MATH 316 Lecture 2

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1 Solving the problem from last class

We started with a simple example that was solvable using integrating factor, and we are now going to do the Taylor expansion of the **answer**, around the point $x = x_0$. THe confusion from last class was that we tried to solve the ODE using a Taylor expansion **of the differential equation**, as opposed to the actual answer.

$$y(x) = y(0) + \frac{y'(0)}{1}x + \frac{y''(0)}{2!}x^2 + ...$$

$$y(x) = C\left[1 - x + \frac{3}{2}x^2 - \frac{7}{6}x^3 + ...\right]$$

From the class notes:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \quad \bigstar$$

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Ex $\quad y' + (1-2x)y = 0$

Integrating factor : $\mu(x) = e^{\int (1-2x)dx} = e^{x-x^2}$

$$\left[y e^{x-x^2}\right]' = 0 \rightarrow y e^{x-x^2} = C \rightarrow \boxed{y = Ce^{-x+x^2}} \rightarrow$$

Taylor expand $y(x)$ about the point $x_0 = 0$

$$y(x) = y(0) + \frac{y'(0)}{1}x + \frac{y''(0)}{2!}x^2 + \dots \qquad (-1+2x)e^{-x}$$

$$*y(x) = C\left[1 - x + \frac{3}{2}x^2 - \frac{7}{6}x^3 + \dots\right] \checkmark \qquad y'(0) = -C$$

Note that we factored C out of the taylor expansion. Moving forward, assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$y' + (1-2x)y = 0$$

$$y' + y - 2xy = 0$$

$$\text{letting } y = \sum_{n=0}^{\infty} a_n x^n, \text{ and } y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$\sum_{n=1}^{\infty} a_n \underbrace{n}_{m=n-1;n=1,m=0} x^{n-1} + \sum_{n=0}^{\infty} \underbrace{a_n}_{m=n} x^n - 2\sum_{n=0}^{\infty} \underbrace{a_n}_{m=n+1;n=0,m=1} x^{n+1} = 0$$

Not exactly sure what these variables are doing with the n and m... I guess they're dummy variables that we're using for each sum.

$$\sum_{m=0}^{\infty} a_{m+1}(m+1)x^m + \sum_{m=0}^{\infty} a_m x^m - 2\sum_{m=1}^{\infty} a_{m-1}x^m = 0$$

Peel-off:

$$a_1x^0 + a_0x^0 + \sum_{m=1}^{\infty} \left[a_{m+1}(m+1) + a_m - 2a_{m-1}\right] x^m = 0$$

Note that $x^0, x, x^2, ..., x^n$ are linearly independent.

$x^0]$ $a_1 + a_0 = 0 \rightarrow a_1 = -a_0$

x^m, $m \geq 1]$ $a_{m+1}(m+1) + a_m - 2a_{m-1} = 0$

Q: Why do we need to shift the indices?

A: We want to get to a single sigma – A single sum. If we don't do it, we aren't able to get to a single relation. Indices must start at the same point to be able to combine sums.

From the relation $a_{m=1}(m+1) + a_m - 2a_{m-1} = 0$, we can find the relation:

$$a_{m+1} = \frac{-a_m + 2a_{m-1}}{m+1}$$

$$m = 1 : a_2 = \frac{-a_1 + 2a_0}{2} = \frac{3}{2}a_0$$

$$m = 2 : a_3 = \frac{-a_2 + 2a_1}{3} = \frac{-7}{6}a_0$$

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ...$$

$$y(x) = a_0 - a_0x + \frac{3}{2}a_0x^2 - \frac{7}{6}a_0x^3 + ...$$

$$y(x) = a_0\left[1 - x + \frac{3}{2}x^2 - \frac{7}{6}x^3 + ...\right]$$

= Taylor expansion of the direct solution

2 Example 2

$$xy' + (2 - x)y = 0$$

$$y' + \frac{2-x}{x}y = 0$$

Solving this using integrating factor method, we find the following:

$$\mu(x) = e^{\int \frac{1-x}{x}dx} = e^{2\ln x - x} = x^2e^{-x}$$

$$[x^2e^{-x}y]' = 0 \to y = \frac{C}{x^2e^{-x}} = Cx^-2e^x$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad , y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$xy' + 2y - xy = 0$$

$$\sum_{n=1}^{\infty} a_n n x^n + 2\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^n + 1 = 0$$

$$\sum_{n=1}^{\infty} a_n n x^n + 2\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$m = n$ $\qquad$ $m = n$ $\qquad$ $m = n+1$

