MATH 316 Lecture 6

Ashtan Mistal

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1 Recap of Frobenius Series Solutions

Assume x_0 is a singular point of the ODE of the form:

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

If x_0 is a regular singular point,

$$\lim_{x \to x_0} \frac{Q(x)}{P(x)} x = p_0$$

and

$$\lim_{x \to x_0} \frac{R(x)}{P(x)} x^2 = q_0$$

The characteristic equation is:

$$r(r-1) + p_0 r + q_0 = 0 \longrightarrow 2 \text{ roots: } r_1, r_2$$

For r_1, we get $y_1(x) = |x|^{r_1} (1 + \sum_{n=1}^{\infty} a_n x^n) = \sum_{n=0}^{\infty} a_n x^{n+r_1}$
a_n is found from a recursion by substitution into the ODE. a_0 is arbitrary.

1) If $r_1 - r_2 \neq 0$ and $r_1 - r_2 \neq N$ (N is an integer), then:

$$y_2 = |x|^{r_2} \left(1 + \sum_{n=1}^{\infty} a_n x^n \right) = \sum_{n=0}^{\infty} a_n x^{n+r_2}$$

2) If $r_1 = r_2$:

$$y_2(c) = y_1(x) \ln(x) + |x|^{r_1} \sum_{n=1}^{\infty} C_n x^n = y_x(x) \ln(x) + \sum_{n=1}^{\infty} C_n x^{n+r_1}$$

Note that $x > 0$.

Where $c_n = a_n' = \frac{da_n}{dr}\Big|_{r=r_1}$
Note 1: What happens of r_1 and r_2 are complex?
If they are, the form of y_2 in 1) (that we discussed), and y_1 are still valid; we just need to convert complex valued to real valued solutions. Needs lots of algebra.
Note 2: A summary of these solutions is given in the formula sheet for the exam.
Note 3: The general solution is in the following format:

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

2 PDEs

Continued from last class's notes.

2.0.1 Heat equation / diffusion equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + k\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

Applications: heat flows, diffusion of chemical substances

2.0.2 Wave equation

$$\frac{\partial^2 u}{\partial t^2} = C^2\frac{\partial^2 u}{\partial x^2} + C^2\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

Applications: Vibrations, acoustics, solid mechanics

2.0.3 Laplace's equation

$$0 = \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial y^2}$$

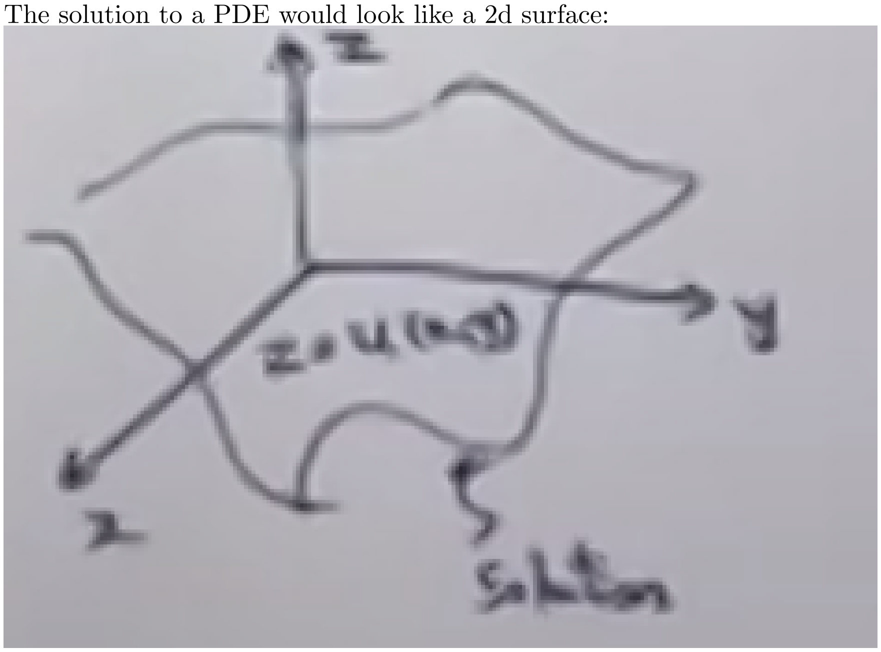
Applications: Heat / wave equations in which there is a steady-state solution (eg potential flow, porous media flow)

