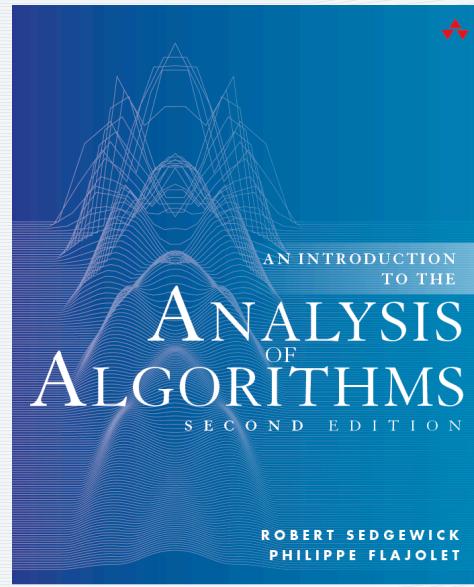


ANALYTIC COMBINATORICS

PART ONE



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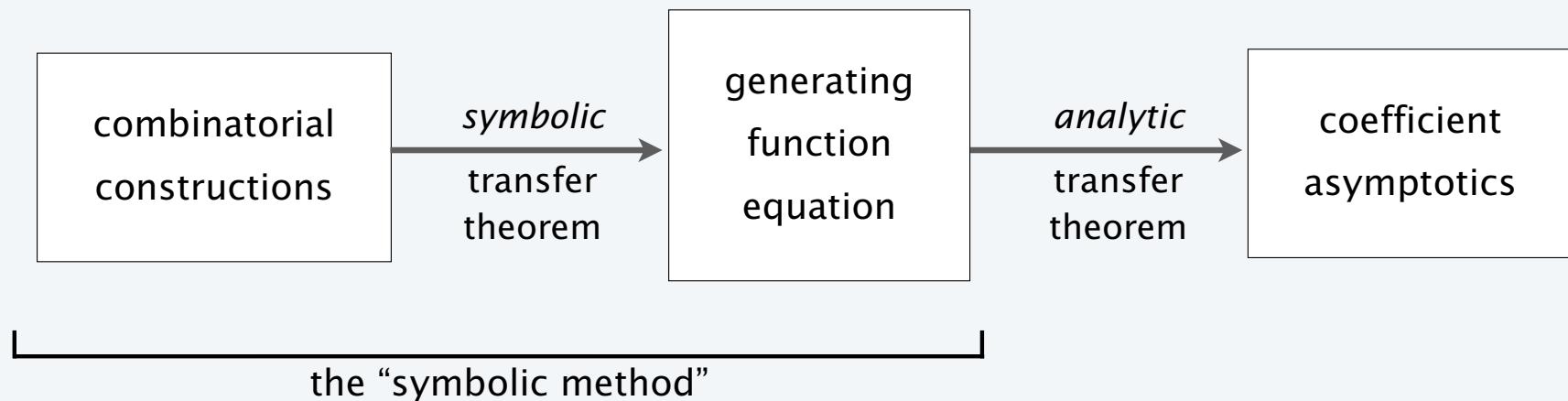
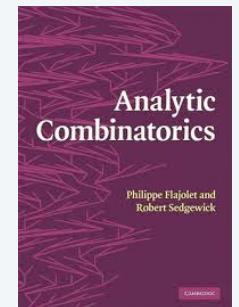
5. Analytic Combinatorics

Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Features:

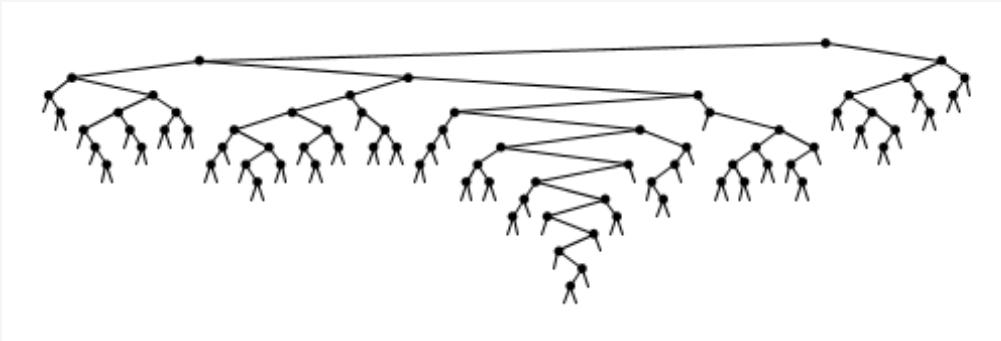
- Analysis begins with formal *combinatorial constructions*.
- The *generating function* is the central object of study.
- *Transfer theorems* can immediately provide results from formal descriptions.
- Results extend, in principle, to any desired precision on the standard scale.
- Variations on fundamental constructions are easily handled.



Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Ex: How many binary trees with N nodes?



$$T = E + Z \times T \times T$$

combinatorial construction

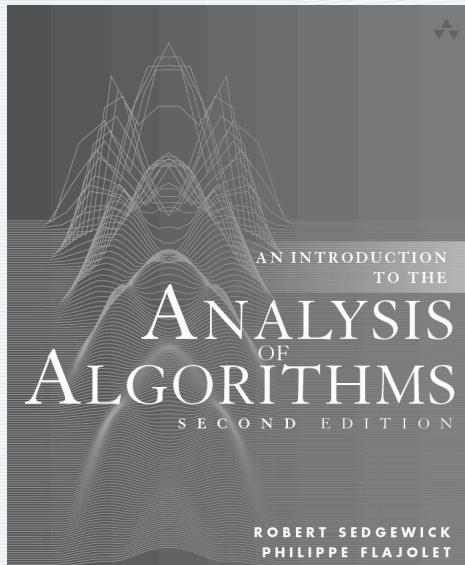
$$T(z) = 1 + zT(z)^2$$

GF equation

$$T_N \sim \frac{4^N}{\sqrt{\pi N^3}}$$

coefficient
asymptotics

ANALYTIC COMBINATORICS PART ONE



5. Analytic Combinatorics

- The symbolic method
- Labelled objects
- Coefficient asymptotics
- Perspective

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5a . AC . Symbolic

The symbolic method

is an approach for translating *combinatorial constructions* to *GF equations*

- Define a *class* of combinatorial objects.
- Define a notion of *size*.
- Define a *GF* whose coefficients count objects of the same size.
- Define *operations* suitable for constructive definitions of objects.
- Develop *translations* from constructions to operations on GFs.

Examples
A, B, Z
$ b $
$A(z)$
$A \times B$
$A(z)B(z)$

Formal basis:

- A *combinatorial class* is a set of objects and a *size function*.
- An *atom* is an object of size 1.
- An *neutral object* is an atom of size 0.
- A *combinatorial construction* uses the union, product, and sequence operations to define a class in terms of atoms and other classes.

Building blocks

notation	denotes	contains
Z	atomic class	an atom
E	neutral class	neutral object
\emptyset	empty class	nothing

Unlabelled class example 1: natural numbers

Def. A *natural number* is a **set** (or a sequence) of atoms.



$$l_1 = 1$$



$$l_2 = 1$$



$$l_3 = 1$$



$$l_4 = 1$$



$$l_5 = 1$$

unary notation

counting sequence

$$l_N = 1$$

OGF

$$\frac{1}{1-z}$$

$$\sum_{N \geq 0} z^N = \frac{1}{1-z}$$

Unlabelled class example 2: bitstrings

Def. A *bitstring* is a **sequence** of 0 or 1 bits.

$$\begin{array}{c} \square \\ 0 \\ 1 \end{array}$$
$$B_0 = 1 \quad B_1 = 2$$

$$\begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array}$$
$$B_2 = 4$$

$$\begin{array}{c} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{array}$$
$$B_3 = 8$$

$$\begin{array}{c} 0000 \\ 0001 \\ 0010 \\ 0011 \\ 0100 \\ 0101 \\ 0110 \\ 0111 \\ 1000 \\ 1001 \\ 1010 \\ 1011 \\ 1100 \\ 1101 \\ 1110 \\ 1111 \end{array}$$
$$B_4 = 16$$

counting sequence

OGF

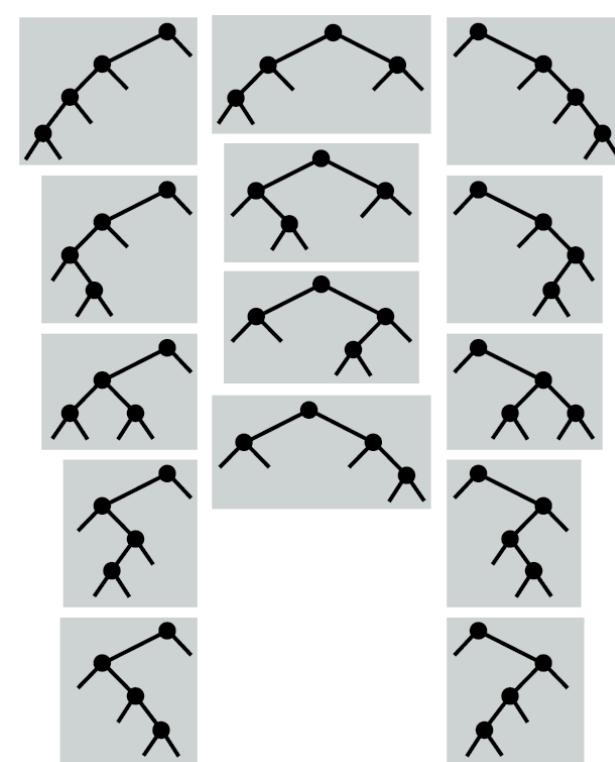
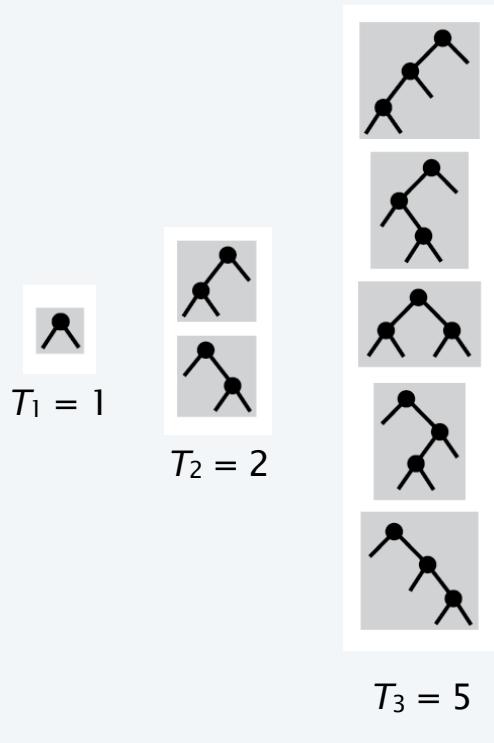
$$B_N = 2^N$$

