

# MATH 316 Lecture 11

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## 1 Introduction

- Check Canvas announcements regarding midterm and such (2 new announcements)

## 2 Recap of Last Lecture

Last lecture, we finished up the heat / diffusion equation.

- Homogeneous equations and boundary conditions (Neumann and Dirichlet)
- Inhomogeneous equations and boundary conditions
- Developed general strategies for splitting inhomogeneous equations into a steady state (Which takes care of inhomogeneous parts, incl. boundary conditions) and a transient part (Satisfying a classic homogeneous diffusion equation)

Today's lecture is about a new equation: Wave equation.

Two methods to solve:

- Applying separation of variables
- We'll talk about the second method tomorrow

## 3 Wave equation

Takes the following form:

$$y_{tt} = a^2 y_{xx}$$

Two derivatives with respect to time, and two derivatives with respect to  $x$ .

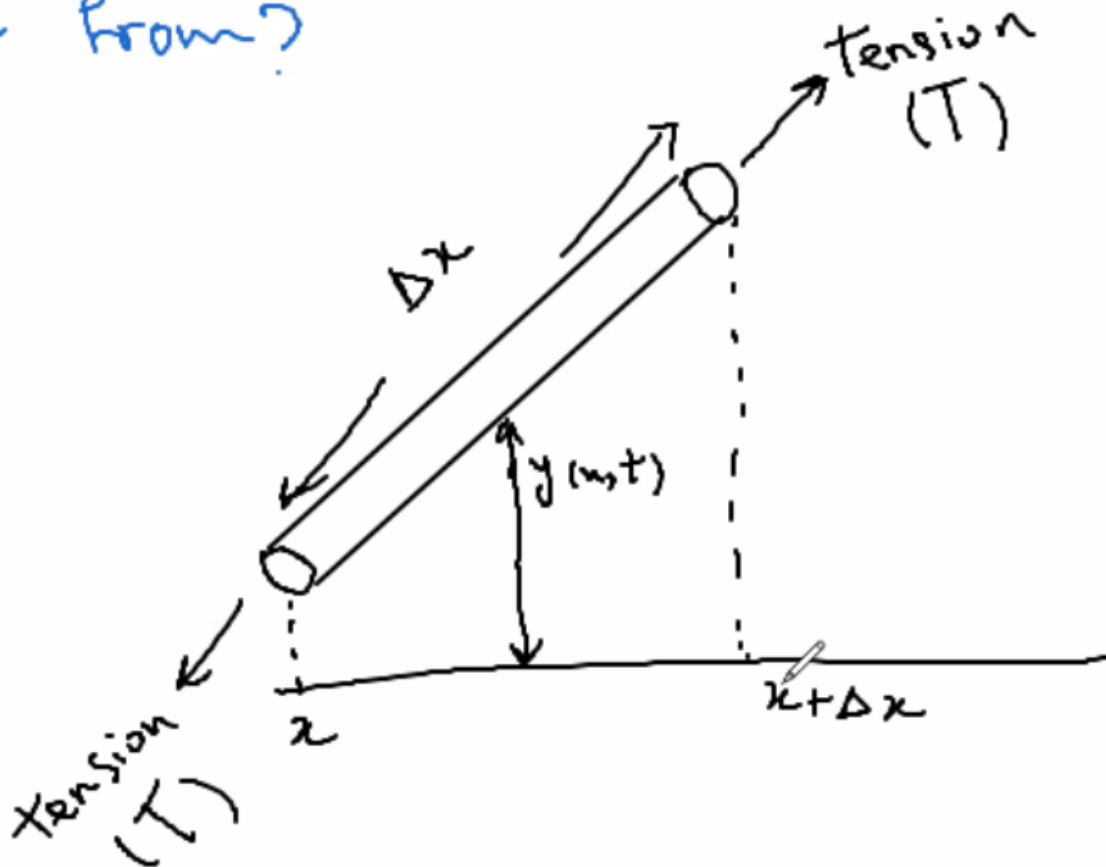
Physically,  $a = \left[ \frac{T}{\rho} \right]^{\frac{1}{2}}$  (A string under tension), where  $T$  is stress / tension, and  $\rho$  is density. Can also be written as  $a = \left[ \frac{E}{\rho} \right]^{\frac{1}{2}}$  (Elastic bar), where  $E$  = elastic stress.

