MATH 316 Notes

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May - June 2021

Contents

Chapter 1

Lecture 1

Some basic, quick notes regarding the course:

- Watch the pre-recorded lectures prior to the actual lectures
- Homeworks: 20%; Midterm: 30%; Final: 50%
- Homeworks are best 4 out of 5
- Exams will be open book, open notes
- Homeworks can be written + scanned, or written using OneNote or LaTeX

1.1 Brief Overview of ODEs

First Order ODEs

- Separable ODEs (ex: $y' = \frac{dy}{dx} = p(x) \cdot q(y)$
- Linear: $Ly = y' + p(x)y = Q(x)$, where $\underbrace{L}_{\text{linear}} = \frac{d}{dx} + p(x)$

Second Order ODEs:

- Constant coefficient ODE: $Ly = y'' + ay' + by = 0$
- Cauchy-Euler, or Equi-dimensional ODEs: $Ly = x^2y'' + axy' + by = 0$
- L is the linear operator. In the constant coefficient, $L = \frac{d^2}{dx^2} + a\frac{d}{dx} + b$

1.1.1 Examples

First Order Separable Equation Examples

$\frac{dy}{dx} = p(x)Q(y) \longrightarrow \int \frac{dy}{Q(y)} = \int P(x)dx + C$

Then, we get:

$\frac{dy}{dx} = y\cos(x) \to \frac{dy}{y} = \cos(x)dx \longrightarrow \ln|y| = \sin(x) + C \to y = C_1 e^{\sin(x)}$

Linear First Order Equation:

$y' + p(x)y = Q(x)$

Multiply both sides by an integrating factor $\mu(x)$

Ex: $y' + \frac{2x}{1+x^2}y = \frac{\cot(x)}{1+x^2}$ $\mu(x)y' + \underbrace{\mu(x)\frac{2x}{1+x^2}y}_{\mu'(x)y} = \frac{\cot(x)}{1+x^2}\mu(x)$ Compare to: $\mu(x)y' + \mu'(x)y = [\mu(x)y(x)]'$

$\frac{d\mu}{dx} = \mu\frac{2x}{1+x^2}$: integrate: $\int \frac{d\mu}{\mu} = \int \frac{2x}{1+x^2}dx + C$

Hence, using integrating factor:

$$\ln|\mu(x)| = \ln(1+x^2) + C$$

with $\mu(x) = C_1(1+x^2)$

Hence,

$$C_1(1+x^2)y' + C_1 2xy = C_1\cot(x)$$

$$\frac{d}{dx}\left[(1+x^2)y\right] = \cot(x)$$

$$(1+x^2)y = \int \cot(x) + C = \ln|\sin(x)| + C$$

Hence,

$$y(x) = \frac{\ln|\sin(x)|}{1+x^2} + \frac{C}{1+x^2}$$

1.1.2 Second Order Constant Coefficient ODE

$$ay'' + by' + cy = 0$$

We start with a guess: e^{rx}, and substitute: $(ar^2 + br + c)\cdot e^{rx} = 0$. Note that $ar^2 + br + c$ is the characteristic equation. We then have two solutions:'

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \Delta = b^2 - 4ac$$

Based on the sign of $Delta$, we have 3 different cases:

- $\Delta > 0$: 2 real, distinct roots
- $\Delta < 0$: 2 complex roots
- $\Delta = 0$: Repeated roots.

Example: $2y'' + 2y' + y = 0$: Guess e^{rx}.

$2r^2 + 2r + 1 = 0 \longrightarrow r_{1,2} = \frac{-2\pm\sqrt{4-8}}{2(2)}, \Delta = -4$. As $-4 < 0$, this is 2 complex roots. We end up with the following solution:

$$e^{\frac{-x}{2}}\left[C_1 e^{\frac{i}{2}x} + C_2 e^{\frac{-i}{2}x}\right]$$

Using Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, we get the following:

$$e^{\frac{-x}{2}}\left[C_1(\cos(\frac{x}{2}) + i\sin(\frac{x}{2})) + C_2(\cos(\frac{x}{2}) - i\sin(\frac{x}{2}))\right]$$

$$e^{\frac{-x}{2}}\left[(C_1 + C_2)\cos(\frac{x}{2}) + i(C_1 + C_2)\sin(\frac{x}{2})\right]$$

$$y(x) = e^{\frac{-x}{2}}\left[A_1\cos\frac{x}{2} + A_2\sin\frac{x}{2}\right]$$

Real form of the solution, A1 & A2: $c_1 = a_1 + ib_1$; $c_2 = a_2 + ib_2$

$A_1 = (a_1 + a_2) + i(b_1 + b_2)$, and $A_2 = i(a_1 - a_2) - (b_1 - b_2)$. $b_1 + b_2 = 0$, $a_1 - a_2 = 0$

Example: $y'' - 2y' + y = 0$. Characteristic equation: $r^2 - 2r + 1 = 0 \longrightarrow (r-1)^2 = 0$, and therefore the roots are $r = 1$ repeated. Hence,

$$y = C_1 e^x + C_2 x e^x$$

is the solution of the equation.

Cauchy - Euler Eqn

$$Ly = x^2 y'' + \alpha x y' + \beta y = 0$$

Guess: $y = x^r$
EX: $2x^2 y'' - xy' + y = 0$. $y(x) = x^r, y'(x) = rx^{r-1}, y'' = r(r-1)x^{r-2}$.
Therefore,

$$2r(r-1)x^r - rx^r + x^r = 0$$

which is equivalent to

$$\left[2r(r-1) - r + 1\right] x^2 = 0$$

$$2r(r-1) - r + 1 = 0 \longrightarrow 2r^2 - 3r + 1 = 0$$

$$r_{1,2} = \frac{3 \pm \sqrt{9-8}}{2(2)} = 1, \frac{1}{2}$$

Hence, the solution to the equation is $y(x) = C_1 x + C_2 x^{\frac{1}{2}}$

EX2: $x^2 y'' - xy' + y = 0$
$y(x) = x^r$ hence, we get: $r(r-1)x^r - rx^r + x^r = 0$.
Therefore $\left[r(r-1) - r + 1\right] x^r = 0$.

$$r^2 - 2r + 1 = (r-1)^2 = 0 \rightarrow r = 1$$

$$y_1 = x, y_2 = \ln(x) \cdot x$$

$$y(x) = C_1 x + C_2 x \ln(x)$$

(?) $Ly = 0, L\frac{d}{dr} y(x,r) = 0$, $\frac{d}{dr}(x^r) = x^r \ln(x)$

EX3: $x^2 y'' - xy' + 5y = 0$. $y(x) = x^r$, and therefore $r(r-1)x^2 - rx^r + 5x^r = 0$
$\left[r(r-1) - r + 5\right] x^r = 0$, hence, $r^2 - 2r + 5 = 0$. We then get the general solution of the following:

$$y(x) = C_1 x^{1+2i} + C_2 x^{1-2i}$$

This can be re-written as the following:

$$y(x) = x\left[C_1 e^{2i\ln(x)} + C_2 e^{-2i\ln(x)}\right]$$

And hence as the following:

$$y(x) = x\left[C_1(\cos(2\ln(x) + i\sin(2\ln(x)) + C_2(\cos(2\ln(x)) - i\sin(2\ln(x)))\right]$$

$$= x\left[(c_1 + c_2)\cos(2\ln(x)) + i(c_1 - c_2)\sin(2\ln(x))\right]$$

$$= x\left[A_1 \cos(2\ln(x)) + A_2 \sin(2\ln(x))\right]$$

A_1 and A_2 are real.

EX4: Solve the IVP

$$x^2y'' - 3xy' + 4y = 0, y(1) = 1, y'(1) = 1$$

If we let $y(x) = x^r$, then we get the following: $y(x) = x^r, y'(x) = rx^{r-1}, y'' = r(r-1)x^{r-2}$
Plugging this into our equations:
$x^2r(r-1)x^{r-2} - 3xrx^{r-1} + 4x^r = 0$
Solving, we discover we have a repeated root of 2. Hence, the general solution is $y(x) = C_1x^2 + C_2x^2\ln(x)$. Plugging in the initial conditions, we find that $C_1 = 1$, and $C_2 = -1$ and hence the solution is $y(x) = x^2 - x^2\ln(x)$

1.2 Series Solutions of ODEs

We use power series expansion to solve variable coefficient linear ODEs. Remember that a function $f(x)$ can be approximated by a polynomial of degree n, such that $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$. As the degree of n increases, the approximation improves. Hence, $f(x) = \sum_{m=0}^{\infty} a_mx^m$. Or, in general, we can approximate $f(x)$ by a power series expanded about a point x_0, writing it as $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$. We can remember that when we had the Taylor series, $a_n = \frac{f^{(n)}(x_0)}{n!}$. So, we have: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}x(0)}{n!}(x-x_0)^n$.

Example: $y' + (1-2x)y = 0$

Using the integrating factor, $\mu(x) = e^{\int(1-2x)dx} = e^{x-x^2}$.
$\left[ye^{x-x^2}\right]' = 0 \longrightarrow ye^{x-x^2} = C \rightarrow y = Ce^{-x+x^2}$.
Use Taylor expansion $y(x)$ about the point $x_0 = 0$.
In order to do this, we write it as the sum described above:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}x(0)}{n!}(x-x_0)^n$$

We can then let $y = \sum_{n=0}^{\infty} \frac{f^{(n)}x(0)}{n!}x^n$ and $y' = \sum_{n=1}^{\infty} \frac{f^{(n)}x(0)}{n!}nx^{n-1}$, and therefore we can write our ODE as

$$\sum_{n=1}^{\infty} \frac{f^{(n)}x(0)}{n!}x^{n-1} + (1-2x)\sum_{n=0}^{\infty} \frac{f^{(n)}x(0)}{n!}x^n = 0$$

Chapter 2

Lecture 2

2.1 Solving the problem from last class

We started with a simple example that was solvable using integrating factor, and we are now going to do the Taylor expansion of the **answer**, around the point $x = x_0$. THe confusion from last class was that we tried to solve the ODE using a Taylor expansion **of the differential equation**, as opposed to the actual answer.

$$y(x) = y(0) + \frac{y'(0)}{1}x + \frac{y''(0)}{2!}x^2 + ...$$

$$y(x) = C\left[1 - x + \frac{3}{2}x^2 - \frac{7}{6}x^3 + ...\right]$$

From the class notes:

May

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \quad \bigstar$$

Ex $\quad y' + (1-2x)y = 0$

Integrating factor : $\mu(x) = e^{\int (1-2x)dx} = e^{x-x^2}$

$$\left[y e^{x-x^2}\right]' = 0 \longrightarrow y e^{x-x^2} = C \longrightarrow \boxed{y = Ce^{-x+x^2}}$$

Taylor expand $y(x)$ about the point $x_0 = 0$

$$y(x) = y(0) + \frac{y'(0)}{1}x + \frac{y''(0)}{2!}x^2 + \dots$$

$$(-1+2x)e^{-x}$$

$$*y(x) = C\left[1 - x + \frac{3}{2}x^2 - \frac{7}{6}x^3 + \dots\right] \checkmark \qquad y'(0) = -C$$

Note that we factored C out of the taylor expansion. Moving forward, assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$y' + (1 - 2x)y = 0$$

$$y' + y - 2xy = 0$$

$$\text{letting } y = \sum_{n=0}^{\infty} a_n x^n, \text{ and } y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$\sum_{n=1}^{\infty} a_n \underbrace{n}_{m=n-1;n=1,m=0} x^{n-1} + \sum_{n=0}^{\infty} \underbrace{a_n}_{m=n} x^n - 2\sum_{n=0}^{\infty} \underbrace{a_n}_{m=n+1;n=0,m=1} x^{n+1} = 0$$

Not exactly sure what these variables are doing with the n and m... I guess they're dummy variables that we're using for each sum.

$$\sum_{m=0}^{\infty} a_{m+1}(m+1)x^m + \sum_{m=0}^{\infty} a_m x^m - 2\sum_{m=1}^{\infty} a_{m-1}x^m = 0$$

Peel-off:

$$a_1x^0 + a_0x^0 + \sum_{m=1}^{\infty} \left[a_{m+1}(m+1) + a_m - 2a_{m-1}\right] x^m = 0$$

Note that $x^0, x, x^2, ..., x^n$ are linearly independent.

$x^0]\quad a_1 + a_0 = 0 \rightarrow a_1 = -a_0$

$x^m,\ m \geq 1]\quad a_{m+1}(m+1) + a_m - 2a_{m-1} = 0$

Q: Why do we need to shift the indices?

A: We want to get to a single sigma – A single sum. If we don't do it, we aren't able to get to a single relation. Indices must start at the same point to be able to combine sums.

From the relation $a_{m=1}(m+1) + a_m - 2a_{m-1} = 0$, we can find the relation:

$$a_{m+1} = \frac{-a_m + 2a_{m-1}}{m+1}$$

$$m = 1 : a_2 = \frac{-a_1 + 2a_0}{2} = \frac{3}{2}a_0$$

$$m = 2 : a_3 = \frac{-a_2 + 2a_1}{3} = \frac{-7}{6}a_0$$

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ...$$

$$y(x) = a_0 - a_0x + \frac{3}{2}a_0x^2 - \frac{7}{6}a_0x^3 + ...$$

$$y(x) = a_0\left[1 - x + \frac{3}{2}x^2 - \frac{7}{6}x^3 + ...\right]$$

$$= \text{Taylor expansion of the direct solution}$$

2.2 Example 2

$$xy' + (2-x)y = 0$$

$$y' + \frac{2-x}{x}y = 0$$

Solving this using integrating factor method, we find the following:

$$\mu(x) = e^{\int \frac{1-x}{x}dx} = e^{2\ln x - x} = x^2e^{-x}$$

$$[x^2e^{-x}y]' = 0 \to y = \frac{C}{x^2e^{-x}} = Cx^-2e^x$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad , y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$xy' + 2y - xy = 0$$

$$\sum_{n=1}^{\infty} a_n n x^n + 2\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^n + 1 = 0$$

$$\sum_{n=1}^{\infty} a_n n x^n + 2\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$m=n \qquad m=n \qquad m=n+1$$

$$m=1, n=0$$

$$\sum_{m=1}^{\infty} a_m m x^m + 2\sum_{m=0}^{\infty} a_m x^m - \sum_{m=1}^{\infty} a_{m-1} x^m = 0$$

peel off first term

$$2a_0x^0 + \sum_{m=1}^{\infty} (a_m m + 2a_m - a_{m-1})x^m = 0$$

$$x^0]\ 2a_0 = 0 \longrightarrow a_0 = 0$$

$$\left.\begin{matrix} x^m \\ m \geq 1 \end{matrix}\right]\ a_m(m+2) = a_{m-1}$$

note that $a_m = \dfrac{a_{m-1}}{m+2}$ for m ≥ 1

$$m = 1 : a_1 = \frac{a_0}{3} = 0$$

$$m = 2 : a_2 = \frac{a_4}{4} = 0$$

This is a trivial solution; they are all zero (and will continue to be).
$y = cx^{-2}e^x$ is the general solution.

$$= C \underbrace{x^{-2}}_{\text{Capture the singularity}} \underbrace{\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right]}_{\sum_{n=0}^{\infty} a_n x^n}$$

$$y(x) = \underbrace{x^r}_{\text{Capture singularity}} \sum_{n=0}^{\infty} a_n x^n$$

This brings us to the **Forbenius Series**.

2.3 Forbenius Series

Let's define ordinary & singular points:
A linear 2nd order ODE:

$$P(x)y'' + Q(x)y' + r(x)y = 0$$

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0$$

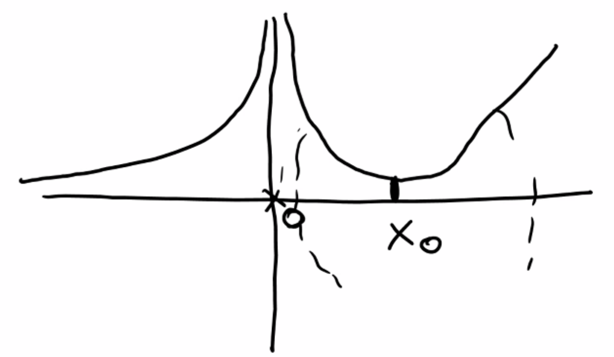
Letting $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$. If they are both **analytic** at the point x_0. i.e. they have Taylor expansions about x_0, x_0 is an ordinary point. Otherwise, x_0 is a singular point.

Example: $\left.\begin{array}{r} p(x) = \frac{1}{x} \\ p(x) = \ln(x) \end{array}\right\}$ At $x = 0$, not analytic.

Analytic means that is is expressible as a power series around x_0. It means that it is infinitely differentiable around x_0.

Quick notes

- A power series solution is possible for all ordinary points (similar to the first example we saw), but not all singular points.
- For singular points, we introduce the Forbenius Series. However, this only works for some singular points.
- Singular points results in the change of the nature of the ODE. Ordinary points exists in the domain of $p(x)$ and $q(x)$.
- The radius of convergence of the power series is at least as large as the distance from the x_0 to the nearest singular point.
  - For example, when we had $y = Cx^{-2}e^x$, we realize that $x = 0$ is a singular point.
  - We can see this in both the answer as well as the ODE.
  - When we plot the function, it looks like this:



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– The dotted circles around x_0 is the radius of convergence.

Example:

$$y' + (1 - 2x)y = 0$$

$$y(x) = Ce^{-x+x^2}$$

There are no singular points. Hence the radius of convergence is infinity.

2.3.1 Singular Points

Singular points are divided into two classes:

- Regular singular points, that we use the Forbenius series solution for
- Irregular singular points (Beyond this course). For these, we **cannot** use Forbenius series.

Chapter 3

Lecture 3

3.1 Singular Points

Singular points are divided into two classes:

- Regular singular points, where we use the Frobenius series solution
- Irregular singular points, which are beyond the scope of the course

If we are to look at the ODE $P(x)y'' + Q(x)y' + R(x)y = 0$, and define the point x_0 as a singular point, we have the following:

The Cauchy-Euler equation is

$$(x - x_0)^2 + \alpha(x - x_0)y' + \beta y = 0 \tag{3.1}$$

We know that $y = (x - x_0)^r$ (As an example).

How do we make $P(x)y'' + Q(x)y' + R(x)y = 0$ look like(3.1)?

If we multiply with $(x - x_0)^2$ and divide by $P(x)$, we may get something similar to the Cauchy-Euler.

Then, we get something like this:

$$(x - x_0)^2 y'' + \left\{ \frac{Q(x)}{P(x)}(x - x_0) \right\} (x - x_0)y' + \left\{ \frac{R(x)}{P(x)}(x - x_0)^2 \right\} y = 0 \tag{3.2}$$

Now, if $\frac{Q(x)}{P(x)}(x - x_0)$ and $\frac{R(x)}{P(x)}(x - x_0)^2$ are analytic at $x = x_0$, then the singularity is not worse than the singularity in the Cauchy-Euler equation (3.1), **and** x_0 is a "regular singular point". Otherwise, x_0 is an "irregular singular point".

If we start to write the Taylor series for (?),

$\frac{Q(x)}{P(x)}(x - x_0) = p_0 + p_1(x - x_0) + ...$

$\frac{R(x)}{P(x)}(x - x_0)^2 = q_0 + q_1(x - x_0) + ...$

Being analytic means that we need to be able to write the series.

An example of a singular point:

$$xy'' + 2\alpha y' = 0$$

$$\left(\frac{2\alpha}{\cancel{x}}\cancel{x}\right)$$

Here, $x = 0$ is a singular point. (We re-wrote the first equation as the second equation, I believe)

As $x \longrightarrow x_0$, our ODE becomes: $(x - x_0)^2 y'' + p_0(x - x_0)y' + q_0 y = 0$

The corresponding Cauchy-Euler equation solution: $y = (x - x_0)^r$. We need to have finite p_0 and q_0. If p_0 and q_0 are both finite, then x_0 is a regular singular point. Otherwise, it is an irregular singular point.

For regular singular points, the solution we are going to write:

$$\underbrace{y(x) = (x - x_0)^r}_{*} \underbrace{\sum_{n=0}^{\infty} a_n (x - x_0)^n}_{\text{Correction}}$$

*: The singular part of the solution to the corresponding Cauchy-Euler.

3.1.1 Example

$$x(1 + x^2)y'' + 2xy' + (1 + x^2)y = 0$$

.

Classify singular points. Here, $p(x) = x(1+x)^2$, $Q(x) = 2x$, and $R(x) = 1+x^2$. Singular points: $\begin{cases} x = 0 \\ x = \pm i \end{cases}$

We need $p(x_0) = 0$ (Take a look at the left hand side if you don't understand!).

Classify them:

$\begin{cases} \lim_{x \to x_0} \frac{Q(x)}{P(x)}(x - x_0) = p_0 \\ \lim_{x \to x_0} \frac{R(x)}{P(x)}(x - x_0)^2 = q_0 \end{cases}$

For $x_0 = 0$, we have $\lim_{x \to 0} \frac{2x}{x(1+x^2)} x = 0 = p_0$

$\lim_{x \to 0} \frac{1+x^2}{x(1+x^2)} x^2 = 0 = q_0$

Now for $x_0 = i$:

$\lim_{x \to i} \frac{2x}{x(1+x^2)}(x - i) = \lim_{x \to i} \frac{2(x-i}{(x-i)(x+i)} = \frac{1}{i} = p_0$

$\lim_{x \to i} \frac{1+x^2}{x(1+x^2)}(x - i)^2 = 0 = q_0$

We see that because both p_0 and q_0 are finite, $x = i$ is also a regular singular point.