$$\frac{1}{1 - 2z}$$

$$\sum_{N \geq 0} 2^N z^N = \sum_{N \geq 0} (2z)^N = \frac{1}{1 - 2z}$$

Unlabelled class example 3: binary trees

Def. A *binary tree* is empty or a **sequence** of a node and two binary trees



counting sequence

$$T_N = \frac{1}{N+1} \binom{2N}{N} \quad \text{OGF}$$

$$\frac{1}{2z}(1 - \sqrt{1 - 4z})$$

Catalan numbers (see Lecture 3)

$$T(z) = 1 + zT(z)^2$$

Combinatorial constructions for unlabelled classes

<i>construction</i>	<i>notation</i>	<i>semantics</i>
disjoint union	$A + B$	disjoint copies of objects from A and B
Cartesian product	$A \times B$	ordered pairs of copies of objects, one from A and one from B
sequence	$SEQ(A)$	sequences of objects from A

A and B are
 combinatorial classes
 of unlabelled objects

Ex 1. $(00 + 01) \times (101 + 110 + 111) = 00101 \quad 00110 \quad 00111 \quad 01101 \quad 01111$

Ex 2. $\bullet \times SEQ(\bullet) = \bullet \quad \bullet \bullet \quad \bullet \bullet \bullet \quad \bullet \bullet \bullet \bullet \quad \bullet \bullet \bullet \bullet \bullet \quad \dots$

Ex 3. $\square \times \bullet \times \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \square \quad \square \end{array} = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \square \quad \bullet \\ \quad \diagdown \quad \diagup \\ \quad \square \quad \square \end{array}$

"unlabelled" ?? Stay tuned.

The symbolic method for unlabelled classes (transfer theorem)

Theorem. Let A and B be combinatorial classes of unlabelled objects with OGFs $A(z)$ and $B(z)$. Then

<i>construction</i>	<i>notation</i>	<i>semantics</i>	<i>OGF</i>
disjoint union	$A + B$	disjoint copies of objects from A and B	$A(z) + B(z)$
Cartesian product	$A \times B$	ordered pairs of copies of objects, one from A and one from B	$A(z)B(z)$
sequence	$SEQ(A)$	sequences of objects from A	$\frac{1}{1 - A(z)}$

Proofs of transfers

are immediate from GF counting

$A + B$

$$\sum_{\gamma \in A+B} z^{|\gamma|} = \sum_{\alpha \in A} z^{|\alpha|} + \sum_{\beta \in B} z^{|\beta|} = A(z) + B(z)$$

$A \times B$

$$\sum_{\gamma \in A \times B} z^{|\gamma|} = \sum_{\alpha \in A} \sum_{\beta \in B} z^{|\alpha|+|\beta|} = \left(\sum_{\alpha \in A} z^{|\alpha|} \right) \left(\sum_{\beta \in B} z^{|\beta|} \right) = A(z)B(z)$$

$SEQ(A)$

$$SEQ(A) \equiv \epsilon + A + A^2 + A^3 + A^4 + \dots$$

$$1 + A(z) + A(z)^2 + A(z)^3 + A(z)^4 + \dots = \frac{1}{1 - A(z)}$$

Symbolic method: binary trees

How many **binary trees** with N nodes?

<i>Class</i>	T , the class of all binary trees
<i>Size</i>	$ t $, the number of internal nodes in t
<i>OGF</i>	$T(z) = \sum_{t \in T} z^{ t } = \sum_{N \geq 0} T_N z^N$

<i>Atoms</i>	<i>type</i>	<i>class</i>	<i>size</i>	<i>GF</i>
	external node	Z_\square	0	1
	internal node	Z_\bullet	1	z

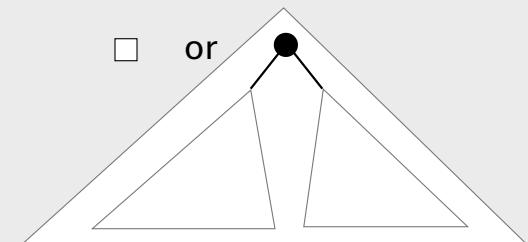
Construction

$$T = Z_\square + T \times Z_\bullet \times T$$

OGF equation

$$T(z) = 1 + zT(z)^2$$

“a binary tree is an external node or an internal node connected to two binary trees”



$$[z^N]T(z) = \frac{1}{N+1} \binom{2N}{N} \sim \frac{4^N}{\sqrt{\pi N^3}}$$

see Lecture 3 and stay tuned.

Symbolic method: binary trees

How many binary trees with N *external* nodes?

<i>Class</i>	T , the class of all binary trees
<i>Size</i>	$[t]$, the number of <i>external</i> nodes in t
<i>OGF</i>	$T^\square(z) = \sum_{t \in T} z^{[t]}$

<i>Atoms</i>	<i>type</i>	<i>class</i>	<i>size</i>	<i>GF</i>
	external node	Z_\square	1	z
	internal node	Z_\bullet	0	1

Construction

$$T = Z_\square + T \times Z_\bullet \times T$$

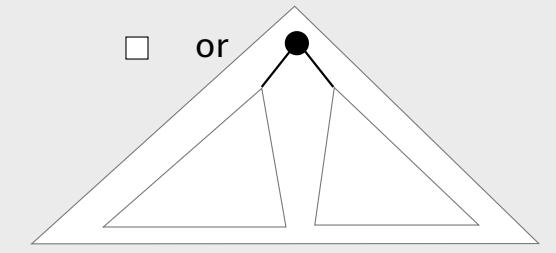
OGF equation

$$T^\square(z) = z + T^\square(z)^2$$

$$T^\square(z) = zT(z)$$

$$[z^N]T^\square(z) = [z^{N-1}]T(z) = \frac{1}{N} \binom{2N-2}{N-1}$$

“a binary tree is an external node or an internal node connected to two binary trees”



← same as # binary trees
with $N-1$ internal nodes

Symbolic method: binary strings

Warmup: How many **binary strings** with N bits?

<i>Class</i>	B , the class of all binary strings
<i>Size</i>	$ b $, the number of bits in b
<i>OGF</i>	$B(z) = \sum_{b \in B} z^{ b } = \sum_{N \geq 0} B_N z^N$

<i>Atoms</i>	<i>type</i>	<i>class</i>	<i>size</i>	<i>GF</i>
0 bit	Z_0	1	z	
1 bit	Z_1	1	z	

Construction

$$B = SEQ(Z_0 + Z_1)$$

“a binary string is a sequence
of 0 bits and 1 bits”

OGF equation

$$B(z) = \frac{1}{1 - 2z}$$

$$[z^N]B(z) = 2^N \quad \checkmark$$

Symbolic method: binary strings (alternate)

Warmup: How many **binary strings** with N bits?

<i>Class</i>	B , the class of all binary strings
<i>Size</i>	$ b $, the number of bits in b
<i>OGF</i>	$B(z) = \sum_{b \in B} z^{ b } = \sum_{N \geq 0} B_N z^N$

<i>Atoms</i>	<i>type</i>	<i>class</i>	<i>size</i>	<i>GF</i>
	0 bit	Z_0	1	z
	1 bit	Z_1	1	z

Construction

$$B = E + (Z_0 + Z_1) \times B$$

“a binary string is empty or
a bit followed by a binary string”

OGF equation

$$B(z) = 1 + 2zB(z)$$

Solution

$$B(z) = \frac{1}{1 - 2z}$$

$$[z^N]B(z) = 2^N \quad \checkmark$$

Symbolic method: binary strings with restrictions

Ex. How many N -bit binary strings have no two consecutive 0s?

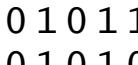

 $T_0 = 1$


 $T_1 = 2$


 $T_2 = 3$


 $T_3 = 5$


 $T_4 = 8$


 $T_5 = 13$

Stay tuned for general treatment (Chapter 8)

Symbolic method: binary strings with restrictions

Ex. How many N -bit binary strings have no two consecutive 0s?

<i>Class</i>	B_{00} , the class of binary strings with no 00
<i>Size</i>	$ b $, the number of bits in b
<i>OGF</i>	$B_{00}(z) = \sum_{b \in B_{00}} z^{ b }$

<i>Atoms</i>	<i>type</i>	<i>class</i>	<i>size</i>	<i>GF</i>
0 bit	Z_0	1	z	
1 bit	Z_1	1	z	

Construction $B_{00} = E + Z_0 + (Z_1 + Z_0 \times Z_1) \times B_{00}$

“a binary string with no 00 is either empty or 0 or it is 1 or 01 followed by a binary string with no 00”

OGF equation $B_{00}(z) = 1 + z + (z + z^2)B_{00}(z)$

solution
$$B_{00}(z) = \frac{1 + z}{1 - z - z^2}$$

$$[z^N]B_{00}(z) = F_N + F_{N+1} = F_{N+2}$$

1, 2, 5, 8, 13, ... ✓

Symbolic method: many, many examples to follow

How many ... with ... ?

