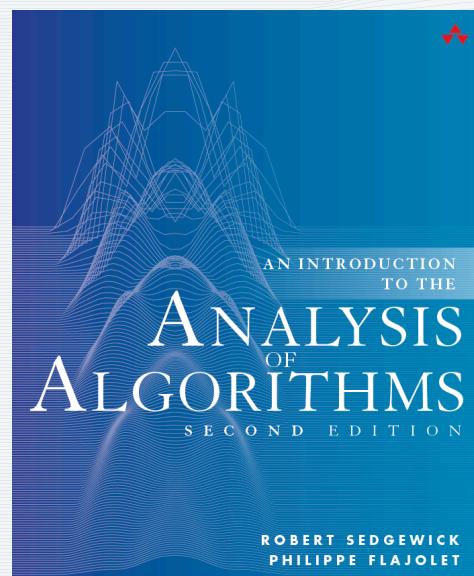


ANALYTIC COMBINATORICS

PART ONE

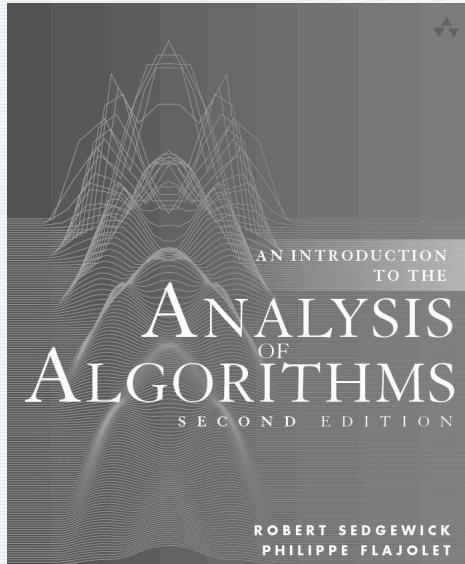


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4. Asymptotic Approximations

ANALYTIC COMBINATORICS

PART ONE



4. Asymptotic Approximations

- Standard scale
- Manipulating expansions
- Asymptotics of finite sums
- Bivariate asymptotics

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4a. Asympt. scale

Asymptotic approximations

Goal: Develop *accurate* and *concise* estimates of quantities of interest

$$O(\log N)$$

 *not accurate*

$$H_N = \sum_{0 \leq k \leq N} \frac{1}{k}$$

 *not concise*

$$\ln N + \gamma + O\left(\frac{1}{N}\right)$$



Informal definition of *concise*:

“easy to compute with constants and standard functions”

Notation (revisited)

“Big-Oh” notation for upper bounds

$$g(N) = O(f(N)) \text{ iff } |g(N)/f(N)| \text{ is bounded from above as } N \rightarrow \infty$$

“Little-oh” notation for lower bounds

$$g(N) = o(f(N)) \text{ iff } g(N)/f(N) \rightarrow 0 \text{ as } N \rightarrow \infty$$

“Tilde” notation for asymptotic equivalence

$$g(N) \sim f(N) \text{ iff } g(N)/f(N) \rightarrow 1 \text{ as } N \rightarrow \infty$$

Notation for approximations

“Big-Oh” approximation

$$g(N) = f(N) + O(h(N))$$

Error will be at most **within a constant factor** of $h(N)$ as N increases.

“Little-oh” approximation

$$g(N) = f(N) + o(h(N))$$

Error will **decrease** relative to $h(N)$ as N increases.

“Tilde” approximation

$$g(N) \sim f(N)$$

Weakest nontrivial o-approximation.

Standard asymptotic scale

Definition. A decreasing series $g_k(N)$ with $g_{k+1}(N) = o(g_k(N))$ is called an *asymptotic scale*.

The series

$$f(N) \sim c_0 g_0(N) + c_1 g_1(N) + c_2 g_2(N) + \dots$$

is called an *asymptotic expansion* of f . The expansion represents the collection of formulae

$$f(N) = O(g_0(N))$$

$$f(N) = c_0 g_0(N) + O(g_1(N))$$

$$f(N) = c_0 g_0(N) + c_1 g_1(N) + O(g_2(N))$$

$$f(N) = c_0 g_0(N) + c_1 g_1(N) + c_2 g_2(N) + O(g_3(N))$$

⋮

The **standard scale** is products of powers of N , $\log N$, iterated logs and exponentials.

Typically, we:

- use only 2, 3, or 4 terms (continuing until unused terms are extremely small)
- use \sim -notation to *drop* information on unused terms.
- use O -notation or o -notation to *specify* information on unused terms.

Methods extend in principle to any desired precision.

Example: Asymptotics of linear recurrences

Theorem. Assume that a rational GF $f(z)/g(z)$ with $f(z)$ and $g(z)$ relatively prime and $g(0)=0$ has a unique pole $1/\beta$ of smallest modulus and that the multiplicity of β is v . Then

$$[z^n] \frac{f(z)}{g(z)} \sim C\beta^n n^{\nu-1} \quad \text{where} \quad C = \nu \frac{(-\beta)^\nu f(1/\beta)}{g^{(\nu)}(1/\beta)}$$

Proof sketch

$$\sum_{0 \leq j < m_1} c_{1j} n^j \beta_1^n + \sum_{0 \leq j < m_2} c_{2j} n^j \beta_2^n + \dots + \sum_{0 \leq j < m_r} c_{rj} n^j \beta_r^n$$

Largest term dominates.

| Ex. | N | 3^N | $3^N + 2^N$ |
|-----|--------|--------|-------------|
| 7 | 2187 | 275 | |
| 8 | 6561 | 17811 | |
| 9 | 19683 | 20195 | |
| 10 | 59049 | 60073 | |
| 11 | 177147 | 179195 | |

Notes:

- Pole of smallest modulus usually dominates.
- Easy to extend to cover multiple poles in neighborhood of pole of smallest modulus.