Try for $x = -i$:

$\lim_{x \to -i} \frac{2x}{x(1+x^2)}(x + i)$

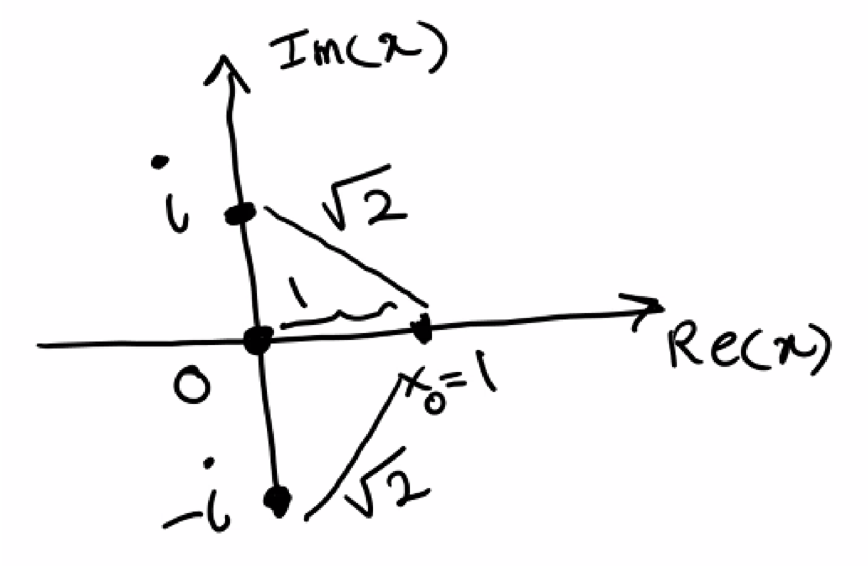
$\lim_{x \to -i} \frac{1+x^2}{x(1+x^2)}(x + i)^2$

yeah uhhhh.... review how to calculate limits.

When we calculate $x_0 = -i$, we get: $p_0 = -\frac{1}{i}$, and $q_0 = 0$. Hence, $x_0 = -i$ is also a regular singular point.

3.2 Radius of Convergence

The radius of convergence of the series solution is at least equal to the distance from the x_0 to the nearest singular point. In the example that we solved:



$y(x) = \sum_{n=0}^{\infty} a_n(x-1)^n$

$\rho = 1$ is the lower bound estimate. ρ is the radius of convergence. Imagine a circle of radius 1 (as it's the distance to the closest singular point)

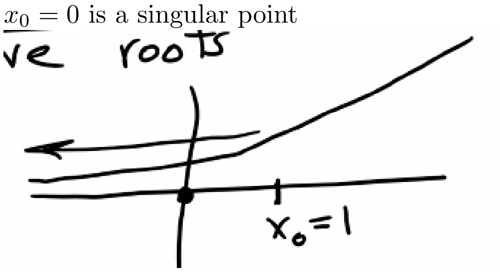
An Example

$$x^2y'' + \alpha xy' + \beta y = 0$$

r_1, r_2 are two positive roots

$$y = C_1x^{r_1} + C_2x^{r_2}$$

$x_0 = 0$ is a singular point



Radius of convergence is infinite. This is a rare case.

3.3 Frobenius Series

EX:

$$6x^2(1+x)y'' + 5xy' - y = 0$$

$P(x) = 6x^2(1+x)$, $Q(x) = 5x$, $R(x) = -1$

Singular points are $x = 0$ and $x = -1$.

For $x = 0$, let's find if it's irregular or regular:

$\lim_{x\to 0} \frac{Q(x)}{P(x)}(x - x_0) = \lim_{x\to 0} \frac{5x}{6x^2(1+x)}x = \frac{5}{6} = p_0$

$\lim_{x\to 0} \frac{-1}{6x^2(1+x)}x^2 = \frac{-1}{6} = q_0$
Therefore $x_0 = 0$ is a regular singular point.

$$(x-x_0)^2y'' + \frac{5}{6}(x-x_0)y' - \frac{1}{6}y = 0$$

This is for $x_0 = 0$. Hence, $x^2y'' + \frac{5}{6}xy' - \frac{1}{6}y = 0$.

Corresponding Cauchy-Euler equation:
$y = x^r$ and therefore $[6r(r-1) + 5r - 1]\,x^r = 0$.
Hence $6r^2 - r - 1 = 0 \longrightarrow r_{1,2} = \frac{1\pm\sqrt{1+24}}{12} \longrightarrow r_{1,2} = \frac{1}{2}, \frac{-1}{3}$

Frobenius Series Solution about $x = 0$:

$$y(x) = x^2\sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} a_nx^{n+r}$$

$$y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}$$

Now, we only need to replace y, y', y'' in the ODE:

$$6x^2(1+x)y'' + 5xy' - y = 0$$

$$6x^2y'' + 6x^3y'' + 5xy' - y = 0$$

$$6\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} \qquad n=m$$

$$+6\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r+1} \qquad n+1=m \quad n=0, m=1$$

$$+5\sum_{n=0}^{\infty} a_n(n+r)x^{n+r} \qquad n=m$$

$$-\sum_{n=0}^{\infty} a_nx^{n+r} \qquad n=m$$

$$6\sum_{m=0}^{\infty} a_m(m+r)(m+r-1)x^{m+r}$$
$$+6\sum_{m=1}^{\infty} a_{m-1}(m+r-1)(m+r-2)x^{m+r}$$
$$+5\sum_{m=0}^{\infty} a_m(m+r)x^{m+r} - \sum_{m=0}^{\infty} a_m x^{m+r}$$

peel off

$6a_0 r(r-1)x^r + 5a_0 r x^r - a_0 x^r +$

$$\sum_{m=1}^{\infty}\left[6a_m(m+r)(m+r-1)+6a_{m-1}(m+r-1)(m+r-2) + 5a_m(m+r) - a_m\right]x^{m+r} = 0$$

$$x^r\Big]\ a_0\underbrace{\left[6r(r-1)+5r-1\right]}_{\text{C-E eqn}}=0 \to r_{1,2} = \tfrac{1}{2}, -\tfrac{1}{3}$$

$$\left.\begin{matrix}x^{m+r}\\ m\geq 1\end{matrix}\right]\ a_m\left[6(m+r)(m+r-1)+5(m+r)-1\right] = -6a_{m-1}(m+r-1)(m+r-2)$$

$$a_m = \frac{-6a_{m-1}(m+r-1)(m+r-2)}{(m+r)\left[6m+6r-6+5\right]-1}$$

recursion for r_1 & r_2

$r=-\frac{1}{3}$, $m\geq 1$

$$a_m = \frac{-6a_{m-1}(m-\frac{4}{3})(m-\frac{7}{3})}{(m-\frac{1}{3})(6m-3)-1}$$

$$m=1:\ a_1 = \frac{-6a_0(-\frac{1}{3})(-\frac{4}{3})}{(+\frac{2}{3})(3)-1} = \frac{-8a_0}{3}$$

$$m=2:\ a_2 = \frac{-32a_0}{42(3)}$$

$$y_1(x) = a_0x^{\frac{-1}{3}}\left[1-\frac{8}{3}x-\frac{32}{3(42)}x^2+...\right]$$

$r=\frac{1}{2}$: If we follow the same steps, we get
$y_2(x) = \underbrace{a_0}_{\text{A different } a_0} x^{\frac{1}{2}}\left[1+\frac{3}{22}x-\frac{22}{22.68}x^2+...\right]$
Hence the general solution is $y = C_1y_1(x)+C_2y_2(x)$

Convergence: ratio test

$$\sum_{m=0}^{\infty} C_m : \lim_{m\to\infty}|\frac{C_{m+1}}{C_m}| < 1$$

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}$$

Ratio test:

$$\lim_{m\to\infty}|\frac{a_{m+1}x^{m+r+1}}{a_mx^{m+r}}| = |x|\lim_{m\to\infty}|\frac{a_{m=1}}{a_m}$$

For this example:

$$|x|\lim_{m\to\infty}|\frac{a_{m=1}}{a_m}, a_m = \frac{-6a_{m-1}(m+r-1)(m+r-2)}{(m+r)(6m+6r-1)-1}$$

$$|x| \lim_{m\to\infty} \frac{-6a_{m-1}(m+r-1)(m+r-2)}{(m+r)(6m+6r-1)-1} = |x| \cdot 1$$

$|x| < 1$

$-1 < x < 1$

$\rho =$ is at least equal to the distance between $x_0 = 0$ & the nearest singular point, $\rho = 1$

$x_{\text{singular}} = 0, -1$

Chapter 4

Lecture 4

4.1 Introduction

4.1.1 Week 2:

- finish power series
- Bessel's function
- Intro to PDEs

4.1.2 Week 3:

- Fourier series and separation of variables
- Heat equations
- Wave equations
- Laplace equations

4.1.3 Week 4:

- Boundary value problems and Sturm-Louiville Theory
- Eigenfunctions and eigenvalues
- Sturm-Louisvill theory for BVP
- Non-homogenous boundary value problems

4.1.4 Week 5:

- Numerical methods for solving PDEs

4.2 Recap of Previous Week

What we're covering next was partly covered in previous lectures by Parisa:

Series Solutions near a regular singular point (RSP)

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

We consider an expansion about a regular singular point x_0 of the ODE.

Next, we take the limits:

$$\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)} = p_0$$

$$\lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)} = q_0$$

When these two limits exist and are finite, then this leads to the following. We divide $P(x)y''+Q(x)y'+R(x)y=0$ by $P(x)$ and multiply it by $(x-x_0)^2$:

$$(x-x_0)^2y''+(x-x_0)\underbrace{\left\{(x-x_0)\frac{Q(x)}{P(x)}\right\}}_{p(x)=p_0+p_1(x-x_0)+\ldots}y'+\underbrace{\left\{(x-x_0)^2\frac{R(x)}{P(x)}\right\}}_{q(x)=q_0+q_1(x-x_0)+\ldots}y=0$$

$$(x-x_0)^2+p_0(x-x_0)y'+q_0y=0$$

is the C-E equation. This equation has a solution of the following format:

$$y_0(x)=(x-x_0)^r$$

To include the effect of the neglected terms, we modify the above solution:

$$y(x)=\underbrace{(x-x_0)^r}_{\text{C-E solution}}\sum_{n=0}^{\infty}\underbrace{a_n(x-x_0)^n}_{\text{The solution}}$$

with $a_0\neq 0$. (2) is the modified equation for $y_0(x)=(x-x_0)^r$ and is Frobenius series.

4.2.1 Solution procedure

1. Find values of r
2. Find the recursive relation for n
3. Find the radius of convergene for $\sum_{n=0}^{\infty}a_n(x-x_0)^{n+r}$

4.2.2 Example 1

$$Ly=2x^2y''-xy'+(1-x)y=0$$

Singular point here is when $x_0=0$
Again, to test if it is a singular point, we take the limits.

$$p_0=\lim_{x\to 0}\frac{-x}{2x^2}x=\frac{-1}{2}$$

Similarly, we check q_0:

$$q_0=\lim_{x\to 0}\frac{(1-x)}{2x^2}x^2=\frac{1}{2}=q_0$$

Hence, the singular point is a RSP.

$$\Rightarrow y(x)=x^r\sum_{n=0}^{\infty}a_nx^n$$

Step 1: Find values of r
Characteristic equation: $r(r-1)+p_0r+q_0=0$
This becomes $r(r-1)-\frac{r}{2}+\frac{1}{2}=0\Rightarrow r^2-\frac{3}{2}r+\frac{1}{2}=0$
Hence, $r=1$ and $r=\frac{1}{2}$ are roots of the indicial equation.

$$y_0 = c_1 x + c_2 x^{\frac{1}{2}}$$

Now, we go to step 2: Find the recursive relation for n.
Take the first and second derivative of the summation (that we've used before) and sub into the equation.

$$2\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

Changing index:

$$2\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - \underbrace{\sum_{n=0}^{\infty} a_n x^{n+r+1}}_{m=n+1} = 0$$

All others are $m = n$.
With that, we get the following:

$$2\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

(Removing $= 0$ due to space)
Peeling off the first terms ($n = 0$), we end up with the following sum:

$$\underbrace{2a_n r(r-1)x^r - a_0 r x^r + a_0 x^r}_{=a_0 x^r(2r^2-3r+1)} \longrightarrow ...$$

$$\hookrightarrow + \sum_{n=1}^{\infty} \left[2a_n(n+r)(n+r-1) - a_n(n+r) + a_{n-1}\right] x^{n+r} = 0$$

Hence, $a_n = \frac{a_{n-1}}{(n+r)(2(n+r)-3)+1}$ is the recursive relation for r values
Finding the recursive relation for r_1 and r_1:

$$r_1 \Rightarrow a_n = \frac{a_{n-1}}{(n+1)(2(n+1)-3)+1} = \frac{1_{n-1}}{2n^2+n}$$

$n = 1 : a_1 = \frac{a_0}{3}; n = 2 : a_2 = \frac{a_1}{10} = \frac{a_0}{30}; n = 3 : a_3 = \frac{a_2}{21} = \frac{a_0}{630}$
Therefore:

$$y(x) = a_0 x^1 (1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + ...)$$

Next, for $r_1 = \frac{1}{2}$:

$$a_n = \frac{a_{n-1}}{(n+\frac{1}{2}(2n+1-3)+1} = \frac{a_{n-1}}{2(n+\frac{1}{2})(n-1)+1}$$

$n = 1 : a_1 = \frac{a_0}{1}; n = 2 : a_2 = \frac{a_1}{6} = \frac{a_0}{6}; n = 3 : a_3 = \frac{a_0}{90}$
So, we have the following:

$$y_2(x) = a_0 x^{\frac{1}{2}} \left[1 + x + \frac{x^2}{6} + \frac{x^3}{90} + ...\right]$$

Now, onto step 3:
Find the radius of convergence.
To find the radius of covergence, we use the ratio test:

$$\lim_{n\to\infty} |\frac{a_n x^n}{a_{n-1}x^{n-1}}|$$

$$= \lim_{n\to\infty} |\frac{a_{n-1}x^n}{(n+r)(2(n+r)-3)+1}\frac{1}{a_{n-1}x^{n-1}}|$$

The above is not correct. Why?

$$\lim_{n\to\infty} |x||\frac{1}{(n+r)(2(n+r)-3)+1}| = 0$$

$\Rightarrow p = \infty$ for all x values
Final solution:

$$y(x) = y_1(x) + y_2(x)$$

$$y(x) = C_1 x\left(1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + ...\right) + C_2 x^{\frac{1}{2}}\left(1 + x + \frac{x^2}{6} + \frac{x^3}{90} + ...\right)$$

4.3 Bessel's equation

Applications: PDEs on circular / cylindrical domain.
e.g. heating and cooling in circular / cylindrical geometries, i.e. pipes and heat exchangers.

The equation format:

$$Ly = x^2y'' + xy' + \left(x^2 - \nu^2\right)y = 0 \tag{4.1}$$

ν is a constant and specifies the order of the equation.
$x_0 = 0$ is a regular singular point, since $\lim_{x\to 0}\frac{x}{x^2}(x) = 0 = p_0$ and $\lim_{x\to 0}\frac{x^2-\nu^2}{x^2}(x^2) = -\nu^2 = q_0$

4.3.1 Step 1: Finding the r values

The characteristic equation is:

$$r(r-1) + p_0 r + q_0 = 0$$

$$r(r-1) + r - \nu^2 = 0 \Rightarrow r = \pm\nu$$

4.3.2 Step 2: Frobenius part – Finding recursive relation

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}$$

Substituting into the equation, we get the following:

$$Ly = x^2y'' + xy' + \left(x^2 - \nu^2\right)y = 0$$

$$Ly = x^2\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2} \to ...$$

$$\hookrightarrow +x\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} \to ...$$

$$\hookrightarrow + \left(x^2-\nu^2\right)\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Shifting index:

$$Ly = x^2y'' + xy' + \left(x^2-\nu^2\right)y = 0$$

$$Ly = x^2\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2} \to ...$$

$$\hookrightarrow +x\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} \to ...$$

$$\hookrightarrow + \underbrace{x^2\sum_{n=0}^{\infty} a_n x^{n+r+2}}_{m=n+2} - \nu^2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Hence, we get the following:

$$Ly = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=2}^{\infty} a_{n-2}x^{n+r}$$

Next, we peel off the first two terms to match the summation.

$$a_0(r(r-1)+r-\nu^2)x^r + a_1((1+r)r+(1+r)-\nu^2)x^{r+1} \to ...$$

$$\hookrightarrow + \sum_{n=2}^{\infty} a_n\left[a_n\left((n+r)^2-\nu^2\right)+a_{n-2}\right]x^{n+r} = 0$$

$$a_0x^r(r^2-\nu^2) + a_1x^{r+1}\left((r+1)^2-\nu^2\right) + \sum_{n=2}^{\infty} a_n\left[a_n\left((n+r)^2-\nu^2\right)+a_{n-2}\right]x^{n+r} = 0$$

Chapter 5

Lecture 5

5.1 Bessel's Equation, continued from last class

$$a_0(r(r-1)+r-\nu^2)x^r + a_1\left((1+r)r+(1+r)-\nu^2\right)x^{r+1} + \sum_{n=2}^{\infty} a_n\,(...)\,x^{n+r} = 0$$

We can use linear independency. This means that the set of the coefficients of all powers of x must be zero.

The following is found from the characteristic equation:

$$x^r | a_0(r^2-\nu^2) = 0 \longrightarrow r = \pm\nu \;\&\; a_0 \neq 0$$

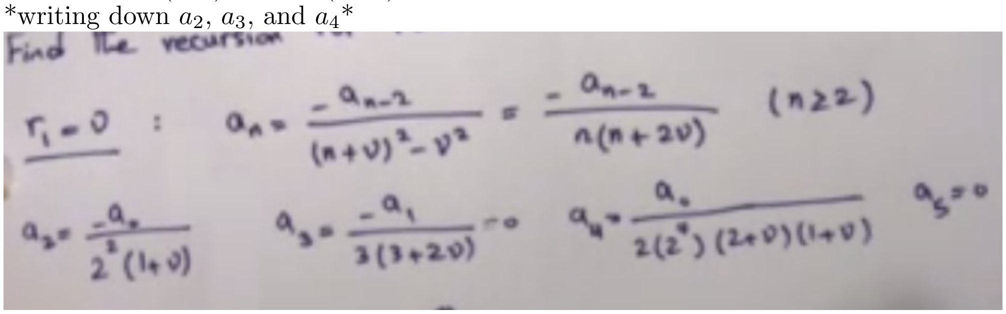
$$x^{r+1} | a_1\left(r^2+2r+1-\nu^2\right) = 0 \underbrace{\longrightarrow}_{\nu^2-r^2} a_1(2\nu+1) = 0$$

$$\hookrightarrow \begin{cases} \nu = \pm\frac{1}{2} & \& \quad q \neq 0 \\ \nu \neq \pm\frac{1}{2} & \& \quad q = 0 \end{cases}$$

$$x^{n+r} | \left((n+r)(n+r-1)+(n+r)-\nu^2\right)a_n + a_{n-2} = 0 \longrightarrow n \geq 2$$

$$(**)\ a_n = \frac{-a_{n-2}}{(n+r)^2-\nu^2}$$

Find the recursive relation for $r = \pm\nu$:
$r_1 = \nu$: $a_n = \frac{-a_{n-2}}{(n+\nu)^2-\nu^2} = \frac{-a_{n-2}}{n(n+2\nu)}$, $(n \geq 2)$
writing down a_2, a_3, and a_4



$\Rightarrow a_{2m} = \frac{(-)^m a_0}{m!2^{2m}(1+\nu)(2+\nu)...(m+\nu)}$

$$y_1(x) = a_0 x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{m!2^{2m}(1+\nu)(2+\nu)...(m+\nu)}$$

Now, for $r_1 = \nu$:

$$a_n = \frac{-1_{n-1}}{(n-\nu)^2 - \nu^2} = -\frac{a_{n-2}}{n(n-2\nu)}; n \geq 2$$

$$a_2 = \frac{-a_0}{2(2-2\nu)} = \frac{-a_0}{2(2)(1-\nu)}$$

$$a_4 = \frac{-a_2}{4(4-2\nu)} = \frac{a_0}{4(2)(2-\nu)(2^2)(1-\nu)}$$

$$a_6 = \frac{-a_4}{6(6-2\nu)} = \frac{-a_0}{6(2)(3-\nu)2^5(2-\nu)(1-\nu)}$$

Note that $a_1 = a_3 = a_5 .... = 0$

$$\Rightarrow a_{2m} = \frac{(-1)^m a_0}{m!2^{2m}(1-\nu)(2-\nu)(3-\nu)...(m-\nu)}$$

$$y_2(x) = a_0 x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m a_0}{m!2^{2m}(1-\nu)(2-\nu)...(m-\nu)}$$

Finally, $y(x)$ is a linear combination of 2 solutions:

$$y(x) = C_1 x^\nu \underbrace{\sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m}}{m(1+\nu)(2+\nu)...(m+\nu)}}_{J_\nu\text{: Bessel Functions of the first kind}} + C_2 \underbrace{x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m a_0}{m!2^{2m}(1-\nu)(2-\nu)...(m-\nu)}}_{Y_\nu\text{: Bessel function of the second kind}}$$

We will be given this in the formula sheet. Note that C_1 and C_2 are not included in J_ν and Y_ν.
For $\nu \neq \pm\frac{1}{2}$: As $x \to 0$, $J_\nu \to 0$ and $x \to 0$, $Y_\nu \to \infty$
What happens when $\nu = 0$?

