MATH 316 Lecture 5

Ashtan Mistal

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1 Bessel's Equation, continued from last class

$$a_0(r(r-1)+r-\nu^2)x^r + a_1\left((1+r)r+(1+r)-\nu^2\right)x^{r+1} + \sum_{n=2}^{\infty} a_n\,(...)\,x^{n+r} = 0$$

We can use linear independency. This means that the set of the coefficients of all powers of x must be zero.

The following is found from the characteristic equation:

$$x^r|a_0(r^2-\nu^2) = 0 \longrightarrow r = \pm\nu \;\&\; a_0 \neq 0$$

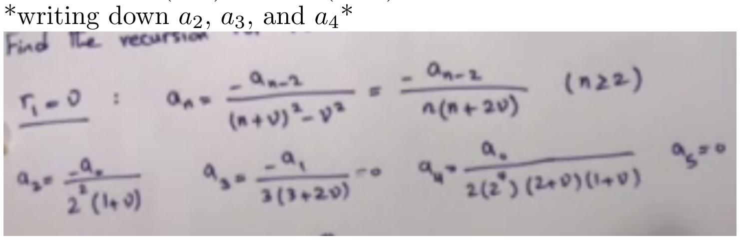
$$x^{r+1}|a_1\left(r^2+2r+1-\nu^2\right) = 0 \underbrace{\longrightarrow}_{\nu^2-r^2} a_1(2\nu+1) = 0$$

$$\hookrightarrow \begin{cases} \nu = \pm\frac{1}{2} & \& \quad q \neq 0 \\ \nu \neq \pm\frac{1}{2} & \& \quad q = 0 \end{cases}$$

$$x^{n+r}|\left((n+r)(n+r-1)+(n+r)-\nu^2\right)a_n + a_{n-2} = 0 \longrightarrow n \geq 2$$

$$(**)\; a_n = \frac{-a_{n-2}}{(n+r)^2-\nu^2}$$

Find the recursive relation for $r = \pm\nu$:
$r_1 = \nu$: $a_n = \frac{-a_{n-2}}{(n+\nu)^2-\nu^2} = \frac{-a_{n-2}}{n(n+2\nu)}$, $(n \geq 2)$
writing down a_2, a_3, and a_4



$\Rightarrow a_{2m} = \frac{(-)^m a_0}{m!2^{2m}(1+\nu)(2+\nu)...(m+\nu)}$

$$y_1(x) = a_0 x^{\nu} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{m! 2^{2m} (1+\nu)(2+\nu)...(m+\nu)}$$

Now, for $r_1 = \nu$:

$$a_n = \frac{-1_{n-1}}{(n-\nu)^2 - \nu^2} = -\frac{a_{n-2}}{n(n-2\nu)}; n \geq 2$$

$$a_2 = \frac{-a_0}{2(2-2\nu)} = \frac{-a_0}{2(2)(1-\nu)}$$

$$a_4 = \frac{-a_2}{4(4-2\nu)} = \frac{a_0}{4(2)(2-\nu)(2^2)(1-\nu)}$$

$$a_6 = \frac{-a_4}{6(6-2\nu)} = \frac{-a_0}{6(2)(3-\nu)2^5(2-\nu)(1-\nu)}$$

Note that $a_1 = a_3 = a_5.... = 0$

$$\Rightarrow a_{2m} = \frac{(-1)^m a_0}{m! 2^{2m}(1-\nu)(2-\nu)(3-\nu)...(m-\nu)}$$

$$y_2(x) = a_0 x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m a_0}{m! 2^{2m}(1-\nu)(2-\nu)...(m-\nu)}$$

Finally, $y(x)$ is a linear combination of 2 solutions:

$$y(x) = C_1 \underbrace{x^{\nu} \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m}}{m(1+\nu)(2+\nu)...(m+\nu)}}_{J_\nu\text{: Bessel Functions of the first kind}} + C_2 \underbrace{x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m a_0}{m! 2^{2m}(1-\nu)(2-\nu)...(m-\nu)}}_{Y_\nu\text{: Bessel function of the second kind}}$$

We will be given this in the formula sheet. Note that C_1 and C_2 are not included in J_ν and Y_ν.

