LC Optics and Waves Lecture Notes

MSci Physics w/ Particle Physics and Cosmology
University of Birmingham

Year 1, Semester 1
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Lectures Index

Wed 01 Oct 2025 11:00

Lecture 1 - Intro to Waves and SHM Recap

Course Objectives

- Have a sound understanding of basic wave properties
- Have a basic understanding of interference effects, inc diffraction
- Be able to use simple geometric optics and understand the fundamentals of optical instruments.

Recommended Textbooks

1. University Physics, Young and Freedman (Ch 15, 16 for Waves, Ch 33-36 for Optics)
2. Physics for Scientists and Engineers (Ch 20, 21 for Waves, Ch 22-24 for Optics)
3. 5e, Tipler and Mosca, (Ch 15, 16 for Waves, 31-33 for Optics)
4. Fundamentals of Optics, Jenkins and White
5. Optics, Hecht and Zajac

What is a wave? Waves occur when a system is disturbed from equilibrium and the disturbance can travel from one region to another region. Waves carry energy, but do not move mass. The course aim is to derive basic equations for describing waves, and learn their physical properties.

Periodic Motion

Waves are very linked to periodic motion. Therefore we recap periodic motion first.

It has these characteristics:

- A period, T (the time for one cycle)
- A frequency, f, the number of cycles per unit time ($f = \frac{1}{T}$)
- An amplitude, A, the maximum displacement from equilibrium.

Periodic motion continues due to the restoring force. When an object is displaced from equilibrium, the restoring force acts back towards the equi point. The object reaches equi with a non-zero speed, so the motion continues past the equi point and continues forever.

Energy

Periodic motion is an exchange between potential and kinetic energy, with no energy loss. Energy is conserved.

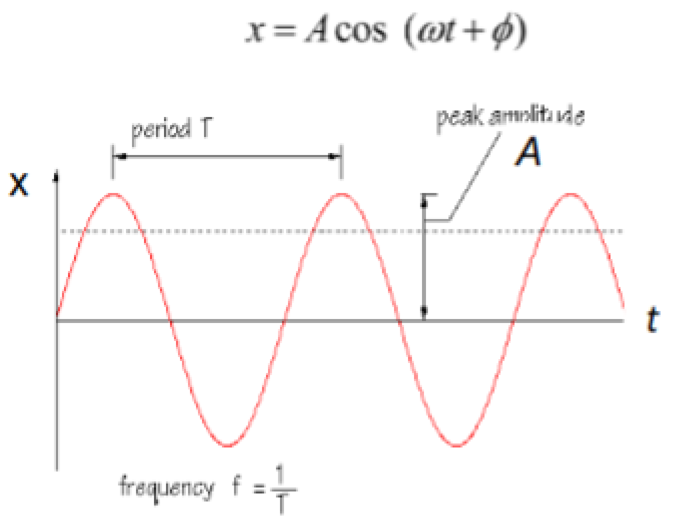
Simple Harmonic Motion

If the restoring force is directly proportional to the displacement $F = -kx$, then the periodic motion becomes Simple Harmonic Motion and the object is called a harmonic oscillator.

In a single dimension, displacement is given by:

$$x = A\cos(\omega t + \phi)$$

Where $\omega = 2\pi f$ is the angular velocity, and ϕ is the phase angle. In cases like this, where the phase angle is $90\deg$ we can simplify to $x = -A\sin(\omega t)$



SHM-t

Figure 1.1: A Phase Angle of 90

More SHM Equations

Velocity

$$v_x = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

Acceleration

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega t + \phi)$$

Both properties are signed to indicate direction, as they are both vectors.

Thu 02 Oct 2025 13:00

Lecture 2 - Wave Functions

Sine Waves

Mechanical Waves A mechanical wave is a disturbance through a medium. It's formed of a single wave pulse or a periodic wave.

Mechanical Waves have the following properties:

1. Transverse: Where displacement of the medium is perpendicular to the direction of propogation.
2. Longitudinal, displaccement of hte medium is in the same direction as propogation.
3. Propogation depends on the medium the wave moves through (i.e. density, rigidity)
4. The medium does not travel with the wave.
5. Waves have a magnitude and a direction.
6. The disturbance travels with a known exact speed.
7. Waves transport energy but not matter throughout the medium.

Wave Functions

We want to define a wave function in terms of two variables, x and t. In any given moment, if we consider a single point on the wave (i.e. $t = 0$), and wait a short while, the wave will have travelled to some $t = t_1 > 0$.

In order to quantify displacement, we therefore want to specify both the time, and the displacement. This will let us find the wave speed, acceleration and the (new) wave number.

We are also able to talk about the velocity and acceleration of individual particles on the wave.

Wave Function for a Sine Wave

Consider a sine wave. We want to find a wave funtion in the form $y(x, t)$. Consider the particle at $x = 0$.

We can express the wave function at this point as $y(x = 0, t) = A \cos \omega t$. However we want to expand this to any general point. Now consider a point (2) which is one wavelength away. We know the behaviour of particle 1 is mirrored by particle 2 (with a time lag).

Since the string is initially at rest, it takes on period (T) for the propogation of the wave to reach point 2, therefore point 2 is lagging behind the motion of point 1. The wave equation is therefore (if particle two has $x = \lambda$) $y(x = \lambda, t) = A \cos(\omega t - 2\pi)$.

For arbitrary x, $y(x, t) = A \cos(\omega t - \frac{x}{\lambda} \cdot 2\pi)$ to account for this delay. This quantity is called the wave number:

$$\text{Wave Number: } k = \frac{2\pi}{\lambda}$$

So:

$$\begin{aligned} y(x, t) &= A \cos(\omega t - kx) \\ &= A \cos(kx - \omega t) \end{aligned}$$

Note the second step is possible as cos is an even function. k can also be signed to indicate direction: if $k > 0$, the wave travels in the positive x. If $k < 0$, the wave travels in the negative x direction. Again, $\omega = 2\pi f$

Displacement Stuff

Considering a point (starting at equi), the time taken for the particle on the sin wave to reach maximum displacement, minimum displacement and back takes the time period T. The speed of the wave is distance travelled over the time taken. We take the distance to be the wavelength λ, as we know the time by definition this takes is one time period T. Therefore wave speed v is:

$$v = \frac{\lambda}{T} = \lambda f$$

Since $\lambda = \frac{2\pi}{k}$ and $f = \frac{\omega}{2\pi}$ (as ω is defined as $\frac{2\pi}{T}$), we can also write:

$$v = \frac{2\pi}{k} \cdot \frac{\omega}{2\pi} = \frac{\omega}{k}$$

Particle Velocity

We can also determine the velocity of individual particles in the medium. We can use this to determine the acceleration.

