

$$-k_1 A \sin -\omega t = -k_2 B \sin -\omega t \tag{9}$$

However this suggests that $A = B = 0$. This is technically a solution, but not really - it doesn't represent an actual wave (just two flat lines with no amplitude ever)

Including Reflection

The previous case did not work as we disregarded reflection at the wave boundary, P. Lets add an extra wave (the C term) to represent the reflection back into the light string:

$$y_1 = A\cos(k_1 x - \omega t) + C\cos(k_1 x + \omega t) = B\cos(k_2 x - \omega t)$$

From y1 = y2 at x = 0 we get:

$$y_1 = A\cos(-\omega t) + C\cos(-\omega t) = B\cos(-\omega t)$$

So: $A + C = B$

From (2) we get:

$$-k_1 A\sin(-\omega t) - k_1 C\sin(\omega t) = -k_2 B\sin(-\omega t)$$

Which has solutions:

$$B = \frac{2k_1}{k_1 + k_2} A$$

$$C = \frac{k_1 - k_2}{k_1 + k_2} A$$

We have the incident wave:

$$y_1 = Acos(k_1 x - \omega t)$$

The transmitted wave:

$$y_2 = B\cos(k_2 x - \omega t) = \frac{2k_1}{k_1 + k_2} A\cos(k_2 x - \omega t)$$

And lastly the reflected wave:

$$y_3 = C\cos(k_1 x + \omega t) = \frac{k_1 - k_2}{k_1 + k_2} A\cos(k_1 x - \omega t)$$

We note:

k_1 is the wave number in the medium where the incident wave comes from.

k_2 is the wave number in the medium where the transmitted wave goes into.

Example

If the wave comes in from the left, then $\mu_2 > \mu_1$ per example diagram, then $k_2 > k_1$, and C is negative.

Note that if the single pulse as a postive y amplitude, then the transmitted pulse in the heavier string will also have a positive y amplitude, but will have a smaller magnitude. The reflected wave will have a negative y-amplitude as the reflection inverts it.

If the wave comes from the right (from thick to thin), then $\mu_1 > \mu_2$, $k_1 > k_2$ and C is positive.

Fri 17 Oct 2025 12:56

Lecture 5

Fri 17 Oct 2025 13:00

Lecture 6 - Standing Waves 2 Electric Boogaloo

Recap

$$\frac{\partial y}{\partial x} = 2Ak\cos(kx)\sin(\omega t)$$

$$\frac{\partial^2 y}{\partial x^2} = 2A(-k^2)\sin(kx)\sin(\omega t)$$

$$\frac{\partial y}{\partial t} = 2A\omega\sin(kx)\cos(\omega t)$$

$$\frac{\partial^2 y}{\partial t^2} = 2A(-\omega^2)\sin(kx)\sin(\omega t)$$

Therefore:

$$\frac{\frac{\partial^2 y}{\partial t^2}}{\frac{\partial^2 y}{\partial x^2}} = \frac{2A(-\omega^2)\sin(kx)\sin(\omega t)}{2A(-k^2)\sin(kx)\sin(\omega t)} = \frac{\omega^2}{k^2} = v^2$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Therefore a standing wave still obeys the wave equation, as it must.

Standing Wave Properties

Wavelength

Consider a horizontal string from $x = 0$ to $x = L$, with both ends fixed. We generate a sinusoidal wave pulse, which must satisfy:

$$y(x,t) = 2A\sin(kx)\sin(\omega t)$$

We know at $x = L$ and $x = 0$, $y = 0$ at all times as fixed at this point. Therefore:

$$kL = n\pi, (n \in \mathbb{N})$$

For $n = 1$, we have half a wavelength on the string:

$$\lambda = \frac{2L}{1} = 2L$$

And this has a general form: $\lambda = \frac{2L}{n}$.

Frequency

$$f_n = \frac{v}{\lambda_n} = \frac{v}{\left(\frac{2L}{n}\right)} = \frac{nv}{2L}$$

Crucially:

$$f_1 = \frac{v}{2L}$$

Is the first harmonic (or fundamental). $f_2 = 2f_1$ is the second harmonic, or first overtone, etc. All of these, f_n where $n \in \mathbb{N}$ are called "normal modes". For each normal mode, the corresponding frequency is called the resonant frequency (natural frequency of the system).

What happens if we try to create a standing wave where $\lambda \neq \frac{2L}{n}$? In short, we cannot. The system will reject any attempts to do so.

Energy

Energy is proportional to ω^2. Energy can only take certain discrete values (corresponding to $f_1, f_2, \ldots, f_n$), we find that the system has quantised possible values for energy.

To generate a wave with a higher frequency we either have to use a shorter L, or a higher v. A higher v is achieved by using a lighter string or placing the system under higher tension.

Sound Waves

Notation

Displacement of a sound wave is denoted:

$$s(x,t) = S_m \cos(kx - \omega t)$$

And pressure is given by:

$$\Delta P(x,t) = \Delta P_m \sin(kx - \omega t)$$

Different Boundary Conditions

The equations for standing waves given is only true for the boundary conditions of both ends fixed. If we vary these (for example left end fixed, right end not, wave initially travelling left) we get a different solution. For example, the first harmonic:

$$L = \frac{1}{4\lambda_1}$$

$$f_1 = \frac{v}{4L}$$

Where the left end forms a node (as required by boundary conditions) and the right end forms an antinode, as it is free to move. For the third harmonic:

$$L = \frac{3}{4\lambda_3}$$

$$f_3 = \frac{3v}{4L} = 3f_1$$

And fifth:

$$L = \frac{5}{4\lambda_5}$$

$$f_5 = \frac{5v}{4L} = 5f_1$$

Notably, this system cannot support even harmonics.

LC Quantum Mechanics

Fri 03 Oct 2025 12:00

Lecture 1 - Atomic Structure

What is the course?

- Quantum mech is weird and unintuative, we will build up a case in the course for why this weird theory was necessary and why we're confident it works.
- Each week will be a self-contained concept and/or historial experiment, working up to the Shroedinger Equation and wave-particle duality.
- Names and dates do not need to be memorised.
- Recommended text: University Physics (Young and Freedman).
- Office hours: 13:00 – 13:50 Fridays (immediately post-lecture), Physics East Rm 207.

Atomic Structure

What actually is an atom? What does it actually look like inside?

Early Clues

- Periodic Table (Mendelev, 1989), periodic patterns in elements properties.
- Radioactivity (Becquerel, 1896, Curie 1898)
- Atoms emit and absorb specific discrete wavelengths, (Balmer, 1884)
- Discovery of the Electron (Thompson 1897). Cathode rays - heating metal in a vacuum with an electric field above it, to strip away electrons from the metal.
  - This showed electrons were negatively charged and extremely light (1/2000th of the atomic mass).

