MATH 316 Lecture 4

Ashtan Mistal

May 17 2021

1 Introduction

1.1 Week 2:

- finish power series

- Bessel's function

- Intro to PDEs

1.2 Week 3:

- Fourier series and separation of variables

- Heat equations

- Wave equations

- Laplace equations

1.3 Week 4:

- Boundary value problems and Sturm-Louiville Theory

- Eigenfunctions and eigenvalues

- Sturm-Louisvill theory for BVP

- Non-homogenous boundary value problems

1.4 Week 5:

- Numerical methods for solving PDEs

2   Recap of Previous Week

What we're covering next was partly covered in previous lectures by Parisa:

Series Solutions near a regular singular point (RSP)

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

We consider an expansion about a regular singular point x_0 of the ODE. Next, we take the limits:

$$\lim_{x \to x_0} (x - x_0)\frac{Q(x)}{P(x)} = p_0$$

$$\lim_{x \to x_0} (x - x_0)^2\frac{R(x)}{P(x)} = q_0$$

When these two limits exist and are finite, then this leads to the following. We divide $P(x)y'' + Q(x)y' + R(x)y = 0$ by $P(x)$ and multiply it by $(x - x_0)^2$:

$$(x - x_0)^2 y'' + (x - x_0) \underbrace{\left\{ (x - x_0)\frac{Q(x)}{P(x)} \right\}}_{p(x) = p_0 + p_1(x - x_0) + \ldots} y' + \underbrace{\left\{ (x - x_0)^2\frac{R(x)}{P(x)} \right\}}_{q(x) = q_0 + q_1(x - x_0) + \ldots} y = 0$$

$$(x - x_0)^2 + p_0(x - x_0)y' + q_0 y = 0$$

is the C-E equation. This equation has a solution of the following format:

$$y_0(x) = (x - x_0)^r$$

To include the effect of the neglected terms, we modify the above solution:

$$y(x) = \underbrace{(x - x_0)^r}_{\text{C-E solution}} \sum_{n=0}^{\infty} \underbrace{a_n(x - x_0)^n}_{\text{The solution}}$$

with $a_0 \neq 0$. (2) is the modified equation for $y_0(x) = (x - x_0)^r$ and is Frobenius series.

2.1   Solution procedure

1. Find values of r

2. Find the recursive relation for n

3. Find the radius of convergene for $\sum_{n=0}^{\infty} a_n(x - x_0)^{n+r}$

2.2 Example 1

$$Ly = 2x^2y'' - xy' + (1-x)y = 0$$

Singular point here is when $x_0 = 0$

Again, to test if it is a singular point, we take the limits.

$$p_0 = \lim_{x \to 0} \frac{-x}{2x^2} x = \frac{-1}{2}$$

Similarly, we check q_0:

$$q_0 = \lim_{x \to 0} \frac{(1-x)}{2x^2} x^2 = \frac{1}{2} = q_0$$

Hence, the singular point is a RSP.

$$\Rightarrow y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$$

Step 1: Find values of r

Characteristic equation: $r(r-1) + p_0 r + q_0 = 0$

This becomes $r(r-1) - \frac{r}{2} + \frac{1}{2} = 0 \Rightarrow r^2 - \frac{3}{2}r + \frac{1}{2} = 0$

Hence, $r = 1$ and $r = \frac{1}{2}$ are roots of the indicial equation.

$$y_0 = c_1 x + c_2 x^{\frac{1}{2}}$$

Now, we go to step 2: Find the recursive relation for n.

Take the first and second derivative of the summation (that we've used before) and sub into the equation.

$$2\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

Changing index:

$$2\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - \underbrace{\sum_{n=0}^{\infty} a_n x^{n+r+1}}_{m=n+1} = 0$$

All others are $m = n$.

With that, we get the following:

$$2\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=1}^{\infty} a_{n-1}x^{n+r}$$

(Removing $= 0$ due to space)

3

Peeling off the first terms ($n = 0$), we end up with the following sum:

$$\underbrace{2a_n r(r-1)x^r - a_0 r x^r + a_0 x^r}_{=a_0 x^r (2r^2 - 3r + 1)} \longrightarrow \dots$$

$$\hookrightarrow + \sum_{n=1}^{\infty} \left[2a_n(n+r)(n+r-1) - a_n(n+r) + a_{n-1}\right] x^{n+r} = 0$$

