# Lecture Notes

Year 1 Semester 2

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# Table of Lectures

# LC Classical Mechanics and Relativity 2

**Tue 17 Feb 2026 11:00**

# Lecture 9 - Special Relativity I: Foundations and Time Dilation

## 1 Introduction

In this lecture:

- Recap from CMR1
- Einstein's two postulates
- Time dilation
- Galilean Transformations

Crucially, there are no special or absolute frames of reference - laws of motion are only sensible when we consider a frame of reference. The only slightly special frame is the one in which we are currently stationary.

Special relativity provides a theory of relative motion between inertial frames of reference. An inertial frame is one which is not accelerating relative to the other frames being considered.

## 2 Frames of Reference

We will generally consider two related frames of reference, denoted $\Sigma$ and $\Sigma'$. $\Sigma$ is our "stationary" frame, i.e the frame taken by an observer sat on earth. $\Sigma'$ is our "moving frame", relative to the stationary observer and is moving with constant speed $v$.

Coordinates in the stationary frame are $(x, y, z, t)$, while in the moving frame they add a prime, so are given by: $(x', y', z', t')$.

We want to compare observations of position/experience of time/velocity in the moving frame with observations of the same categories in the stationary frame.

The frame in which an object is stationary is denoted its rest frame $\Sigma_0$.

## 3 Galilean Transformations

**"Galilean Invariance":** The laws of physics are invariant in all inertial (non-accelerating) frames.

**"Galilean Transformation":** Time is invariant and universal in all frames.

In Classical Mechanics, we treat time as invariant, but this stops being true in special relativity. We can transform between frames in such a manner that time is kept constant, this is a Galilean Transformation, but this is not true by default.

Consider our two reference frames $\Sigma$ and $\Sigma'$, where the latter moves with speed $v$ relative to the former. Our coordinates in $(x, y)$ become coordinates in $(x', y')$.

An object moves at speed $u'$ in the x-direction in reference frame $\Sigma'$. If we assume that time is invariant (as we're still currently doing classical physics without having introduced special relativity), then:

$$t = t'$$

And as the motion is entirely in the x-axis:

$$y = y'$$

After time $t$:

$$x = x' + \text{distance moved by } \Sigma' \text{ in time } t \text{ relative to } \Sigma$$

$$x = x' + vt$$

The object velocity is defined as:

$$u = \frac{dx}{dt} = \frac{d}{dt}(x' + vt)$$
$$= \frac{dx'}{dt} + v = u' + v$$

This holds if any only if time is the quantity we treat as being invariant. This agrees with our classical understanding.

### 3.1 Where does this break down?

Lets assume that the moving object is a photon, moving in $\Sigma'$ with velocity $c'$, hence, according to the previous derivation:

$$c = c' + v$$

Therefore observers in different frames will measure different values for the speed of light…oh no!

If we assume Galilean invariance, the laws of physics are the same in all reference frames, and yet the speed of light is included as a constant in many laws (i.e. electromagnetism). Therefore observers must measure the same speed of light in all reference frames.

This is a contradiction - we cannot assume that both time and the laws of physics are invariant.

### 3.2 Experimental Verification

The earth travels around the sun extremely quickly, and the sun is travelling even faster around the galactic centre. Therefore, the earth is moving in space.

If Galilean relativity is correct, we would measure the speed of light in one direction as $c + u$ and measure the speed of light in the other direction as $c - u$.

This was tested by the Michelson-Morley experiment, where incoming light was split by a half-silvered mirror. Half the light travels in one direction, and half the light is split off by 90°. They reflect off a pair of mirrors and are recombined in a splitter to be observed.

If the speed of light was different in different directions, the two beams would be out of phase and we would observe an interference pattern on combination. We would expect to see phase difference that varies with time, i.e. turning from destructive to constructive etc.

This was not observed and the interference pattern generated was constant. Therefore, the results showed no variation in the speed of light.[1]

## 4 The Solution - Einstein's Special Relativity

To fix this contradiction, Einstein came up with two postulates:

1. The laws of physics are the same in any inertial frame of reference.
    - This is the same as Galilean invariance and is easily believable.
2. The speed of light is constant in every frame of reference.
    - This was revolutionary and is much less intuitive.

The last postulate is difficult to understand intuitively, but makes everything work if we accept it as true!

[1]This is the same idea used in the LIGO experiment to discover gravitational waves, except this was used to show that the path length changed, and not the speed of light. If the path lengths changed, this was due to a gravitational wave (ripple in space time) propagating to the earth and interfering with the measurement.

## 4.1 Proving Time Dilation

Consider an observer on the earth in frame $\Sigma$. This observer watches two rockets both travelling side by side away from the earth in the x-direction with speed $v$. They are both travelling in $x$, with a constant y-difference $y_0$ between them.

Suppose the upper rocket A fires a laser beam directly at B (i.e. directly in the y-direction downwards).

**Frame $\Sigma$:** This is the rest frame of the earth (and the observer on earth) with coordinates $(x, y)$.

**Frame $\Sigma'$:** This is the rest frame of the rockets, with coordinates $(x', y')$.

In the rockets rest frame $\Sigma'$, the rockets are stationary and the time taken for a laser pulse to travel between them is:

$$t_0 = \frac{y_0}{c}$$

Or equivalently:

$$y_0 = ct_0$$

In the earth's rest frame $\Sigma$, the observer on the earth doesn't just see the light travelling in the y-direction. It also sees the light moving in the x-direction, as the whole $\Sigma'$ frame containing the rockets relativistically move away.

Effectively, we have:

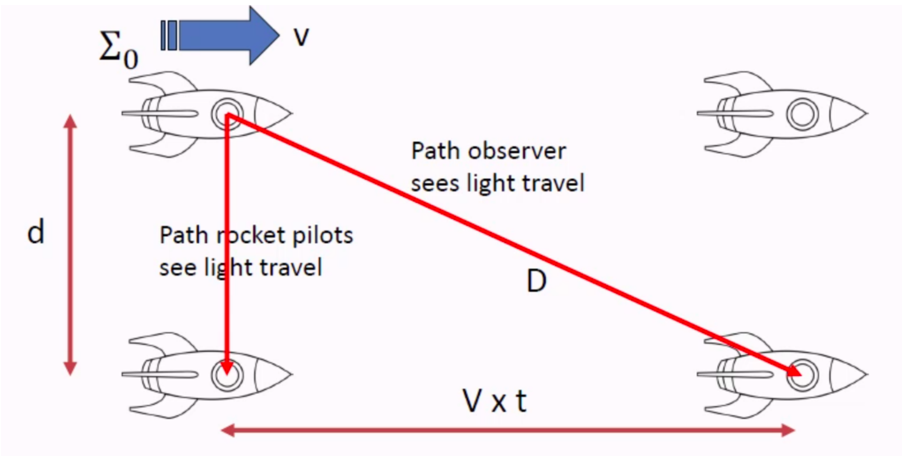



Figure 1.1

B moves distance $x$ horizontally while it waits for the laser to hit it, and the length of the path of the laser beam in time $t$ is given by $D$. If the time taken for light to reach B in $\Sigma$ is $t$, then:

$$D = ct$$

As the speed of light is constant in all frames. As the rockets are moving away:

$$x = vt$$

This gives us a Pythagorean triangle, where:

$$D^2 = x^2 + y^2 = x^2 + y_0^2$$

And substituting in:

$$c^2t^2 = v^2t^2 + c^2t_0^2$$
$$t^2(c^2 - v^2) = t_0^2c^2$$
$$t = t_0\sqrt{\frac{c^2}{c^2 - v^2}}$$

Dividing through by $c_2$:

$$t = t_0\sqrt{\frac{1}{1-\frac{v^2}{c^2}}}$$

Or:

$$t = t_0\gamma, \qquad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$\gamma > 1$ for all object speeds that don't exceed the speed of light (which is required), so the time observed in the frame $\Sigma$ is longer than the "proper time" observed by the rocket.

# 5 The Twin Paradox

Consider two twins, an astronaut and a physics professor. The astronaut goes on a long trip close to the speed of light and reunites with his brother. Which brother is older?

Special relativity says that because there is no absolute frame of reference, the brother travelling away also sees the same problem in reverse, i.e. he sees the physics professor brother travelling away in the opposite direction at equal and opposite speed, i.e. they are the same age.

In practice, the astronaut is non-inertial and special rel isn't sufficient, so we can't solve it directly.

**Thu 19 Feb 2026 15:00**

# Lecture 10 - Special Relativity II: Length Contraction

## 1 Proving Length Contraction

Consider a horizontal laser cavity. We fire a laser from the left of the cavity (point A) to the right of the cavity (point B). The laser bounces off a mirror at point B and bounces back.

The cavity has length $L_0$ as measured in the rest frame $\Sigma_0$. In this frame:

$$t_0 - \frac{2L_0}{c}, \qquad \text{Where } t_0 \text{ is the time taken for the A->B->A journey in } \Sigma_0$$

Now suppose the cavity moves at speed $v$ relative to a stationary observer in a different frame ($\Sigma$).

The observer sees the cavity as having length $L$, how long does the A->B->A journey take as observed by a stationary observer in $\Sigma$.

$$t = t_{A\to B} + t_{B\to A}$$

The cavity is moving away from the observer, so the laser beam doesn't just travel $L$ when going from $A \to B$. It must travel $L$ plus the distance the whole cavity has moved away in this time (as $B$ is moving away from the pulse). This extra distance is $vt_{A\to B}$, hence the distanced travelled when going from A to B is $L + vt_{A\to B}$.

As $c$ is constant in all frames, we can say:

$$c = \frac{L + vt_{A\to B}}{t_{A\to B}}$$
$$t_{A\to B}(c - v) = L$$
$$t_{A\to B} = \frac{L}{c - v}$$

And now on the return journey from $B \to A$, as it travels back towards $A$, the cavity is moving in the same direction as the light, so $A$ "catches up" to the pulse and results in a smaller required distance to be travelled, $L - vt_{B\to A}$

$$c = \frac{L - vt_{B\to A}}{t_{B\to A}} \implies t_{B\to A} = \frac{L}{c + v}$$

And returning to the total time:

$$t = t_{A\to B} + t_{B\to A}$$
$$t = \frac{L}{c - v} + \frac{L}{c + v}$$
$$= L\left(\frac{1}{c - v} + \frac{1}{c + v}\right)$$
$$= L\left(\frac{c + v + c - v}{(c - v)(c + v)}\right)$$
$$= L\frac{2c}{c^2 - v^2}$$
$$= \frac{2L}{c}\frac{c^2}{c^2 - v^2}$$
$$= \frac{2L}{c}\frac{1}{1 - \frac{v^2}{c^2}}$$

$$= \frac{2L}{c}\gamma^2$$

We now apply time dilation, which says that $t = \gamma t_0$, hence:

$$\gamma t_0 = \frac{2L}{c}\gamma^2$$

$$t_0 = \frac{2L}{c}\gamma$$

From the rest frame, we know that $t_0 = 2L_0/c$, so:

$$\frac{2L_0}{c} = \frac{2L\gamma}{c}$$

$$L_0 = L\gamma$$

As $\gamma > 1$, $\forall(u < c)$, the proper length measured from the rest frame will always be greater than the length measured by an observer, hence length is contracted for a moving object.

# 2 Example

In a particle accelerator, protons are accelerated to a measly $0.9c$. These protons pass through a tunnel of length 2km as viewed from the a laboratory rest frame at the accelerator.

Recall that:

$$t = \gamma t_0$$

$$L = \frac{L_0}{\gamma}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \text{where: } \beta = u/c$$

## 2.1 How long would the journey take, according to the rest frame?

Viewed in the lab frame, and $v = d/t$ so:

$$t = \frac{2 \times 10^3\text{m}}{0.9 \times 3 \times 10^8} = 7.4 \times 10^{-6}\text{s} = 7.4\mu\text{s}$$

No relativity needed!

## 2.2 How long would the journey take, according to the proton's frame?

We do need to consider relativity here, and use time dilation:

$$t = \gamma t_0$$

$$\gamma = \frac{1}{\sqrt{1-0.9^2}} = 2.3$$

$$t = \gamma t_0 \implies t_0 = \frac{t}{\gamma} = \frac{7.4\mu\text{s}}{2.3} = 3.2\mu\text{s}$$

## 2.3 How long is the tunnel, according to the proton's frame?

The rest frame now corresponds to the tunnel. It has proper length in its rest frame of 2km.

As viewed by the protons in their rest frame, it is the tunnel that is moving and is rushing towards them at $0.9c$. This is therefore a length contraction problem.

$$L = \frac{L_0}{\gamma} = \frac{2\text{km}}{2.3} = 870\text{m}$$

### 2.4 Checking Values

To check, we can reuse $v = d/t$ but this time in the proton rest frame.

$$d = 870\text{m}, \quad t = 3.2 \times 10^{-6}\text{s}$$

Hence

$$v = \frac{870\text{m}}{3.2 \times 10^{-6}\text{s}} = 2.7 \times 10^{8}\text{ms}^{-1} = 0.9c \text{ as required!}$$

## 3 Example II: Cosmic Rays

The highest energy cosmic rays are protons with massive energies $E \sim 10^{20}$eV, compared to the LHC with $E \sim 10^{12}$eV

This corresponds to a $\gamma = 10^{11}$, and our galaxy is $\sim 10^{20}$m across. [1]

As viewed on earth, these cosmic protons travel at $v \approx c$, hence,

$$t \approx \frac{d}{v} \approx \frac{10^{20}}{3 \times 10^{8}} = 3 \times 10^{11}\text{s} \approx 10^{4}\text{years}$$

As viewed by the protons:

$$t_0 = \frac{t}{\gamma} = \frac{3 \times 10^{11}}{10^{11}} \approx 3\text{s}$$

Which is starkly different!

## 4 Lorentz Transformations

In general, transforming between coordinates in two different frames can get extremely messy, more than can easily be handed in simple applications of length contraction or time dilation.

The Lorentz Transformations provide a general set of coordinate transformations between $\Sigma$ and $\Sigma'$, i.e. $(x, y, z, t) \to (x', y', z', t')$.

These transformations must be:

1. Symmetric about a change in sign of $u$, i.e. the transformation from $\Sigma \to \Sigma'$ with $u$ and the transformation from $\Sigma' \to \Sigma$ with $-u$ must be the same.
2. They must be linear as otherwise a constant speed in one would not be constant in the other (and therefore not inertial frames).
3. If relative velocity is only in the $x$ direction, the transformations must be independent of $y$ and $z$.

These transformations are (assuming motion only in $x$):

$$x = \gamma(x' + vt')$$
$$y = y'$$
$$z = z'$$
$$t = \gamma\left(t' + \frac{x'v}{c^2}\right)$$

And:

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{xv}{c^2}\right)$$

We do not need to know the derivations, but do need to know the results.

[1] $\gamma = E/mc^2$ - will be expanded on in later lectures.

**Tue 24 Feb 2026 11:00**

# Lecture 11 - Special Relativity III: Lorentz Transformations

Recap of the first two relativity lectures:

- Recap from Semester 1 Special Rel:
  - Background
  - Einstein's postulates.
  - Time dilation.
  - Lorentz contraction.
- Galilean transformations and when they fail.
- Lorentz transformations for space and time.

## 1 Galilean vs. Lorentz Transformations

In a standard setup with two frames, $\Sigma$ and $\Sigma'$, where $\Sigma'$ moves with speed $v$ relative to $\Sigma$.

In $\Sigma$, we have coordinates $(x, y, t)$, in $\Sigma'$ we have coordinates $(x', y',')$.

In a Galilean transformations, we kept time invariant across both frames, so $t = t'$. This lead to $x' = x - vt \implies u' = u - v$ but raised a contradiction with the speed of light as observers in different frames would take two measurements of the speed of light.

In the second lecture, we consider the speed of light as being invariant across all frames instead of time to resolve the contradiction. This gave us the Lorentz Transformations:

$$x' = \gamma(x - vt)$$

$$t' = \gamma\left(t = \frac{xv}{c^2}\right)$$

Where $y = y'$, $z ='$.

As we cannot use $u' = u - v$ due to the contradiction with the speed of light, we in this lecture will consider:

- Lorentz transformations for velocity.
- Space-time, Lorentz invariance, causality.

## 2 Lorentz Transformation for Velocity

We can revisit the question from the first lecture, but in the Lorentz transformation perspective rather than a Galilean perspective.

Again, we have our two standard frames $\Sigma, \Sigma'$. An object is moving with speed $u'$ in frame $\Sigma'$. We want to find an expression for the velocity of the object as viewed from $\Sigma$, given by $u$. The prime frame moves at speed $v$ wrt the rest frame and the objects motion is entirely along the $x$ acis.

By definition, we have:

$$u = \frac{dx}{dt} = \frac{dx}{dt'}\frac{dt'}{dt} = \frac{\left(\frac{dx}{dt'}\right)}{\left(\frac{dt'}{dt}\right)} \tag{1}$$

And:

$$\frac{dx}{dt'} = \gamma\left(\frac{dx'}{dt'} + v\right) = \gamma(y' + v) \tag{2}$$

$$\frac{dt}{dt'} = \gamma\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right) = \gamma\left(1 + \frac{u'v}{c^2}\right) \tag{3}$$

And substituting (3) and (2) into (1):

$$u = \frac{\gamma(u'+v)}{\gamma\left(1 + \frac{u'v}{c^2}\right)} \implies \boxed{u = \frac{u'+v}{1 + \frac{u'v}{c^2}}}$$

To get the reverse transformation, we invoke forward-backward symmetry, i.e. swap $x \to x'$, $t \to t'$ etc, where $v \to -v'$:

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

## 2.1 Limiting Cases

Firstly, we'll look at the non-relativistic case, i.e. where $u, v \ll c$:

$$u = \frac{u'+v}{1 + \frac{u'v}{c^2}} = \frac{u'+v}{1 + \text{small}} \approx u' + v$$

This is the classical Galilean approach, which is what we'd expect for non-relativistic behaviour.

What about an ultra-relativistic limit where $u, u' \to c$ with $v \ll c$. Now we have:

$$u = \frac{u' + \text{small}}{1 + \text{small}} \approx u'$$

As the speed of light is invariant, speeds close to the speed of light do not change between reference frames.

## 2.2 Example

An observer on a rocket sees a second rocket moving away from them at $v_1 = 0.8c$. Another observer on a planet sees the first rocket moving at $0.7c$. Assuming all motion is in the same direction, how fast does the planetary observer see the second rocket?

- From $\Sigma'$ (the first rockets rest frame), the second rocket moves with $0.8c$.
- From $\Sigma$ (the planetary rest frame), the first rocket moves with $0.7c$, and the second rocket moves with some unknown speed $u$.

We apply the formula:

$$u = \frac{u'+v}{1 + \frac{u'v}{c^2}}$$

In $\Sigma'$, $u' = 0.8c$. The two frames move with relative velocity equal to the change in speed of the first rocket from $\Sigma \to \Sigma'$: $v = 0.7c$.

In frame $\Sigma$ we are tying to do:

$$u = \frac{0.8c + 0.7c}{1 + \frac{0.8c \times 0.7c}{c^2}}$$

$$= \frac{1.5c}{1.56} = 0.96c$$

# 3 Space-Time and Lorentz Invariants

From length contraction and time dilation, we can see that in special relativity, intervals in time and space are not fixed but vary from frame-to-frame.

This means that the order of events may even be different for different observers.

There are some quantities however which are invariant across frames, such as the speed of light. One of these is space-time.

## 3.1 Space-Time Invariance

As usual, we have $\Sigma$ and $\Sigma'$ with relative speed $v$. The axes of these frames coincide at $t = 0, x = 0$ and $t' = 0, x' = 0$.

At $t = t' = 0$, there is a flash of light at the origin which propagates isotropically.

In $\Sigma$, the light forms a spherical shell of radius $\Delta r$ at time $\Delta t$, and in $\Sigma'$ it reaches a radial distance $\Delta r'$ at time $\Delta t'$.

After time $\Delta T$ in $\Sigma$ we have $\Delta r = c\delta T$. Similarly in $\Sigma'$ we have $\Delta r' = c\Delta t'$.

Squaring these we have:

$$(\Delta r)^2 = c^2\Delta t^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$$
$$(\Delta r')^2 = c^2\Delta t'^2 = \Delta x'^2 + \Delta y'^2 + \Delta z'^2$$

Hence:

$$c^2\Delta t - \Delta x^2 - \Delta y^2 - \Delta z^2 = 0 = c^2\Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2$$

This means that, **for light**:

$$c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = 0$$

We call this quantity $\Delta s^2$:

What about for objects moving at a speed less than the speed of light? We want to apply the Lorentz transformations to $\Delta s'2$ (again treating the "boost" as being in $x$):

$$\begin{aligned}
\Delta s'^2 &= c'^2\Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2 \\
&= c^2\gamma^2\left(\Delta t - \frac{v}{c^2}\Delta x\right)^2 - \gamma^2\left(\Delta x - v\Delta t\right)^2 - \Delta y^2 - \Delta z^2 \\
&= c^2\gamma^2\left(\Delta t^2 + \frac{v^2}{c^4}\Delta x^2 - 2\frac{v}{c^2}\Delta t\Delta x\right) - \gamma^2\left(\Delta x^2 + v^2\Delta t^2 - 2v\Delta x\Delta t\right) - \Delta y^2 - \Delta z^2 \\
&= \Delta t^2\left(c^2\gamma^2 - v^2\gamma^2\right) + \Delta x^2\left(\frac{v^2\gamma^2}{c^2} - \gamma^2\right) + \Delta t\Delta x\left(-2v\gamma^2 + 2v\gamma^2\right) - \Delta y^2 - \Delta z^2 \\
&= \Delta t^2\gamma^2\left(c^2 - v^2\right) + \Delta x^2\gamma^2\left(\frac{v^2}{c^2} - 1\right) - \Delta y^2 - \Delta z^2 \\
&= \Delta t^2\frac{c^2 - v^2}{1 - \frac{v^2}{c^2}} + \Delta x^2\frac{(\frac{v^2}{c^2}) - 1}{1 - \frac{v^2}{c^2}} - \Delta y^2 - \Delta z^2 \\
\Delta s'^2 &= \boxed{c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = \Delta s^2}
\end{aligned}$$

Hence $\Delta s^2$ is invariant under the Lorentz transformations regardless of speed.

# LC Electric Circuits

**Fri 13 Feb 2026 11:00**

# Lecture 3

**Fri 20 Feb 2026 11:00**

# Lecture 5

# LC Electromagnetism I

**Mon 19 Jan 2026 11:00**

# Lecture 1 - EM1 Intro and Electric Fields

## 1 Course Intro

Course Materials:

- Background material and derivations etc on PowerPoint.
- Worked examples etc are handwritten on visualiser, these are the bits we really need to know.

Why is EM important?

- Foundations of modern technology and the modern world.
- What gives elements their properties.
- Responsible for life itself.
- Everyday materials are held together by EM forces.
- Optics can only be understood through EM theory.

The course aim is to lay down the foundations, eventually leading us to Maxell's Laws.

### 1.1 Maxwell's Laws

Maxwell's four equations are:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \wedge \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \wedge \boldsymbol{B} = \mu_0 \left( \boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right)$$

Where:

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$$

Together, these show that the electric and magnetic fields are related and are two aspects of a single force, the electromagnetic force. We don't have to properly understand them yet, but cannot learn them in EMII unless we fundamentally understand E and B fields from this module.

### 1.2 Course Structure

**Part I: Electric Fields**

- Charge and Coulomb's Law.
- The electric field.
- Gauss' Law.
- Capacitors.

**Part II: Magnetic Fields**

- Magnetic Fields

- Charged Particles in B-Fields
- Electromagnetic Induction.
- Magnetic Dipoles

In this lecture:

- Introduction to EM.
- Electric charge.
- Force between charges.
- The concept of the Electric Field (E-Field).

# 2 Electric Charge

First attributed to Thales circa. 624 - 546 BC. Experiments by Franklin and Coulomb expanded and showed that there was two types of charge, which they called positive and negative.

The "positive electricity" came from rubbing a glass rod with silk, and the negative from rubbing an ebonite (early plastic) rod with fur. They found that like charges repel and opposite charges attract.

We know that the elementary unit charge is the magnitude of charge of an electron/proton and everything else is a multiple of this [1]:

$$e = 1.6 \times 10^{-19} C$$

This has units of the Coulomb.

