

MATH 316 Lecture 3

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1 Singular Points

Singular points are divided into two classes:

- Regular singular points, where we use the Frobenius series solution
- Irregular singular points, which are beyond the scope of the course

If we are to look at the ODE $P(x)y'' + Q(x)y' + R(x)y = 0$, and define the point x_0 as a singular point, we have the following:

The Cauchy-Euler equation is

$$(x - x_0)^2 + \alpha(x - x_0)y' + \beta y = 0 \quad (1)$$

We know that $y = (x - x_0)^r$ (As an example).

How do we make $P(x)y'' + Q(x)y' + R(x)y = 0$ look like (1)?

If we multiply with $(x - x_0)^2$ and divide by $P(x)$, we may get something similar to the Cauchy-Euler.

Then, we get something like this:

$$(x - x_0)^2 y'' + \left\{ \frac{Q(x)}{P(x)}(x - x_0) \right\} (x - x_0)y' + \left\{ \frac{R(x)}{P(x)}(x - x_0)^2 \right\} y = 0 \quad (2)$$

Now, if $\frac{Q(x)}{P(x)}(x - x_0)$ and $\frac{R(x)}{P(x)}(x - x_0)^2$ are analytic at $x = x_0$, then the singularity is not worse than the singularity in the Cauchy-Euler equation (1), **and** x_0 is a "regular singular point". Otherwise, x_0 is an "irregular singular point".

If we start to write the Taylor series for (?),

$$\frac{Q(x)}{P(x)}(x - x_0) = p_0 + p_1(x - x_0) + \dots$$

$$\frac{R(x)}{P(x)}(x - x_0)^2 = q_0 + q_1(x - x_0) + \dots$$

Being analytic means that we need to be able to write the series.

An example of a singular point:

$$xy'' + 2\alpha y' = 0$$

$$\left(\frac{2\alpha}{x}\right)$$

Here, $x = 0$ is a singular point. (We re-wrote the first equation as the second equation, I believe)

As $x \rightarrow x_0$, our ODE becomes: $(x - x_0)^2 y'' + p_0(x - x_0)y' + q_0 y = 0$

The corresponding Cauchy-Euler equation solution: $y = (x - x_0)^r$. We need to have finite p_0 and q_0 . If p_0 and q_0 are both finite, then x_0 is a regular singular point. Otherwise, it is an irregular singular point.

For regular singular points, the solution we are going to write:

$$\underbrace{y(x) = (x - x_0)^r}_{*} \underbrace{\sum_{n=0}^{\infty} a_n (x - x_0)^n}_{\text{Correction}}$$

*: The singular part of the solution to the corresponding Cauchy-Euler.

1.1 Example

$$x(1 + x^2)y'' + 2xy' + (1 + x^2)y = 0$$

Classify singular points. Here, $p(x) = x(1 + x^2)$, $Q(x) = 2x$, and $R(x) = 1 + x^2$. Singular points: $\begin{cases} x = 0 \\ x = \pm i \end{cases}$ We need $p(x_0) = 0$ (Take a look at the left hand side if you don't understand!).

Classify them:

$$\begin{cases} \lim_{x \rightarrow x_0} \frac{Q(x)}{P(x)}(x - x_0) = p_0 \\ \lim_{x \rightarrow x_0} \frac{R(x)}{P(x)}(x - x_0)^2 = q_0 \end{cases}$$

For $x_0 = 0$, we have $\lim_{x \rightarrow 0} \frac{2x}{x(1+x^2)}x = 0 = p_0$

$$\lim_{x \rightarrow 0} \frac{1+x^2}{x(1+x^2)}x^2 = 0 = q_0$$

Now for $x_0 = i$:

$$\lim_{x \rightarrow i} \frac{2x}{x(1+x^2)}(x - i) = \lim_{x \rightarrow i} \frac{2(x-i)}{(x-i)(x+i)} = \frac{1}{i} = p_0$$

$$\lim_{x \rightarrow i} \frac{1+x^2}{x(1+x^2)}(x - i)^2 = 0 = q_0$$

We see that because both p_0 and q_0 are finite, $x = i$ is also a regular singular point.