Example 1.
 $3^N + 2^N \sim 3^N$



Example 2.
 $2^N + 1.99999^N \sim 2^N$



Asymptotics of linear recurrences

Theorem. Assume that a rational GF $f(z)/g(z)$ with $f(z)$ and $g(z)$ relatively prime and $g(0)=0$ has a unique pole $1/\beta$ of smallest modulus and that the multiplicity of β is v . Then

$$[z^n] \frac{f(z)}{g(z)} \sim C\beta^n n^{\nu-1} \quad \text{where} \quad C = \nu \frac{(-\beta)^\nu f(1/\beta)}{g^{(\nu)}(1/\beta)}$$

Example from earlier lectures.

$$a_n = 5a_{n-1} - 6a_{n-2} \quad \text{for } n \geq 2 \text{ with } a_0 = 0 \text{ and } a_1 = 1$$

Make recurrence valid for all n .

$$a_n = 5a_{n-1} - 6a_{n-2} + \delta_{n1}$$

Multiply by z^n and sum on n .

$$A(z) = 5zA(z) - 6z^2A(z) + z$$

Solve.

$$A(z) = \frac{z}{1 - 5z + 6z^2}$$

Smallest root of denominator is $1/3$.

$$a_n \sim 3^n \quad C = 1 \frac{(-3)(1/3)}{-5 + 12/3} = 1$$

Fundamental asymptotic expansions

are immediate from Taylor's theorem.

| | |
|-------------|---|
| exponential | $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4)$ |
| logarithmic | $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)$ |
| binomial | $(1 + x)^k = 1 + kx + \binom{k}{2}x^2 + \binom{k}{3}x^3 + O(x^4)$ |
| geometric | $\frac{1}{1 - x} = 1 + x + x^2 + x^3 + O(x^4)$ |

as $x \rightarrow 0$.

Fundamental asymptotic expansions

are immediate from Taylor's theorem.

Substitute $x = 1/N$ to get expansions as $N \rightarrow \infty$.

| | |
|-------------|--|
| exponential | $e^{1/N} = 1 + \frac{1}{N} + \frac{1}{2N^2} + \frac{1}{6N^3} + O(\frac{1}{N^4})$ |
| logarithmic | $\ln(1 + \frac{1}{N}) = \frac{1}{N} - \frac{1}{2N^2} + \frac{1}{3N^3} + O(\frac{1}{N^4})$ |
| binomial | $(1 + \frac{1}{N})^k = 1 + \frac{k}{N} + \binom{k}{2} \frac{1}{N^2} + \binom{k}{3} \frac{1}{N^3} + O(\frac{1}{N^4})$ |
| geometric | $\frac{1}{N-1} = \frac{1}{N} + \frac{1}{N^2} + \frac{1}{N^3} + O(\frac{1}{N^4})$ |

as $N \rightarrow \infty$.

Inclass exercise

Develop the following asymptotic approximations

$$\ln\left(1 + \frac{1}{N}\right) + \ln\left(1 - \frac{1}{N}\right) \quad \text{to} \quad O(1/N^3)$$

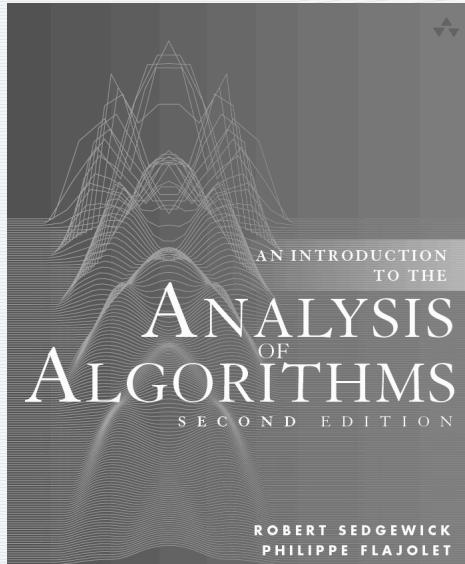
$$\begin{aligned} &= \frac{1}{N} - \frac{1}{2N^2} + O\left(\frac{1}{N^3}\right) - \frac{1}{N} - \frac{1}{2N^2} + O\left(\frac{1}{N^3}\right) \\ &= -\frac{1}{N^2} + O\left(\frac{1}{N^3}\right) \end{aligned}$$

$$\ln\left(1 + \frac{1}{N}\right) - \ln\left(1 - \frac{1}{N}\right) \quad \text{to} \quad O(1/N^3)$$

$$\begin{aligned} &= \frac{1}{N} - \frac{1}{2N^2} + O\left(\frac{1}{N^3}\right) + \frac{1}{N} + \frac{1}{2N^2} + O\left(\frac{1}{N^3}\right) \\ &= \frac{2}{N} + O\left(\frac{1}{N^3}\right) \end{aligned}$$

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4. Asymptotic Approximations

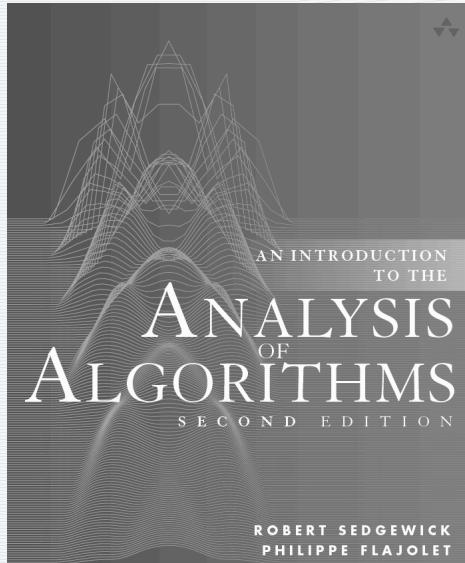
- Standard scale
- Manipulating expansions
- Asymptotics of finite sums
- Bivariate asymptotics

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4a. Asympt. scale

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4. Asymptotic Approximations

- Standard scale
- **Manipulating expansions**
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4b. Asympt. manip

Manipulating asymptotic expansions

Goal.

Develop expansion **on the standard scale** for any given expression.

$$\frac{1}{N^2 + N}$$

$$\frac{H_N}{\ln(N+1)}$$

$$e^{H_N}$$

$$(1 - \frac{1}{N})^N$$

$$(H_N)^2$$

$$\binom{2N}{N}$$

Techniques.

- simplification
- substitution
- factoring
- multiplication
- division
- composition
- exp/log

Why?