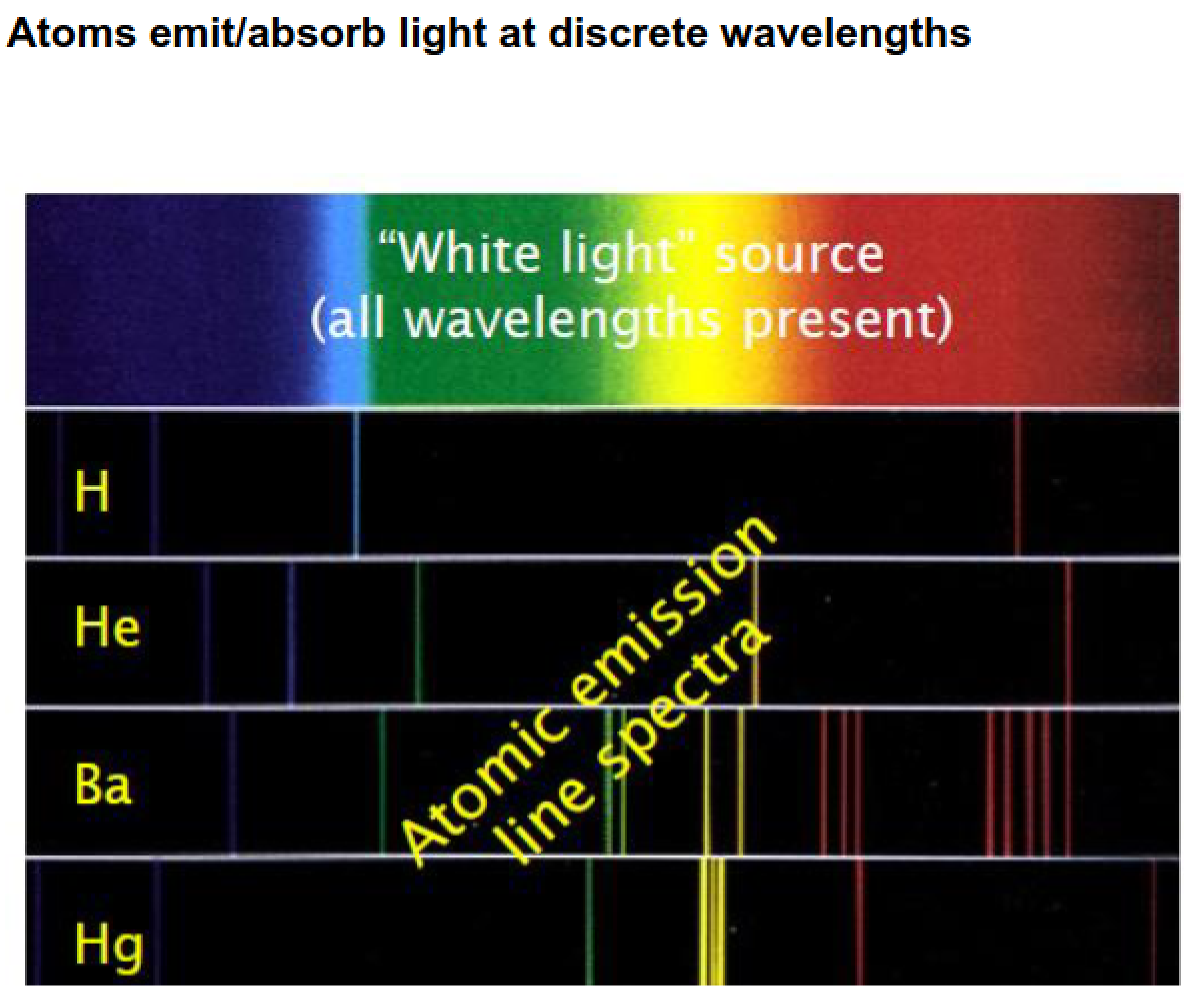


Figure 6: Absorption Spectra

Plum Pudding Model

A solid, uniform lump of positively charged matter, approximately 10^{-10}m across. This had evenly distributed negative charges (electrons) scattered throughout.

Discovery of the Nucleus

Geiger and Marsden (1908-1913), fired alpha particles (He nuclei) at thin gold foil and measured the deflection / scattering.

The alpha particles had a mass of 4u, a charge of +2e and an energy of approximately 5MeV.

They found that most α were scattered only by small angles, but (surprisingly) a small number were scattered right back towards to emitter (through $\theta > 90\,\text{deg}$). The distribution of the angles is approximately Normally distributed, with a mean of 0. Only approximately 1 in 8,000 fired αs were scattered by $\theta > 90$ ("back-scattering").

Can this be explained with the Plum Pudding Model? No, it cannot. This was used to demonstrate that atoms cannot be evenly distributed.

Demonstrating by Calculation

Lets work out the work done to take an α from infty to the pudding centre. If the electrostatic repulsion is not enough to overcome this, we cannot stop the α and cannot back scatter.

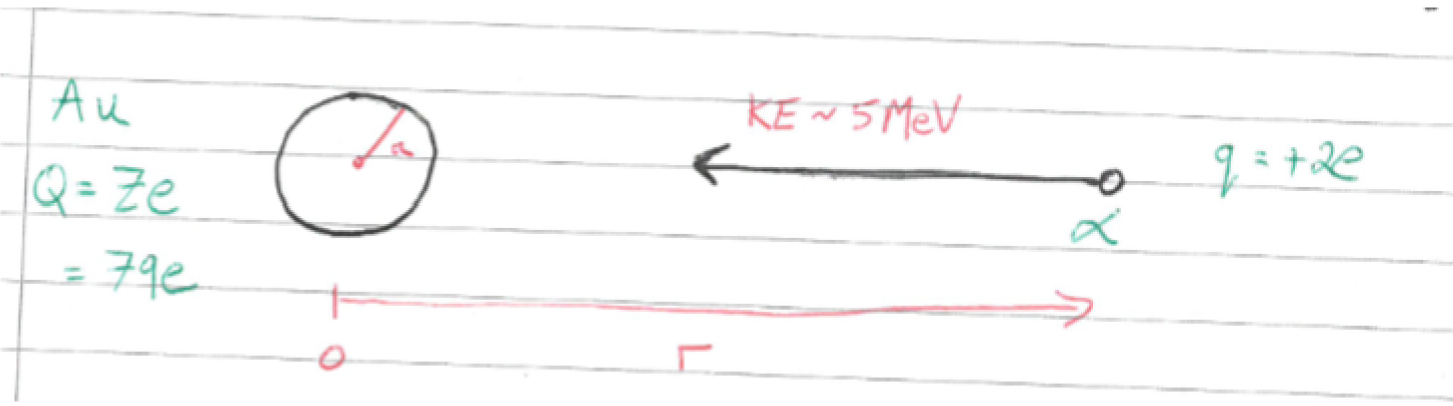


Figure 7: The experiment

Assumptions

- The atom stays still.
- Ignore the gold electrons (this is fine, as they would cancel some positive charge and make repulsion weaker, which would be even worse. If we can't do it without them, it would be equally impossible to do it with.)

Recap and Eqns

Coulomb Potential Energy is:

$$u(r) = \frac{qQ}{4\pi\epsilon_0 r}$$

Force is:

$$F(r) = -\frac{du}{dr} = \frac{qQ}{4\pi\epsilon_0 r^2}$$

Change in potential energy $(u_2 - u_1)$ is work done:

$$\int_{u_1}^{u_2} du = -\int_{r_1}^{r_2} F(r)dr$$

From outside the atomic radius, we treat the atomic pudding as a point charge of charge Q. From inside the atomic radius, we treat it as a smaller point charge $Q'(r)$, where we only consider the charge inside the portion of the pudding where $r < a$, where a is the current position inside the sphere.