Try for $x = -i$:

$$\lim_{x \rightarrow -i} \frac{2x}{x(1+x^2)}(x+i)$$

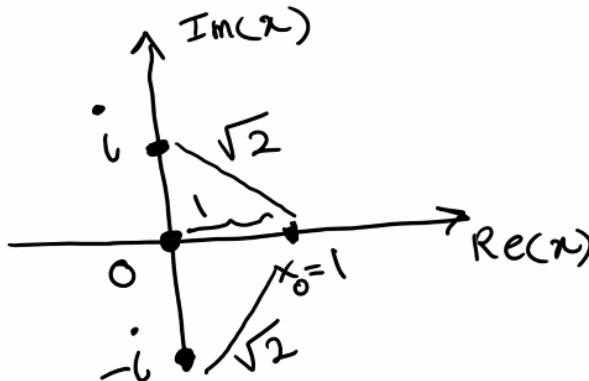
$$\lim_{x \rightarrow -i} \frac{1+x^2}{x(1+x^2)}(x+i)^2$$

yeah uhhhh.... review how to calculate limits.

When we calculate $x_0 = -i$, we get: $p_0 = -\frac{1}{i}$, and $q_0 = 0$. Hence, $x_0 = -i$ is also a regular singular point.

2 Radius of Convergence

The radius of convergence of the series solution is at least equal to the distance from the x_0 to the nearest singular point. In the example that we solved:



$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$\rho = 1$ is the lower bound estimate. ρ is the radius of convergence. Imagine a circle of radius 1 (as it's the distance to the closest singular point)

2.0.1 An Example

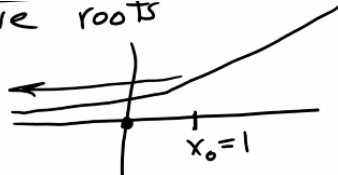
$$x^2 y'' + \alpha x y' + \beta y = 0$$

r_1, r_2 are two positive roots

$$y = C_1 x^{r_1} + C_2 x^{r_2}$$

$x_0 = 0$ is a singular point

ve roots



Radius of convergence is infinite. This is a rare case.

3 Frobenius Series

EX:

$$6x^2(1+x)y'' + 5xy' - y = 0$$

$$P(x) = 6x^2(1+x), Q(x) = 5x, R(x) = -1$$

Singular points are $x = 0$ and $x = -1$.

For $x = 0$, let's find if it's irregular or regular:

$$\lim_{x \rightarrow 0} \frac{Q(x)}{P(x)}(x - x_0) = \lim_{x \rightarrow 0} \frac{5x}{6x^2(1+x)}x = \frac{5}{6} = p_0$$

$$\lim_{x \rightarrow 0} \frac{-1}{6x^2(1+x)}x^2 = \frac{-1}{6} = q_0$$

Therefore $x_0 = 0$ is a regular singular point.

$$(x - x_0)^2 y'' + \frac{5}{6}(x - x_0)y' - \frac{1}{6}y = 0$$

This is for $x_0 = 0$. Hence, $x^2 y'' + \frac{5}{6}xy' - \frac{1}{6}y = 0$.

Corresponding Cauchy-Euler equation:

$$y = x^r \text{ and therefore } [6r(r-1) + 5r - 1]x^r = 0.$$

$$\text{Hence } 6r^2 - r - 1 = 0 \longrightarrow r_{1,2} = \frac{1 \pm \sqrt{1+24}}{12} \longrightarrow r_{1,2} = \frac{1}{2}, \frac{-1}{3}$$

Frobenius Series Solution about $x = 0$:

$$y(x) = x^2 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+2}$$

$$y' = \sum_{n=0}^{\infty} a_n (n+2) x^{n+1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+2)(n+1) x^{n+0}$$

