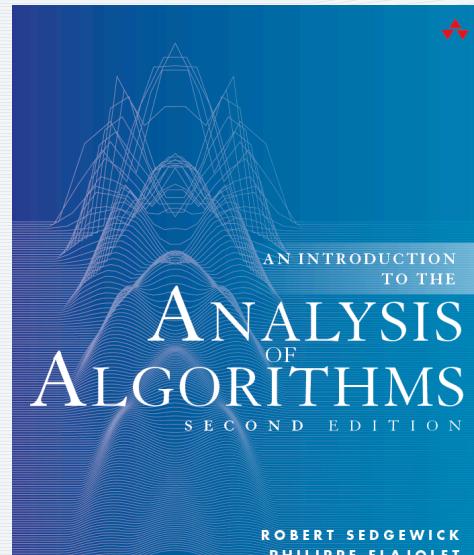


ANALYTIC COMBINATORICS

PART ONE

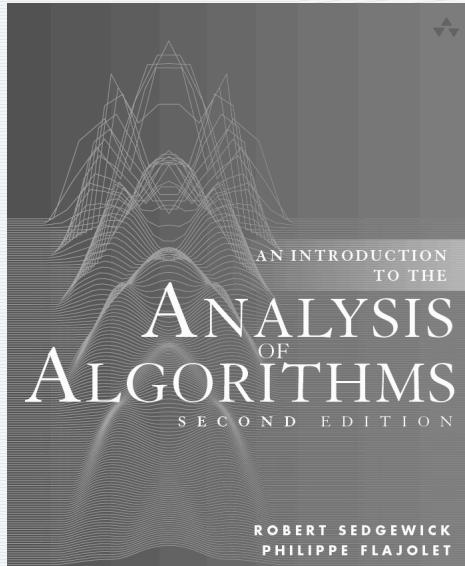


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3. Generating Functions

ANALYTIC COMBINATORICS

PART ONE



3. Generating Functions

- OGFs
- Solving recurrences
- Catalan numbers
- EGFs
- Counting with GFs

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Ordinary generating functions

Definition.

$$A(z) = \sum_{k \geq 0} a_k z^k \quad \text{is the ordinary generating function (OGF)}$$

of the sequence $a_0, a_1, a_2, \dots, a_k, \dots$

Notation. $[z^n]A(z)$ is “the coefficient of z^n in $A(z)$ ”

sequence	OGF
$1, 1, 1, 1, 1, \dots$	$\sum_{N \geq 0} z^N = \frac{1}{1-z}$
$1, 1/2, 1/6, 1/24, \dots$	$\sum_{N \geq 0} \frac{z^N}{N!} = e^z \quad \leftarrow [z^N]e^z = 1/N!$

Significance. Can represent an entire sequence with a single function.

Operations on OGFs: Scaling

If $A(z) = \sum_{k \geq 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \dots, a_k, \dots$

then $A(cz) = \sum_{k \geq 0} a_k c^k z^k$ is the OGF of $a_0, ca_1, c^2 a_2, c^3 a_3, \dots$

sequence	OGF
1, 1, 1, 1, 1, ...	$\sum_{N \geq 0} z^N = \frac{1}{1-z}$
1, 2, 4, 8, 16, 32, ...	$\sum_{N \geq 0} 2^N z^N = \frac{1}{1-2z}$ ← $[z^N] \frac{1}{1-2z} = 2^N$

Operations on OGFs: Addition

If $A(z) = \sum_{k \geq 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \dots, a_k, \dots$

and $B(z) = \sum_{k \geq 0} b_k z^k$ is the OGF of $b_0, b_1, b_2, \dots, b_k, \dots$

then $A(z) + B(z)$ is the OGF of $a_0 + b_0, a_1 + b_1, a_2 + b_2, \dots, a_k + b_k \dots$

Example:

sequence	OGF
1, 1, 1, 1, 1, ...	$\sum_{N \geq 0} z^N = \frac{1}{1-z}$
1, 2, 4, 8, 16, 32, ...	$\sum_{N \geq 0} 2^N z^N = \frac{1}{1-2z}$
0, 1, 3, 7, 15, 31, ...	$\frac{1}{1-2z} - \frac{1}{1-z}$

Operations on OGFs: Differentiation

If $A(z) = \sum_{k \geq 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \dots, a_k, \dots$

then $zA'(z) = \sum_{k \geq 1} k a_k z^k$ is the OGF of $0, a_1, 2a_2, 3a_3, \dots, k a_k, \dots$

OGF	sequence
$\frac{1}{1-z} = \sum_{N \geq 0} z^N$	1, 1, 1, 1, 1, ...
$\frac{z}{(1-z)^2} = \sum_{N \geq 1} N z^N$	0, 1, 2, 3, 4, 5, ...
$\frac{z^2}{(1-z)^3} = \sum_{N \geq 2} \binom{N}{2} z^N$	0, 0, 1, 3, 6, 10, ...
$\frac{z^M}{(1-z)^{M+1}} = \sum_{N \geq M} \binom{N}{M} z^N$	0, ..., 1, M+1, (M+2)(M+1)/2, ...
$\frac{1}{(1-z)^{M+1}} = \sum_{N \geq 0} \binom{N+M}{M} z^N$	1, M+1, (M+2)(M+1)/2, ...

Operations on OGFs: Integration

If $A(z) = \sum_{k \geq 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \dots, a_k, \dots$

then $\int_0^z A(t) dt = \sum_{n \geq 1} \frac{a_{n-1}}{n} z^n$ is the OGF of $0, a_0, \frac{a_1}{2}, \frac{a_2}{3}, \dots, \frac{a_{k-1}}{k}, \dots$

Example:

OGF	sequence
$\frac{1}{1-z} = \sum_{N \geq 0} z^N$	1, 1, 1, 1, 1, ...
$\ln \frac{1}{1-z} = \sum_{N \geq 1} \frac{z^N}{N}$	0, 1, 1/2, 1/3, 1/4, 1/5, ...

Operations on OGFs: Partial sum

If $A(z) = \sum_{k \geq 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \dots, a_k, \dots$

then $\frac{1}{1-z} A(z)$ is the OGF of $a_0, a_0 + a_1, a_0 + a_1 + a_2, \dots$

Proof.

$$\frac{1}{1-z} A(z) = \sum_{k \geq 0} z^k \sum_{n \geq 0} a_n z^n$$

Distribute

$$= \sum_{k \geq 0} \sum_{n \geq 0} a_n z^{n+k}$$

Change n to $n-k$

$$= \sum_{k \geq 0} \sum_{n \geq k} a_{n-k} z^n$$

Switch order of summation.

$$= \sum_{n \geq 0} \left(\sum_{0 \leq k \leq n} a_{n-k} \right) z^n$$

Change k to $n-k$

$$= \sum_{n \geq 0} \left(\sum_{0 \leq k \leq n} a_k \right) z^n$$

Operations on OGFs: Partial sum

If $A(z) = \sum_{k \geq 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \dots, a_k, \dots$

then $\frac{1}{1-z} A(z)$ is the OGF of $a_0, a_0 + a_1, a_0 + a_1 + a_2, \dots$

Example:

OGF	sequence
$\frac{1}{1-z} = \sum_{N \geq 0} z^N$	1, 1, 1, 1, 1, ...
$\ln \frac{1}{1-z} = \sum_{N \geq 1} \frac{z^N}{N}$	0, 1, 1/2, 1/3, 1/4, 1/5, ...
$\frac{1}{1-z} \ln \frac{1}{1-z} = \sum_{N \geq 1} H_N z^N$	1, 1 + 1/2, 1 + 1/2 + 1/3, ...