We know that

$$y(x,t) = A\cos(kx - \omega t)$$

The vertical velocity v_y is therefore given by:

$$v_y = \frac{dy(x,t)}{dt}$$

Which is unhelpful (as we can't differentiate two variables at once), we can slightly cheat this by looking at purely a certain value of x, and therefore treating x as constant (to get a single variable derivative).

$$v_y = \frac{dy(x,t)}{dt}\Big|_{x=\text{const.}}$$

However this is notationally yucky, so we therefore use the notation:

$$\frac{\partial y(x,t)}{\partial t}$$

To represent the same thing. Finally (carrying out the partial derivative):

$$v_y(x,t) = \frac{\partial y(x,t)}{\partial t} = \omega A \sin(kx - \omega t)$$

Particle Acceleration

We can work out particle acceleration (transverse acceleration) by differentiating in the same manner again:

$$a_y(x,t) = \frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial y(x,t)}{\partial t}\right) = -\omega^2 A\cos(kx - \omega t) = -\omega^2 y(x,t).$$

Wed 08 Oct 2025 11:00

Lecture 3 - Generalised Wavefunctions

Recap: For a sine wave, the wavefunction is:

$$y(x,t) = A\cos(kx - \omega t)$$

Where k is the wave number, $k = \frac{2\pi}{\lambda}$ and omega is $\frac{2\pi}{T} = 2\pi f$

More Wavefunctions

What is the general form of a triangular shaped wave? What about a square stepped wave? TODO: Diagram

We know that each particle (i.e. along a string) copies the motion of its immediate left-hand neighbour (for a particle moving in the positive x), with a time delay proportional to their distance. Every wave is describable $y(x,t)$ wave function, and every mechanical wave relies on a medium (i.e. a piece of string or water, concrete etc) to travel.

Ideally, we want to be able to find a general form of a wave function.

In a single fixed instant of time, the moving wave pulse is stationary, so purely a function of $y = f(x)$. We want a wave function where we can input any value of t, so we need a moving frame of reference. We define this frame of reference as O' (for the origin) and x', y' axes. This frame of reference moves with the wave pulse and at the same speed, therefore y' is a function of x' only, independant of speed.

New System

x is the distance from the origin O to the relevant point, while x' is the distance from O'.

$$x = x' + vt$$

$$x' = x - vt$$

$$y' = f(x') = f(x - vt)$$

However, as the wave is moving purely in one direction (along x), $y = y'$, so:

$$y = f(x - vt)$$

Back to Basics

Going back to:

$$\begin{aligned} y(x,t) &= A\cos(kx - \omega t) \\ &= A\cos\left[k\left(x - \frac{\omega}{k}t\right)\right] \\ &= A\cos\left[k\left(x - vt\right)\right] \end{aligned}$$

(Note, this is true for a wave in the positive x, for a wave moving in the negative x this would be x + vt)

Equivalent Rrepresentations

There are some equivalent representations for a sine wave:

$$y(x,t) = A\cos\left(2\pi f\frac{x}{v} - \omega t\right) = A\cos\left(2\pi\left[\frac{x}{\lambda} - \frac{t}{T}\right]\right) = TODO$$

More Differentiation

We've already looked at differentiating with respect to t, but what about x? This would give us the slope of the string at that point:

$$\frac{\partial y(x,t)}{\partial x}$$

And the curvature of the string:

$$\frac{\partial^2 y(x,t)}{\partial x^2} = -k^2 A\cos(kx - \omega t) = -k^2 y(x,t)$$

Note the similarities here with the equation for transverse acceleration:

$$(1)\frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 y(x,t)$$

$$(2)\frac{\partial^2 y(x,t)}{\partial x^2} = -k^2 y(x,t)$$

Dividing (1) by (2):

$$\frac{\partial^2 y(x,t)/\partial t^2}{\partial^2 y(x,t)/\partial x^2} = \frac{\omega^2}{k^2} = v^2$$

Therefore:

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

We call this the 'wave equation' as every wave function y(x, t) must satisfy it, regardless of whether or not it is periodic or its direction of travel. If y(x, t) does not satisfy this, it is not a wave function.

An Example

$$y(x,t) = \frac{x^3 - vt^2}{e^t}$$

TODO, the example in our own time

Wave Equation for a String

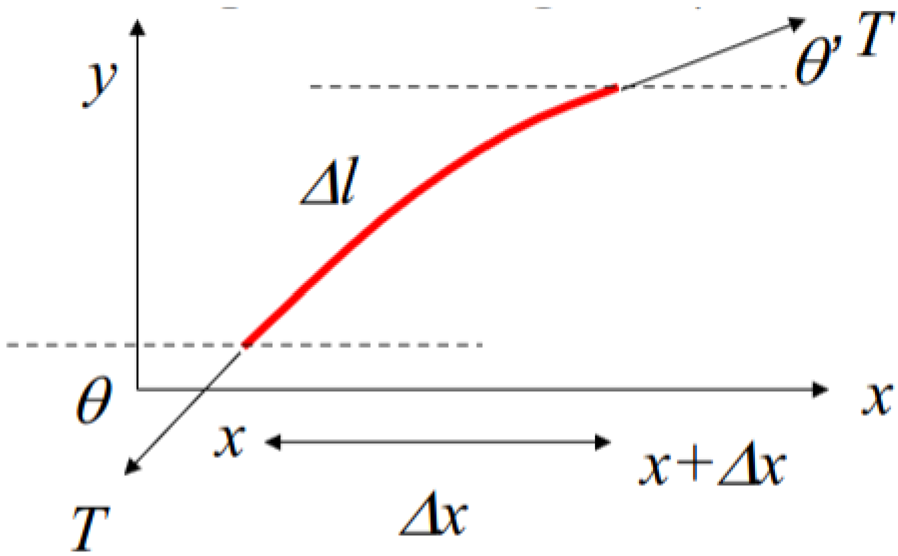


Figure 3.1: A snapshot of the wave.

Lets say we have some string, suspended horizontally under tension. We generate a single wave pulse and allow it to propogate down the string.

We assume the string is 1D, under tension T (constant throughout) and has mass per unit length μ.

Consider a small segment of string of length ΔL from x to $x + \Delta x$. This string makes angle θ with the horizontal at the bottom of the string, and angle θ' with the horizontal at the top of the string.

Net force (transverse in y) is:

$$F_y = T\sin\theta' - T\sin\theta$$

And using the small angle approx $\sin\theta \approx \tan\theta = \frac{dy}{dx}$:

$$F_y = T\left(\frac{dy}{dx}\Big|_{x+\Delta x} - \frac{dy}{dx}\Big|_{x}\right)$$

Using differentiation by first principles:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

We can say:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \lim_{\Delta x \to 0} \frac{\frac{dy}{dx}\Big|_{x+\Delta x} - \frac{dy}{dx}\Big|_{x}}{\Delta x}$$

Therefore:

$$F_y = T\frac{d^2y}{dx}\Delta x$$

The mass of this section of string is $\mu\Delta x$, and considering the acceleration in the y direction we can plug into $F = ma$ to get:

$$F = ma$$

$$T\frac{d^2y}{dx^2}\Delta x = \mu\Delta x\frac{d^2y}{dt^2}$$

$$\frac{d^2y}{dx^2} = \frac{\mu}{T}\frac{d^2y}{dt^2}$$

The LHS is the rate of change of the string's gradient, which as mentioned is the curvature of the string. The RHS includes the transverse acceleration of the string, therefore acceleration is proportional to curvature. To evaluate this at a fixed time/position we should write:

$$\frac{\partial^2y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2y}{\partial t^2}$$

And comparing to the wave equation we get:

$$\frac{1}{v^2}\frac{\partial^2y}{\partial t^2} = \frac{\mu}{T}\frac{\partial^2y}{\partial t^2}$$

$$\frac{1}{v^2} = \frac{\mu}{T}$$

$$v = \sqrt{\frac{T}{\mu}}$$

So a wave travels faster under a higher tension with a lower mass per unit length.

Thu 09 Oct 2025 13:00

Lecture 4 - Waves at Boundaries

Recap

We previously derived:

$$v = \sqrt{\frac{T}{\mu}}$$

And we also have:

$$v = \frac{\omega}{k}$$

The former is useful explictly for a wave travelling over a string, while the latter is applicable to the movement of any wave. On a string, a higher tension yields a higher restoring force and therefore a higher speed, while a higher mass per unit length gives a higher mass for some arbitrary length of string, therefore a lower acceleration and lower speed.