<i>Class</i>	
<i>Size</i>	
<i>OGF</i>	

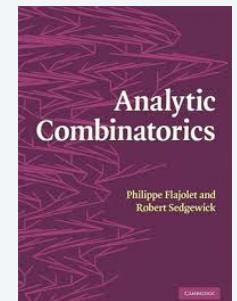
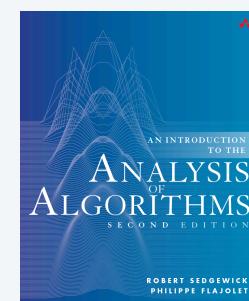
<i>Atoms</i>	<i>type</i>	<i>class</i>	<i>size</i>	<i>GF</i>

Construction

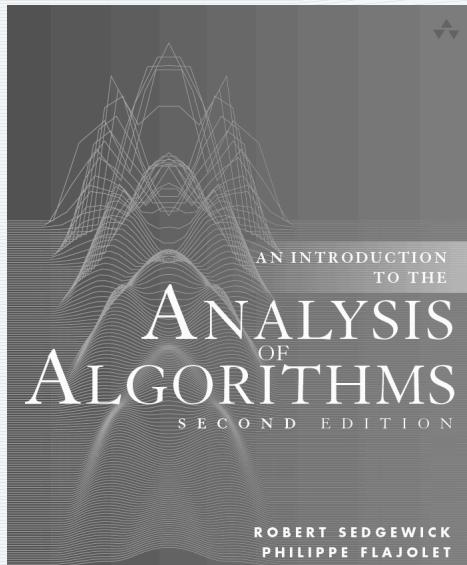
“a ... is either ...
or ... and ...”

OGF equation

solution



ANALYTIC COMBINATORICS PART ONE



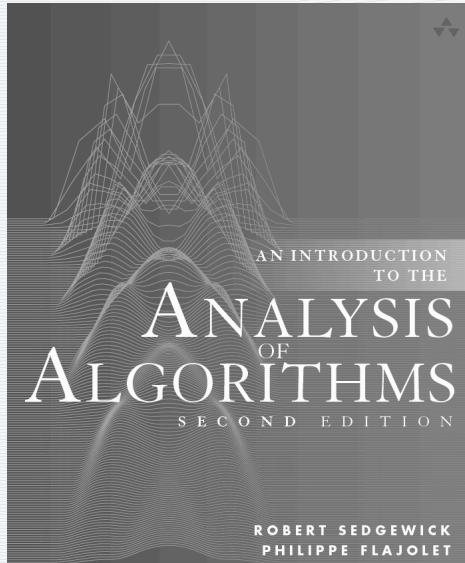
5. Analytic Combinatorics

- The symbolic method
- Labelled objects
- Coefficient asymptotics
- Perspective

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5a . AC . Symbolic

ANALYTIC COMBINATORICS PART ONE



5. Analytic Combinatorics

- The symbolic method
- **Labelled objects**
- Coefficient asymptotics
- Perspective

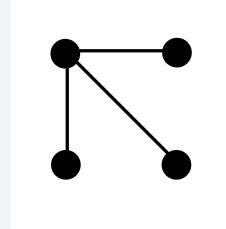
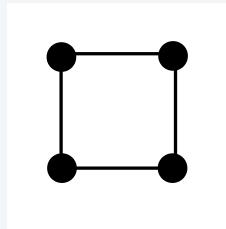
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5b . AC . Labelled

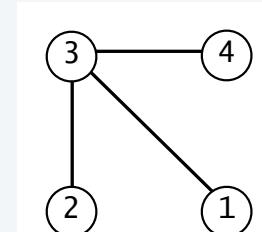
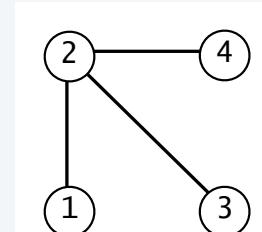
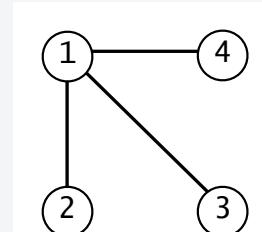
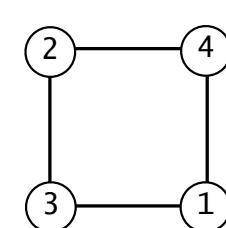
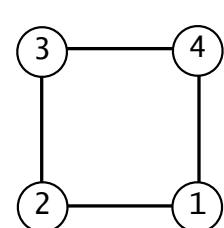
Labelled combinatorial classes

have objects composed of N atoms, labelled with the integers 1 through N .

Ex. Different unlabelled objects



Ex. Different labelled objects

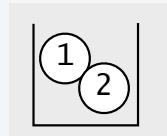


Labelled class example 1: urns

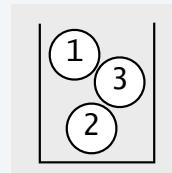
Def. An *urn* is a **set** of labelled atoms.



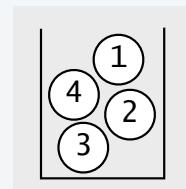
$$U_1 = 1$$



$$U_2 = 1$$



$$U_3 = 1$$



$$U_4 = 1$$

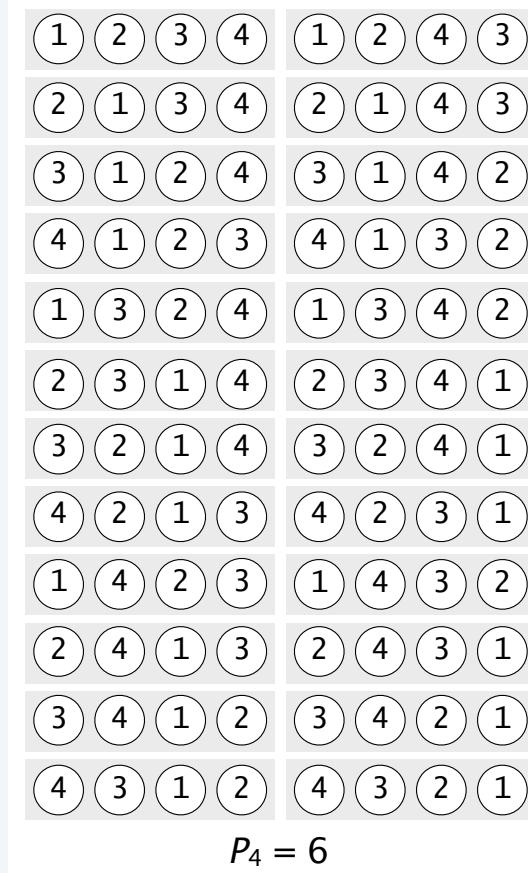
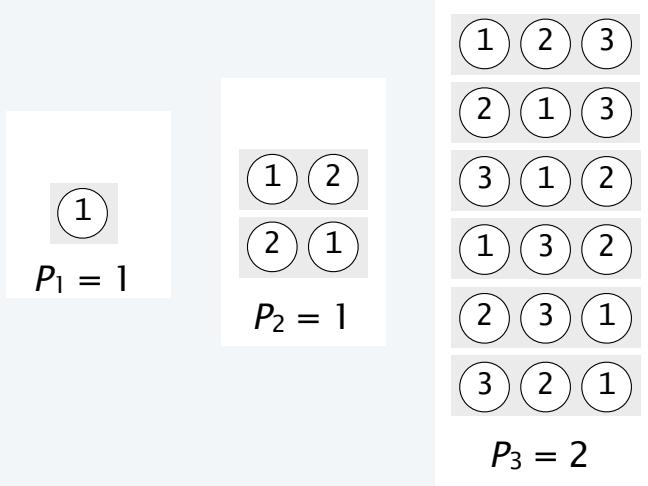
counting sequence

$U_N = 1$	e^z
-----------	-------

$$\sum_{N \geq 0} \frac{z^N}{N!} = e^z$$

Labelled class example 2: permutations

Def. A *permutation* is a **sequence** of labelled atoms.



counting sequence

EGF

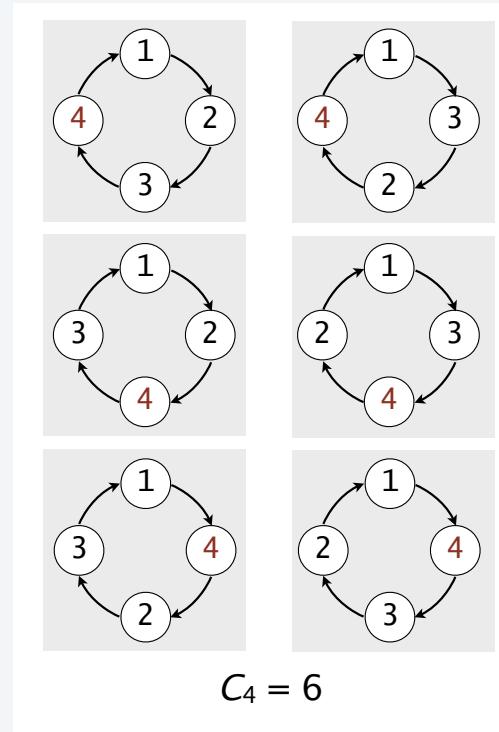
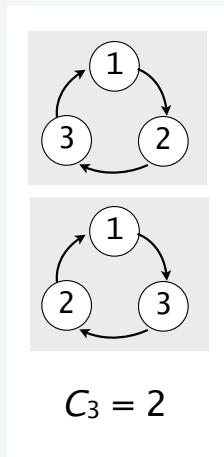
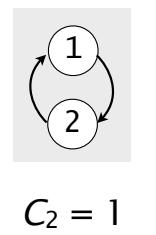
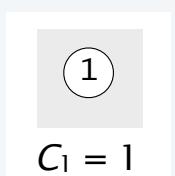
$$P_N = N!$$

$$\frac{1}{1-z}$$

$$\sum_{N \geq 0} \frac{N! z^N}{N!} = \sum_{N \geq 0} z^N = \frac{1}{1-z}$$

Labelled class example 3: cycles

Def. A *cycle* is a **cyclic sequence** of labelled atoms



counting sequence

$$C_N = (N - 1)!$$

EGF

$$\ln \frac{1}{1 - z}$$

$$\sum_{N \geq 1} \frac{(N - 1)!z^N}{N!} = \sum_{N \geq 1} \frac{z^N}{N} = \ln \frac{1}{1 - z}$$

Star product operation

Analog to Cartesian product requires *relabelling in all consistent ways*.