$$x^r | a_0(r^2 - \nu^2) = 0 \to r = \pm\nu, a_0 \neq 0$$

Two solutions are the same. Therefore, $r_{1,2} = 0$
Then, $J_\nu(x) = C_1 x^0 \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2}\right)^{2m}$
How about $Y_\nu(x)$?
Similar to Euler's equation (Refer to section 5.4 of the textbook), the second solution for repeated roots is:

$$y_2(x) = y_1(x)\ln(x) + x^{r_1} \sum_{n=1}^{\infty} a_n'(r_1) x^n$$

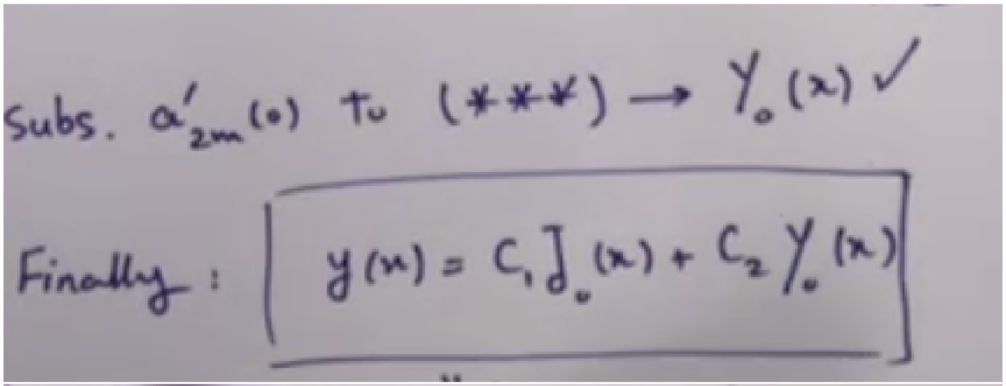
where $a_n'(r_1) = \frac{da_1}{dr}\Big|_{r=r_1}$
According to this formula, $y_0 = J_0(x)\ln(x) + x^0 \sum_{n=1}^{\infty} a_n'(0)x^n$ (***)
The recursion for this case (**) is found to be

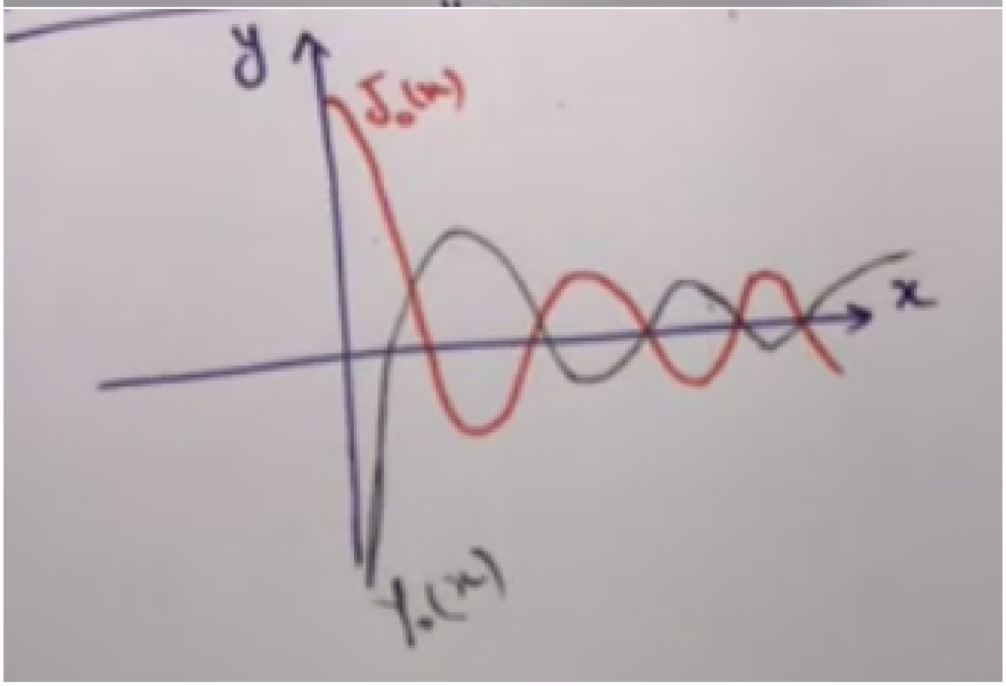
$$a_{2m}(r) = -\frac{a_{2m-2}}{(r+2m)^2}, m = 1, 2, 3, ...$$

$$a_{2m}(r) = \frac{(-1)^m a_0}{(r+2)^2....(r+2m)^2}$$

$$a_{2m}'(r) = -2\left(\frac{1}{r+2} + \frac{1}{r+4} + ... + \frac{1}{r+2m}\right) a_{2m}(r)$$

$$a_{2m}'(0) = -2\left(\frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2m}\right) a_{2m}(0) = -\left(\frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2m}\right) \frac{(-1)^m a_0}{2^{2m}(m!)^2}$$





As $x \to 0$, $y_0(x) \to -\infty$, i.e. if the solution, $y(x)$, is finite at zero, then $C_2 = 0$

5.2 Bessel Equation of Order of $\pm\frac{1}{2}$

$$Ly = x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$$

$$x^r|a_0(r^2 - \nu^2) = 0 \longrightarrow a_0 \neq 0 \text{ and } r = \pm\nu$$

For $\nu = \frac{1}{2} \Rightarrow r = \pm\frac{1}{2}$

$$x^{r+1}|a_1(r^2 + 2r + 1 - \nu^2) = 0 \Rightarrow a_1(1 \pm 2\nu) = 0$$

If $\nu = \pm\frac{1}{2}$, a_1 is arbitrary

$$x^{m+r}| - a_m(r^2 + 2r + 1 - \nu^2) = a_{m=2} \Rightarrow a_m = \frac{-a_{m-2}}{(m+r)^2 - \nu^2}$$

For $\nu \pm \frac{1}{2}, a_m = \frac{-a_{m-2}}{(m+\frac{1}{2})^2 - \frac{1}{4}} = \frac{-a_{m-2}}{m(m+1)}$
Let $r_1 = \frac{1}{2}$: $a_1(1 + 2(\frac{1}{2})) = 0 \Rightarrow a_1 = 0$
$a_2 = -\frac{a_0}{2(3)}$, $a_4 = \frac{-a_2}{3(4)} = \frac{a_0}{5!}$
Therefore:

$$y_1(x) = a_0 x^{\frac{1}{2}} \underbrace{\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + ...\right) \frac{x}{x}}_{\text{Taylor series for } \sin(x)}$$

$$y_1(x) = a_0 x^{-\frac{1}{2}} \sin(x)$$

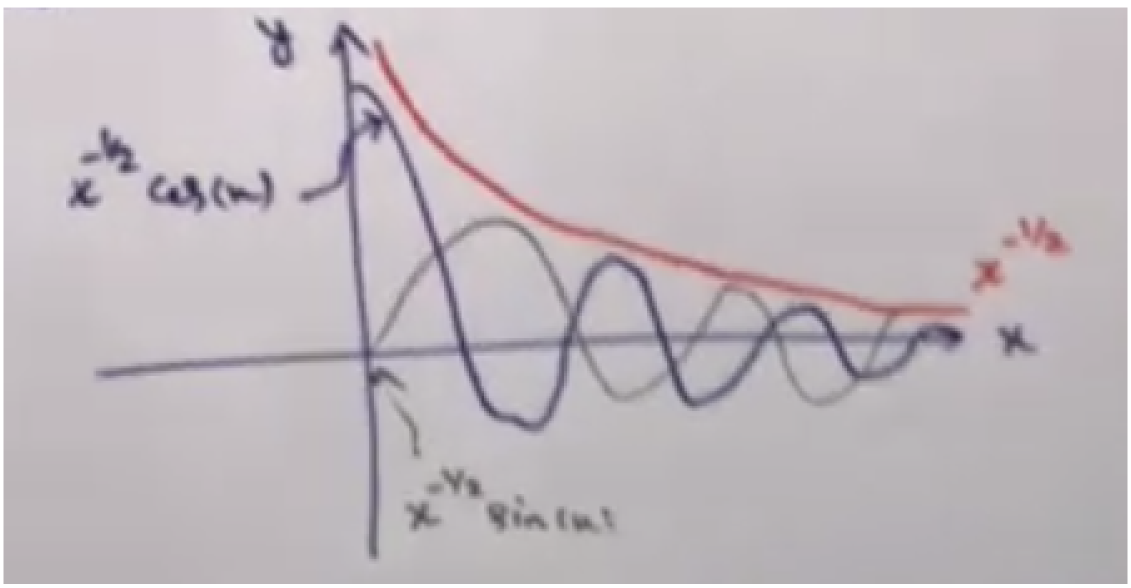
let $r_2 = \frac{-1}{2}$: $a_1(1 - 2\frac{1}{2}) = 0 \Rightarrow a_1$ is arbitrary. it could be another solution.
for this case, $a_m = \frac{-a_{m-2}}{(m=\frac{1}{2})^2 - \frac{1}{4}} = \frac{-a_{m-2}}{m(m-1)}$
$a_2 = \frac{-a_0}{2(1)}$, $a_4 = \frac{-a_2}{4(3)} = \frac{a_0}{4!}$

$$\Rightarrow y_x(x) = a_0 x^{\frac{-1}{2}} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...\right) = a_0 x^{-\frac{1}{2}} \cos(x)$$

Let's check for $a_1 \neq 0$:
$a_3 = \frac{-a_1}{3 \cdot 2}$, $a_5 = \frac{-a_3}{5 \cdot 4} = \frac{a_1}{5!}$
$y_3(x) = a_1 x^{-\frac{1}{2}} \sin(x)$. But this doesn't give us another solution – This is the same as y_1; they are not independent.

Hence we write the final solution as:

$$y(x) = a_0 x^{\frac{-1}{2}} \cos(x) + a_1 x^{\frac{-1}{2}} \sin(x)$$



End of series functions

5.3 Introduction to PDE Classification

What is a PDE?

A differential equation that includes partial derivatives with respect to all independent variables.
$u(x,t) \longrightarrow$ PDEs include $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x \partial t}$, $\frac{\partial^2 u}{\partial x^2}$, ...

- Heat equation
- Wave equation
- Laplace equation

Chapter 6

Lecture 6

6.1 Recap of Frobenius Series Solutions

Assume x_0 is a singular point of the ODE of the form:

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

If x_0 is a regular singular point,

$$\lim_{x \to x_0} \frac{Q(x)}{P(x)} x = p_0$$

and

$$\lim_{x \to x_0} \frac{R(x)}{P(x)} x^2 = q_0$$

The characteristic equation is:

$$r(r-1) + p_0 r + q_0 = 0 \longrightarrow 2 \text{ roots: } r_1, r_2$$

For r_1, we get $y_1(x) = |x|^{r_1} \left(1 + \sum_{n=1}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty} a_n x^{n+r_1}$
a_n is found from a recursion by substitution into the ODE. a_0 is arbitrary.

1) If $r_1 - r_2 \neq 0$ and $r_1 - r_2 \neq N$ (N is an integer), then:

$$y_2 = |x|^{r_2} \left(1 + \sum_{n=1}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty} a_n x^{n+r_2}$$

2) If $r_1 = r_2$:

$$y_2(c) = y_1(x)\ln(x) + |x|^{r_1} \sum_{n=1}^{\infty} C_n x^n = y_x(x)\ln(x) + \sum_{n=1}^{\infty} C_n x^{n+r_1}$$

Note that $x > 0$.

Where $c_n = a_n' = \frac{da_n}{dr}\Big|_{r=r_1}$
Note 1: What happens of r_1 and r_2 are complex?
If they are, the form of y_2 in 1) (that we discussed), and y_1 are still valid; we just need to convert complex valued to real valued solutions. Needs lots of algebra.
Note 2: A summary of these solutions is given in the formula sheet for the exam.
Note 3: The general solution is in the following format:

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

6.2 PDEs

Continued from last class's notes.

Heat equation / diffusion equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + k\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

Applications: heat flows, diffusion of chemical substances

Wave equation

$$\frac{\partial^2 u}{\partial t^2} = C^2\frac{\partial^2 u}{\partial x^2} + C^2\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

Applications: Vibrations, acoustics, solid mechanics

Laplace's equation

$$0 = \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial y^2}$$

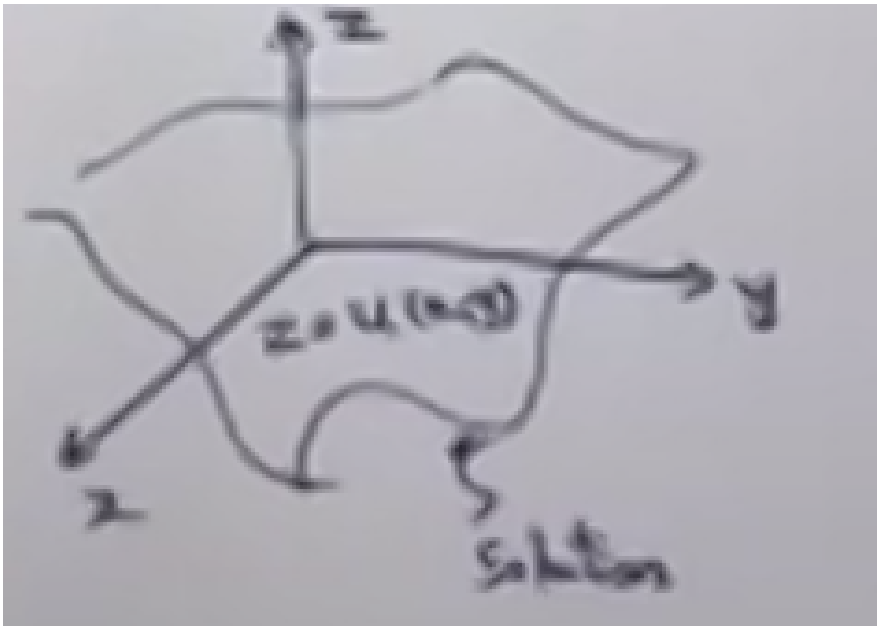
Applications: Heat / wave equations in which there is a steady-state solution (eg potential flow, porous media flow)

6.2.1 Classification of PDEs

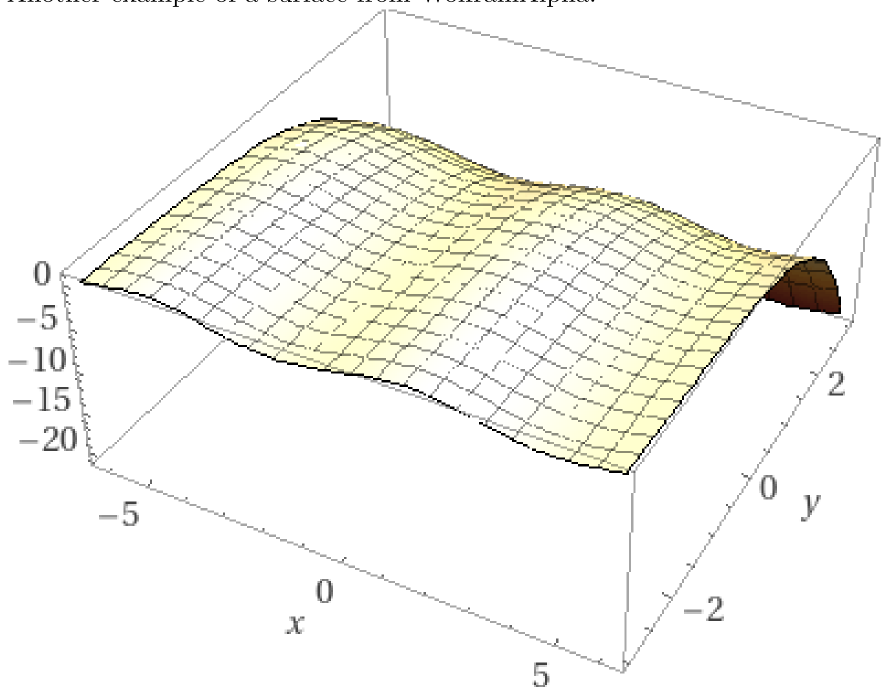
ODEs: $f(x, u(x), u'(x)) = 0$. e.g. $u' = e^u$

PDEs: $\underbrace{a(x,y)u_x + b(x,y)u_y = c(x,y)u}_{\text{First order, linear PDE}}$

The solution to a PDE would look like a 2d surface:



Another example of a surface from WolframAlpha:



This course primarily focuses on second order linear PDEs.

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{6.1}$$

A, B, C, D, E, F, G can either be constants or functions of (x, y).
The examples that we saw (heat equation, wave eq, etc) are all examples of the above (1).

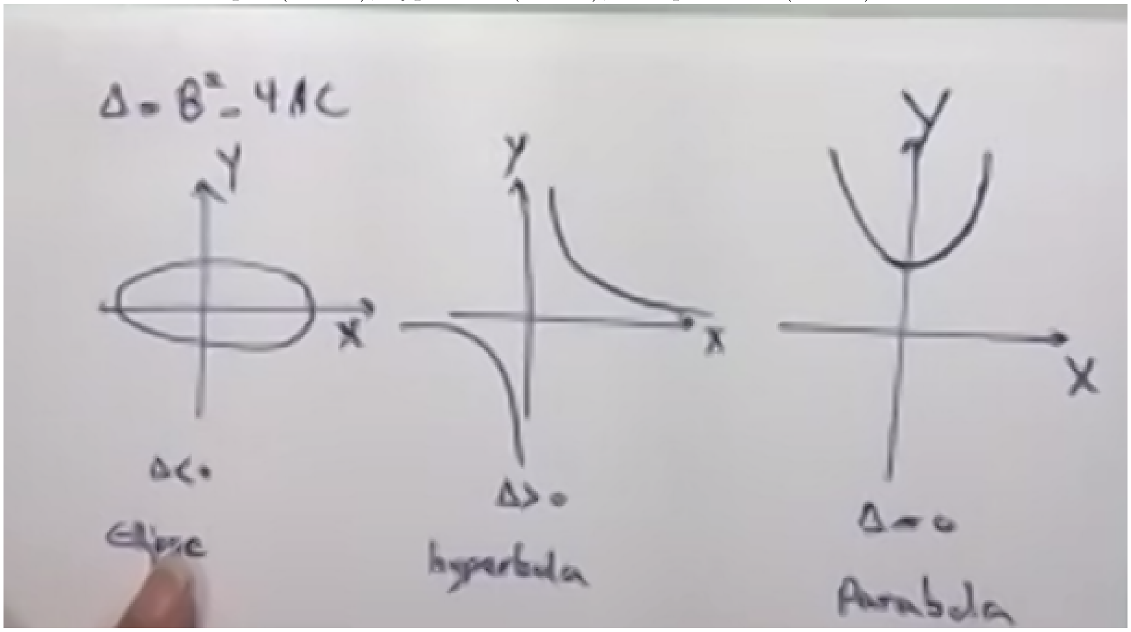
If $G = 0$, the PDE is homogeneous. Else, it is non-homogeneous.
To classify PDEs we use the analogy with corresponding quadratic surfaces:

$$AX^2 + BXY + CY^2 + DX + EY = K$$

To classify, we use the discriminant:

$$\Delta = B^2 - 4AC$$

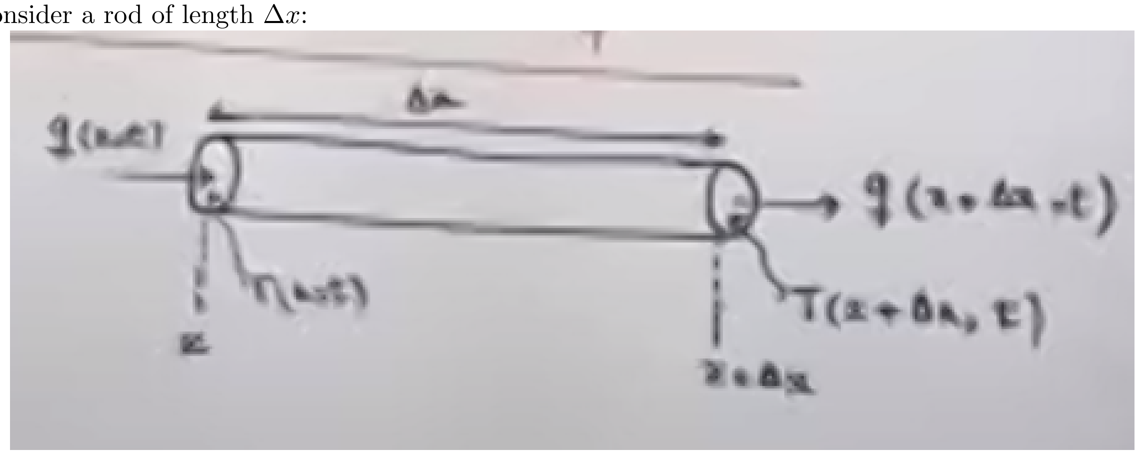
It tells us either ellipse ($\Delta < 0$), hyperbola ($\Delta > 0$), or a parabola ($\Delta = 0$)



Δ	Type	PDE	Note
$\Delta = 0$	parabolic	$u_t = u_{xx}$	Heat eq
$\Delta < 0$	elliptic	$u_{xx} + u_{yy} = 0$	Laplace eq
$\Delta < 0$	elliptic	$u_{xx} + u_{yy} = G$	Poisson's eq
$\Delta > 0$	Hyperbolic	$u_{tt} = c^2 u_{xx}$	Wave eq

6.2.2 Heat / Diffusion Equation

Consider a rod of length Δx:



(The equations that are a bit blurry are the following, left to right and top to bottom: $q(x,t)$, Δx, $q(x+\Delta x, t)$, $T(x,t)$, $T(x+\Delta x, t)$, x, $x+\Delta x$)

- $T(x,t)$: Temperature at (x,t)
- $q(x,t)$: The heat flux (heat energy per unit area)

- C: The specific heat capacity
- ρ: density of material
- A: The cross sectional area

Energy conservation: The increase in the thermal energy of the bar is equal to the (influx - outflux) of heat. (Physical description, not mathematical description).