## 2.1 Charge Conservation

Electrons and protons are both stable (protons decay with a life greater than $10^{31}$ years). This means that the total charge of an isolated system is constant and can be conserved.

They have the same magnitude of charge, exactly:

$$|q_p| = |q_e| = e$$

## 2.2 Electrostatic Force

Like charges repeal and opposite charges attract, along the line of action given by a line drawn between the two charges. The force is proportional to the product of charges so:

$$F \propto q_1 q_2$$

Here, a negative force means attraction and a positive force means repulsion. Newton called this "force at a distance". Like gravity, two charges will exert a force on each other at a distance without any contact.

There must, therefore, be something between them that mediates this force. Later physics gives this as "virtual particles" which isn't a Y1 topic, so classically we say that this medium is the Electric Field.

## 2.3 Electric Field

A charge produces a field around it. Another charge also interacts with this field, and this interaction is what causes a force:

$$\underline{F} = \underline{E} q$$

Where $F$ is the force exerted on a test charge of charge $q$ by a charge $Q$ producing a field $E$. The magnitude of the electric field has units $NC^{-1}$ (force per units charge).

$$|\underline{E}| \propto Q \qquad |\underline{F}| \propto Qq$$

[1] While quarks have fractional charge, we don't get free quarks

Consider a point charge with a spherical electric field spreading out around it. As the distance from the charge increases, the surface area increases as $4\pi r^2$. Therefore the magnitude of the electric field must decrease with $4\pi r^2$

Therefore:

$$|\underline{E}| \propto \frac{Q}{4\pi r^2}$$

We need a (inverse) constant of proportionality. This depends on the medium, but for a vacuum we call it the permittivity of free space $\epsilon_0$:

$$\epsilon_0 = 8.854 \times 10^{-12} C^2 m^{-2} N^{-1}$$

Hence:

$$\boxed{E = |\underline{E}| = \frac{Q}{4\pi\epsilon_0 r^2}}$$

## 2.4 Direction of the E-Field

Force is a vector, so the E-field must be too. Consider an E-field from a charge $Q$ at distance $r$. We give $E$ components $E_r$, $E_\theta$, $E_\phi$, where, since it's a sphere $\phi$ and $\theta$ represent the unit vectors in the two possible tangential directions.

If there was a component in $E_\theta$ this would be clockwise from one perspective, but anticlockwise from another (walking behind it). This is not possible, as the field must behave in the same manner from all viewpoints. Therefore:

$$E_\theta = E_\phi = 0$$

$$\underline{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{t}$$

## 2.5 Force between two charges

Consider two charges $q_1$ and $q_2$. The force on $q_2$ due to the e-field from $q_1$ is:

$$\underline{F_1} = \underline{E_1} q_2$$

This is equal to the force on $q_1$ due to the e-field from $q_2$, given by:

$$\underline{F_2} = \underline{E_2} q_1$$

So the force between two charges is:

$$\underline{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{r}_{12}$$

## 2.6 Force between many charges

If we have more than two positive charges, we use the "principle of superposition". Effectively, you consider each pair of charges at the same time and vector sum of the forces together. I.e. if we have three points and we care about the net force on one, we take the vector sum of the two vectors from that point to the others.

In general:

$$\begin{aligned} \underline{F} &= \underline{F_1} + \underline{F_2} + \underline{F_3} + \cdots \\ &= q \sum_i \frac{q_i}{4\pi\epsilon_0 r^2}\hat{r_j} \end{aligned}$$

Where $r_j$ is the distance between $q_i$ and $q$, with unit vector $\hat{r_j}$ between them.

Since $\underline{F} = q\underline{E}$, the electric field at a test charge $q$ must be:

$$\underline{E} = \sum_i \frac{q_i}{4\pi\epsilon_0 r_i^2}\hat{r_j}$$

## 2.7 Example

Say we have a square of side length $a$. Clockwise, these corners have charge $Q$, $q$, $-2Q$, $3Q$.

What is the net force exerted on the $q$ charge?

**Thu 26 Feb 2026 11:00**

# Lecture 12 - Currents and Magnetic Force

Today we start Part II, Magnetism:

- Definition of current and current density.
- Magnetic force on a moving charge.
- The Lorentz Force.
- Field lines for a magnetic field.

## 1 Current

### 1.1 Definition of Current

Suppose a current carries a current $I$, this is defined in terms of the rate of flow of charge past a given cross section:

$$I = \frac{dQ}{dt}$$

The SI unit of current is the *ampere*, defined as the flow of one coulomb per second.

### 1.2 Nature of Current

There's two experimentally observed effects of a current flow:

- Heating
- Creation of magnetic fields.

Consider a current through a conductor. This is caused by electrons moving (with average drift velocity $v$) as a result of the induced E-field.

In some time, they travel $v\Delta t$ through the cross section, and if this cross section has area $A$ they sweep out an area of $Av\Delta t$.

If a unit volume has $n$ conducting free electrons, the number of charges is given by:

$$N = n(Av\Delta t)$$

And hence the total charge is:

$$\Delta Q = nAv\Delta T(-e)$$

Hence:

$$\hat{\imath} = \frac{\Delta Q}{\Delta t} = -nAe\underline{v}$$

### 1.3 Current Density

More usefully, we talk about the current per unit cross-section, which is the current density $J$:

$$\hat{\jmath} = \frac{\hat{\imath}}{\underline{A}} = -ne\underline{v}$$

The negative sign arises as we define current as flowing from positive to negative. Electrons, being negatively charged, actually physically flow from negative to positive. Current flow and actual electron flow are therefore in the opposite direction. This is because current was initially understood and defined before the electron was defined.

Typically, $v < 1\text{mms}^{-1}$ when a current is flowing.

# 2 Magnetism

## 2.1 Magnetic Force on a Charge

A magnetic field exerts a force on a moving charge that is present in the magnetic field.

Suppose a particle of charge $+q$ moving with some velocity $\underline{v}$ in a magnetic field $\underline{B}$ experiences some force $\underline{F}_m$.

We cannot derive from first principles, but experimentally have observed:

- $\underline{F}_m \perp \underline{v}, \underline{B}$.
- $\underline{F}_m \propto \underline{v}$.
- $\underline{F}_m \propto \underline{B}$.
- $\underline{F}_m \propto q$

If $\theta$ is the angle between the velocity and the B-field direction, we hae:

$$\boxed{\underline{F}_m = Bqv \sin\theta}$$

In vector form:

$$\boxed{\underline{F}_m = q\underline{v} \wedge \underline{B}}$$

Here, $\underline{B}$ has the unit Tesla, T. In base units this is $\text{NC}^{-1}\text{m}^{-1}\text{s}$. A one Tesla magnetic field is really quite powerful, so we also define the Gauss, $1\text{G} = 10^{-4}\text{T}$ to move to a nicer scale.

By definition, if a 1C charge moving at 1m/s perpendicular to a magnetic field experiences a force 1N, the field is 1T.

- Earths magnetic field is $\sim 0.5\text{G}$
- Poles of a large electromagnetic: $\sim 2\text{T}$.
- Surface of a neutron star: $\sim 10^8\text{T}$.
- The maximum pulsed magnetic field that can create in a lab: $\sim 50\text{T}$.

The Earth's field is believed to be generated by electron currents in the iron alloys in its core. It's not completely understood yet, but convection currents causing these conductive alloys to flow and move is the working theory. This movement causes the North and South poles to swap places on average every $300,000$ years.

## 2.2 The Lorentz Force

In a region where we have both an E-field and a B-field, we have the total force as the vector sum of both:

$$\underline{F} = q\left(\underline{E} + \underline{v} \wedge B\right)$$

This is called the Lorentz force.

Suppose Earth's magnetic field is given by:

$$\underline{B} = B\cos 70^\circ \hat{\jmath} - B\sin 70^\circ \hat{k} \qquad \text{where: } B = 5 \times 10^{-5}\text{T}$$

A proton moves in this field with:

$$\underline{v} = 10^7 \hat{\jmath}\text{ms}^{-1}$$

$$\underline{F}_m = +e\underline{v} \wedge \underline{B} = 1.6 \times 10^{19} \times 10^7 \times \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 0 \\ 0 & B_y & B_z \end{bmatrix}$$
$$= 1.6 \times 10^{-12} \left[\hat{\imath}\,(B_z - 0) + \hat{\jmath}\,(0-0) + \hat{k}\,(0-0)\right]$$
$$= 1.6 \times 10^{-12} B_Z \hat{\imath}$$
$$= -1.6 \times 10^{-12} \times 5 \times 10^{-5} \sin 70° \hat{\imath}$$
$$-7.5 \times 10^{-17} \hat{\imath} \mathrm{N}$$

## 2.3 Magnetic Field Lines

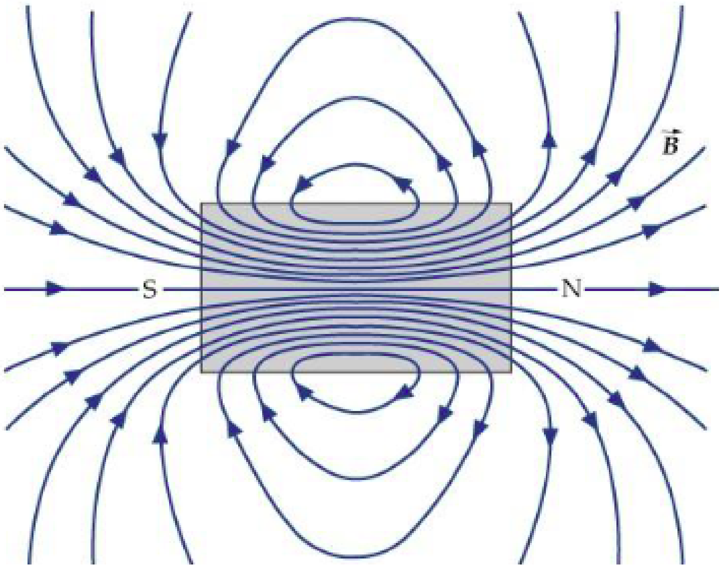



Figure 7.1: Magnetic Field Lines Around a Bar Magnetic

Magnetic field lines are different to electrical field lines. They do not point in the direction of force and travel out of North poles and into South poles. Magnetic field lines are always continuous so there is no magnetic monopoles.

Physicists have searched for a potential magnetic monopoles, and the current upper limit for the number of magnetic monopoles per body is $< 10^{-29}$.

The tangent to a field line at some point gives the direction of $B$ at that point. The number of field lines drawn per unit cross section $\propto$ B.

## 2.4 Magnetic Flux

The magnetic flux $\Delta\phi_B$ has the same definition as electric flux in an E-field. For a small area $\Delta A$:

$$\Delta\phi_B = B\cos\theta\Delta A$$

Where $\theta$ is the angle between the B-field and the surface normal vector.

The total magnetic flux is therefore the integral of this:

$$\phi_B = \int_S \underline{B} \cdot d\underline{S}$$

As there is no monopoles, and exiting flux must also return. Therefore, the net flux over any enclosed surface is zero:

$$\int_S \underline{B} \cdot d\underline{S} = 0$$

While this does form one of Maxwell's equations and is very useful for EMII next year, it's not useful in EMI...

# LC Introduction to Particle Physics and Cosmology

**Thu 22 Jan 2026 16:00**

# Lecture 1 - Start of Particle Physics: The Standard Model of Particle Physics

## 1 Course Introduction

**Course Structure**

- Particle Physics: 6 lectures in weeks 1 to 6.
  1. Introduction and the standard model.
  2. Experimental measurements.
  3. Interactions with matter
  4. Tracking detectors I
  5. Tracking detectors II
  6. Calorimeters and Particle Identification.
- Cosmology: 4 lectures in weeks 7 to 11.

**Course Aims**

- Overview of current methods in Particle Physics experiments.
- An emphasis placed on the questions and challenges.

For example, the LHC has already been programmed with experiments all the way up to 2041. Therefore, any detectors we design today only become relevant in over a decade, which makes good detector design decisions incredibly important. This course will equip us to understand what drives those design choices.

The course is assessed by a single one hour long exam, weighted half particle physics and half cosmology.

**Recommended Texts**

- Detectors for particle radiation (2nd edition), K. Kleinknecht (1998)
- Particle Physics, Martin and Shaw.
- High Energy Physics, D. H. Perkins (2nd through 4th editions)
- Feynman Lectures.
- Modern Particle Physics, M. Thomson.

## 2 Matter Particles

Fermions all have quantum spin 1/2. Spin is a purely inherent quantum property (like mass or charge) and has no classical representation, but is analogous to angular momentum. They are subject to Fermi-Dirac statistics, which means that no identical fermion in a system of fermions can have the same quantum number as any other. Fermions are divided into two types, quarks and leptons.

## 2.1 Quarks

There are three generations of quarks:

**First Generation**

- Up Quark ($u$), mass of $\approx 0.001$GeV
- Down Quark ($d$), mass of $\approx 0.001$GeV

**Second Generation**

- Charm Quark ($c$), mass of $\approx 1.3$GeV
- Strange Quark ($s$) mass of $\approx 4.3$GeV

**Third Generation**

- Top Quark ($t$), mass of $\approx 175$GeV
- Bottom Quark ($b$), mass of $\approx 4.3$GeV

"up-type quarks", $u, c, t$ have electromagnetic charge $+2/3$ and "down-type quarks", $d, s, b$, have electromagnetic charge $-1/3$ (charges in units of e). Quarks do not ever exist alone in isolation.

## 2.2 Leptons

There are three generations of leptons, given by the electron $e^-$, muon, $\mu^-$ and tau $\tau^-$. These all have charge of $-1$ (in units of e). These have masses (in MeV) of approximately $0.5, 105, 1800$.

These also have associated neutrinos, the electron neutrino, the mu neutrino and the tau neutrino, $\nu_e, \nu_\mu, \nu_\tau$. These are not massless, but have a tiny mass many orders of magnitude smaller than their corresponding non-neutrino counterparts. These are neutrally charged.

Leptons are not subject to the strong interaction.

## 2.3 Hadrons

Quarks do not exist in isolation, but form bound states subject to the strong force called hadrons. There are two types of hadrons - baryons and mesons.

Baryons are formed from three quarks, which may be the same or different $q_1 q_2 q_2$.

Mesons are formed from a quark-antiquark pair, $q_1 \bar{q}_2$. Examples of baryons include the proton and the neutrino, given by $uud$ and $udd$.

# 3 Forces

As far as we know, there are four fundamental forces:

- Gravity
- Electromagnetism
- Strong
- Weak

For considering particle interactions, we disregard gravity as it becomes incredibly weak for small masses. Creating a complete theory that incorporates all four is an open question in physics. It's okay to neglect it, but it is unsatisfying.

We consider these forces as arising by the exchange (between two particles subject to a force between them) of particles called bosons. These have spin-1, so are called "gauge bosons". These are subject to Bose-Einstein statistics, which does not impose the same restriction as Fermi-Dirac for quantum numbers in a system.

**For EM:** The exchange particle is a photon, $\gamma$. This is represented on a Feynman diagram as a wiggly line. They are massless and couple to electric charge.

**For Weak:** The exchange particle is a $W^\pm$ or $Z^0$ boson. This is represented by a wiggly line or a dotted straight line. They are not massless, and have masses of approx. 80 and 90GeV respectively.

**For Strong:** The exchange particle is a gluon $g$. This is represented by a series of curls on a Feynman diagram. They are massless. They couple to "colour charge" which is just another quantum number analogous to electric charge. Just like electric charge has values $\pm$, colour charge has values we denote $r$, $g$, $b$

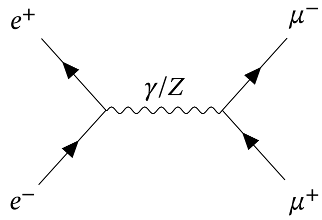
Quarks are subject to the strong, electromagnetic and weak interactions

Leptons are not subject to the strong interaction, but the $e, \mu, \tau$ are subject to EM ($\nu$ is not as it is neutrally charged), and all are subject to the weak interaction. This makes neutrinos very difficult to detect as they are only affected by the weak interaction.

# 4 Feynman Diagrams

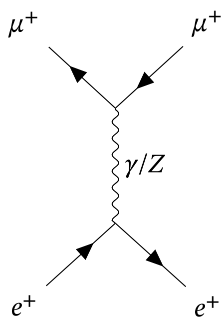
Feynman diagrams are space (y-axis), time (x-axis) diagrams to show allowed interactions between particles.

Consider a simple example of electron-positron annihilation. They travel towards each other, meeting and annihilating into either a photon or a Z boson. This is called a 'time-like exchange'. The boson then decays and we see pair production of two muons (one $\mu^-$ muon and one $\mu^+$ antimuon).




Arrows in Feynman diagrams convey "fermion flow". This means that for a matter particle, the arrows aligns to the time axis. For antimatter particles, they antialign. Some conservation laws (i.e. charge) apply at the vertex level, while others only apply across whole processes.

We now consider a space-like exchange where the exchange is aligned with the vertical (space) axis. An antimuon scatters off a position like such:




# 5 Luminosity

We can determine the rate of interactions with the following:

$$W = \mathcal{L}\sigma$$

Where $W\,(\mathrm{s}^{-1})$ is interaction rate, $\mathcal{L}\,(\mathrm{cm}^{-2}\mathrm{s}^{-1})$ represents the luminosity (an attribute of the accelerator being used), and $\sigma\,(\mathrm{cm}^{-2})$ is the cross section, representing the underlying physics of the interaction.

These are investigated in greater detail in Lecture 03.

**Mon 02 Feb 2026 12:00**

# Lecture 2 - Luminosity and Particle Signatures

## 1 Cross Sections

Consider a proton-proton interaction, producing some unknown particle $X$:

$$pp \to ppX$$

We have said that the rate of interaction is given by:

$$W = \mathcal{L}\sigma$$

Where $\mathcal{L}$ in $\mathrm{cm}^{-2}\mathrm{s}^{-1}$ is (coarsely) a parameter of the accelerator, describing its ability to produce collisions, and $\sigma$ in $cm^{-2}$ is a measure of interaction probability. Even though the particles are point-like, we treat them as having an effective area, and the magnitude of that area dictates how likely an interaction is to take place.

In this interaction, we have two protons (modelled as solid balls) passing immediately next to each other (one travelling clockwise and one counter-clockwise) around the accelerator. Assuming they pass immediately next to each other, and we model them as having radius $10^{-15}\mathrm{m}$, we have a a separation between the centres of each proton as $2 \times 10^{-15}\mathrm{m}$, therefore a cross section of:

$$\pi \left(2 \times 10^{-28}\mathrm{m}^2\right) \sim 0.12 \times 10^{-28}\mathrm{m}^2$$

To move this to a less annoying length scale, we define a new unit, the barn:

$$1\mathrm{barn} \equiv 10^{-28}\mathrm{m}^2 = 10^{-24}cm^2$$

In reality, this model may approximate a cross section, but it's not accurate. In reality, there's a much wider variation:

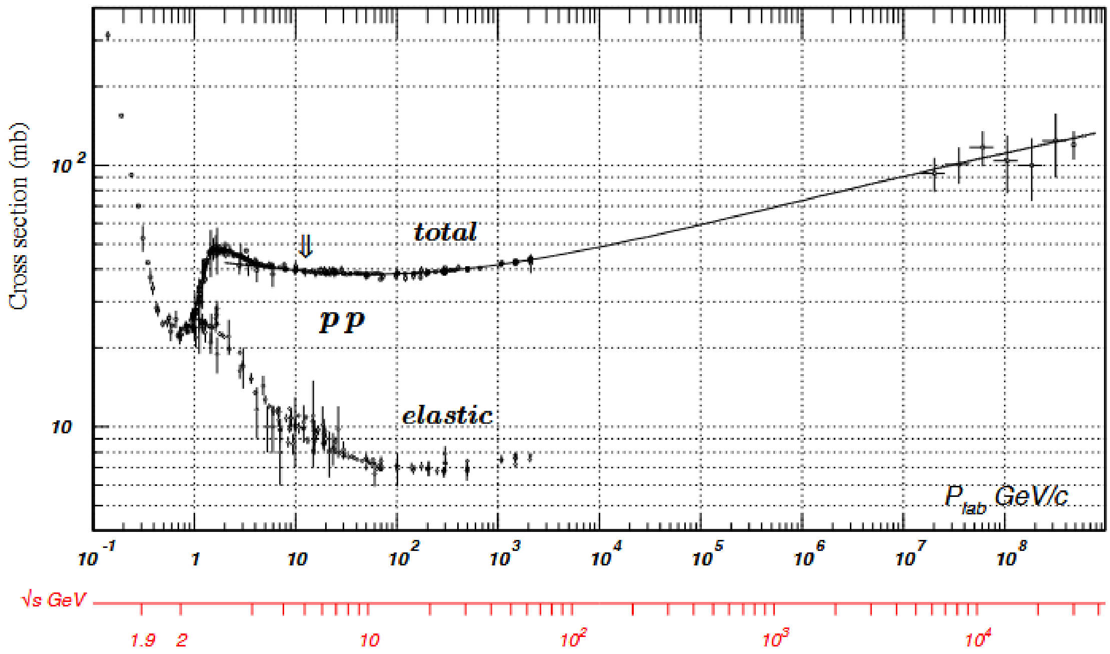



Figure 9.1

Here we have two x-axes running in parallel - the black axis is the momentum in a lab frame (as it hits some fixed target) while the red axis is the corresponding "centre of mass energy". How do we relate these two?

We want to know what the maximum mass of the particle we can generate is. In the lab frame, this requires us to take the momentum of the incoming and generated particle into account. We then have to take the final momentum of the system into account to conserve momentum, as the whole system must continue moving in the direction of motion for conservation.

Translating into the frame of reference given by the system centre of mass gives us a system where the two masses can be thought of as approaching each other with equal and opposite momenta. Since the total momentum is zero, the objects (incoming particle and the target) can theoretically hit each other and come to a complete stop. Because the system does not have to keep moving after the collision, all of the energy in this frame is free to be converted into the mass of a new particle.

While this may not be accurate in practice, it gives us a hard upper maximum for the possible energy available for production.

We can show relativistically that the energy in the centre of mass frame (labelled $\sqrt{s}$, the energy available for particle production) is:

$$E_{\text{com}} = \sqrt{s} \propto \sqrt{p_{\text{lab}}}$$

This can be seen in the final values of the x-axis, which (both starting approx. 1) are $10^8$ and $10^4$. This tells us that we reach diminishing returns with a fixed target collider - increasing energies by 8 orders of magnitude only increases the energy available for particle production by 4 orders. This is an inherent inefficiency of a fixed target collider.

This is much cheaper as its easier to align, we just fire a beam at a block of (for example) lead. It also makes it quite easy to change the target material. Changing materials in a colliding-beam collider (where two beams are fired in opposite directions, one clockwise and one anticlockwise, and collide with each other), i.e. to fire lead nuclei requires an extensive recalibration process.

In a dual-beam collider, only a very small proportion of the accelerated material from each beam actually interact with each other. In a fixed target collider, the target is much more dense, so we see a higher rate of interactions.

In summary, the advantages of a fixed target collider are:

1. Easier to collide.
2. Easier to change the target.

3. Very high density

# 2 Luminosity

## 2.1 Fixed Target Case

Consider a fixed target collider. We want to build an expression for the luminosity of this setup.

We have some incoming flux of particles (per second per unit area), $J$, incident on the block of material with density $\rho$, thickness $t$ and mass of one nucleus $m_A$. The beam, modelled by a cylinder, illuminates some circular portion of the block, with area $A$.

Consider our interaction rate $W$. This is given by (where $V$ is the volume of a cylinder from the illuminated beam, of thickness equal to the target):

$$W = \underbrace{JA}_{\substack{\text{incident particles} \\ \text{per unit time}}} \times \underbrace{\frac{\rho}{m_A} V}_{\substack{\text{total number of} \\ \text{target particles}}} \times \underbrace{\frac{\sigma}{A}}_{\substack{\text{probablity of} \\ \text{interaction}}}$$

$$W = JA \times \frac{\rho}{m_A} At \times \frac{\sigma}{A}$$

$$W = J \times \frac{\rho}{m_A} At \times \sigma$$

$$W = \frac{J\rho At}{m_A}\sigma$$

Comparing to $W = \mathcal{L}\sigma$ gives the luminosity as:

$$\mathcal{L} = \frac{J\rho At}{m_A}$$

## 2.2 Colliding Beam Case

In a colliding beam case, the derivation is more (and too) complex. It is equal to:

$$\mathcal{L} = \frac{f_{\text{rep}} n_b N_1 N_2}{4\pi\sigma_x\sigma_y}$$

Where:

- $f_{\text{rep}}$ is the repetition frequency, i.e the rate of the beam passing the collision point.
- $n_b$ is the number of bunches in the beam.
  - A beam can be thought of as as string of pearls, rather than a single discrete constant beam - i.e. clusters of particles "bunches", followed by empty space between them.
- $N_1, N_2$ are the number of particles per bunch for each beam
- $\sigma_x, \sigma_y$ are the dimensions of the beam in the x- and y-direction, not a cross section as previously.

