Laplace's Equation in 2-Dimensional Regions

Laplace's Equation arises in many situations, e.g.

- Steady Heat Flow in a 2-D region

$$\rho c_p \frac{\partial T}{\partial t} = K\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \qquad (x,y) \in [0,L] \times [0,L]$$
$$T(0,y,t) = 10, \quad T(L,y,t) = 20$$
$$T(x,0,t) = 10 + 10x/L, \quad T(L,y,t) = 10 + 10y/L$$

  At sufficiently long times we have seen the solutions tend to decay exponentially fast, to a steady temperature solution:

$$0 = K\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \qquad (x,y) \in [0,L] \times [0,L]$$
$$T(0,y) = 10, \quad T(L,y) = 20$$
$$T(x,0) = 10 + 10x/L, \quad T(L,y) = 10 + 10y/L$$

- Steady diffusion problems (as above, with T replaced by a concentration C)
- Steady wave problems
- Potential Flow (e.g. irrotational inviscid flow), modelling for example, flows around aerofoils, cylinders, etc.. $\boldsymbol{u} = \nabla\varphi$, where the velocity potential φ satisfies:

$$0 = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \qquad + \text{ BC's}$$

Two-dimensional region is typically a rectangle or a circle, (or even outside of a circle = aerofoil), but could (in principle) be a more arbitrary shape, denoted Ω.

Therefore, we consider:

$$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad (x, y) \in \Omega$$

The boundary of Ω is denoted $\partial\Omega$. Two types of boundary conditions are prescribed on $\partial\Omega$:

1. **Dirichlet conditions**: this means that u is given on $\partial\Omega$
2. **Neumann conditions**: this means that $\frac{\partial u}{\partial n}$ is given on $\partial\Omega$, where $\boldsymbol{n}$ denotes the unit normal vector to $\partial\Omega$.

The method we use to solve Laplace's equation in symmetric regions is separation of variables

Example 12: Find the solution to Laplace's equation in the rectangle: $\Omega = [0, a] \times [0, b]$, satisfying the following boundary conditions:

$$\begin{aligned} u(0, y) &= 0, \quad & u(a, y) &= 0, \quad & y &\in [0, b] \\ u(x, 0) &= f(x), \quad & u(x, b) &= 0, \quad & x &\in [0, a] \end{aligned}$$

Example 13: Find the solution to Laplace's equation in the rectangle: $\Omega = [0, a] \times [0, b]$, satisfying the following boundary conditions:

$$\frac{\partial u}{\partial x}(0, y) = 0, \quad \frac{\partial u}{\partial x}(a, y) = g(y), \quad y \in [0, b]$$
$$u(x, 0) = f(x), \quad u(x, b) = 0, \quad x \in [0, a]$$

Example 14: Laplace's equation in cylindrical polar coordinates is:

$$0 = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \qquad (r,\theta) \in \Omega$$

Solve Laplace's equation inside the circular region $\Omega = \{r : r \in (0, a)\}$ subject to the following boundary conditions:

$$u(a,\theta) = f(\theta) \quad \theta \in [0,2\pi]$$
$$u(r,\theta) \text{ bounded as } r \to 0$$

Example 15: Show that $\varphi(r,\theta) = -U\left(r + \frac{a^2}{r}\right) cos\,\theta + \frac{\kappa\theta}{2\pi}$, is a solution to Laplace's equation in cylindrical polar coordinates:

$$0 = \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} \qquad (r,\theta) \in \Omega$$

where $\Omega = \{r : r > a\}$. What is the condition satisfied by $\varphi(r,\theta)$ at $r = a$. This solution represents the potential flow around a moving cylinder (a circular aerofoil). Find the velocity field corresponding to the potential $\varphi(r,\theta)$.