Operations on OGFs: Convolution

If $A(z) = \sum_{k \geq 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \dots, a_k, \dots$

and $B(z) = \sum_{k \geq 0} b_k z^k$ is the OGF of $b_0, b_1, b_2, \dots, b_k, \dots$

then $A(z)B(z)$ is the OGF of $a_0b_0, a_1b_0 + a_0b_1, \dots, \sum_{0 \leq k \leq n} a_k b_{n-k}, \dots$

Proof.

$$A(z)B(z) = \sum_{k \geq 0} a_k z^k \sum_{n \geq 0} b_n z^n$$

Distribute

$$= \sum_{k \geq 0} \sum_{n \geq 0} a_k b_n z^{n+k}$$

Change n to $n-k$

$$= \sum_{k \geq 0} \sum_{n \geq k} a_k b_{n-k} z^n$$

Switch order of summation.

$$= \sum_{n \geq 0} a_k b_{n-k} z^n$$

Operations on OGFs: Convolution

If $A(z) = \sum_{k \geq 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \dots, a_k, \dots$

and $B(z) = \sum_{k \geq 0} b_k z^k$ is the OGF of $b_0, b_1, b_2, \dots, b_k, \dots$

then $A(z)B(z)$ is the OGF of $a_0b_0, a_1b_0 + a_0b_1, \dots, \sum_{0 \leq k \leq n} a_k b_{n-k}, \dots$

Example:

OGF	sequence
$\frac{1}{1-z} = \sum_{N \geq 0} z^N$	1, 1, 1, 1, 1, ...
$\frac{1}{(1-z)^2} = \sum_{N \geq 0} (N+1)z^N$	1, 2, 3, 4, 5, ...

Expanding a GF (summary)

The process of expressing an unknown GF as a power series (finding the coefficients) is known as **expanding** the GF.

Techniques we have been using:

1. Taylor theorem: $f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \frac{f''''(0)}{4!}z^4 + \dots$

Example.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

2. Reduce to known GFs.

Example.

$$[z^N] \frac{1}{(1-z)} \ln \frac{1}{1-z} = \sum_{1 \leq k \leq N} \frac{1}{k} = H_N$$

Integrate $\frac{1}{1-z}$ to get $\ln \frac{1}{1-z}$
then convolve $\frac{1}{1-z}$ with $\ln \frac{1}{1-z}$

In-class exercise

Exercise 3.4 Prove that $\sum_{1 \leq k \leq N} H_k = (N+1)(H_{N+1} - 1)$

1. Find GF for LHS (convolve $\frac{1}{1-z}$ with $\frac{1}{1-z} \ln \frac{1}{1-z}$)

$$\frac{1}{(1-z)^2} \ln \frac{1}{1-z}$$

2. Expand GF to find RHS coefficients (convolve $\ln \frac{1}{1-z}$ with $\frac{1}{(1-z)^2}$)

$$[z^N] \frac{1}{(1-z)^2} \ln \frac{1}{1-z} = \sum_{1 \leq k \leq N} \frac{1}{k} (N+1-k)$$

Operations on OGFs: Partial sum

If $A(z) = \sum_{k \geq 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \dots, a_k, \dots$

then $\frac{1}{1-z} A(z)$ is the OGF of $a_0, a_0 + a_1, a_0 + a_1 + a_2, \dots$

Example:	OGF	sequence
	$\frac{1}{1-z} = \sum_{N \geq 0} z^N$	1, 1, 1, 1, 1, ...
	$\ln \frac{1}{1-z} = \sum_{N \geq 1} \frac{z^N}{N}$	0, 1, 1/2, 1/3, 1/4, 1/5, ...
	$\frac{1}{1-z} \ln \frac{1}{1-z} = \sum_{N \geq 1} H_N z^N$	1, 1 + 1/2, 1 + 1/2 + 1/3, ...

3. Do some math

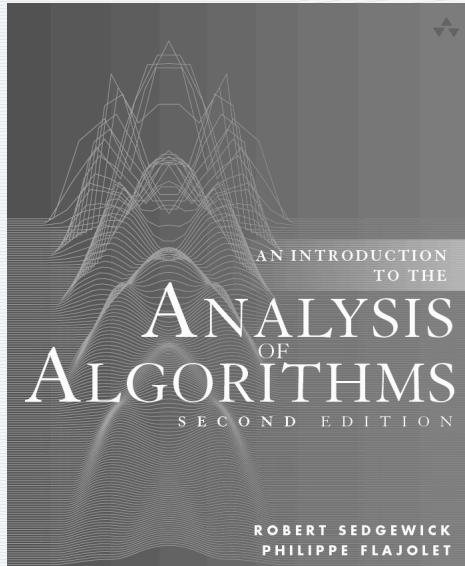
$$= (N+1)H_N - N$$

$$= (N+1)(H_{N+1} - \frac{1}{N+1}) - N$$

$$= (N+1)(H_{N+1} - 1)$$

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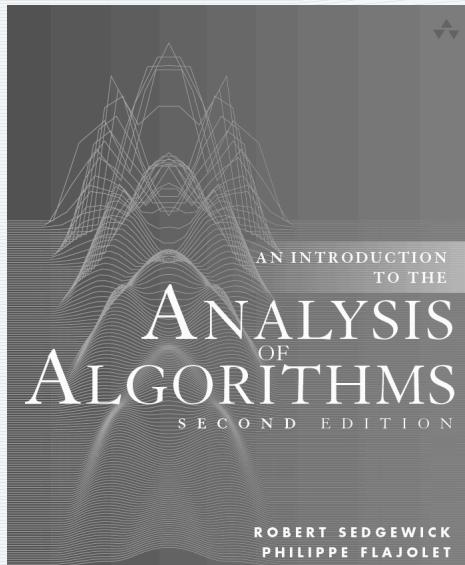
3. Generating Functions

- OGFs
- Solving recurrences
- Catalan numbers
- EGFs
- Counting with GFs

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ANALYTIC COMBINATORICS

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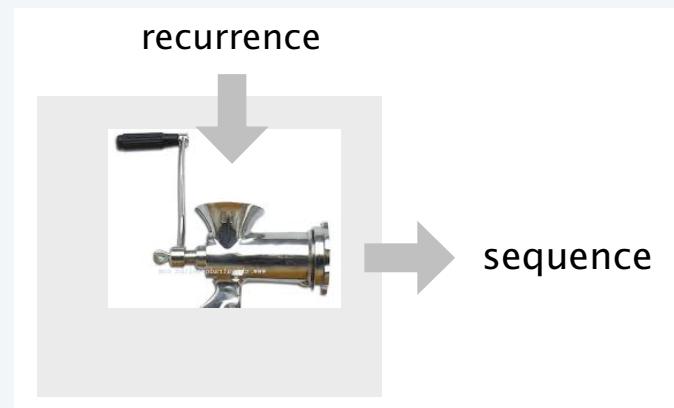
3. Generating Functions

- OGFs
- **Solving recurrences**
- Catalan numbers
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Solving recurrences with OGFs

General procedure:

- Make recurrence valid for all n .
- Multiply both sides of the recurrence by z^n and sum on n .
- Evaluate the sums to derive an equation satisfied by the OGF.
- Solve the equation to derive an explicit formula for the OGF.
(Use the initial conditions!)
- Expand the OGF to find coefficients.



Solving recurrences with GFs

For linear recurrences with constant coefficients, the GF equation is a polynomial, so the general procedure is a **algorithm**.

Example 4 from previous lecture.

$$a_n = 5a_{n-1} - 6a_{n-2} \quad \text{for } n \geq 2 \text{ with } a_0 = 0 \text{ and } a_1 = 1$$

Make recurrence valid for all n .

$$a_n = 5a_{n-1} - 6a_{n-2} + \delta_{n1}$$

Multiply by z^n and sum on n .

$$A(z) = 5zA(z) - 6z^2A(z) + z$$

Solve.

$$A(z) = \frac{z}{1 - 5z + 6z^2}$$

Use partial fractions:
solution must be of the form

$$A(z) = \frac{c_0}{1 - 3z} + \frac{c_1}{1 - 2z}$$

Solve for coefficients.

$$c_0 + c_1 = 0$$

$$2c_0 + 3c_1 = -1$$

Solution is $c_0 = 1$ and $c_1 = -1$

$$A(z) = \frac{1}{1 - 3z} - \frac{1}{1 - 2z}$$

Expand.

$$a_n = 3^n - 2^n$$

Solving linear recurrences with GFs

For linear recurrences with constant coefficients, the GF equation is a polynomial, so the general procedure is a **algorithm**.

