

# MATH 316 Lecture 4

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## 1 Introduction

### 1.1 Week 2:

- finish power series
- Bessel's function
- Intro to PDEs

### 1.2 Week 3:

- Fourier series and separation of variables
- Heat equations
- Wave equations
- Laplace equations

### 1.3 Week 4:

- Boundary value problems and Sturm-Liouville Theory
- Eigenfunctions and eigenvalues
- Sturm-Liouville theory for BVP
- Non-homogenous boundary value problems

### 1.4 Week 5:

- Numerical methods for solving PDEs

## 2 Recap of Previous Week

What we're covering next was partly covered in previous lectures by Parisa:

Series Solutions near a regular singular point (RSP)

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

We consider an expansion about a regular singular point  $x_0$  of the ODE.

Next, we take the limits:

$$\lim_{x \rightarrow x_0} (x - x_0) \frac{Q(x)}{P(x)} = p_0$$

$$\lim_{x \rightarrow x_0} (x - x_0)^2 \frac{R(x)}{P(x)} = q_0$$

When these two limits exist and are finite, then this leads to the following. We divide  $P(x)y'' + Q(x)y' + R(x)y = 0$  by  $P(x)$  and multiply it by  $(x - x_0)^2$ :

$$(x - x_0)^2 y'' + (x - x_0) \underbrace{\left\{ (x - x_0) \frac{Q(x)}{P(x)} \right\}}_{p(x)=p_0+p_1(x-x_0)+\dots} y' + \underbrace{\left\{ (x - x_0)^2 \frac{R(x)}{P(x)} \right\}}_{q(x)=q_0+q_1(x-x_0)+\dots} y = 0$$

$$(x - x_0)^2 + p_0(x - x_0)y' + q_0y = 0$$

is the C-E equation. This equation has a solution of the following format:

$$y_0(x) = (x - x_0)^r$$

To include the effect of the neglected terms, we modify the above solution:

$$y(x) = \underbrace{(x - x_0)^r}_{\text{C-E solution}} \sum_{n=0}^{\infty} \underbrace{a_n (x - x_0)^n}_{\text{The solution}}$$

with  $a_0 \neq 0$ . (2) is the modified equation for  $y_0(x) = (x - x_0)^r$  and is Frobenius series.

### 2.1 Solution procedure

1. Find values of  $r$
2. Find the recursive relation for  $n$
3. Find the radius of convergence for  $\sum_{n=0}^{\infty} a_n (x - x_0)^{n+r}$

## 2.2 Example 1

$$Ly = 2x^2y'' - xy' + (1-x)y = 0$$

Singular point here is when  $x_0 = 0$

Again, to test if it is a singular point, we take the limits.

$$p_0 = \lim_{x \rightarrow 0} \frac{-x}{2x^2} = \frac{-1}{2}$$

Similarly, we check  $q_0$ :

$$q_0 = \lim_{x \rightarrow 0} \frac{(1-x)}{2x^2} = \frac{1}{2} = q_0$$

Hence, the singular point is a RSP.

$$\Rightarrow y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$$

Step 1: Find values of  $r$

Characteristic equation:  $r(r-1) + p_0r + q_0 = 0$

This becomes  $r(r-1) - \frac{r}{2} + \frac{1}{2} = 0 \Rightarrow r^2 - \frac{3}{2}r + \frac{1}{2} = 0$

Hence,  $r = 1$  and  $r = \frac{1}{2}$  are roots of the indicial equation.

$$y_0 = c_1x + c_2x^{\frac{1}{2}}$$

Now, we go to step 2: Find the recursive relation for  $n$ .

Take the first and second derivative of the summation (that we've used before) and sub into the equation.

$$2 \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

Changing index:

$$2 \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - \underbrace{\sum_{n=0}^{\infty} a_n x^{n+r+1}}_{m=n+1} = 0$$

All others are  $m = n$ .