Facilitate comparisons of different quantities.
Simplify computations.

$$\text{Ex. } \frac{1}{4^N} \binom{2N}{N}$$



$N = 10^6 ??$

Manipulating asymptotic expansions

Simplification. An asymptotic series is only as good as its O-term.

Discard smaller terms.

$$\ln N + \gamma + O(1) \quad \times$$

$$\ln N + O(1) \quad \checkmark$$

Substitution. Change variables in a known expansion.

Taylor series $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)$ as $x \rightarrow 0$

Substitute $x = 1/N$ $\ln\left(1 + \frac{1}{N}\right) = \frac{1}{N} - \frac{1}{2N^2} + \frac{1}{3N^3} + O\left(\frac{1}{N^4}\right)$ as $N \rightarrow \infty$

Manipulating asymptotic expansions

Factoring. Estimate the leading term, factor it out, expand the rest.

$$\frac{1}{N^2 + N}$$

Factor out $1/N^2$.

$$= \frac{1}{N^2} \frac{1}{1 + 1/N}$$

Expand the rest.

$$= \frac{1}{N^2} \left(1 - \frac{1}{N} + O\left(\frac{1}{N^2}\right) \right)$$

Distribute.

$$= \frac{1}{N^2} - \frac{1}{N^3} + O\left(\frac{1}{N^4}\right)$$

Manipulating asymptotic expansions

Multiplication. Do term-by-term multiplication, simplify, collect terms.

Ex.

Term-by-term multiplication.

Collect terms.

$$(H_N)^2 = \left(\ln N + \gamma + O\left(\frac{1}{N}\right) \right) \left(\ln N + \gamma + O\left(\frac{1}{N}\right) \right)$$

$$= \left((\ln N)^2 + \gamma \ln N + O\left(\frac{\log N}{N}\right) \right)$$

$$+ \left(\gamma \ln N + \gamma^2 + O\left(\frac{1}{N}\right) \right)$$

$$+ \left(O\left(\frac{\log N}{N}\right) + O\left(\frac{1}{N}\right) + O\left(\frac{1}{N^2}\right) \right)$$

slight improvement
in precision

$$= (\ln N)^2 + 2\gamma \ln N + \gamma^2 + O\left(\frac{\log N}{N}\right)$$

big improvement
in precision

| | $(H_N)^2$ | $(\ln N)^2$ | $+2\gamma \ln N$ | $+\gamma^2$ |
|--------|-----------|-------------|------------------|-------------|
| 1000 | 56.032 | 47.717 | 55.692 | 56.025 |
| 10000 | 95.797 | 84.830 | 95.463 | 95.796 |
| 100000 | 146.172 | 132.547 | 145.838 | 146.172 |

May need trial-and-error to get desired precision.



Manipulating asymptotic expansions

Division. Expand, factor denominator, expand $1/(1-x)$, multiply.

Ex.

$$\frac{H_N}{\ln(N+1)}$$

Expand.

$$= \frac{\ln N + \gamma + O(\frac{1}{N})}{\ln N + O(\frac{1}{N})}$$

Factor denominator.

$$= \frac{1 + \frac{\gamma}{\ln N} + O(\frac{1}{N})}{1 + O(\frac{1}{N})}$$

OK to simplify by replacing
 $O(1/N \log N)$ by $O(1/N)$

Expand $1/(1-x)$.

$$= \left(1 + \frac{\gamma}{\ln N} + O(\frac{1}{N})\right) \left(1 + O(\frac{1}{N})\right)$$

Multiply.

$$= 1 + \frac{\gamma}{\ln N} + O(\frac{1}{N})$$

Manipulating asymptotic expansions

Composition. Substitute an expansion.

$$e^{H_N}$$

Substitute H_N expansion.

$$= e^{\ln N + \gamma + O(1/N)}$$

Lemma.

$$e^{O(1/N)} = 1 + O\left(\frac{1}{N}\right)$$

Expand e^x .

$$= Ne^\gamma \left(1 + O\left(\frac{1}{N}\right) + O\left(O\left(\frac{1}{N}\right)^2\right)\right)$$

Simplify.

$$= Ne^\gamma \left(1 + O\left(\frac{1}{N}\right)\right)$$

Distribute.

$$= Ne^\gamma + O(1)$$

big improvement
in precision

| N | e^{H_N} | Ne^γ |
|---------|-----------|-------------|
| 1000 | 1782 | 1781 |
| 10000 | 17812 | 17811 |
| 100000 | 178108 | 178107 |
| 1000000 | 1781073 | 1781072 |



Manipulating asymptotic expansions

Exp/log. Start by writing $f(x) = \exp(\ln(f(x)))$.

$$\left(1 - \frac{1}{N}\right)^N$$

Exp/log.

$$= \exp\left\{\ln\left(\left(1 - \frac{1}{N}\right)^N\right)\right\}$$

Expand $\ln(1+x)$

$$= \exp\left\{N \ln\left(1 - \frac{1}{N}\right)\right\}$$

Distribute.

$$= \exp\left\{-1 + O\left(\frac{1}{N}\right)\right\}$$

$$= 1/e + O\left(\frac{1}{N}\right)$$


big improvement
in precision

Q. How would you
compute values
for large N?

Lemma.
 $e^{O(1/N)} = 1 + O\left(\frac{1}{N}\right)$

| N | $(1 - \frac{1}{N})^N$ | $1/e$ |
|---------|-----------------------|----------|
| 10000 | 0.367861 | 0.367879 |
| 100000 | 0.367878 | 0.367879 |
| 1000000 | 0.367879 | 0.367879 |



Inclass exercises

Develop asymptotic approximations for

$$\ln(N - 2) \quad \text{to} \quad O(1/N^2)$$

$$= \ln N + \ln\left(1 - \frac{2}{N}\right) \quad \text{Factor out } \ln N.$$

$$= \ln N - \frac{2}{N} + O\left(\frac{1}{N^2}\right) \quad \text{Expand the rest.}$$