If charge is spread uniformly, the total charge is proportional to the volume of the sphere. So:

$$\frac{Q'}{Q} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi a^3}$$

$$Q' = Q\frac{r^3}{a^3}$$

Inside the Pudding

$$F = \frac{qQ'}{4\pi\epsilon_0 r^2}$$

$$F = \frac{qQr^3}{4\pi\epsilon_0 r^2 a^3}$$

$$F = \frac{qQr}{4\pi\epsilon_0 a^3}$$

$$F = \frac{qQ}{4\pi\epsilon_0 a^3} \times r$$

Hence inside, $F \propto r$

Outside the Pudding

$$F = \frac{Qq}{4\pi\epsilon_0 r^2}$$

$$F = \frac{Qq}{4\pi\epsilon_0} \times \frac{1}{r^2}$$

Hence outside, $F \propto \frac{1}{r^2}$

To integrate, we are therefore integrating the area under this (almost) triangle:

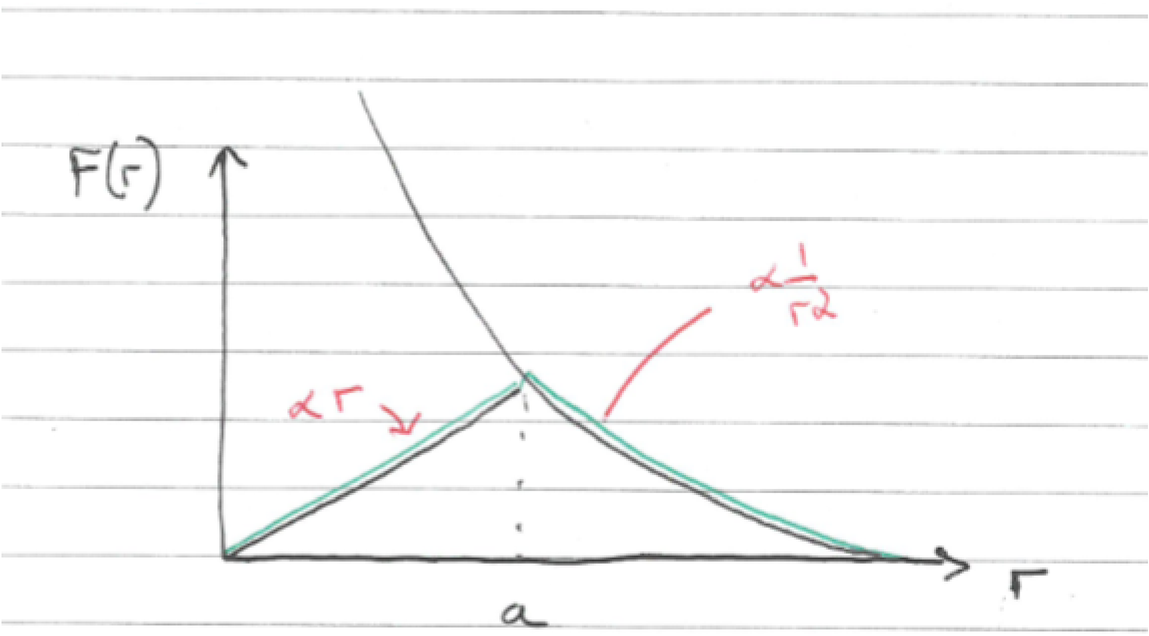


Figure 8: Radius vs electrostatic repulsion force

$$\Delta u = -\int_{r_1}^{r_2} F(r)\,dx$$

Splitting into two sections (prior to, and after the point where r = atomic radius, $r = a$), and integrating across all r (as we are attempting to work out the work done to bring an α from infinity to the charge, at which poind distance is 0):

$$= -\int_{\infty}^{a} \frac{qQ}{4\pi\epsilon_0 r^2}\,dx - \int_{a}^{0} \frac{qQr}{4\pi\epsilon_0 a^3}\,dx$$

$$= -\frac{qQ}{4\pi\epsilon_0}\int_{\infty}^{a} \frac{1}{r^2}\,dx - \frac{qQ}{4\pi\epsilon_0 a^3}\int_{a}^{0} r\,dx$$

$$= -\frac{qQ}{4\pi\epsilon_0}\lim_{x\to\infty}\int_{x}^{a} \frac{1}{r^2}\,dx - \frac{qQ}{4\pi\epsilon_0 a^3}\int_{a}^{0} r\,dx$$

$$= -\frac{qQ}{4\pi\epsilon_0}\lim_{x\to\infty}\left[-\frac{1}{r}\right]_{x}^{a} - \frac{qQ}{4\pi\epsilon_0 a^3}\left[\frac{1}{2}r^2\right]_{a}^{0}$$

$$= -\frac{qQ}{4\pi\epsilon_0}\left(-\frac{1}{a} - 0\right) - \frac{qQ}{4\pi\epsilon_0 a^3}\left(\frac{1}{2}0^2 - \frac{1}{2}a^2\right)$$

$$= -\frac{qQ}{4\pi\epsilon_0}\left(-\frac{1}{a}\right) - \frac{qQ}{4\pi\epsilon_0 a^3}\left(-\frac{1}{2}a^2\right)$$

$$= \frac{qQ}{4\pi\epsilon_0 a} + \frac{qQa^2}{8\pi\epsilon_0 a^3}$$

$$= \frac{qQ}{4\pi\epsilon_0 a} + \frac{qQ}{8\pi\epsilon_0 a}$$

$$= \frac{qQ}{4\pi\epsilon_0 a} + \frac{1}{2}\frac{qQ}{4\pi\epsilon_0 a}$$

$$= \frac{3}{2}\frac{qQ}{4\pi\epsilon_0}$$

As required! Plugging in values gives us:

$$\Delta u = \frac{3}{2}\frac{(2e)(79e)}{4\pi(8.854 \times 10^{-12}) \times 10^{-10}}$$

$$= 5.45 \times 10^{-16} J = 3.41 KeV$$

This is much less than the kinetic energy of the 5MeV alpha particle, therefore (as this value is maximum work done against the repulsive force) a plum pudding could not backscatter a 5MeV alpha particle. However, since $\Delta u \propto 1/a$, a smaller volume of charge could. How small, however?

$$\Delta u = 5\text{MeV} = -\int_{\infty}^{r_{max}} \frac{qQ}{4\pi\epsilon_0 r^2}\,dr$$

$$-5\text{MeV} = \frac{qQ}{4\pi\epsilon_0}\int_{\infty}^{r_{max}} \frac{1}{r^2}\,dr$$

$$-5\text{MeV} = \frac{qQ}{4\pi\epsilon_0}\lim_{x\to\infty}\left[-\frac{1}{r}\right]_x^{r_{max}}$$

$$-5\text{MeV} = \frac{qQ}{4\pi\epsilon_0}\left[-\frac{1}{r_{max}} - \lim_{x\to\infty}\frac{1}{x}\right]$$

$$-5\text{MeV} = \frac{qQ}{4\pi\epsilon_0}\left[-\frac{1}{r_{max}}\right]$$

$$5\text{MeV} = \frac{qQ}{4\pi\epsilon_0 r_{\text{max}}}$$

Subsitution and rearrangement gives $r_{\text{max}} = 4.5 \times 10^{-14} m = 45 fm$. We accept the 10^{-10}m $= 100,000$fm figure as the total width of the atom, but this demonstrates that there must exist a nucleus of no larger than 45fm.

Next Idea: The Solar System Model

Therefore, the next idea was an orbiting solar system model, where electrons orbit in fixed paths around a central nucleus. However, accelerating charges (i.e. a charge in circular motion) radiates energy, so this orbiting electron would be on a decaying path to crash into the nucleus. We can observe this does not happen, so need another idea...