For $\nu \neq \pm\frac{1}{2}$: As $x \to 0$, $J_\nu \to 0$ and $x \to 0$, $Y_\nu \to \infty$

What happens when $\nu = 0$?

$$x^r | a_0(r^2 - \nu^2) = 0 \to r = \pm\nu, a_0 \neq 0$$

Two solutions are the same. Therefore, $r_{1,2} = 0$

Then, $J_\nu(x) = C_1 x^0 \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2}\right)^{2m}$

How about $Y_\nu(x)$?

Similar to Euler's equation (Refer to section 5.4 of the textbook), the second solution for repeated roots is:

$$y_2(x) = y_1(x)\ln(x) + x^{r_1} \sum_{n=1}^{\infty} a_n'(r_1) x^n$$

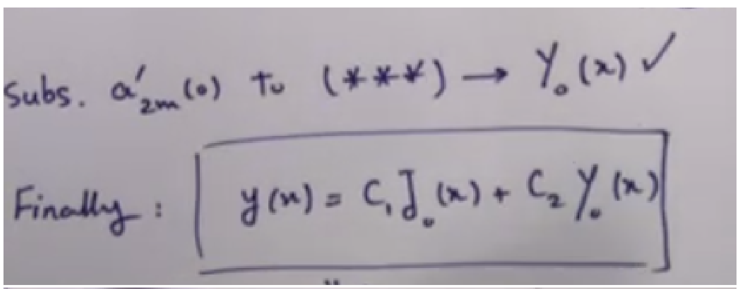
where $a'_n(r_1) = \frac{da_1}{dr}\Big|_{r=r_1}$
According to this formula, $y_0 = J_0(x)\ln(x) + x^0 \sum_{n=1}^{\infty} a'_n(0)x^n$ (***)
The recursion for this case (**) is found to be

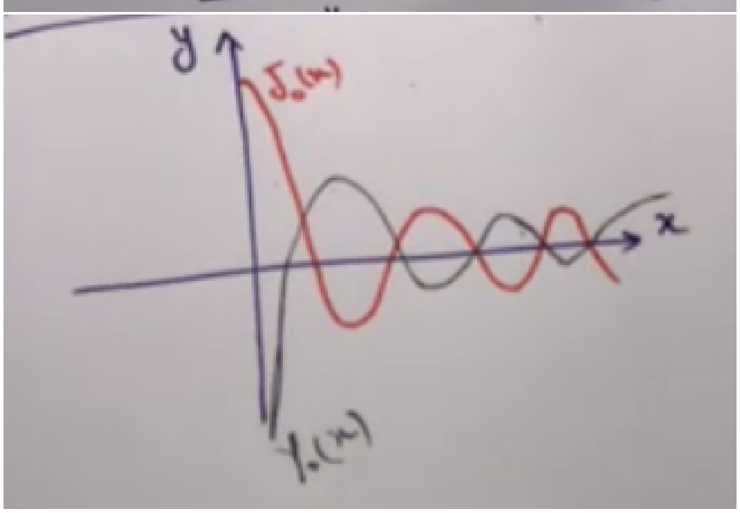
$$a_{2m}(r) = -\frac{a_{2m-2}}{(r+2m)^2}, m = 1, 2, 3, ...$$

$$a_{2m}(r) = \frac{(-1)^m a_0}{(r+2)^2....(r+2m)^2}$$

$$a'_{2m}(r) = -2\left(\frac{1}{r+2} + \frac{1}{r+4} + ... + \frac{1}{r+2m}\right) a_{2m}(r)$$

$$a'_{2m}(0) = -2\left(\frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2m}\right) a_{2m}(0) = -\left(\frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2m}\right) \frac{(-1)^m a_0}{2^{2m}(m!)^2}$$





As $x \to 0$, $y_0(x) \to -\infty$, i.e. if the solution, $y(x)$, is finite at zero, then $C_2 = 0$