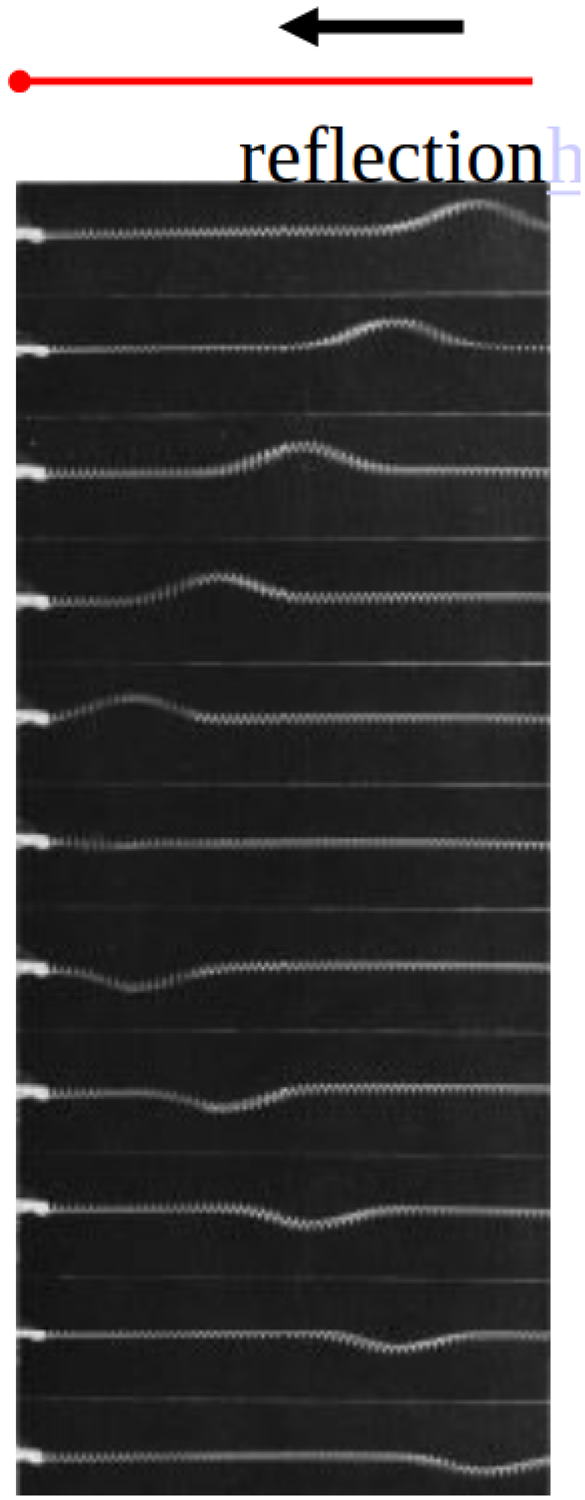
A Quick Interlude

The speed of a mechanical wave has the general form:

$$v = \sqrt{\frac{\text{Restoring force returning to equilibrium}}{\text{Inertia resisting return to equilibrium}}}$$

Reflection

When a wave hits a fixed boundary, it is reflected and inverted. Lets consider a case where a string is fixed on the LHS and is reflected back:


reflection

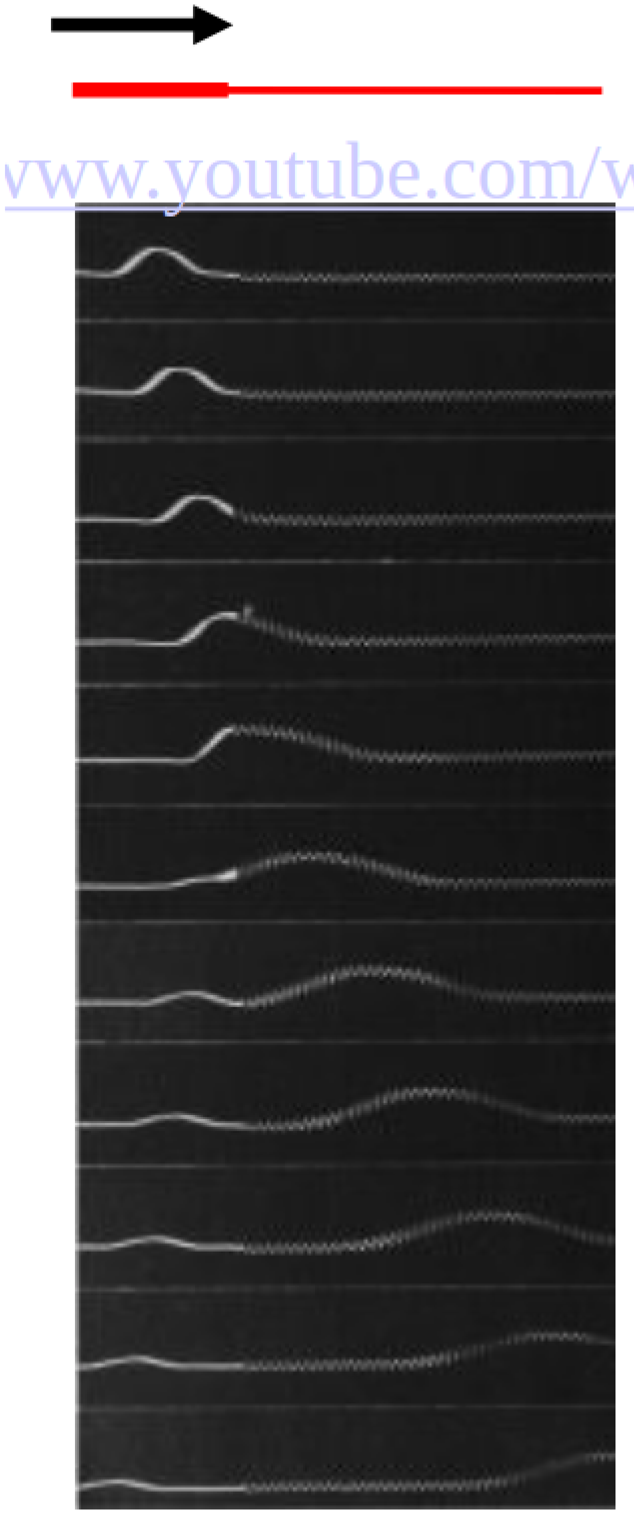


Figure 4.1: Or with two strings, going from thick to thin:

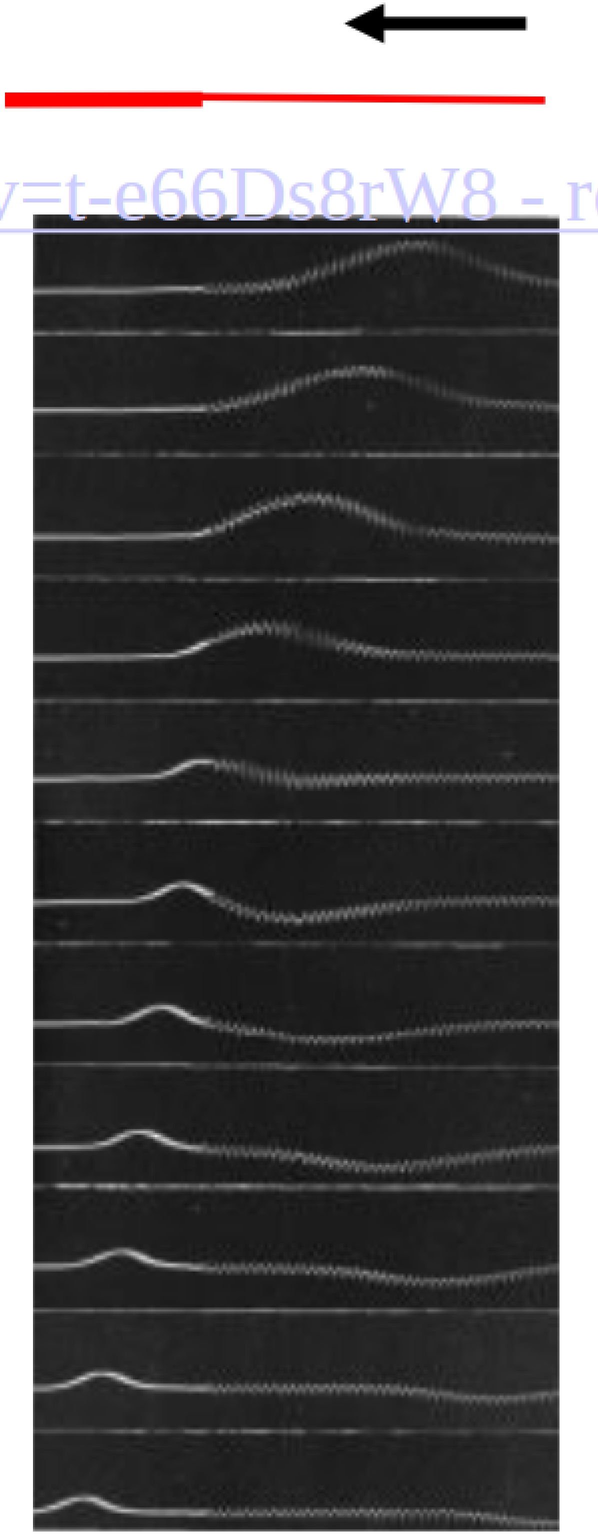


Figure 4.2: Or from thin to thick.

Waves Interacting at Boundaries

Say we have two pieces of string connected to each other, one thin string with mass per unit length μ_1 and a thicker string with μ_2 (both under the same tension, T). If a wave pulse is passed along from the thin string to the thicker string, whath happens at the point of connection, P?

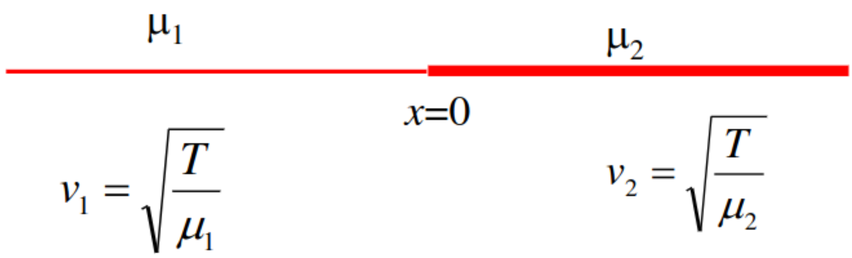


Figure 4.3: The two connected strings.

Consider a travelling wave coming from the left:

$$y_1 = A\cos(k_1 x - \omega_1 t)$$

$$y_2 = B\cos(k_2 x - \omega_2 t)$$

At the point of connection, we'll define this as $x = 0$. Here (as the string is not broken, so it must be connected):

$$y_1 = y_2 \tag{4.1}$$

Also, force is finite, therefore curvature must be finite (as force is proportional to curvature). Therefore we cannot have any discontinuities in curvature.

$$\frac{\partial y_1}{\partial x_1} = \frac{\partial y_2}{\partial x_2} \tag{4.2}$$

Disregarding non-linear effects, so assuming that the frequency with which the waves travel down the string is the same for both parts of the string, $\omega_1 = \omega_2 = \omega$.

From (1) and noting $x = 0$:

$$A\cos(-\omega t) = B\cos(\omega t) \tag{4.3}$$

And from (2) with the same note:

$$-k_1 A \sin -\omega t = -k_2 B \sin -\omega t \tag{4.4}$$

However this suggests that $A = B = 0$. This is technically a solution, but not really - it doesn't represent an actual wave (just two flat lines with no amplitude ever)

Including Reflection

The previous case did not work as we disregarded reflection at the wave boundary, P. Lets add an extra wave (the C term) to represent the reflection back into the light string:

$$y_1 = A\cos(k_1 x - \omega t) + C\cos(k_1 x + \omega t) = B\cos(k_2 x - \omega t)$$

From y1 = y2 at x = 0 we get:

$$y_1 = A\cos(-\omega t) + C\cos(-\omega t) = B\cos(-\omega t)$$

So: $A + C = B$

From (2) we get:

$$-k_1 A\sin(-\omega t) - k_1 C\sin(\omega t) = -k_2 B\sin(-\omega t)$$

Which has solutions:

$$B = \frac{2k_1}{k_1 + k_2} A$$

$$C = \frac{k_1 - k_2}{k_1 + k_2} A$$

We have the incident wave:

$$y_1 = Acos(k_1 x - \omega t)$$

The transmitted wave:

$$y_2 = B\cos(k_2 x - \omega t) = \frac{2k_1}{k_1 + k_2} A\cos(k_2 x - \omega t)$$

And lastly the reflected wave:

$$y_3 = C\cos(k_1 x + \omega t) = \frac{k_1 - k_2}{k_1 + k_2} A\cos(k_1 x - \omega t)$$

We note:

k_1 is the wave number in the medium where the incident wave comes from.

k_2 is the wave number in the medium where the transmitted wave goes into.

Example

If the wave comes in from the left, then $\mu_2 > \mu_1$ per example diagram, then $k_2 > k_1$, and C is negative.

Note that if the single pulse as a postive y amplitude, then the transmitted pulse in the heavier string will also have a positive y amplitude, but will have a smaller magnitude. The reflected wave will have a negative y-amplitude as the reflection inverts it.

If the wave comes from the right (from thick to thin), then $\mu_1 > \mu_2$, $k_1 > k_2$ and C is positive.

Wed 15 Oct 2025 11:00

Lecture 5 - Wave Applications and Introduction to Standing Waves

1 Wave Applications

Reflection and transmission have some very important real word applications:

- Fibre Optics.
- Vision and Photography.
- X-Ray Imaging.
- Sonar Imaging.
- Ultrasound Imaging
- Police Speed Checks.

We will look at a few in a bit more detail.

1.1 Electron Microscopes

We have two types:

- Scanning Electron Microscope (SEM)
  - We take some sample, and generate a very narrow beam of electrons. Some of these electrons will scatter off the sample back up. We take a detector, "scan" it around many angles and detect angles of scattering, to build a picture of the sample's surface.
- Transmission Electron Microscope (TEM)
  - We take a very thin sample, fire the electron beam at it. Some of the electron beam will pass through the sample, where we can detect it on the other side. Areas of the sample which are thinner act as being more transparent, with a greater rate of transmission. We can measure this difference.

1.2 Solar Cells

Consider a solar cell with light incident on it at some angle. We want to maximise absorption of light and minimise reflection, to maximise the energy efficiency of the cell.

1.3 Ultrasound Imaging

So far, we have considered a wave to only be able to transmit or reflect at a boundary. What we have not considered yet is attenuation. This is where a wave passing through a medium loses some energy to the medium. As the wave propagates, it loses energy and hence decreases in amplitude.