Bohr made two postulates:

- The electron in hydrogen moves in a set non-radiating circular orbit.
- Radiation is only emitted or abosrbed when an electron moves from one orbit to another.

This works (at least for hydrogen) and explains the absoption spectra, but for now lacks a physical grounding.

Fri 10 Oct 2025 12:00

Lecture 2 - The Ultraviolet Catastrophe

In this lecture:

- How classical theories fail to explain black body radiation ("The Ultraviolet Catastrophe").
- How quantising light into photons gives predictions that fit this observation.

Black Body Radiation

A 'black body' is an idealised perfect object, that does not reflect, and absorbs internally all light (regardless of wavelength) incident upon it. No light is transmitted, so nothing shines out the other side. The object is perfectly black.

All bodies emit electromagnetic energy, usually outside the visible portion of the spectrum. For example, Paul Hollywood (and other humans) emit at about 300 Kelvin, which is infrared (at the temperature which night vision goggles are tuned to).

For the black body, emission spectrum is **only** from this thermal emission (no reflection, no flourescence, etc). Hotter objects are brighter and bluer (hotter means higher energy, and therefore a shorter wavelength)

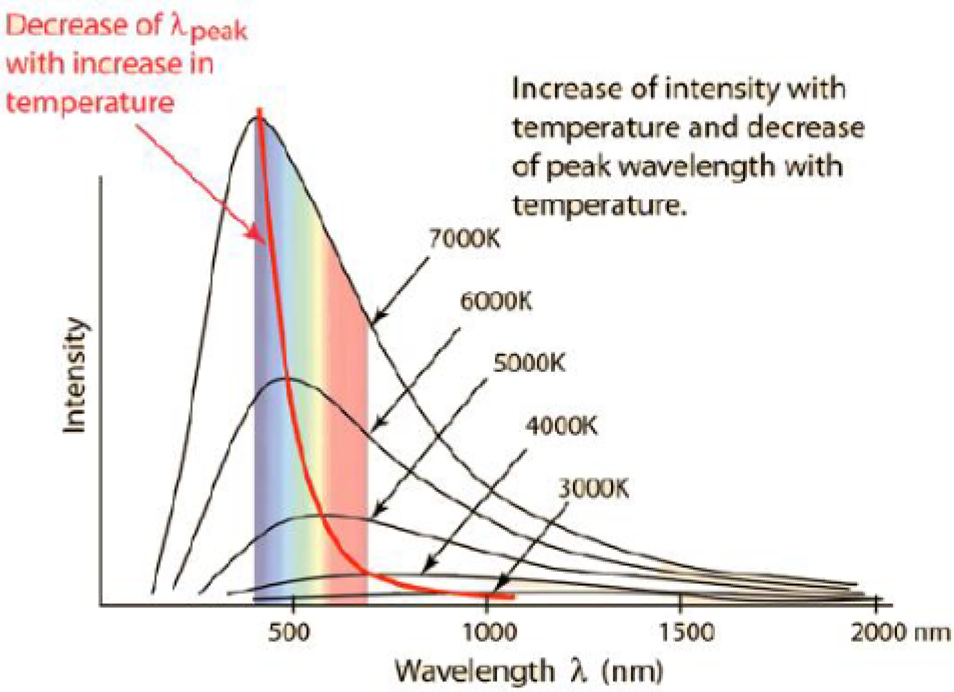


Figure 9: Observed Emission Spectra

However, we run into a problem. If we plot the spectra predicted by classical thermodynamics, vs the observed spectra for a given temperature object, the classical prediction gets it totally wrong, especially at shorter wavelengths.

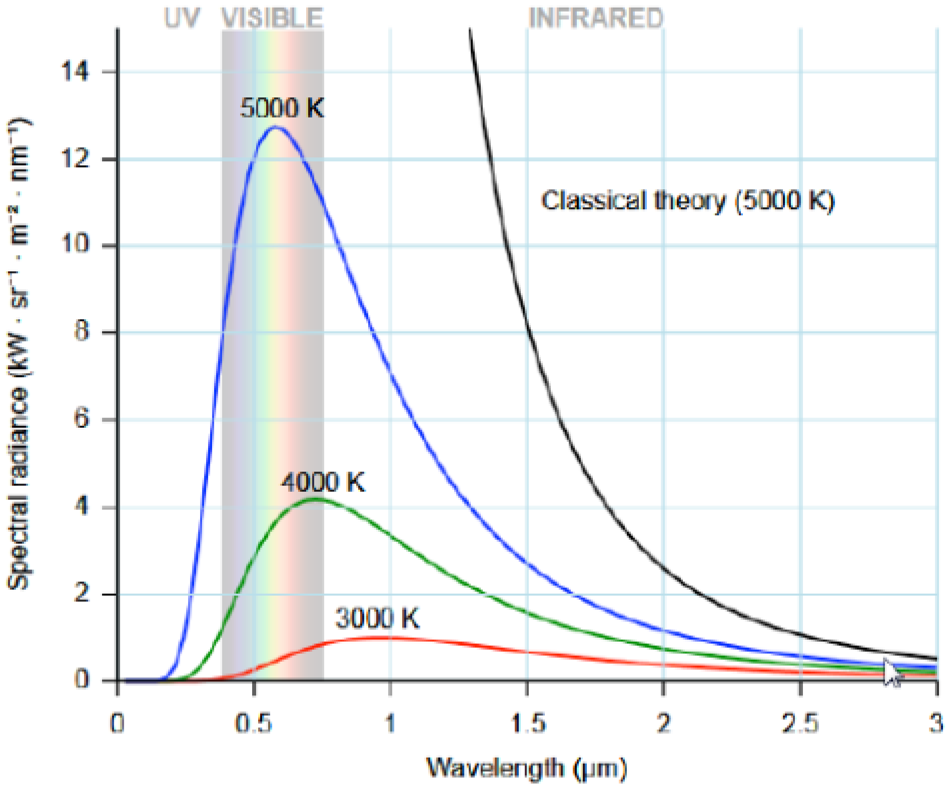


Figure 10: Predicted and Observed Spectra for 5000K (and Observed for 4k K and 3k K)

Notation

$I(\lambda)$ is the intensity per wavelength for an emitted wavelength λ. I is the total intensity across all wavelengths per unit time (in W/m^2, power per unit area).

$$I = \int_0^\infty I(\lambda)\,d\lambda$$

I is the total area under the $I(\lambda)$ curve, i.e. the sum of intensity per wavelength, across every wavelength.

The Ultraviolet Catastrophe

Empirical Results

The Stefan-Boltzmann Law gives $I = \sigma T^4$, where σ is the Stefan-Boltzmann constant, $\sigma = 6.57 \times 10^{-8} W m^{-2} K^{-4}$

Wien's Displacement Law gives $\lambda_{\text{peak}} = \frac{b}{T}$, where $b = 2.898 \times 10^{-3} Km$.

Why does classical mechanics break?

Lets model the $I(\lambda)$ spectrum by slotting standing waves into a cavity. Inside the blackbody, EM waves form standing waves (in a limited number of possible configurations).

We can simplify by considering a 1D cavity of length L. We can consider 'cavity modes' as the possible standing waves that can exist in this cavity. As we know the wave is bound at each end, the displacement at each end of the cavity must be 0. Therefore, the only possible waves must obey this, and these are cavity modes.