Use variables: $C(T(x,t+\Delta t)-T(x,t))\rho\Delta xA = (q(x,t)-q(x+\Delta x,t))A\Delta t$

Divide by $\Delta t \cdot \Delta x$:

$$\rho C \frac{T(x,t+\Delta t)-T(x,t)}{\Delta t} = \frac{q(x,t)-q(x,t+\Delta t)}{\Delta x}$$

As $\Delta t \to 0$ and $\Delta x \to 0$:

$$\rho C \frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x}$$

The energy conservation equation is hence:

$$\frac{\partial q}{\partial x} + \rho C \frac{\partial T}{\partial t} = 0 \tag{6.2}$$

In order to reduce the number of dependent variables, we need a constitutive equation between q and T. Can we relate the heat flux to the temperature?

Yes. The heat transfer through conduction is formulated as:

$$q = -k\frac{\partial T}{\partial x} \tag{Fourier's Law}$$

where k is the thermal conductivity of the material. What does this tell us?

- Heat flux will flow from high temperature to low temperature.

We can substitute Fourier's Law in the energy conservation equation:

$$-k\frac{\partial^2 T}{\partial x^2} + \rho c\frac{\partial T}{\partial t} = 0$$

$$\hookrightarrow \frac{\partial T}{\partial t} = \alpha^2 \frac{\partial T}{\partial x^2}$$

Where $\alpha^2 = \frac{k}{\rho c}$ (diffusion coefficient).

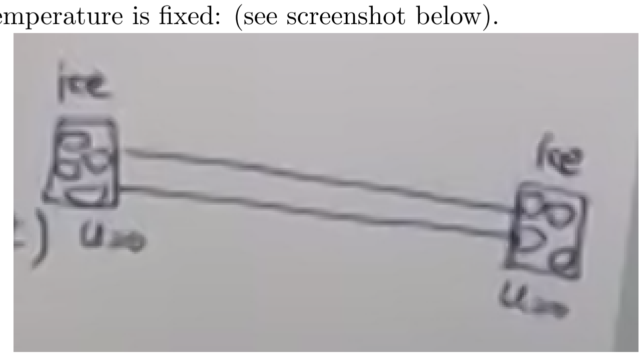
6.2.3 Solving diffusion equations using separation of variables

The initial boundary value problems, $u_t = \alpha^2 u_{xx}$, needs one initial condition (IC) and two boundary conditions (BC).

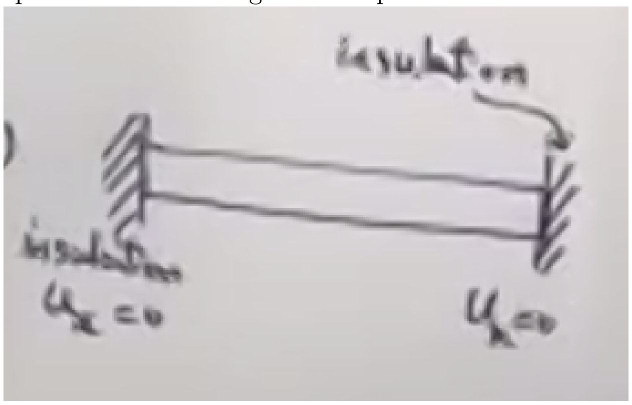
Initial condition: $u(x,t=0) = f(x)$ on the domain $0 < x < L$

Boundary conditions

(1) Dirichlet boundary conditions $u(0,t) = 0 = u(L,t)$ (i.e. same temperature on either side of the rod). Temperature is fixed: (see screenshot below).



(2) Neumann boundary conditions: $u_x(0,t) = 0 = u_x(L,t)$. i.e. insulation on either side of a rod. Temperature won't change with respect to x.



(3): Mixed boundary conditions. $u(0,t) = 0$ and $u_x(L,t) = 0$

Example 1

$$u_t = \alpha^2 u_{xx},\ 0 < x < L,\ t > 0$$

Boundary conditions:

$$u(0,t) = 0$$
$$u(L,t) = 0$$

Initial conditions:

$$u(x,0) = f(x)$$

To solve, we use the method of separation of variables. We will do this next class.

Chapter 7

Lecture 7

7.1 Boundary value Problems

Is there a similar setup for BVPs/ Let's consider 3 different BVPs:

1. P1: $y'' + \lambda y = 0$, for $x \in [0, L]$, with $y(0) = 0 = y(L)$
2. P2: $y'' + \lambda y = 0$, for $x \in [0, L]$, with $y'(0) = 0 = y'(L)$
3. P3: $y'' + \lambda y = 0$, for $x \in [0, L]$, with $y(0) = y(L)$ and $y'(0) = y'(L)$

Any value of λ for which P1 (P2 or P3) has a non-zero solution is called an **eigenvalue** of P1 (P2 or P3) and the corresponding solution is called and **eigenfunction** of P1 (P2 or P3).

Exercise: find the eigenvalues and eigenfunctions of problems P1, P2 and P3.

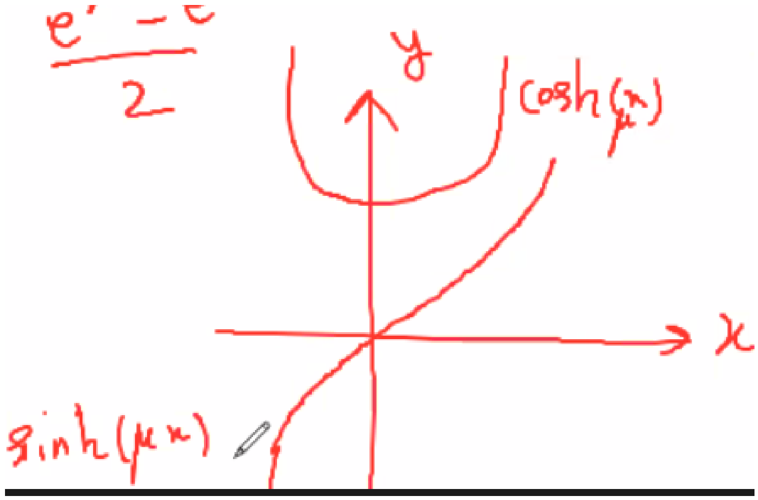
7.1.1 Solving BVP (P1,P2,P3)

P1:

$$y'' + \lambda y = 0, \text{ and } y(0) = 0 = y(L)$$

λ: Eigenvalue. There are three categories that we have to investigate each time we solve such a problem:

1. If λ is negative ($\lambda < 0$):
   - $\lambda = -\mu^2 \rightarrow y'' - \mu^2 y = 0$
   - $r^2 - \mu^2 = 0 \rightarrow r_1 = \mu, r_2 = -\mu \longrightarrow y(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$
   - Note that $\cosh(\mu x) = \frac{e^{\mu x} + e^{-\mu x}}{2}$ and $\sinh(\mu x) = \frac{e^{\mu x} - e^{-\mu x}}{2} \longrightarrow$
   - $\hookrightarrow y(x) = A\sinh(\mu x) + B\cosh(\mu x)$

- 

- Note that we have the boundary conditions $y(0) = 0 \rightarrow B = 0$ and $y(L) = 0 \rightarrow A\sinh(\mu L) = 0$
- $\Rightarrow A = 0$ and $\Rightarrow y(x) = 0$ which is trivial

2. If λ is zero: $y'' = 0 \longrightarrow y(x) = Ax + B$
   - $y(0) = 0 \longrightarrow B = 0$
   - $y(L) = 0 \longrightarrow AL = 0 \rightarrow A = 0 \rightarrow y(x) = 0$
   - Therefore it's a trivial solution.
3. If $\lambda > 0$: $\lambda = \mu^2 \rightarrow y'' + \mu^2 y = 0$
   - $r^2 + \mu^2 = 0 \longrightarrow r = \pm i\mu$
   - $y(x) = A\sin(\mu x) + B\cos(\mu x)$
   - $y(0) = 0 \rightarrow B = 0$
   - $y(L) = 0 \rightarrow A\sin(\mu L) = 0 \rightarrow \begin{matrix} A = 0(\text{trivial}) \\ \sin(\mu L) = 0 \rightarrow \mu L = n\pi \end{matrix}$ therefore $\mu = \frac{n\pi}{L}$
   - Eigenvalue: $\lambda = \left(\frac{n\pi}{L}\right)^2$
   - Eigenfunction: $y_n(x) = A_n \sin(\frac{n\pi}{L}x)$

7.1.2 P2: $y'' + \lambda y = 0$

$y'(0) = 0 = y'(L)$
$r^2 + \lambda = 0$

1. If $\lambda > 0$, $\lambda = \mu^2$
   - $r^2 + \mu^2 = 0 \longrightarrow r = \pm i\mu$
   - $y(x) = A\sin(\mu x) + B\cos(\mu x)$
   - Sub boundary conditions:
   - $y'(0) = A = 0$ and $y'(L) = 0 \longrightarrow -B\mu\sin(\mu L) = 0$
   - This gives us two solutions: $\begin{matrix} B = 0 \text{ which is trivial} \\ \underbrace{\sin(\mu L) = 0}_{\mu = \frac{n\pi}{L}} \end{matrix}$
   - $\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$ is the eigenvalue
   - $y_n(x) = B\cos\left(\frac{n\pi}{L}x\right)$ is the eigenfunction

2. If $\lambda < 0$: $\longrightarrow \lambda = -\mu^2$
   - $r^2 - \mu^2 = 0 \longrightarrow r = \pm\mu \longrightarrow y(x) = C_1 e^{\mu x} + C_2 e^{-\mu x} \Rightarrow y(x) = A\sinh(\mu x) + B\cosh(\mu x)$
   - Substituting the boundary conditions into $y(x) = A\sinh(\mu x) + B\cosh(\mu x)$, we get:
   - $y'(0) = 0 \longrightarrow A = 0$
   - $y'(L) = 0 \longrightarrow -\mu B\sinh(\mu L) = 0 \longrightarrow B = 0$ (which means that $y(x) = 0$ which is trivial)
3. If $\lambda = 0 \longrightarrow y'' = 0 \rightarrow y = Ax + B$
   - $y'(0) = 0 \rightarrow A = 0$ and $y'(L) = 0 \rightarrow A = 0$: $\longrightarrow y(x) = B$
   - $\lambda = 0$ is an eigenvalue $\longrightarrow y(x) = 1$ is the eigenfunction
   - For P2 problems: eigenvalues are: $0, \frac{n^2\pi^2}{L^2}$ and eigenfunctions are: $1, \cos\left(\frac{n\pi}{L}x\right)$

7.1.3 P3: $y'' + \lambda y = 0$

Periodic boundary conditions: $y(0) = y(L)$

1. If $\lambda > 0$: $\lambda = \mu^2$
   - $r = \pm\mu i \longrightarrow y(x) = A\sin(\mu x) + B\cos(\mu x)$
   - $y(0)\ y(L) \longrightarrow B = A\sin(\mu L) + B\cos(\mu L)$
   - $y'(0) = y'(L) \longrightarrow A\mu = A\mu\cos(\mu L) - B\mu\sin(\mu L)$

$$r = \pm\mu i \longrightarrow y(x) = A\sin(\mu x) + B\cos(\mu x)$$

$$\left.\begin{array}{lll} y(0)=y(L) \longrightarrow & B = A\sin(\mu L) + B\cos(\mu L) & \xrightarrow{\times \frac{A}{B}} A = \frac{A^2}{B}\sin(\mu L) + A\cos(\mu L) \\ y'(0)=y'(L) \longrightarrow & A\mu = A\mu\cos(\mu L) - B\mu\sin(\mu L) & \longrightarrow A = A\cos(\mu L) - B\sin(\mu L) \end{array}\right.$$

$$\left(\frac{A^2}{B} + B\right)\sin(\mu L) = 0$$

   -
   - We get the last term by using the previous two terms (on the right) and cancelling out.
   - $\sin(\mu L) = 0 \rightarrow \mu L = n\pi$. $a \neq 0$ and $B \neq 0$ and therefore $\mu = \frac{n\pi}{L}$
   - Eigenvalues: $\lambda = \left(\frac{n\pi}{L}\right)^2$
   - Eigenfunction is $y_n(x) = A_n\sin(\frac{n\pi}{L}x) + B_n\cos(\frac{n\pi}{L}x)$
2. If $\lambda < 0$: $\lambda = -\mu^2$
   - $y(x) = A\sinh(\mu x) + B\cosh(\mu x)$
   - $y(0) = y(L) \longrightarrow B = A\sinh(\mu L) + B\cosh(\mu L)$
   - $y'(0) = y'(L) \longrightarrow A = A\cosh(\mu L) - B\sinh(\mu L)$
   - Multiplying $B = A\sinh(\mu L) + B\cosh(\mu L)$ by:

$$= -\mu \qquad y(x) = A\sinh(\mu x) + B\cosh(\mu x) \quad ①$$

$$B = A\sinh(\mu L) + B\cosh(\mu L) \quad \times \frac{A}{B} \longrightarrow A = \frac{A^2}{B}\sinh(\mu L) + A\cosh(\mu L)$$

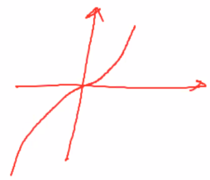
   -
   - Okay this is going way too fast... screenshots it is.

- (ii) if $\lambda < 0$: $\lambda = -\mu^2$ $\quad y(x) = A\sinh(\mu x) + B\cosh(\mu x)$

  $y(0) = y(L) \longrightarrow$ $B = A\sinh(\mu L) + B\cosh(\mu L)$ ① $\times \frac{A}{B} \longrightarrow A = \frac{A^2}{B}\sinh(\mu L) + A\cosh(\mu L)$

  $y'(0) = y'(L) \longrightarrow$ ② $A = A\cosh(\mu L) - B\sin(\mu L)$

  ① − ② $\longrightarrow$ $\underbrace{\left(\frac{A^2}{B} + B\right)}\underbrace{\sinh(\mu L)} = 0$ ✗

  No solution $\longrightarrow$ No eigenvalues

3. $\lambda = 0$:

   - View screenshot below.

   - (iii) $\lambda = 0$: $y'' = 0 \longrightarrow y = Ax + B$

     $y(0) = B$ & $y(L) = AL + B \longrightarrow A = 0$ $\Big\} \longrightarrow y(x) = B$ ✓✓

     $y'(0) = y'(L) = A$

     $\lambda = 0$ is an eigenvalue

     $y(x) = 1$ is an eigenfunction

7.2 Fourier Series

Fourier series arise in 3 different situations of relevance to this course:1.Simple boundary value problems, e.g. P1-P32.Partial differential equations that describe heat flow, waves and diffusion (more later).3.Some initial value problems with less simple periodic forcing, e.g. we are very unlikely to have exactly: $f(t) = F_0\cos(\omega t)$, in any real system, but might have a periodic forcing function.

For what follows, let the interval in P1-P3 be the interval $[a, b] = [-L, L]$. The key idea is that an arbitrary function, $f(t)$, defined on $[-L, L]$ can be represented in the following form:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi t}{L}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi t}{L}) \tag{7.1}$$

Note that these are the eigenfunctions of problem P3. Outside of the interval,because each function above has period $2L$, the above series must converge to a periodic extension of $f(t)$ of period $2L$.

Two immediate questions:

1. Can all functions $f(t)$ be represented in this way, i.e. which functions?
2. How do we find the coefficients a_n and b_n?

Definition:If the series on the right-hand side of (1) converges to a function $f(t)$ then this is called the Fourier series of $f(t)$.

Comments:

Firstly, in order for $f(t)$ to have Fourier series representation(1), that is valid for all t it is necessary that $f(t)$ is periodic, with period $2L$, i.e.

$$f(t + 2L) = f(t)\ \forall t$$

Secondly, suppose that $f(t)$ has a Fourier series representation (1). Then a_n and b_n are determined straightforwardly. See below for a_n:

1. Multiply (1) by $\cos(\frac{m\pi t}{L})$
2. Integrate both sides of the equation between $[-L, L]$:

$$\int_{-L}^{L} f(t)\cos\frac{m\pi t}{L}dt = \int_{-L}^{L}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}\right)\cos\frac{m\pi t}{L}dt$$

Note that:

$$\begin{array}{l}\int_{-L}^{L}\cos\frac{n\pi t}{L}\cos\frac{m\pi t}{L}dt = \left\{\begin{array}{ll}0 & m\neq n\\ L & m=n\end{array}\right.\\ \int_{-L}^{L}\cos\frac{n\pi t}{L}\sin\frac{m\pi t}{L}dt = 0\\ \int_{-L}^{L}\sin\frac{n\pi t}{L}\sin\frac{m\pi t}{L}dt = \left\{\begin{array}{ll}0 & m\neq n\\ L & m=n\end{array}\right.\end{array}$$

Therefore, interchanging summation and integration:

$$\int_{-L}^{L} f(t)\cos\frac{m\pi t}{L}dt = a_m\int_{-L}^{L}\cos\frac{m\pi t}{L}\cos\frac{m\pi t}{L}dt = a_m L$$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{m\pi t}{L}dt$$

For the coefficients b_n a similar procedure is possible (exercise).
Thus, we finally have:

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(t)dt\\ a_m &= \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{m\pi t}{L}dt \quad m=1,2,3,\ldots\\ b_m &= \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{m\pi t}{L}dt \quad m=1,2,3,\ldots\end{aligned}$$

which are known as the **Euler-Fourier series**.

7.2.1 Example 1

Assumer that the function $f(t)$, defined by $(t) = \begin{cases} t & -L\leq t<0\\ 0 & 0\leq t<L\end{cases}$ with $f(t+2L) = f(t)$, has a fourier series. Sketch the function and find the fourier series.

Solution:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(\frac{n\pi t}{L}) + \sum_{n=1}^{\infty} b_n\sin(\frac{n\pi t}{L})$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt = \frac{1}{L}\int_{-L}^{0} tdt = \frac{-L}{2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos(\frac{n\pi t}{L})dt = \frac{1}{L}\int_{-L}^{0} t\cos(\frac{n\pi t}{L})dt$$

.

Using integration by parts, with $u = t$ $(du = dt)$ and $dv = \cos(\frac{n\pi t}{L})dt$, with $v = \frac{L}{n\pi}\sin(\frac{n\pi t}{L})$, we get the following:

$$a_n = \frac{1}{L}\left[\frac{tL}{n\pi}\frac{\sin(n\pi t)}{L}\Big|_{-L}^{0} - \int_{-L}^{0}\frac{L}{n\pi}\frac{\sin(n\pi t}{L}dt\right.$$
$$= \frac{1}{n\pi}\left[\frac{L}{n\pi}\cos(\frac{n\pi t}{L}|_{-L}^{0}\right] = \frac{L}{n^2\pi^2}\left(1-\cos(n\pi)\right)$$

Chapter 8

Lecture 8

8.1 Introduction

- Midterm is on Tuesday, June 8th 12:30 to 2pm.
- Send an email (on Canvas), and explain if you have difficulties regarding the exam including timezone differences.
- Homeworks can be on Webwork. For the weeks that we have Webwork homework, it will be **instead** of written work.
- It will be a mixture of webwork homework and written homework for the rest of the course; one week could we webwork and the next could be written.