# 3 Examples of Detectors

We have some interaction point producing a spray of particles, and surround this with a series of different detector layers. Each produced particle will trigger a different subset of these layers, defining a unique signature we can use to identify produced particles.

Broadly, in some generic detector, we have:

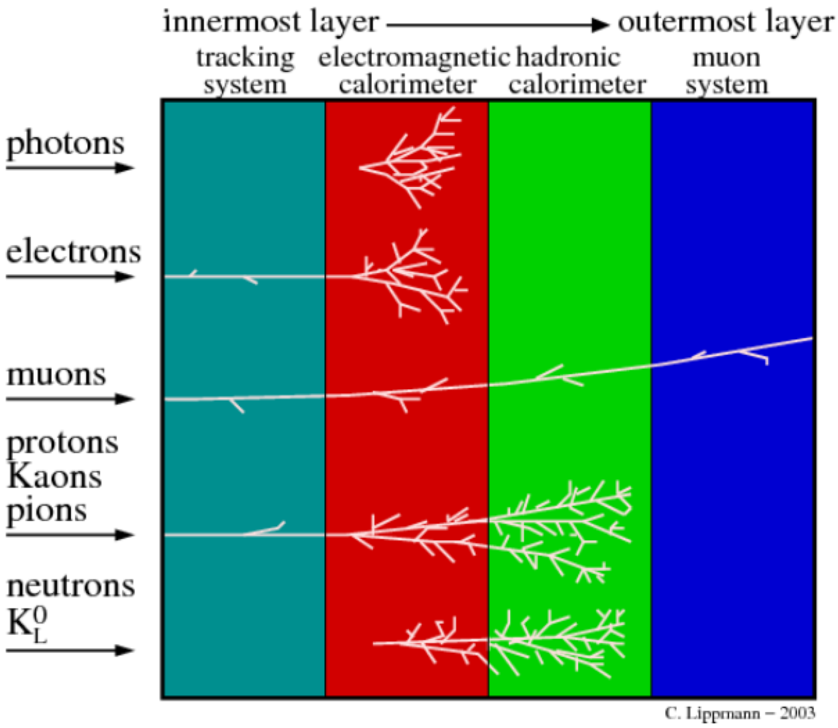



Figure 9.2

The tracking system is non-destructive. Formed of layers of silicon, a charged particle with ionise small portions of each layer. This can be turned into a signal. Neutral hadrons and photons will pass straight through, but charged particles will leave a deposit of charge and pass through unaffected.

We then move to destructive layers. Particles leave tree-like structures as they pass through these layers and create a shower of particles.

## 3.1 Decays

Most decays take place over a very low time scale, $< 10^{-8}$s and produce a final state made up of some subset of the following: $\gamma, \pi^+, \pi^-, \kappa^+, \kappa^-, p, n, \pi^0, e^+, e^-, \mu^+, \mu^-$.

So, in order to detect some exotic particle, we assume that it will either persist long enough to be detected itself, or decay into some subset of these known particles which will reach out detector.

Consider a parent particle $A$, for example $B^0$ ($\bar{b}d$) decaying into two child particles $B, C$, given by a "J Psi" ($c\bar{c}$), $J/\psi$ and a "K Short", $K^0_s$ ($d\bar{s}$):

$$\begin{aligned} B &\to J/\psi \quad K^0_s \\ \bar{b}d &\to c\bar{c} \quad d\bar{s} \end{aligned}$$

This decay has a lifetime of $10^{-12}$s, via the weak interaction due to the change in quark flavour. The J Psi decays into $e^+e^-$ or $\mu^+\mu^-$ via the EM interaction very rapidly in $10^{-21}$s (its lifetime is governed by the strong interaction, which it may also use to decay via, even though we detect it via the EM decay path). The K Short decays into $\pi^+\pi^-$ or $\pi^0\pi^0$ again via the weak interaction with lifetime in $10^{-10}$s.

The range of a particle is given by:

$$\text{Range} = \beta\gamma c\tau$$

Where $\tau$ is a time scale (lifetime), $c$ is the speed of light, $\gamma$ is the Lorentz Factor and $\beta$ scales the range based on the speed actually being travelled. We also have:

$$E = \gamma m$$

$$p = \beta\gamma m$$

And familiarly:

$$E^2 - p^2 = m^2$$

If the $B^0$ has energy 20GeV and mass 5GeV, we have $\gamma = 4$ and this gives a range of $\approx$ 1mm. This is so small we will never observe it directly. The $K_s^0$ however has range $\approx$ 30cm, so is detectable.

Crucially:

- SI lifetimes: $\sim 10^{-21} - 10^{-24}$s
- EM lifetimes: $\sim 10^{-16} - 10^{-20}$s
- WI lifetimes: $\sim 10^{-12}$s

**Thu 05 Feb 2026 16:00**

# Lecture 3 - Particle/Matter Interactions

In order to look at specific detectors and how they work, we need to consider how particles interact with matter. We'll consider categories of particles and their standard interactions, and use this to build a model for how we can build a detector.

We've looked at this previously:

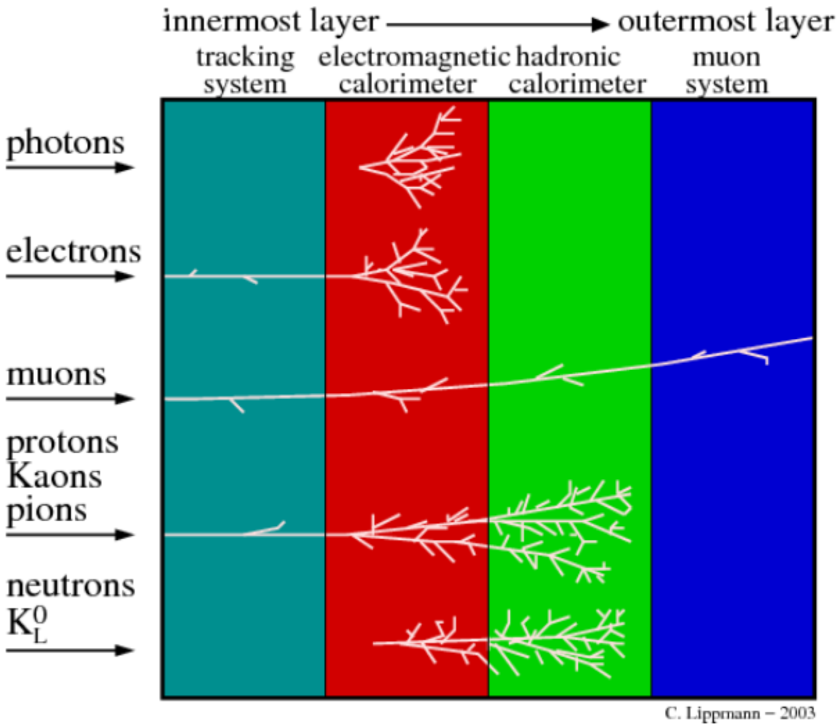



Figure 10.1

For example:

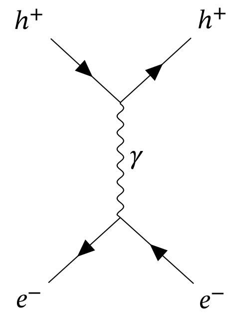
- Charged particles produce ionisation in the non-destructive tracking system, leaving deposits we can detect as signal.
- Electrons and photons leave distinctive signatures (shape of the shower of particles) in the electromagnetic calorimeters.
- Hadrons leave deposits in the electromagnetic calorimeter, but they dump all their energy (and are mostly identifiable by) the hadronic calorimeter.
- Muons make it through all of the previous layers, and are picked up at the very end by the outermost muon system.

# 1 Charged Particles

## 1.1 Ionisation

This is the process that powers the tracking system. A charged particle, $h$ comes into the system (it could be a hadron or a lepton). It exchanges a photon with an electron in the system and excites the electron, ejecting it from the atom.

Strictly:




We can to try to characterise the rate of energy loss of the charged particle. The average rate of energy loss per unit distance is:

$$-\left\langle \frac{dE}{dl} \right\rangle \propto \ln E$$

Where $E$ is energy, and the sign is negative as energy is being lost here.

For example, a real detector here might be two charged plates with a high voltage sandwiching a gas mixture. As the gas mixture is ionised, the ionised portion drifts towards one of the electrodes where it induces a detectable current.

## 1.2 Bremsstrahlung

Translates to "braking radiation". Consider a free electron radiating a photon. This is impossible for a free particle (as if we consider the electron's rest frame, emission would violate conservation of energy). We therefore need a source of external interference, in this case matter.

An electron is accelerated by nuclear change as it passes through material and is scattered. This scattering causes bremsstrahlung photon emission. The average rate of energy loss is given by:

$$-\left\langle \frac{dE}{dx} \right\rangle \propto \frac{E}{m^2} \propto \frac{E}{X_0}$$

An electron will then generate more bremsstrahlung than a muon, due to its much smaller mass. $X_0$ is called radiation length, and is covered in future lectures.

## 1.3 Cherenkov Radiation

Consider a charged particle moving through a (non-vacuum) material. It emits photons at some angle $\theta_c$ if the particle is moving faster than the speed of light in the medium (note this does not violate relativity, as the speed of light in a medium is less than the speed of light in a vacuum).

These emitted photons cause a coherent wavefront to form around the particle's trajectory, forming a cone around the direction of travel. The angle $\theta_c$ is given by:

Geometrically, after time $t$, the emitted photon has travelled $ct/n$ and the particle $vt$, hence:

$$\cos\theta_c = \frac{ct/n}{vt} = \frac{c}{nv} = \frac{1}{n\beta}$$

Where $n$ is the refractive index of the material. This is analogous to shock waves forming when an object goes faster than the speed of sound.

A planar detector will take a single cross-section through this cone, detecting rings around the point the charged particle passed through the material. By measuring this ring, we can work out the speed of the particle, and use this along with a measured momentum (in a tracking detector) to work out the mass of the particle.

Again:

$$-\left\langle \frac{dE}{dx} \right\rangle \propto z^2 \sin^2 \theta_c$$

Where $z$ is the particle charge in units of $|e|$. It is important to note that this is a very small energy loss for the particle. It may emit $10^3\gamma$/cm, and only lose a few keV/cm

# 2 Photons

## 2.1 Photoelectric Effect

We have a photon strike a atom, transferring energy and forcing an electron to be ejected. The max kinetic energy of the electron is given by the photon energy minus some amount of work to eject it:

$$E_{kmax} = hf - \phi$$

Where $\phi$, the work function, is the energy required to liberate one electron from the atom's surface.

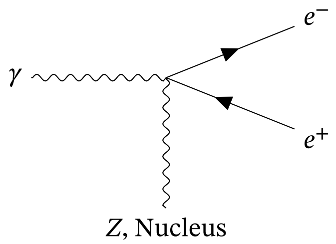
## 2.2 Compton Scattering

See QM1. A photon scatters off a quasi-free electron in an atom. The photon and electron are scattered with a change in energy:

$$\gamma + e^- \to \gamma' + e^{-\prime}$$

Thomson scattering is a low-energy form of scattering where the energies do not change, and Rayleigh scattering is a very low energy limit, where the interaction takes place between a photon and multiple atomic electrons.

## 2.3 Pair Production




A photon interacting with a nucleus can (leaving the nucleus unscathed) produce a particle and the corresponding antiparticle, typically a positron and an electron. This has a minimum photon energy of $E_\gamma > 2m_ec^2$[1] to ensure conservation of energy isn't violated and must occur in the presence of matter (for the same reason as Bremsstrahlung). The photon is not present in the final state.

However, say a photon has precisely the energy required to create a pair. This would (as it stands) create a pair of electrons with zero momentum (or very small momentum). However, a photon always has a momentum given by its energy, so momentum is not conserved. If the pair travel to attempt to resolve this, they now have some kinetic energy too, which means the photon must have a higher energy, and hence a higher momentum. This creates mismatch, we cannot conserve both energy and momentum in this situation.

By this interaction taking place in the presence of a nucleus, the nucleus can absorb some recoil (via exchange of a virtual photon) to ensure conservation of energy and momentum are satisfied.

This must take place in a Coulomb field to contribute this photon. The present nucleus has charge $z|e|$.

Lets consider the cross section of this interaction:

$$\sigma_{\text{pprod}} \propto \frac{7}{9} \frac{A}{N_A} \frac{1}{X_0}$$

[1] $m_ec^2 = 0.511$MeV

$X_0$ is the radiation length. It is a complex property of the material but is proportional to $1/z^2$.

# 3 Neutrinos

Neutrinos are really difficult to detect, as they are only effected by the weak interaction and have no ionisation/pair production etc - they are effectively non-interacting. We detect them by detecting the products of a weak force interaction.

For example:

- A muon neutrino can exchange a $W$ boson with a proton, which becomes a neutron. The muon neutrino becomes a muon and the up quark in the proton becomes a down quark in the neutron.
  - We can detect the muon that's produced as the neutrino passes through the detector and interacts.
  - We build a massive multi-tonne detector that puts a large amount of mass in the way of the neutrino, often water.
  - We do this in the hope that it will interact with some of the matter and produce the more-detectable muon.
- The neutrino can scatter off a nucleus.
  - A mall amount of energy is exchanged with the nucleus, depositing a small amount of heat energy in the detector's matter.
  - Nothing changes/decays/is produced etc other than a small amount of heat.
  - We build a very sensitive detector capable of detecting this very small amount of heat.

# 4 Hadrons

Hadrons are really complex in their interactions. They can have inelastic nuclear interactions, with large energy deposits at a small number of sites (compared to an ECal shower where we'd expect to see many smaller deposits). This arises as a result of the nucleus becoming excited or breaking apart entirely and producing a chain of secondary hadrons/particles and a change in the nucleus.

These secondary particles can go on to interact again. This are complex and messy objects which can fragment off to cause many secondary impacts.

# 5 Conclusion

In summary:

- The key interactions we care about are ionisation, Bremsstrahlung and Cherenkov radiation for charged particles.
- For photons, we care about pair production.
- For hadrons, hadron showers are messy and complex. We add inelastic nuclear interactions on top of an electromagnetic component from ionisation and everything becomes rather tricky.

**Mon 09 Feb 2026 12:00**

# Lecture 4 - Vertexing and Tracking Systems

In our familiar layered model:

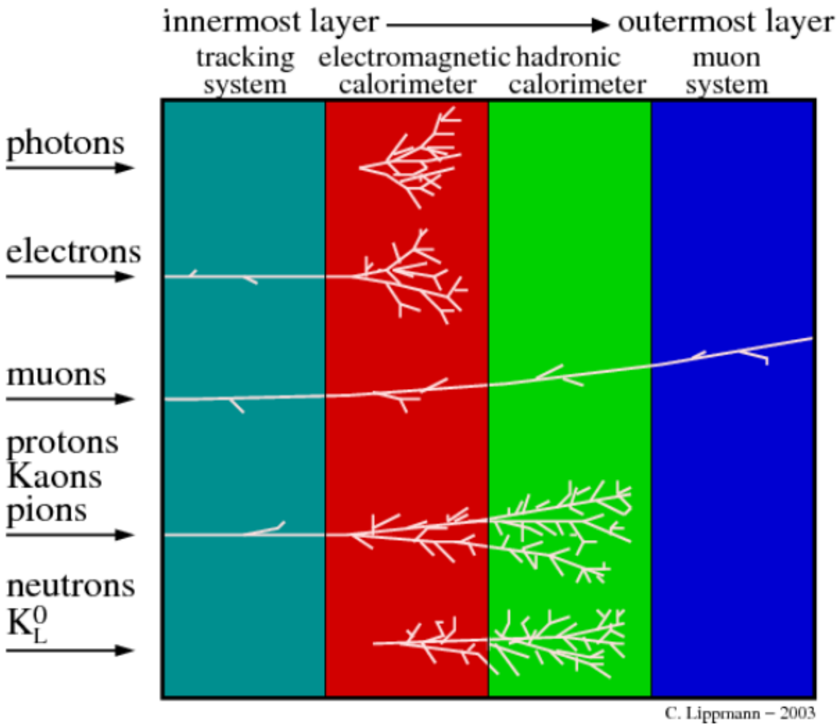



Figure 11.1

The first subdetector type is the initial tracking system which provides a non-destructive estimate of a particle's trajectory.

## 1 Tracking Systems

**Purpose:** To determine the trajectory of charged particles (usually in the presence of a magnetic field), in order to infer the momentum of the particle. This happens by ionisation, whereby the particle leaves small deposits of charge in either a gas or layers of a semiconductor sensor.

**Position:** As the following layers are destructive (i.e. calorimeters will absorb the particle entirely in order to make an energy measurement, while a muon system will block any other particle), we need to place the tracking system first for it to be effective.

We also want it placed near the primary interaction point. Since we want to use position measurements in the tracking system to extrapolate backwards and determine the origin point (where the collision took place), keeping them as close as possible to this origin point reduces the overall uncertainty of the track.

For a long path, a small uncertainty in position points propagates to a much larger uncertainty in track the further away we are from the initial collision point.

### 1.1 Material Budget

- In an ideal world, the tracking detector would have a perfectly massless and lightweight material, in order to reduce the risk of scattering. However, we need some mass in order to measure ionisation, so it's a constant trade off between the two.
- We want as small a number of radiation lengths $X_0$s within / upstream of the tracking system as possible.
- Some material is unavoidable, for example in the LHC there needs to be a conductive shield around the beam to prevent the large magnetic fields from inducing currents that would interfere with the sensitive measurement equipment. It also separates the highly pure vacuum of the beam pipe from the slightly less pure vacuum of the outer portion containing the LHCs electronics.

**Radiation Length** This is an inherent property of each material and is a measure of energy loss. A particle of energy $E_0$ passes through a distance of one radiation length and loses a factor of energy $1/e$.

$$E(x) = E_o \exp\left(\frac{-x}{X_0}\right)$$

For example:

- For Cu: $X_0 = 15$mm.
- For Be: $X_0 = 35.2$cm.

The processes by which energy is lost depends on the particle species in question. For example, an electron loses energy by Bremsstrahlung much more rapidly compared to a muon, due to the differences in mass (average energy loss $\propto 1/m^2$). To define radiation length, we use an electron as a scale.

This lets us talk about "material budget", as adding more material causes a higher uncertainty. For each particle path we care about, we want the fewest radiation lengths possible.

## 2 Measuring Trajectories

For a particle passing through a tracking system, we reconstruct its trajectory by measuring individual energy deposits (called "hits") caused by ionisation in a large volume (typically 1m $\times$ 1m $\times$ 1m)

This volume is made up often of many layers, so we can gain an idea of the particles position as it passes through each layer and use this to extrapolate. A tracking system has a resolution of a few 100µm and may leave $10 - 100$ hits.

In a "vertex detector", we use a smaller system to reconstruct tracks at/around the interaction point with a higher precision - typically aiming for $\sim$ 10µm. It is typically a silicon detector (layers of silicon detector sheets) and generally records fewer hits ($< 10$).

In a "tracking detector" we use a much larger system commonly further away from the primary interaction point. In a vertex system, there's no magnetic field so while we can use it to extrapolate back to find the PIP, we cannot use it to estimate momentum.

The LHC-b experiment for example has both a vertexer very close to the PIP (to determine the PIP location), and a much more substantial tracking system behind a magnet further away (to determine the deflected track in a magnetic field and hence the momentum.)

## 3 Measuring Momentum

The motion of a charged particle in a magnetic arises due to the Lorentz force and is proportional to charge:

$$\underline{F} = q(\underline{E} + \underline{v} \times \underline{B})$$

Where $\underline{F}$ is the force, $\underline{E}$ is the electric field, $\underline{v}$ is the particle velocity and $\underline{B}$ is the magnetic field. Assuming we are only dealing with a magnetic field, and taking magnitudes, we have:

$$\boxed{\underbrace{p}_{\text{GeV}} = 0.3 \times \underbrace{B}_{\text{T}} \times \underbrace{q}_{|e|} \times \underbrace{r}_{\text{m}}}$$

Where the magnetic field produces curvature of radius $r$ in the plane perpendicular to the $B$-field. Motion parallel to $\underline{B}$ is unchanged.

In 3D, and assuming we are contained within the magnetic field, the particle will follow a helix of constant radius of curvature. This assumes that there are no non-conservative forces (i.e. scattering). In the ideal case, there is no work done, meaning in our force:

$$\underline{F} = q\underline{v} \times \underline{B}$$

The force and the velocity are perpendicular, so $\underline{F} \cdot \underline{v} = 0$.

## 3.1 Quantitatively

Consider a particle path with a constant radius of curvature $R$. Consider three hits, with vertical separation between the top and the bottom being $L/2$. The distance between the top/bottom hit and the origin is $l$, and the final distance between this straight line and the deflected path is the "sagitta" $s$.

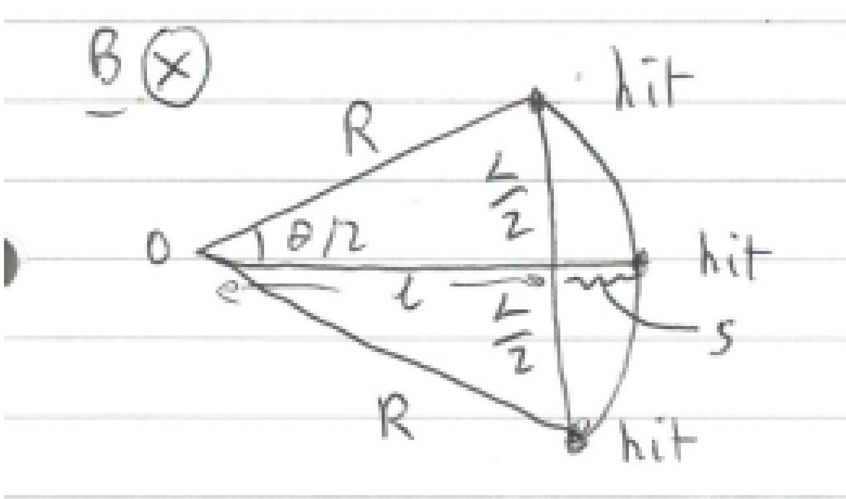



Figure 11.2

We know $L/2$ (the vertical separation between hits) as we've build the detector so know the resolution, and we want to measure the sagitta, the deviation from a straight line.