Ex 1. $\boxed{1} \star \boxed{1 \ 2 \ 3} = \boxed{1 \ 2 \ 3 \ 4} \quad \boxed{2 \ 1 \ 3 \ 4} \quad \boxed{3 \ 1 \ 2 \ 4} \quad \boxed{4 \ 1 \ 2 \ 3}$

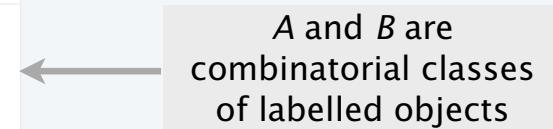
Ex 2.

$$\boxed{\begin{array}{c} 1 \\ \swarrow \searrow \\ 2 \end{array}} \star \boxed{\begin{array}{c} 1 \\ \swarrow \searrow \\ 3 \ 2 \end{array}} = \boxed{\begin{array}{c} 1 \\ \swarrow \searrow \\ 2 \end{array}} \quad \boxed{\begin{array}{c} 3 \\ \swarrow \searrow \\ 5 \ 4 \end{array}} \quad \boxed{\begin{array}{c} 2 \\ \swarrow \searrow \\ 3 \end{array}} \quad \boxed{\begin{array}{c} 1 \\ \swarrow \searrow \\ 5 \ 4 \end{array}} \quad \boxed{\begin{array}{c} 2 \\ \swarrow \searrow \\ 4 \end{array}} \quad \boxed{\begin{array}{c} 1 \\ \swarrow \searrow \\ 5 \ 3 \end{array}} \quad \boxed{\begin{array}{c} 3 \\ \swarrow \searrow \\ 4 \end{array}} \quad \boxed{\begin{array}{c} 1 \\ \swarrow \searrow \\ 5 \ 2 \end{array}} \quad \boxed{\begin{array}{c} 2 \\ \swarrow \searrow \\ 5 \end{array}} \quad \boxed{\begin{array}{c} 1 \\ \swarrow \searrow \\ 4 \ 3 \end{array}} \quad \boxed{\begin{array}{c} 4 \\ \swarrow \searrow \\ 5 \end{array}} \quad \boxed{\begin{array}{c} 1 \\ \swarrow \searrow \\ 3 \ 2 \end{array}}$$

Combinatorial constructions for labelled classes

<i>construction</i>	<i>notation</i>	<i>semantics</i>
disjoint union	$A + B$	disjoint copies of objects from A and B
labelled product	$A \star B$	ordered pairs of copies of objects, one from A and one from B
sequence	$SEQ(A)$	sequences of objects from A
set	$SET(A)$	sets of objects from A
cycle	$CYC(A)$	cyclic sequences of objects from A

A and B are
combinatorial classes
of labelled objects



The symbolic method for labelled classes (transfer theorem)

Theorem. Let A and B be combinatorial classes of **labelled** objects with EGFs $A(z)$ and $B(z)$. Then

<i>construction</i>	<i>notation</i>	<i>semantics</i>	<i>EGF</i>
disjoint union	$A + B$	disjoint copies of objects from A and B	$A(z) + B(z)$
labelled product	$A \star B$	ordered pairs of copies of objects, one from A and one from B	$A(z)B(z)$
sequence	$SEQ_k(A)$	k - sequences of objects from A	$A(z)^k$
	$SEQ(A)$	sequences of objects from A	$\frac{1}{1 - A(z)}$
set	$SET_k(A)$	k -sets of objects from A	$A(z)^k/k!$
	$SET(A)$	sets of objects from A	$e^{A(z)}$
cycle	$CYC_k(A)$	k -cycles of objects from A	$A(z)^k/k$
	$CYC(A)$	cycles of objects from A	$\ln \frac{1}{1 - A(z)}$

The symbolic method for labelled classes: basic constructions

<i>class</i>	<i>construction</i>	<i>EGF</i>	<i>counting sequence</i>	<i>construction</i>	<i>notation</i>	<i>EGF</i>
urns	$U = SET(Z)$	$U(z) = e^z$	$U_N = 1$	disjoint union	$A + B$	$A(z) + B(z)$
cycles	$C = CYC(Z)$	$C(z) = \ln \frac{1}{1-z}$	$C_N = (N-1)!$	labelled product	$A \star B$	$A(z)B(z)$
permutations	$P = SEQ(Z)$	$P(z) = \frac{1}{1-z}$	$P_N = N!$	sequence	$SEQ_k(A)$	$A(z)^k$
	$P = E + Z \star P$				$SEQ(A)$	$\frac{1}{1-A(z)}$
				set	$SET_k(A)$	$A(z)^k/k!$
					$SET(A)$	$e^{A(z)}$
				cycle	$CYC_k(A)$	$A(z)^k/k$
					$CYC(A)$	$\ln \frac{1}{1-A(z)}$

Proofs of transfers

are immediate from GF counting

$A + B$

$$\sum_{\gamma \in A+B} \frac{z^{|\gamma|}}{|\gamma|!} = \sum_{\alpha \in A} \frac{z^{|\alpha|}}{|\alpha|!} + \sum_{\beta \in B} \frac{z^{|\beta|}}{|\beta|!} = A(z) + B(z)$$

$A \star B$

$$\sum_{\gamma \in A \times B} \frac{z^{|\gamma|}}{|\gamma|!} = \sum_{\alpha \in A} \sum_{\beta \in B} \binom{|\alpha| + |\beta|}{|\alpha|} \frac{z^{|\alpha|+|\beta|}}{(|\alpha| + |\beta|)!} = \left(\sum_{\alpha \in A} \frac{z^{|\alpha|}}{|\alpha|!} \right) \left(\sum_{\beta \in B} \frac{z^{|\beta|}}{|\beta|!} \right) = A(z)B(z)$$

Notation. We write A^2 for $A \star A$, A^3 for $A \star A \star A$, etc.

Proofs of transfers

are immediate from GF counting

$$A(z)^k = \sum_{N \geq 0} \{\#k\text{-sequences of size } N\} \frac{z^N}{N!} = \sum_{N \geq 0} k \{\#k\text{-cycles of size } N\} \frac{z^N}{N!} = \sum_{N \geq 0} k! \{\#k\text{-sets of size } N\} \frac{z^N}{N!}$$

$$\frac{A(z)^k}{k} = \sum_{N \geq 0} \{\#k\text{-cycles of size } N\} \frac{z^N}{N!}$$

$$\frac{A(z)^k}{k!} = \sum_{N \geq 0} \{\#k\text{-sets of size } N\} \frac{z^N}{N!}$$

<i>class</i>	<i>construction</i>	<i>EGF</i>
k-sequence	$SEQ_k(A)$	$A(z)^k$
sequence	$SEQ_k(A) = SEQ_0(A) + SEQ_1(A) + SEQ_2(A) + \dots$	$1 + A(z) + A(z)^2 + A(z)^3 + \dots = \frac{1}{1 - A(z)}$
k-cycle	$CYC_k(A)$	$\frac{A(z)^k}{k}$
cycle	$CYC_k(A) = CYC_0(A) + CYC_1(A) + CYC_2(A) + \dots$	$1 + \frac{A(z)}{1} + \frac{A(z)^2}{2} + \frac{A(z)^3}{3} + \dots = \ln \frac{1}{1 - A(z)}$
k-set	$SET_k(A)$	$\frac{A(z)^k}{k!}$
set	$SET_k(A) = SET_0(A) + SET_1(A) + SET_2(A) + \dots$	$1 + \frac{A(z)}{1!} + \frac{A(z)^2}{2!} + \frac{A(z)^3}{3!} + \dots = e^{A(z)}$

Labelled class example 4: sets of cycles

Q. How many sets of cycles of labelled atoms?

$$P^*_1 = 1$$

$$P^*_2 = 2$$

$$P^*_3 = 6$$

$$P^*_4 = 24$$

Symbolic method: sets of cycles

How many **sets of cycles** of length N ?

<i>Class</i>	P^* , the class of all sets of cycles of atoms	<i>Atom</i>	<i>type</i>	<i>class</i>	<i>size</i>	<i>GF</i>
<i>Size</i>	$ p $, the number of atoms in p		labelled atom	Z	1	z
<i>EGF</i>	$P^*(z) = \sum_{p \in P^*} \frac{z^{ p }}{ p !} = \sum_{N \geq 0} P_N^* \frac{z^N}{N!}$					

Construction

$$P^* = SET(CYC(Z))$$

OGF equation

$$P^*(z) = \exp\left(\ln \frac{1}{1-z}\right) = \frac{1}{1-z}$$

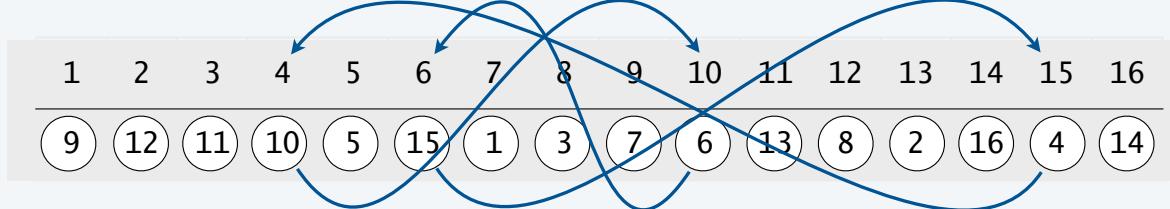
Counting sequence

$$P_N^* = N![z^N]P^*(z) = N!$$

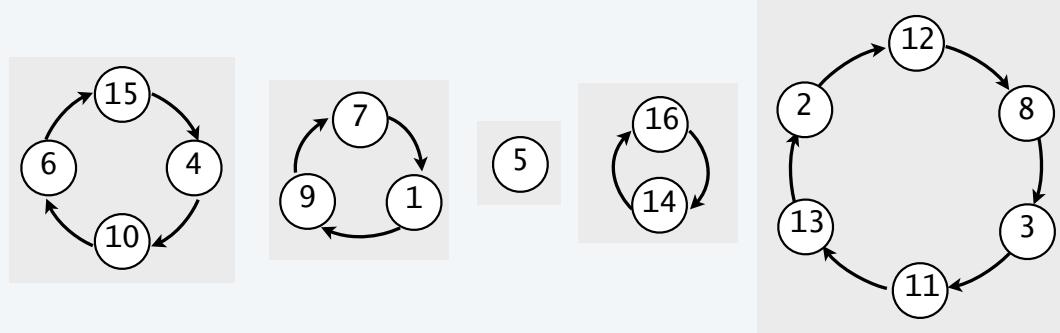
Aside: A combinatorial bijection

A permutation is a set of cycles.