$m = 1, n = 0$

$$\sum_{m=1}^{\infty} a_m m x^m + 2\sum_{m=0}^{\infty} a_m x^m - \sum_{m=1}^{\infty} a_{m-1} x^m = 0$$

peel off first term

$$2a_0x^0 + \sum_{m=1}^{\infty} (a_m m + 2a_m - a_{m-1}) x^m = 0$$

$$x^0]\ 2a_0 = 0 \longrightarrow a_0 = 0$$

$$\left.\begin{matrix} x^m \\ m \geq 1 \end{matrix}\right]\ a_m(m+2) = a_{m-1}$$

note that $a_m = \frac{a_{m-1}}{m+2}$ for m ≥ 1

$$m = 1 : a_1 = \frac{a_0}{3} = 0$$

$$m = 2 : a_2 = \frac{a_4}{4} = 0$$

This is a trivial solution; they are all zero (and will continue to be).
$y = cx^{-2}e^x$ is the general solution.

$$= C \underbrace{x^{-2}}_{\text{Capture the singularity}} \underbrace{\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right]}_{\sum_{n=0}^{\infty} a_n x^n}$$

$$y(x) = \underbrace{x^r}_{\text{Capture singularity}} \sum_{n=0}^{\infty} a_n x^n$$

This brings us to the **Forbenius Series**.

3 Forbenius Series

Let's define ordinary & singular points:
A linear 2nd order ODE:

$$P(x)y'' + Q(x)y' + r(x)y = 0$$

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0$$

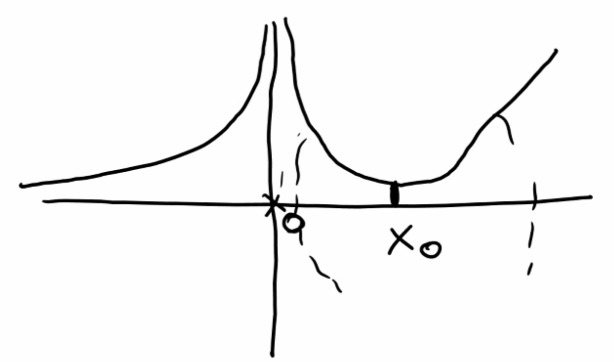
Letting $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$. If they are both **analytic** at the point x_0. i.e. they have Taylor expansions about x_0, x_0 is an ordinary point. Otherwise, x_0 is a singular point.

Example: $\left.\begin{matrix} p(x) = \frac{1}{x} \\ p(x) = \ln(x) \end{matrix}\right\}$ At $x = 0$, not analytic.

Analytic means that is is expressible as a power series around x_0. It means that it is infinitely differentiable around x_0.

3.0.1 Quick notes

- A power series solution is possible for all ordinary points (similar to the first example we saw), but not all singular points.
- For singular points, we introduce the Forbenius Series. However, this only works for some singular points.
- Singular points results in the change of the nature of the ODE. Ordinary points exists in the domain of $p(x)$ and $q(x)$.
- The radius of convergence of the power series is at least as large as the distance from the x_0 to the nearest singular point.
  - For example, when we had $y = Cx^{-2}e^x$, we realize that $x = 0$ is a singular point.
  - We can see this in both the answer as well as the ODE.
  - When we plot the function, it looks like this:



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– The dotted circles around x_0 is the radius of convergence.

Example:

$$y' + (1 - 2x)y = 0$$

$$y(x) = Ce^{-x+x^2}$$

There are no singular points. Hence the radius of convergence is infinity.

3.1 Singular Points

Singular points are divided into two classes:

- Regular singular points, that we use the Forbenius series solution for
- Irregular singular points (Beyond this course). For these, we **cannot** use Forbenius series.