For example, a sound wave propagating though a medium causes particles in the medium to oscillate. This causes a heating effect on the medium, where some of the wave energy is lost thermally resulting in a decrease in amplitude (volume).

Ultrasound is sound with very high frequencies, compared to audible sound:

$$f_{\text{ultrasound}} : 2 - 20\text{MHz}$$

$$f_{\text{sound}} : 20\text{Hz} - 20\text{kHz}$$

Wave speed is given by the below, where ρ is density and B is bulk modulus (resistance to compression):

$$v = \sqrt{\frac{B}{\rho}}$$

This means:

- Ultrasound's wavelength in air (10MHz) is $33\mu m$.
- Sound's wavelength in air (at 300Hz) is $\approx 1m$

Ultrasound has a longer wavelength in fat (human tissue) of about $150\mu m$ as while fat does have a higher density, it has a much greater bulk modulus, and a greater B/ρ ratio.

Lets consider transmission of ultrasound at an air-fat boundary. Since the wavelength in air is smaller, and the density of air is much smaller, $C \approx A$, so $B \approx 0$ and there is (almost) zero transmission from air to fat.

This is why ultrasound gel is used, to ensure that there is transmission from air to gel and gel to fat, effectively bridging the gap.

2 Introduction to Standing Waves

Consider a string (of indefinite length) attached to the wall on the left. We send series of wave pulses (y_1) into the string from the right, which travel left and reflect. The reflected wave (y_2) travels back to the right.

If we continue sending a wave signal from $y_1(x,t)$ we would have a continuous wave signal back in the form of $y_2(x,t)$.

This causes the two waves to overlap and interfere with each other. We can say that the resultant amplitude for any point on the string is given by:

$$y = y_1(x,t) + y_2(x,t)$$

This works because the wave equation is a linear partial differential equation. If two functions y_1, y_2 are solutions to the wave equation, then this means their sum will be two (hence y is a valid final wave).

The incident wave is:

$$y_1 = -A\cos(kx + \omega t)$$

And the reflected wave is (assuming the two waves have equal amplitude, period and wavelength, with the second travelling in the opposite direction and being inverted due to the reflection):

$$y_2 = A\cos(kx - \omega t)$$

We add these together:

$$y_{\text{total}} = y_1 + y_2 = A\left[\cos(\underbrace{kx}_{A} - \underbrace{\omega t}_{B}) - \cos(kx + \omega t)\right]$$

$$\text{using: } \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$y_{\text{total}} = A\left[-\cos(kx)\cos(\omega t) + \sin(kx)\sin(\omega t)\right] + A\left[\cos(kx)\cos(\omega t) + \sin(kx)\sin(\omega t)\right]$$

$$y_{\text{total}} = 2A\sin(kx)\sin(\omega t)$$

This gives us the general form of a sinusoidal standing wave function.

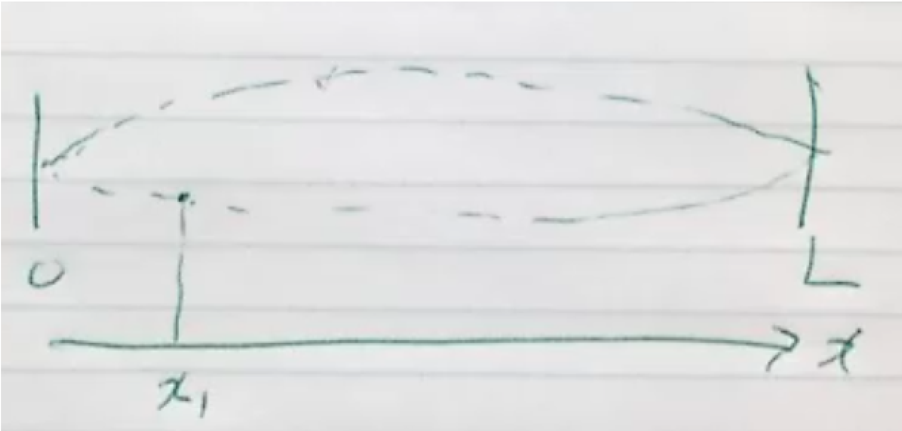


Figure 5.1

At some point $x = x_1$:

$$y_t(x = x_1, t) = \boxed{2A\sin(kx_1)}\sin(\omega t)$$

As x_1 is a constant, the boxed term is now another constant too. Let this constant be E.

$$y_t(x = x_1, t) = E\sin\omega t$$

This is the equation for a harmonic oscillator. This point of the string oscillates in simple harmonic motion with a maximum amplitude dependant on position.

If we consider a higher frequency wave, we may encounter points where $\sin kx = 0$. These are called nodes.

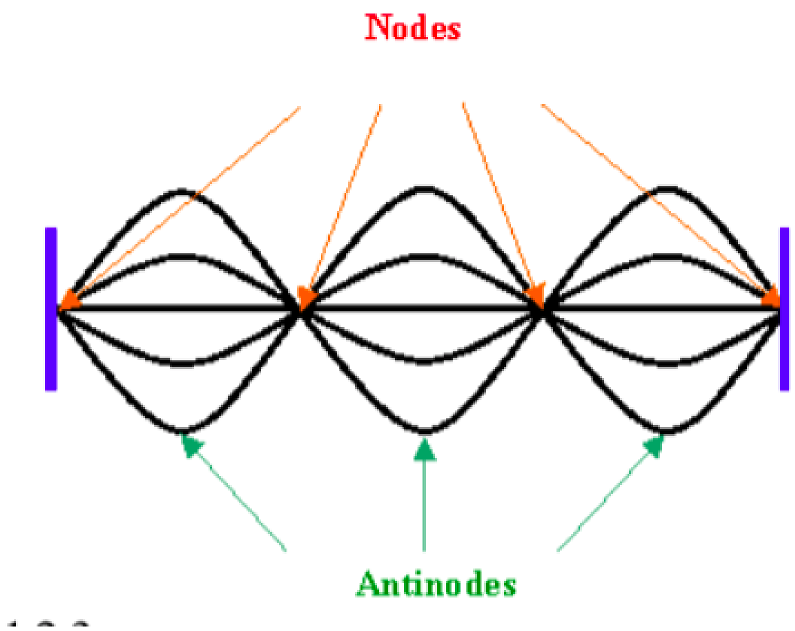


Figure 5.2

The amplitude at a node is always $y = 0$, regardless of time, as the sine term is zero regardless of time.

This wave does not propagate, it is "stationary".

3 Standing Waves and the Wave Equation

It's important to note that standing waves also satisfy the wave equation. Carrying out derivatives, we can get:

$$\frac{\partial y}{\partial x} = 2Ak\cos kx\sin\omega t$$

$$\frac{\partial^2 y}{\partial x^2} = 2A(-k^2)\sin kx \sin \omega t$$
$$\frac{\partial y}{\partial t} = 2A\omega \sin kx \cos \omega t$$
$$\frac{\partial^2 y}{\partial t^2} = 2A(-\omega^2)\sin kx \sin \omega t$$

And substituting into the wave equation:

$$\frac{\left(\dfrac{\partial^2 y}{\partial t^2}\right)}{\left(\dfrac{\partial^2 y}{\partial x^2}\right)} = \frac{2A(-\omega^2)\sin kx \sin \omega t}{2A(-k^2)\sin kx \sin \omega t} = \frac{\omega^2}{k^2} = v^2$$
$$\therefore \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

So the wave equation is satisfied.