Standard representation



Set of cycles representation



Derangements

N people go to the opera and leave their hats on a shelf in the cloakroom.
When leaving, they each grab a hat at random.

Q. *What is the probability that nobody gets their own hat ?*



Definition. A **derangement** is a permutation with no singleton cycles

Derangements (various versions)

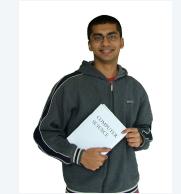
A group of N people go to the opera and leave their hats in the cloakroom. When leaving, they each grab a hat at random.

Q. *What is the probability that nobody gets their own hat ?*



A professor returns exams to N students by passing them out at random.

Q. *What is the probability that nobody gets their own exam ?*



A group of N sailors go ashore for revelry that leads to a state of inebriation. When returning, they each end up sleeping in a random cabin.

Q. *What is the probability that nobody sleeps in their own cabin ?*



A group of N students who live in single rooms go to a party that leads to a state of inebriation. When returning, they each end up in a random room.

Q. *What is the probability that nobody ends up in their own room ?*

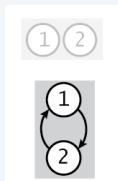


Derangements

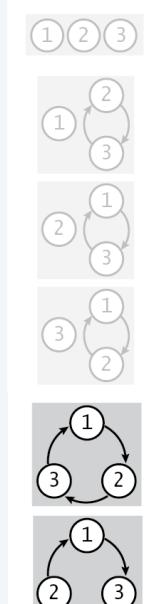
are permutations with no singleton cycles.



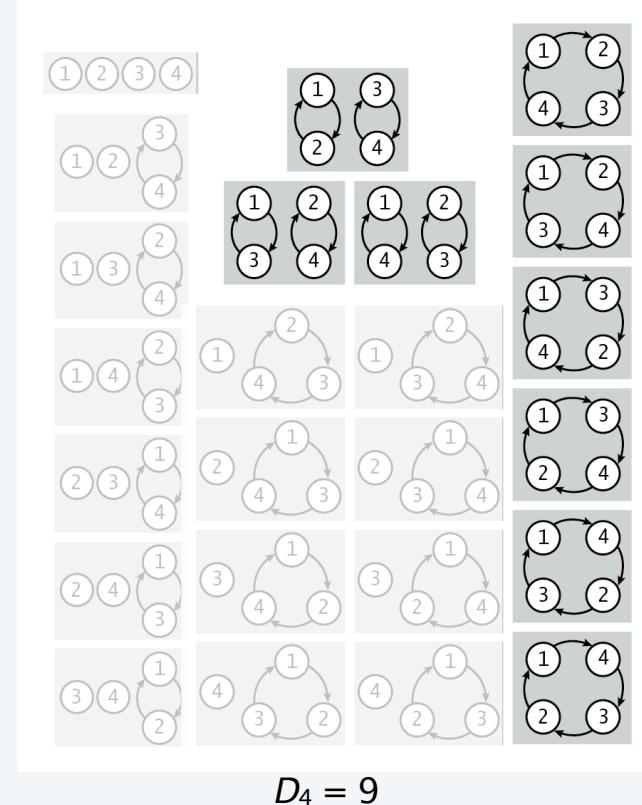
$$D_1 = 0$$



$$D_2 = 1$$



$$D_3 = 2$$



$$D_4 = 9$$

Symbolic method: derangements

How many **derangements** of length N ?

<i>Class</i>	D , the class of all derangements
<i>Size</i>	$ p $, the number of atoms in p
<i>EGF</i>	$D(z) = \sum_{d \in D} \frac{z^{ d }}{ d !} = \sum_{N \geq 0} D_N \frac{z^N}{N!}$

<i>Atom</i>	<i>type</i>	<i>class</i>	<i>size</i>	<i>GF</i>
labelled atom	Z	1	z	

Construction

$$D = SET(CYC_{>1}(Z))$$

"Derangements are permutations with no singleton cycles"

OGF equation

$$D(z) = e^{z^2/2 + z^3/3 + z^4/4 + \dots} = \exp\left(\ln \frac{1}{1-z} - z\right) = \frac{e^{-z}}{1-z}$$

Expansion

$$[z^N]D(z) \equiv \frac{D_N}{N!} = \sum_{0 \leq k \leq N} \frac{(-1)^k}{k!} \sim \left(\frac{1}{e}\right)$$

probability that a random permutation is a derangement

simple convolution

see "Asymptotics" lecture

Alternate derivation

$$\begin{aligned} Set(Z) * D &= P \\ e^z D(z) &= \frac{1}{1-z} \end{aligned}$$

Derangements

A group of N students who live in single rooms go to a party that leads to a state of inebriation. When returning, they each end up in a random room.

Q. *What is the probability that nobody ends up in their own room ?*



A. $\frac{1}{e} \doteq 0.36788$

Derangements

A group of N graduating seniors each throw their hats in the air and each catch a random hat.

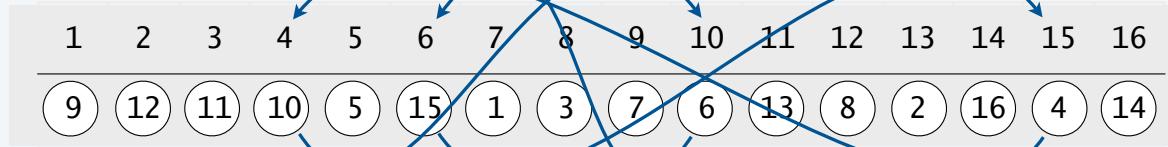
Q. *What is the probability that nobody gets their own hat back ?*



A. $\frac{1}{e} \doteq 0.36788$

Generalized derangements

In the hats-in-the-air scenario, a student can get her hat back by "following the cycle".



Q. What is the probability that all cycles are of length $> M$?

Symbolic method: generalized derangements

How many permutations of length N have no cycles of length $\leq M$?

<i>Class</i>	D_M , the class of all generalized derangements
<i>Size</i>	$ d $, the number of atoms in d
<i>EGF</i>	$D_M(z) = \sum_{d \in D_M} \frac{z^{ d }}{ d !} = \sum_{N \geq 0} D_M N \frac{z^N}{N!}$

<i>Atom</i>	<i>type</i>	<i>class</i>	<i>size</i>	<i>GF</i>
labelled atom	Z	1	z	

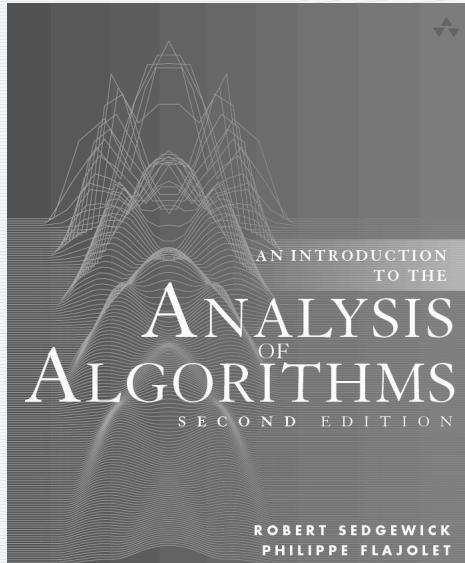
Construction $D_M = SET(CYC_{>M}(Z))$

OGF equation
$$D_M(z) = e^{\frac{z^{M+1}}{M+1} + \frac{z^{M+2}}{M+2} + \dots} = \exp\left(\ln \frac{1}{1-z} - z - z^2/2 - \dots - z^M/M\right)$$

$$= \frac{e^{-z - \frac{z^2}{2} - \frac{z^3}{3} - \dots - \frac{z^M}{M}}}{1-z}$$

Expansion $D_{MN} = ?? M\text{-way convolution (stay tuned)}$

ANALYTIC COMBINATORICS PART ONE



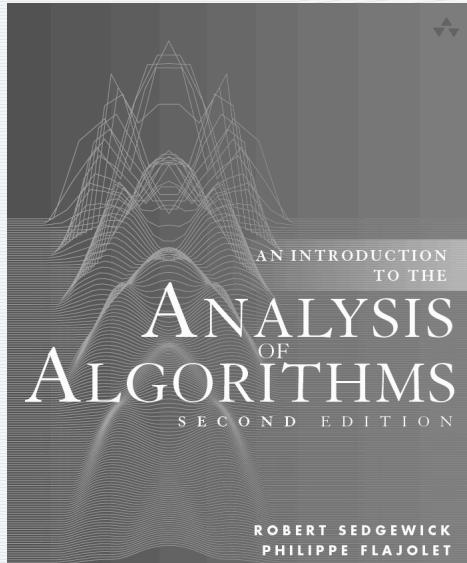
5. Analytic Combinatorics

- The symbolic method
- **Labelled objects**
- Coefficient asymptotics
- Perspective

<http://aofa.cs.princeton.edu>

5b . AC . Labelled

ANALYTIC COMBINATORICS PART ONE



5. Analytic Combinatorics

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5c . AC . Asymptotics

Generating coefficient asymptotics

are often *immediately* derived via general "analytic" transfer theorems.