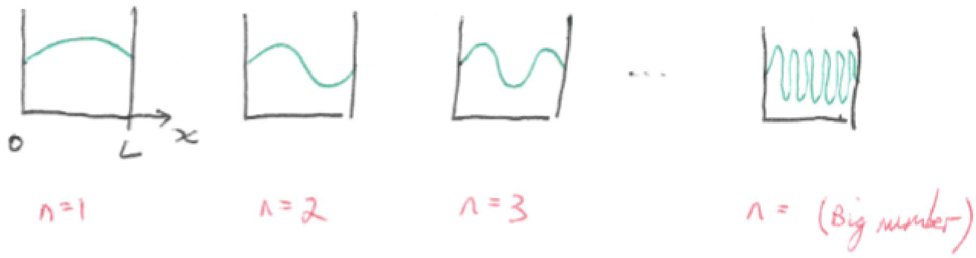


Figure 11: Possible cavity modes

The amplitude $a(x)$ can be given by this:

$$a(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, 3, \dots n \quad \text{(mode number)}$$

And by inspection from the figures:

$$\lambda = \frac{2L}{n}$$

And therefore the number of nodes per wavelength is:

$$n(\lambda) = \frac{2L}{\lambda}$$

So, classically:

$$I(\lambda) \propto \frac{n(\lambda)}{\lambda} \times k_B T \propto \frac{1}{\lambda^2}$$

Where the first term is the density of nodes at lamda, and the second is the average energy of nodes. As we head to UV and $\lambda \to 0$, $I(\lambda) \to \infty$... which is not accurate. This is the UV Catastrophe!

Where did it go wrong?

The issue was assuming that all cavity modes have average energy $k_B T$ - the "Equipartition Theorem" (which we'll meet in later courses).

In brief: the probability distribution of energies is a "Boltzmann Distribution":

$$p(E) = \frac{\exp\left(-\frac{E}{k_B T}\right)}{k_B T}$$

Average energy:

$$\bar{E} = \int_0^\infty E p(E)\, dE = k_B T$$

Which the UV Catastrophe says is incorrect... We therefore need another model to replace the idea of the Boltzmann Distribution.

Plank's Hypothesis

A rather desperate Plank hypothesised that energy was quantised, i.e. it comes in discrete packets, called quanta. The energy of these quanta is proportional to frequency. This was radical at the time, even though we accept it now.

$$\Delta E = hf = \frac{hc}{\lambda}$$

Where $h = 6.626 \times 10^{-34} Js$ is Plank's constant.

Sticking this into the Partition Function from statistical mechanics (which we will properly encounter later on, for now don't worry!), we get an average energy:

$$\bar{E}(\lambda) = \frac{hc/\lambda}{\exp(hc/\lambda k_B T) - 1}$$

Looking at limits:

$$\text{For } \bar{E}(\lambda \to \infty): \quad \frac{hc}{\lambda k_B T} << 1$$

And the Taylor Series of e^x:

$$\exp\left(\frac{hc}{\lambda k_B T}\right) \approx 1 + \frac{hc}{\lambda k_B T} + \ldots$$

Yields:

$$\bar{E}(\lambda) \approx \frac{hc/\lambda}{1 + (hc/\lambda k_B T) - 1} = k_B T$$

This is a good sign, because it means that Plank's Hypothesis holds the correct classically predicted and empirically observed behaviour for higher wavelengths.

Now what about lower wavelengths, where the classical behaviour broke?

$$\text{For } \bar{E}(\lambda \to 0): \quad \exp\left(\frac{hc}{\lambda k_B T}\right) \to \infty \quad \text{(very quickly)}$$

$$\text{For } \bar{E}(\lambda \to 0): \quad \frac{hc}{\lambda} \to \infty \quad \text{(slower)}$$

Therefore, in the expression:

$$\bar{E}(\lambda) = \frac{hc/\lambda}{\exp(hc/\lambda k_B T) - 1}$$

The numerator and denominator both tend to infinity, but the denominator does so much faster. Therefore (and this can be done in a less handwavy manner via L'Hopital):

$$\bar{E}(\lambda \to 0) \to \frac{1}{\infty} \to 0$$

Which recovers the behaviour at UV wavelengths, so no Catastrophe!

Conclusion

- This strange quantisation hypothesis actually fits the data.
- Quantising energy means that the average energy of each cavity mode is wavelength dependant, and not fixed $k_B T$ as seen at larger wavelengths.
- This solves the UV Catastrophe!

Thu 17 Oct 2025 12:00

Lecture 3 - Particle Nature of Light

In this lecture:

- The photoelectric effect.
- Compton scattering.

Which are two examples where classical theory (light as a wave) break down.

The Photoelectric Effect

When shining ultraviolet light on a metal surface, electrons are emitted. This is the photoelectric effect.

Why are we not bombarded by electrons in daily life? For the electron to fly off, we must be in a vacuum. Otherwise, it'll immediately strike an air molecule and be absorbed.

Photoelectric Effect Background

- Discovered by Hertz, 1887
- Thomson (1889) went further, so did Lenard (1902) and others.
- Einstein won his Nobel Prize for explaining this, not from relativity.

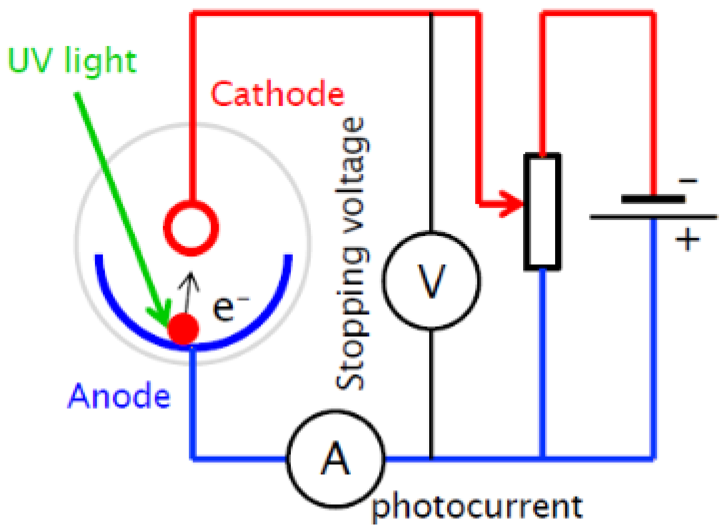


Figure 12: A circuit diagram for measuring the photoelectric effect.