We know that:

$$p = 0.3BqR$$

And from the diagram:

$$R = l + s$$

$$l = R\cos\left(\frac{\theta}{2}\right)$$

Putting these together:

$$s = R\left(1 - \cos\frac{\theta}{2}\right)$$

Assuming the bending is gentle in a detector, so $\theta$ and $s$ are small. Hence we apply a small angle approximation:

$$\cos\frac{\theta}{2} \approxeq 1 - \left(\frac{\theta}{2}\right)^2 \frac{1}{2!} + \left(\frac{\theta}{2}\right)^4 \frac{1}{4!} + \dots$$

Taking the first order:

$$s \approxeq \left(\left(\frac{\theta}{2}\right)^2 \frac{1}{2!}\right) = \frac{R\theta^2}{8}$$

We can also use the small angle approximation for sine:

$$\sin\frac{\theta}{2} = \frac{L}{2R}$$

For small theta:

$$\sin\frac{\theta}{2} \approxeq \frac{\theta}{2}$$

$$\theta \approxeq \frac{L}{R}$$

So:

$$s = \frac{L^2}{8R}$$

Hence, finally, we have:

$$p = 0.3Bq\frac{L^2}{8s}$$

### 3.2 Uncertainties

We want to find the uncertainty on momentum, $\sigma_p/p$:

$$\frac{\sigma_p}{p} = \frac{\sigma_s}{s} = \frac{8p}{0.3BqL^2}\sigma_s$$

However we want to use our uncertainties on individual hits, $x_1, x_2, x_3$ etc. We have (and will not derive):

$$\frac{\sigma_p}{p} = \frac{\sigma_{xy}P}{0.3BL^2}\sqrt{\frac{720}{N+4}}$$

**Mon 09 Feb 2026 12:00**

# Lecture 5 - Calorimetry

Now that we've finished talking about determining the principal interaction point and the momentum of produced particles in a tracking detector, we move onto calorimeters:

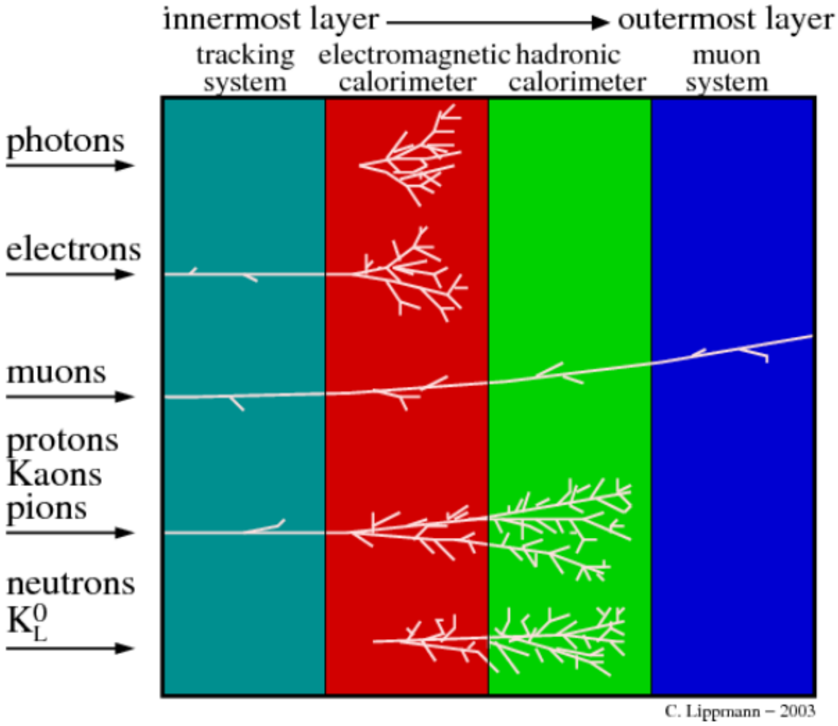



Figure 12.1

## 1 Materials and Energy Loss

Energy loss in a material is described by the Bethe formula, which gives mean energy loss by ionisation for unit path length:

$$\left\langle -\frac{dE}{dx} \right\rangle = Kz^2 \frac{1}{\beta^2} \left[ \frac{1}{2} \ln \frac{2m_e c^2 \beta^2 \gamma^2 W_{\max}}{I^2} - \beta^2 - \frac{\delta(\beta\gamma)}{2} \right]$$

This is:

- Valid for $\beta\gamma < 1000$ within a few percent precision.
- Includes dependence on the medium, $I, Z, A$ etc.

The good news is that we do not need to know this formula. We do however need to know some key features:

- $\frac{dE}{dx} \propto \frac{1}{\beta^2}$ below some minimal value of $\frac{dE}{dx}$
- $\frac{dE}{dx} \propto \ln(\beta^2\gamma^2)$ above this minimal value of $\frac{dE}{dx}$. This is called "relativistic rise".
- This minima happens at approximately $\beta\gamma \sim 3 - 4$.

- At large $\beta\gamma$, polarisation of the medium causes saturation (effectively a plateau).

Broadly, we expect to see a fall up until some minimal value, then a relativistic rise, and then a plateau:

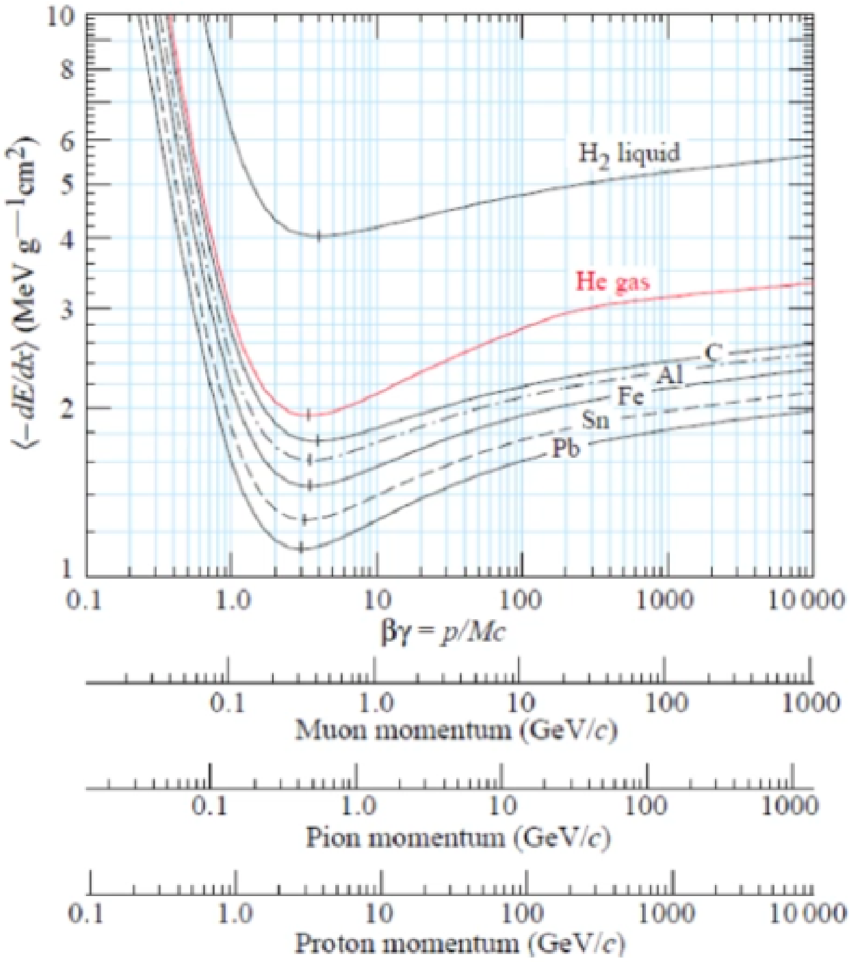



Figure 12.2

## 2 Calorimeters

The purpose of a calorimeter is to determine the total energy of an incident particle. It does so by absorbing the particle entirely, creating a measurable signal which is proportional to the energy of the incident particle. As it is destructive, it is always located after the non-destructive tracking system. It has no need for the B-field of the tracking detector, and may be called a "calo", "ECAL" or "HCAL".

There are two types of calorimeter:

- Electromagnetic Calorimeter: ECAL.
- Hadronic Calorimeter: HCAL.

We characterise the length of an ECAL in terms of radiation length $X_0$, typically $0.15X_0 - 0.3X_0$. It works by bremsstrahlung, and emitted bremsstrahlung photons causing pair production and more bremsstrahlung emission etc.

A HCAL works on nuclear hadronic interactions, so we instead define an interaction length $\lambda_{\text{int}}$, typically $5\lambda_{\text{int}} - 8\lambda_{\text{int}}$.

They don't measure the total energy of a particle if particles produced in the process pass straight through, so not all the energy in an ECAL will be captured, as some will pass straight through. Muons, neutrinos and sometimes pions pass straight through and escape, they are not (may not for a pion) be detected by the calorimeters.

An ideal calorimeter has an output signal "response" which is proportional to the energy of the input particle. We would ideally like a directly proportional linear relationship, but this may not always be realistic.

If we already know mass (from Cherenkov rings, to be discussed later in the course) and momentum from the tracking detector, why do we need a calo when we can just use $E^2 - p^2 = m^2$ to determine energy?

- Cherenkov rings and tracking systems only work with a charged particle. The calculation method is not sensitive enough, as the Cherenkov+tracker setup cannot detect $\gamma, \nu$ or neutral hadrons.
- Having a direct energy measurement adds another constraint to our system - the more information we can get the better. A direct energy measurement improves resolution and is more likely to be accurate than determining it from two other uncertain quantities.
- At very high momentum, (with little Coulomb scattering) the relative uncertainty of the transverse momentum of a tracking system is given by:

  $$\frac{\sigma_{p_\perp}}{p_\perp} \propto \frac{p_\perp}{BL^2}$$

  Therefore at a higher momentum, we have a higher uncertainty. At high momenta, there is a smaller deflection, so a high momentum track will just pass rapidly through the field with very little deflection and is therefore much more difficult to measure accurately - the tracking system degrades at higher momenta.

  For a calorimeter, we have nicer behaviour at high momenta:

  $$\frac{\sigma_E}{E} = \frac{a}{\sqrt{E}} \oplus b \oplus \frac{e}{C}$$

  Where:

  - $a$ arises statistical fluctuations inherent to the measurement.
  - $b$ arises from calibration effects (i.e. from non-linearity)
  - $c$ arises from electronic noise.
  - $\oplus$ means to "add in quadrature", $(p \oplus q \oplus r) = \sqrt{p^2 + q^2 + r^2}$
- Trackers are relatively slow as they require ionisation clouds drifting towards a collection point and involve quite a computationally complex problem to recognise the patterns and reconstruct the track. A calorimeter relies on "scintillation" to convert energy to a response, which is much faster.

  This is important for triggering. Every 25ns, we need to decide whether or not to keep an event (as there is nowhere near enough storage/processing power) to perform detailed track reconstruction for every single event. We need a calorimeter to participate in this.

## 2.1 EM Calorimeters

There are two types of electromagnetic calorimeters, sampling or homogeneous. A sampling calorimeter is layered, with a layer of absorber producing a shower of particles (i.e. lead) in front of a collector layer with a scintillator that generates a signal when particles pass through.

HCALs are always sampling-type calorimeters, as it takes a lot more mass to slow down/break apart a hadron compared to a lighter charged particle.

Lets build a simple model of an electromagnetic shower that might be detected by a calorimeter:

We have a high energy photon (so we can mostly disregard scattering etc) enter the detector. This pair produces (i.e. an electron and a positron) which emits photons via bremsstrahlung emission. These emitted photons can then go on to pair produce themselves, generating (i.e.) an electron and a positron again, which emit more photons via bremsstrahlung, etc.

The maximum energy is deposited when the average particle energy (of particles developing in the shower) is the "critical energy".

Here, a photon has entered from the left, and the first few radiation lengths have left the material intact.

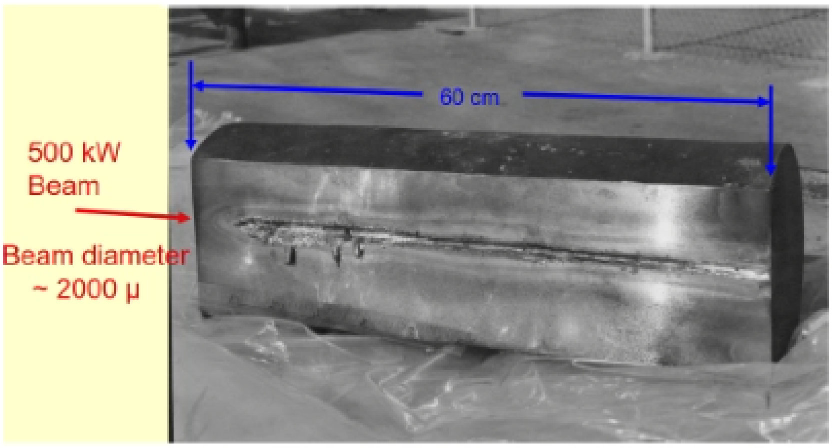



Figure 12.3: Damage to a copper block that has had energy dumped into it via this process.

After the first few radiation lengths, we have the "critical point", where a large amount of energy has been dumped. This point is defined as the point where the probability of bremsstrahlung and ionisation are equal.

The shower deposits a constant amount of energy per $X_0$ travelled (by definition of radiation length). Each step in path length, in terms of $X_0$ has a doubling number of particles and a halving individual particle energy. The angle of photon emission is fairly small, so the shower is narrow. This is unlike a hadronic calorimeter that has a wider shower. The width of the shower is given by the multiple scattering of the $e^+$ and $e^-$.

When the energy of the particle is less than the critical energy, ionisation dominates the interactions and the shower development stops rapidly. When the particle energy is $> E_c$, pair production and bremsstrahlung dominate.

After depth $t$ (in units of $X_0$), the number of particles is $2^t$, and the average particle energy is $E_0/2t$. The shower stops developing after $E = \frac{E_0}{2t} < E_c$. This happens when $E = 2t = E_0/E_c$. Hence:

$$t_{\max} \log 2 = \log\left(\frac{E_0}{E_c}\right)$$

$$t_{\max} = \frac{\log(E_0/E_c)}{\log 2}$$

This is what defines how deep our calorimeter needs to be in order to collect the energies we are expecting to require.

**Mon 16 Feb 2026 16:00**

# Lecture 6 - End of Particle Physics: Calorimetry II and PID

## 1 EM Calorimetry: Shower Shape

The energy deposited vs the depth in material can be expressed as:

$$\frac{dE}{dt} = E_0 b \frac{(bt)^{a-1} e^{-bt}}{\Gamma(a)}$$

Where $t$ is expressed as a length in units of radiation length $t = x/X_0$ and $\Gamma(a)$ is the standard gamma function $\Gamma(x) = \int_0^\infty t^{x-1} e^{-x}\, dx$. $a, b$ are both constants.

**For small $t$ (early in the detector):** $t^{a-1}$ dominates as $e^{-bt} \approx 1$

**For large $t$ (later in the detector):** $e^{-bt}$ dominates.

Considering a plot of $\frac{dE}{dt}$ against $t$, we initially have polynomial growth, reaching a peak. We then from that peak decay exponentially with an infinite tail.

The maximum point of the shower is given at $t = t_{\text{max}}$ and is achieved by further differentiating $dE/dt$ and is given by:

$$t_{\text{max}} = \frac{a-1}{b} = \ln\left(\frac{E_0}{E_c}\right) + C_{\gamma \text{ or } e^-}$$

For a photon, $C_\gamma = 0.5$ and for an electron, $C_e = -1$. Hence, the shower maximum is at a smaller value of $t$ for an electron/positron compared to photon.

Crucially, we want to find the total energy deposited by the particle. We get that by integrating $dE/dt$:

$$\int_0^{\text{calo thickness}} \frac{dE}{dt}\, dt$$

While the tail is an infinitely long decaying exponential, we only care about the portion of the tail which is inside the calorimeter. Any energy that would be deposited past this point is lost:

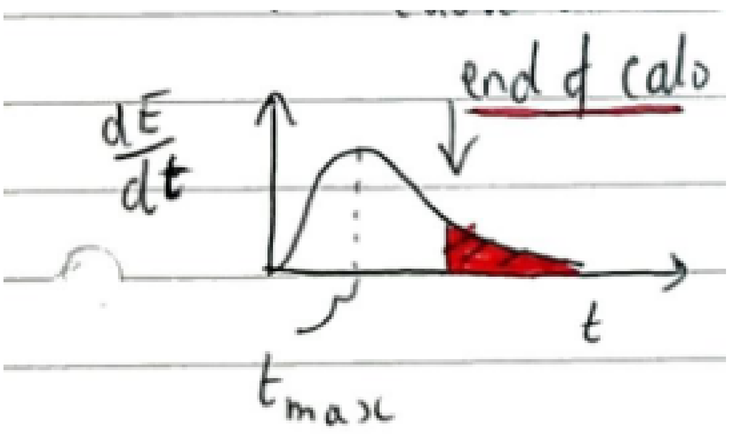



Figure 13.1

If the calorimeter is too thin, we need to make a large correction to adjust for this and we therefore have a bigger uncertainty. We want to ensure that the calorimeter is thick enough to capture the vast majority of deposited energy with minimal loss.

### 1.1 Resolution

The resolution for the calorimeter varies heavily based on the material used to construct it. Typically, however:

- ECAL resolution: $2 - 10\%$
- HCAL resolution: $50 - 100\%$

Different materials will have different characteristic radiation lengths,

For example:

- W has $X_0 = 0.35\text{cm}$, $\lambda_{\text{int}} = 9.9\text{cm}$, $R_M = 0.93\text{cm}$
- Pb has $X_0 = 0.56\text{cm}$, $\lambda_{\text{int}} = 17.6\text{cm}$, $R_M = 1.6\text{cm}$
- Fe has $X_0 = 1.8\text{cm}$, $\lambda_{\text{int}} = 16.7\text{cm}$, $R_M = 1.7\text{cm}$
- Cu has $X_0 = 1.4\text{cm}$, $\lambda_{\text{int}} = 15.3\text{cm}$, $R_M = 1.8\text{cm}$

$R_M$ is the "Moliere Radius" and is the radial width in which 90% of the shower's energy deposit is confined. Note that we have different design considerations for an HCAL vs an ECAL, for example the size of an HCAL is generally much larger than that of an ECAL.

## 2 Particle Identification (PID)

We can now measure a particle's momentum and energy, and we now want to determine the mass of the particle. Surely we can do that with:

$$E^2 - p^2 = m^2$$

Unfortunately not…the resolution (especially for energy on a calo) is pretty poor, so we would end up with an unreasonably large uncertainty on our mass, too large to do reasonable particle identification with. Instead, we:

- Use information from the whole ensemble of detectors.
- Use hypothesis tests: proposing a specific particle type and then using all the information we have (i.e. momentum) see if this is a reasonably likely outcome.
- For example, we can use information on whether or not the particle reached the muon chambers to help narrow down what it could be. Alternatively, what is the pattern of energy deposit observed in the ECAL or HCAL?
- We can also use dedicated PID, such as:
  - A time of flight detector, determining the particle velocity (hence mass, with momentum measurements from a tracker).
  - Cherenkov detectors (not useful at high momenta, i.e >100GeV): As a recap, the particle emits a cone of photons at angle $\theta_c$ which depends on the refractive index of the material and $\beta$. Measuring this cone allows us to determine $\theta_c$ hence $\beta$ hence $v$.

Here is an example of a detector setup:

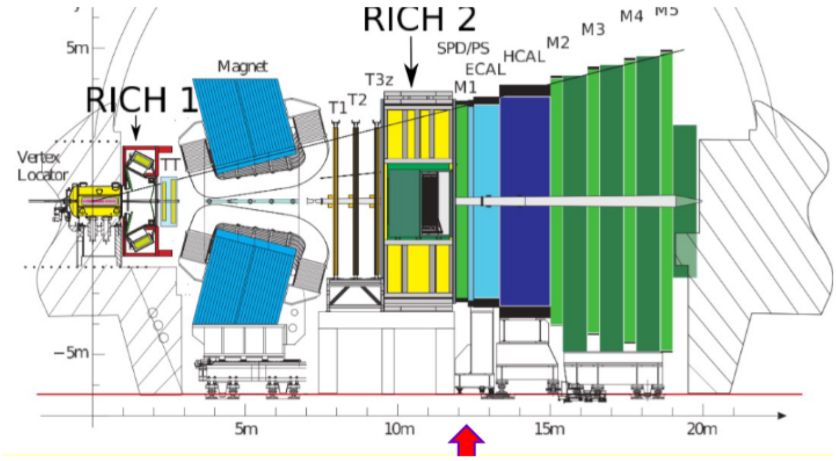



Figure 13.2

We have covered the vertex locator, magnet, tracking stations T1, T2, T3, the HCAL, ECAL and (while we haven't yet touched on them), we have 5 muon systems.

We are yet to discuss RICH1 and RICH2. These are called "Ring Imaging Cherenkov Detectors". These cones of Cherenkov emissions are focussed using a magnet onto a plane, where they form rings. The radii of these rings can be measured and the detector geometry can be used to determine the Cherenkov angle. The two detectors work similarly, but cover different ranges of measurable momenta.

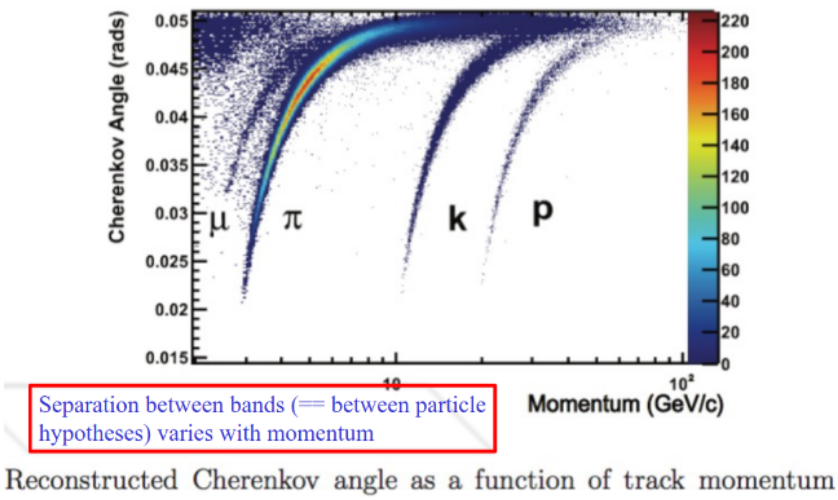



Figure 13.3

This Cherenkov angle can therefore be used to discriminate against particles, up to a limit of momentum, past which they become fairly useless.

NOTE: There was a practice exam question here that would be good to come back to during revision.

## End of Particle Physics

**Mon 23 Feb 2026 12:00**

# Lecture 1 - Start of Cosmology: A Brief Introduction to Cosmology

**Primary Textbook:** "An Introduction to Modern Cosmology", Andrew Liddle. Each lecture will have a chapter or two as recommended reading from this book and is available as an ePDF from the library.

## 1 The origin of Cosmology

Comes from the Ancient Greek word $\kappa o \sigma \mu o \sigma$, "kosmos" meaning order. Cosmology is a branch of astronomy concerned with studying the origin and evolution of the universe. We won't really consider anything on a smaller scale than clusters of galaxies.

The earliest form of this is *celestial mechanics* which goes back to Ancient Greek philosophers sch as Aristotle and Ptolemy.

### 1.1 Geocentric Ptolemaic System

This was the prevailing theory until the 16th century where the earth was at the centre of the universe and other stars/planet etc revolved around the earth in "epicycles". Here, the planets sat on rotating spheres, which orbited the earth.

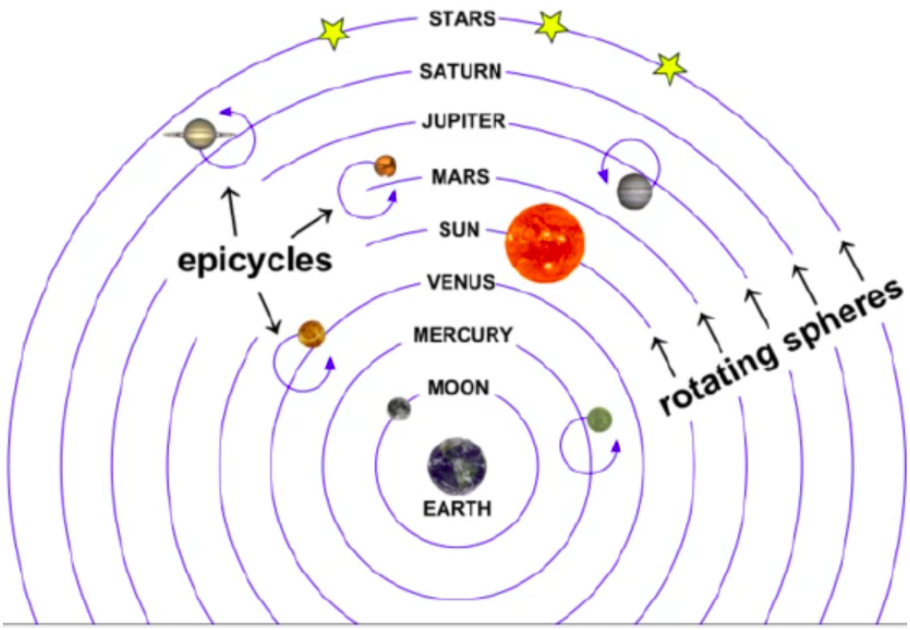



Figure 14.1

The epicycles explain retrograde motion. Observational power was too limited to get more accurate measurements, and they noticed that some objects appeared to go backwards as they were moving forwards overall in the orbit. We can solve this by placing another smaller circular orbit superimposed on the larger orbit.

We now know that this is not accurate, but it explained the motion at the time.