Example 1. Taylor's theorem

Theorem. If $f(z)$ has N derivatives, then $[z^N]f(z) = f^{(N)}(0)/N!$

Example 2. Rational functions transfer theorem (see "Asymptotics" lecture)

Theorem. If $f(z)$ and $g(z)$ are polynomials, then

$$[z^n] \frac{f(z)}{g(z)} = -\frac{\beta f(1/\beta)}{g'(\beta)} \beta^n$$

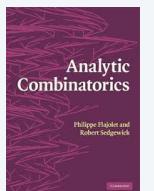
where $1/\beta$ is the largest root of g (provided that it has multiplicity 1).

see "Asymptotics" lecture for general case

Example 3. Radius-of-convergence transfer theorem

[see next slide]

Most are based on complex asymptotics.
Stay tuned for Part 2



Radius-of-convergence transfer theorem

Theorem. If $f(z)$ has radius of convergence > 1 with $f(1) \neq 0$, then

$$[z^n] \frac{f(z)}{(1-z)^\alpha} \sim f(1) \binom{n+\alpha-1}{n} \sim \frac{f(1)}{\Gamma(\alpha)} n^{\alpha-1}$$

for any real $\alpha \notin 0, -1, -2, \dots$

↑
convolution,
 $f_1 + f_2 + \dots + f_n \sim f(1)$

↑
standard asymptotics
with generalized
binomial coefficient

*Gamma function
(generalized factorial)*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

$$\Gamma(N+1) = N!$$

$$\Gamma(1) = 1$$

$$\Gamma(1/2) = \sqrt{\pi}$$

Corollary. If $f(z)$ has radius of convergence $> \rho$ with $f(\rho) \neq 0$, then

$$[z^n] \frac{f(z)}{(1-z/\rho)^\alpha} \sim \frac{f(\rho)}{\Gamma(\alpha)} \rho^n n^{\alpha-1}$$

for any real $\alpha \notin 0, -1, -2, \dots$

Radius-of-convergence transfer theorem: applications

Corollary. If $f(z)$ has radius of convergence $>\rho$ with $f(\rho) \neq 0$, then

$$[z^n] \frac{f(z)}{(1-z/\rho)^\alpha} \sim \frac{f(\rho)}{\Gamma(\alpha)} \rho^n n^{\alpha-1}$$

for any real $\alpha \notin 0, -1, -2, \dots$

Ex 1: Catalan

$$T(z) = \frac{1}{2z}(1 - \sqrt{1 - 4z})$$

$$[z^N]T(z) \sim \frac{4^N}{\sqrt{\pi N^3}}$$

$$\begin{aligned} \rho &= 1/4 & \alpha &= -1/2 & f(z) &= -1/2 \\ \Gamma(-1/2) &= -2\Gamma(1/2) & & & &= -2\sqrt{\pi} \end{aligned}$$

Ex 2: Derangements

$$D_M(z) = \frac{e^{-z-z^2/2-\dots-z^M/M}}{1-z}$$

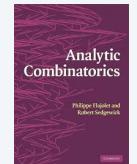
$$[z^N]D_M(z) \sim \frac{N!}{e^{H_M}}$$

$$\rho = 1 \quad \alpha = 1 \quad f(z) = e^{-z-z^2/2-\dots-z^M/M}$$

Transfer theorems based on complex asymptotics

provide *universal laws* of sweeping generality

Example: Context-free constructions



A system of combinatorial constructions

$$\begin{aligned} \langle G_0 \rangle &= OP_0(\langle G_0 \rangle, \langle G_1 \rangle, \dots, \langle G_t \rangle) \\ \langle G_1 \rangle &= OP_1(\langle G_0 \rangle, \langle G_1 \rangle, \dots, \langle G_t \rangle) \\ &\dots \\ \langle G_t \rangle &= OP_t(\langle G_0 \rangle, \langle G_1 \rangle, \dots, \langle G_t \rangle) \end{aligned}$$

symbolic method

transfers to a system of GF equations

$$\begin{aligned} G_0(z) &= F_0(G_0(z), G_1(z), \dots, G_t(z)) \\ G_1(z) &= F_1(G_0(z), G_1(z), \dots, G_t(z)) \\ &\dots \\ G_t(z) &= F_t(G_0(z), G_1(z), \dots, G_t(z)) \end{aligned}$$

Grobner basis elimination

that reduces to a *single* GF equation

$$G_0(z) = F(G_0(z), G_1(z), \dots, G_t(z))$$

Drmota-Lalley-Woods theorem

that has an *explicit* solution

$$G(z) \sim c - a\sqrt{1 - bz}$$

singularity analysis

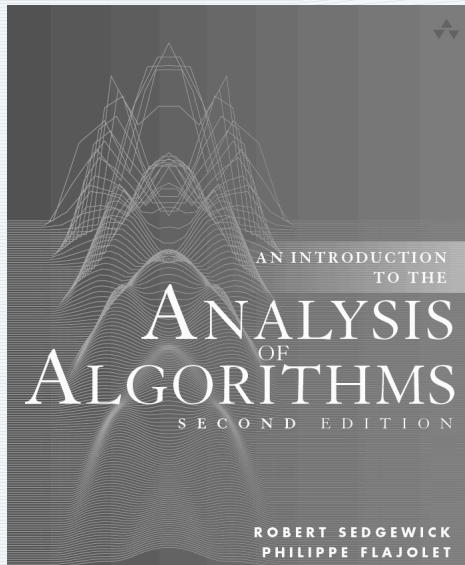
that transfers to a
simple asymptotic form

$$G_N \sim \frac{a}{2\sqrt{\pi N^3}} b^N$$

!!

Stay tuned for many more (in Part 2).

ANALYTIC COMBINATORICS PART ONE



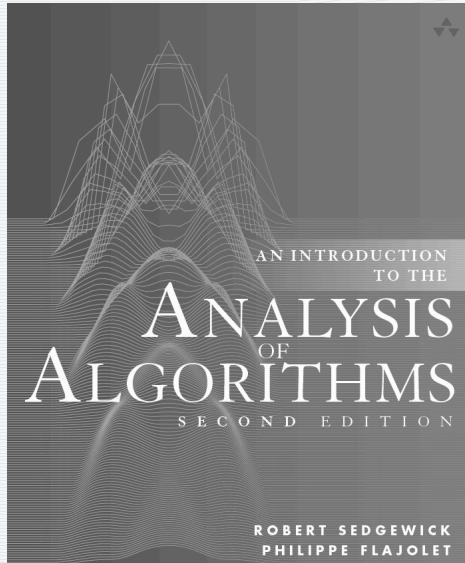
5. Analytic Combinatorics

- The symbolic method
- Labelled objects
- **Coefficient asymptotics**
- Perspective

<http://aofa.cs.princeton.edu>

5c . AC . Asymptotics

ANALYTIC COMBINATORICS PART ONE



5. Analytic Combinatorics

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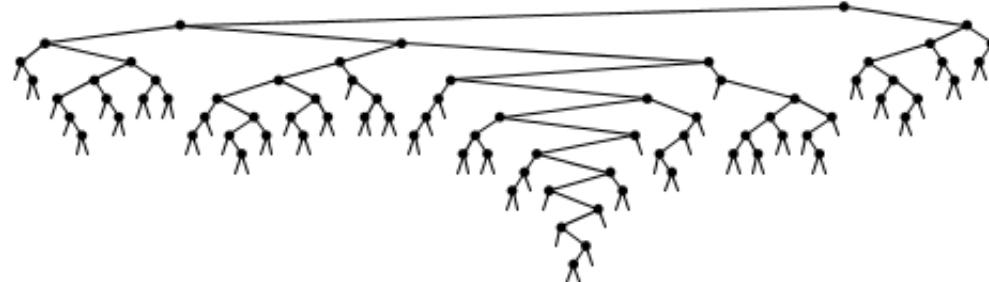
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5d . AC . Perspective

Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Ex: How many binary trees with N nodes?



$$T = E + Z \times T \times T$$

combinatorial construction

$$T(z) = \frac{1}{2z}(1 - \sqrt{1 - 4z})$$

GF

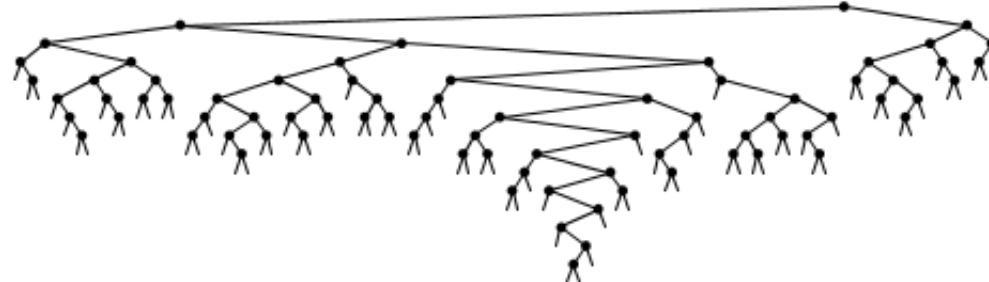
$$T_N \sim \frac{4^N}{\sqrt{\pi N^3}}$$

coefficient
asymptotics

Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Ex: How many binary trees with N nodes?



$$T = E + Z \times T \times T$$

combinatorial construction

$$T(z) = 1 + zT(z)^2$$

GF equation

$$T_N \sim \frac{4^N}{\sqrt{\pi N^3}}$$

coefficient
asymptotics

Note: With complex asymptotics, we can transfer directly from GF equation (no need to solve it). See Part 2.