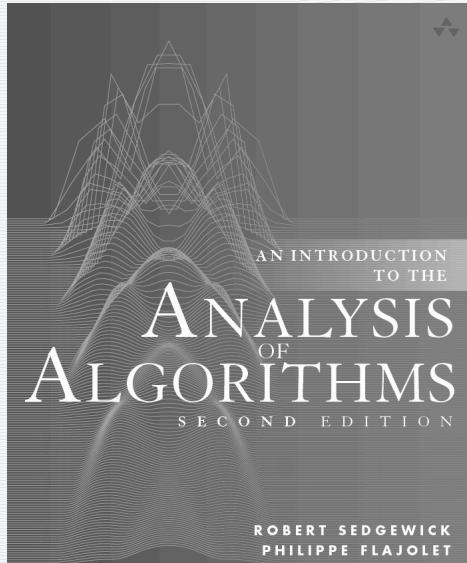
$$(H_N)^2 \quad \text{to} \quad O(1/N)$$

$$(H_N)^2 = \left(\ln N + \gamma + \frac{1}{2N} + O\left(\frac{1}{N^2}\right)\right) \left(\ln N + \gamma + \frac{1}{2N} + O\left(\frac{1}{N^2}\right)\right)$$

$$= (\ln N)^2 + 2\gamma \ln N + \gamma^2 + \frac{\ln N}{N} + O\left(\frac{1}{N}\right)$$

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4. Asymptotic Approximations

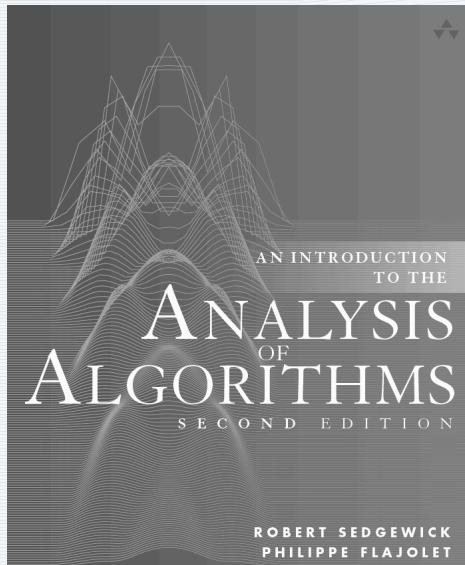
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4b. Asympt. manip

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4c. Asympt. sums

Asymptotics of finite sums

Bounding the tail. Make a rapidly decreasing sum infinite.

$$N! \sum_{0 \leq k \leq N} \frac{(-1)^k}{k!} = N!e^{-1} - R_N \quad \text{where} \quad R_N = N! \sum_{k>N} \frac{(-1)^k}{k!}$$
$$= \frac{N!}{e} + O\left(\frac{1}{N}\right) \quad |R_N| < \frac{1}{N+1} + \frac{1}{(N+1)^2} + \frac{1}{(N+1)^3} + \dots = \frac{1}{N}$$

Using the tail. The last term of a rapidly increasing sum may dominate.

$$\sum_{0 \leq k \leq N} k! = N! \left(1 + \frac{1}{N} + \sum_{0 \leq k \leq N-2} \frac{k!}{N!} \right) = N! \left(1 + O\left(\frac{1}{N}\right) \right)$$

N-1 terms, each less than $1/N(N-1)$

Approximating with an integral.

$$H_N = \sum_{1 \leq k \leq N} \frac{1}{k} \sim \int_1^N \frac{1}{x} dx = \ln N$$

$$\ln N! = \sum_{1 \leq k \leq N} \ln k \sim \int_1^N \ln x dx = N \ln N - N + 1$$

see text for proofs;
stay tuned for
better approximations

Euler-Maclaurin Summation

is a classic formula for estimating sums with integrals.

Theorem. (Euler-Maclaurin summation). Let f be a function defined on $[1, \infty)$ whose derivatives exist and are absolutely integrable. Then

$$\sum_{1 \leq k \leq N} f(k) = \int_1^N f(x)dx + \frac{1}{2}f(N) + C_f + \frac{1}{12}f'(N) - \frac{1}{720}f'''(N) + \dots$$

Asymptotic series diverges; need to check bound on last term (see text for many details).
BUT this form is useful for many applications.

Classic example 1. $H_N = \ln N + \gamma + \frac{1}{2N} - \frac{1}{12N^2} + O\left(\frac{1}{N^4}\right)$

Classic example 2. $\ln N! = N \ln N - N + \ln \sqrt{2\pi N} + \frac{1}{12N} + O\left(\frac{1}{N^3}\right)$

Inclass exercise

Given Stirling's approximation $\ln N! = N \ln N - N + \ln \sqrt{2\pi N} + O(\frac{1}{N})$

Develop an asymptotic approximation for $\binom{2N}{N}$ to $O(1/N)$ (relative error)

$$\begin{aligned}\binom{2N}{N} &= \exp(\ln(2N!) - 2 \ln N!) \\ &= \exp(2N \ln(2N) - 2N + \ln \sqrt{4\pi N} + O(1/N) \\ &\quad - 2(N \ln(N) - N + \ln \sqrt{2\pi N} + O(1/N))) \\ &= \exp(2N \ln 2 - \ln \sqrt{\pi N} + O(1/N))\end{aligned}$$

$$\begin{aligned}\ln \sqrt{4\pi N} - 2 \ln \sqrt{2\pi N} &= \ln 2 - 2 \ln \sqrt{2} - \ln \sqrt{\pi N} \\ &= -\ln \sqrt{\pi N}\end{aligned}$$