2.1 Classification of PDEs

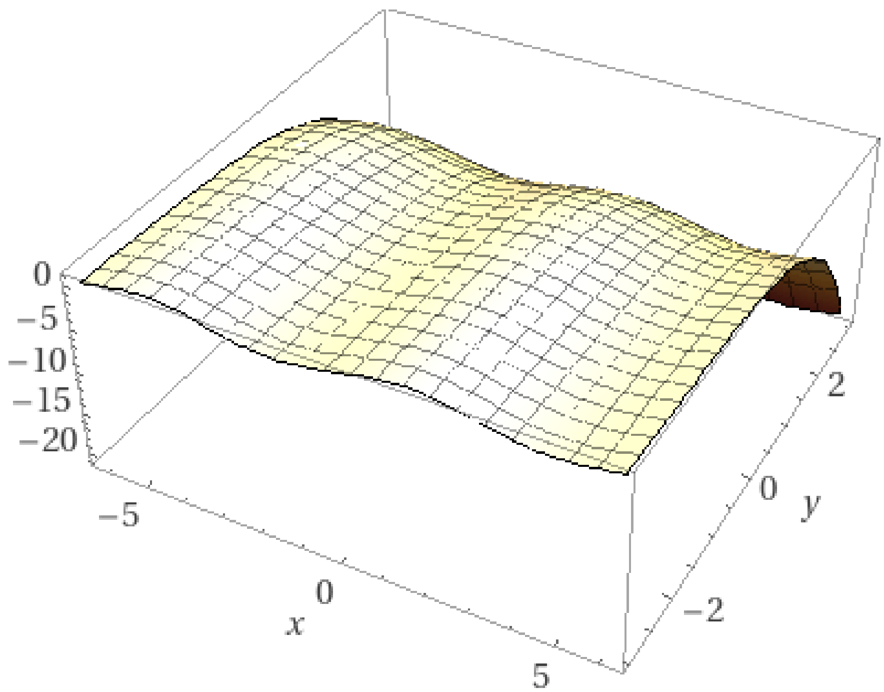
ODEs: $f(x, u(x), u'(x)) = 0$. e.g. $u' = e^u$

PDEs: $\underbrace{a(x,y)u_x + b(x,y)u_y = c(x,y)u}_{\text{First order, linear PDE}}$

The solution to a PDE would look like a 2d surface:



Another example of a surface from WolframAlpha:



This course primarily focuses on second order linear PDEs.

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{1}$$

A, B, C, D, E, F, G can either be constants or functions of (x, y).
The examples that we saw (heat equation, wave eq, etc) are all examples of the above (1).

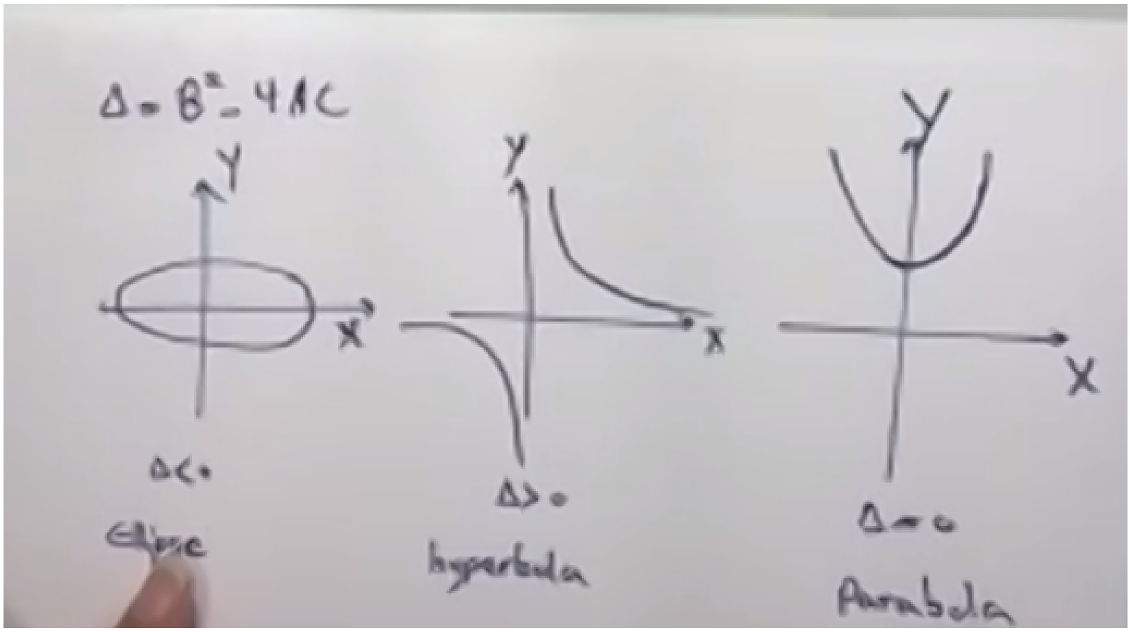
If $G = 0$, the PDE is homogeneous. Else, it is non-homogeneous.
To classify PDEs we use the analogy with corresponding quadratic surfaces:

$$AX^2 + BXY + CY^2 + DX + EY = K$$

To classify, we use the discriminant:

$$\Delta = B^2 - 4AC$$

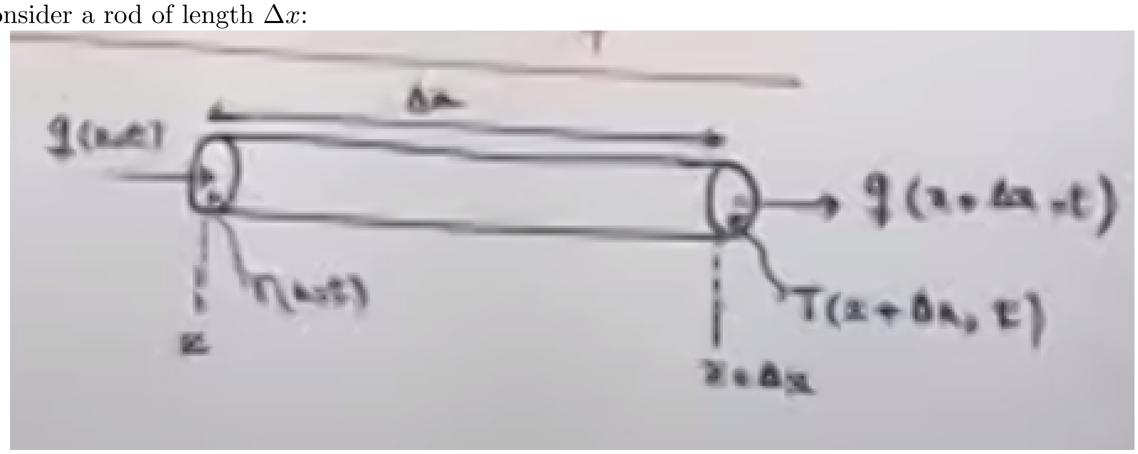
It tells us either ellipse ($\Delta < 0$), hyperbola ($\Delta > 0$), or a parabola ($\Delta = 0$)



Δ	Type	PDE	Note
$\Delta = 0$	parabolic	$u_t = u_{xx}$	Heat eq
$\Delta < 0$	elliptic	$u_{xx} + u_{yy} = 0$	Laplace eq
$\Delta < 0$	elliptic	$u_{xx} + u_{yy} = G$	Poisson's eq
$\Delta > 0$	Hyperbolic	$u_{tt} = c^2 u_{xx}$	Wave eq

2.2 Heat / Diffusion Equation

Consider a rod of length Δx:



(The equations that are a bit blurry are the following, left to right and top to bottom: $q(x,t)$, Δx, $q(x+\Delta x, t)$, $T(x,t)$, $T(x+\Delta x, t)$, x, $x+\Delta x$)

- $T(x,t)$: Temperature at (x,t)
- $q(x,t)$: The heat flux (heat energy per unit area)
- C: The specific heat capacity
- ρ: density of material

- A: The cross sectional area

Energy conservation: The increase in the thermal energy of the bar is equal to the (influx - outflux) of heat. (Physical description, not mathematical description).