Boundary conditions: We have two  $x$  derivatives  $\rightarrow$  2 conditions needed.

Initial conditions: We also need 2 initial conditions (Because we have 2 time derivatives). This is the main difference between the wave equation and the heat equation.  $y(x, 0) = C$  (Initial displacement), and  $y_t(x, 0) = k$  (initial velocity)

Where did the wave equation come from?

from?



Derivation for small  $\left| \frac{\partial y}{\partial x} \right|$

String of density  $\rho$  (with units of  $\left[ \frac{kg}{m} \right]$ )

Force balance in the  $y$  direction:

$$\rho \Delta x \frac{\partial^2 y}{\partial t^2} = m \cdot \vec{a} = \sum F_y$$

$$\rho \Delta x \frac{\partial^2 y}{\partial t^2} = T \frac{\partial y}{\partial x}(x + \Delta x, t) - T \frac{\partial y}{\partial x}(x, t)$$

Divide by  $\Delta x$  and let  $\Delta x \rightarrow 0$ :

$$\Rightarrow \rho \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2}$$

N.B length of element is  $\Delta x(1 + (\frac{\partial y}{\partial x})^2)^{\frac{1}{2}} \approx \Delta x$

Wave equation:

$$y_{tt} = a^2 y_{xx}$$

$$\text{BC: } y(0, t) = y(L, t) = 0$$

$$\text{IT: } y(x, 0) = f(x) \text{ and } y_t(x, 0) = g(x)$$

The idea is to split the solution into two parts.

- Problem 1: Initial velocity, but no displacement of string

- $w_{tt} = a^2 w_{xx}$
- $w(0, t) = w(L, t)$
- $w(x, 0) = 0$  for  $0 < x < L$
- $w_t(x, 0) = g(x)$  for  $0 < x < L$

- Problem 2: Initial displacement, but no velocity of spring

- $z_{tt} = a^2 z_{xx}$
- $z_t(x, 0) = f(x)$  for  $0 < x < L$
- $z_t(x, 0) = 0$  for  $0 < x < L$

Solve problems 1 and 2:  $y(x, t) = w(x, t) + z(x, t)$

### 3.1 Step 1: Solving Problem 1

$$w(x, t) = X(x)T(t) \quad \underbrace{\Rightarrow}_{\text{Substitute into PDE}} \quad X\ddot{T} = a^2 X''T$$

Divide by  $a^2 XT$ :

$$\Rightarrow \frac{1}{a^2} \frac{\ddot{T}}{T} = \frac{X''}{X} = -\lambda$$

where  $\lambda$  is a constant.

First, boundary value problem:

$$X'' + \lambda X = 0$$

Boundary conditions are  $w(0, t) = X(0)T(t) \Rightarrow X(0) = 0$ , and  $w(L, t) = X(L)T(t) = 0 \Rightarrow X(L) = 0$   
It is a P1 eigenvalue problem.

Therefore, the eigenvalue problem for  $X(x)$  is exactly as for heat / diffusion equation.  
Therefore:

$$\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

and

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

where  $n \in N$  (Natural numbers; 1,2,3,...)

IVP is:

$$\frac{1}{a^2} \frac{\ddot{T}_n}{T_n} = -\lambda_n \Rightarrow \ddot{T}_n + \left(\frac{an\pi}{L}\right)^2 T_n = 0$$

Therefore:

$$T_n(t) = A_n \cos\left(\frac{an\pi}{L}t\right) + B_n \sin\left(\frac{an\pi}{L}t\right)$$

How about the initial condition?

$$w(x, 0) = 0 \longrightarrow X(x)T(0) = 0 \Rightarrow T(0) = 0 \text{ and } w_t(x, 0) = g(x)$$

$$T_n(0) = 0 \Rightarrow A_n = 0$$

As a result:

$$T_n(t) = B_n \sin\left(\frac{an\pi}{L}t\right)$$

Now, note that PDE, boundary conditions, and  $w(x, 0)$  are homogeneous. Therefore, we can superimpose solutions.

$$w(x, t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{an\pi}{L}t\right)$$

We need to find  $B_n$ . To find this, we use the second initial condition.

$$w_t(x, 0) = \sum_{n=1}^{\infty} B_n \left(\frac{an\pi}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{an\pi}{L}t\right)$$

$$w_t(x, 0) = \sum_{n=1}^{\infty} B_n \left(\frac{an\pi}{L}\right) \sin\left(\frac{n\pi x}{L}\right) = g(x)$$