Example with multiple roots.

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3} \quad \text{for } n \geq 3 \text{ with } a_0 = 0, a_1 = 1 \text{ and } a_2 = 4$$

Make recurrence valid for all n .

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3} + \delta_{n1} - \delta_{n2}$$

Multiply by z^n and sum on n .

$$A(z) = 5zA(z) - 8z^2A(z) + 4z^3A(z) + z - z^2$$

Solve.

$$A(z) = \frac{z - z^2}{1 - 5z + 8z^2 - 4z^3}$$

Simplify.

$$A(z) = \frac{z(1-z)}{(1-z)(1-2z)^2} = \frac{z}{(1-2z)^2}$$

Expand.

$$a_n = n2^{n-1}$$

multiplicity 3 gives terms
of the form $n^2\beta^n$, etc.

Solving linear recurrences with GFs

For linear recurrences with constant coefficients, the GF equation is a polynomial, so the general procedure is a **algorithm**.

Example with complex roots.

$$a_n = 2a_{n-1} - a_{n-2} + 2a_{n-3} \quad \text{for } n \geq 3 \text{ with } a_0 = 1, a_1 = 0 \text{ and } a_2 = -1$$

Make recurrence valid for all n .

$$a_n = 2a_{n-1} - a_{n-2} + 2a_{n-3} + \delta_{n0} - 2\delta_{n1}$$

Multiply by z^n and sum on n .

$$A(z) = 2zA(z) - z^2A(z) + 2z^3A(z) + 1 - 2z$$

Solve.

$$A(z) = \frac{1 - 2z}{1 - 2z + z^2 - 2z^3}$$

Simplify.

$$A(z) = \frac{1 - 2z}{(1 - 2z)(1 + z^2)} = \frac{1}{(1 + z^2)}$$

Use partial fractions.

$$A(z) = \frac{1}{2} \left(\frac{1}{1 - iz} + \frac{1}{1 + iz} \right)$$

Expand.

$$a_n = \frac{1}{2}(i^n + (-i)^n) = \frac{1}{2}i^n(1 + (-1)^n)$$

$$1, 0, -1, 0, 1, 0, -1, 0, 1 \dots$$

Solving linear recurrences with GFs (summary)

Solution to $a_n = x_1 a_{n-1} + x_2 a_{n-2} + \dots + x_t a_{n-t}$

is a linear combination of t terms.

Suppose the roots of the polynomial $1 - x_1 z + x_2 z^2 + \dots + x_t z^t$

are $\beta_1, \beta_2, \dots, \beta_r$ where the multiplicity of β_i is m_i so $m_1 + m_2 + \dots + m_r = t$

Solution is

$$\sum_{0 \leq j < m_1} c_{1j} n^j \beta_1^n + \sum_{0 \leq j < m_2} c_{2j} n^j \beta_2^n + \dots + \sum_{0 \leq j < m_r} c_{rj} n^j \beta_r^n \quad \leftarrow t \text{ terms}$$

The t constants c_{ij} are determined from the initial conditions.

Note: complex roots (and -1) introduce periodic behavior.

Solving the Quicksort recurrence with OGFs

$$C_N = N + 1 + \frac{2}{N} \sum_{1 \leq k \leq N} C_{k-1}$$

Multiply both sides by N .

$$NC_N = N(N+1) + 2 \sum_{1 \leq k \leq N} C_{k-1}$$

Multiply by z^N and sum.

$$\sum_{N \geq 1} NC_N z^N = \sum_{N \geq 1} N(N+1)z^N + 2 \sum_{N \geq 1} \sum_{1 \leq k \leq N} C_{k-1} z^N$$

Evaluate sums to get an
ordinary differential equation

$$C'(z) = \frac{2}{(1-z)^3} + 2 \frac{C(z)}{1-z}$$

homogeneous equation
 $\rho'(z) = 2\rho(z)/(1-z)$
solution (integration factor)
 $\rho(z) = 1/(1-z)^2$

Solve the ODE.

$$\begin{aligned} ((1-z)^2 C(z))' &= (1-z)^2 C'(z) - 2(1-z)C(z) \\ &= (1-z)^2 \left(C'(z) - 2 \frac{C(z)}{1-z} \right) = \frac{2}{1-z} \end{aligned}$$

Integrate.

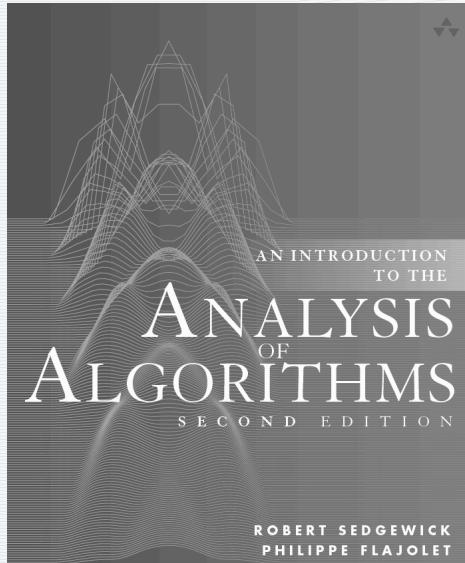
$$C(z) = \frac{2}{(1-z)^2} \ln \frac{1}{1-z}$$

Expand.

$$C_N = [z^N] \frac{2}{(1-z)^2} \ln \frac{1}{1-z} = 2(N+1)(H_{N+1} - 1)$$

ANALYTIC COMBINATORICS

PART ONE



3. Generating Functions

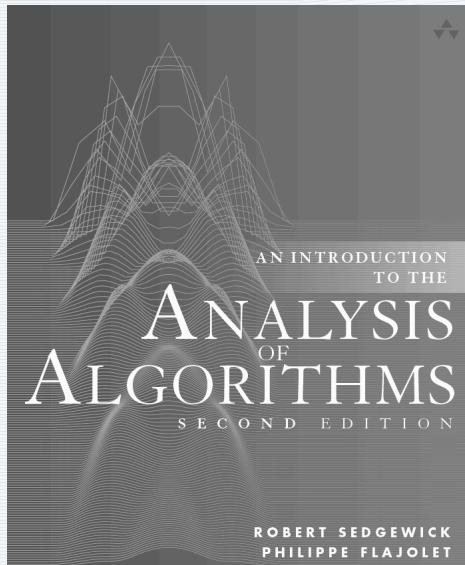
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3b. GFs . recurrences

ANALYTIC COMBINATORICS

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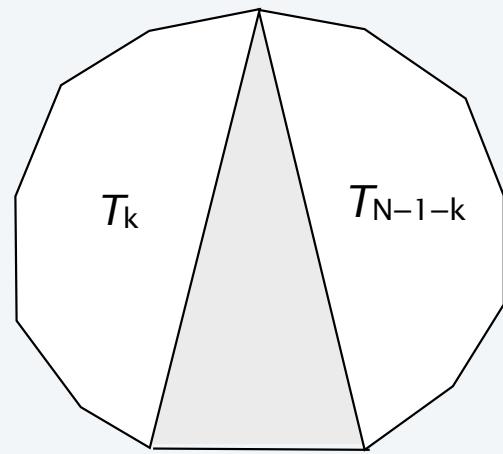
3. Generating Functions

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Catalan numbers

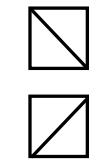
How many **triangulations** of an $(N+2)$ -gon?