Thu 16 Oct 2025 13:00

Lecture 6 - Standing Waves 2 Electric Boogaloo

Recap

$$\frac{\partial y}{\partial x} = 2Ak\cos(kx)\sin(\omega t)$$

$$\frac{\partial^2 y}{\partial x^2} = 2A(-k^2)\sin(kx)\sin(\omega t)$$

$$\frac{\partial y}{\partial t} = 2A\omega\sin(kx)\cos(\omega t)$$

$$\frac{\partial^2 y}{\partial t^2} = 2A(-\omega^2)\sin(kx)\sin(\omega t)$$

Therefore:

$$\frac{\frac{\partial^2 y}{\partial t^2}}{\frac{\partial^2 y}{\partial x^2}} = \frac{2A(-\omega^2)\sin(kx)\sin(\omega t)}{2A(-k^2)\sin(kx)\sin(\omega t)} = \frac{\omega^2}{k^2} = v^2$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Therefore a standing wave still obeys the wave equation, as it must.

Standing Wave Properties

Wavelength

Consider a horizontal string from $x = 0$ to $x = L$, with both ends fixed. We generate a sinusoidal wave pulse, which must satisfy:

$$y(x,t) = 2A\sin(kx)\sin(\omega t)$$

We know at $x = L$ and $x = 0$, $y = 0$ at all times as fixed at this point. Therefore:

$$kL = n\pi, (n \in \mathbb{N})$$

For $n = 1$, we have half a wavelength on the string:

$$\lambda = \frac{2L}{1} = 2L$$

And this has a general form: $\lambda = \frac{2L}{n}$.

Frequency

$$f_n = \frac{v}{\lambda_n} = \frac{v}{\left(\frac{2L}{n}\right)} = \frac{nv}{2L}$$

Crucially:

$$f_1 = \frac{v}{2L}$$

Is the first harmonic (or fundamental). $f_2 = 2f_1$ is the second harmonic, or first overtone, etc. All of these, f_n where $n \in \mathbb{N}$ are called "normal modes". For each normal mode, the corresponding frequency is called the resonant frequency (natural frequency of the system).

What happens if we try to create a standing wave where $\lambda \neq \frac{2L}{n}$? In short, we cannot. The system will reject any attempts to do so.

Energy

Energy is proportional to ω^2. Energy can only take certain discrete values (corresponding to $f_1, f_2, \ldots, f_n$), we find that the system has quantised possible values for energy.

To generate a wave with a higher frequency we either have to use a shorter L, or a higher v. A higher v is achieved by using a lighter string or placing the system under higher tension.

Sound Waves

Notation

Displacement of a sound wave is denoted:

$$s(x,t) = S_m \cos(kx - \omega t)$$

And pressure is given by:

$$\Delta P(x,t) = \Delta P_m \sin(kx - \omega t)$$

Different Boundary Conditions

The equations for standing waves given is only true for the boundary conditions of both ends fixed. If we vary these (for example left end fixed, right end not, wave initially travelling left) we get a different solution. For example, the first harmonic:

$$L = \frac{1}{4\lambda_1}$$

$$f_1 = \frac{v}{4L}$$

Where the left end forms a node (as required by boundary conditions) and the right end forms an antinode, as it is free to move. For the third harmonic:

$$L = \frac{3}{4\lambda_3}$$

$$f_3 = \frac{3v}{4L} = 3f_1$$

And fifth:

$$L = \frac{5}{4\lambda_5}$$

$$f_5 = \frac{5v}{4L} = 5f_1$$

Notably, this system cannot support even harmonics.

Thu 04 Sep 2025 01:00

Lecture 7 - Energy and Power of Waves

Kinetic Energy for a Sine Wave

Assuming a sine wave created by a harmonic oscillator.

$$KE = \frac{1}{2}mv^2$$
$$(KE)_{\text{max}} = \frac{1}{2}mv_{\text{max}}^2$$
$$(KE)_{\text{max}} = \frac{1}{2}(\mu dx)(\omega A)^2$$

And then to get the total wave energy, we can integrate over lamda: TODO

And Power

$$\text{Power} = \frac{\text{Energy}}{\text{Time}}$$
$$P = \frac{\frac{1}{2}(\omega A)^2\mu\lambda}{T}$$
$$= \frac{1}{2}(\omega A)^2\mu v$$

Example

Considering a standing wave with $\lambda = 2L$:

$$y_t = y_1 + y_2$$
$$\implies (KE)_t = (KE)_1 + (KE)_2$$

Where y_1 and y_2 are identical except for their direction, but they carry the same kinetic energy. The KE of a standing wave is the sum of the two waves that make it up.

Note: Normally, A is the amplitude of the travelling wave (hence 2A is the amplitude of the standing wave created by them), however it is sometimes ambigous what is being referred to by A.

Interference

Superposition: When waves overlap in the same region, the resulting wave is the algebraic sum of waves (they interfere)

Consider two waves:

Thu 23 Oct 2025 13:00

Lecture 8 - EM Standing Waves and Lasers

Lasers

A traditional laser has two mirrors on either side of a cavity. One is (ideally) 100% reflecting, while another is almost perfectly reflecting (say 99% reflective), but allows some transmission. This causes EM waves to reflect back and forth (with some transmitted to actually cause the visible laser).

These reflecting waves cause a standing wave inside the laser cavity. This standing wave oscillates with the equation given in Lec 07.

A Quick Audio Interlude

Listening to various sounds from a frequency generator, we notice two things:

- The human ear is very sensitive to changes in frequency, even in the single Hz range.
- A square wave has a higher percieved pitch than a sine wave of the same frequency.
  - This is because a sqaure wave can be decomposed into the sum of many sine waves, and some of these sine wave components have a higher frequency compared to the square wave itself, which out ears can detect.

And Back to Superposition

Given two travelling waves:

$$\begin{aligned} y_1 &= A\cos(k_1 x - \omega_1 x) \\ y_2 &= A\cos(k_2 x - \omega_2 x) \\ y &= y_1 + y_2 \end{aligned}$$

Hence:

$$y = A\cos(k_1 x - \omega_1 x) + A\cos(k_2 x - \omega_2 x)$$

And using a double angle formula:

$$y = 2A\cos\left(\frac{k_1 - k_2}{2}x - \frac{\omega_1 - \omega_2}{2}t\right)\cos\left(\frac{k_1 + k_2}{2}x - \frac{\omega_1 + \omega_2}{2}t\right)$$

Or briefly:

$$y = 2A\,{\color{blue}\cos(\Delta k x - \Delta\omega t)}\,{\color{red}\cos(k_{avg}x - \omega_{avg}t)}$$

Noting that we define the delta portions as including the /2.

We define the blue portion as the group and the red portion as the carrier. These are important and will be covered properly later.

We define the phase velocity as:

$$v_{\text{phase}} = \frac{\omega_{avg}}{k_{avg}}$$

And the group velocity as:

$$v_{\text{group}} = \frac{\Delta\omega}{\Delta k}$$

Carriers and Groups

What actually is a carrier and a group? Lets take radio as an example. Radio waves travel for potentially hundreds of kilometers, and it's not possible to send sound waves remotely close to that distance because of high attenuation.

The idea therefore arose of using a radio frequency wave to carry sound. Lets consider an example EM RF wave at $500kHz$. We then approximate our sound wave as a sine wave of pressure variations, say at $1000Hz$.

The first thing we can do is to modulate the amplitude of the RF wave by the audio wave. The RF wave keeps the same frequency, but changes amplitude depending on the amplitude of the audio wave.

Upon receipt of the wave, we filter out the high frequency oscillations of the radio wave itself, but we extract a lower frequency wave defined by a wave that passes through the maximum points of oscillation. This effectively restores the sound wave.