8.2 Recap of last lecture

We covered two categories last week:

We discussed eigenvalue problems, or boundary value problems, of (P1, P2, P3), which had the following form:

$$y'' + \lambda y = 0$$

with different boundary conditions:

- P1 are Dirichlet boundary conditions (Fourier sine series)
- P2 are Neumann boundary conditions (Fourier cosine series)
- P3 are periodic boundary conditions (Mixture of Fourier sine and cosine series)

8.2.1 Fourier series

A periodic function on $(-L, L)$ and integrable can be written as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi t}{L}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi t}{L})$$

a_0 can be found through the following formula:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt$$

a_n can be found through

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos(\frac{n\pi t}{L})dt$$

And b_n can be found from:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin(\frac{n\pi t}{L})dt$$

8.2.2 Example 1 (Continued from last class)

$$f(t) = \begin{matrix} t & -L \le t < 0 \\ 0 & 0 \le t < L \end{matrix}, f(t+2L) = f(t)$$

$a_0 = \frac{-L}{2}$; $a_n = \frac{L}{n^2\pi^2}\left(1-\cos(n\pi)\right) = \frac{L}{n^2\pi^2}\left(1-(-1)^n\right)$ for $n \in \mathbb{N}$

$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin(\frac{n\pi t}{L})dt = \frac{1}{L}\int_{-L}^{0} t\sin(\frac{n\pi t}{L})dt$
Using integration by parts, we get the following:

$b_n = \frac{L}{n\pi}\left(\cos(n\pi) - \underbrace{\frac{\sin(n\pi)}{n\pi}}_{=0}\right)$
$b_n = \frac{L}{n\pi}\cos(n\pi) = \frac{L}{n\pi}(-1)^n$ for $n \in \mathbb{N}$
Substitute (a_0, a_n, b_n) into the equation:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(\frac{n\pi t}{L}) + \sum_{n=1}^{\infty} b_n\sin(\frac{n\pi t}{L})$$

$$\Rightarrow f(t) = \frac{-L}{4} + L\sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{n^2\pi^2}\cos(\frac{n\pi t}{L}) + L\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\sin(\frac{n\pi t}{L})$$

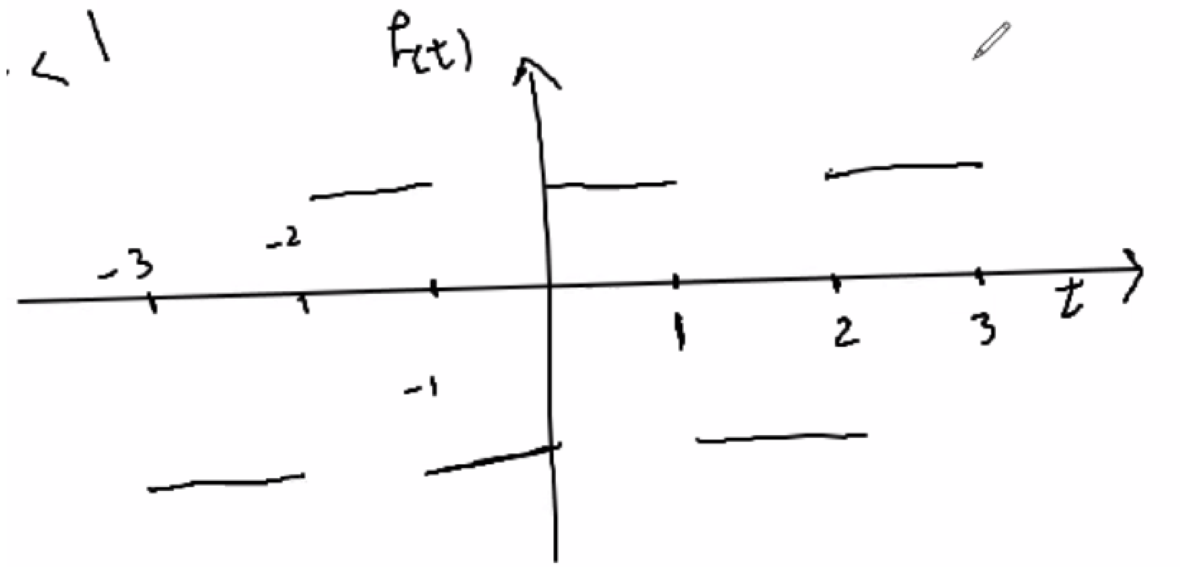
If $L = \pi$:

$$f(t) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{n^2\pi}\cos(nt) + \sum_{n=1}^{\infty}\sin(nt)$$

8.2.3 Example 2

Fourier series example.

$$f(t) = \begin{cases} -1 & -1 < t < 0 \\ 1 & 0 < t < 1 \end{cases} \quad f(t+2) = f(t)$$



This is a square wave function, and it is odd.
$f_{odd}(x) \cdot f_{even}(x) = f_{odd}(x)$: An odd function multiplied by an even funciton is an odd function.
$f_{odd}(x) \cdot f_{odd}(x) = f_{even}(x)$: An odd funciton multiplied by an odd function is an even function.
If $f(t)$ is an even function:

$$\int_{-L}^{L} f(t)dt = 2\int_{0}^{L} f(t)dt$$

If $f(t)$ is an odd function:

$$\int_{-L}^{L} f(t)dt = 0$$

Now, let's find out what the coefficients are of the fourier series.

$$a_0 = \frac{1}{L}\int_{-L}^{L} \underbrace{f(t)}_{odd}\, dt = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(t)}_{odd} \underbrace{\cos(\frac{n\pi t}{L})}_{even}\, dt = 0$$

(note that L = 1)

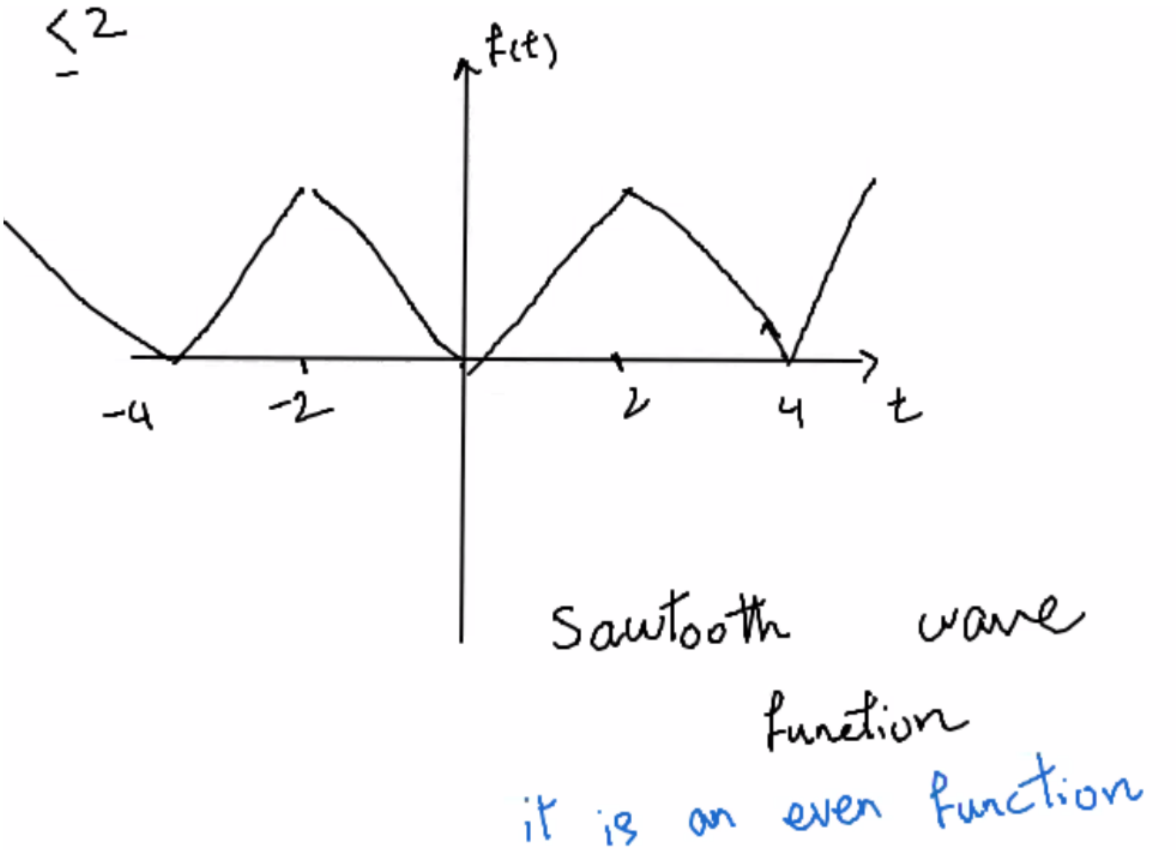
$$b_n = \int_{-1}^{1} f(t)\sin(n\pi t)dt = 2\int_{0}^{1}(1)\sin(n\pi t)dt = -\frac{2}{n\pi}\cos(n\pi t)\bigg|_0^1$$

$$b_n = -\frac{2}{n\pi}\left(\cos(n\pi) - 1\right) = \frac{4}{(2k-1)\pi}, \text{with } n = 2k-1 \text{ for } k \in \mathbb{N}$$

$$\Rightarrow f(t) = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi}\sin\left((2k-1)\pi t\right) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin\left((2k-1)\pi t\right)}{2k-1}$$

8.2.4 Example 2, part B

$$f(t) = \begin{cases} -t & -2 < t < 0 \\ t & 0 \le t \le 2 \end{cases} \quad f(t+4) = f(t)$$



[Note that this is a triangle wave, not a sawtooth wave, but it does not matter for the problem]

$$b_n = 0$$

$$a_0 = \frac{2}{2}\int_0^2 t dt = \frac{t^2}{2}\bigg|_0^2 = 2$$

$$a_n = \frac{2}{L}\int_0^2 t\cos(\frac{n\pi t}{2})dt$$

Using integration by parts, with the following:

$$\begin{array}{ll} u = t & dv = \cos(\frac{n\pi t}{2})dt \\ du = dt & v = \frac{2}{n\pi}\sin(\frac{n\pi t}{2}) \end{array}$$

$$= \underbrace{\frac{2}{n\pi}t\sin(\frac{n\pi t}{2})\bigg|_0^2}_{=0} - \frac{2}{n\pi}\int_0^2 \sin(\frac{n\pi t}{2})dt = \frac{4}{n^2\pi^2}\cos(\frac{n\pi t}{2})\bigg|_0^2 = \frac{4}{n^2\pi^2}(\cos(n\pi) - 1)$$

$$\Rightarrow a_n = \frac{-8}{(2k-1)^2\pi^2}$$

(substituting $n = 2k-1$ above)

$$f(t) = 1 - \frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{(2k-1)\pi t}{2}\right)}{(2k-1)^2}$$

See slides; handwritten before page 6 and pages 6 up to 9 are covered in moderate depth(page numbers based on the numbers at the bottom of the page), and an overview up to the end. Example 3 is an exercise. The last slide is important.

8.3 Fourier Series Slides

Included here for reference; also available on Canvas under Modules

Boundary value problems

Is there a similar setup for BVPs? Let's consider 3 different BVPs:

P1: $y'' + \lambda y = 0$, for $x \in [0, L]$, with $y(0) = 0 = y(L)$

P2: $y'' + \lambda y = 0$, for $x \in [0, L]$, with $y'(0) = 0 = y'(L)$

P3: $y'' + \lambda y = 0$, for $x \in [0, L]$, with $y(0) = y(L)$ and $y'(0) = y'(L)$

Any value of λ for which P1 (P2 or P3) has a non-zero solution is called an **eigenvalue** of P1 (P2 or P3) and the corresponding solution is called and **eigenfunction** of P1 (P2 or P3).

Exercise: find the eigenvalues and eigenfunctions of problems P1, P2 and P3

Fourier Series

Fourier series arise in 3 different situations of relevance to this course:

1. Simple **boundary value problems**, e.g. P1-P3
2. **Partial differential equations** that describe heat flow, waves and diffusion (more later).
3. Some **initial value problems** with less simple periodic forcing, e.g. we are very unlikely to have exactly: $f(t) = F_0\, cos\, \omega\, t$, in any real system, but might have a periodic forcing function

---

For what follows, let the interval in P1-P3 be the interval $[a. b] = [-L, L]$. The key idea is that an arbitrary function, $f(t)$, defined on $[-L, L]$ can be represented in the following form:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\, cos\, \frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n\, sin\, \frac{n\pi t}{L} \qquad (1)$$

Note that these are the eigenfunctions of problem P3. Outside of the interval, because each function above has period $2L$, the above series must converge to a periodic extension of $f(t)$ of period $2L$

Two immediate questions:

1. Can all functions $f(t)$ be represented in this way, i.e. which functions?
2. How do we find the coefficients a_n and b_n?

Definition: If the series on the right-hand side of (1) converges to a function $f(t)$, then this is called the **Fourier series** of $f(t)$.

Comments:

Firstly, in order for $f(t)$ to have **Fourier series representation** (1), that is valid for all t, it is **necessary** that $f(t)$ be periodic, with period $2L$, i.e.

$$f(t+2L) = f(t) \qquad \forall t$$

Secondly, suppose that $f(t)$ has a Fourier series representation (1). The a_n & b_n are then determined straightforwardly, (see below for a_n).

1. Multiply (1) by: $cos\,\frac{m\pi t}{L}$
2. Integrate both sides of the equation between $[-L, L]$:

$$\int_{-L}^{L} f(t)\, cos\,\frac{m\pi t}{L}\, dt = \int_{-L}^{L} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\, cos\,\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n\, sin\,\frac{n\pi t}{L} \right) cos\,\frac{m\pi t}{L}\, dt$$

Note that:

$$\int_{-L}^{L} cos\,\frac{n\pi t}{L}\, cos\,\frac{m\pi t}{L}\, dt = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

$$\int_{-L}^{L} cos\,\frac{n\pi t}{L}\, sin\,\frac{m\pi t}{L}\, dt = 0$$

$$\int_{-L}^{L} sin\,\frac{n\pi t}{L}\, sin\,\frac{m\pi t}{L}\, dt = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

$$\boxed{\text{Trig identity: } \cos(A)\cos(B) = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)}$$

So for $m \neq n$: $$\int_{-L}^{L} \cos\left(\frac{n\pi t}{L}\right)\cos\left(\frac{m\pi t}{L}\right) dt$$

$$= \int_{-L}^{L} \frac{1}{2}\left[\cos\left(\frac{(n+m)\pi t}{L}\right) + \cos\left(\frac{(n-m)\pi t}{L}\right)\right] dt$$

$$= \frac{L}{2\pi(n+m)}\left(\sin\left(\frac{(n+m)\pi t}{L}\right)\Big|_{-L}^{L} + \sin\left(\frac{(n-m)\pi t}{L}\right)\Big|_{-L}^{L}\right) = 0$$

$$\boxed{\cos^2(A) = \frac{1}{2}\left(1 + \cos(2A)\right)}$$

So for $m = n$: $$\int_{-L}^{L} \cos^2\left(\frac{n\pi t}{L}\right) dt = \frac{1}{2}\int_{-L}^{L}\left(1 + \cos\left(\frac{n\pi t}{L}\right)\right) dt$$

$$= \frac{1}{2}\left(t\Big|_{-L}^{L} + \frac{L}{2n\pi} \cancel{\sin\left(\frac{2n\pi t}{L}\right)\Big|_{-L}^{L}}^{\,0} \right) = L$$

Subs. into $*$:

$$\int_{-L}^{L} f(t)\cos\left(\frac{m\pi t}{L}\right) dt = a_0 \frac{L}{m\pi} \sin\left(\frac{m\pi t}{L}\right)\Big|_{-L}^{L} + \underbrace{a_m L}_{\text{for } n=m} + 0$$

$$\Rightarrow \quad a_m = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{m\pi t}{L}\right) dt \qquad \therefore$$

$$\boxed{\text{trig identity: } \quad \sin A \cdot \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)}$$

if $m \neq n$: $\displaystyle\int_{-L}^{L} \sin\left(\frac{n\pi t}{L}\right)\sin\left(\frac{m\pi t}{L}\right) dt$

$$= \int_{-L}^{L} \frac{1}{2}\left(\cos\left(\frac{(n-m)\pi t}{L}\right) - \cos\left(\frac{(n+m)\pi t}{L}\right)\right) dt$$

$$= \frac{1}{2}\left[\frac{L}{(n-m)\pi}\sin\left(\frac{\pi t(n-m)}{L}\right)\Big|_{-L}^{L} - \frac{L}{(n+m)\pi}\sin\left(\frac{(n+m)\pi t}{L}\right)\Big|_{-L}^{L}\right]$$

$= 0$

$$\boxed{\sin^2 A = \frac{1}{2}\left(1 - \cos 2A\right)}$$

if $m = n$: $\displaystyle\int_{-L}^{L} \sin^2\left(\frac{n\pi t}{L}\right) dt = \frac{1}{2}\int_{-L}^{L}\left(1 - \cos\left(\frac{2n\pi}{L}t\right)\right) dt =$

$$= \frac{1}{2}\left(t\Big|_{-L}^{L} - \underbrace{\frac{L}{2n\pi}\sin\left(\frac{2\pi n}{L}t\right)\Big|_{-L}^{L}}_{0}\right) = L$$

if you multiply eq (1) by $\sin\left(\frac{m\pi}{L}t\right)$ and integrate $\int_{-L}^{L}$:

$$\Rightarrow \int_{-L}^{L} f(t)\sin\left(\frac{m\pi t}{L}\right)dt = \int_{-L}^{L}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}\right)\sin\frac{m\pi t}{L}\,dt$$

$$\Rightarrow \int_{-L}^{L} f(t)\sin\left(\frac{m\pi t}{L}\right)dt = 0 + 0 + b_m\int_{-L}^{L}\sin^2\left(\frac{m\pi t}{L}\right)dt + 0 + 0 + \cdots$$

only the mth terms are nonzero

$$\Rightarrow \int_{-L}^{L} f(t)\sin\left(\frac{m\pi t}{L}\right)dt = b_m L \Rightarrow b_m = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{m\pi t}{L}\right)dt$$

Therefore, interchanging summation and integration:

$$\int_{-L}^{L} f(t)\cos\frac{m\pi t}{L}\,dt = a_m \int_{-L}^{L} \cos\frac{m\pi t}{L}\cos\frac{m\pi t}{L}\,dt = a_m L$$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{m\pi t}{L}\,dt$$

For the coefficients b_n a similar procedure is possible (exercise).

Thus, we finally have:

$$\boxed{\begin{aligned} a_0 &= \tfrac{1}{L}\textstyle\int_{-L}^{L} f(t)dt \\ a_m &= \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{m\pi t}{L}dt \qquad m = 1,2,3,\ldots \\ b_m &= \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{m\pi t}{L}dt \qquad m = 1,2,3,\ldots \end{aligned}}$$

which are known as the **Euler-Fourier** formulas.

Example 1: Assume that the function $f(t)$, defined by

$$f(t) = \begin{cases} t & -L \leq t < 0 \\ 0 & 0 \leq t < L \end{cases}$$

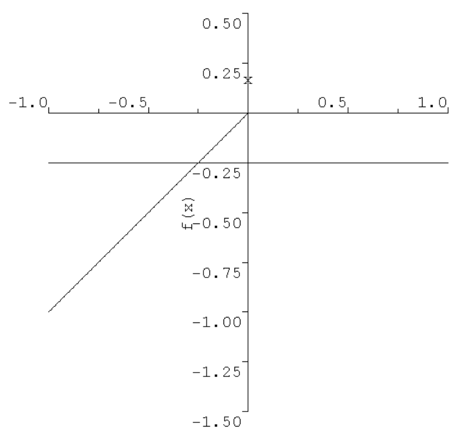
with $f(t + 2L) = f(t)$, has a Fourier series. Sketch the function and find the Fourier series.

Why are we doing this?

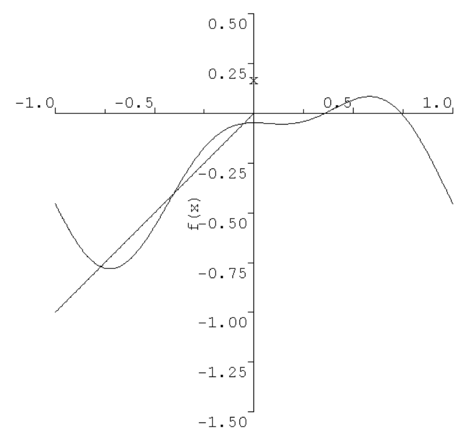
Lets fix $L = 1$ in the above example and plot the partial sums:

$$f(t) \sim -\frac{1}{4} + \sum_{n=1}^{k} \frac{1-(-1)^n}{(n\pi)^2} \cos n\,\pi t + \sum_{n=1}^{k} \frac{(-1)^{n+1}}{n\pi} \sin n\,\pi t$$

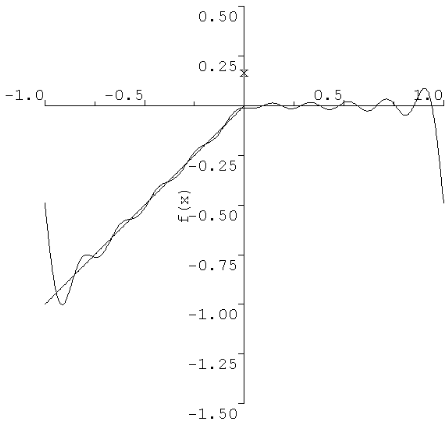
k=0 Constant term only



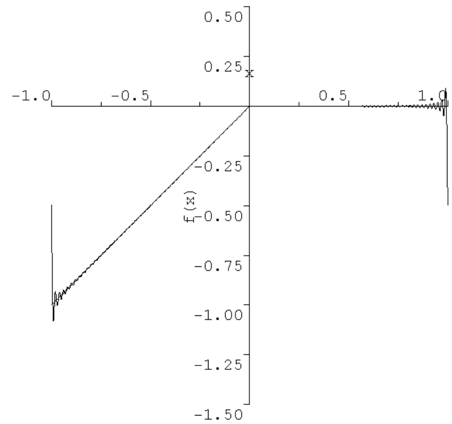
k=2 First 2 trignometric terms



k=10 First 10 terms

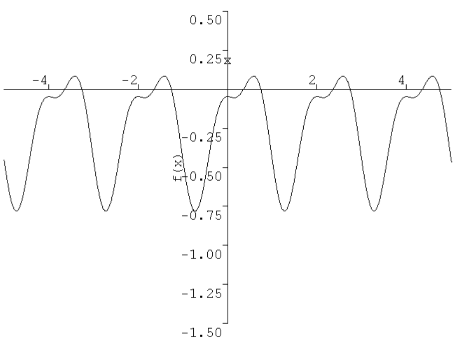


k=100 First 100 terms

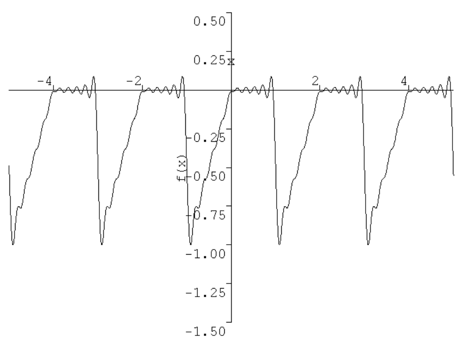


What's happening over longer interval of t?