### 1.2 Heliocentric System

Copernicus in 1543 radically changed the understanding of orbits. He suggested that the sun, not the earth, was near the centre of the universe. The planets, including the earth revolved in circles around the sun, with other stars orbiting much further away.

In order to match the observed motion, we still need epicycles in this initial version of the theory.

This was the foundation of modern astronomy, and lead to:

- Kepler discovering elliptical orbits. This got rid of the need for epicentres and described the motion of planets with the data collected at the time.
- Galilei discovering the moons of Jupiter and phases of Venus. Before Galilei, these telescopes were not sufficiently powerful to observe these phases. The existence of these moons provided additional evidence for the heliocentric system as we now had a celestial body which did not orbit the Earth. This suggested that the Earth wasn't special, which was one of the beliefs that drove the geocentric model.
- Newton's theory of universal gravity in 1678.

This theory, and the implications it had on humanity's role in the universe was deeply unpopular with religious leaders and this caused opposition to its widespread acceptance.

This was a better way to think about the solar system (which was about the limit of what was practically observable). Copernicus therefore needed to provide evidence to justify this. These later from Kepler, Galilei and Newton acted as this evidence.

## 2 From Herschel to Hubble

In order to go from scales on a solar system level to a cosmological level, we need more powerful measurement equipment to be able to gather as much data as possible on much larger scales. William and Caroline Herschel pioneered this by creating the most powerful telescopes to date in 1877.

They probed deeper and deeper past the solar system, into the "deep sky". They investigated the contents of the Milky Way to build a catalogue of objects and try to find the edge of the galaxy.

This was in an attempt to work out whether or not there was some kind of edge to the universe. They discovered diffuse objects ("nebulae") beyond the milky way, and began to be able to resolve individual stars with improved technology.

This lead to the question of whether these nebulae (named "island galaxies" at the time) belong as components of the milky way or existed in their own right as galaxies.

### 2.1 Shapley-Curtis Debate

In the 1920s, the "Great Debate of Astronomy" investigated whether these spiral shape nebulae were small object on the outskirts of the Milky Way (advocated for by Harlow Shapley) or were independent small galaxies outside the gravitational attraction of the MIlky Way (advocated for by Heber Curtis).

In 1924, Edwin Hubble established that these bright nebulae were not part of the Milky Way and were instead of extragalactic origin. This told us that the observable universe extended beyond the Milky Way.

He discovered Cepheid Variable Stars, which are stars of periodic varying brightness. The period of pulsation is linked to intrinsic luminosity, and they act as distance indicators based on the perceived brightness and period on Earth.

These observations estimated the distance of these CVSes away from the Earth, and proved that they were too distant to sit within the Milky Way. This profoundly changed our understanding of the scale of brightness.

These CVSes are called "standard candles" which are astronomical objects with known intrinsic luminosity, $M$. We observe some different apparent brightness $m$, and can measure the pulsation period to determine the actual intrinsic luminosity.

Using $M$ (determined from period) and $m$ (measured) directly, we can use this formula:

$$5\log D = m - M - 10$$

$$\implies D = 10^{(m-M-10)/5}$$

This made it apparent that these distances were much larger than our understanding of the size of the Milky Way, by in excess of 2 orders of magnitude.

We can also know the intrinsic luminosity of a Type 1a supernovae, when a star explodes and collapses into a white dwarf.

## 2.2 Cosmic Ladder

We can form the "Cosmic Distance Ladder". Since traditional distance measurements become infeasible as scales get larger, we need to use different ranges for different distances. For example:

- Cepheid Variable Stars are not that bright, so stop being useful at longer ranges.
- Supernovae are brighter, so stay useful for longer.

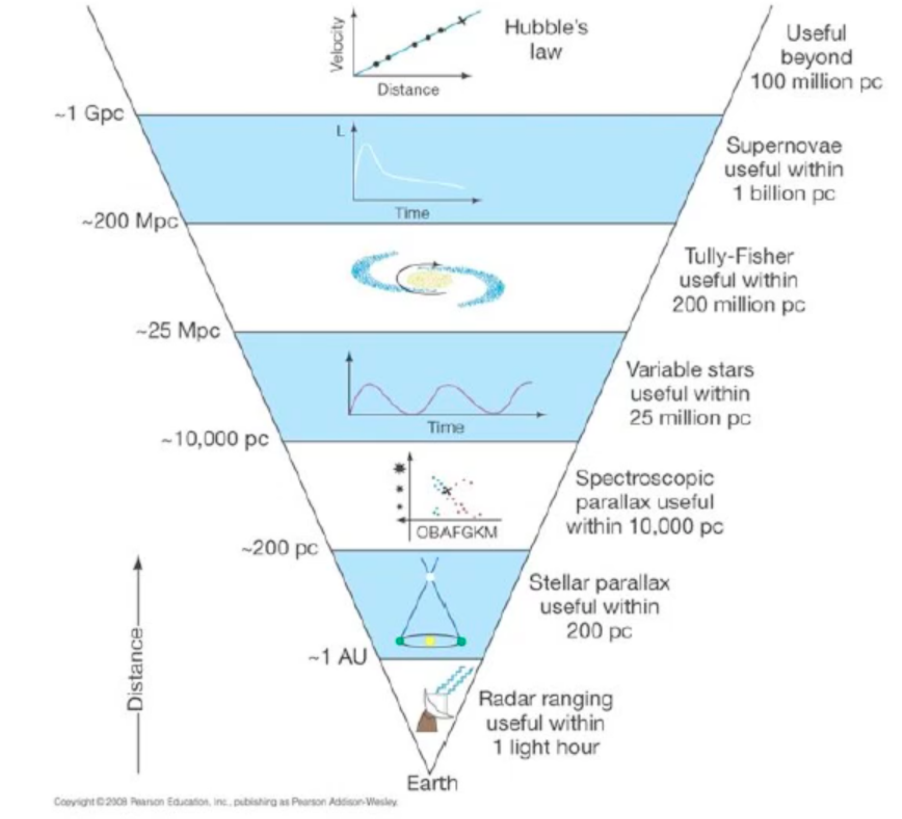



Figure 14.2: The Cosmic Distance Ladder

# 3 Expansion of the Universe

In 1929, Hubble analysed the emission spectra of galaxies. As a source moves, wavelengths (hence the positions of spectral lines) are impacted by redshift:

$$z = \frac{\lambda_o - \lambda_e}{\lambda_e} \approx \frac{v}{c} \qquad \text{approximation valid for non-relativistic limits.}$$

From this, noting the changes of the spectral lines vs known non-redshifted calibration points on earth allowed for the velocity of each galaxy to be determined.

Plotting $v$ against $r$ yields a straight line, suggesting $v \propto r$. Hence galaxies further away are moving faster. The sign of the redshift (i.e. is it indeed redshift and not blueshift) confirms that they are moving away from us.

This is very strong evidence that the universe is not static and is actually expanding. This gives us a dramatically new understanding of the universe, where it grows in size as time goes on.

# 4 Key Observations

We have a number of key objectives, and want to build a theory that fits these and makes predictions. Most of these observations are made using electromagnetic radiation, although additional information is provided by particle physics (and more recently gravitational wave astronomy).

Some of these key observations are:

- The universe is expanding (Hubble).
- There is an abundance of light elements (nucleosynthesis).
- Large-scale structure (LSS).
- Cosmic microwave background (CMB).
- Galaxy rotation curves (implying the existence of dark matter).
- Acceleration of the rate of expansion of the universe (dark energy).

## 4.1 Nucleosynthesis

Stars are very good and producing heavy elements via nuclear fusion. Here, stars start with lighter elements (H, He) and fuse upwards through increasing mass numbers up to Iron. Heavier elements require a stellar explosion to form.

It turns out that young stars have an abundance of light elements. These start from gas clouds that have collapsed gravitationally. In this very early stage, they started out with some contamination of heavier lighter elements (Helium isotopes and other Hydrogen isotopes) and couldn't just start with Hydrogen. They need some of these other elements to kickstart the process.

There has to be some process that took place before stars were formed that provided these elements to kickstart star formation. Stars could not have formed without this.

Theorists (George Gamov 1948) hypothesised reversing the expanding universe, i.e. stepping back in time to a much smaller, denser, hotter early universe. This would have formed a plasma-like state which could have provided these elements.

Nuclear fusion requires a very large energy, i.e. nuclear binding energy for these interactions is $\sim$ 1MeV. This plasma state lasted about 3 minutes, and allows for the formation up to Hydrogen-4. Heavier elements are still formed in stars.

## 4.2 Large Scale Structure

# LC Mathematics for Physicists 1B

**Wed 21 Jan 2026 11:00**

# Lecture 1 - Course Welcome and Introduction to Partial Differentiation

## 1 Course Welcome

### 1.1 Recommended books:

- Mathematical Techniques 4e, Jordan & Smith
- Engineering Mathematics 8e, Stroud
- Calculus (Schaum), 6e, Ayres & Mendelson
- Advanced Calculus (Schaum), 6e, Ayres & Mendelson

### 1.2 Assessment details:

- Maths 1A/1B form a single 20 credit module.
- 80% assessed by a 3 hour exam - Section 1 is 36% with 6 short questions and Section 2 is 64% with 4 long questions.
- 20% assessed by problem sheets.

### 1.3 Course structure:

1. Partial Differentiation
   - Definition, total differential, chain rule, gradient.
   - Taylor series, stationary points, Lagrange multipliers.
2. Differential Equations
   - Definition, 1st order separable, exact and homogenous.
   - Linear equations: general solution, 1st order and constant coefficients.
3. Integration
   - Definition as area under the curve, fundamental theorem of calculus.
   - Integration by: substitution, parts, partial fractions and tricks.
4. Multiple Integrals
   - Multiple and repeated integrals. Change of order of integration.
   - Change of variables and the Jacobian. Arc length. Solids of revolution.

## 2 Multivariate Functions

Lots of physics involves functions of more than one variable. A physical quantity defined at every point in space is called a field. We can have both scalar fields and vector fields.

For example, some scalar fields are:

- $V(x, y, z)$: Electrostatic potential. This is often easier to work with compared to the full electric (vector) field.
- $T(x, y, z)$: Temperature.
- $p(x, y, z)$: Pressure.

While some vector fields are:

- $\underline{E}(x, y, z)$: Electric Field.
- $\underline{B}(x, y, z)$: Magnetic Field.
- $\underline{v}(x, y, z)$: Velocity Field (i.e in fluid mechanics).

## 2.1 Partial Derivatives

Consider a function of two variables. The partial derivative of a function with respect to one variable is the rate of change of a function wrt that variable, while keeping other variables constant. Effectively, we carry out a derivative while treating the other variables as if they were constants.

Suppose we have a function $f(x, y)$. The definition of a partial derivative is:

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h\to 0} \frac{f((x_0 + h), y_0) - f(x_0, y_0)}{h}$$
$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{k\to 0} \frac{f(x_0, (y_0 + k)) - f(x_0, y_0)}{k}$$

Just like we denote $\frac{df}{dx}$ as the derivative of a function of a single variable, we denote $\frac{\partial f}{\partial x}$ as the partial derivative of a function of several variables.

Note that this is not delta f and delta y, i.e. not $\frac{\delta f}{\delta x}$

In theory, we'd explicitly notate:

$$\left(\frac{\partial f}{\partial x}\right)_y$$

With the subscript $y$ explicitly stating that $y$ is being kept constant. This is rarely, but sometimes, needed.

Consider $f(x, y, z) = x^2 \sin yz$. We have:

$$\frac{\partial f}{\partial x} = 2x \sin yz$$
$$\frac{\partial f}{\partial y} = x^2 z \cos yz$$
$$\frac{\partial f}{\partial z} = x^2 y \cos yz$$

## 2.2 Higher Orders

Higher derivatives are defined as they were previously, but they can now be mixed. For example, with $f(x, y) = x^2 \sin y$, we can write:

$$\frac{\partial f}{\partial x} = 2x \sin y \qquad \frac{\partial f}{\partial y} = x^2 \cos y$$
$$\frac{\partial^2 f}{\partial x^2} = 2 \sin y$$

We can also have:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2x \cos y$$

Shorthand notation exists, i.e. $f_{xx} = \frac{\partial^2 f}{\partial x^2}$ or $f_{yx} = \frac{\partial^2 f}{\partial y \partial x}$

For most cases, but not all, mixed derivatives are often independent of the order of partial derivatives, so: $f_{xy} = f_{yx}$

**Thu 22 Jan 2026 09:00**

# Lecture 2 - Partial Differentiation II

## 1 The Total Differential

In order to generalise the chain rule, we need to define the total differential. Consider the change in a function of two variables, $f(x, y)$ when we move from some point $(x, y)$ to some point $(x + dx, y + dy)$.

The partial derivative only tells us what happens when we change one variable, but we're changing two here. The total differential sums these two in order to get the full total change.

$$df = f(x + dx, y + dy) - f(x, y)$$

We have to look at the change in a single variable at a time, so we split it into two pieces where only $x$ changes in the first, and only $y$ changes in the second.

$$df = \underbrace{[f(x + dx, y + dy) - f(x, y + dy)]}_{\text{isolates change in } x} + \underbrace{[f(x, y + dy) - f(x, y)]}_{\text{isolates change in } y}$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

More generally for a function of $f(x_1, x_1, \dots, x_n)$:

$$df = \frac{\partial f}{x_1}dx_1 + \frac{\partial f}{x_2}dx_2 + \cdots + \frac{\partial f}{x_n}dx_n = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}dx_i$$

The total change in the function $f(x_1, x_1, \dots, x_n)$ is the sum of partial changes due to changing a single variable.

## 2 The Chain Rule

Recall that if $y = y(x)$, $x = x(t)$, then:

$$dy = \frac{dy}{dx}dx = \frac{dy}{dx}\frac{dx}{dt}dt \implies \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Now, if $f = f(x, y)$ and $x = x(t)$, $y = y(t)$, we can adjust the chain rule to say:

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\ &= \frac{\partial f}{\partial x}\frac{dx}{dt}dt + \frac{\partial f}{\partial y}\frac{dy}{dt}dt \\ \implies \frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \end{aligned}$$

**Example**

$$f(x, y) = x^2 + y^2, \qquad x(t) = t^2, \qquad y(t) = t^3$$

Hence:

$$f(t) = t^4 + t^5 \implies \frac{df}{dt} = 4t^3 + 6t^5$$

By rewriting in terms of one variable:

$$\frac{\partial f}{\partial x} = 2x = 2t^2 \qquad \frac{\partial f}{\partial y} = 2y = 2t^3$$

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 3t^2$$

And instead using the new chain rule:

$$\begin{aligned} &\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= (2t^2 + 2t) + (2t^3)(3t^2) \\ &= 4t^3 + 6t^5 \end{aligned}$$

Hence the new chain rule works!

## 2.1 Polar Coordinates

Suppose our $x$ and $y$ are now functions of two different variables themselves, so:

$$f = f(x, y) \qquad x = x(r, \theta) \qquad y = y(r, \theta)$$

From $r, \theta$ we want to calculate $x, y$ and then from $x, y$ we want to calculate $f$.

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\ dx &= \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta \\ dy &= \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta \end{aligned}$$

Hence:

$$df = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial r}\,dr + \frac{\partial x}{\partial \theta}\,d\theta\right) + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial r}\,dr + \frac{\partial y}{\partial \theta}\,d\theta\right)$$

$$df = \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}\right)dr + \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}\right)d\theta$$

We also know (if we substitute $x$, $y$ into the original function to get a function of $r, \theta$):

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta$$

We can read off the final partial derivatives:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial x}{\partial r}, \qquad \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

As expected!

## 2.2 Generalising

Suppose we have two functions which map $\mathbb{R}^m \to \mathbb{R}^p$, and $\mathbb{R}^p \to \mathbb{R}^n$, respectively:

$$(x_1, x_2, \dots, x_m) \to (y_1, y_2, \dots, y_p) \to (z_1, z_2, \dots, z_n)$$

Then we have:

$$\begin{aligned} dz_i &= \sum_{k=1}^{p} \frac{\partial z_i}{\partial y_k}dy_k \\ dy_k &= \sum_{l=1}^{m} \frac{\partial y_k}{\partial x_l}dx_l \end{aligned}$$

And substituting:

$$dz_i = \sum_{k=1}^{p} \sum_{l=1}^{m} \frac{\partial z_i}{\partial y_k} \frac{\partial y_k}{\partial x_l} dx_l$$

And (as the two sums are independent), we can pull out the inner sum:

$$= \sum_{l=1}^{m} \left[ \sum_{k=1}^{p} \frac{\partial z_i}{\partial y_k} \frac{\partial y_k}{\partial x_l} \right] dx_l$$

$$= \sum_{l=1}^{m} \frac{\partial z_i}{\partial x_l} dx_l$$

The partial derivatives $\partial z_i / \partial x_j$ are therefore given by:

$$\frac{\partial z_i}{\partial x_j} = \sum_{k=1}^{p} \frac{\partial z_i}{\partial y_k} \frac{\partial y_k}{\partial x_j}$$

**Fri 23 Jan 2026 12:00**

# Lecture 3 - Partial Differentiation III

**Fri 13 Feb 2026 12:00**

# Lecture 12 - End of Partial Differentiation & Start of ODEs

Recap of lecture 11:

- The tangent plane to a surface $f(x, y, z) = 0$ at $(x_0, y_0, z_0)$ is given by:

$$\left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0) + \left(\frac{\partial f}{\partial z}\right)_0 (z - z_0) = 0$$

  Such that $\underline{\nabla} f(x_0, y_0, z_0)$ is the normal vector to the plane
- The parametric representation of a curve $\underline{r}(t)$ has:
  - Unit tangent: $\underline{\hat{T}} = \frac{dr}{dt} / \left|\frac{dr}{dt}\right|$
  - Arc length $s(t)$: $\frac{ds}{dt} = \left|\frac{dr}{dt}\right| \to \underline{\hat{T}} = \frac{dr}{ds}$.
  - Unit normal and curvature:

## 1 Orthonormal Triads

We can create an *orthonormal triad* by introducing a new normal vector called the unit binormal, $\underline{\hat{B}} = \underline{\hat{T}} \times \underline{\hat{N}}$. Since $\underline{\hat{N}} \times \underline{\hat{N}}$, differentiating wrt $s$ gives:

$$\underline{\hat{N}} \cdot \frac{d\underline{\hat{N}}}{ds} = 0$$

TODO

We have:

$$\frac{d\underline{\hat{T}}}{ds} = \kappa \underline{\hat{N}}$$

$$\frac{d\underline{\hat{N}}}{ds} = -\kappa \underline{\hat{T}} + \tau \underline{\hat{B}}$$

Hence:

$$\begin{aligned}
\frac{d\underline{\hat{B}}}{ds} &= \frac{d}{ds}\left(\underline{T} \hat{\times} \underline{\hat{N}}\right) \\
&= \frac{d\underline{\hat{T}}}{ds} \times \underline{\hat{N}} + \underline{\hat{T}} \times \frac{d\underline{\hat{N}}}{ds} \\
&= \kappa \underline{\hat{N}} \times \underline{\hat{N}} + \underline{\hat{T}} \times \left(-\kappa \underline{\hat{T}} + \tau \underline{\hat{B}}\right) \\
&= \tau \underline{\hat{T}} \times \underline{\hat{B}} = \tau \underline{\hat{T}} \times \left(\underline{\hat{T}} \times \underline{\hat{N}}\right) \\
&= \tau \left[\left(\underline{\hat{T}} \cdot \underline{\hat{N}}\right) \underline{\hat{T}} - \left(\underline{\hat{T}} \cdot \underline{\hat{T}}\right) \underline{\hat{N}}\right] \\
\\
&= \tau \underline{\hat{N}}
\end{aligned}$$

This gives the Frenet-Serret Formulae:

**This concludes partial differentiation! :D**

# 2 Ordinary Differential Equations

A differential equation is any equation that involves derivatives. We care, because most laws of physics manifest themselves in the form of differential equations. For example:

$$\text{Newton's Second Law:} \qquad \underline{F} = m\frac{d^2 \underline{r}}{dt^2}$$

$$\text{3D Time-Independent Schrödinger Eqn:} \qquad -\frac{\hbar}{2m}\left(\frac{\partial^2\psi}{dx^2} + \frac{\partial^2\psi}{dy^2} + \frac{\partial^2\psi}{dz^2}\right) + V(x,y,z)\psi = E\psi$$

$$\begin{aligned}\text{3D Wave Eqn:} \qquad & \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \\ & = \nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}\end{aligned}$$

$$\text{Gauss' Law:} \qquad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0}$$

$$\text{Navier-Stokes Eqn:} \qquad \rho\left(\frac{\partial \underline{v}}{\partial t} + \left(\underline{v}\cdot\underline{\nabla}\right)\underline{v}\right) = -\underline{\nabla}p + \rho\underline{g} + \mu\nabla^2\underline{v}$$

In this course, we will only solve DEs of a single variable, i.e. Ordinary Differential Equations (ODEs). We don't look at Partial DEs of multiple variables yet.

In order to think about solving these, we need to classify them. Most DEs aren't soluble in closed form with elementary functions and need to be solved numerically. Here, we only consider nice soluble functions, but this is a vast minority in reality. We want to identify classes of DEs we can reasonable solve with a method for each.

We can generally solve linear equations by breaking them into small chunks and solving them individually, for example.

# 3 Types of DEs

## 3.1 Partial vs. Ordinary

In the examples above, only the first was an ODE, and the rest PDEs. Ordinary Differential Equations (ODEs) involve only a single variable.

Consider a vector $\underline{r}(t) = (x(t), y(t), z(t))$. $t$ is called the independent variable, with $x, y, z$ being dependant variables. While we have 3 dependant variables, we only have one independent variable (so only one thing to differentiate wrt), so this would end up being ordinary.

PDEs involve equations of two or more variables and hence involve partial derivatives.

## 3.2 Order

The order of a DE is given by the order of the highest derivative involved, so Newton's 2nd Law is a second order DE, as the highest order derivative is a second derivative.

## 3.3 Degree

The degree of a DE is a less important measure than the others. It is given by the highest power of the highest order derivative. For example, Newton's 2nd is a first degree, while an equation containing $a^3$ would be third degree (and second order, as $a$ is a second derivative).

Ideally, we want this to be 1 for ease of solving, and higher degrees are rare but they do exist. For example, from Lagrangian Mechanics we have:

$$\frac{1}{2m}\left[\left(\frac{\partial s}{\partial x}\right)^2 + \left(\frac{\partial s}{\partial y}\right)^2 + \left(\frac{\partial s}{\partial z}\right)^2\right] + V(x,y,z) = \frac{ds}{dt}$$

### 3.4 Homogenous and Inhomogeneous

A homogenous DE is a DE that does not have any terms of only the independent variable(s), while an inhomogeneous DE does.

For example, Newton's 2nd is homogenous as there is no term that involves $t$ alone. This would be inhomogeneous:

$$\frac{\partial^2 x}{\partial t^2} = t + x$$

While this would be homogenous:

$$\frac{\partial^2 x}{\partial t^2} = tx$$

As $t$ is a coefficient and not a pure term in its own right.

### 3.5 Linear and Non-Linear

A DE is linear if the dependant variable(s) and all of its/their derivatives occur purely as linear functions. For example:

$$\left(1 - x^2\right)\frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

This is linear, as the dependant variable $y$ never has a power greater than 1.

$$\frac{dy}{dx} + xy = 0$$

Is also linear, while this is not:

$$\frac{dy}{dx} + xy^2 = 0$$

This is also non-linear (as shown by the Taylor Expansion of sine):

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

### 3.6 Examples

$$(1) \quad \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = u^2$$

Homogenous first-order second-degree non-linear PDE.

$$(2) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = x^2 + y^2 + z^2$$

Inhomogeneous second-order first-degree linear PDE.

$$(3) \quad \frac{\partial y}{\partial x} + y^2 = x$$

Inhomogeneous first-order second-degree non-linear ODE.

**Wed 18 Feb 2026 11:00**

# Lecture 13 - ODEs II

Summary of Lecture 12:

- Frenet-Serret Equations for a curve $\underline{r}(s)$, where $\hat{T} = d\underline{r}/ds$.
- Classification of differential equations:
  - Partial vs Ordinary
  - Order
  - Degree
  - Homogeneous vs Inhomogeneous
  - Linear vs Non-Linear

## 1 Equations Soluble By Direct Integration

Given some:

$$\frac{dy}{dx} = f(x) \implies y(x) = \int^x f(x')dx' + c$$

We can generalise this to some repeated derivative:

$$\frac{d^n y}{dx^n} = f(x)$$

Instead of having one undetermined constant here, we now have $n$. As each round of integration picks up a factor of the integration subject, these constants will form an $n$th degree polynomial.