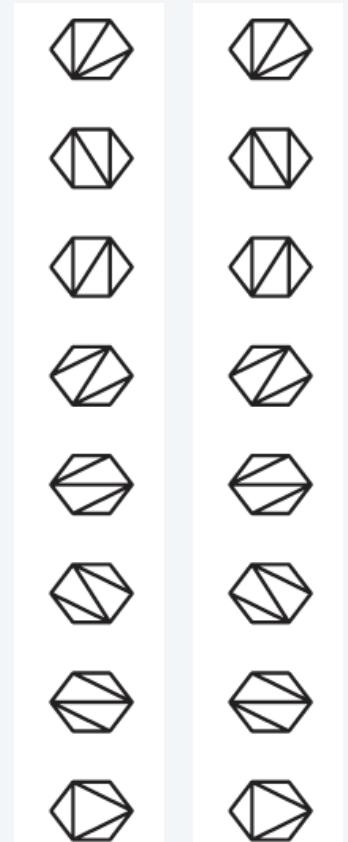
$$T_N = \sum_{0 \leq k < N} T_k T_{N-1-k} + \delta_{N0}$$

$$\begin{array}{c|cc} & T_0 = 1 & T_1 = 1 \\ & \triangle & \square \end{array}$$

$$T_2 = 2$$



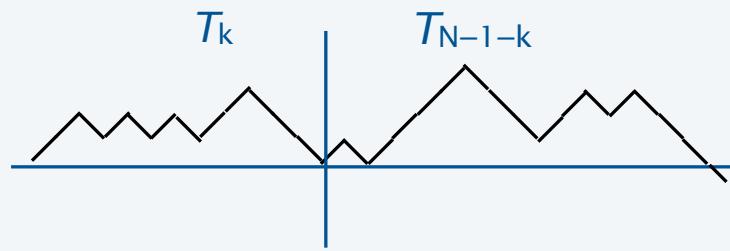
$$T_3 = 5$$



$$T_4 = 14$$

Catalan numbers

How many gambler's ruin sequences with N wins?



$$T_N = \sum_{0 \leq k < N} T_k T_{N-1-k} + \delta_{N0}$$

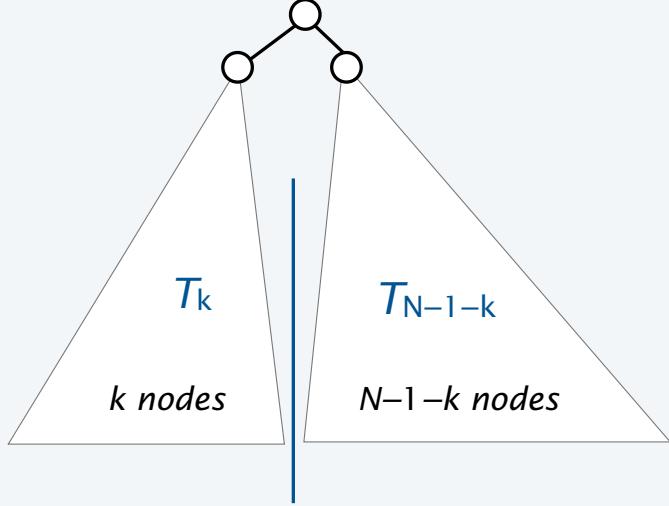
		$T_0 = 1$		$T_1 = 1$		$T_2 = 2$		$T_3 = 5$
--	--	-----------	--	-----------	--	-----------	--	-----------



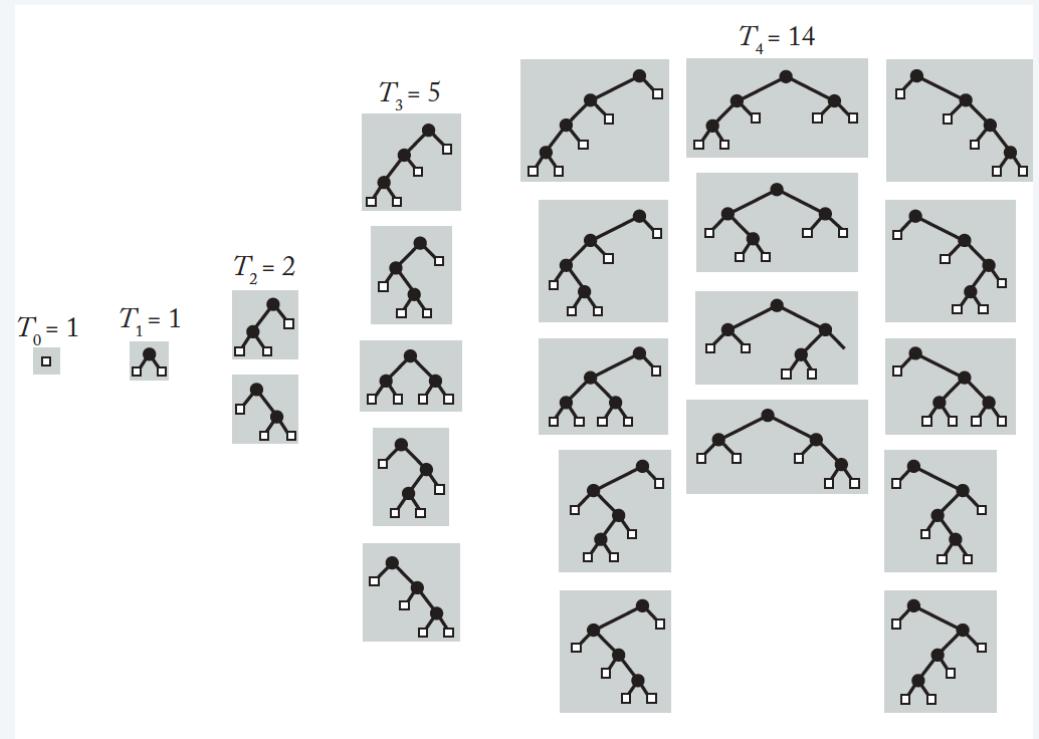
$$T_4 = 14$$

Catalan numbers

How many binary trees with N nodes?

