MATH 316 Lecture 12

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1 Wave Equations: Neumann Boundary Conditions

Solve IBVP: $y_{tt} = a^2 y_{xx}$

Boundary conditions: $y_x(0,t) = y_x(L,t) = 0 \Rightarrow$ Homogeneous Neumann boundary conditions.

Initial conditions: $y(x,0) = f(x)$, $y_t(x,0) = 0 = g(x)$ (Therefore zero initial velocity, with specified initial displacement).

Split into two problems (w and z). Because $g(x) = 0$, we only have the z equation. For the solution, we use separation of variables. [1]

$X'' + \lambda X = 0$	$\ddot{T} + a^2\lambda T = 0$
$\Rightarrow X_n(x) = \cos\left(\frac{n\pi x}{L}\right)$	$\dot{T}_n(0) = 0$
$\lambda_n = \left(\frac{n\pi}{L}\right)^2$	$T_n(t) = A_n \cos\left(\frac{n\pi a}{L}t\right)$
$X_0(x) = 1 \Rightarrow \lambda_0 = 0$	$T_0(t) = 1$

The solution would be:

$$y(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi a}{L}t\right)$$

Note that A_0 is the multiplication of $X_0 T_0$ terms.

$$\text{At } t = 0 \longrightarrow y(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) = f(x)$$

We construct Fourier cosine series for $f(x)$.

$$f(x) = \frac{a_0}{2} + \sum n = 1^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow A_0 = \frac{a_0}{2}, A_n = a_n$$

Note: We can do a similar solution if initial velocity is given.

1.1 Recap

- Introduced wave equation
- Developed separation of variables method to find its solution
  - Dirichlet and Neumann boundary conditions
  - Examples and normal modes

Now: New method.

- New look at the wave equation and we solve the wave equation using **D'Alembert's solution**.

[1] Note that left and right side of table are entirely separate.

2 D'Alembert's Solution

$$y_{tt} = a^2 y_{xx}$$

Let's see if we can guess a solution of exponential format.

$$y(x,t) = e^{ikx+\sigma t}$$

where k and σ are constants. [2]
Substitute the guessed solution into the PDE.

$$y_t t = \sigma^2 e^{ikx+\sigma t}$$

$$y_{xx} = -k^2 e^{ikx+\sigma t}$$

Now, substitute this into the PDE:

$$\left(\sigma^2 + a^2 k^2\right) e^{ikt+\sigma t} = 0$$

$$\Rightarrow \sigma = \pm ika$$

$$y_1(x,t) = e^{ik(x-at)}$$

$$y_2(x,t) = e^{ik(x+at)}$$

$x \pm at$ are known as characteristics, these are lines in x and t along which the initial conditions (and general information) travels.

The question here is this: Can this form of solution be more general such that we can apply it to any wave equation?

$$y_1(x,t) = F(x-at), y_2(x,t) = G(x+at)$$

Can we find a general equation that satisfies the wave equation?

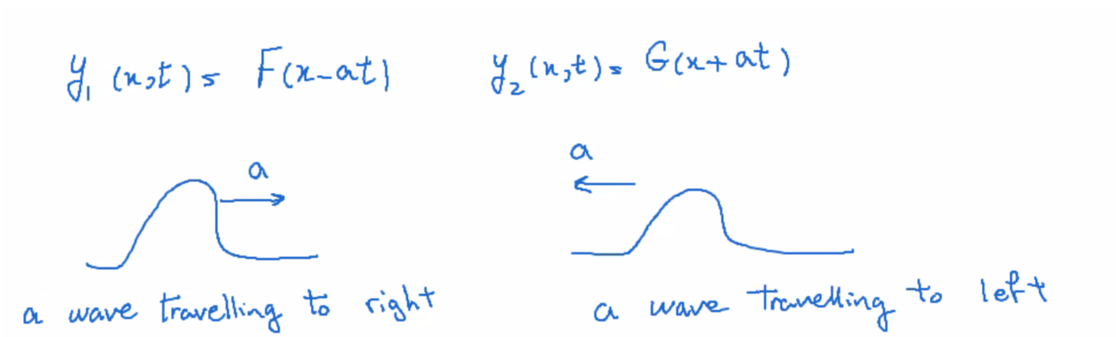

Figure 1: $F(x-at)$ is a wave travelling to the right with a speed of a. $G(x+at)$ is a wave travelling to the left with a speed of a.