### 1.1 Example: Particle Falling

$$\frac{d^2 z}{dt^2} = -g$$
$$\implies \frac{dz}{dt} = -gt + v_0$$
$$z = -\frac{1}{2}gt^2 + v_0 t + z_0$$

Here we consider our unknown constants as boundary conditions, i.e. the initial velocity and initial height.

## 2 Separable Equations

These are equations in the form:

$$\frac{dy}{dx} = \frac{f(x)}{g(x)}$$

$$g(y)dy = f(x)dx \implies \int g(y)dy = \int f(x)dx + c$$

### 2.1 Example I: Falling Particle with Air Resistance

$$\frac{dv}{dt} = -g - kv$$
$$\int \frac{dv}{g + kv} = -\int dt$$
$$\implies \frac{1}{k} \ln (g + kv) = -t + c$$
$$\implies k + gv = Ae^{-kt}$$
$$\implies v(t) = -\frac{g}{k} + \left(v_0 + \frac{g}{k}\right) e^{-kt}$$

### 2.2 Example II

$$\frac{dy}{dx} - x^2y^2 = x^2$$
$$\frac{dy}{dx} = x^2 + x^2y^2 = x^2(y^2 + 1)$$
$$\implies \frac{1}{y^2 + 1} dy = x^2 dx$$
$$\implies \arctan y = \frac{1}{3}x^3 + c$$
$$\implies y = \tan\left(\frac{1}{3}x^3 + c\right)$$

### 2.3 Example III

$$\frac{dy}{dx} = -\frac{x}{y}$$
$$ydy = -xdx$$
$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + c$$
$$y^2 + x^2 = 2c$$

This is the equation for a circle.

## 3 Exact Equations

Suppose a function $y(x)$ is implicitly defined such that $f(x, y) = c$. It follows that the total differential:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

As $f(x, y)$ is a constant for all values of $x, y$, the constant differentiates to zero.

We can set:

$$M(x, y) = \frac{\partial f}{\partial x} \qquad N(x, y) = \frac{\partial f}{\partial y}$$

To try to solve this equation:

$$M(x, y)dx + N(x, y)dy = 0$$

If we can do this (i.e. if the function can be decomposed into gradient components $M$ and $N$), the equation is called "exact". For this to be true, we need:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

If this relationship holds, we have an exact equation and can integrate M and N to get $f(x, y)$ to solve the equation in the form $f(x, y) = c$

### 3.1 Example

$$\frac{dy}{dx} = -\frac{2x+y}{x+2y}$$
$$(2x+y)dx + (x+2y)dy = 0$$

So:

$$M(x,y) = 2x + y \qquad N(x,y) = x + 2y$$

And:

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 1$$

So yes, it is an exact function. It follows that:

$$\frac{\partial f}{\partial x} = 2x + y \implies f(x,y) = x^2 + xy + c(y)$$
$$\frac{\partial f}{\partial y} = x + 2y \implies f(x,y) = xy + y^2 + d(x)$$
$$\implies f(x,y) = x^2 + xy + y^2 = c$$

**Thu 19 Feb 2026 09:00**

# Lecture 14

Recap of last lecture:

- Soluble by direct integration:

$$\frac{dy}{dx} = f(x) \implies y(x) = \int f(x)dx + c$$

- Separable equations:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \implies \int f(x)dx = \int g(y)dy$$

- Exact equations:

  Given in the form (or re-arrangeable to the form):

$$M(x, y)dx + N(x, y)dy = 0$$

  These are soluble if:

$$M(x, y) = \frac{df}{dx} \qquad N(x, y) = \frac{df}{dy}$$

  With $f(x, y) = c$.

  The condition for an equation being exact is:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

## 1 "Homogeneous" Equations

Here, the word homogenous has a different meaning to classification of differentiating equations. If we have some function that satisfied this:

$$g(\lambda x, \lambda y) = \lambda^p g(x, y)$$

We call it a homogeneous function of order $p$.

"Homogeneous" differential equations in this case are functions of the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Here, we make the substitution $v(x) = y(x)/x$. We can write:

$$y(x) = v(x)x$$

And substitute back in. We use the product rule where:

$$\frac{dy}{dx} = x\frac{dv}{dx} + v = f(v)$$

$$x\frac{dv}{dx} = f(v) - v$$

This is now separable, so:

$$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x}$$

## 1.1 Example

$$2xydy - (x^2 + y^2)dx = 0$$

This is not exact, so we put it into a standard form and substitute:

$$2xydy = (x^2 + y^2)dx$$

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy} = \frac{(y/x)^2 + 1}{2(y/x)}$$

Let $u(x) = y/x$:

**Fri 20 Feb 2026 11:56**

# Lecture 15 - Differential Equations III

Summary of last lecture:

- "Homogeneous" Equations
$$\frac{dy}{dx} = f\left(\frac{x}{y}\right) \to x\frac{dv}{dx} = f(v) - v \text{ where } y(x) = xv(x)$$
- Linear Equations
$$\sum_{k=0}^{n} a_k(x)\frac{d^k y}{dx^k} = f(x) \to y(x) = \sum_{k=1}^{n} \alpha_k y_k(x) + y_{PI}(x)$$
The solution is a sum of the complementary function (the general solution of the homogenous equation) and the particular integral (one solution of the inhomogeneous equation).
- First Order Linear Equations
$$\frac{dy}{dx} = P(x)y = Q(x) \to \frac{d}{dx}\left[y(x)e^{\int P(x)dx}\right] = Q(x)e^{\int P(x)dx}$$

## 1 Examples

### 1.1 Example I

$$(1-x^2)\frac{dy}{dx} - xy = 1$$

Rewriting in the standard form:
$$\frac{dy}{dx} - \frac{x}{1-x^2}y = \frac{1}{1-x^2}$$

Hence:
$$P(x) = \frac{-x}{1-x^2}$$
$$I(x) = \exp\left(-\int \frac{xdx}{1-x^2}\right) = \exp\left(\frac{1}{2}\ln(1-x^2)\right) = \sqrt{1-x^2}$$

Multiplying through by the integrating factor:
$$\sqrt{1-x^2}\frac{dy}{dx} - \frac{x}{\sqrt{1-x^2}}y = \frac{1}{\sqrt{1-x^2}}$$

The L.H.S is now the derivative of a product:
$$\frac{d}{dx}\left(y\sqrt{1-x^2}\right) = \frac{1}{\sqrt{1-x^2}}$$

And integrating both sides:
$$y\sqrt{1-x^2} = \arcsin x + c$$
$$\implies \boxed{y = \frac{c}{\sqrt{1-x^2}} + \frac{\arcsin x}{\sqrt{1-x^2}}}$$

### 1.2 Example II

## 2 Linear ODEs with Constant Coefficients

In the most general form, we have:

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = f(x)$$

Firstly, we solve the homogenous equation:

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots a_1 \frac{dy}{dx} + a_0 y = 0$$

Let $y(x) = e^{\lambda x}$:

$$e^{\lambda x} \left( a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 \right) = 0$$

This reduces to finding the zeroes of the corresponding nth order polynomial. If this polynomial has $n$ distinct zeroes, then the complimentary function is given by:

$$y_{CF}(x) = \alpha_1 e^{\lambda_1 x} + \alpha_2 e^{\lambda_2 x} + \cdots + \alpha_n e^{\lambda_n x}$$

If these roots contain a repeated root, this will reduce the number of unique solutions by one. If we have any complex solutions, they wil come in complex conjugate pairs:

$$a^{(\pm ib)x} = e^{ax} (\cos bx \pm i \sin bx)$$

This therefore has independent real solutions:

$$e^{ax} \cos bx$$

$$e^{ax} \sin bx$$

## 3 Equidimensional Equations

These have coefficients which do depend on $x$, but where the coefficients are functions of $x$ such that the $n$th derivative has a coefficient of $a_n x^n$

The general form of a homogeneous equidimensional equation is:

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = f(x)$$

This has a solution in the general form $y(x) = x^{\lambda}$. When we differentiate $k$ times with respect to $x$ we lose $k$ powers of $x$, as each differentiation decreases the power. We need to restore this therefore with a $x^k$ prefactor.

## 4 Mass on a Spring (SHM)

A mass, $m$ on a spring is displaced from its equilibrium position by some distance $x$. There is a restoring force given by $F = -kx$.

$$F = ma \implies F = m \frac{d^2}{dt}$$

$$m \frac{d^2 x}{dt^2} + kx = 0$$

$$\frac{d^2 x}{dt^2} + \omega^2 X = 0 \qquad \text{where } \omega^2 = \frac{k}{m}$$

We have:

$$x(t) = e^{\lambda t}$$

$$(\lambda^2 + \omega^2)e^{\lambda t} = 0$$
$$\lambda^2 + \omega^2 = 0$$

Hence:

$$\lambda = e^{\pm i\omega t}$$

# LC Temperature and Matter

**Tue 20 Jan 2026 12:00**

# Lecture 1

**Tue 20 Jan 2026 12:00**

# Lecture 1

**Tue 20 Jan 2026 12:00**

# Lecture 1

**Tue 20 Jan 2026 12:00**

# Lecture 1

**Tue 20 Jan 2026 12:00**

# Lecture 1

**Tue 20 Jan 2026 12:00**

# Lecture 1

**Mon 09 Feb 2026 11:00**

# Lecture 7 - Introduction to Temperature

We're starting the portion of the module on thermodynamics, looking at temperature, pressure etc and transitions involving these.

## 1 What is Temperature?

Particle for a single particle isn't well defined for the definition of temperature we want to use. There is two definitions:

- One is statistical, looking at average kinetic energy of particles.
- One is based on entropy and isn't looked at until second year.

We define temperature as *a statistical collection of energies for particles that make up the system for which we are measuring the temperature.* A material will have a range of kinetic energies, some very large, some very small, so we care about a statistical average. Thermalisation happens as a result of these high kinetic energy particles striking lower kinetic energy particles.

**Temperature:** A statistical collection of energies of the particles that comprise the system in question.

We can express temperature as using the Maxwell-Boltzmann distribution:

$$P(v) = 4\pi\left(\frac{M}{2\pi RT}\right)^{\frac{3}{2}} v^2 \exp\left(\frac{-Mv^2}{2RT}\right)$$

We look at this in more depth when we cover statistical physics later, but for now we care about the key features. If we plot $P(v)$ (the probability of finding a particle at certain velocity) against these velocities for a range of temperatures:

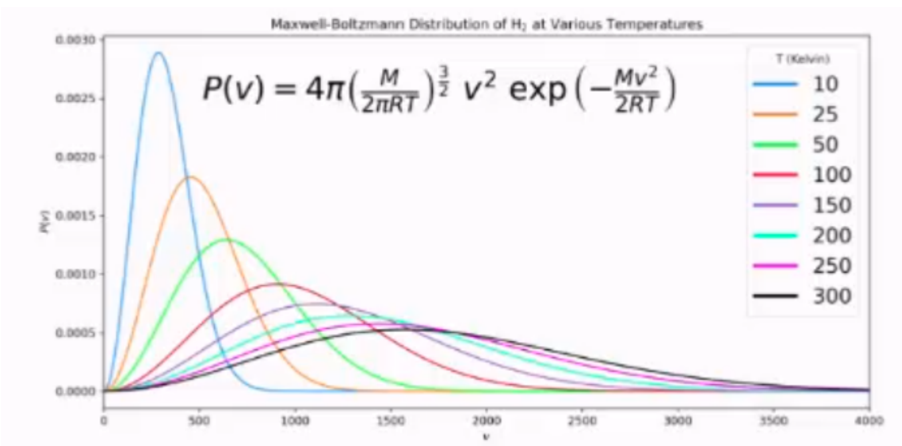



Figure 28.1

This has exponential decay, so theoretically we can find a particle at any velocity within a gas of any temperature. Particles with higher temperatures have a higher peak, so a higher average velocity.

## 2 Temperature Scales

We can use a number of different temperature scales:

- The Fahrenheit scale is set such that the freezing point of water is 98.6°F, and the boiling point of water at 212°F. It's designed to be based around the range of temperatures that a person could realistically could experience in the core range from 0-200. It's not a useful unit and won't be used in exams.
- Celsius is the standard unit.
- Kelvin is the common unit of thermodynamics. It has the same graduations as degrees celsius but is shifted such that absolute zero is at 0K. Note that Kelvin has no degrees symbol. °C → K can be converted between using:

$$T(\mathrm{K}) = T(°\mathrm{C}) + 273.15$$

## 3 Thermal Expansion

As a solid is heated, it expands in all directions. Consider a cuboid of height, width, depth $W, L, D$, we have new dimensions of $D + \Delta W$, $L + \Delta L$, $D + \Delta D$ after a small change in temperature. We can define the linear expansion coefficient (a material property) $\alpha_L$ based on these dimensions and the change in temperature:

$$\alpha_L \Delta T = \frac{\Delta L}{L} = \frac{\Delta W}{W} = \frac{\Delta D}{D}$$

If we consider the area instead, we have the area expansion coefficient:

$$\frac{\Delta A}{A} = \alpha_A \Delta T \approx 2\alpha_L \Delta T$$

The latter approximation can be derived as follows. Consider one dimension, i.e. $L$:

$$\frac{\Delta}{L} = \alpha_L \Delta T$$
$$\implies \Delta L = L\alpha_L \Delta T$$

So:

$$L_{\text{new}} = L + \Delta L = L\left(1 + \alpha_L \Delta T\right)$$

Pairing this with width to find an area (of a face):

$$W_{\text{new}} = W + \Delta W = W(1 + \alpha_L \Delta T)$$
$$A_{\text{old}} = L \times W$$
$$A_{\text{new}} = L_n \times W_n$$
$$A_n = LW(1 + \alpha_L \Delta T)^2$$
$$= LW\left(1 + 2\alpha_L \Delta T + \alpha_L^2 \Delta T^2\right)$$

For a small change in temperature, the second order term is very small, hence:

$$\frac{\Delta(LW)}{LW} \approx 2\alpha_L \Delta T$$

The linear expansion coefficient has units of $\mathrm{K}^{-1}$.

### 3.1 Thermal Stress

Consider a rod fixed between two immovable walls. The rod attempts to expand, but is fixed. This provides a thermal stress on the wall. If the rod has cross-sectional area $A$ and Young Modulus $Y$, the thermal stress is given by:

$$\frac{F}{A} = Y\alpha_L \Delta T$$

This is why bridges etc need to have thermal expansion joints, otherwise they would buckle/break under this thermal stress.

### 3.2 Lennard Jones and Thermal Expansion

**Thu 12 Feb 2026 13:00**

# Lecture 8 - Thermodynamics I

## 1 Temperature Scales II

In order to measure temperature, we need to define a quantity which directly depends on temperature - ideally linearly.

We can't measure temperature directly, we need this extra intermediary quantity. For example, thermometers use volume and measure the volume of a known quantity of a substance which behaves nicely at known temperatures.

This means that:

$$T + T_0 + k\frac{x - x_0}{x_0}$$

Where $T_0$ is a calibration point, at which point our quantity has value $x_0$. For the celsius scale, we use $T_0 = 0°C$ and we choose a value for the constant $k$ such that when water boils, $T = 100°C$.

However, $PV = nRT$ says that this is dependant on pressure too which isn't ideal, as for different pressures the relationship between volume and temperature differ and our scale is only consistent for a single pressure....

Instead of using the boiling/freezing point of water at 1ATM has our calibration point, we can use the "triple point" of water. This point only happens at a single pressure/volume, so is more consistent:

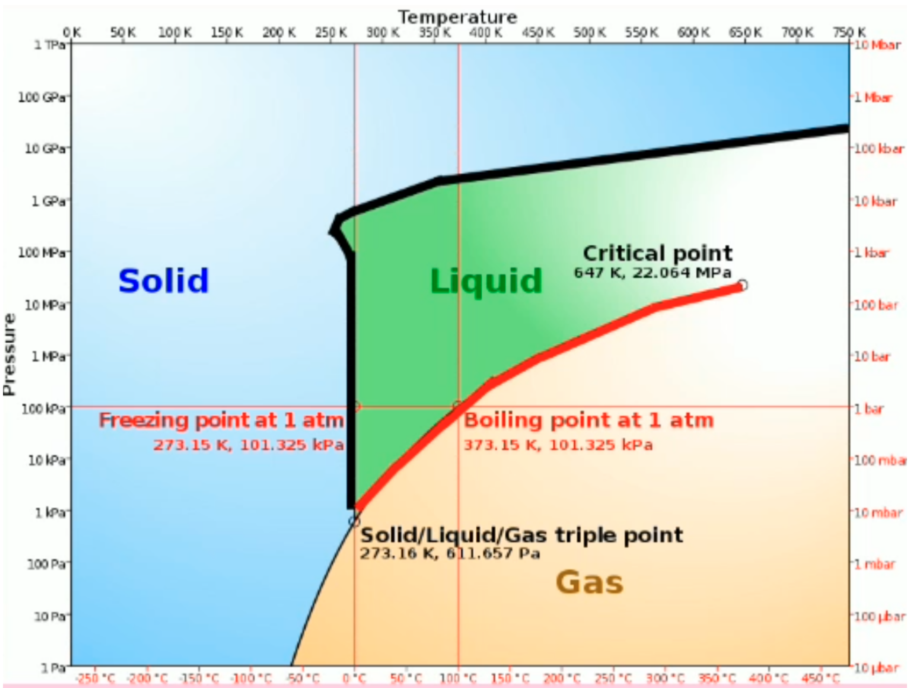



Figure 29.1

Alternatively, we could use the technique employed by the Kelvin scale and use absolute zero as our calibration point. Regardless of pressure/volume/amount of substance (provided its kept constant), they will all reach absolute zero at exactly the same theoretical temperature. We can therefore calibrate our scale off of this point.

# 2 Thermodynamics

**Thermodynamics:** The science of the relationship between heat, work, temperature and energy. Thermodynamics broadly deals with the transfer of energy from one place/form to another. Heat is a form of energy corresponding to a definite amount of mechanical work.

## 2.1 What is heat?

Heat is measure of thermal energy transfer between two systems. This shouldn't be confused with the colloquial understanding of heat i.e. being hot or cold, which is relative.

## 2.2 Systems

A system is a part of the universe we are investigating or discussing eg a room, bottle, galaxy etc. We often consider the ideal physics case of a closed system where the system doesn't exchange any heat with outside the system however this is not realistic.

We often discuss systems comprised of a container of some kind of fluid which either closed (with a lid) or open (no lid).

We can consider two types of walls:

- Adiabatic walls: No heat transfer possible (does not exist in practice).
- Diathermal walls: Heat transfer is possible through the wall.

## 2.3 Thermal Equilibrium

A system is in thermal equilibrium if it has a constant energy density throughout, so the system is in internal thermal equilibrium.

Two systems are in thermal equilibrium if there is no net transfer of heat energy between them. There will be some amount of heat transfer between both systems, and thermal equilibrium does not mean that there is no exchange of heat energy between the two. Some of the particles in one part of the system may statistically have higher energy and exchange energy, but there is no *net* transfer.

Consider some gas in a piston with volume, pressure and temperature $V_I, P_I, T_I$. As we pull out the piston, the volume of the gas changes and we have a new final temperature volume (and as pressure/volume are related to pressure), pressure and temperature $V_F, P_F, T_F$.

This does not happen instantly and there will be some time required to reach thermal equilibrium (a few seconds) after we have finished pulling out the system.

We can bring a system in contact with another system, i.e. placing a rubber duck (system B) in a bath (system A) or the ocean (system C). Which of these two pairs (A and B vs C and B) will thermalise first? While the duck will reach the temperature of the water faster when in the ocean than the bath, thermal equilibrium requires the entire system to thermalise. Therefore, as the ocean has much higher volume it takes longer for the ocean to entirely thermalise to the newer duck-adjusted temperature.

# 3 Definitions

- **Isothermal:** A change to a system which takes place at a **constant temperature**.
- **Isobaric:** A change to a system which takes place at a **constant pressure**.
- **Isochoric:** A change to a system which takes place at a **constant volume**.
- **Adiabatic:** A change to a system which takes place **without transfer of heat**. Temperature may change, but heat cannot enter or leave the system.

# 4 Heat Capacity

What is the relationship between heat energy $\Delta Q$ transferred to a system and the corresponding increase in system temperature $\Delta T$? This is given by:

$$\Delta Q = C\Delta T$$

Where $C$ is the heat capacity in J/K, hence:

$$C = \frac{dQ}{dT}$$

$C$ varies with the amount of material, so we additionally define specific and molar heat capacity:

- Specific Heat Capacity $J/kgK$: $C = mc$
- Molar Heat Capacity $J/molK$: $C = nc$

A higher heat capacity means a smaller increase in temperature for a given heat (i.e. more energy required to raise the temperature of the system by some amount).

To make things more difficult, heat capacity also varies with the system temperature. We also need to specify whether we are measuring isobaric or isochoric, as these will give different values:

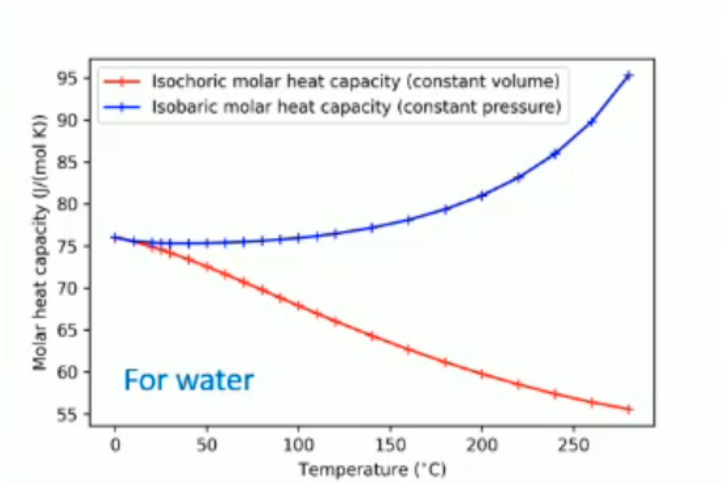



Figure 29.2

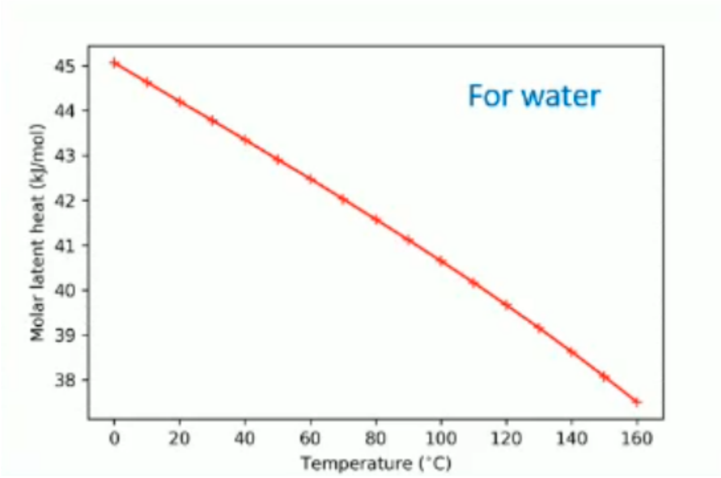



Figure 29.3

Notably, isobaric heat capacity is mostly constant around the standard values of liquid water at standard temperature:

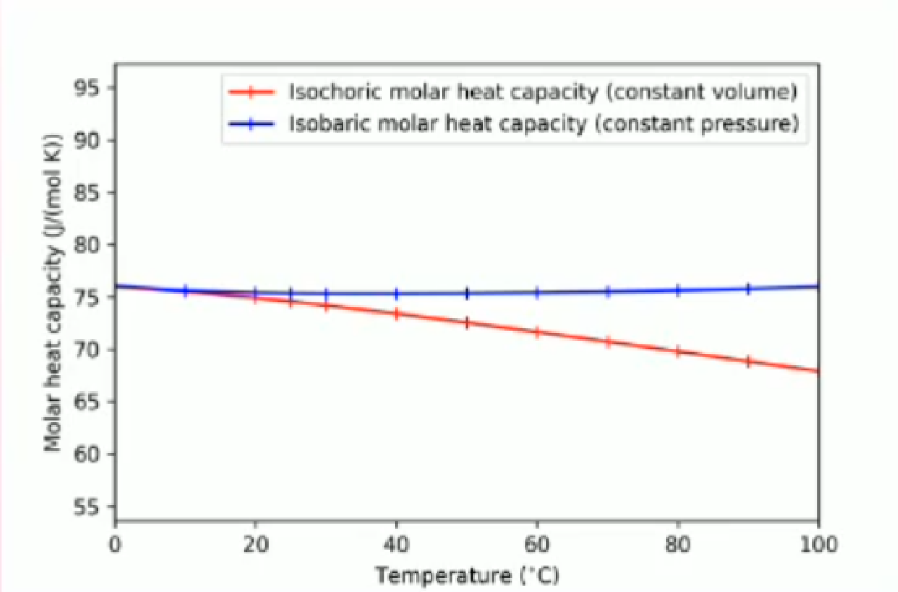



Figure 29.4

It is therefore not unreasonable to treat heat capacity as a constant in some circumstances, provided we state this as an assumption and justify it.