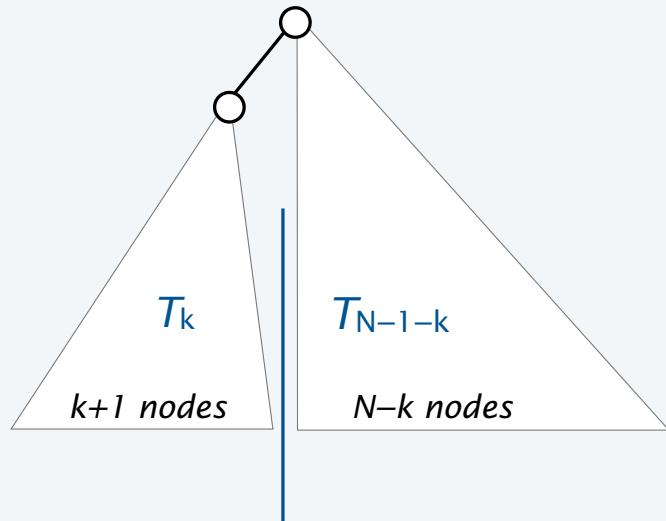


$$T_N = \sum_{0 \leq k < N} T_k T_{N-1-k} + \delta_{N0}$$

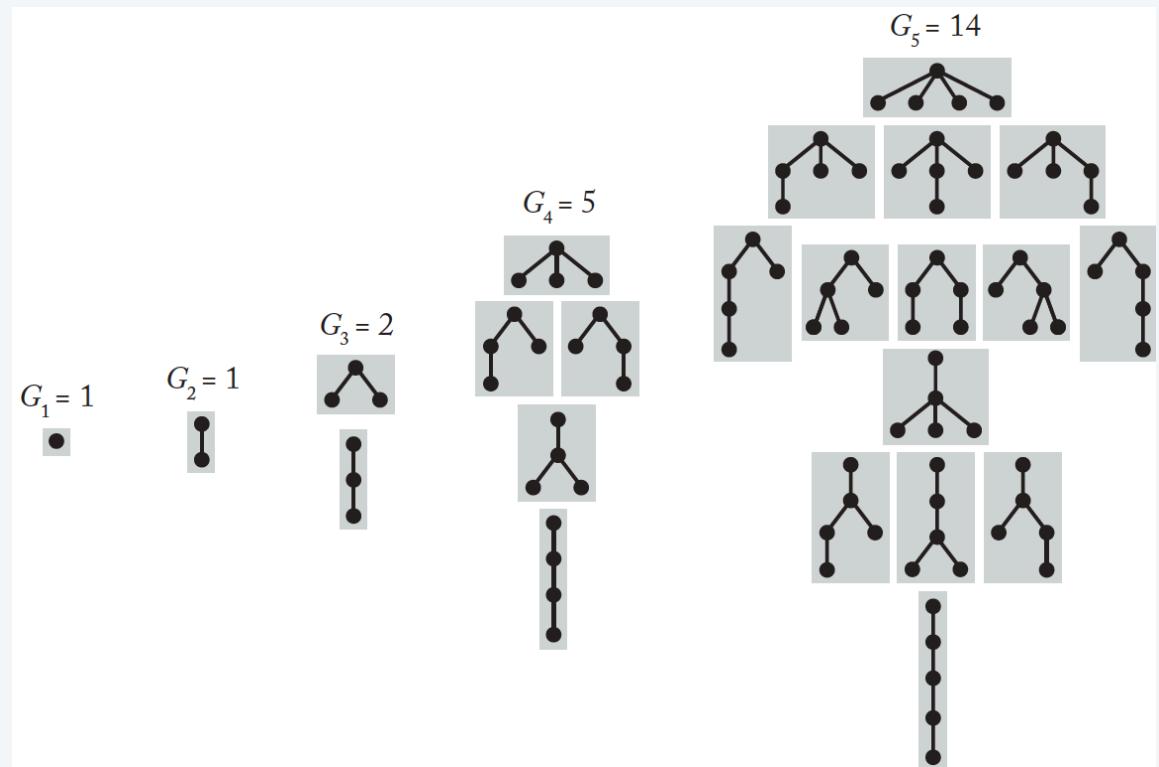


Catalan numbers

How many **trees** with $N+1$ nodes?



$$T_N = \sum_{0 \leq k < N} T_k T_{N-1-k} + \delta_{N0}$$



Solving the Catalan recurrence with GFs

Recurrence that holds for all N .

$$T_N = \sum_{0 \leq k < N} T_k T_{N-1-k} + \delta_{N0}$$

Multiply by z^N and sum.

$$T(z) \equiv \sum_{N \geq 0} T_N z^N = \sum_{N \geq 0} \sum_{0 \leq k < N} T_k T_{N-1-k} z^N + 1$$

Switch order of summation

$$T(z) = 1 + \sum_{k \geq 0} \sum_{N > k} T_k T_{N-1-k} z^N$$

Change N to $N+k+1$

$$T(z) = 1 + \sum_{k \geq 0} \sum_{N \geq 0} T_k T_{N+k+1} z^{N+k+1}$$

convolution
(backwards)

Distribute.

$$T(z) = 1 + z \left(\sum_{k \geq 0} T_k z^k \right) \left(\sum_{N \geq 0} T_N z^N \right)$$

$$T(z) = 1 + z T(z)^2$$

Common-sense rule for working with GFs

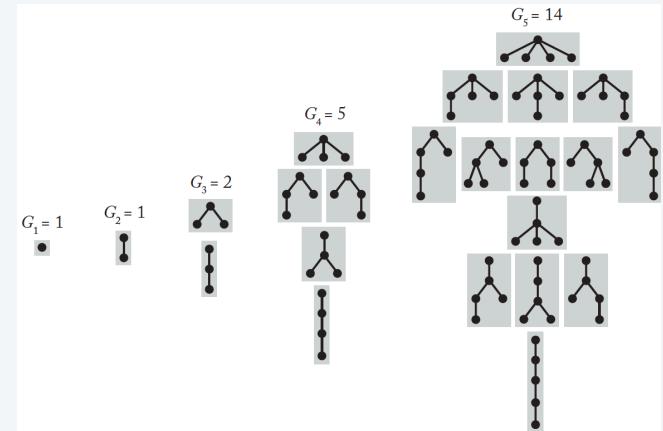
It is **always** worthwhile to check your math with your computer.

Known from initial values:

$$T(z) = 1 + z + 2z^2 + 5z^3 + 14z^4 + \dots$$

Check:

$$T(z) = 1 + zT(z)^2$$



```
sage: ZP.<z> = ZZ[]
sage: 1 + z*(1+z+2*z^2+5*z^3+14*z^4)*(1+z+2*z^2+5*z^3+14*z^4)
196*z^9 + 140*z^8 + 81*z^7 + 48*z^6 + 42*z^5 + 14*z^4 + 5*z^3 + 2*z^2 + z + 1
```



not valid because
 z^5 and beyond
missing in factors



Solving the Catalan recurrence with GFs (continued)

Functional GF equation.

$$T(z) = 1 + zT(z)^2$$

Solve with quadratic formula.

$$zT(z) = \frac{1}{2}(1 \pm \sqrt{1 - 4z})$$

Expand via binomial theorem.

$$zT(z) = -\frac{1}{2} \sum_{N \geq 1} \binom{\frac{1}{2}}{N} (-4z)^N$$

Set coefficients equal

$$T_N = -\frac{1}{2} \binom{\frac{1}{2}}{N+1} (-4)^{N+1}$$

Expand via definition.

$$= -\frac{1}{2} \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\dots(\frac{1}{2}-N)(-4)^{N+1}}{(N+1)!}$$
$$= \frac{1 \cdot 3 \cdot 5 \cdots (2N-1) \cdot 2^N}{(N+1)!}$$

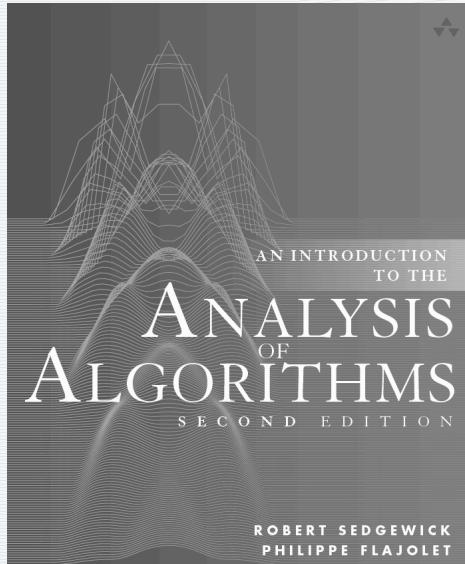
Distribute $(-2)^N$ among factors.

$$= \frac{1}{N+1} \frac{1 \cdot 3 \cdot 5 \cdots (2N-1)}{N!} \frac{2 \cdot 4 \cdot 6 \cdots 2N}{1 \cdot 2 \cdot 3 \cdots N}$$
$$= \frac{1}{N+1} \binom{2N}{N}$$

Substitute $(2/1)(4/2)(6/3)\dots$ for 2^N .