With that, we get the following:

$$2 \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

(Removing  $= 0$  due to space)

Peeling off the first terms ( $n = 0$ ), we end up with the following sum:

$$\underbrace{2a_n r(r-1)x^r - a_0 r x^r + a_0 x^r}_{=a_0 x^r(2r^2-3r+1)} \rightarrow \dots$$

$$\hookrightarrow + \sum_{n=1}^{\infty} [2a_n(n+r)(n+r-1) - a_n(n+r) + a_{n-1}] x^{n+r} = 0$$

Hence,  $a_n = \frac{a_{n-1}}{(n+r)(2(n+r)-3)+1}$  is the recursive relation for  $r$  values  
Finding the recursive relation for  $r_1$  and  $r_1$ :

$$r_1 \Rightarrow a_n = \frac{a_{n-1}}{(n+1)(2(n+1)-3)+1} = \frac{1_{n-1}}{2n^2+n}$$

$$n=1 : a_1 = \frac{a_0}{3}; n=2 : a_2 = \frac{a_1}{10} = \frac{a_0}{30}; n=3 : a_3 = \frac{a_2}{21} = \frac{a_0}{630}$$

Therefore:

$$y(x) = a_0 x^1 \left(1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \dots\right)$$

Next, for  $r_1 = \frac{1}{2}$ :

$$a_n = \frac{a_{n-1}}{(n+\frac{1}{2})(2(n+\frac{1}{2})-3)+1} = \frac{a_{n-1}}{2(n+\frac{1}{2})(n-1)+1}$$

$$n=1 : a_1 = \frac{a_0}{1}; n=2 : a_2 = \frac{a_1}{6} = \frac{a_0}{6}; n=3 : a_3 = \frac{a_0}{90}$$

So, we have the following:

$$y_2(x) = a_0 x^{\frac{1}{2}} \left[1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \dots\right]$$

Now, onto step 3:

Find the radius of convergence.

To find the radius of convergence, we use the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{a_n x^n}{a_{n-1} x^{n-1}} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{a_{n-1} x^n}{(n+r)(2(n+r)-3)+1} \frac{1}{a_{n-1} x^{n-1}} \right|$$

**The above is not correct. Why?**

$$\lim_{n \rightarrow \infty} |x| \left| \frac{1}{(n+r)(2(n+r)-3)+1} \right| = 0$$

$\Rightarrow p = \infty$  for all  $x$  values

Final solution:

$$y(x) = y_1(x) + y_2(x)$$

$$y(x) = C_1 x \left(1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \dots\right) + C_2 x^{\frac{1}{2}} \left(1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \dots\right)$$

### 3 Bessel's equation

Applications: PDEs on circular / cylindrical domain.

e.g. heating and cooling in circular / cylindrical geometries, i.e. pipes and heat exchangers.

The equation format:

$$Ly = x^2 y'' + xy' + (x^2 - \nu^2) y = 0 \quad (1)$$

$\nu$  is a constant and specifies the order of the equation.

$x_0 = 0$  is a regular singular point, since  $\lim_{x \rightarrow 0} \frac{x}{x^2}(x) = 0 = p_0$  and  $\lim_{x \rightarrow 0} \frac{x^2 - \nu^2}{x^2}(x^2) = -\nu^2 = q_0$

#### 3.1 Step 1: Finding the $r$ values

The characteristic equation is:

$$r(r-1) + p_0 r + q_0 = 0$$

$$r(r-1) + r - \nu^2 = 0 \Rightarrow r = \pm \nu$$

#### 3.2 Step 2: Frobenius part – Finding recursive relation

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

Substituting into the equation, we get the following:

$$Ly = x^2 y'' + xy' + (x^2 - \nu^2) y = 0$$

$$Ly = x^2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} \rightarrow \dots$$

$$\hookrightarrow +x \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} \rightarrow \dots$$

$$\hookrightarrow + (x^2 - \nu^2) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Shifting index:

$$\begin{aligned}
Ly &= x^2 y'' + xy' + (x^2 - \nu^2) y = 0 \\
Ly &= x^2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} \rightarrow \dots \\
&\hookrightarrow +x \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} \rightarrow \dots \\
&\hookrightarrow +x^2 \underbrace{\sum_{n=0}^{\infty} a_n x^{n+r+2}}_{m=n+2} - \nu^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0
\end{aligned}$$

Hence, we get the following:

$$Ly = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Next, we peel off the first two terms to match the summation.

$$\begin{aligned}
&a_0(r(r-1) + r - \nu^2)x^r + a_1((1+r)r + (1+r) - \nu^2)x^{r+1} \rightarrow \dots \\
&\hookrightarrow + \sum_{n=2}^{\infty} a_n [a_n ((n+r)^2 - \nu^2) + a_{n-2}] x^{n+r} = 0 \\
&a_0 x^r (r^2 - \nu^2) + a_1 x^{r+1} ((r+1)^2 - \nu^2) + \sum_{n=2}^{\infty} a_n [a_n ((n+r)^2 - \nu^2) + a_{n-2}] x^{n+r} = 0
\end{aligned}$$