If we write a Fourier sine series for  $g(x)$ , we can match up the coefficients:  
To make this work, represent  $g(x)$  as a Fourier sine series on  $[0, L]$ :

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \Rightarrow b_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

We know this series converges, so we match up the coefficients.

$$B_n = b_n \frac{L}{n\pi a} \text{ for } n \in N.$$

$$\Rightarrow w(x, t) = \sum_{n=1}^{\infty} b_n \frac{L}{n\pi a} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi a t}{L}\right)$$

### 3.2 Step 2: Solving Problem 2

$$z_{tt} = a^2 z_{xx}$$

Initial boundary conditions:  $z(0, t) = Z(L, t) = 0$  and  $Z(x, 0) = f(x)$ ;  $z_t(x, 0) = 0$  for  $0 < x < L$

Solution: Similarly, we use separation of variables and we assume that  $z$  is a product of  $X$  and  $T$ :

$$z(x, t) = X(x)T(t)$$

(Note that these are different  $X$  and  $T$  than in step 1!!)

The solution for the eigenvalue problem is a P1 problem again.

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \text{ for } n \in N$$

For the IVP part,

$$\ddot{T}_n + \left(\frac{n\pi a}{L}\right)^2 T_n = 0$$

$$\Rightarrow T_n(t) = A_n \cos\left(\frac{n\pi a}{L}t\right) + B_n \sin\left(\frac{n\pi a}{L}t\right)$$

Now,  $z_t(x, 0) = X(x)\dot{T}(0) = 0 \Rightarrow \dot{T}_n(0) = 0 \Rightarrow B_n = 0$

So, let's superimpose the solution:

$$z_n(x, t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi a}{L}t\right)$$

Use the other initial condition and find  $A_n$ :

The other initial condition tells us:

$$f(x) = z(x, 0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

Suppose we compute the Fourier sine series for  $f(x)$ , Then,

$$b'_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

**Note that prime is NOT a derivative, just used to denote that it's a different  $b_n$ .**

$$\Rightarrow z(x, t) = \sum_{n=1}^{\infty} b'_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi a}{L}t\right)$$

### 3.3 Step 3

$$y(x, t) = x(x, t) + z(x, t)$$

$$y(x, t) = \sum_{n=1}^{\infty} \left[ b_n \frac{L}{n\pi a} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi at}{L}\right) + b'_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi a}{L}t\right) \right]$$

We can factor<sup>1</sup>:

$$y(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[ \frac{b_n L}{n\pi a} \sin\left(\frac{n\pi a}{L}t\right) + b'_n \cos\left(\frac{n\pi a}{L}t\right) \right]$$

For Neumann boundary conditions, the procedure is exactly the same as Dirichlet boundary conditions. (Using the PDF file posted on Canvas – Wave Equations, under week 4. Posted at the bottom of this document.)

## 4 Example 8

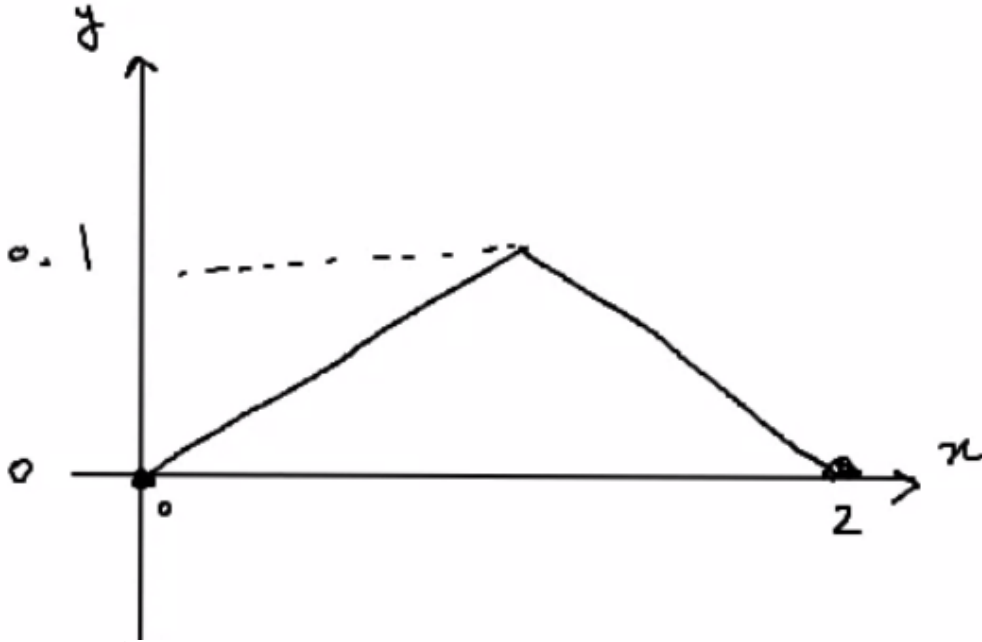
(Example 8 of the pdf)