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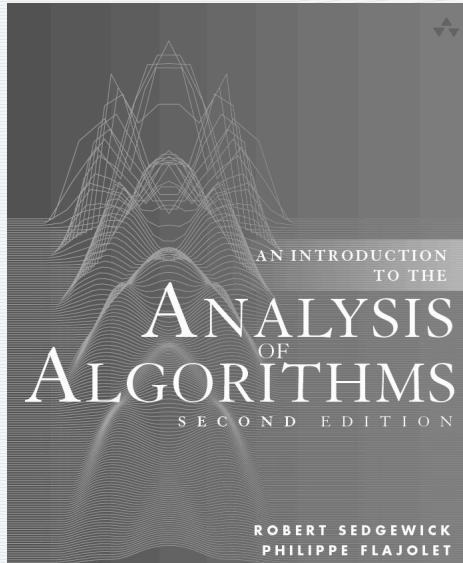
3. Generating Functions

- OGFs
- Solving recurrences
- **Catalan numbers**
- EGFs
- Counting with GFs

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Exponential generating functions (EGFs)

Definition.

$A(z) = \sum_{k \geq 0} a_k \frac{z^k}{k!}$ is the exponential generating function (EGF)

of the sequence $a_0, a_1, a_2, \dots, a_k, \dots$

sequence	EGF
1, 1, 1, 1, 1, 1, ...	$\sum_{N \geq 0} \frac{z^N}{N!} = e^z$
1, 2, 4, 8, 16, 32, ...	$\sum_{N \geq 0} 2^N \frac{z^N}{N!} = e^{2z}$
1, 1, 2, 6, 24, 120 ...	$\sum_{N \geq 0} N! \frac{z^N}{N!} = \frac{1}{1-z}$

Operations on EGFs: Binomial convolution

If $A(z) = \sum_{k \geq 0} a_k \frac{z^k}{k!}$ is the EGF of $a_0, a_1, a_2, \dots, a_k, \dots$

and $B(z) = \sum_{k \geq 0} b_k \frac{z^k}{k!}$ is the EGF of $b_0, b_1, b_2, \dots, b_k, \dots$

then $A(z)B(z)$ is the EGF of $a_0b_0, a_0b_1 + a_1b_0, \dots, \binom{n}{k} a_k b_{n-k}, \dots$

Proof.

$$A(z)B(z) = \sum_{k \geq 0} a_k \frac{z^k}{k!} \sum_{n \geq 0} b_n \frac{z^n}{n!}$$

Distribute.

$$= \sum_{k \geq 0} \sum_{n \geq 0} \frac{a_k}{k!} \frac{b_n}{n!} z^{n+k}$$

Change n to $n-k$

$$= \sum_{k \geq 0} \sum_{n \geq k} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!} z^n$$

Multiply and divide by $n!$

$$= \sum_{k \geq 0} \sum_{n \geq k} \binom{n}{k} a_k b_{n-k} \frac{z^n}{n!}$$

Switch order of summation.

$$= \sum_{n \geq 0} \left(\sum_{0 \leq k \leq n} \binom{n}{k} a_k b_{n-k} \right) \frac{z^n}{n!}$$

Solving recurrences with EGFs

Choice of EGF vs. OGF is typically dictated naturally from the problem.

Example.

$$f_n = \sum_k \binom{n}{k} \frac{f_k}{2^k}$$

Multiply by $z^n / n!$ and sum on n .

$$f(z) = \sum_{n \geq 0} \sum_k \binom{n}{k} \frac{f_k}{2^k} \frac{z^n}{n!}$$

Switch order of summation.

$$f(z) = \sum_{k \geq 0} \sum_{n \geq k} \binom{n}{k} \frac{f_k}{2^k} \frac{z^n}{n!}$$

Change n to $n+k$.

$$f(z) = \sum_{k \geq 0} \sum_{n \geq 0} \binom{n+k}{k} \frac{f_k}{2^k} \frac{z^{n+k}}{(n+k)!}$$

Simplify.

$$f(z) = \sum_{k \geq 0} \sum_{n \geq 0} f_k \frac{(z/2)^k}{k!} \frac{z^n}{n!}$$

Distribute.

$$f(z) = \left(\sum_{k \geq 0} f_k \frac{(z/2)^k}{k!} \right) \left(\sum_{n \geq 0} \frac{z^n}{n!} \right)$$

Evaluate and telescope.

$$f(z) = e^z f(z/2) = e^{z+z/2+z/4+z/8+\dots} = e^{2z}$$

Expand.

$$f_n = 2^n$$

Check.

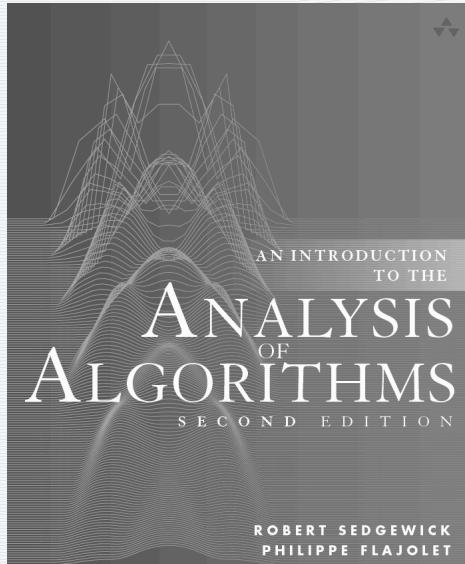
$$2^n = \sum_k \binom{n}{k} \frac{2^k}{2^k}$$

binomial convolution
(backwards)

convergence
not assured

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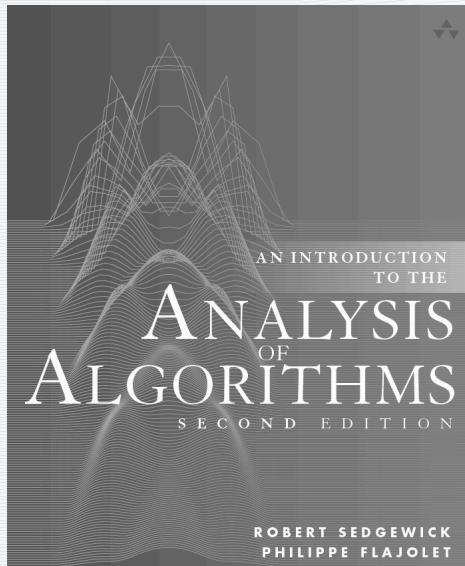
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Counting with generating functions

An alternative (combinatorial) view of GFs

- Define a *class* of combinatorial objects with associated *size* function.
- GF is sum over all members of the class.

Example.

$T \equiv$ set of all binary trees

$|t| \equiv$ number of *internal* nodes in $t \in T$

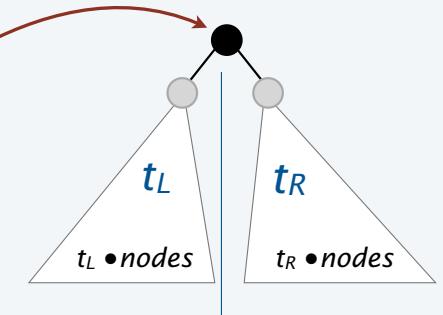
$T_N \equiv$ number of $t \in T$ with $|t| = N$

$$T(z) \equiv \sum_{t \in T} z^{|t|} = \sum_{N \geq 0} T_N z^N$$

Decompose from definition

$$\begin{aligned} T(z) &= 1 + \sum_{t_L \in T} \sum_{t_R \in T} z^{|t_L|+|t_R|+1} \\ &= 1 + z \left(\sum_{t_L \in T} z^{|t_L|} \right) \left(\sum_{t_R \in T} z^{|t_R|} \right) \\ &= 1 + z T(z)^2 \end{aligned}$$

Distribute



Combinatorial view of Catalan GF

Each term z^N in the GF corresponds to an object of size N .

Collect all the terms with the same exponent to expose counts.