The above setup would be encased in a glass ball (containing a vacuum), with a setup like this:

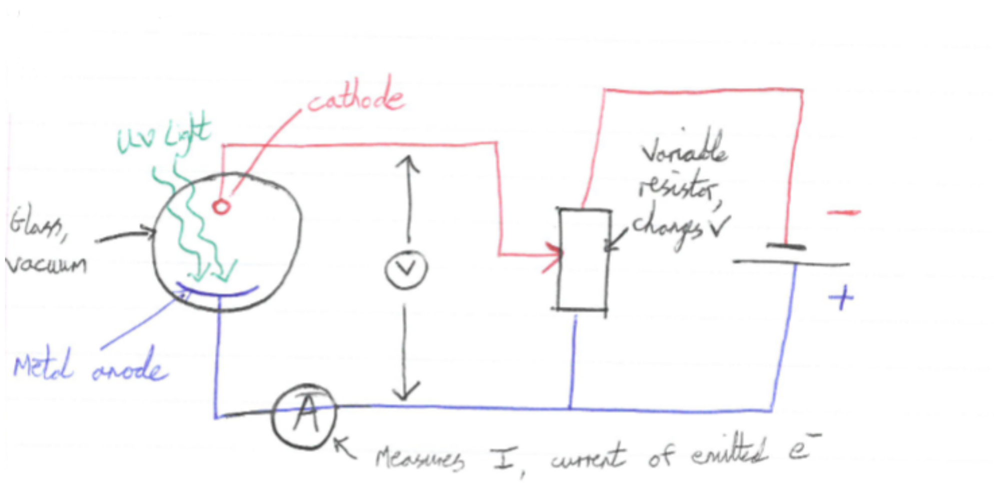


Figure 13: Experimental Setup

Results

Result One - Changing Intensity

For fixed UV wavelength, increasing the intensity of light increases the measured photocurrent:

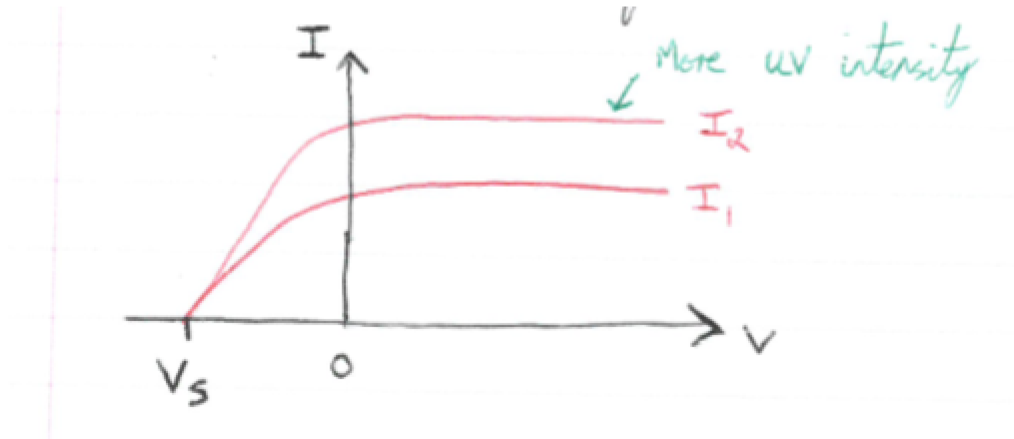


Figure 14

- Increasingly negative potential the cathode decreases photocurrent. At some potential v_s (the "stopping potential") this current drops to zero.
- Potential does not affect electron emission, however adding potential causes an electric field which effectively blows electrons back towards the anode. The stopping potential is when this electric field is perfectly strong to prevent electrons from reaching the cathode and causing a current.
- The fact this can happen consistently (i.e. no current means no electrons made it through) implies that there must be some maximum kinetic energy these electrons can have ($KE_{max} = eV_S$).
- The stopping potential is independant of UV intensity. More UV makes current increase, but does not change stopping potential (i.e. it does not give more energy to each electron, they each have the same energy). This does not make sense classically. Classically we would expect adding more energy to cause emitted electrons to have more energy, therefore changing the stopping potential.

Result Two - Changing Wavelength

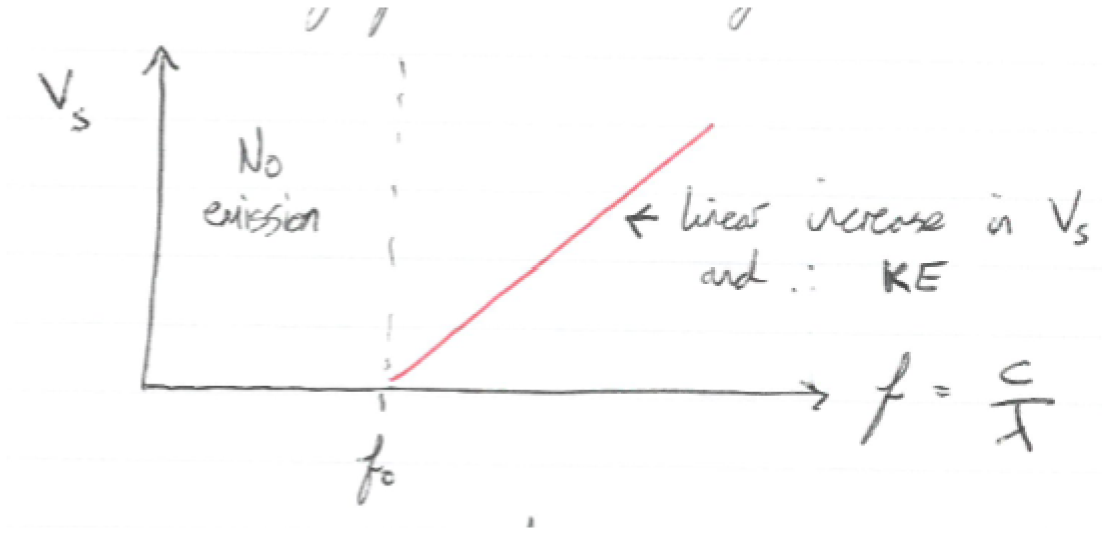


Figure 15

We have to reach some baseline threshold frequency f_0 before we see any photocurrent. After this, increasing wavelength increases photocurrent (and hence KE of emitted electrons) linearly.

- For a given metal, we find the threshold frequency f_0, below which there is no emission of electrons (no current). If below the frequency f_0, intensity is irrelevant. This contradicts classical mechanics which would suggest that turning up the light intensity would supply more (and potentially sufficient) energy.
- Above the threshold, the energy of individual emitted photons depends on UV frequency and not intensity (by result one).

Conclusions

Classically

Classically, we expect energy to be proportional to intensity, $\therefore v_s$ should increase with greater intensity. We also expect there to be no link between between frequency and energy, hence no threshold frequency. We'd expect no threshold frequency, instead being a time delay as electrons "soak up" energy to reach the required threshold.

In theory, great, in practice *this is not observed.*

Einstein's Proposal

Energy in light comes from phots with energy $E = hf$. There is a minimum energy required for an electron to be able to escape from the metal. This minimum energy is called the work function ϕ.

$$KE_{\max} = hf - \phi = eV_s$$

Now:

- Higher intensity means more of the same particles (more photons), but the energy of each is unchanged.
- $E = hf$ so frequency changes energy (as observed).
- The Bohr model says that an electron can only have certain electron energy transitions when the correct energy is supplied (an electron cannot gradually soak up energy). This explains why there is a cutoff below the work function, and no observed time delay (as the "soaking up" that casues the delay does not happen). Either an incoming photon has sufficient energy, or it does not. Having more photons does not help.
- The first incoming photons immediately releases an electron (assuming the incoming light has sufficient energy), therefore there's no time delay.