Solve the IBVP  $y_{tt} = y_{xx}$

Initial conditions:  $y(0, t) = y(2, t)$

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<sup>1</sup>Again,  $b'_n$  is not a derivative!



$$y(x,0) = \begin{cases} 0.1x & 0 \leq x \leq 1 \\ 0.1(2-x) & 1 \leq x \leq 2 \end{cases}$$

$$y_t(x,0) = 0 = g(x) \Rightarrow b_n = 0 \quad \forall n$$

Solution:  $a = 1$  and  $L = 2$ . (There is only initial displacement  $z(x,t)$  problem)

$$b'_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx = 0.1 \int_0^1 x \sin\left(\frac{n\pi x}{2}\right) + 0.1 \int_1^2 (2-x) \sin\left(\frac{n\pi x}{2}\right)$$

$$b'_n = -\frac{0.2}{n\pi} \cos\left(\frac{n\pi}{2}\right) + 0 + \frac{0.4}{(n\pi)^2} \cdot \sin\left(\frac{n\pi x}{2}\right) \Big|_0^1 - 0 + \frac{0.2}{n\pi} \cos\left(\frac{n\pi}{2}\right) - \frac{0.4}{(n\pi)^2} \sin\left(\frac{n\pi x}{2}\right) \Big|_1^2$$

$$b'_n = \frac{0.8}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right)$$

$$y(x,t) = \sum_{n=1}^{\infty} \frac{0.8}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{2}\right) \cos\left(\frac{n\pi}{2}t\right)$$

N.B<sup>2</sup>:

$$\sin\left(\frac{n\pi}{2}\right) = 1, 0, -1, 0, 1, 0, \dots \text{ for } n \in \mathbb{N}$$

Thus, we could write  $n = 2k - 1$  and  $b'_k = (-1)^{k+1}$  for  $k \in \mathbb{N}$

$$y(x,t) = \sum_{k=1}^{\infty} \frac{0.8(-1)^{k+1}}{(2k-1)^2\pi^2} \sin\left(\frac{(2k-1)\pi}{2}x\right) \cos\left(\frac{(2k-1)\pi}{2}t\right)$$

## 5 Example 9

$$y_{tt} = y_{xx}$$

$$y(0,t) = 0 \text{ and } y(1,t) = 0$$

$$y(x,0) = 0 \text{ and } y_t(x,0) = \sin(5\pi x)$$

Solution:  $a = 1$  and  $L = 1$  and  $f(x) = 0 \Rightarrow$  we only have the velocity problem to solve.

Need to find the Fourier sine series for  $g(x)$ :

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<sup>2</sup>Nota bene. Used to denote an important point.

$$\sum_{n=1}^{\infty} b_n \sin(n\pi x) = g(x) = \sin(5\pi x)$$

But this is already a fourier series. For any  $n$  value  $\neq 5$ ,  $b_n = 0$ .  $b_5 = 1$ .  
Therefore, the general solution

$$y(x, t) = \frac{1}{5\pi} \sin(5\pi x) \sin(5\pi t)$$

## 6 Wave Equations PDF

## Wave Equation

The wave equation takes the form:

$$y_{tt} = a^2 y_{xx}$$

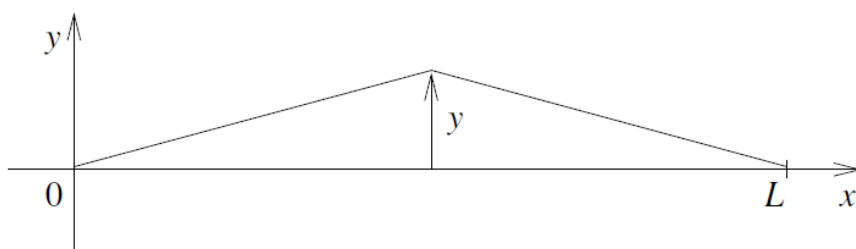
Physically,  $a = [T/\rho]^{0.5}$  (a string under tension) or  $a = [E/\rho]^{0.5}$  (elastic bar)