Each term $z^i z^j$ in a product corresponds to an object of size $i + j$.

$$T(z) = 1 + z + z^2 + z^2 + z^3 + z^3 + z^3 + z^3 + z^3 + \dots$$
$$= 1 + z + 2z^2 + 5z^3 + \dots$$

$$T(z) = 1 + zT(z)^2$$
$$= 1 + z(1 + z + z^2 + z^2 + \dots)(1 + z + z^2 + z^2 + \dots)$$
$$= 1 + z + z^2 + z^2 + z^3 + z^3 + z^3 + z^3 + z^3 + \dots$$

\square \square \square \square \square \square \square \square

$z \times 1 \times 1$ $z \times 1 \times z$ $z \times z \times 1$ $z \times 1 \times z^2$ $z \times 1 \times z^2$ $z \times z \times z$ $z \times z^2 \times 1$ $z \times z^2 \times 1$

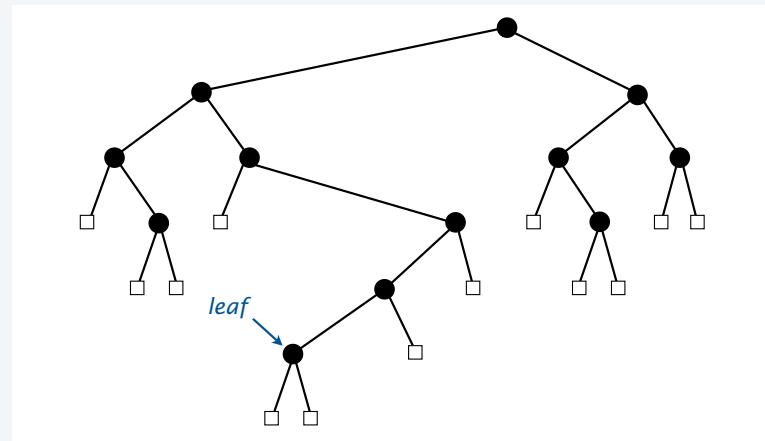
Values of parameters ("costs")

are often the object of study in the analysis of algorithms.

How many 1 bits in a random bitstring? (Easy)

01110100100010001110101000001010000

How many leaves in a random binary tree? (Not so easy)



Computing expected costs by counting

An alternative (combinatorial) view of probability

- Define a *class* of combinatorial objects.
- Model: All objects of size N are equally likely

$\mathcal{P} \equiv$ set of all objects in the class

$|p| \equiv$ size of $p \in \mathcal{P}$

$P_N \equiv$ number of $p \in \mathcal{P}$ with $|p| = N$

$\text{cost}(p) \equiv$ cost associated with p

$P_{Nk} \equiv$ number of $p \in \mathcal{P}$ with $|p| = N$ and $\text{cost}(p) = k$

Expected cost of an object of size N

$$C_N \equiv \sum_{k \geq 0} k \frac{P_{Nk}}{P_N}$$

P_{Nk}/P_N is the probability that the cost of an object of size N is k

$$= \frac{\sum_{k \geq 0} k P_{Nk}}{P_N}$$

“cumulated cost”

Def. *Cumulated cost* is total cost of all objects of a given size.

Expected cost is cumulated cost divided by number of objects.

Counting with generating functions: cumulative costs

An alternative (combinatorial) view of GFs

- Define a *class* of combinatorial objects.
- Model: All objects of size N are equally likely
- GF is sum over all members of the class.

$\mathcal{P} \equiv$ set of all objects in the class

$|p| \equiv$ size of $p \in \mathcal{P}$

$P_N \equiv$ number of $p \in \mathcal{P}$ with $|p| = N$

$\text{cost}(p) \equiv$ cost associated with p

Counting GF

$$P(z) \equiv \sum_{p \in \mathcal{P}} z^{|p|} = \sum_{N \geq 0} P_N z^N$$

Cumulative cost GF

$$C(z) \equiv \sum_{p \in \mathcal{P}} \text{cost}(p) z^{|p|} = \sum_{N \geq 0} \sum_{k \geq 0} k P_{Nk} z^N$$

Average cost

$$[z^N]C(z)/[z^N]P(z)$$

Bottom line: Reduces computing expectation to GF counting

Warmup: How many 1 bits in a random bitstring?

B is the set of all bitstrings.

$|b|$ is the number of bits in b .

$\text{ones}(b)$ is the number of 1 bits in b .

B_N is the # of bitstrings of size N (2^N).

C_N is the total number of 1 bits in all bitstrings of size N .

Counting GF.

$$B(z) = \sum_{b \in B} z^{|b|} = \sum_{N \geq 0} 2^N z^N = \frac{1}{1 - 2z}$$

Cumulative cost GF.

$$\begin{aligned} C(z) &= \sum_{b \in B} \text{ones}(b) z^{|b|} \\ &= \sum_{b' \in B} (1 + 2 \cdot \text{ones}(b')) z^{|b'|+1} \\ &= zB(z) + 2zC(z) \\ &= \frac{z}{(1 - 2z)^2} \end{aligned}$$

0	b'
1	b'

$$\frac{2z}{(1 - 2z)^2} = \sum_{N \geq 1} N(2z)^N$$

Average # 1 bits in a random bitstring of length N .

$$\frac{[z^N]C(z)}{[z^N]B(z)} = \frac{N2^{N-1}}{2^N} = \frac{N}{2} \quad \checkmark$$

Leaves in binary trees

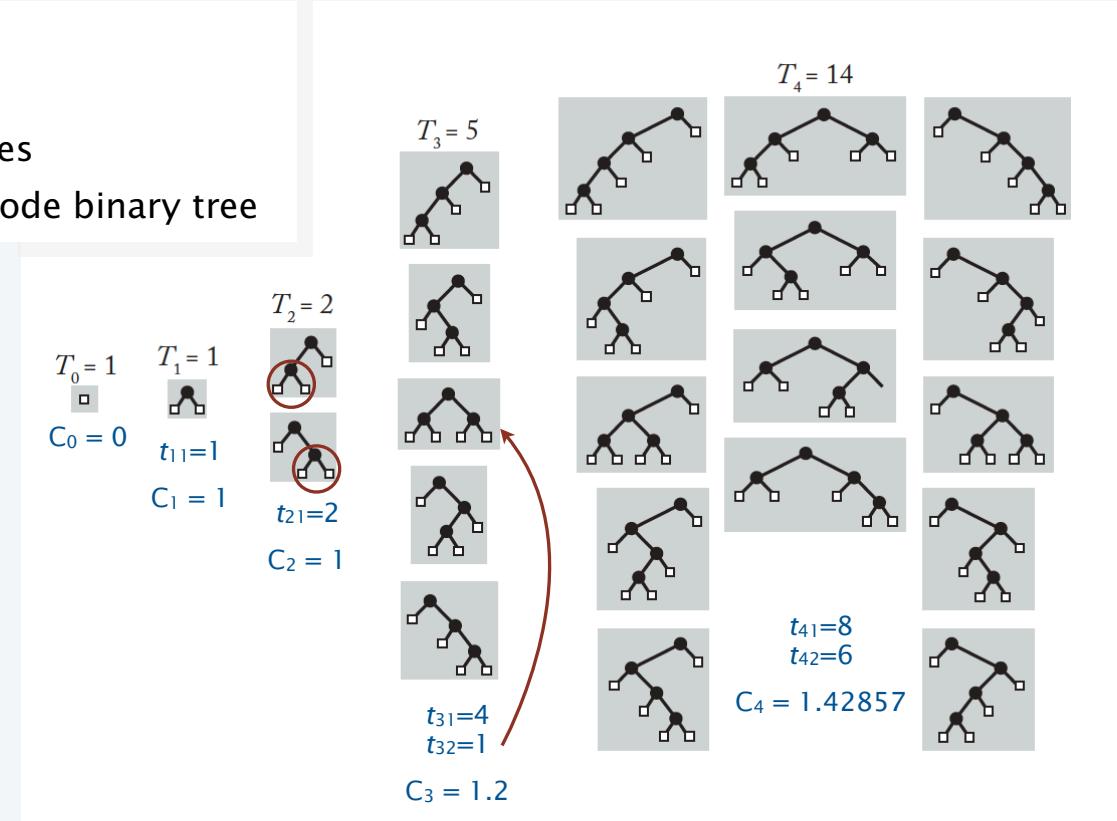
are internal nodes whose children are both external.