In Practice

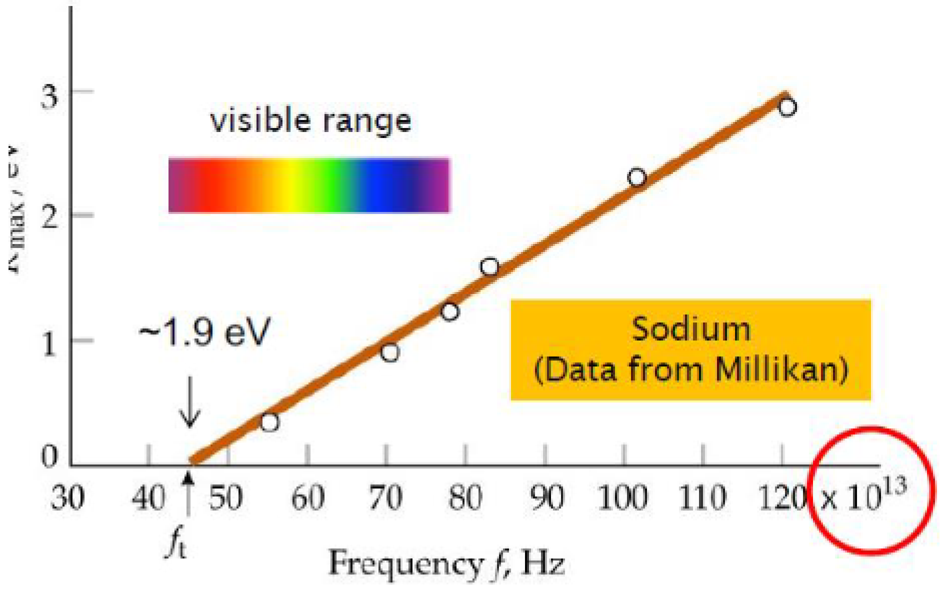


Figure 16: Sodium photocurrent measurements by Robert Millikan

Compton Scattering

Compton Scattering is the scattering of x-rays off carbon atoms.

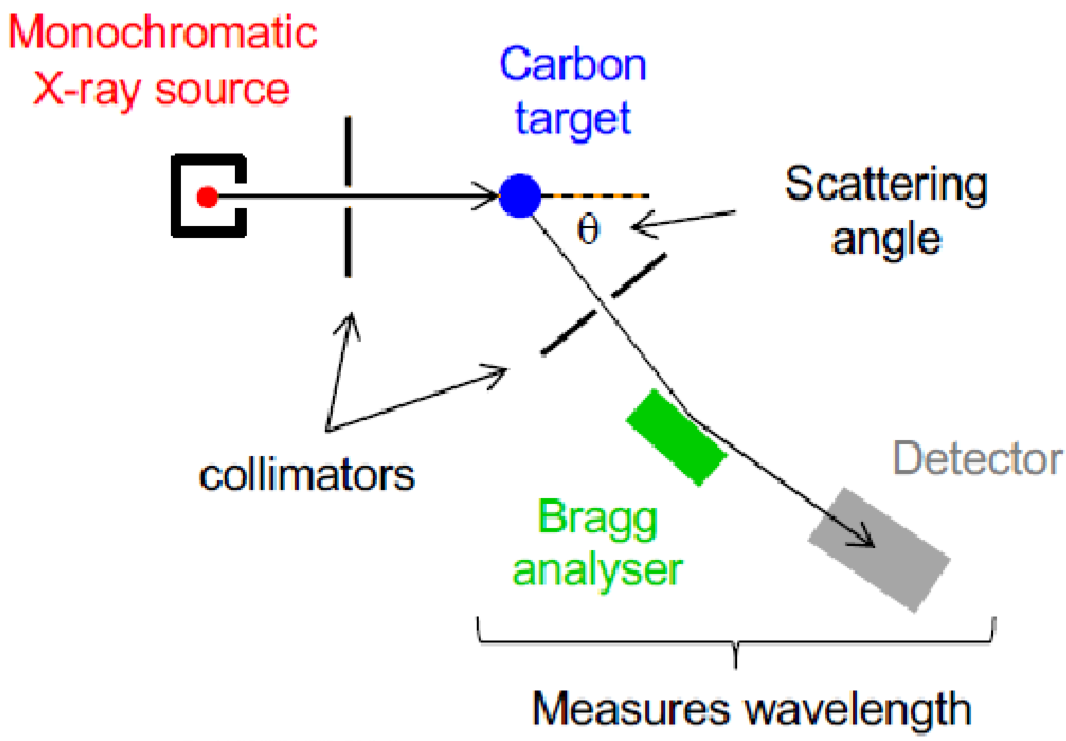


Figure 17: A Compton Scattering experimental setup.

The surprising result is that two wavelengths were observed (not just the original) - λ_1, λ_2, where λ_1 is the original and λ_2 is different. Classically this is hard to exlain and λ should not change.

The difference between these two wavelengths increases with scattering angle θ. This can be explained if the x-ray beam is a stream of photons, but not classically.

Possible options when a photon collides:

- Elastic: No loss of energy, hence no change in wavelength.
- Inelastic: Change of energy, so the wavelength also changes. An electron is fired out of the target, carrying energy and momentum, so the photon loses energy.