$$= \frac{4^N}{\sqrt{\pi N}} \left(1 + O\left(\frac{1}{N}\right)\right)$$

Ex. $\frac{1}{4^N} \binom{2N}{N} \sim \frac{1}{\sqrt{\pi N}}$

Asymptotics of the Catalan numbers: an application

Q. How many bits to represent a binary tree with N internal nodes?

A. At least

$$\lg \frac{1}{N+1} \binom{2N}{N}$$

\times not concise

$N = 10^6 ??$

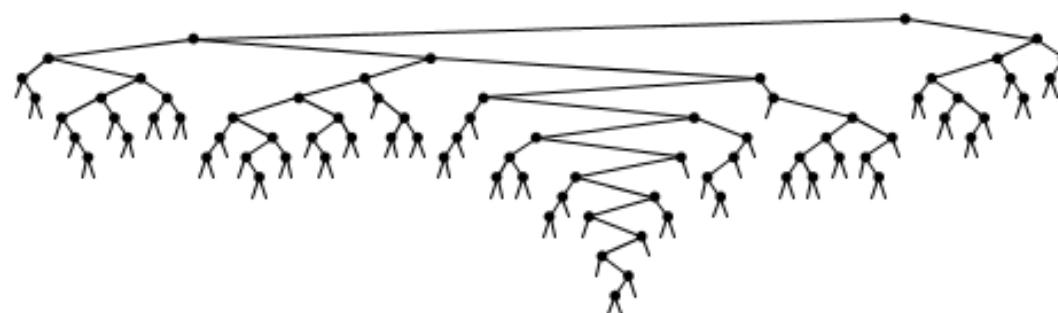


A. At least

$$\sim \lg \frac{4^N}{\sqrt{\pi N^3}} \sim 2N - 1.5 \lg N$$

✓

Note: Can do it with $2N$ bits



preorder traversal

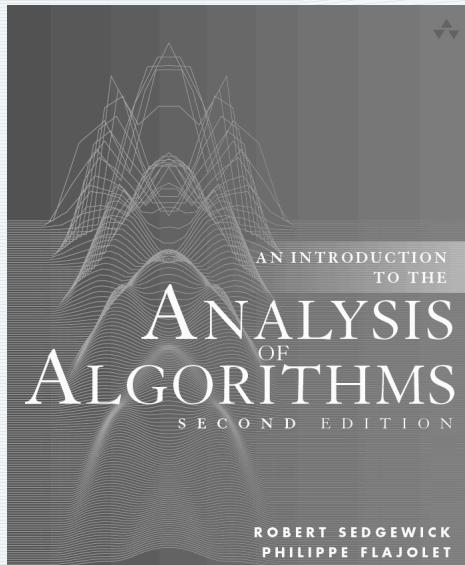
0 for internal nodes

1 for external nodes

0000101100010101101011001101100000111001011011001011011011000001111000011
0110000111001001001111011000101111010000110110110010110000110011001110101100111

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4. Asymptotic Approximations

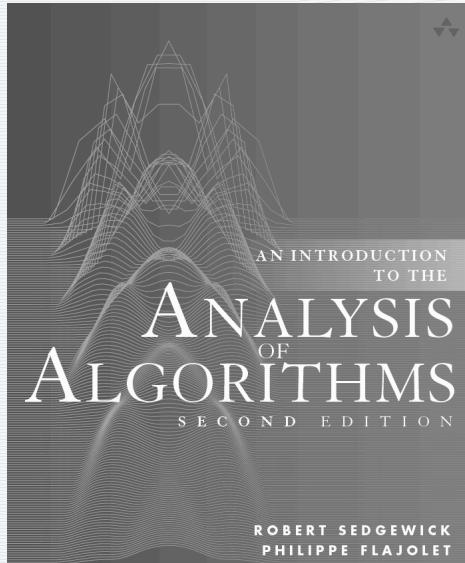
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4c. Asympt. sums

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4d. Asympt. bivariate

Bivariate asymptotics

is often required to analyze functions of **two** variables.

Ex. applications in analysis of algorithms involve

- N (size)
- k (cost)

Challenges:

- asymptotics depends on **relative** values of variables
- may need to approximate **sums** over whole range of relative values.