This is known as AM radio. The carrier is the high frequency wave, and the group is the lower frequency constructed wave.

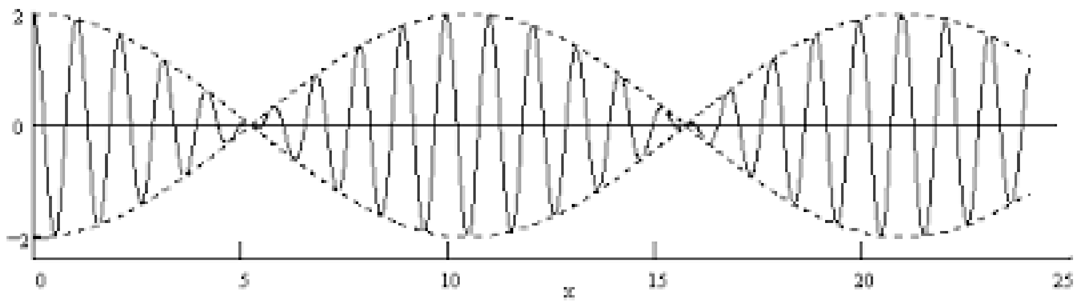


Figure 8.1: Carrier and Group waves. The lower freq. dotted line is the extracted wave (group), the higher frequency wave is the carrier.

Velocities

Which of the two waves have a higher speed?

[TODO]

In a medium where the wave speed depends on frequency, we call this medium dispersive.

Air is non-dispersive (i.e. sound), and in a vacuum the same applies (i.e. speed of light).

What about the speed of light in glass? Does this depend on frequency? We'll investigate later in the optics section.

A Few More Definitions

Coherence: Two (or more) sources that are in phase, or have a constant phase difference (Δ) are said to be coherent.

Adding Coherent Waves

Constant phase difference means we can nicely add amplitudes, so total intensity:

$$I = |A|^2 = (A_1 + A_2 + A_2, \ldots, A_n)$$

Adding Incoherent Waves

TODO

The Decibel Scale

$$\beta = 10 \log \left(\frac{I}{I_0} \right)$$

Where I is the intensity of the source, and I_0 is a reference level approximately at threshold of hearing equal to $10^{-12} W/m^2$

At the threshold of hearing:

$$\beta = 10 \log \left(\frac{10^{-12}}{10^{-12}} \right) = 10 \log(1) = 0dB$$

Crucially, 0dB is not zero sound intensity.

At the upper pain threshold:

$$\beta = 10 \log \left(\frac{1}{10^{-12}} \right) = 120dB$$

But, how do we calculate I here?

$$I = \frac{\text{Power}}{\text{Area}}$$

Power is related to energy, so:

$$I = \frac{\text{energy}}{\text{time} \times \text{area}}$$

And to get intensity for an actual volume:

$$I = \frac{energy \times length}{time \times volume}$$

$$I = \frac{energy}{volume} \times wavespeed$$

What displacement on the eardrum do we actually have? For 120dB, 10^{-5}m, or 10^{-11} for 0dB. The latter measurement is smaller than an atom. The ear is a very sensitive instrument.

Human Senses

Interestingly, we see, hear and smell all on a log scale. Physical feeling is not very scientific and can't easily be quantified. Emotional feeling is also not quantifiable...

Thu 30 Oct 2025 13:00

Lecture 9 - Optics Part Two

Key Principles

There are two main principles that we'll use:

1. **Hugen's Principle**
   - Each point on a wavefront serves as a source of spherical secondary *wavelets* that advance with speed and frequency identical to the primary wave.
   - If we consider each point on a wave as a spherical wavelet source, the wavefront is given by a line tangent to all of these.
2. **Fermat's Principle**
   - The actual path taken by a beam of light is the one which takes the least time to traverse.
   - i.e. $dt/dl = 0$
   - Light could theoretically take an infinite number of possible routes between any two points, but in practice (in the same material) it will travel in a straight line as this is the fastest route.
   - When light is travelling through an interface (i.e. an air to glass boundary), the fastest point is no longer a straight line, as refraction appears.

Reflection

Lets consider some mirror as a reflecting surface, with light travelling from point A to point B. There are multiple different theoretical paths that the light could take, so Fermat's principle requires the shortest possible time taken.

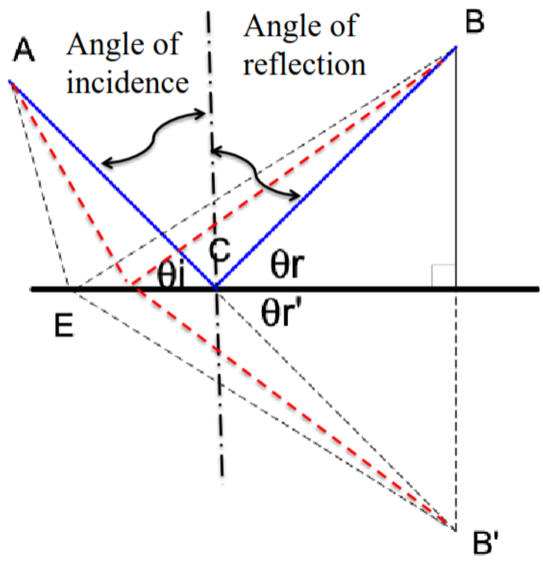


Figure 9.1

We require, by Fermat's Principle (as same material, so shortest time is simply shortest path):

$$AC + CB = \text{min}$$

By considering congruent triangles (let O be the midpoint of B and B'), therefore:

$$AC + CB = AC + CB'$$

Provided ACB' is a straight line, $\theta_i = \theta_r (= \theta_{r'})$. Therefore the angles of incidence and reflection must be equal.

Refraction

When light is travelling through a material, the speed of light in each is different. Therefore, a straight line is no longer the fastest path.