**Boundary conditions?**

**Initial conditons?**

**Typical IBVP** for the wave equation looks like this:

$$\begin{aligned} y_{tt} &= a^2 y_{xx} \\ y(0, t) &= 0, \quad y(L, t) = 0, \\ y(x, 0) &= f(x) \\ y_t(x, 0) &= g(x) \end{aligned}$$





## Superposition and separation of variables

1. Split  $y(x, t)$  and the initial “data” into 2 problems:

$$y(x, t) = w(x, t) + z(x, t)$$

Problem 1: initial velocity, but no displacement of string

$$\begin{aligned}w_{tt} &= a^2 w_{xx}, \\w(0, t) &= w(L, t) = 0, \\w(x, 0) &= 0 && \text{for } 0 < x < L, \\w_t(x, 0) &= g(x) && \text{for } 0 < x < L.\end{aligned}$$

Problem 2: initial displacement, but no velocity of string

$$\begin{aligned}z_{tt} &= a^2 z_{xx}, \\z(0, t) &= z(L, t) = 0, \\z(x, 0) &= f(x) && \text{for } 0 < x < L, \\z_t(x, 0) &= 0 && \text{for } 0 < x < L.\end{aligned}$$

2. Solve each problem by separation of variables

Exactly analogous procedure for Neumann boundary conditions

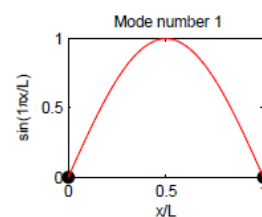
## Period and frequency of the nth mode:

Modes of vibration:

- Note these are standing waves of wavelength  $\lambda_n = 2L/n$
- Each mode:  $n+1$  positions at which displacement is zero

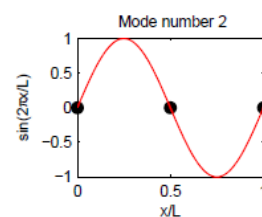
I: The fundamental mode of vibration with 2 nodes

$$X_1(x) = \sin\left(\frac{\pi x}{L}\right)$$



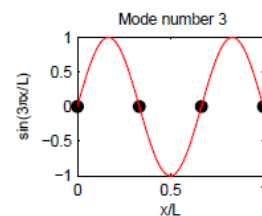
II: The second mode of vibration or first overtone with 3 nodes

$$X_2(x) = \sin\left(\frac{2\pi x}{L}\right)$$



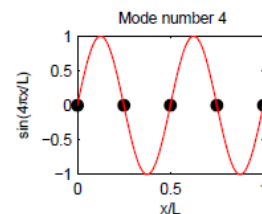
III: The third mode of vibration with 4 nodes

$$X_3(x) = \sin\left(\frac{3\pi x}{L}\right)$$



IV: The fourth mode of vibration with 5 nodes

$$X_4(x) = \sin\left(\frac{4\pi x}{L}\right)$$



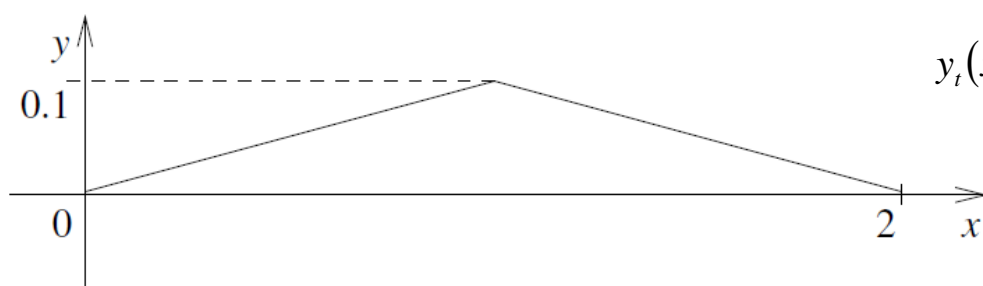
**Example 8:** Solve the IBVP

$$y_{tt} = y_{xx}$$

$$y(0, t) = 0, \quad y(2, t) = 0,$$

$$y(x, 0) = \begin{cases} 0.1x & 0 \leq x \leq 1 \\ 0.1(2 - x) & 1 < x \leq 2 \end{cases}$$