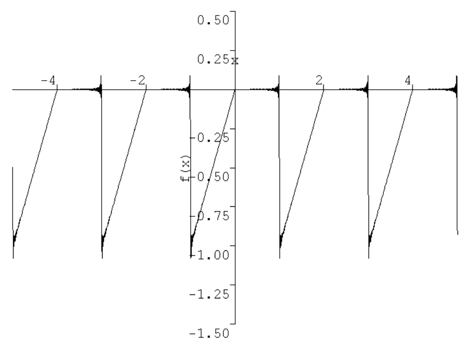
k=2



k=10



k=100



Observations:

1. Take more terms in the series it appears to converge to $f(t)$, (even if $f(t)$ has discontinuities!)
2. The coefficients a_n & b_n that we calculated decrease as $n \to \infty$.
3. Initial coefficient $a_0/2$ is the mean value of $f(t)$
4. Appears to be a slight overshoot at the points of discontinuity of the function $f(t)$

The above are common observations for Fourier series expansions with arbitrary functions $f(t)$.

Fourier Sine and Cosine Series

Our main usage for Fourier series will be in representing a function $f(x)$, over a finite interval $[0, L]$, e.g. the initial temperature in a heat conduction problem. It turns out that there are many possible ways to do this, depending on the particular function.

Odd and even functions:

Suppose that $f(x)$ is defined at $-x$ whenever it is defined at x

- The function $f(x)$ is an **even** function if $f(x) = f(-x)$. Examples: 1, x^2, x^{2n}, $|x|$, $cos\, x$
- The function $f(x)$ is an **odd** function if $f(x) = -f(-x)$. Examples: x, x^3, x^{2n+1}, $sin\, x$

Note: Most functions are neither odd nor even

Simple properties:

1. The sum (difference) and product (quotient) of 2 even functions is an even function
2. The sum (difference) of 2 odd functions is an odd function
3. The product (quotient) of 2 odd functions is an even function
4. The product (quotient) of an odd and an even function is an odd function
5. The sum (difference) of an odd and an even function is neither odd nor even

Integral properties:

1. If $f(x)$ is an even function then: $\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$
2. If $f(x)$ is an odd function then: $\int_{-L}^{L} f(x)\,dx = 0$

The form of the Fourier series for $f(x)$ is different, if $f(x)$ is an odd or an even function.

Fourier Cosine series: Assume that $\boldsymbol{f(x)}$ is piecewise differentiable on $[\boldsymbol{-L, L}]$ and $\boldsymbol{f(x)}$ is an even function. Then $\boldsymbol{f(x)}$ has Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

called the **Fourier cosine series**, with coefficients a_n given by:

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 0,1,2,3,\ldots$$

Fourier Sine series: Assume that $\boldsymbol{f(x)}$ is piecewise differentiable on $[\boldsymbol{-L, L}]$ and $\boldsymbol{f(x)}$ is an odd function. Then $\boldsymbol{f(x)}$ has Fourier series:

$$f(x) = \sum_{n=1}^{\infty} b_n \, cos \frac{n\pi x}{L}$$

called the **Fourier sine series**, with coefficients b_n given by:

$$b_n = \frac{2}{L} \int_0^L f(x) \, sin \frac{n\pi x}{L} dx \qquad n = 1,2,3, \ldots$$

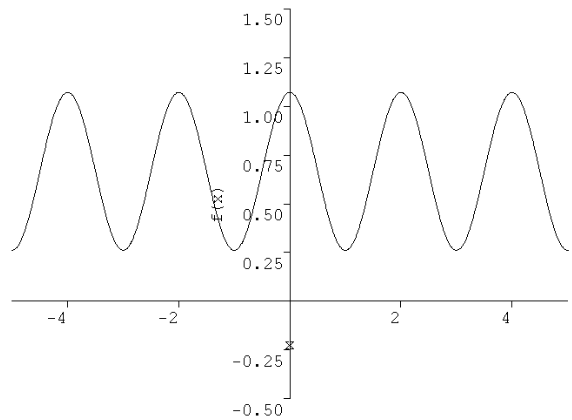
Example 2: Sketch the following functions $f(t)$ & find the Fourier series:

(a) $$f(t) = \begin{Bmatrix} -1 & -1 < t < 0 \\ 1 & 0 \le t \le 1 \end{Bmatrix} \quad f(t+2) = f(t)$$

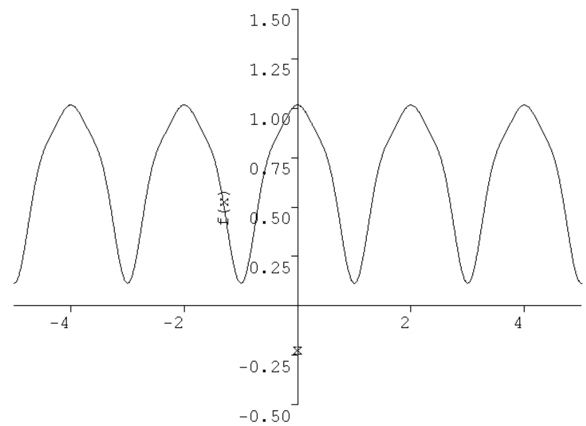
(b) $$f(t) = \begin{Bmatrix} -t & -2 < t < 0 \\ t & 0 \le t \le 2 \end{Bmatrix} \quad f(t+4) = f(t)$$

Example 3: Consider the function $f(t) = 1 - t^2$ for $-1 \leq t \leq 1$ with $f(t+2) = f(t)$. Find the Fourier series expansion and plot the k -th partial sums of the Fourier series for $k = 1,3,10,100$

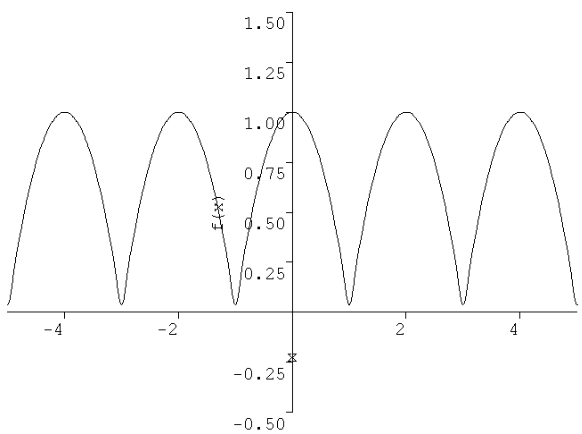
k=1



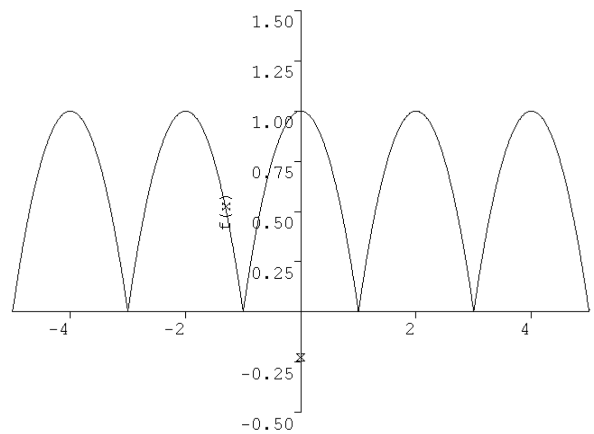
k=3



k=10



k=100



Example 4:

Find the Fourier series for $f(x) = x: \quad -L \leq x \leq L; \;\; f(x+2L) = f(x)$

Example 5:

Find the Fourier series for $f(x) = |x|: \quad -L \leq x \leq L; \;\; f(x+2L) = f(x)$

Suppose we wish to represent $f(x)$ on $[0, L]$, but don't care what form it has outside $[0, L]$. Many alternatives exist:

1. Use the Fourier cosine series. This series will converge to the function $g(x)$:
$$g(x) = \begin{cases} f(x) & 0 \le x \le L \\ f(-x) & -L < x < 0 \end{cases}$$
$$g(x + 2L) = g(x)$$
which is the even periodic extension of $f(x)$.
2. Use the Fourier sine series. This function will converge to the function $h(x)$:
$$h(x) = \begin{cases} f(x) & 0 < x < L \\ 0 & x = 0, L \\ -f(-x) & -L < x < 0 \end{cases}$$
$$h(x + 2L) = h(x)$$
which is the odd periodic extension of $f(x)$.
3. Define any function $k(x)$ that is piecewise differentiable on $[-L, L]$ and for which: $k(x) = f(x): \quad 0 \le x \le L$. Find the Fourier series for $k(x)$. Note that there are infinitely many choices for $k(x)$!

Factors affecting your choice of Fourier series representation:

- Speed of convergence. Generally, slow convergence results from discontinuities; the smoother the function, the faster the convergence.
- Sometimes the problem at hand dictates directly the choice

Chapter 9

Lecture 9

9.1 Recap

We learned how to write a function as a fourier series, in the following format:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L})$$

We have the following formulas:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})dx$$

9.2 Solving the Heat / Diffusion Equation

Examples are posted in the pdf slides posted.

9.2.1 Example 1

Solve the initial boundary value problem (IBVP)

$$\tfrac{\partial u}{\partial t} = \alpha \tfrac{\partial^2 u}{\partial x^2} \quad 0 < x < L; \quad t > 0 \tag{9.1}$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = f(x)$$

We need to use the method of separation of variables.

$$u(x,t) = X(x)T(t)$$

Taking the partial derivative with respect to t:

$$\rightarrow u_t = X(x)\dot{T}(t)$$

where dots are derivatives with respect to time.

$$u_x = X'(x)T(t)$$

$$u_{xx} = X''(x)T(t)$$

Now, we substitute this into the PDE equation 9.1

$$X\dot{T} = \alpha X''T$$

Now, we divide by αXT:

$$\frac{\dot{T}}{\alpha T} = \frac{X''}{X}$$

The left hand side of the equation is a function of t, and the right hand side is a function of x. In what condition are they equal?

The only way that theycan both be equal is if:

$$\frac{1}{\alpha}\frac{\dot{T}}{T} = \frac{X''}{X} = -\lambda \tag{9.2}$$

Where λ is a constant.

Now, let's work on boundary conditions:

$$u(0, t) = X(0)T(t) = 0$$

$$u(L, t) = X(L)T(t) = 0$$

$$\text{Hence, } X(0) = X(L) = 0$$

From 9.2, we get two equations:

- 1 - BVP
- 2 - IVP

BVP

$$\frac{X''}{X} = -\lambda \Rightarrow X'' + \lambda X = 0$$

with $X(0) = X(L) = 0$. This is a dirichlet boundary condition (BVP type P1)

The solution to P1:

$$X(x) = X_n(x) = C_n \sin(\frac{n\pi x}{L})$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, n \in \mathbb{N}$$

Therefore, there is a countably infinite set of $\lambda_n, X_n(x)$ as a solutiojn for the BVP. For each λ_n we find an IVP separately:

IVP

$$\frac{\dot{T}}{\alpha T} = -\lambda_n \longrightarrow T_n(t) = e^{-\lambda_n \alpha t}$$

is the solution to the IVP.

Summary

We found, for $n = 1, 2, 3, ...$ ($n \in \mathbb{N}$), we found:

$$u_n(x,t) = X_n(x)T_n(t) = C_n \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$$

This satisfies $u_t = \alpha u_{xx}$ with the conditions $u(0,t) = u(L,t) = 0$
Since the PDE and boundary conditions are homogeneous, we can superimpose solution, i.e.

$$C_k u_k + C_m u_m$$

also satisfies this problem for any constants of C_k and C_m.
Let's extend this idea to ∞:

$$u(x,t) = \sum_{n=1}^{\infty} C_n u_n(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$$

for constants $C_1, C_2, C_3, ...$
How abut initial conditions $u(x,0) = f(x)$?

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \tag{9.3}$$

How do we find C_n to meet this condition?
We need to find C_n such that $u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$ holds.
If we write $f(x)$ as a fourier sine series, we can match the coefficients.
Let's write $f(x)$ as a fourier sine series on $[0, L]$: i.e.

$$f(x) \approx \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}) \tag{9.4}$$

where $b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$

$\Rightarrow$ With comparing (3) and (4) $\rightarrow C_n = b_n$
Finally, the solution for IBVP is:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}) e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$$

- Homogeneous boundary conditions (Dirichlet: $u(0,t) = u(L,t) = 0$)
- Neumann: $u_x(0,t) = U_x(L,t) = 0$
- if it's not equal to 0 it's inhomogeneous
- If $u_t = \alpha u_{xx} + G$ it's inhomogeneous

9.2.2 Example 2

Same as example 1: $f(x) = x(L - x)$, $0 < x \leq L$
To solve, we use the method of separation of variables: $u(x,t) = X(x)T(t)$
Step 1: $u_t = X\dot{T}$; $U_x = X'T$; $U_{xx} = X''T$
Substitute into PDE and separate variables:

$$X\dot{T} = \alpha X''T \longrightarrow \frac{\dot{T}}{\alpha T} = \frac{X''}{X} = -\lambda$$

where λ is a constant.

Step 2: Boundary conditions.

$$u(0,t) = 0 \longrightarrow X(0) = 0$$
$$u(L,t) = 0 \longrightarrow X(L) = 0$$

Step 3: Solve the eigenvalue problem for $X(x)$:
$X'' + \lambda X = 0$, $X(0) = 0 = X(L)$
Hence, the solution:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

$$X_n = \sin(\frac{n\pi}{L}x)$$

where $n \in \mathbb{N}$
Step 4: For each λ_n find $T_n(t)$:

$$\frac{1}{\alpha}\frac{\dot{T}_n}{T_n} \rightarrow T_n(t) = e^{-\alpha\lambda_n t} = e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$$

Step 5: use superposition and linearity to construct a general series:

$$u(x,t) = \sum_{n=1}^{\infty} C_n u_n(x,t) = \underbrace{\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}}_{X_n(x)T_n(t)}$$

Step 6: Apply initial conditions:

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi x}{L}) = x(L-x)$$

Write $x(L-x)$ as a Fourier sine series:

$$x(L-x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L})$$

$$b_n = \frac{2}{L}\int_0^L x(L-x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = -\frac{2}{L}x(L-x)\frac{L}{n\pi}\cos(\frac{n\pi x}{L})\Big|_0^L + \frac{2}{n\pi}\int_0^L (L-2x)\cos(\frac{n\pi x}{L})dx$$

$$= 0 + \frac{2}{n\pi}\int_0^L (L-2x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2L}{(n\pi)^2}(L-2x)\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{4L}{(n\pi)^2}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{-4L^2}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L = \frac{4L^2}{(n\pi)^3}\left((-1)^{n+1}+1\right)$$

Step 7: Match the initial condition of the series solution ($C_n = b_n$)

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4L^2}{(n\pi)^3}\left((-1)^{n+1}+1\right)\sin\left(\frac{n\pi x}{L}\right)e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$$

Note that all $\sin(\frac{n\pi x}{L})$ terms are linearly independent (orthogonal)

9.2.3 Example 3

Similar to example 1 but with Neumann boundary conditions.

Please find the examples in the pdf "Heat / diffusion examples" on Canvas

Solution:

Step 1:

$$u(x,t) = X(x)T(t) \to \frac{\dot{T}}{\alpha T} = \frac{X''}{X} = -\lambda$$

Step 2:

$$u_x(0,t) = X'(0)T(t) = 0 \to X'(0) = 0$$

$$u_x(L,t) = X'(L)T(t) = 0 \to X'(L) = 0$$

Step 3: Solve the BVP with the conditions

$$X'' + \lambda X = 0$$

$$X'(0) = 0 = X'(L)$$

$\Rightarrow$ P2 problem. Cosine series.

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

and

$$X_n(x) = \cos(\frac{n\pi x}{L})$$

for $n \in \mathbb{N}$

and $\lambda_0 = 0; X_0(x) = 1$

Step 4: Solving the IVP

For each λ_n and X_n, there is a T_n such that

$$\frac{\dot{T}_n}{T_n} = -\alpha\lambda_n \to T_n = e^{-\alpha\lambda_n t}$$

Step 5:

For $\lambda_0 = 0 \to u_0(x,t) = 1$

For $\lambda_n = \left(\frac{n\pi}{L}\right)^2 \to u_n(x,t) = \cos\left(\frac{n\pi x}{L}\right) e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$

PDE is linear and homogeneous $\Rightarrow$ we may superimpose the solutions in a linear combination:

$$u(x,t) = \sum_{n=0}^{\infty} d_n u_n(x,t) = d_0 + \sum_{n=1}^{\infty} d_n \cos\left(\frac{n\pi x}{L}\right) e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$$

Step 6: Initial conditions

$$u(x,0) = f(x) = d_0 + \sum_{n=1}^{\infty} d_n \cos\left(\frac{n\pi x}{L}\right)$$

If we rewrite as a fourier series (cosine), we find that $d_0 = a_0/2$ and that $d_n = a_n$

if we take the even extension of $f(x)$, to $[-L, 0]$ interval, then we know $f(x)$ has a fourier cosine series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow d_0 = \tfrac{a_0}{2}; \quad d_n = a_n = \tfrac{2}{L}\int_0^L f(x)\cos\left(\tfrac{n\pi x}{L}\right)dx$$

Step 7:
Thus, the solution is

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$$

Chapter 10

Lecture 10

10.1 Recap of Last Lecture

Last class, we solved homogeneous heat / diffusion PDEs. We used the method of separation of variables, where we assumed $u(x,t)$ can be separated into two functions X_x and T_t, such that $u(x,t) = X_x T_t$:

We then substituted that into the PDE, getting two equations; IVP and a BVP. We represented the solutions as a superposition of solutions for each eigenvalue and eigenfunction.

We then found the coefficients of the Fourier series, or the series of the solution, by writing the initial condition in terms of Fourier series.

Then we can match up the coefficients and match up the coefficients of the Fourier series into the solution that we found for the PDE.

Today, we will continue to do more examples.

10.2 Examples

10.2.1 Example 4

Note that these are the continued examples from the pdf file, named in last class. 1-3 are from last class also.

$$u_t = 0.003 u_{xx} \quad 0 < x < 1; \quad t > 0$$

$$u_x(0,t) = u_x(1,t) = 0 \quad t > 0$$

$$u(x,0) = 50x(1-x) \quad 0 \le x \le 1$$

How long does it take for $u(0.5,t)$ to obtain its steady-state value, with 1% error?

The solution must be of this form: Note that this is a Neumann boundary condition.

$$u(x,t) = d_0 + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{L}\right) e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t} \tag{10.1}$$

Given that $\alpha = 0.003$, $L = 1$, and that $f(x) = 50x(1-x)$, we substitute:
We need to write the Fourier cosine series for $f(x)$:

$$a_0 = \frac{2}{1}\int_0^1 50x(1-x)dx = 100\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{100}{6}$$

$$a_n = 2 \cdot 50 \int_0^1 x(1-x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{100}{n\pi}x(1-x)\sin\left(\frac{n\pi x}{1}\right)\Big|_0^1 - \frac{100}{n\pi}\int_0^1 (1-2x)\sin(n\pi x)dx$$

$$a_n = \frac{100}{(n\pi)^2}(1-2x)\cos(n\pi x)\Big|_0^1 + \frac{200}{(n\pi)^2}\int_0^1 \cos(n\pi x)dx$$

$$a_n = \frac{100}{(n\pi)^2}\left((-1)^{n+1}-1\right)$$

Hence:

$$f(x) = \frac{100}{6} + \sum_{n=1}^{\infty}\frac{100}{(n\pi)^2}\left[(-1)^{n+1}-1\right]\cos(n\pi x)$$

$d_0 = \frac{50}{6}$ and $d_n = a_n$. This we substitute into (1) and simplify.

$$u(x,t) = \frac{50}{6} + \frac{100}{\pi^2}\sum_{n=1}^{\infty}\frac{\left[(-1)^{n+1}-1\right]}{n^2}\cos(n\pi x)e^{-0.003(n\pi)^2 t}$$

As $t \to \infty$, $u(x,t) \to \frac{50}{6}$. This is the steady-state solution. This is also the mean value of the initial condition. (why?)

(n = 2)

At $x - 0.5$: $u(x,t) - u(x,\infty) \approx \frac{50}{\pi}e^{-0.0012\pi^2 t}$, which is $< \frac{1}{100}(\frac{50}{6})$

If $0.0012\pi^2 t > \ln(\frac{600}{\pi^2}) \Rightarrow t > \frac{1}{0.012\pi^2}\ln\left(\frac{600}{\pi^2}\right) = 34.68$s

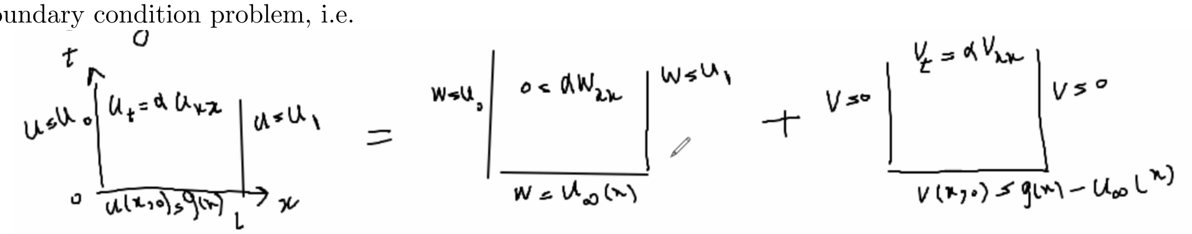
The plot for the general solution is on the pdf file.

10.3 Inhomogeneous Equations

In the previous examples, we solved heat / diffusion equations using separation of variables for homogeneous boundary conditions (Neumann and Dirichlet). Now, we are moving on to inhomogeneous equations.

Either eq: $\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2} + g(x,t)$ or a boundary condition of $u(0,t) = a(t)$ (boundary condition is time dependent), or $u(L,t) = C \neq 0$.