Example 1: Binomial distribution

$$\frac{1}{4^N} \binom{2N}{N-k} \sim \frac{1}{\sqrt{\pi N}} \text{ for } k = 0$$

exponentially small for k close to N

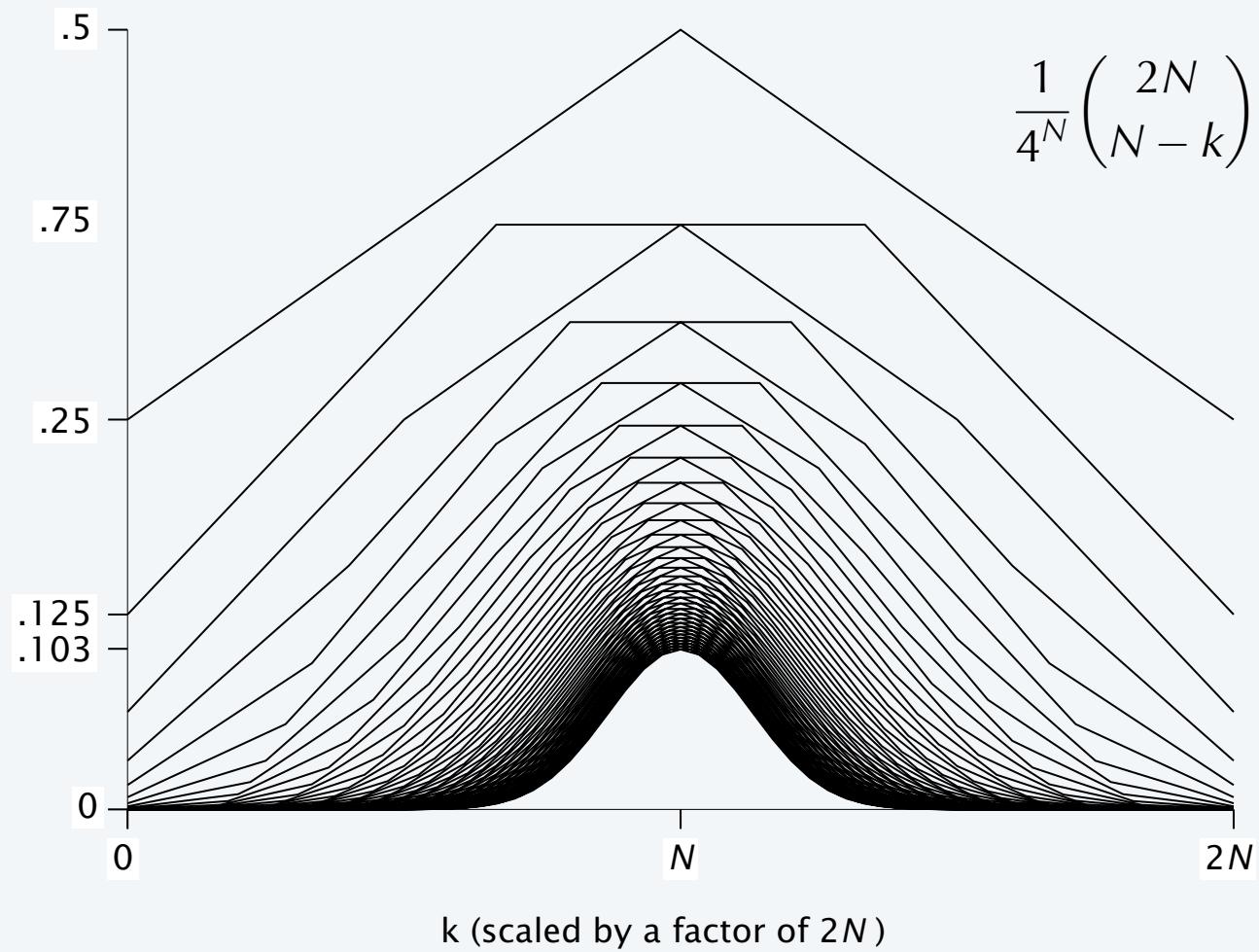
Example 2: Ramanujan Q-distribution

$$\frac{N!}{(N-k)!N^k}$$

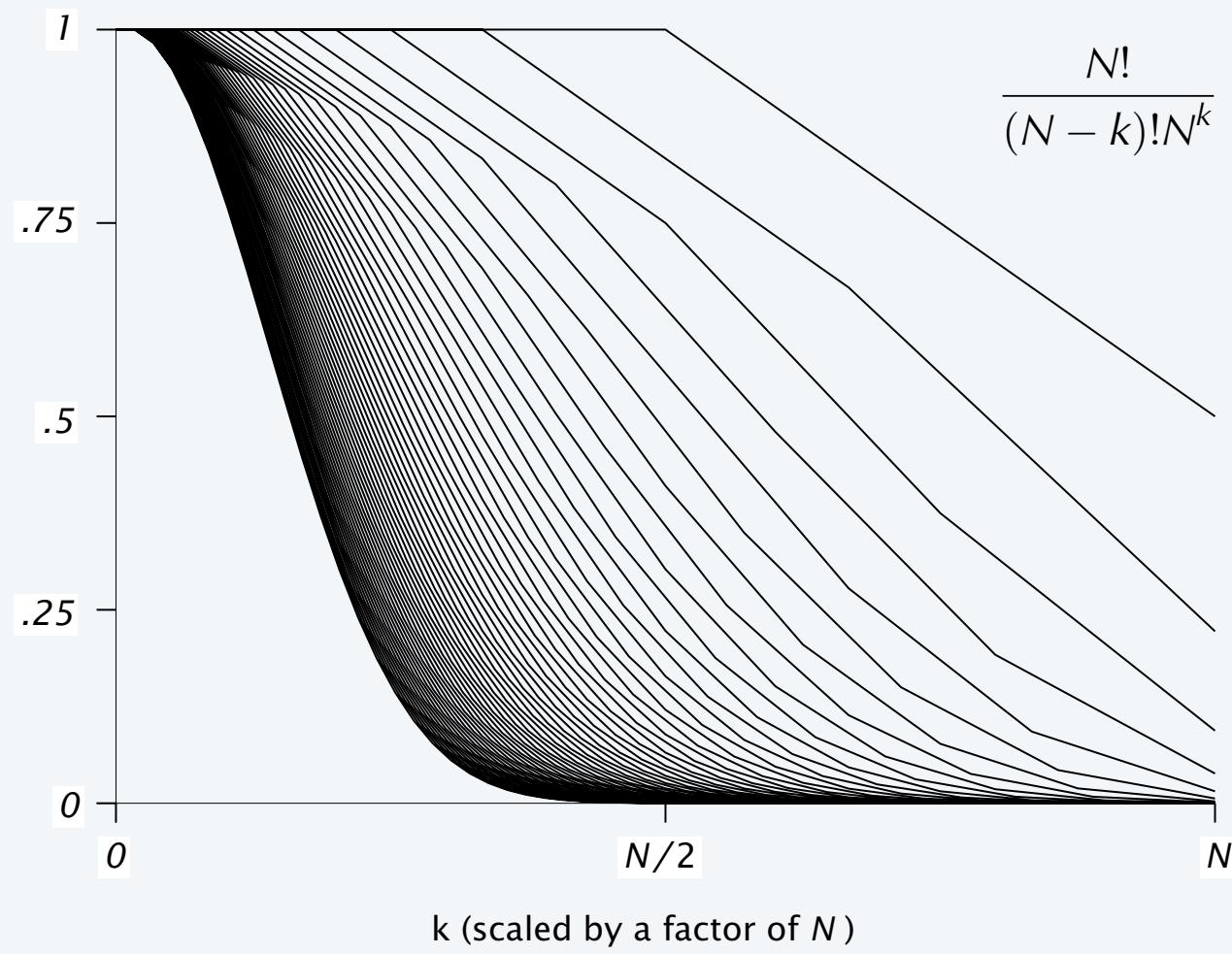
1 for $k = 0$

exponentially small for k close to N

Binomial distribution



Ramanujan Q-distribution



Ramanujan Q-distribution

$$\frac{N!}{(N-k)!N^k} = \exp(\ln N! - \ln(N-k)! - k \ln N)$$

$$\ln N! = (N + \frac{1}{2}) \ln N - N + \ln \sqrt{2\pi} + O(\frac{1}{N})$$

Stirling's approximation.

$$= \exp\left((N + \frac{1}{2}) \ln N - N + \ln \sqrt{2\pi} - (N - k + \frac{1}{2}) \ln(N - k) + N - k - \ln \sqrt{2\pi} - k \ln N + O(\frac{1}{N})\right)$$

Collect terms

$$= \exp\left(-(N - k + \frac{1}{2}) \ln(1 - \frac{k}{N}) - k + O(\frac{1}{N})\right)$$

$$\ln(1 - \frac{k}{N}) = -\frac{k}{N} - \frac{k^2}{2N^2} + O(\frac{k^3}{N^3})$$

Expand $\ln(1 - k/N)$

$$= \exp\left(k + \frac{k^2}{2N} - \frac{k^2}{N} - k + O(\frac{k^3}{N^2}) + O(\frac{k}{N})\right)$$