Definitions:

T_N is the # of binary trees with N nodes.

t_{Nk} is the # of N -node binary trees with k leaves

C_N is the average # of leaves in a random N -node binary tree



Q. How many leaves in a random binary tree?

How many leaves in a random binary tree?

T is the set of all binary trees.

$|t|$ is the number of internal nodes in t .

$\text{leaves}(t)$ is the number of leaves in t .

T_N is the # of binary trees of size N (Catalan).

C_N is the total number of leaves in all binary trees of size N .

Counting GF.

$$T(z) = \sum_{t \in T} z^{|t|} = \sum_{N \geq 0} T_N z^N = \sum_{N \geq 0} \frac{1}{N+1} \binom{2N}{N} z^N$$

Cumulative cost GF.

$$C(z) = \sum_{t \in T} \text{leaves}(t) z^{|t|}$$

Average # leaves in a random N -node binary tree.

$$\frac{[z^N]C(z)}{[z^N]T(z)} = \frac{[z^N]C(z)}{T_N}$$

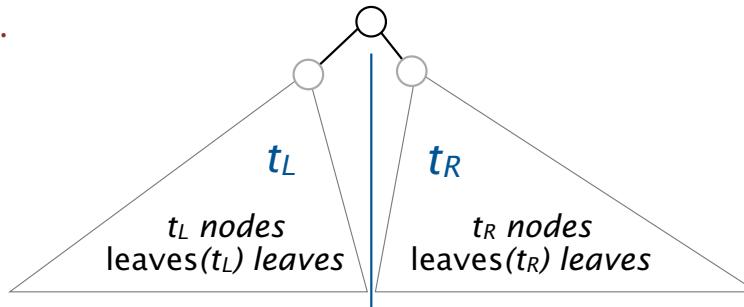
Next: Derive a functional equation for the CGF.

CGF functional equation for leaves in binary trees

CGF.

$$C(z) = \sum_{t \in T} \text{leaves}(t) z^{|t|}$$

Decompose from definition.



$$\begin{aligned} C(z) &= z + \sum_{t_L \in T} \sum_{t_R \in T} (\text{leaves}(t_L) + \text{leaves}(t_R)) z^{|t_L| + |t_R| + 1} \\ &= z + z \sum_{t_L \in T} \text{leaves}(t_L) z^{|t_L|} \sum_{t_R \in T} z^{|t_R|} + z \sum_{t_L \in T} z^{|t_L|} \sum_{t_R \in T} \text{leaves}(t_R) z^{|t_R|} \\ &= z + 2zC(z)T(z) \end{aligned}$$

How many leaves in a random binary tree?

CGF.

$$C(z) = \sum_{t \in T} \text{leaves}(t) z^{|t|}$$

Decompose from definition.

$$\begin{aligned} C(z) &= z + \sum_{t_L \in T} \sum_{t_R \in T} (\text{leaves}(t_L) + \text{leaves}(t_R)) z^{|t_L| + |t_R| + 1} \\ &= z + 2zC(z)T(z) \end{aligned}$$

Compute number of trees T_N .

Catalan numbers

$$\begin{aligned} T(z) &= zT(z)^2 - z \\ &= \frac{1}{2z}(1 - \sqrt{1 - 4z}) \end{aligned}$$

$$\begin{aligned} T_N &= [z^N] \frac{1}{2z}(1 - \sqrt{1 - 4z}) \\ &= \frac{1}{N+1} \binom{2N}{N} \end{aligned}$$

Compute cumulated cost C_N .

$$\begin{aligned} C(z) &= z + 2zT(z)C(z) \\ &= \frac{z}{1 - 2zT(z)} = \frac{z}{\sqrt{1 - 4z}} \end{aligned}$$

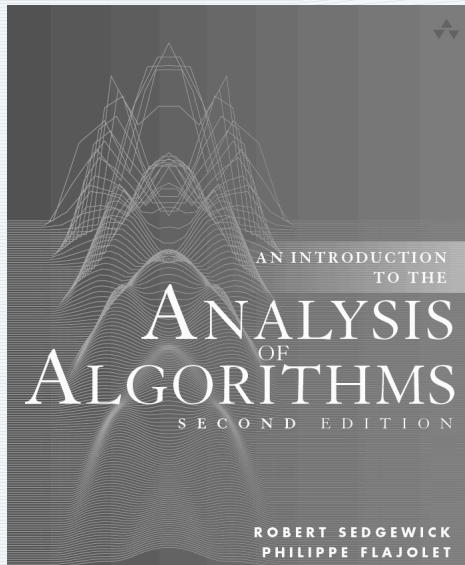
$$\begin{aligned} C_N &= [z^N] \frac{z}{\sqrt{1 - 4z}} \\ &= \binom{2N-2}{N-1} \end{aligned}$$

Compute *average number of leaves*.

$$C_N/T_N = \frac{\binom{2N-2}{N-1}}{\frac{1}{N+1} \binom{2N}{N}} = \frac{(N+1) \cdot N \cdot N}{2N(2N-1)} \sim \boxed{N/4}$$

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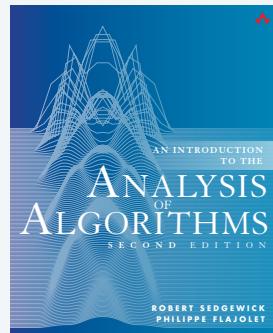
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Exercise 3.20

Solve a linear recurrence. Initial conditions matter.



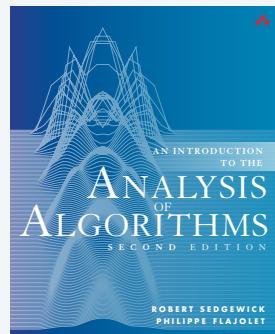
Exercise 3.20 Solve the recurrence

$$a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3} \quad \text{for } n > 2 \text{ with } a_0 = a_1 = 0 \text{ and } a_2 = 1.$$

Solve the same recurrence with the initial condition on a_1 changed to $a_1 = 1$.

Exercise 3.28

The art of expanding GFs.



Exercise 3.28 Find an expression for

$$[z^n] \frac{1}{\sqrt{1-z}} \ln \frac{1}{1-z}.$$

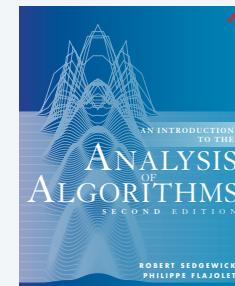
(Hint: Expand $(1-z)^{-\alpha}$ and differentiate with respect to α .)

Assignments for next lecture

1. Use a symbolic mathematics system
to check initial values for $C(z) = z + 2C(z)T(z)$.

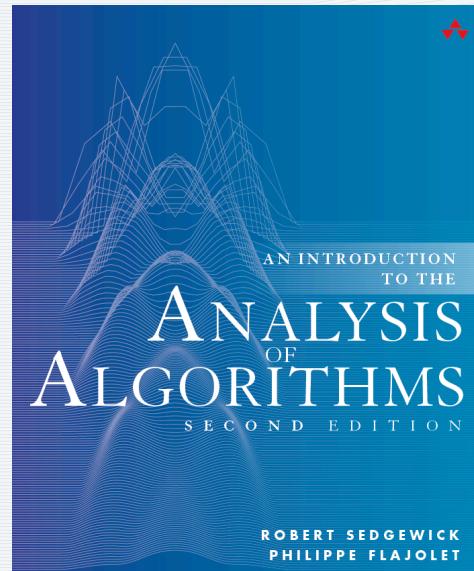
2. Read pages 89-147 in text.

3. Write up solutions to Exercises 3.20 and 3.28.



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3. Generating Functions