The idea is to solve these problems by decomposing it into a steady state problem and a homogeneous boundary condition problem, i.e.



Middle portion is the steady-state problem, and the right hand side is the transient problem with homogeneous boundary conditions.

The general solution is thus:

$$u(x,t) = w(x) + v(x,t)$$

10.3.1 Example 6

$$u_t = \alpha u_{xx} \quad 0 < t < 1; \quad t > 0$$

$u(0,t) = 1$ and $u(1,t) = 3$ for $t > 0$ and $u(x,0) = x(1-x)$ for $0 \leq x \leq 1$

Step 1

We need to decompose the solution into two parts:

$$u(x,t) = \underbrace{u_s(x)}_{\text{Steady-state}} + \underbrace{v(x,t)}_{\text{Transient}}$$

Step 2

$$0 = \alpha \frac{\partial^2 u_s}{\partial x^2}$$

$u_s(0) = 1$ and $u_s(1) = 3 \to u_s = Ax + B$
$u_s(0) = 1 \to B = 1$
$u_s(1) = 3 \to A = 2$
Therefore $u_s = 2x + 1$

Step 3

Formulate $v(x,t)$ and find boundary conditions and initial conditions.
Boundary conditions:
$v(0,t) = u(0,t) - u_s(0) = 0$
$v(1,t) = u(1,t) - u_s(1) = 0$
Initial conditions:

$$v(x,0) = u(x,0) - u_s(x) = x(1-x) - (2x+1) = -(x^2 + x + 1)$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(u_s(x) + v(x,t)\right) = \frac{\partial v}{\partial t}$$

$$\alpha \frac{\partial^2 u}{\partial x^2} = \alpha \frac{\partial^2}{\partial x^2}\left(u_s(x) + v(x,t)\right) = \alpha \frac{\partial^2 v}{\partial x^2}$$

$$\Rightarrow \frac{\partial v}{\partial t} = \alpha \frac{\partial^2 v}{\partial x^2}$$

note: Similar to example 1 and 2, it is a standard (homogeneous) Dirichlet problem.

Step 4

Solve the transient problem.

$$v(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-\alpha(n\pi)^2 t}$$

(Refer to example 1 and 2)

$$b_n = \frac{-2}{1}\int_0^1 (1 + x + x^2)\sin(n\pi x)dx = \frac{2}{n\pi}\left[3(-1)^n - 1\right] - \frac{4}{(n\pi)^3}\left[(-1)^n - 1\right]$$

$$\Rightarrow v(x,t) = \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\left[3(-1)^n - 1\right] - \frac{4}{(n\pi)^3}\left[(-1)^n - 1\right]\right)\sin(n\pi x) e^{-\alpha(n\pi)^2 t}$$

Step 5

Sum the steady state and transient parts of the solution.

$$u(x,t) = u_x(x) + v(x,t)$$

$$u(x,t) = 1 + 2x + \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\left[3(-1)^n - 1\right] - \frac{4}{(n\pi)^3}\left[(-1)^n - 1\right]\right)\sin(n\pi x)e^{-\alpha(n\pi)^2 t}$$

10.3.2 Example 7

Inhomogeneous equation and boundary conditions

Equation is given to be the heat equation:

$$\rho c_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + Q(x)$$

Q is a source (or sink) of heat.

Let $p = 8940\frac{kg}{m^3}$, $c_p = 914\frac{J}{kg\cdot C}$, $k = 930\frac{W}{m\cdot C}$ $L = 10m$

Boundary conditions:

$$T(10,t) = 30$$
$$T(0,t) = 30$$

Initial conditons:

$$T(x,0) = 30$$

$$Q(x) = 80000x$$

Solution:

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2} + q(x)$$

$$\alpha = \frac{k}{\rho c_p} \approx 10^{-4}$$

$$q(x) = \frac{Q(x)}{\rho c_p} = 0.01x$$

We need to write $u = T(x,t)$

Step 1

We need to divide into two parts:

$$u(x,t) = \underbrace{u_s(x)}_{\text{Steady-state}} + \underbrace{v(x,t)}_{\text{Transient}}$$

Step 2

$$0 = \alpha \frac{\partial^2 u_s}{\partial x^2} + q(x)$$

(note that the boundary conditions are $u_s(0) = 30 = u_s(10)$

$$\frac{\partial^2 u_s}{\partial x^2} + 100x = 0 \longrightarrow \frac{\partial u_s}{\partial x} + 50x^2 = C_1$$

$$u_s = -\frac{50}{3}x^3 + C_1 x + C_2$$

Given boundary conditions: $u_s(0) = 30 \longrightarrow C_2 = 30$ and $u_s(10) = 30 \longrightarrow C_1 = \frac{10^4}{6}$

$$\Rightarrow u_s(x) = -\frac{50}{3}x^3 + \frac{10^4}{6}x = 30$$

10.3.3 Step 3

Formulate the transient part:
Boundary conditions:

$$\begin{cases} v(0,t) = u(0,t) - u_s(0) = 0 \\ v(10,t) = u(10,t) - u_s(10) = 0 \end{cases}$$

Initial conditions:

$$v(x,0) = 30 - u_s(x) = \frac{10^2}{6}x(x^2 - 100)$$

Double check these!!
Now, we need to solve for $v(x,t)$, which satisfies a standard Dirichlet problem.

Step 4

$$v(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{10}x\right) e^{-10^{-6}(n\pi)^2 t}$$

$$b_n = \frac{2}{10}\int_0^{10} \frac{10^2}{6}x(x^2 - 100)\sin(\frac{n\pi}{10}x)dx$$

Step 5

$$u(x,t) = u_s + v(x,t)$$

Chapter 11

Lecture 11

11.1 Introduction

- Check Canvas announcements regarding midterm and such (2 new announcements)

11.2 Recap of Last Lecture

Last lecture, we finished up the heat / diffusion equation.

- Homogeneous equations and boundary conditions (Neumann and Dirichlet)
- Inhomogeneous equations and boundary conditions
- Developed general strategies for splitting inhomogeneous equations into a steady state (Which takes care of inhomogeneous parts, incl. boundary conditions) and a transient part (Satisfying a classic homogeneous diffusion equation)

Today's lecture is about a new equation: Wave equation.
Two methods to solve:

- Applying separation of variables
- We'll talk about the second method tomorrow

11.3 Wave equation

Takes the following form:

$$y_{tt} = a^2 y_{xx}$$

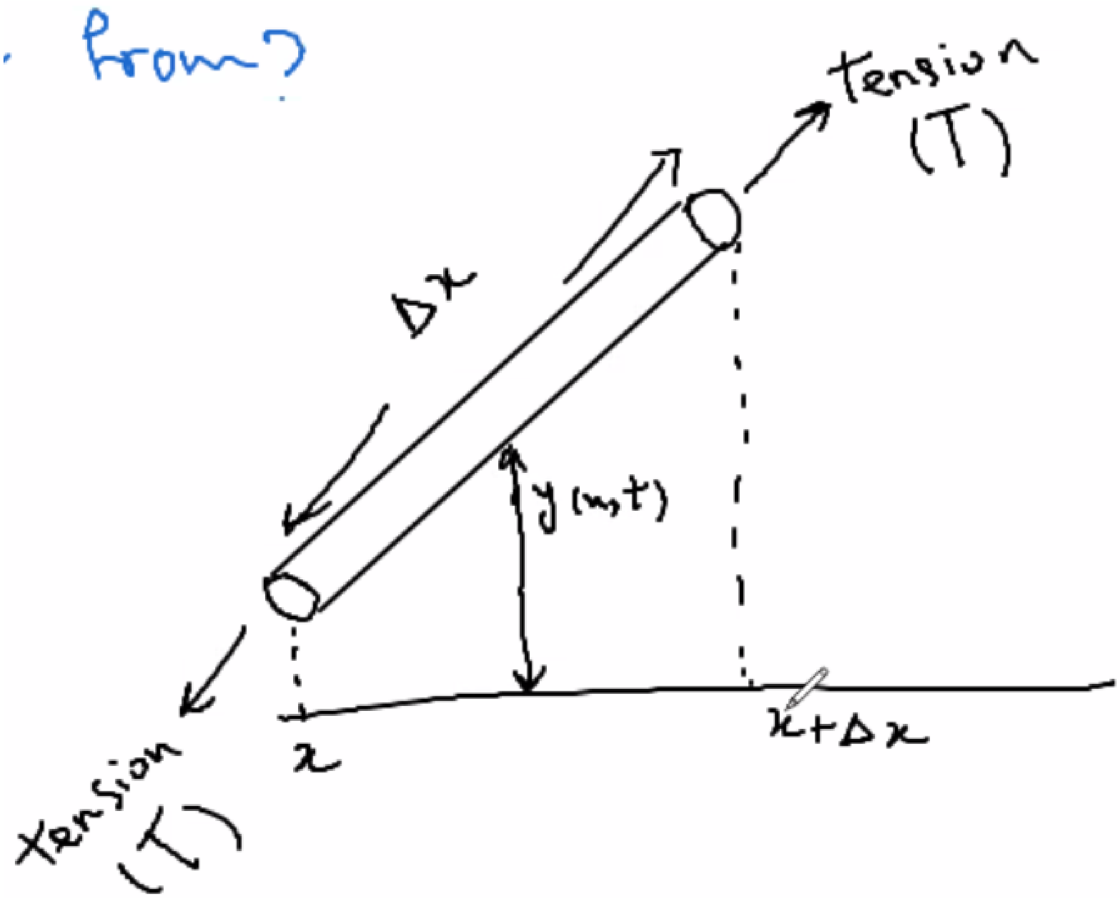
Two derivatives with respect to time, and two derivatives with respect to x.

Physically, $a = \left[\frac{T}{\rho}\right]^{\frac{1}{2}}$ (A string under tension), where T is stress / tension, and ρ is density. Can also be written as $a = \left[\frac{E}{\rho}\right]^{\frac{1}{2}}$ (Elastic bar), where $E =$ elastic stress.

Boundary conditions: We have two x derivatives $\rightarrow$ 2 conditions needed.

Initial conditions: We also need 2 initial conditions (Because we have 2 time derivatives). This is the main difference between the wave equation and the heat equation. $y(x, 0) = C$ (Initial displacement), and $y_t(x, 0) = k$ (initial velocity)

Where did the wave equation come from?



Derivation for small $\left|\frac{\partial y}{\partial x}\right|$
String of density ρ (with units of $\left[\frac{kh}{m}\right]$)
Force balance in the y direction:

$$\rho \Delta x \frac{\partial^2 y}{\partial t^2} = m \cdot \vec{a} = \sum F_y$$

$$\rho \Delta x \frac{\partial^2 y}{\partial t^2} = T \frac{\partial y}{\partial x}(x + \Delta x, t) - T \frac{\partial y}{\partial x}(x, t)$$

Divide by Δx and let $\Delta x \to 0$:

$$\Rightarrow \rho \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2}$$

N.B length of element is $\Delta x(1 + (\frac{\partial y}{\partial x})^2)^{\frac{1}{2}} \approx \Delta x$

Wave equation:

$$y_{tt} = a^2 y_{xx}$$

BC: $y(0, t) = y(L, t) = 0$
IT: $y(x, 0) = f(x)$ and $y_t(x, 0) = g(x)$

The idea is to split the solution into two parts.

- Problem 1: Initial velocity, but no displacement of string
  - $w_{tt} = a^2 w_{xx}$
  - $w(0,t) = w(L,t)$
  - $w(x,0) = 0$ for $0 < x < L$
  - $w_t(x,0) = g(x)$ for $0 < x < L$
- Problem 2: Initial displacement, but no velocity of spring
  - $z_{tt} = a^2 z_{xx}$
  - $z_t(x,0) = f(x)$ for $0 < x < L$
  - $z_t(x,0) = 0$ for $0 < x < L$

Solve problems 1 and 2: $y(x,t) = w(x,t) + z(x,t)$

11.3.1 Step 1: Solving Problem 1

$$w(x,t) = X(x)T(t) \quad \underbrace{\Rightarrow}_{\text{Substitute into PDE}} \quad X\ddot{T} = a^2 X'' T$$

Divide by a^2XT:

$$\Rightarrow \frac{1}{a^2}\frac{\ddot{T}}{T} = \frac{X''}{X} = -\lambda$$

where λ is a constant.
First, boundary value problem:

$$X'' + \lambda X = 0$$

Boundary conditions are $w(0,t) = X(0)T(t) \Rightarrow X(0) = 0$, and $w(L,t) = X(L)T(t) = 0 \Rightarrow X(L) = 0$
It is a P1 eigenvalue problem.
Therefore, the eigenvalue problem for $X(x)$ is exactly as for heat / diffusion equation.
Therefore:

$$\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

and

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

where $n \in \mathbb{N}$ (Natural numbers; 1,2,3,...)
IVP is:

$$\frac{1}{a^2}\frac{\ddot{T}_n}{T_n} = -\lambda_n \Rightarrow \ddot{T}_n + \left(\frac{an\pi}{L}\right)^2 T_n = 0$$

Therefore:

$$T_n(t) = A_n \cos\left(\frac{an\pi}{L}t\right) + B_n \sin\left(\frac{an\pi}{L}t\right)$$

How about the initial condition?
$w(x,0) = 0 \longrightarrow X(x)T(0) = 0 \Rightarrow T(0) = 0$ and $w_t(x,0) = g(x)$
$T_n(0) = 0 \Rightarrow A_n = 0$

As a result:

$$T_n(t) = B_n \sin\left(\frac{an\pi}{L}t\right)$$

Now, note that PDE, boundary conditions, and $w(x,0)$ are homogeneous. Therefore, we can superimpose solutions.

$$w(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{an\pi}{L}t\right)$$

We need to find B_n. To find this, we use the second initial condition.

$$w_t(x,0) = \sum_{n=1}^{\infty} B_n \left(\frac{an\pi}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{an\pi}{L}t\right)$$

$$w_t(x,0) = \sum_{n=1}^{\infty} B_n \left(\frac{an\pi}{L}\right) \sin\left(\frac{n\pi x}{L}\right) = g(x)$$

If we write a Fourier sine series for $g(x)$, we can match up the coefficients:
To make this work, represent $g(x)$ as a Fourier sine series on $[0, L]$:

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \Rightarrow b_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

We know this series converges, so we match up the coefficients.

$$B_n = b_n \tfrac{L}{n\pi a} \text{ for } n \in \mathbb{N}.$$

$$\Rightarrow w(x,t) = \sum_{n=1}^{\infty} b_n \frac{L}{n\pi a} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi a t}{L}\right)$$

11.3.2 Step 2: Solving Problem 2

$$z_{tt} = a^2 z_{xx}$$

Initial boundary conditions: $z(0,t) = Z(L,t) = 0$ and $Z(x,0) = f(x)$; $z_t(x,0) = 0$ for $0 < x < L$
Solution: Similarly, we use separation of variables and we assume that z is a product of X and T:

$$z(x,t) = X(x)T(t)$$

(Note that these are different X and T than in step 1!!)
The solution for the eigenvalue problem is a P1 problem again.

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

$$\lambda_n = \left(\tfrac{n\pi}{L}\right)^2 \text{ for } n \in \mathbb{N}$$

For the IVP part,

$$\ddot{T}_n + \left(\frac{n\pi a}{L}\right)^2 T_n = 0$$

$$\Rightarrow T_n(t) = A_n \cos\left(\frac{n\pi a}{L}t\right) + B_n \sin\left(\frac{n\pi a}{L}t\right)$$

Now, $z_t(x,0) = X(x)\dot{T}(0) = 0 \Rightarrow \dot{T}_n(0) = 0 \Rightarrow B_n = 0$
So, let's superimpose the solution:

$$z_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi a}{L}t\right)$$

Use the other initial condition and find A_n:

The other initial condition tells us:

$$f(x) = z(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

Suppose we compute the Fourier sine series for $f(x)$, Then,

$$b_n' = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Note that prime is NOT a derivative, just used to denote that it's a different b_n.

$$\Rightarrow z(x,t) = \sum_{n=1}^{\infty} b_n' \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi a}{L}t\right)$$

11.3.3 Step 3

$$y(x,t) = x(x,t) + z(x,t)$$

$$y(x,t) = \sum_{n=1}^{\infty}\left[b_n \frac{L}{n\pi a}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi at}{L}\right) + b_n' \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi a}{L}t\right)\right]$$

We can factor[1]:

$$y(x,t) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[\frac{b_n L}{n\pi a}\sin\left(\frac{n\pi a}{L}t\right) + b_n' \cos\left(\frac{n\pi a}{L}t\right)\right]$$

For Neumann boundary conditions, the procedure is exactly the same as Dirichlet boundary conditions. (Using the PDF file posted on Canvas – Wave Equations, under week 4. Posted at the bottom of this document.)

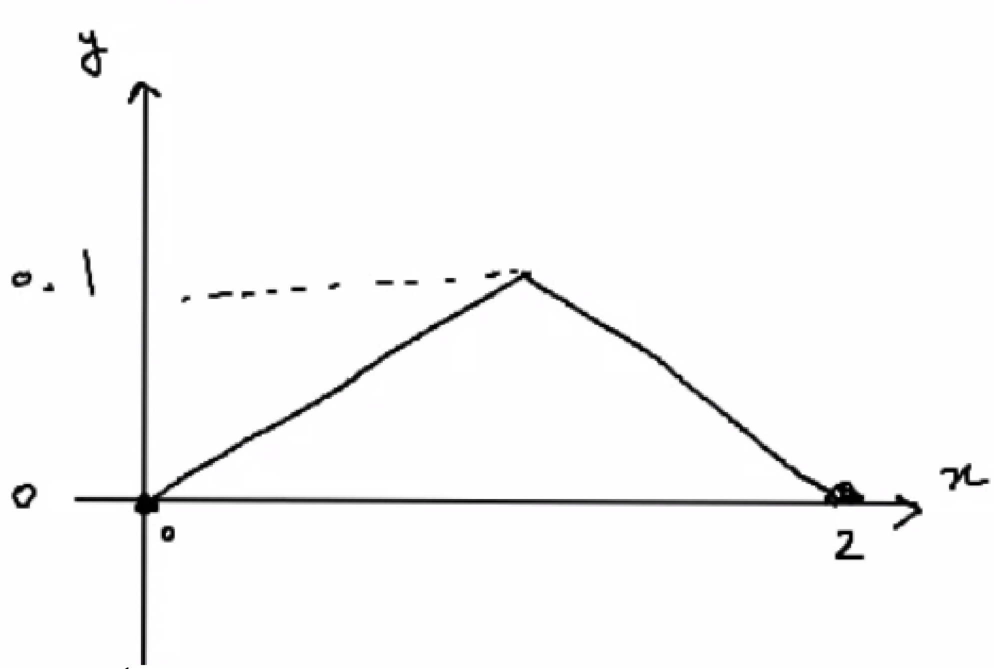
11.4 Example 8

(Example 8 of the pdf)

Solve the IBVP $y_{tt} = y_{xx}$

Initial conditions: $y(0,t) = y(2,t)$

[1]Again, b_n' is not a derivative!



$y(x,0) = \begin{cases} 0.1x & 0 \leq x \leq 1 \\ 0.1(2-x) & 1 \leq x \leq 1 \end{cases}$
$y_t(x,0) = 0 = g(x) \Rightarrow b_n = 0 \ \forall n$
Solution: a = 1 and L = 2. (There is only initialdisplacement $z(x,t)$ problem)

$$b'_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx = 0.1\int_0^1 x\sin\left(\frac{n\pi x}{2}\right) + 0.1\int_1^2 (2-x)\sin\left(\frac{n\pi x}{2}\right)$$

$$b'_n = -\frac{0.2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + 0 + \frac{0.4}{(n\pi)^2}\cdot\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1 - 0 + \frac{0.2}{n\pi}\cos(\frac{n\pi}{2}) - \frac{0.4}{(n\pi)^2}\sin\left(\frac{n\pi x}{2}\right)\Big|_1^2$$

$$b'_n = \frac{0.8}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

$$y(x,t) = \sum_{n=1}^{\infty}\frac{0.8}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{2}\right)\cos\left(\frac{n\pi}{2}t\right)$$

N.B[2]:
$\sin(\frac{n\pi}{2}) = 1, 0, -1, 0, 1, 0, ...$ for $n \in \mathbb{N}$
Thus, we could write $n = 2k-1$ and $b'_k = (-1)^{k+1}$ for $k \in \mathbb{N}$

$$y(x,t) = \sum_{k=1}^{\infty}\frac{0.8(-1)^{k+1}}{(2k-1)^2\pi^2}\sin\left(\frac{(2k-1)\pi}{2}x\right)\cos\left(\frac{(2k-1)\pi}{2}t\right)$$

11.5 Example 9

$$y_{tt} = y_{xx}$$

$y(0,t) = 0$ and $y(1,t) = 0$
$y(x,0) = 0$ and $y_t(x,0) = \sin(5\pi x)$
Solution: a = 1 and L = 1 and $f(x) = 0 \Rightarrow$ we only have the velocity problem to solve.
Need to find the Fourier sine series for $g(x)$:

[2]Nota bene. Used to denote an important point.

$$\sum_{n=1}^{\infty} b_n \sin(n\pi x) = g(x) = \sin(5\pi x)$$

But this is already a fourier series. For any n value $\neq 5$, $b_n = 0$. $b_5 = 1$.
Therefore, the general solution

$$y(x,t) = \frac{1}{5\pi} \sin(5\pi x) \sin(5\pi t)$$

11.6 Wave Equations PDF

Wave Equation

The wave equation takes the form:

$$\boldsymbol{y_{tt} = a^2 y_{xx}}$$

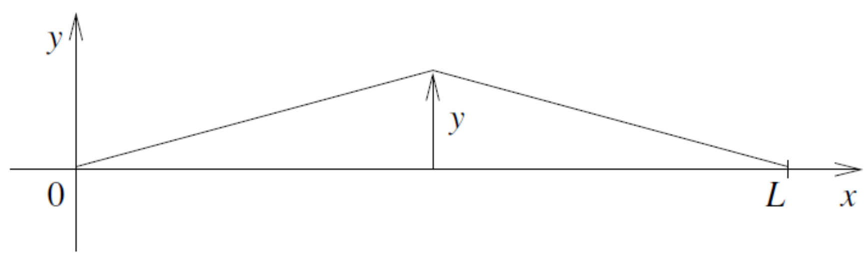
Physically, $a = [T/\rho]^{0.5}$ (a string under tension) or $a = [E/\rho]^{0.5}$ (elastic bar)

Boundary conditions?