$$y_t(x, 0) = 0$$



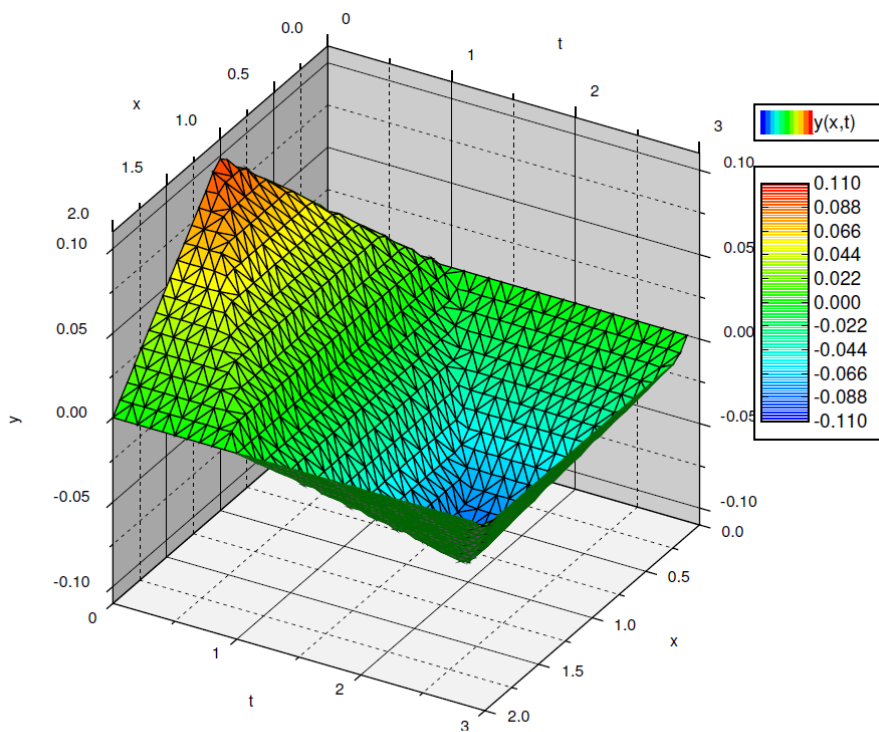


Figure 4.20: Shape of the plucked string for  $0 < t < 3$ .

**Example 9:** Solve the IBVP – what makes this one simple?

$$\begin{aligned}y_{tt} &= y_{xx} \\ y(0, t) &= 0, \quad y(1, t) = 0, \\ y(x, 0) &= 0 \\ y_t(x, 0) &= \sin 5\pi x\end{aligned}$$

**Wave Equation with Neumann boundary condition:**

$$\begin{aligned}y_{tt} &= a^2 y_{xx} \\ y_x(0, t) &= 0, \quad y_x(L, t) = 0, \\ y(x, 0) &= f(x) \\ y_t(x, 0) &= g(x)\end{aligned}$$

**D'Alembert's solution:**

$$y_{tt} = a^2 y_{xx}$$

Return to wave equation and see if we can guess a solution of exponential form:

$$y(x, t) = e^{ikx + \sigma t}$$

Why this form?

$$y_1(x, t) = e^{ik(x+at)}$$

$$y_2(x, t) = e^{ik(x-at)}$$

Is this form of solution more general – how about:

$$y_1(x, t) = F(x - at), \quad y_2(x, t) = G(x + at)$$

Consider a change of variables:  $\xi=x-at$ ,  $\eta=x+at$

Suppose initial conditions:

$$\begin{aligned}y(x, 0) &= f(x) \\ y_t(x, 0) &= g(x)\end{aligned}$$

Finally, D'Alembert's solution:

$$y(x, t) = \frac{1}{2} [f(x - at) + f(x + at)] + \frac{1}{2a} \int_{x-at}^{x+at} g(s) ds$$



**Above analysis has no boundary conditions!**

Let  $F_o(x)$  and  $G_o(x)$  be the odd  $2L$ -periodic extensions of  $f(x)$  and  $g(x)$ , respectively.