**Tue 17 Feb 2026 12:00**

# Lecture 9 - Laws of Thermodynamics

## 1 0th Law of Thermodynamics

Consider two blocks next to each other, one with $T_h$ and one with $T_c$. The blocks with thermalise to their mean $(T_h + T_c)/2$ in the trivial case where the block are identical.

Net heat will flow between thw two systems until they have the same energy density and hence temperature.

If we have three systems: A, B, C. If we know that A and B are in thermal equilibrium, and A and C are in thermal equilibrium, then B and C are also in thermal equilibrium.

It's a fairly trivial axiom, so we denote it the zeroth law. Effectively, all objects in a system in equilibrium share the same temperature.

## 2 Ideal Gases

An ideal gas is defined as a collection of molecules (or atoms, if monatomic) that are non-interacting with each other (no interatomic forces) and collide elastically with each other.
The internal energy of the gas is dependant on the velocities of the molecules, and hence on the temperature, and not on pressure or volume.

**Boyle's Law:** "The absolute pressure exerted by a given mass of an ideal gas is inversely proportional to the volume is occupies, if the temperature and amount of gas remain unchanged within a closed system."

Effectively:

$$P \propto \frac{1}{V}$$

**Charles' Law:** "When the pressure of a sample of an ideal gas is held constant, the Kelvin temperature and volume will be in direct proportion."

Effectively:

$$T \propto V$$

**Ideal Gas Law:** Since $P \propto \frac{1}{V}$ and $T \propto V$, we have: $PV = kT$, where $k$ varies with context, for example when considering moles:

$$\boxed{PV = nRT}$$

Where $n$ is the number of moles and $R = 8.314\text{Jmol}^{-1}\text{K}^{-1}$ is the gas constant.
Or for molecules:

$$\boxed{PV = Nk_bT}$$

Where $N = N_a n$ is the number of molecules, and $k_b = R/N_a$ is the Boltzmann Constant.

This is called an equation of state and allows us to describe the gas' state macroscopically. We generally express this on a P/V diagram, where each point on the plot represents a specific gas state. If we keep the amount of gas present constant, we can use any two of the variables to determine the third.

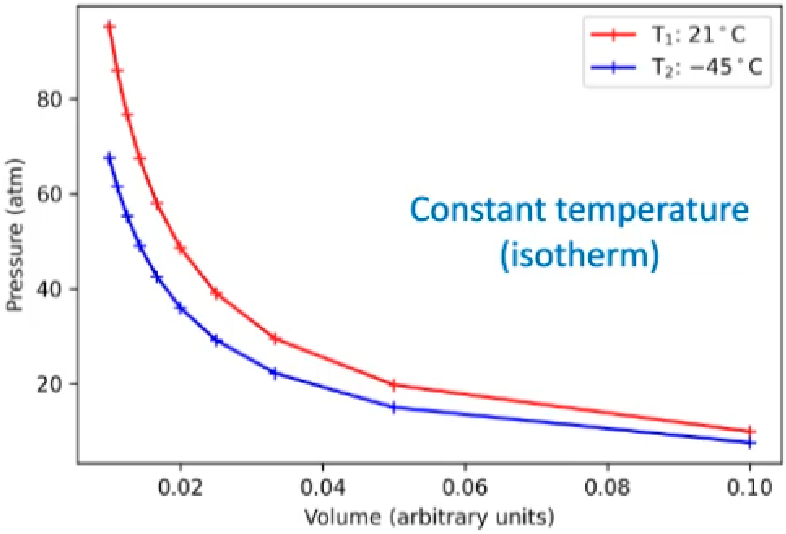



Figure 30.1

## 2.1 Joule's Second Law

Consider a system with a box divided in two. An ideal gas is contained in the leftmost half, and the rightmost half is a vacuum. We separate the two with a divider constituting an impermeable membrane.

We remove the membrane laterally, doing no work on the gas. Joule observed that the gas stayed at the same temperature as it diffused into the vacuum.

Consider the internal energy of the gas, denoted $U$, assuming that U is a function of two of the state variables. We know the temperature did not change, and the volume did, so $U(T, V)$.

We would expect:

$$dU = \frac{\partial U}{\partial T} dT + \frac{\partial U}{\partial V} dV$$

Temperature was observed experimentally to not change, hence $dT = 0$, so:

$$dU = 0 + \frac{\partial U}{\partial V} dV$$

Volume did change, so $dV \neq 0$. We did no work on the gas, as the divider was removed laterally, hence there was no change in internal energy, so:

$$dU = \frac{dU}{dV} dV = 0$$

And since $dU = 0$, we must have:

$$\frac{dU}{dV} \neq 0$$

This means that there is no dependence on volume for internal energy, hence $U$ is dependent only on $T$, as found by Joule. This means that the gas Joule chose was well approximated by an ideal gas.

This is easier at higher temperatures, as a high temperature leads to high kinetic energies, so the interatomic forces becomes less significant and easier to disregard in reality.

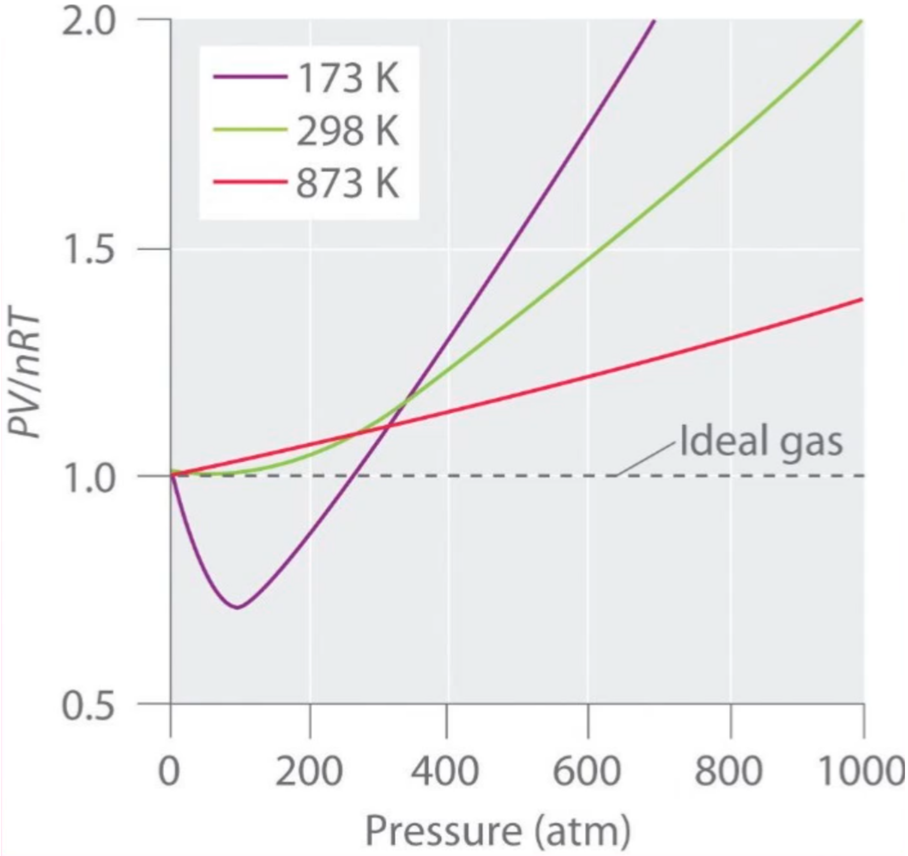



Figure 30.2

We see here that it the ideal gas law becomes a better approximation has we increase temperature, and a worse and worse approximation as pressure increases.

From a LJP perspective, increasing the pressure decreases the average distance between gas molecules. This means that the potential between gas molecules is no longer negligible, and the assumption of zero potential no longer holds.

If we increase pressure even further, the force becomes repulsive and $PV/nRT$ returns to being positive. At higher temperatures, this little dip isn't observed as a higher kinetic energy makes any interatomic forces relatively more negligible.

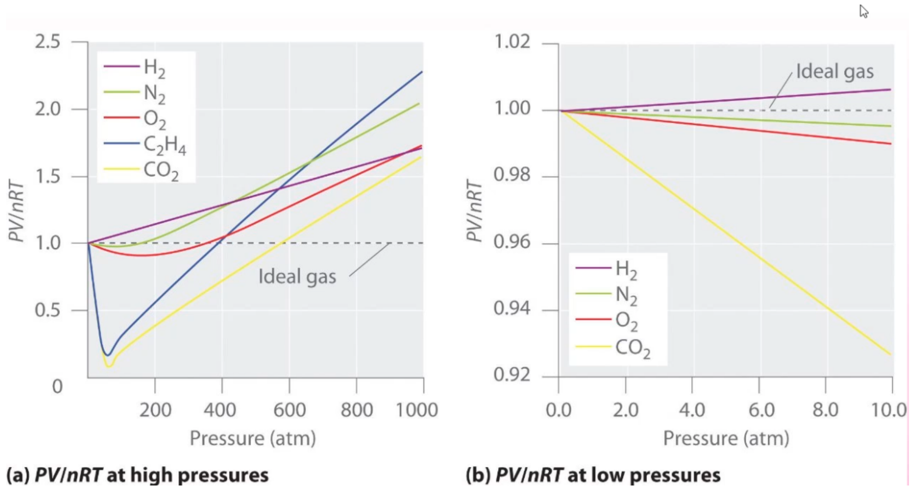



Figure 30.3

At low pressure or high temperature, the ideal gas law is a good assumption as:

- At low pressure, interatomic spacing is large enough to disregard interatomic forces.

- At high temperature, kinetic energy is large enough to comparatively disregard interatomic forces.

We deal solely with Maxwell-Boltzmann gases in this source that follow classical laws. We also have Fermi and Bose gases (made entirely of fermions and bosons respectively), but they're quantum mechanical, exotic and outside of this course.

## 2.2 Changing Energies

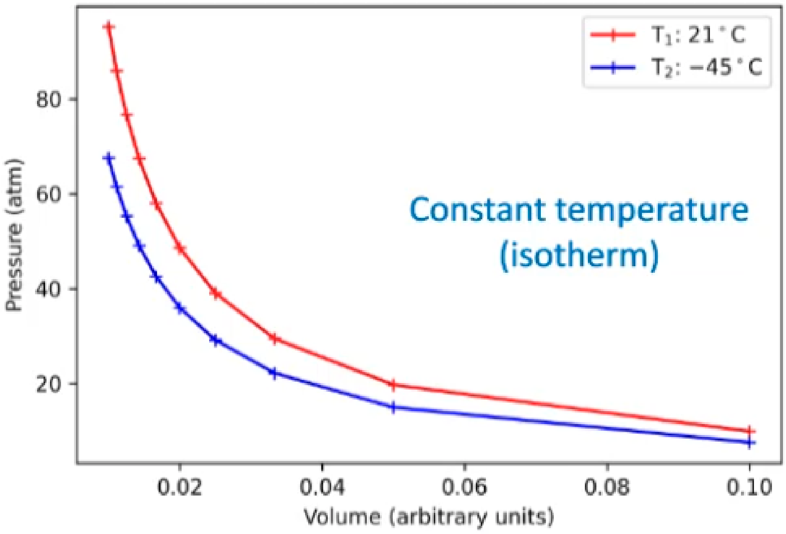



Figure 30.4

Say we want to go from $T_1$ to $T_2$. We need to change the internal energy of the gas somehow. We have two general ways to change this:

- Add heat to the system by transferring heat to the gas at a constant volume.
- Do some work on the gas, i.e. by crushing it and reducing volume.

Consider a piston of area $A$, applying constant force $F$ with extension $\Delta L$. The pressure on the gas is $P = F/A$. The work done on the gas is:

$$\Delta W = F\Delta L$$

$$\implies \Delta W = P(A\Delta L)$$

$$\implies \Delta W = P\Delta V$$

Moving from infinitesimal Delta values:

$$W_{\text{on gas}} = \int dW = \int P\,dV$$

In the opposite case considering work done by the gas, conservation of energy says it must be oppositely signed:

$$W_{\text{by gas}} = -\int P\,dV$$

## 2.3 Mass Drop Experiment

The falling mass spins the paddles in the water, the work as potential energy is converted to kinetic energy raising the temperature of the water.

This tells us that mechanical work done and heat energy must be related in some manner.

# 3 1st Law of Thermodynamics

The change in internal energy of a system, $\Delta U$ is increased with increasing heat transfer into the system, $Q_{\text{in}}$ and work done on the system, $W_{\text{on}}$:

$$\Delta U = Q_{\text{in}} + W_{\text{on}}$$

This is just conservation of energy.

**Thu 19 Feb 2026 13:00**

# Lecture 10 - Thermodynamic Transitions

## 1 Thermodynamic Transitions

### 1.1 Isothermal (Fixed Temperature)

This may look like:

- A curve on a PV plot. As $PV = nRT$, a PV plot shows a $1/x$ relationship for constant $T$.
- A vertical straight line on a PT plot.
- A vertical straight line on a VT plot.

If the temperature change is 0, $\Delta U$ (which we derived last time only depends on temperature) follows:

$$\Delta T = \Delta U = 0$$

Hence:

$$0 = Q_{\text{in}} + W_{\text{on}}$$

### 1.2 Isochoric (Fixed Volume)

This may look like:

- A vertical straight line on a PV plot.
- A proportional relationship on a PT plot.
- A horizontal straight line on a VT plot.

For an isochoric transition, $\Delta U = Q_i n$, as:

$$W_{\text{on}} = (-) \int PdV$$

And as $dV = 0$, $W_{\text{on}} = 0$

### 1.3 Isobaric (Fixed Pressure)

This may look like:

- A horizontal straight line on a PV plot.
- A horizontal straight line on a PT plot.
- A proportional relationship on a VT plot.

For an isobaric transition, $P$ is just a constant, so the integral:

$$W_{\text{on}} = \int_{V_1}^{V_2} P\,dV = [PV]_{v_1}^{v_2}$$

Where $P$ is constant, which is nice and easy. Generally:

$$W_{\text{on}} = -\int PdV = (-)P\Delta V$$

$$\Delta U = \Delta Q - P\Delta V$$

# 2 Heat Capacities of Ideal Gases

Recall that:

$$\Delta Q = C\Delta T \implies C = \frac{dQ}{dT}$$

At a fixed volume:

$$C_V = \left(\frac{dQ_{\text{in}}}{dT}\right)_T$$

And at a fixed pressure:

$$C_P = \left(\frac{dQ_{\text{in}}}{dT}\right)_P$$

Mayer's Relation says that:

$$C_P = C_V + nR$$

This is an examinable derivation.

## 2.1 Deriving Mayer's Relation

*Proof.* Consider a gas piston. We have state variables $V, P, T$, and $N$ is constant as the piston is sealed. The first law of thermodynamics says:

$$\Delta U = Q_{\text{in}} + W_{\text{on}}$$

We can consider this for both a constant volume or a constant pressure. For a constant volume:

$$\Delta U = \Delta Q_1$$

And for a constant pressure:

$$\Delta U = \Delta Q_2 - P\Delta V$$

Noting that the two $Q$s are different as we have two different transitions. We can create a slightly different expression for constant volume by multiplying by $\Delta T/\Delta T$:

$$\Delta U = \frac{\Delta Q_1}{\Delta T} \times \Delta_T = C_V \Delta T$$

And for constant pressure:

$$\Delta Q_2 = \Delta U + P\Delta V = \frac{\Delta Q_2}{\Delta T}\Delta T = C_P \Delta T$$

Hence:

$$C_P\Delta T = \Delta U + P\Delta V$$
$$C_V\Delta T = \Delta U$$

Setting equal to each other based on $\Delta U$:

$$\Delta U = \boxed{C_V\Delta T = C_P\Delta T - P\Delta V}$$
$$C_P\Delta T - C_V\Delta T = P\Delta V$$
$$\Delta T(C_P - C_V) = P\Delta V$$
$$\implies C_P - C_v = \frac{\Delta V}{\Delta T} \times P \qquad \text{NB: } P \text{ is still constant.}$$

Using $PV = nRT$, for constant pressure we have $\frac{P\Delta V}{\Delta T} = nR$. Hence:

$$C_P - C_V = nR$$
$$C_P = C_V + nR$$

$\square$

It is clear that $C_P > C_V$.

Note: We tend to use heat capacity at a constant volume unless specified.

# 3 Back to Thermodynamic Transitions

## 3.1 Adiabatic Transitions (No Heat Transfer)

These seem quite a bit like isothermal transitions, but look different on a PV plot:

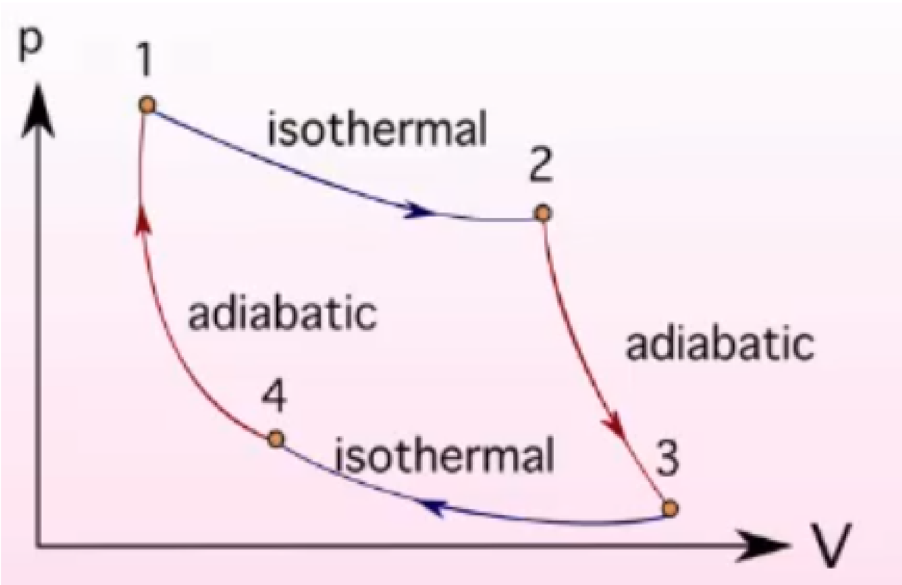



Figure 31.1: Differences in transitions on a PV Plot

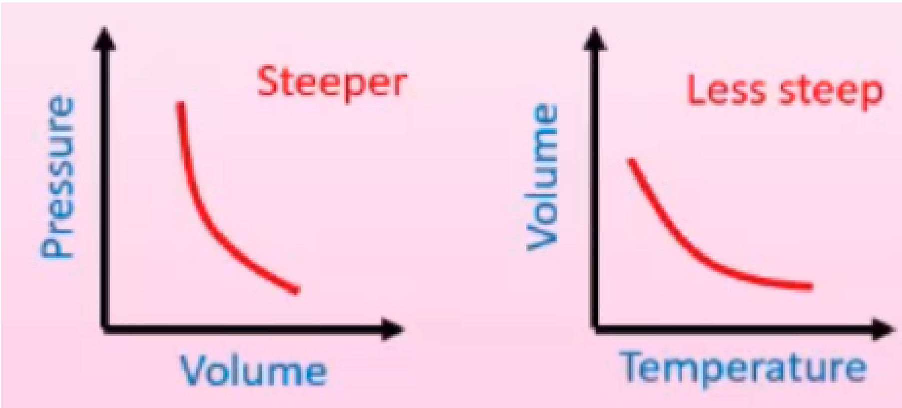



Figure 31.2: Adiabatic Transitions on a PV and VT Plot

For isothermal transitions, $PV = \text{const}$.

For adiabatic transitions, $PV^\gamma = \text{const}$ where $\gamma = \frac{C_P}{C_V}$

In a PV plane, adiabatic processes appear steeper than isothermal processes and adiabatic processing are steeper in the PV plane than the VT plane.

For an adiabatic transition, we can reduce the first law of thermodynamics like so:

$$\Delta U = Q_{\text{in}} + W_{\text{on}}$$

$Q_{\text{in}} = 0$ as there is no heat transfer, hence:

$$\Delta U = W_{\text{on}}$$

## 3.2 Derivations

*Proof.* For adiabatic transitions, where $Q = 0$.

The First Law becomes $\Delta U = Q_{\text{in}} + W_{\text{on}}$. Since $Q_{\text{in}} = 0$:

$$dU = 0 + dW_{\text{on}}$$

And assuming work is being done *on* the gas, the sign becomes negative:

$$dU = -PdV$$
$$\frac{dUdT}{dT} = -PdV$$

We slightly abuse the definition of heat capacity here, and this will be elaborated on in a later lecture:

$$C_V dT = -PdV$$

Using $PV = nRT$, we have: $d(PV) = nRdT$:

$$nRdT = PdV + VdP$$
$$dT = \frac{PdV + VdP}{nR}$$
$$C_V dT = -PdV$$
$$0 = C_V \frac{PdV + VdP}{nR} + PdV$$
$$0 = C_V (PdV + VdP) + nRpdV$$
$$0 = PdV(C_v + nR) + C_V VdP$$
$$0 = PdV(C_P) + VdP(C_V)$$
$$0 = \frac{C_P}{C_V} PdV + VdP$$

Let $\gamma = \frac{C_P}{C_V}$, and dividing through by pressure and volume:

$$\gamma \frac{dV}{V} + \frac{dP}{Ps}$$
$$\int \gamma \frac{dV}{V} = -\int \frac{dP}{P}$$
$$\gamma \ln V + c_1 = -\ln(P) + c_2$$

Finally:

$$\gamma \ln(v) + \ln(p) = \text{const}$$
$$\ln(PV^\gamma) = \text{const}$$
$$PV^\gamma = e^{\text{const}} = \text{another constant}$$

Hence:

$$PV^\gamma = \text{const}$$

As required! □

Note that in a TV plane, we instead have:

$$TV^{\gamma - 1} = \text{const}$$

**Tue 24 Feb 2026 12:00**

# Lecture 11 - PV Planes and Transitions

**Note RE: Last Lecture**
To avoid confusion, the Physics convention is:

$$W_{\text{by}} = \int PdV$$

$$W_{\text{on}} = -\int PdV$$

## 1 PV Diagrams

We can move between points on a PV plane through processes which usually involve some kind of heat transfer and work. For example:

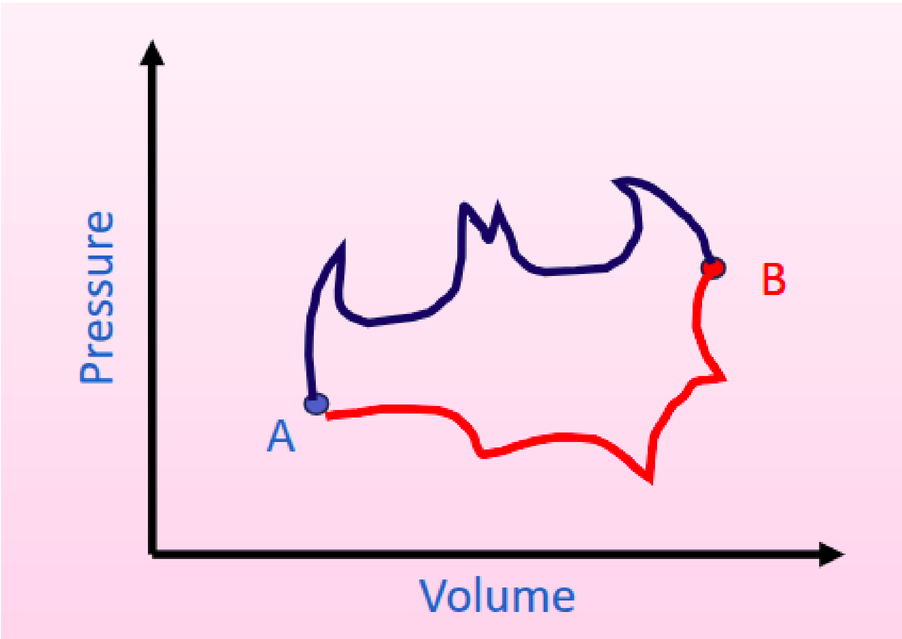



Figure 32.1

The two transitions from $A \to B$ are processes, and the nature of the process depends on the shape of the transition on the PV plot.