Initial conditons?

Typical IBVP for the wave equation looks like this:

$$\begin{aligned}
&y_{tt} = a^2 y_{xx} \\
&y(0,t) = 0, \quad y(L,t) = 0, \\
&y(x,0) = f(x) \\
&y_t(x,0) = g(x)
\end{aligned}$$



Superposition and separation of variables

1. Split *y(x,t)* and the initial "data" into 2 problems:

$$y(x,t) = w(x,t) + z(x,t)$$

Problem 1: initial velocity, but no displacement of string

$$\begin{aligned} &w_{tt} = a^2 w_{xx}, && \\ &w(0,t) = w(L,t) = 0, && \\ &w(x,0) = 0 && \text{for } 0 < x < L, \\ &w_t(x,0) = g(x) && \text{for } 0 < x < L. \end{aligned}$$

Problem 2: initial displacement, but no velocity of string

$$\begin{aligned} &z_{tt} = a^2 z_{xx}, && \\ &z(0,t) = z(L,t) = 0, && \\ &z(x,0) = f(x) && \text{for } 0 < x < L, \\ &z_t(x,0) = 0 && \text{for } 0 < x < L. \end{aligned}$$

2. Solve each problem by separation of variables

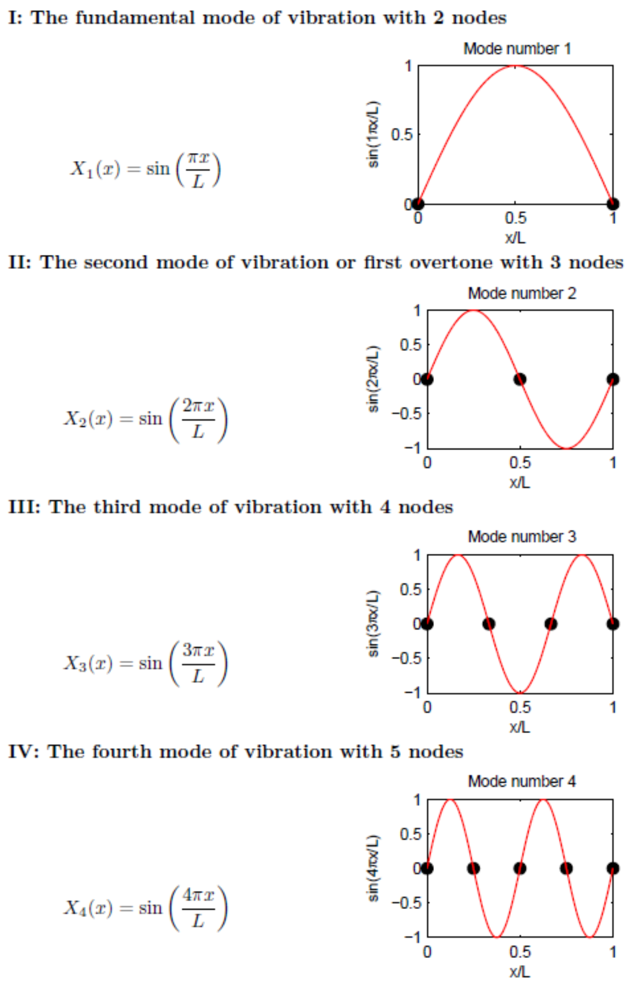
Exactly analogous procedure for Neumann boundary conditions

Period and frequency of the nth mode:

Modes of vibration:

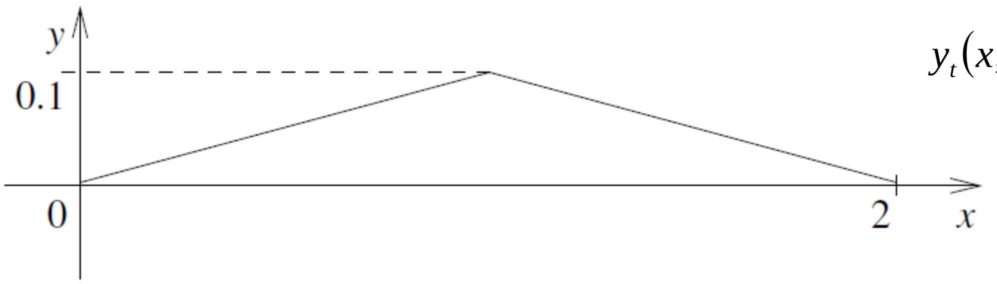
- Note these are standing waves of wavelength $\lambda_n = 2L/n$

- Each mode: $n+1$ positions at which displacement is zero



Example 8: Solve the IBVP

$$y_{tt} = y_{xx}$$
$$y(0,t) = 0, \quad y(2,t) = 0,$$
$$y(x,0) = \begin{cases} 0.1x & 0 \le x \le 1 \\ 0.1(2-x) & 1 < x \le 2 \end{cases}$$
$$y_t(x,0) = 0$$



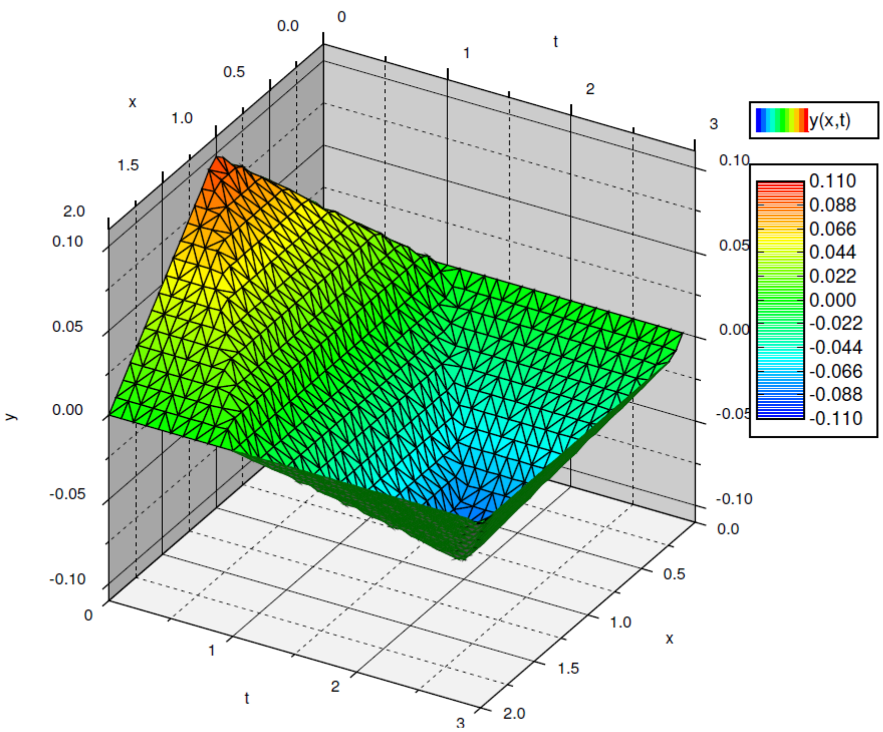


Figure 4.20: Shape of the plucked string for $0 < t < 3$.

Example 9: Solve the IBVP – what makes this one simple?

$$
\begin{aligned}
&y_{tt} = y_{xx} \\
&y(0,t) = 0, \quad y(1,t) = 0, \\
&y(x,0) = 0 \\
&y_t(x,0) = sin\, 5\, \pi x
\end{aligned}
$$

Wave Equation with Neumann boundary condition:

$$
\begin{aligned}
&y_{tt} = a^2 y_{xx} \\
&y_x(0,t) = 0, \quad y_x(L,t) = 0, \\
&y(x,0) = f(x) \\
&y_t(x,0) = g(x)
\end{aligned}
$$

D'Alembert's solution:

$$y_{tt} = a^2 y_{xx}$$

Return to wave equation and see if we can guess a solution of exponential form:

$$y(x,t) = e^{ikx+\sigma t}$$

Why this form?

$$y_1(x,t) = e^{ik(x+at)}$$
$$y_2(x,t) = e^{ik(x-at)}$$

Is this form of solution more general – how about:

$$y_1(x,t) = F(x-at), \qquad y_2(x,t) = G(x+at)$$

Consider a change of variables: $\xi = x - at$, $\eta = x + at$

Suppose initial conditions:

$$y(x,0) = f(x)$$
$$y_t(x,0) = g(x)$$

Finally, D'Alembert's solution:

$$y(x,t) = \frac{1}{2}[f(x-at) + f(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds$$

Above analysis has no boundary conditions!

Let $F_o(x)$ and $G_o(x)$ be the odd 2L-periodic extensions of *f(x)* and *g(x)*, respectively.

$$y(x,t) = \frac{1}{2}[F_o(x-at) + F_o(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} G_o(\zeta)\, d\zeta$$

What is the relationship between d'Alembert's formula and our separation of variables solution?

$$\boxed{\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} b_n \frac{L}{n\pi a} \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{n\pi a}{L} t\right) + c_n \sin\left(\frac{n\pi}{L} x\right) \cos\left(\frac{n\pi a}{L} t\right) \\ &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L} x\right) \left[b_n \frac{L}{n\pi a} \sin\left(\frac{n\pi a}{L} t\right) + c_n \cos\left(\frac{n\pi a}{L} t\right)\right]. \end{aligned}}$$

Region of influence & domain of dependence:

Example 10: Solve the following IVP using D'Alembert's method

$$y_{tt} = y_{xx}, \qquad -\infty < x < \infty$$
$$y(x,0) = \begin{cases} 1, & |x| < 1 \\ 0, & \text{otherwise} \end{cases}$$
$$y_t(x,0) = 0$$

Example 11: Solve the following IVP using D'Alembert's method

$$y_{tt} = y_{xx},$$
$$y(0,t) = 0, \quad y(1,t) = 0,$$
$$y(x,0) = \begin{cases} 0, & 0 \le x < 0.45 \\ 20(x - 0.45), & 0.45 \le x < 0.5 \\ 20(0.55 - x), & 0.5 \le x < 0.55 \\ 0, & 0.55 \le x \le 1 \end{cases}$$
$$y_t(x,0) = 0$$

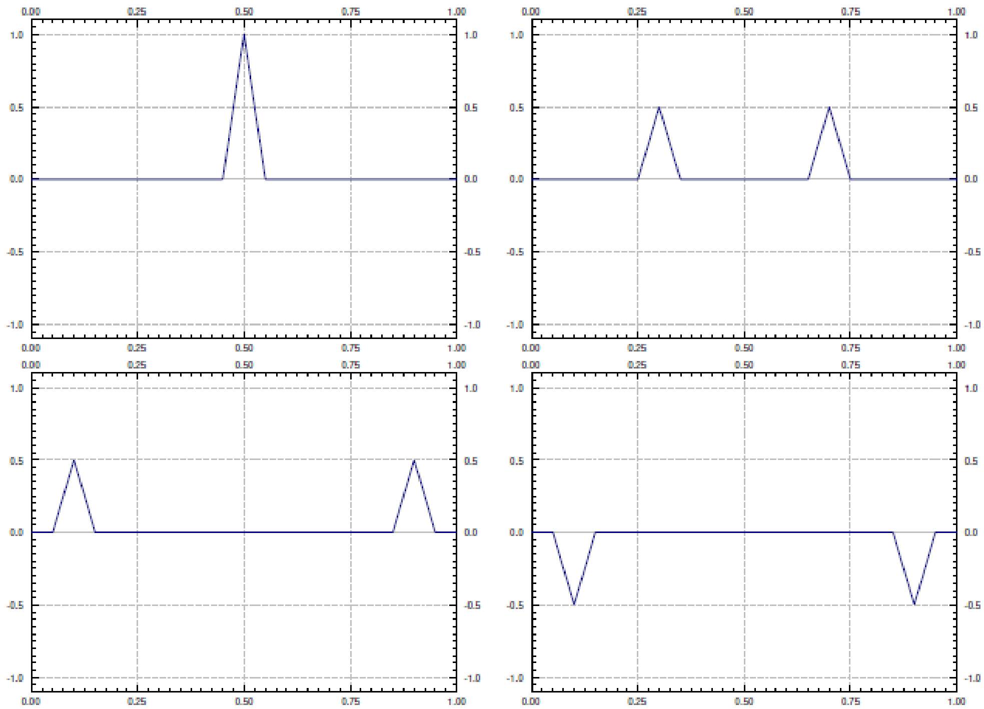


Figure 4.21: Plot of the d'Alembert solution for $t = 0$, $t = 0.2$, $t = 0.4$, and $t = 0.6$.

Chapter 12

Lecture 12

12.1 Wave Equations: Neumann Boundary Conditions

Solve IBVP: $y_{tt} = a^2 y_{xx}$

Boundary conditions: $y_x(0,t) = y_x(L,t) = 0 \Rightarrow$ Homogeneous Neumann boundary conditions.

Initial conditions: $y(x,0) = f(x)$, $y_t(x,0) = 0 = g(x)$ (Therefore zero initial velocity, with specified initial displacement).

Split into two problems (w and z). Because $g(x) = 0$, we only have the z equation. For the solution, we use separation of variables. [1]

$X'' + \lambda X = 0$	$\ddot{T} + a^2 \lambda T = 0$
$\Rightarrow X_n(x) = \cos\left(\frac{n\pi x}{L}\right)$	$\dot{T}_n(0) = 0$
$\lambda_n = \left(\frac{n\pi}{L}\right)^2$	$T_n(t) = A_n \cos\left(\frac{n\pi a}{L} t\right)$
$X_0(x) = 1 \Rightarrow \lambda_0 = 0$	$T_0(t) = 1$

The solution would be:

$$y(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi a}{L} t\right)$$

Note that A_0 is the multiplication of $X_0 T_0$ terms.

$$\text{At } t = 0 \longrightarrow y(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) = f(x)$$

We construct Fourier cosine series for $f(x)$.

$$f(x) = \frac{a_0}{2} + \sum n = 1^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow A_0 = \frac{a_0}{2}, A_n = a_n$$

Note: We can do a similar solution if initial velocity is given.

12.1.1 Recap

- Introduced wave equation
- Developed separation of variables method to find its solution

[1] Note that left and right side of table are entirely separate.

- Dirichlet and Neumann boundary conditions
- Examples and normal modes

Now: New method.

- New look at the wave equation and we solve the wave equation using **D'Alembert's solution**.

12.2 D'Alembert's Solution

$$y_{tt} = a^2 y_{xx}$$

Let's see if we can guess a solution of exponential format.

$$y(x,t) = e^{ikx+\sigma t}$$

where k and σ are constants. [2]
Substitute the guessed solution into the PDE.

$$y_t t = \sigma^2 e^{ikx+\sigma t}$$

$$y_{xx} = -k^2 e^{ikx+\sigma t}$$

Now, substitute this into the PDE:

$$\left(\sigma^2 + a^2 k^2\right) e^{ikt+\sigma t} = 0$$

$$\Rightarrow \sigma = \pm ika$$

$$y_1(x,t) = e^{ik(x-at)}$$

$$y_2(x,t) = e^{ik(x+at)}$$

$x \pm at$ are known as characteristics, these are lines in x and t along which the initial conditions (and general information) travels.

The question here is this: Can this form of solution be more general such that we can apply it to any wave equation?

$$y_1(x,t) = F(x-at), y_2(x,t) = G(x+at)$$

Can we find a general equation that satisfies the wave equation?
Hence, a general solution:

$$y(x,t) = F(x-at) + G(x+at)$$

Does it satisfy the PDE?

$$\begin{aligned} y(x,0) = f(x) \quad &\Rightarrow \quad F(x) + G(x) = f(x) \quad (1) \\ y_t(x,0) = g(x) \quad &\Rightarrow \quad -aF'(x) + aG'(x) = g(x) \quad (2) \end{aligned}$$

We get (2) from:

$$-aF(x) + aG(x) = \int_0^x g(s)ds + A$$

[2] Try this guess solution with heat solution! You will find that it does work for heat equations.

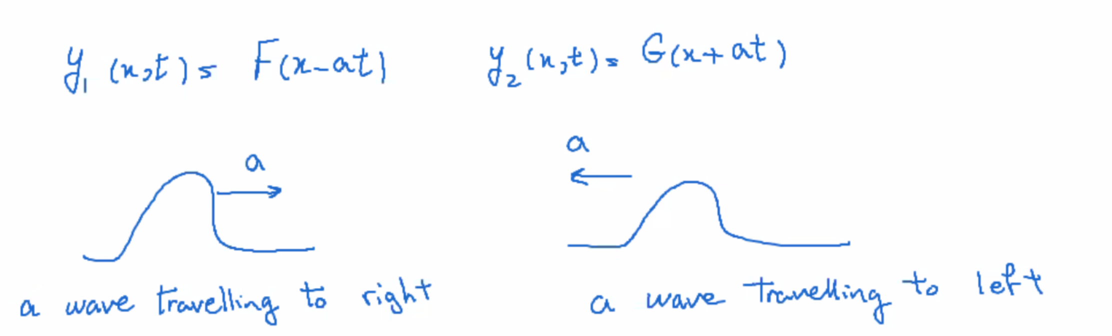


Figure 12.1: $F(x-at)$ is a wave travelling to the right with a speed of a. $G(x+at)$ is a wave travelling to the left with a speed of a.

$$(1)xa + 2 \Rightarrow 2aG(x) + f(x) = \int_0^x g(s)ds + A$$

$$\Rightarrow G(x) = \frac{1}{2}f(x) + \frac{1}{2a}\int_0^x g(s)ds + \frac{A}{2a}$$

To find $F(x)$:

$$(1)xa - (2) \Rightarrow 2aF(x) = af(x) - \int_0^x g(s)ds - A$$

$$\Rightarrow F(x) = \frac{1}{2}f(x) - \frac{1}{2a}\int_0^x g(s)ds - \frac{A}{2a}$$

Now, substitute these into the general solution: (plug into $y(x,t) = F(x-at) + G(x+at)$)
This gives us:

$$\frac{1}{2}f(x-at) - \frac{1}{2a}\int_0^{x-at} g(s)ds + \frac{1}{2}f(x+at) + \frac{1}{2a}\int_0^{x+at} g(s)ds$$

Note that $-\frac{A}{2a}$ and $\frac{A}{2a}$ cancel.

$$y(x,t) = \frac{1}{2}\left[\underbrace{f(x-at)}_{\text{half of init cond travels right}} + \underbrace{f(x+at)}_{\text{half of init cond travels left}}\right] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)ds \tag{12.1}$$

N.B. Above analysis has no boundary conditions: $-\infty < x < \infty$

What if the problem has boundary condition?
Let $F^o(x)$ and $G^o(x)$ be the odd[3] 2L-periodic extension of $f(x)$ and $g(x)$ respectively:

$$y(x,t) = \frac{1}{2}\left[F^o(x-at) + F^o(x+at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} G^o(s)ds$$

Boundary conditions: [4]

$$y(0,t) = \underbrace{\frac{1}{2}\left[F^o(-at) + F^o(at)\right]}_{=0} + \underbrace{\frac{1}{2a}\int_{-at}^{at} G^o(s)ds}_{=0} = 0$$

[3](Assumes Dirichlet boundary conditions)
[4]Note that we are using the properties of odd functions to cancel out both F and G.

What's the relationship between D'Alembert's formula and the separation of variables?

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[b_n \frac{L}{n\pi a}\sin\left(\frac{n\pi a}{L}t\right) + b_n' \cos\left(\frac{n\pi a}{L}t\right)\right]$$

Recall trig formulae:

$$\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\sin(A)\cos(B) = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

Let's apply these:

$$y(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{b_n L}{n\pi a} \underbrace{\left[\cos\left(\frac{n\pi}{L}(x-at)\right) - \cos\left(\frac{n\pi}{L}(x+at)\right)\right]}_{\text{Let's write this in integral format}} \rightarrow$$

$$\hookrightarrow +b_n'\left[\sin\left(\frac{n\pi}{L}(x-at)\right) - \sin\left(\frac{n\pi}{L}(x+at)\right)\right]$$

$$y(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} b_n'\left[\sin\left(\frac{n\pi}{L}(x-at)\right) - \sin\left(\frac{n\pi}{L}(x+at)\right)\right] \rightarrow$$

$$\hookrightarrow +\frac{1}{2a}\sum_{n=1}^{\infty} b_n \int_{x-at}^{x+at} \sin\left(\frac{n\pi s}{L}\right) ds$$

Recall:

$$\sum_{n=1}^{\infty} b_n' \sin(\frac{n\pi x}{L}) = f(x)$$

and

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = g(x)$$

$\Rightarrow$ d'Alembert's solution:

$$y(x,t) = \frac{1}{2}\left[f(x-at) + f(x+at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)ds$$

Both methods give similar solution.