$$y(x, t) = \frac{1}{2} [F_o(x - at) + F_o(x + at)] + \frac{1}{2a} \int_{x-at}^{x+at} G_o(\zeta) d\zeta$$

**What is the relationship between d'Alembert's formula and our separation of variables solution?**

$$\begin{aligned} y(x, t) &= \sum_{n=1}^{\infty} b_n \frac{L}{n\pi a} \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{n\pi a}{L} t\right) + c_n \sin\left(\frac{n\pi}{L} x\right) \cos\left(\frac{n\pi a}{L} t\right) \\ &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L} x\right) \left[ b_n \frac{L}{n\pi a} \sin\left(\frac{n\pi a}{L} t\right) + c_n \cos\left(\frac{n\pi a}{L} t\right) \right]. \end{aligned}$$

**Region of influence & domain of dependence:**

**Example 10:** Solve the following IVP using D'Alembert's method

$$\begin{aligned}y_{tt} &= y_{xx}, & -\infty < x < \infty \\y(x, 0) &= \begin{cases} 1, & |x| < 1 \\ 0, & \text{otherwise} \end{cases} \\y_t(x, 0) &= 0\end{aligned}$$

**Example 11:** Solve the following IVP using D'Alembert's method

$$\begin{aligned}y_{tt} &= y_{xx}, \\y(0, t) &= 0, \quad y(1, t) = 0, \\y(x, 0) &= \begin{cases} 0, & 0 \leq x < 0.45 \\ 20(x - 0.45), & 0.45 \leq x < 0.5 \\ 20(0.55 - x), & 0.5 \leq x < 0.55 \\ 0, & 0.55 \leq x \leq 1 \end{cases} \\y_t(x, 0) &= 0\end{aligned}$$

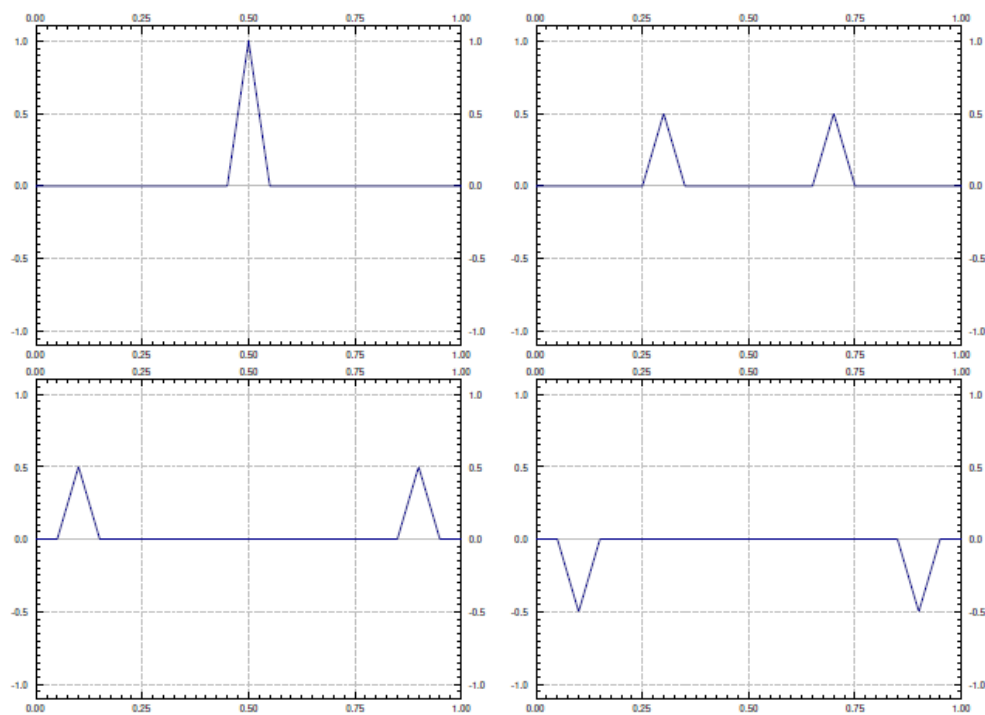


Figure 4.21: Plot of the d'Alembert solution for  $t = 0$ ,  $t = 0.2$ ,  $t = 0.4$ , and  $t = 0.6$ .