Old vs. New: Two ways to count binary trees

Old

Recurrence → GF

Expand GF

Asymptotics

Solving the Catalan recurrence with GFs

Recurrence that holds for all N .

$$T_N = \sum_{0 \leq k < N} T_k T_{N-1-k} + \delta_N$$

Multiply by z^N and sum.

$$T(z) \equiv \sum_{N \geq 0} T_N z^N = \sum_{N \geq 0} \sum_{0 \leq k < N} T_k T_{N-1-k}$$

Switch order of summation

$$T(z) = 1 + \sum_{k \geq 0} \sum_{N > k} T_k T_{N-1-k}$$

Change N to $N+k+1$

$$T(z) = 1 + \sum_{k \geq 0} \sum_{N \geq 0} T_k T_{N+k+1}$$

Distribute.

$$T(z) = 1 + z \left(\sum_{k \geq 0} T_k z^k \right) \left(\sum_{N \geq 0} T_{N+k+1} z^{N+k+1} \right)$$

$$T(z) = 1 + z T(z)^2$$

Solving the Catalan recurrence with GFs (continued)

Functional GF equation.

$$T(z) = 1 + z T(z)^2$$

Solve with quadratic formula.

$$zT(z) = \frac{1}{2} (-1 \pm \sqrt{1 - 4z})$$

Expand via binomial theorem.

$$zT(z) = -\frac{1}{2}$$

Set coefficients equal

$$T_N = -\frac{1}{2}$$

Expand via definition.

$$= -\frac{1}{2}$$

Distribute $(-2)^N$ among factors.

$$= \frac{1}{2} \cdot \frac{(-1)^N}{N!} \cdot 2^N$$

Substitute $(2/1)(4/2)(6/3)\dots$ for 2^N .

$$= \frac{1}{N!} \cdot \frac{(-1)^N}{2^N}$$

Inclass exercise

Given Stirling's approximation $\ln N! = N \ln N - N + \ln \sqrt{2\pi N} + O(\frac{1}{N})$

Develop an asymptotic approximation for $\binom{2N}{N}$ to $O(1/N)$ (relative error)

$$\binom{2N}{N} = \exp(\ln(2N!) - 2 \ln N!)$$

$$= \exp(2N \ln(2N) - 2N + \ln \sqrt{4\pi N} + O(1/N))$$

$$- 2(N \ln(N) - N + \ln \sqrt{2\pi N} + O(1/N))$$

$$= \exp(2N \ln 2 - \ln \sqrt{\pi N} + O(1/N))$$

$$= \frac{4^N}{\sqrt{\pi N}} (1 + O(\frac{1}{N}))$$

$$\ln \sqrt{4\pi N} - 2 \ln \sqrt{2\pi N} = \ln 2 - 2 \ln \sqrt{2} - \ln \sqrt{\pi N}$$

$$= -\ln \sqrt{\pi N}$$

$$\text{Ex. } \frac{1}{4^N} \binom{2N}{N} \sim \frac{1}{\sqrt{\pi N}}$$

New

$$T = E + Z \times T \times T$$

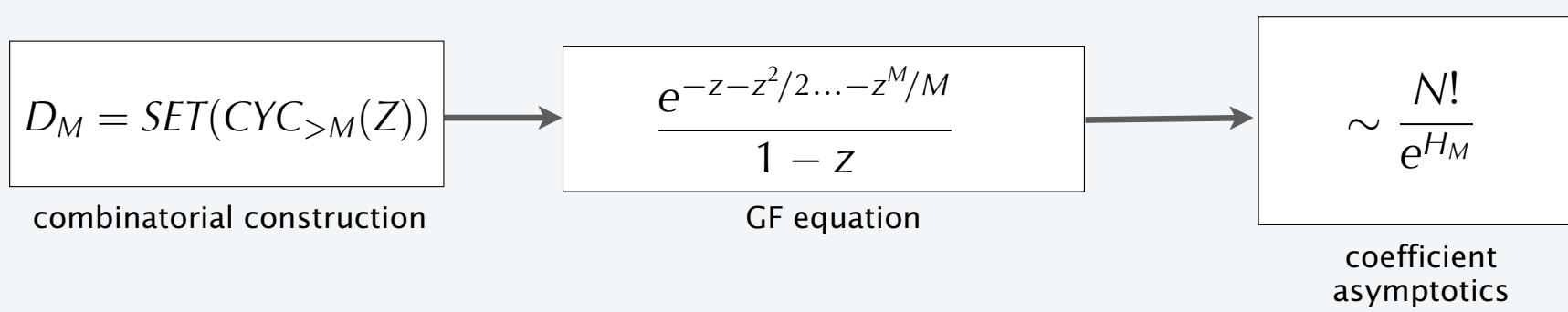
$$T(z) = 1 + z T(z)^2$$

$$T_N \sim \frac{4^N}{\sqrt{\pi N^3}}$$

Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

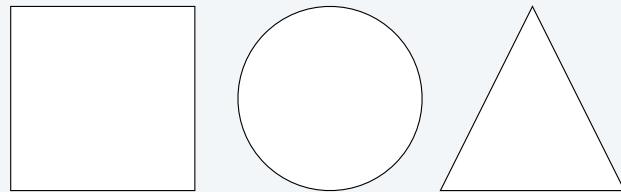
Ex: How many generalized derangements?



A standard paradigm for analytic combinatorics

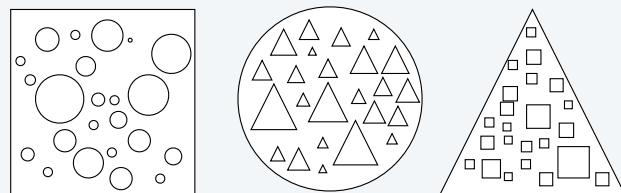
Fundamental constructs

- elementary or trivial
- confirm intuition



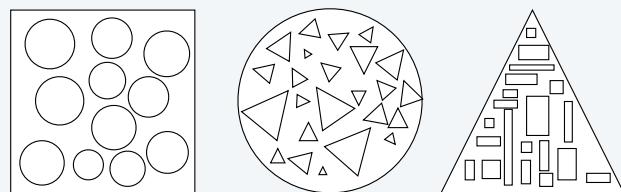
Compound constructs

- many possibilities
- classical combinatorial objects
- expose underlying structure



Variations

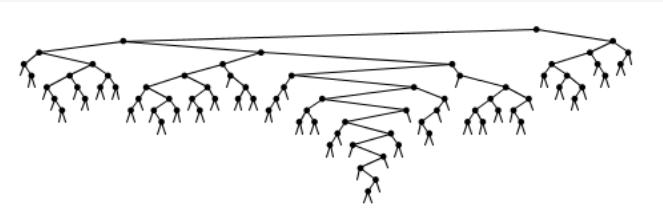
- unlimited possibilities
- *not* easily analyzed otherwise



Combinatorial parameters

are handled as two counting problems via cumulated costs.

Ex: How many leaves in a random binary tree?



1. Count trees

$$T = E + Z \times T \times T$$

$$T(z) = \frac{1}{2z}(1 - \sqrt{1 - 4z})$$

$$T_N \sim \frac{4^N}{\sqrt{\pi N^3}}$$

2. Count leaves in all trees

$$T = E + Z \times T \times T$$

$$T_u(1, z) = \frac{z}{\sqrt{1 - 4z}}$$

$$C_N \sim \frac{4^{N-1}}{\sqrt{\pi N}}$$

Symbolic method works for BGFs (see text)

3. Divide

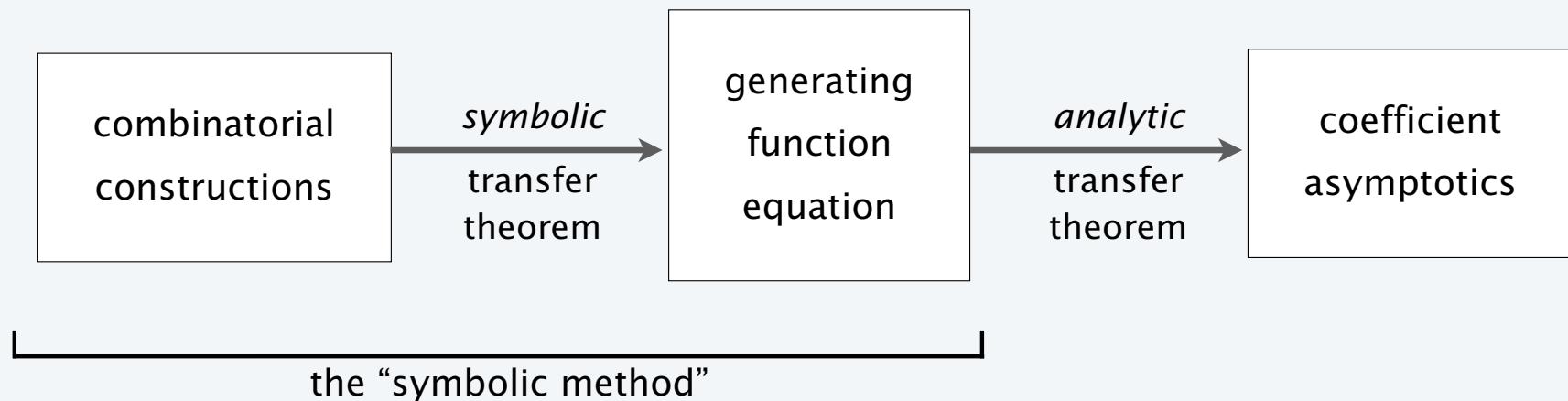
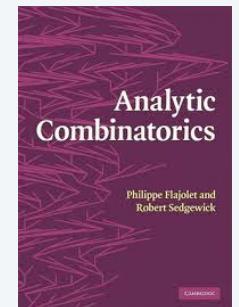
$$\frac{C_N}{T_N} \sim \frac{N}{4}$$

Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Features:

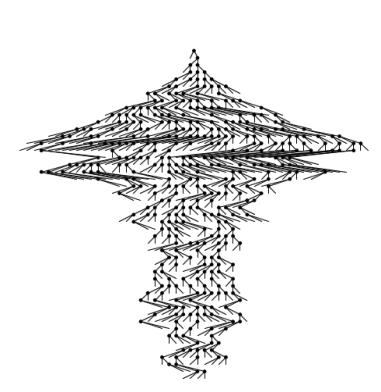
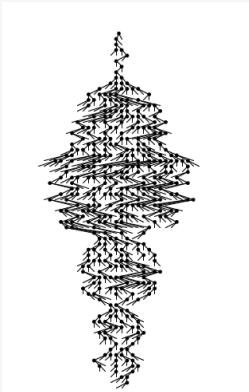
- Analysis begins with formal *combinatorial constructions*.
- The *generating function* is the central object of study.
- *Transfer theorems* can immediately provide results from formal descriptions.
- Results extend, in principle, to any desired precision on the standard scale.
- Variations on fundamental constructions are easily handled.