Use variables: $C(T(x,t+\Delta t)-T(x,t))\rho\Delta x A = (q(x,t)-q(x+\Delta x,t))A\Delta t$

Divide by $\Delta t \cdot \Delta x$:

$$\rho C\frac{T(x,t+\Delta t)-T(x,t)}{\Delta t} = \frac{q(x,t)-q(x,t+\Delta t)}{\Delta x}$$

As $\Delta t \to 0$ and $\Delta x \to 0$:

$$\rho C\frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x}$$

The energy conservation equation is hence:

$$\frac{\partial q}{\partial x} + \rho C\frac{\partial T}{\partial t} = 0 \tag{2}$$

In order to reduce the number of dependent variables, we need a constitutive equation between q and T. Can we relate the heat flux to the temperature?

Yes. The heat transfer through conduction is formulated as:

$$q = -k\frac{\partial T}{\partial x} \qquad \text{(Fourier's Law)}$$

where k is the thermal conductivity of the material. What does this tell us?

- Heat flux will flow from high temperature to low temperature.

We can substitute Fourier's Law in the energy conservation equation:

$$-k\frac{\partial^2 T}{\partial x^2} + \rho c\frac{\partial T}{\partial t} = 0$$

$$\hookrightarrow \frac{\partial T}{\partial t} = \alpha^2\frac{\partial T}{\partial x^2}$$

Where $\alpha^2 = \frac{k}{\rho c}$ (diffusion coefficient).

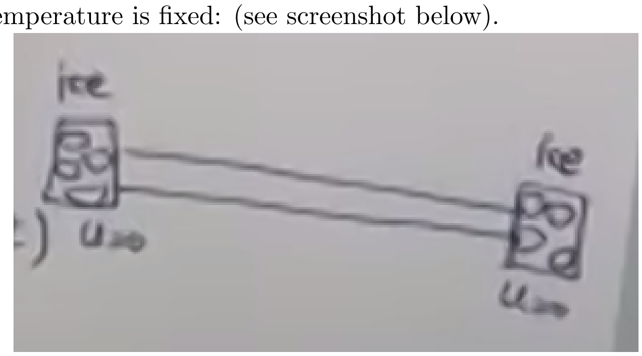
2.3 Solving diffusion equations using separation of variables

The initial boundary value problems, $u_t = \alpha^2 u_{xx}$, needs one initial condition (IC) and two boundary conditions (BC).

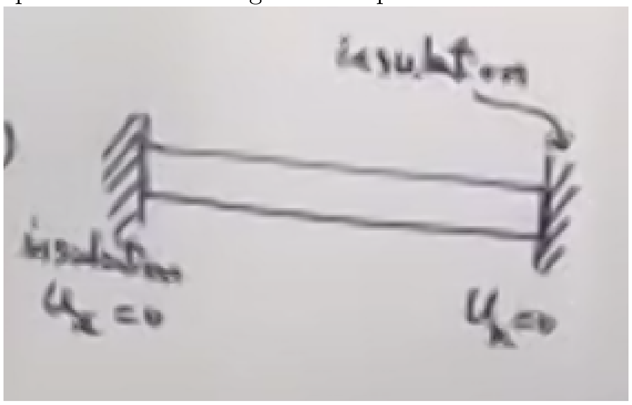
Initial condition: $u(x, t=0) = f(x)$ on the domain $0 < x < L$

2.3.1 Boundary conditions

(1) Dirichlet boundary conditions $u(0,t) = 0 = u(L,t)$ (i.e. same temperature on either side of the rod). Temperature is fixed: (see screenshot below).



(2) Neumann boundary conditions: $u_x(0,t) = 0 = u_x(L,t)$. i.e. insulation on either side of a rod. Temperature won't change with respect to x.



(3): Mixed boundary conditions. $u(0,t) = 0$ and $u_x(L,t) = 0$

2.3.2 Example 1

$$u_t = \alpha^2 u_{xx},\ 0 < x < L,\ t > 0$$

Boundary conditions:

$$u(0,t) = 0$$
$$u(L,t) = 0$$

Initial conditions:

$$u(x,0) = f(x)$$

To solve, we use the method of separation of variables. We will do this next class.