- Since the two possible transitions between A and B share the same start and end point, the change in internal energy $U_{AB}$ is the same for both. Since we go from $T_A$ to $T_B$ in both cases (as $U$ depends on $T$, which in turn depends on $P, V$ so for the same $P, V$ we have the same $T$), both paths have the same start/end internal energy, so the same change between them.
- The work done by the gas in the two paths is not the same, however. $W_{\text{by}}$ is larger for the blue path as $W$ is an integral of $\int PdV$, which represents the area under the curve. The blue line is further up the pressure axis, so the integral is larger and more work is done.

- What about heat input to the gas, $Q_{\text{in}}$? Since the First Law says $\Delta U = Q_{\text{in}} + W_{\text{on}} \implies \Delta U = Q_{\text{in}} - W_{\text{by}}$. $\Delta U$ is constant, and $-W_{\text{by}}$ has a larger magnitude for blue, so must $Q_{\text{in}}$ to keep $\Delta U$ constant.

These transitions are called a *PV Cycle* as we can start at $A$, go to $B$ and end back up at $A$.

Consider a simple example with straight lines:

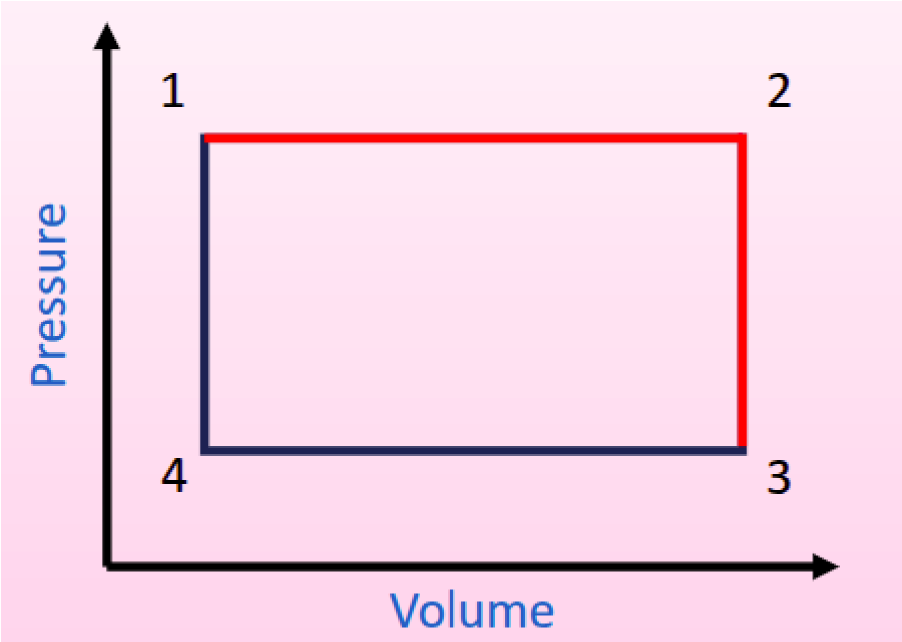



Figure 32.2

Where $1 \to 2$ and $3 \to 4$ are isochoric processes and $2 \to 3$ and $4 \to 1$ are isobaric. We can also look at isothermal and adiabatic processes:

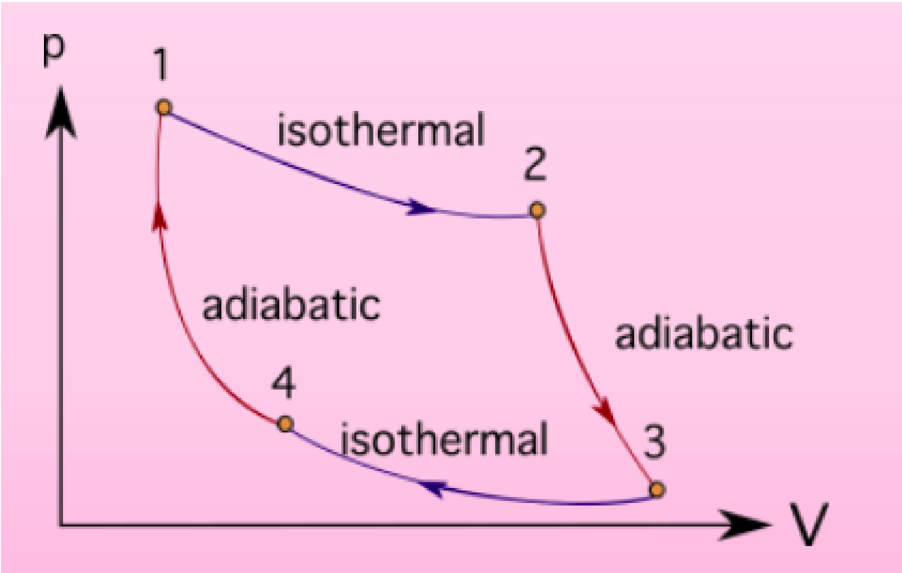



Figure 32.3

A cycle must follow these rules:

1. $\Delta U = 0$, for a full cycle.
2. Total work done is given by the area contained by the cycle.
3. Clockwise cycle for positive work, anticlockwise for negative work.

4. $Q_{\text{in}} + W_{\text{on}} = 0$ as $\Delta U = 0$.

## 2 Otto Cycle

This describes a petrol internal combustion engine:

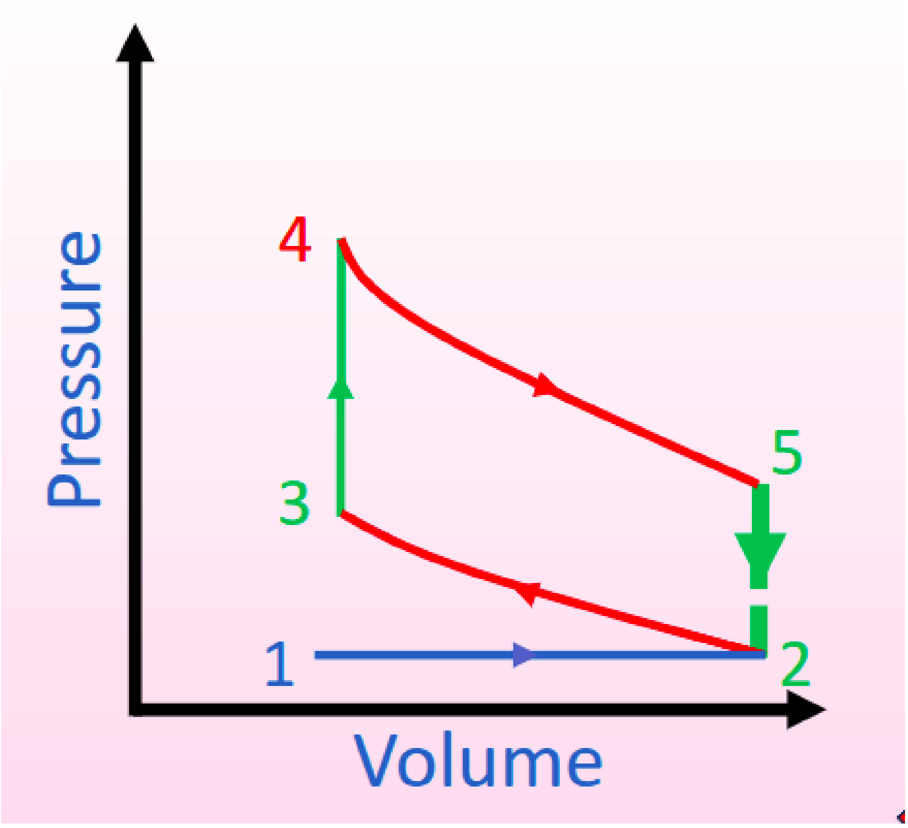



Figure 32.4: Blue: Isobaric, Green: Isochoric, Red: Adiabatic

This has the following components:

1. 1 → 2: Intake stroke: Isobaric introduction of air and fuel to the system.
2. 2 → 3: Adiabatic compression of the fluid as the piston compresses it.
3. 3 → 4: Isochoric introduction of heat energy to the system (ignition of the fluid mixture).
4. 4 → 5: Adiabatic expansion of the fluid mixture as the gas does work on the piston (pushing it down).
5. 5 → 2: Isochoric expulsion of heat from the system.
6. 2 → 1: Mass of air and exhaust gases released at a constant pressure.

## 3 Diesel Cycle

Here, the ignition and combustion process is performed at a constant pressure where work is done by the gas:

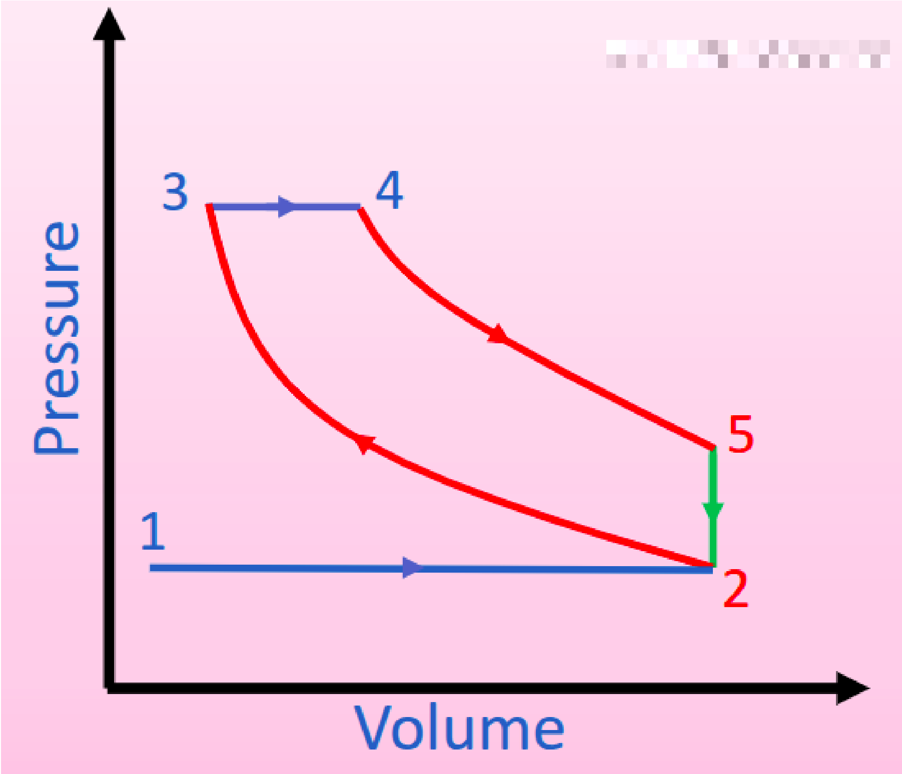



Figure 32.5: Blue: Isobaric, Green: Isochoric, Red: Adiabatic

In both cases, heat is introduced (by combustion) from $3 \to 4$, denoted $Q_1$ and heat is lost from $5 \to 2$ when heat is expelled from the system. The useful work done is $Q_1 - Q_2$ and is given by the area enclosed by the loop.

## 3.1 Example

Consider the adiabatic gas compression stroke in a diesel engine. (This is the adiabatic stroke where pressure is increasing/volume is decreasing, i.e. from $2 \to 3$.)

$10^{-3}\text{m}^3$ of nitrogen gas is initially at atmospheric pressure and 298K and is compressed to 1/15th of its original volume.

You may assume that nitrogen is an ideal diatomic gas with $\gamma = 1.4$ and $C_V = 20.85J/Kmol$.

Calculate:

1. The final pressure.
2. The final temperature.
3. The number of moles in the gas.
4. The change in internal energy.
5. The work done by the gas.

**Part I**
Atmospheric pressure is $101325 \approx 10^5$Pa. As this is an adiabatic transition, $PV^\gamma = \text{const}$. Hence:

$$P_1 V_1^\gamma = P_2 V_2^\gamma$$

$$\implies P_2 = P_1 \left(\frac{V_1}{V_2}\right)^\gamma$$

$$\implies P_2 = P_1 \left(\frac{V_1}{(V_1/15)}\right)^\gamma$$

$$\implies P_2 = P_1 (15)^\gamma = 4.5\text{MPa}$$

**Part II**
We can use either $PV = nRT$ twice, using the initial conditions to determine $n$ or we can use:

$$T_1 V_1^{(\gamma-1)} = T_2 V_2^{(\gamma-1)}$$

Using the same process gives:

$$T_2 = T_1 (15)^{(\gamma-1)} = 880\text{K}$$

**Part III**
Since we know $P_1, V_1$ or $P_2, V_2$ we can use $PV = nRT$, as mentioned above. It would be a good sanity check to do both and compare the values.

**Part IV**
It's an adiabatic transition, we can related $C_V = \frac{\partial U}{\partial T}$.

Integrating both sides gives:

$$\int dT = \int \frac{\partial U}{\partial T}$$

**Thu 26 Feb 2026 13:00**

# Lecture 12 - Heat Transfer

## 1 Thermal Transport

**Newton's Law of Cooling:** "The rate of heat loss of a body is directly proportional to the difference in the temperatures between the body and the environment"

There are three means of thermal transport:

- **Conduction:** Heat is transmitted directly from one material to an adjoining material (or across parts of the same material) if there is a temperature difference between the two. There is no movement of the material.
- **Convection:** Movement of particles through a substances (often particles that comprise a fluid) that transport their heat energy along with them.
  - The Archimedes principle describes buoyancy based on the material's density $\rho$, $g$ and the volume of the the fluid displaced by the object, $V$:

$$F_b = \rho g V$$

  - In convection, heating a particle causes an increase in bond size (by the LJP), hence the density decreases. This causes it to displace gas of a larger density, hence it rises.
- **Radiation:** Emission of electromagnetic radiation by all bodies which have heat (i.e. all bodies full stop), depending on their temperature.

### 1.1 Heat Baths

A heat bath is an object with a sufficiently large heat capacity, $C$ such that a large change in its heat energy will result in a negligible change in temperature.

For example, a hot object placed in a very large bath of water (i.e. a whole swimming pool), or a building placed on a planet.

## 2 Conduction

Suppose we have a small uniform rod of length $L$ and cross-sectional area $A$ linking two heat baths. The rod is conducting and the heat baths have temperature $T_1, T_2$.

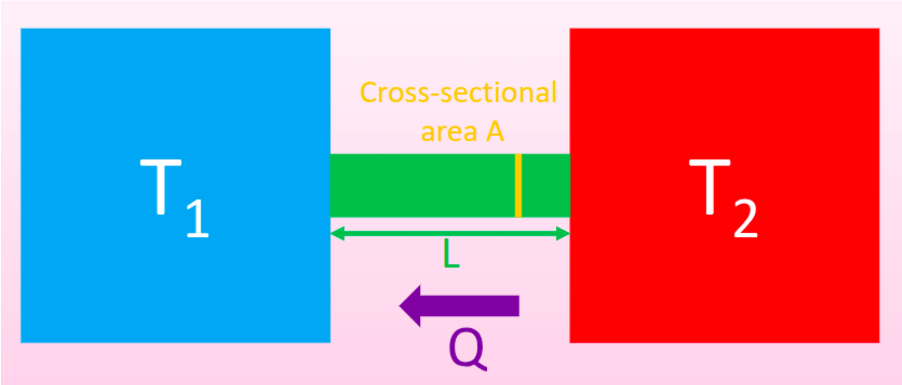



Figure 33.1

Eventually, we reach a "steady state" where all points on the road have a $Q$ which does not change with time.

$$\frac{dQ}{dt} = \dot{Q} = \text{constant}, \qquad \forall \text{x along rod (in steady state)}$$

What does this value of $\dot{Q}$ depend on? For a small section $dx$ of the rod:

- Temperature change of the small section, $dT$.
- Cross-sectional area of the rod $A$.
- The "thermal conductivity" of the rod, $\varkappa$[1]. It has units $\text{Wm}^{-1}\text{K}^{-1}$.

Putting these together gives us Fourier's Law:

$$\dot{Q} = -\varkappa A \frac{dT}{dX}$$

As $T_2$ is hotter, the flow of heat is from right to left. Therefore until thermalisation occurs, temperature increases from left to right across the rod ($\frac{dT}{dx} > 0$). However, the heat flow is in the opposite direction from right to left, hence the negative sign in the formula for $\dot{Q}$.

We want to show that $\frac{dT}{dx} = \frac{T_2 - T_1}{L}$:

$$\begin{aligned}
\dot{Q} &= -\varkappa A \frac{dT}{dX} \\
\frac{dT}{dx} &= -\frac{\dot{Q}}{\varkappa A} \\
-\int \frac{\dot{Q}}{\varkappa A} dx &= \int dT \\
-\int_0^L \frac{\dot{Q}}{\varkappa A} &= \int_{T_1}^{T_2} dT \\
-\frac{\dot{Q}}{\varkappa A} [x]_0^L &= T_2 - T_1 \\
\frac{dT}{dx} &= \frac{T_2 - T_1}{L}
\end{aligned}$$

## 2.1 Thermal Conductivity

A higher value of $\varkappa$ means a material is more thermally conductive, for example:

- Diamond (natural): $\varkappa = 2200\text{Wm}^{-1}\text{K}^{-1}$.
- Copper: $\varkappa = 400\text{Wm}^{-1}\text{K}^{-1}$.
- Air: $\varkappa = 0.02\text{Wm}^{-1}\text{K}^{-1}$.

[1]We also have another very similar quantity labelled $K$ which will be covered later and may be a source of confusion

- Aerogel: $\kappa = 0.003\text{Wm}^{-1}\text{K}^{-1}$.

Diamond is highly thermally conductive due to high frequency vibrations ("phonons") being able to propagate through the material as a result of its rigidity.

There are two sources of heat conductivity, one being as a result of electrons and one as a result of phonons.

- Inside the material, we have two bands - valence and conduction. These bands are a thick collection of many energy levels very close together. As the Pauli exclusion principle says that no two electrons can share a quantum number, each level must shift infinitesimally to not overlap. The levels within each band are so close that we treat electron as being able to move freely between them.
- The valence band is where electrons are still bound to an atom, the conduction band is where they are bound to the material as a whole and are free to move through the material.
- For an insulator, the difference between the two bands is $\sim$ 10eV. This represents a very small proportion of particles with sufficient energy and results in very little thermal excitation (hence little thermal conduction). As an electron is excited, it leaves a hole behind (that we treat as a particle $h^+$ for ease). We treat this hole as being able to move (as electrons themselves move).
- For an electron to fall back to the valence band, it must fall into a hole. In an insulator the conduction band is effectively empty. For a semiconductor, this gap is more like $\sim$ 1eV, so a much higher number of electrons can jump, but this is still temperature dependant.
- In metals, the conduction and valence bands have little to no gap. Therefore, all electrons are free conduction electrons.

Diamonds sit on the boundary between being an insulator and a semiconductor. They work by phonons. As we heat a material, bonds in the material vibrate. As diamond is very rigid, vibrating one bond causes a ripple effect - effectively an wave of excitation (where one of these waves is called a phonon). This phonon propagation causes the transfer of vibrations hence the transfer of heat energy.

## 2.2 Thermal Properties

We can also define the:

- Thermal Resistivity: $\rho = 1/\kappa$.
- Thermal Conductance $K = \kappa A/L$. [2]
- Thermal Resistance: $R = L/(\kappa A)$.

This are the heat analogues of the equivalent terms for electrical conductance in a circuit.

Annoyingly, thermal conductivity also varies with temperature…It also does so in a really weird way depending on the material:

[2]This is the aforementioned annoying symbol overlap between $\kappa$ and K

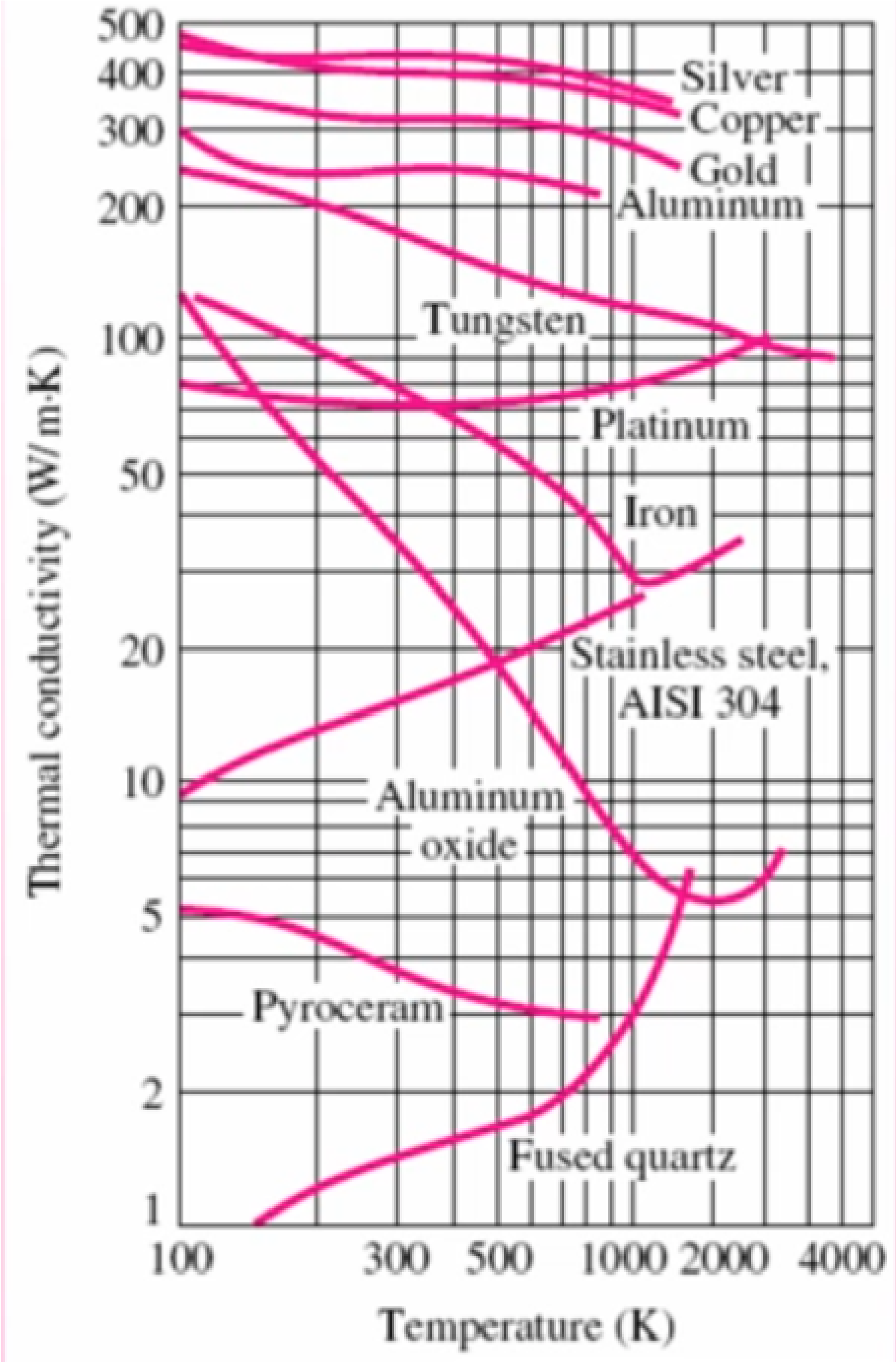



Figure 33.2: $\kappa$ for various materials, as a function of temperature.

This comes from phonon excitation and electron excitation being two different competing process that both have different degrees of contribution at different temperatures.

This means that Newton's Law of Cooling no longer holds for $\kappa = f(T)$, as the proportionality no longer holds...