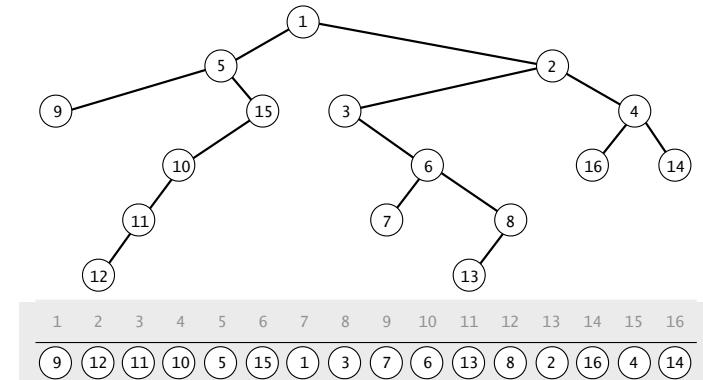
Stay tuned

for many applications of analytic combinatorics *and applications to the analysis of algorithms*

Trees



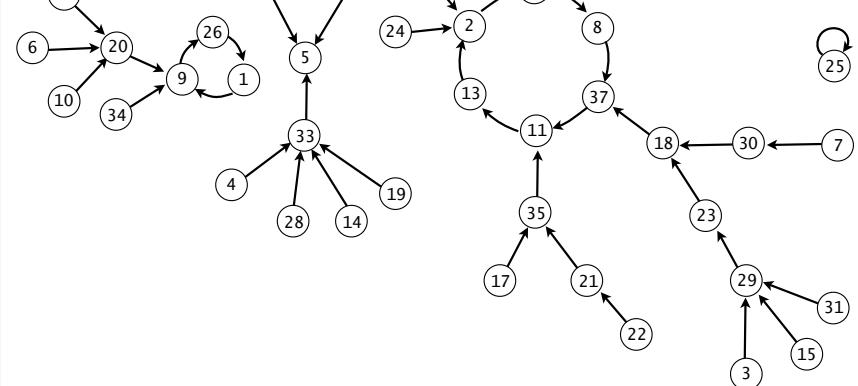
Permutations



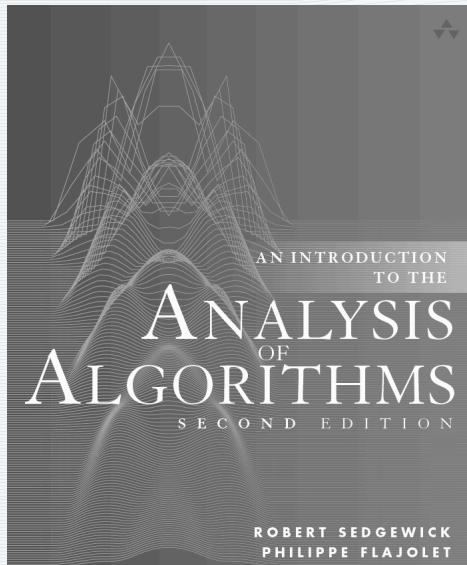
Mappings

Bitstrings

1011111101001010011001111000100111110110100000111100001100111011101111101011000
110100101010001111010011110011010011101101011111000001011011100110100000111001110
11101110101100111010111001101000011100111001010111110011001000011001000101010001
1011110000110110001100111011100110101110111100111101011000001100110010100000110
1010110011101000110101110110010010110100101001101111100110000001111101000001111
10000001001100000110001100010000111100111001111000001100111110011010000100100111
1000101010111000111010100000110000011110101010000101100010011011111001111010001
0011101100101110010000010011110100110000011001110010011000000101000001010000111
001111111001101110101111010100110110111000111111110101110010011000000101011110
1010100011110000010100000011001000001100101001010000011001010010101011011011111110
11000000010111101011000001010101100101000000110011100100000011001010000000101000
11101111010101111101111111101001110100101111110101110100111010001100000010010001010
00111111100111010110111111000010000100001110000011101011110010101111100111010101111



ANALYTIC COMBINATORICS PART ONE



5. Analytic Combinatorics

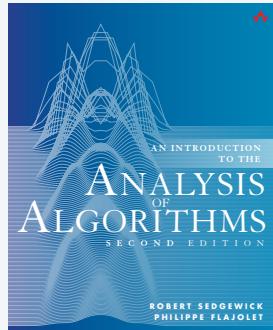
- The symbolic method
- Labelled objects
- Coefficient asymptotics
- Perspective

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5d . AC . Perspective

Exercise 5.1

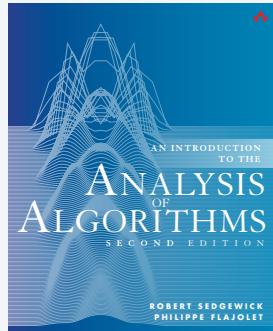
Practice with counting bitstrings.



Exercise 5.1 How many bitstrings of length N have no 000?

Exercise 5.3

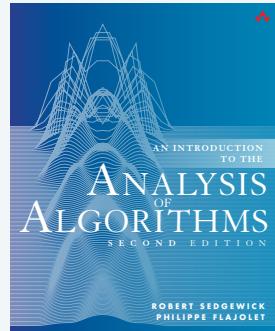
Practice with counting trees.



Exercise 5.3 Let \mathcal{U} be the set of binary trees with the size of a tree defined to be the total number of nodes (internal plus external), so that the generating function for its counting sequence is $U(z) = z + z^3 + 2z^5 + 5z^7 + 14z^9 + \dots$. Derive an explicit expression for $U(z)$.

Exercise 5.7

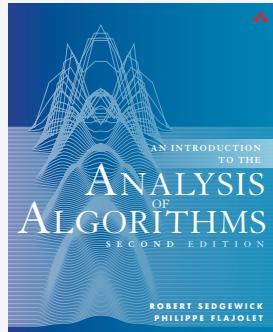
Practice with counting permutations.



Exercise 5.7 Derive an EGF for the number of permutations whose cycles are all of odd length.

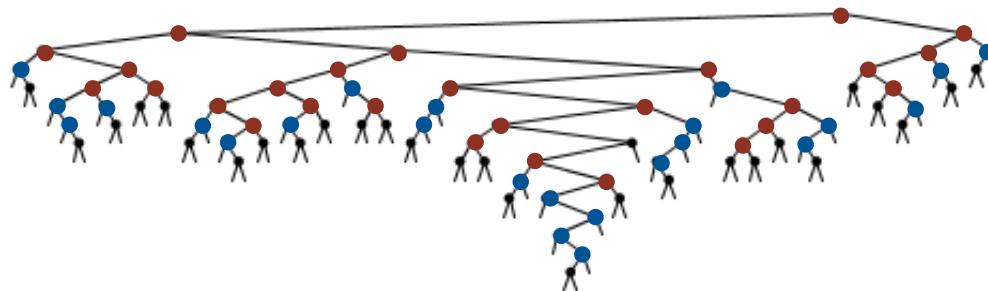
Exercises 5.15 and 5.16

Practice with tree parameters.



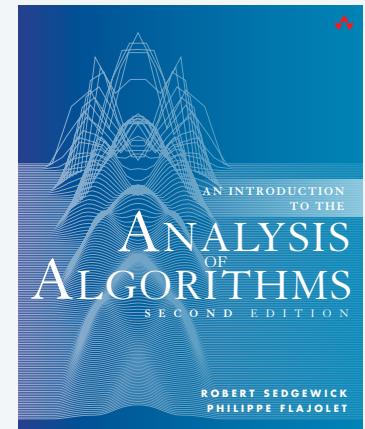
Exercise 5.15 Find the average number of internal nodes in a binary tree of size n with both children internal. ●

Exercise 5.16 Find the average number of internal nodes in a binary tree of size n with one child internal and one child external. ●



Assignments for next lecture

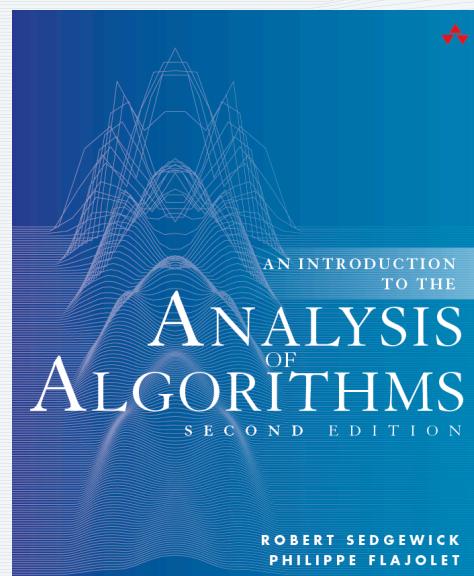
1. Read pages 219-255 in text.



2. Write up solutions to Exercises 5.1, 5.3, 5.7, 5.15, and 5.16.

ANALYTIC COMBINATORICS

PART ONE



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5. Analytic Combinatorics