Hence, a general solution:

$$y(x,t) = F(x-at) + G(x+at)$$

Does it satisfy the PDE?

[2]Try this guess solution with heat solution! You will find that it does work for heat equations.

$$\begin{array}{lcll} y(x,0) = f(x) & \Rightarrow & F(x) + G(x) = f(x) & (1) \\ y_t(x,0) = g(x) & \Rightarrow & -aF'(x) + aG'(x) = g(x) & (2) \end{array}$$

We get (2) from:

$$-aF(x) + aG(x) = \int_0^x g(s)ds + A$$

$$(1)xa + 2 \Rightarrow 2aG(x) + f(x) = \int_0^x g(s)ds + A$$

$$\Rightarrow G(x) = \frac{1}{2}f(x) + \frac{1}{2a}\int_0^x g(s)ds + \frac{A}{2a}$$

To find $F(x)$:

$$(1)xa - (2) \Rightarrow 2aF(x) = af(x) - \int_0^x g(s)ds - A$$

$$\Rightarrow F(x) = \frac{1}{2}f(x) - \frac{1}{2a}\int_0^x g(s)ds - \frac{A}{2a}$$

Now, substitute these into the general solution: (plug into $y(x,t) = F(x-at) + G(x+at)$)
This gives us:

$$\frac{1}{2}f(x-at) - \frac{1}{2a}\int_0^{x-at} g(s)ds + \frac{1}{2}f(x+at) + \frac{1}{2a}\int_0^{x+at} g(s)ds$$

Note that $-\frac{A}{2a}$ and $\frac{A}{2a}$ cancel.

$$y(x,t) = \frac{1}{2}\left[\underbrace{f(x-at)}_{\text{half of init cond travels right}} + \underbrace{f(x+at)}_{\text{half of init cond travels left}}\right] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)ds \qquad (1)$$

N.B. Above analysis has no boundary conditions: $-\infty < x < \infty$

What if the problem has boundary condition?
Let $F^o(x)$ and $G^o(x)$ be the odd[3] 2L-periodic extension of $f(x)$ and $g(x)$ respectively:

$$y(x,t) = \frac{1}{2}\left[F^o(x-at) + F^o(x+at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} G^o(s)ds$$

Boundary conditions: [4]

$$y(0,t) = \underbrace{\frac{1}{2}\left[F^o(-at) + F^o(at)\right]}_{=0} + \underbrace{\frac{1}{2a}\int_{-at}^{at} G^o(s)ds}_{=0} = 0$$

What's the relationship between D'Alembert's formula and the separation of variables?

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[b_n \frac{L}{n\pi a}\sin\left(\frac{n\pi a}{L}t\right) + b_n' \cos\left(\frac{n\pi a}{L}t\right)\right]$$

Recall trig formulae:

$$\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

[3](Assumes Dirichlet boundary conditions)
[4]Note that we are using the properties of odd functions to cancel out both F and G.

$$\sin(A)\cos(B) = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

Let's apply these:

$$y(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{b_n L}{n\pi a} \underbrace{\left[\cos\left(\frac{n\pi}{L}(x-at)\right) - \cos\left(\frac{n\pi}{L}(x+at)\right)\right]}_{\text{Let's write this in integral format}} \rightarrow$$

$$\hookrightarrow +b_n' \left[\sin\left(\frac{n\pi}{L}(x-at)\right) - \sin\left(\frac{n\pi}{L}(x+at)\right)\right]$$

$$y(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} b_n' \left[\sin\left(\frac{n\pi}{L}(x-at)\right) - \sin\left(\frac{n\pi}{L}(x+at)\right)\right] \rightarrow$$

$$\hookrightarrow +\frac{1}{2a}\sum_{n=1}^{\infty} b_n \int_{x-at}^{x+at} \sin\left(\frac{n\pi s}{L}\right) ds$$

Recall:

$$\sum_{n=1}^{\infty} b_n' \sin(\frac{n\pi x}{L}) = f(x)$$

and

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = g(x)$$

$\Rightarrow$ d'Alembert's solution:

$$y(x,t) = \frac{1}{2}\left[f(x-at) + f(x+at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)ds$$

Both methods give similar solution.