Hence, $a_n = \frac{a_{n-1}}{(n+r)(2(n+r)-3)+1}$ is the recursive relation for r values
Finding the recursive relation for r_1 and r_1:

$$r_1 \Rightarrow a_n = \frac{a_{n-1}}{(n+1)(2(n+1)-3)+1} = \frac{1_{n-1}}{2n^2+n}$$

$n = 1 : a_1 = \frac{a_0}{3}; n = 2 : a_2 = \frac{a_1}{10} = \frac{a_0}{30}; n = 3 : a_3 = \frac{a_2}{21} = \frac{a_0}{630}$
Therefore:

$$y(x) = a_0 x^1 \left(1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \dots\right)$$

Next, for $r_1 = \frac{1}{2}$:

$$a_n = \frac{a_{n-1}}{(n+\frac{1}{2}(2n+1-3)+1)} = \frac{a_{n-1}}{2(n+\frac{1}{2})(n-1)+1}$$

$n = 1 : a_1 = \frac{a_0}{1}; n = 2 : a_2 = \frac{a_1}{6} = \frac{a_0}{6}; n = 3 : a_3 = \frac{a_0}{90}$
So, we have the following:

$$y_2(x) = a_0 x^{\frac{1}{2}} \left[1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \dots\right]$$

Now, onto step 3:
Find the radius of convergence.
To find the radius of covergence, we use the ratio test:

$$\lim_{n \to \infty} \left| \frac{a_n x^n}{a_{n-1} x^{n-1}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{a_{n-1} x^n}{(n+r)(2(n+r)-3)+1} \frac{1}{a_{n-1} x^{n-1}} \right|$$

The above is not correct. Why?

$$\lim_{n \to \infty} |x| \left| \frac{1}{(n+r)(2(n+r)-3)+1} \right| = 0$$

$\Rightarrow p = \infty$ for all x values
Final solution:

$$y(x) = y_1(x) + y_2(x)$$

$$y(x) = C_1 x \left(1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \dots\right) + C_2 x^{\frac{1}{2}} \left(1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \dots\right)$$

4

3 Bessel's equation

Applications: PDEs on circular / cylindrical domain.

e.g. heating and cooling in circular / cylindrical geometries, i.e. pipes and heat exchangers.

The equation format:

$$Ly = x^2 y'' + xy' + \left(x^2 - \nu^2\right) y = 0 \tag{1}$$

ν is a constant and specifies the order of the equation.

$x_0 = 0$ is a regular singular point, since $\lim_{x \to 0} \frac{x}{x^2}(x) = 0 = p_0$ and $\lim_{x \to 0} \frac{x^2 - \nu^2}{x^2}(x^2) = -\nu^2 = q_0$

3.1 Step 1: Finding the r values

The characteristic equation is:

$$r(r-1) + p_0 r + q_0 = 0$$

$$r(r-1) + r - \nu^2 = 0 \Rightarrow r = \pm \nu$$

3.2 Step 2: Frobenius part − Finding recursive relation

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

Substituting into the equation, we get the following:

$$Ly = x^2 y'' + xy' + \left(x^2 - \nu^2\right) y = 0$$

$$Ly = x^2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} \rightarrow \dots$$

$$\hookrightarrow + x \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} \rightarrow \dots$$

$$\hookrightarrow + \left(x^2 - \nu^2\right) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Shifting index:

$$Ly = x^2 y'' + xy' + \left(x^2 - \nu^2\right) y = 0$$

$$Ly = x^2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r-2} \rightarrow \dots$$

$$\hookrightarrow +x \sum_{n=0}^{\infty} a_n (n+r)x^{n+r-1} \rightarrow \dots$$

$$\hookrightarrow +x^2 \underbrace{\sum_{n=0}^{\infty} a_n x^{n+r+2}}_{m=n+2} -\nu^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Hence, we get the following:

$$Ly = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r)x^{n+r} + \sum_{n=2}^{\infty} a_{n-2}x^{n+r}$$

Next, we peel off the first two terms to match the summation.

$$a_0(r(r-1)+r-\nu^2)x^r + a_1((1+r)r + (1+r) - \nu^2)x^{r+1} \rightarrow \dots$$

$$\hookrightarrow + \sum_{n=2}^{\infty} a_n \left[a_n \left((n+r)^2 - \nu^2\right) + a_{n-2}\right] x^{n+r} = 0$$

$$a_0 x^r (r^2 - \nu^2) + a_1 x^{r+1} \left((r+1)^2 - \nu^2\right) + \sum_{n=2}^{\infty} a_n \left[a_n \left((n+r)^2 - \nu^2\right) + a_{n-2}\right] x^{n+r} = 0$$