Simplify.

$$= e^{-k^2/2N} \left(1 + O(\frac{k^3}{N^2}) + O(\frac{k}{N})\right)$$

| k | k/N | k^3/N^2 |
|-----------|-------------|-------------|
| $N^{2/5}$ | $1/N^{3/5}$ | $1/N^{4/5}$ |
| $N^{1/2}$ | $1/N^{1/2}$ | $1/N^{1/2}$ |
| $N^{3/5}$ | $1/N^{2/5}$ | $1/N^{1/5}$ |

Normal approximation to the binomial distribution

$$\binom{2N}{N-k} = \exp(\ln(2N!) - \ln(N-k)! - \ln(N+k)!)$$

$$\ln N! = (N + \frac{1}{2}) \ln N - N + \ln \sqrt{2\pi} + O(\frac{1}{N})$$

Stirling's approximation.

$$\begin{aligned} &= \exp\left((2N + \frac{1}{2}) \ln(2N) - 2N + \ln \sqrt{2\pi} + O(1/N)\right. \\ &\quad - (N - k + \frac{1}{2}) \ln(N - k) - N + k - \ln \sqrt{2\pi} + O(1/N) \\ &\quad \left. - (N + k + \frac{1}{2}) \ln(N + k) - N - k - \ln \sqrt{2\pi} + O(1/N)\right) \end{aligned}$$

Collect terms

$$= \exp\left((2N) \ln 2 - \ln \sqrt{\pi N} - (N - k + \frac{1}{2}) \ln(1 - \frac{k}{N}) - (N + k + \frac{1}{2}) \ln(1 + \frac{k}{N}) + O(1/N)\right)$$

Rearrange terms

$$\begin{aligned} &= \exp\left(2N) \ln 2 - \ln \sqrt{\pi N}\right. \\ &\quad \left. - (N + \frac{1}{2})(\ln(1 - \frac{k}{N}) + \ln(1 + \frac{k}{N})) + k(\ln(1 - \frac{k}{N}) - \ln(1 + \frac{k}{N})) + O(1/N)\right) \end{aligned}$$

Expand $\ln(1 - k/N)$ and $\ln(1 + k/N)$.

$$= \exp\left((2N) \ln 2 - \ln \sqrt{\pi N} - \frac{k^2}{N} + O(\frac{k^4}{N^3}) + O(\frac{1}{N})\right)$$

$$\begin{aligned} \ln(1 - \frac{k}{N}) + \ln(1 + \frac{k}{N}) &= -\frac{k^2}{N^2} + O(\frac{k^3}{N^3}) \\ \ln(1 - \frac{k}{N}) - \ln(1 + \frac{k}{N}) &= -\frac{2}{N} + O(\frac{k^3}{N^3}) \end{aligned}$$

$$\frac{1}{4^N} \binom{2N}{N-k} = \frac{e^{-k^2/N}}{\sqrt{\pi N}} \left(1 + O(\frac{k^4}{N^3}) + O(\frac{1}{N})\right)$$

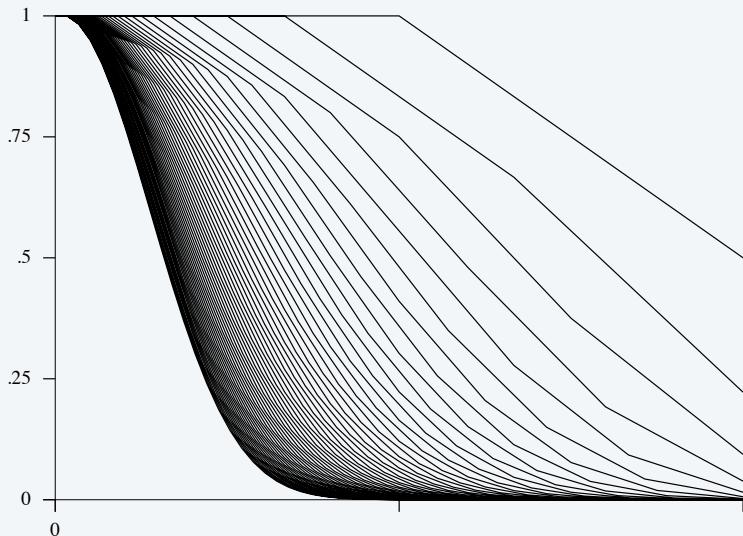
Fundamental bivariate approximations

| | | uniform | central |
|---------|--|--|--|
| normal | $\binom{2N}{N-k}$ | $\frac{e^{-k^2/N}}{\sqrt{\pi N}} + O(\frac{1}{N^{3/2}})$ | $\frac{e^{-k^2/N}}{\sqrt{\pi N}} \left(1 + O(\frac{1}{N}) + O(\frac{k^4}{N^3})\right)$ |
| Poisson | $\binom{N}{k} \left(\frac{\lambda}{N}\right)^k \left(1 - \frac{\lambda}{N}\right)^{N-k}$ | $\frac{\lambda^k e^{-\lambda}}{k!} + o(1)$ | $\frac{\lambda^k e^{-\lambda}}{k!} \left(1 + O(\frac{1}{N}) + O(\frac{k}{N})\right)$ |
| Q | $\frac{N!}{(N-k)!N^k}$ | $e^{-k^2/(2N)} + O(\frac{1}{\sqrt{N}})$ | $e^{-k^2/(2N)} \left(1 + O(\frac{k}{N}) + O(\frac{k^3}{N^2})\right)$ |

Next challenge: Approximating sums via bivariate asymptotics

Example: Ramanujan Q-function

$$Q(N) \equiv \sum_{1 \leq k \leq N} \frac{N!}{(N-k)!N^k}$$



What is the area under this curve?