Now, we only need to replace y, y', y'' in the ODE:

$$6x^2(1+x)y'' + 5xy' - y = 0$$

$$6x^2 y'' + 6x^3 y'' + 5x y' - y = 0$$

$$\begin{aligned}
& 6 \sum_{n=0}^{\infty} a_n (n+r) (n+r-1) x^{n+r} \quad n=m \\
& + 6 \sum_{n=0}^{\infty} a_n (n+r) (n+r-1) x^{n+r+1} \quad n+1=m \\
& \quad n=0, m=1 \\
& + 5 \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} \quad n=m \\
& - \sum_{n=0}^{\infty} a_n x^{n+r} \quad n=m
\end{aligned}$$

$$\begin{aligned}
& 6 \sum_{m=0}^{\infty} a_m (m+r) (m+r-1) x^{m+r} \\
& + 6 \sum_{m=1}^{\infty} a_{m-1} (m+r-1) (m+r-2) x^{m+r} \\
& + 5 \sum_{m=0}^{\infty} a_m (m+r) x^{m+r} - \sum_{m=0}^{\infty} a_m x^{m+r}
\end{aligned}$$

peel off

$$\begin{aligned}
& 6a_0 r(r-1)x^r + 5a_0 r x^r - a_0 x^r + \\
& \sum_{m=1}^{\infty} \left[\frac{6a_m (m+r) (m+r-1) + 6a_{m-1} (m+r-1) (m+r-2)}{+ 5a_m (m+r) - a_m} \right] x^{m+r} = 0
\end{aligned}$$

$x^r a_0 [6r(r-1) + 5r - 1] = 0 \Rightarrow r_{1,2} = \frac{1}{2}r - \frac{1}{3}$

C-E eqn

$$\begin{aligned}
& x^{m+r} \left[a_m [6(m+r) (m+r-1) + 5(m+r) - 1] - \right. \\
& \quad \left. - 6a_{m-1} (m+r-1) (m+r-2) \right] = 0
\end{aligned}$$

recursion for r_1 & r_2

$$\begin{aligned}
& r = -\frac{1}{3}, m \geq 1 \\
& a_m = \frac{-6a_{m-1} (m - \frac{4}{3}) (m - \frac{7}{3})}{(m - \frac{1}{3}) (6m - 3) - 1} \\
& m=1: a_1 = \frac{-6a_0 (-\frac{1}{3}) (-\frac{4}{3})}{(\frac{2}{3})(3) - 1} \\
& = \frac{-8a_0}{3} \\
& m=2: a_2 = \frac{-82a_0}{42(3)}
\end{aligned}$$

$$y_1(x) = a_0 x^{\frac{-1}{3}} \left[1 - \frac{8}{3}x - \frac{32}{3(42)}x^2 + \dots \right]$$

$r = \frac{1}{2}$: If we follow the same steps, we get

$$y_2(x) = \underbrace{a_0}_{\text{A different } a_0} x^{\frac{1}{2}} \left[1 + \frac{3}{22}x - \frac{22}{22.68}x^2 + \dots \right]$$

Hence the general solution is $y = C_1 y_1(x) + C_2 y_2(x)$

3.0.1 Convergence: ratio test

$$\sum_{m=0}^{\infty} C_m : \lim_{m \rightarrow \infty} \left| \frac{C_{m+1}}{C_m} \right| < 1$$

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}$$

Ratio test:

$$\lim_{m \rightarrow \infty} \left| \frac{a_{m+1} x^{m+r+1}}{a_m x^{m+r}} \right| = |x| \lim_{m \rightarrow \infty} \left| \frac{a_{m+1}}{a_m} \right|$$

For this example:

$$|x| \lim_{m \rightarrow \infty} \left| \frac{a_{m+1}}{a_m} \right|, a_m = \frac{-6a_{m-1}(m+r-1)(m+r-2)}{(m+r)(6m+6r-1)-1}$$

$$|x| \lim_{m \rightarrow \infty} \frac{-6a_{m-1}(m+r-1)(m+r-2)}{(m+r)(6m+6r-1)-1} = |x| \cdot 1$$

$$|x| < 1$$

$$-1 < x < 1$$

$\rho =$ is at least equal to the distance between $x_0=0$ & the nearest singular point, $\rho=1$

$$x_{\text{singular}} = 0, -1$$