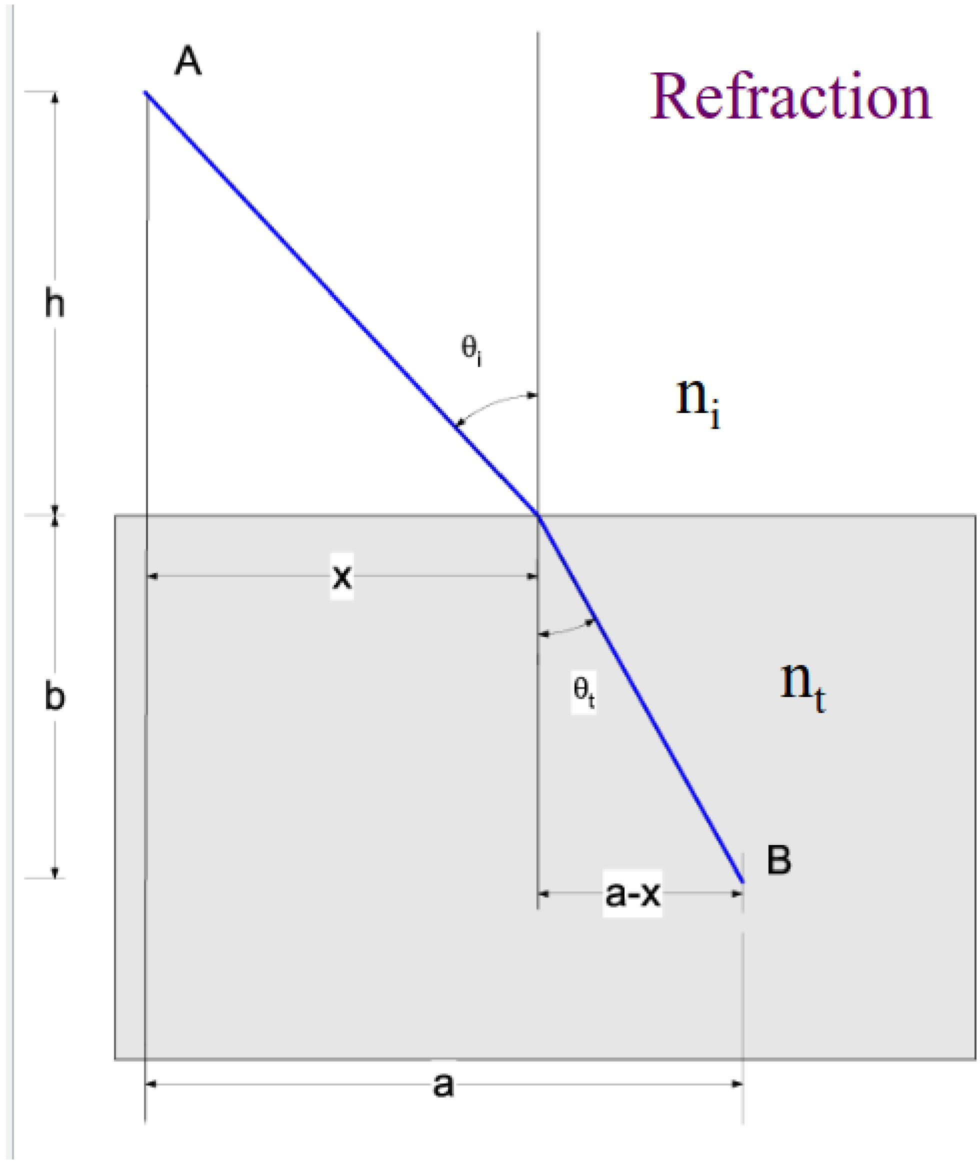


Figure 9.2

Where:

- n_i is the incident index.
- n_t is the post-refraction index.

- θ_i is the angle of incidence.
- θ_t is the angle of refraction.
- Light is travelling from point A to point B.
- Distances are given per diagram.

The aim is to be able to find the true trajectory (per Fermat) that takes the shortest amount of time. By s = d/t, for given distance values:

$$t = \frac{\sqrt{h^2 + x^2}}{v_i} + \frac{\sqrt{b^2 + (a - x)^2}}{v_t}$$

Where v_i, v_t are speeds of light in the respective media. t is a function of x, with constants, so to find a min value we differentiate wrt x and solve for minima:

$$\frac{dt}{dx} = \frac{x}{v_i\sqrt{h^2 + x^2}} + \frac{-(a - x)}{v_t\sqrt{b^2 + (a - x)^2}} = 0$$

Subsituting values for distances, we get:

$$\frac{\sin\theta_i}{v_i} = \frac{\sin\theta_t}{v_t}$$

And since $n = c/v$, we get:

$$n_i \sin\theta_i = n_t \sin\theta_t \tag{9.1}$$

The shortest (wrt time) path must have angles which satisfies this. This is called Snell's Law.

Connecting Fermat and Huygen

Speed of light in a medium is less than in a vacuum, this is categorised by the index of refraction, n:

$$n = \frac{c}{v}$$

Where c is speed of light in a vacuum, and v is speed of light in the medium. i.e. for water, $n = 1.333$, for air, $n = 1.0003$.

Suppose we have a wavefront travelling as a series of wavelets. As soon as each wavelet hits the glass, they slow down. This leads to an uneven distribution of speeds across the wavelength, as some have slowed down and some have not. Therefore, the wave bends around and we get a new wavefront at a different angle to the previous wavefront.

Note the condition where the light travels perpendicular to the interface, i.e. $\theta_i = \theta_t = 0$. In this case, all wavelets hit the medium and change speed at exactly the same time, therefore there is no difference in velocity across the wavefront, so no direction change.

But how?

How does light actually "choose" which path to take? Apparently it just does and we have to wait to find out...

Combining Reflection and Refraction

When light strikes a boundary surface, there is two components - both reflected and transmitted (just like on a string). To determine this, we have to use the wave equation:

$$E = E_0 \cos(kx - \omega t)$$

Noting that instead of traditional mechanical amplitude we use the amplitude of the electromagnetic field at this point, E_0. For now, we ignore polarisation as a potential scenario.

More Index

We note that since $v = f\lambda$, the index of refraction will change depending on the frequency of light:

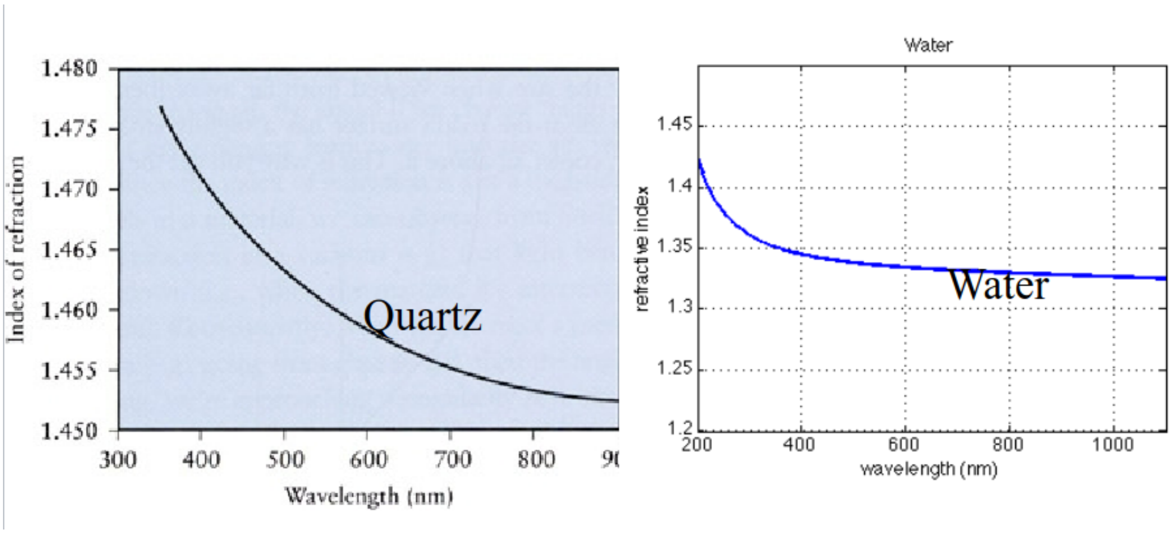


Figure 9.3

When we shine light of multiple frequencies (i.e. white light), the higher frequency light bends further than the lower frequency light. This causes dispersion of light, i.e. the formation of a rainbow out of a prism. This arises because each frequency has a different speed and therefore a different change in θ.

If we shine pulses of light of multiple frequencies down a fibre optic line, the higher frequencies will have a higher n, hence a lower speed and will arrive later.

This allows us to explain the formation of sunsets and rainbows etc. In the case of a sunset, when we see the sun just above the horizon, the sun has actually set just below the horizon. We cannot therefore see the sun via direct line of sight, and yet we can still see it as if we could?

This is because the light from the sun is refracted and bends towards us. The atmosphere is higher density at the bottom, and lower density as altitude increases. The amount of refraction depends on density, hence changing the index of refraction. We can say that n is a function of y, where y is height.

We imagine it as being actually present precisely where we see it, because our brain does not account for this refraction and extrapolates the light as a straight line. We therefore see the sun has just about to set, when in reality the sun physically has set below the horizon.