2 Bessel Equation of Order of $\pm\frac{1}{2}$

$$Ly = x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$$

$$x^r|a_0(r^2 - \nu^2) = 0 \longrightarrow a_0 \neq 0 \text{ and } r = \pm\nu$$

For $\nu = \frac{1}{2} \Rightarrow r = \pm\frac{1}{2}$

$$x^{r+1}|a_1(r^2 + 2r + 1 - \nu^2) = 0 \Rightarrow a_1(1 \pm 2\nu) = 0$$

If $\nu = \pm\frac{1}{2}$, a_1 is arbitrary

$$x^{m+r}| - a_m(r^2 + 2r + 1 - \nu^2) = a_{m=2} \Rightarrow a_m = \frac{-a_{m-2}}{(m+r)^2 - \nu^2}$$

For $\nu \pm \frac{1}{2}, a_m = \frac{-a_{m-2}}{(m+\frac{1}{2})^2 - \frac{1}{4}} = \frac{-a_{m-2}}{m(m+1)}$
Let $r_1 = \frac{1}{2}$: $a_1(1 + 2(\frac{1}{2})) = 0 \Rightarrow a_1 = 0$
$a_2 = -\frac{a_0}{2(3)}$, $a_4 = \frac{-a_2}{3(4)} = \frac{a_0}{5!}$
Therefore:

$$y_1(x) = a_0x^{\frac{1}{2}} \underbrace{\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + ...\right) \frac{x}{x}}_{\text{Taylor series for } \sin(x)}$$

$$y_1(x) = a_0x^{-\frac{1}{2}}\sin(x)$$

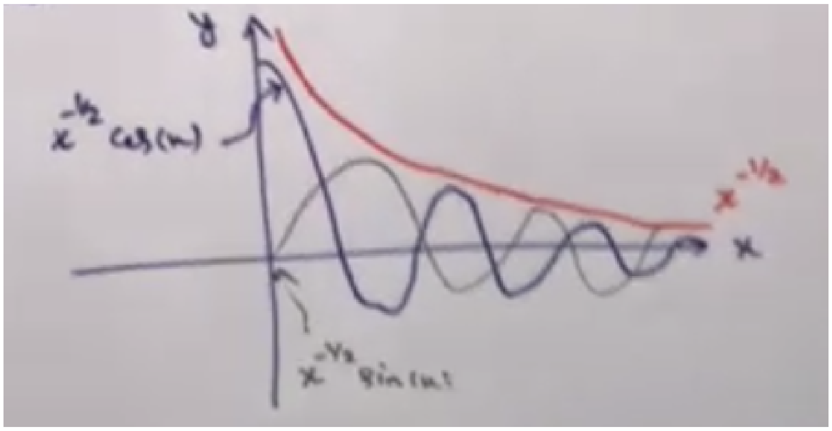
let $r_2 = \frac{-1}{2}$: $a_1(1 - 2\frac{1}{2}) = 0 \Rightarrow a_1$ is arbitrary. it could be another solution.
for this case, $a_m = \frac{-a_{m-2}}{(m=\frac{1}{2})^2 - \frac{1}{4}} = \frac{-a_{m-2}}{m(m-1)}$
$a_2 = \frac{-a_0}{2(1)}$, $a_4 = \frac{-a_2}{4(3)} = \frac{a_0}{4!}$

$$\Rightarrow y_x(x) = a_0x^{\frac{-1}{2}}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...\right) = a_0x^{-\frac{1}{2}}\cos(x)$$

Let's check for $a_1 \neq 0$:
$a_3 = \frac{-a_1}{3\cdot2}$, $a_5 = \frac{-a_3}{5\cdot4} = \frac{a_1}{5!}$
$y_3(x) = a_1x^{-\frac{1}{2}}\sin(x)$. But this doesn't give us another solution – This is the same as y_1; they are not independent.

Hence we write the final solution as:

$$y(x) = a_0x^{\frac{-1}{2}}\cos(x) + a_1x^{\frac{-1}{2}}\sin(x)$$



End of series functions

3 Introduction to PDE Classification

What is a PDE?

A differential equation that includes partial derivatives with respect to all independent variables.

$u(x,t) \longrightarrow$ PDEs include $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x \partial t}$, $\frac{\partial^2 u}{\partial x^2}$, ...

- Heat equation
- Wave equation
- Laplace equation