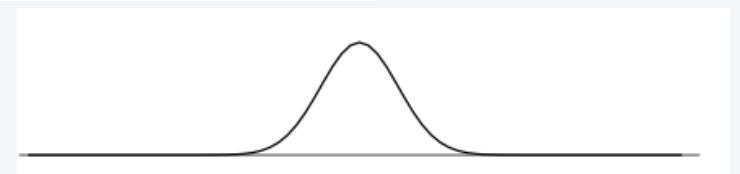
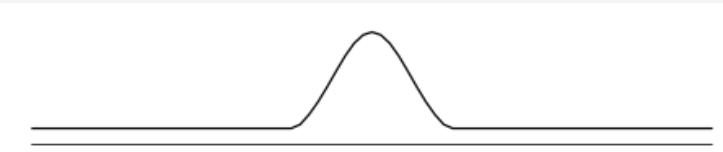
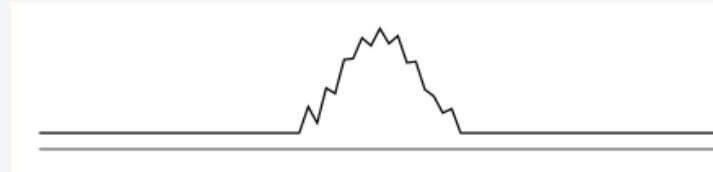
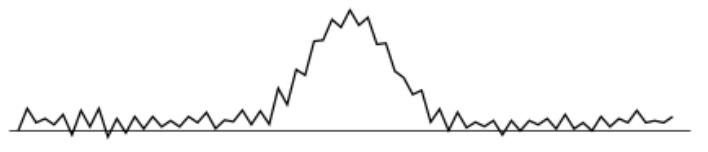
Observations:

- nearly 1 for small k
- negligible for large k
- bivariate asymptotics needed to give different estimates in different ranges.

Laplace method

To approximate a sum:

- Restrict the range to an area that contains the largest summands.
- Approximate the summand.
- Extend the range by bounding the tails to get a simpler sum.
- Approximate the new sum with an integral.



Laplace method for Ramanujan Q-function

$$Q(N) \equiv \sum_{1 \leq k \leq N} \frac{N!}{(N-k)!N^k}$$

Restrict the range to an area that contains the largest summands.

$$Q(N) = \sum_{1 \leq k \leq k_0} \frac{N!}{(N-k)!N^k} + \sum_{k_0 < k \leq N} \frac{N!}{(N-k)!N^k}$$

Take $k_0 = o(N^{2/3})$ to make tail exponentially small.

Approximate the summand.

$$\sum_{1 \leq k \leq k_0} \frac{N!}{(N-k)!N^k} \sim \sum_{1 \leq k \leq k_0} e^{-k^2/2N}$$

Q-distribution approximation.

Extend the range by bounding the tails to get a simpler sum.

$$Q(N) \sim \sum_{k \geq 1} e^{-k^2/2N}$$

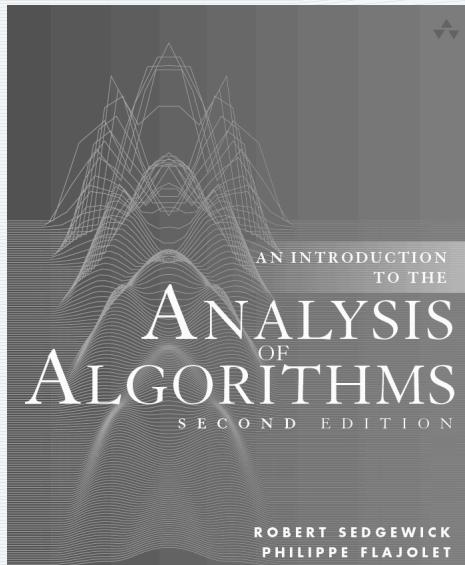
Tail is also exponentially small for this sum.

Approximate the new sum with an integral.

$$Q(N) \sim \sqrt{N} \int_0^\infty e^{-x^2/2} dx = \sqrt{\pi N/2}$$

ANALYTIC COMBINATORICS

PART ONE



4. Asymptotic Approximations

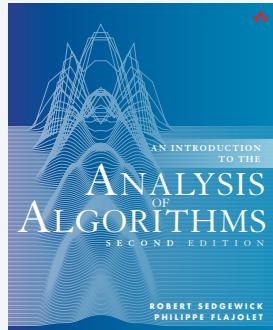
- Standard scale
- Manipulating expansions
- Asymptotics of finite sums
- **Bivariate asymptotics**

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4d. Asympt. bivariate

Exercise 4.9

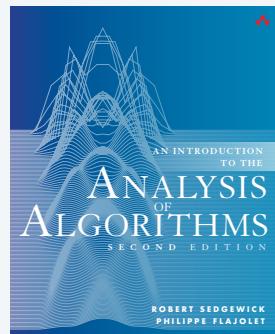
How small is "exponentially small"?



Exercise 4.9 If $\alpha < \beta$, show that α^N is exponentially small relative to β^N . For $\beta = 1.2$ and $\alpha = 1.1$, find the absolute and relative errors when $\alpha^N + \beta^N$ is approximated by β^N , for $N = 10$ and $N = 100$.

Exercise 4.71

Asymptotics of another Ramanujan function.



Exercise 4.71 Show that

$$P(N) = \sum_{k \geq 0} \frac{(N-k)^k (N-k)!}{N!} = \sqrt{\pi N/2} + O(1)$$

Assignments for next lecture

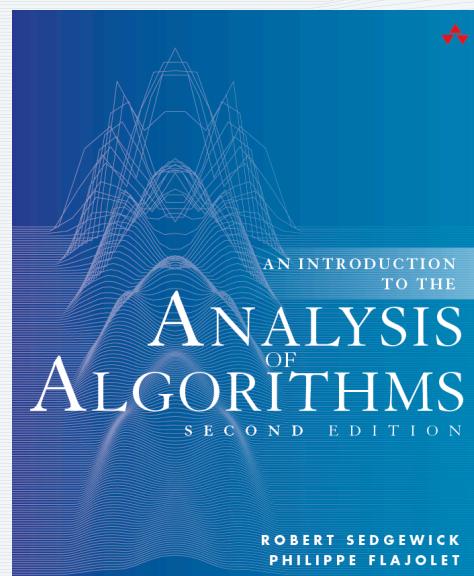
1. Write a program that takes N and k from the command line and prints

$$\lg \binom{N}{k}$$

2. Write up solutions to Exercises 4.9 and 4.70.
3. Read pages 149-215 (Asymptotic Approximations) in text.

ANALYTIC COMBINATORICS

PART ONE



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4. Asymptotic Approximations