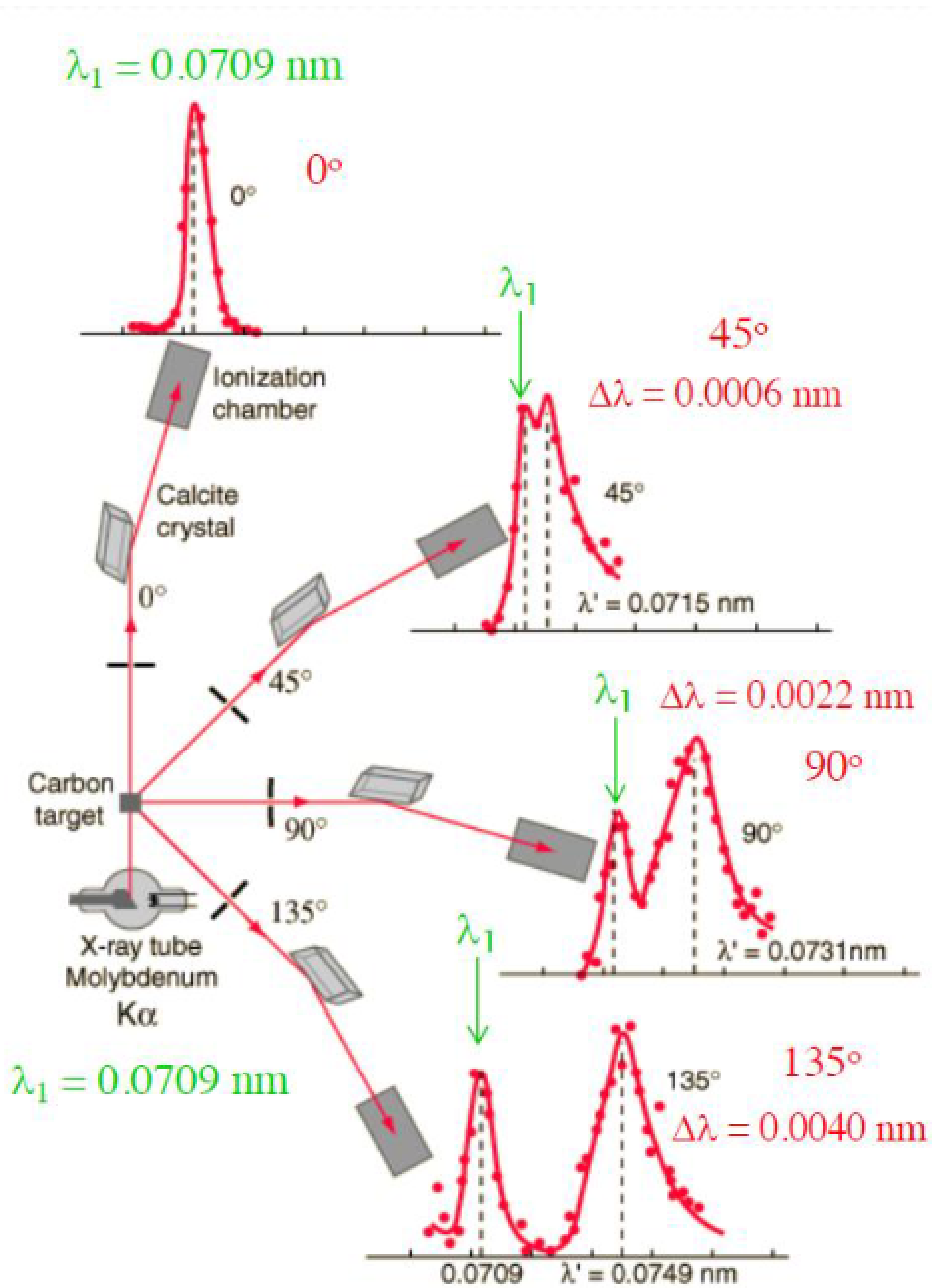


Figure 18: The observed results

Deriving Compton's Equation

Given an incoming photon with with energy E_1, wavelength λ_1 and momentum $\underline{p}_1$. This strikes a carbon atom and is deflected by angle θ. There is also an emitted electron at angle ϕ which must be in the oppsite direction (angled up vs down) to conserve momentum. The new deflected photon has E_2, λ_2, $\underline{p}_2$.

We must consider relativistic effects here given the high speed ($E^2 = p^2c^2 + m^2c^4$)

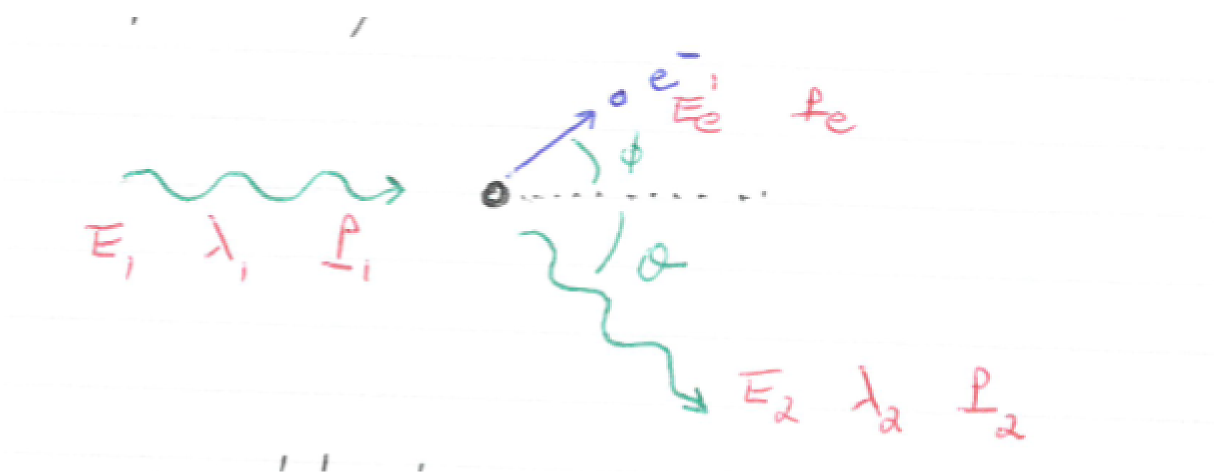


Figure 19

Setup

For a massless photon ($m = 0$):

$$E = pc$$

And:

$$E = \frac{hc}{\lambda}$$

So:

$$p = \frac{h}{\lambda} \tag{10}$$

Conservation and Relativity

Conserving momentum (underlines ommitted for speed):

$$p_1 = p_e + p_2$$

$$p_e = p_1 - p_2$$

Squaring both sides:

$$p_e^2 = p_1^2 + p_2^2 - 2p_1 \cdot p_2$$
$$p_e^2 = p_1^2 + p_2^2 - 2p_1 p_2 \cos\theta \tag{11}$$

And then by conservation of energy:

$$E_1 + E_e = E_2 + E_e'$$

Where E_1 is the incoming photon energy, E_e is the energy of the electron at rest in atom before collision, E_2 is the deflected photon energy and finally E_e' is the deflected electron's energy.

Using this:

$$p_1 c + m_e c^2 = p_2 c + \sqrt{p_e^2 c^2 + m_e^2 c^4}$$
$$\implies p_1 - p_2 + m_e c = \sqrt{p_e^2 + m_e^2 c^2}$$

And squaring both sides:

$$(p_1 - p_2)^2 + \cancel{m_e^2 c^2} + 2m_e c(p_1 - p_2) = p_e^2 + \cancel{m_e^2 c^2}$$

Subsituting in Eqn 11 for p_e^2

$$(p_1 - p_2)^2 + 2m_e c(p_1 - p_2) = p_e^2$$
$$(p_1 - p_2)^2 + 2m_e c(p_1 - p_2) = p_1^2 + p_2^2 - 2p_1 p_2 \cos\theta$$

And rearranging:

$$\begin{aligned}
(p_1 - p_2)^2 + 2m_e c(p_1 - p_2) &= p_1^2 + p_2^2 - 2p_1 p_2 \cos\theta \\
p_1^2 + p_2^2 - 2p_1 p_2 + 2m_e c(p_1 - p_2) &= p_1^2 + p_2^2 - 2p_1 p_2 \cos\theta \\
-2p_1 p_2 + 2m_e c(p_1 - p_2) &= -2p_1 p_2 \cos\theta \\
-p_1 p_2 + m_e c(p_1 - p_2) &= -p_1 p_2 \cos\theta \\
m_e c(p_1 - p_2) &= -p_1 p_2 \cos\theta + p_1 p_2 \\
m_e c(p_1 - p_2) &= p_1 p_2 (1 - \cos\theta)
\end{aligned}$$

Subsituting Eqn 10:

$$\begin{aligned}
m_e c(p_1 - p_2) &= p_1 p_2 (1 - \cos\theta) \\
m_e c\left(\frac{h}{\lambda_1} - \frac{h}{\lambda_2}\right) &= \frac{h}{\lambda_1}\frac{h}{\lambda_2}(1 - \cos\theta) \\
m_e c\left(\frac{h}{\lambda_1} - \frac{h}{\lambda_2}\right) &= \frac{h^2}{\lambda_1 \lambda_2}(1 - \cos\theta) \\
m_e c\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) &= \frac{h}{\lambda_1 \lambda_2}(1 - \cos\theta) \\
m_e c\left(\frac{\lambda_1 \lambda_2}{\lambda_1} - \frac{\lambda_1 \lambda_2}{\lambda_2}\right) &= h(1 - \cos\theta) \\
m_e c\,(\lambda_2 - \lambda_1) &= h(1 - \cos\theta) \\
(\lambda_2 - \lambda_1) &= \frac{h}{m_e c}(1 - \cos\theta)
\end{aligned}$$

Which is the Compton Equation. This shows that the change in wavelength is proportional to $1 - \cos\theta$.

Conclusions

The photoelectric effect and Compton scattering are two more physical phenomena that cannot be explained using traditional classical mechancics with EM waves alone. They both require assuming photons of energy $E = hf$ to be adequately explained.