Wave Equation

The wave equation takes the form:

$$\boldsymbol{y_{tt} = a^2 y_{xx}}$$

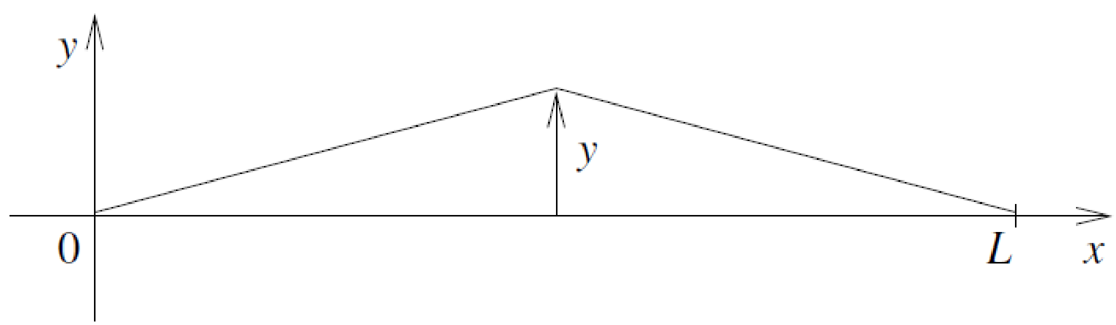
Physically, $a = [T/\rho]^{0.5}$ (a string under tension) or $a = [E/\rho]^{0.5}$ (elastic bar)

Boundary conditions?

Initial conditons?

Typical IBVP for the wave equation looks like this:

$$
\begin{aligned}
&y_{tt} = a^2 y_{xx} \\
&y(0,t) = 0, \quad y(L,t) = 0, \\
&y(x,0) = f(x) \\
&y_t(x,0) = g(x)
\end{aligned}
$$



Superposition and separation of variables

1. Split $y(x,t)$ and the initial "data" into 2 problems:

$$y(x,t) = w(x,t) + z(x,t)$$

Problem 1: initial velocity, but no displacement of string

$$\begin{aligned} &w_{tt} = a^2 w_{xx}, \\ &w(0,t) = w(L,t) = 0, \\ &w(x,0) = 0 && \text{for } 0 < x < L, \\ &w_t(x,0) = g(x) && \text{for } 0 < x < L. \end{aligned}$$

Problem 2: initial displacement, but no velocity of string

$$\begin{aligned} &z_{tt} = a^2 z_{xx}, \\ &z(0,t) = z(L,t) = 0, \\ &z(x,0) = f(x) && \text{for } 0 < x < L, \\ &z_t(x,0) = 0 && \text{for } 0 < x < L. \end{aligned}$$

2. Solve each problem by separation of variables

Exactly analogous procedure for Neumann boundary conditions

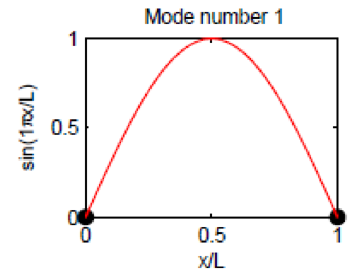
Period and frequency of the nth mode:

Modes of vibration:

- Note these are standing waves of wavelength *$\lambda_n = 2L/n$*
- Each mode: *$n+1$* positions at which displacement is zero

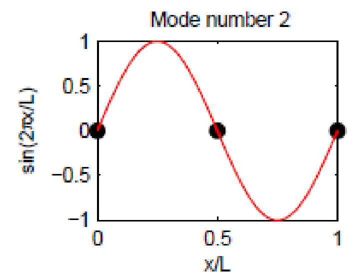
I: The fundamental mode of vibration with 2 nodes

$$X_1(x) = \sin\left(\frac{\pi x}{L}\right)$$



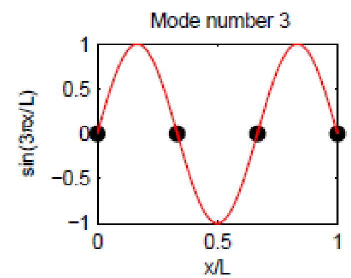
II: The second mode of vibration or first overtone with 3 nodes

$$X_2(x) = \sin\left(\frac{2\pi x}{L}\right)$$



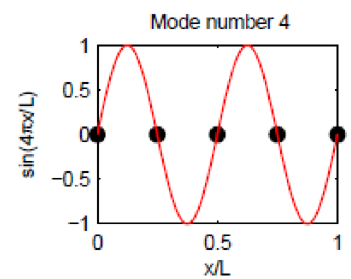
III: The third mode of vibration with 4 nodes

$$X_3(x) = \sin\left(\frac{3\pi x}{L}\right)$$



IV: The fourth mode of vibration with 5 nodes

$$X_4(x) = \sin\left(\frac{4\pi x}{L}\right)$$



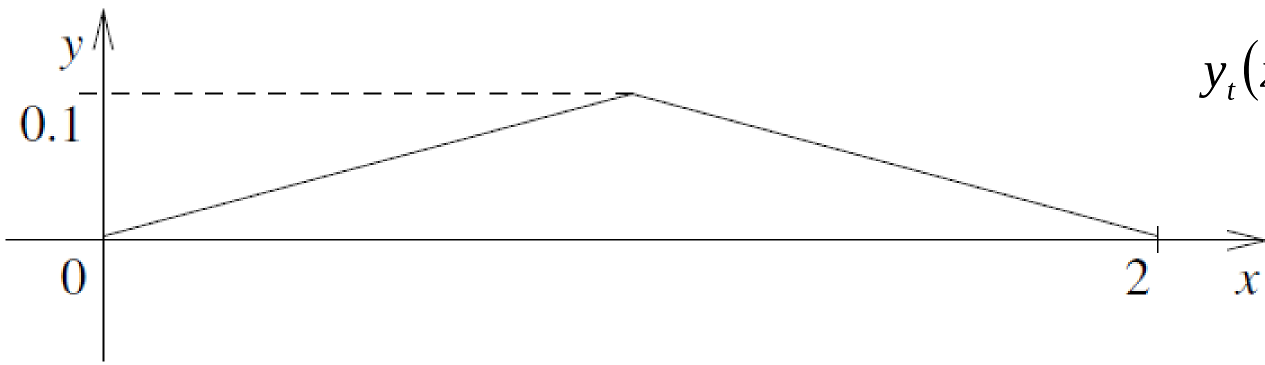
Example 8: Solve the IBVP

$$y_{tt} = y_{xx}$$

$$y(0,t) = 0, \quad y(2,t) = 0,$$

$$y(x,0) = \begin{cases} 0.1x & 0 \le x \le 1 \\ 0.1(2-x) & 1 < x \le 2 \end{cases}$$

$$y_t(x,0) = 0$$



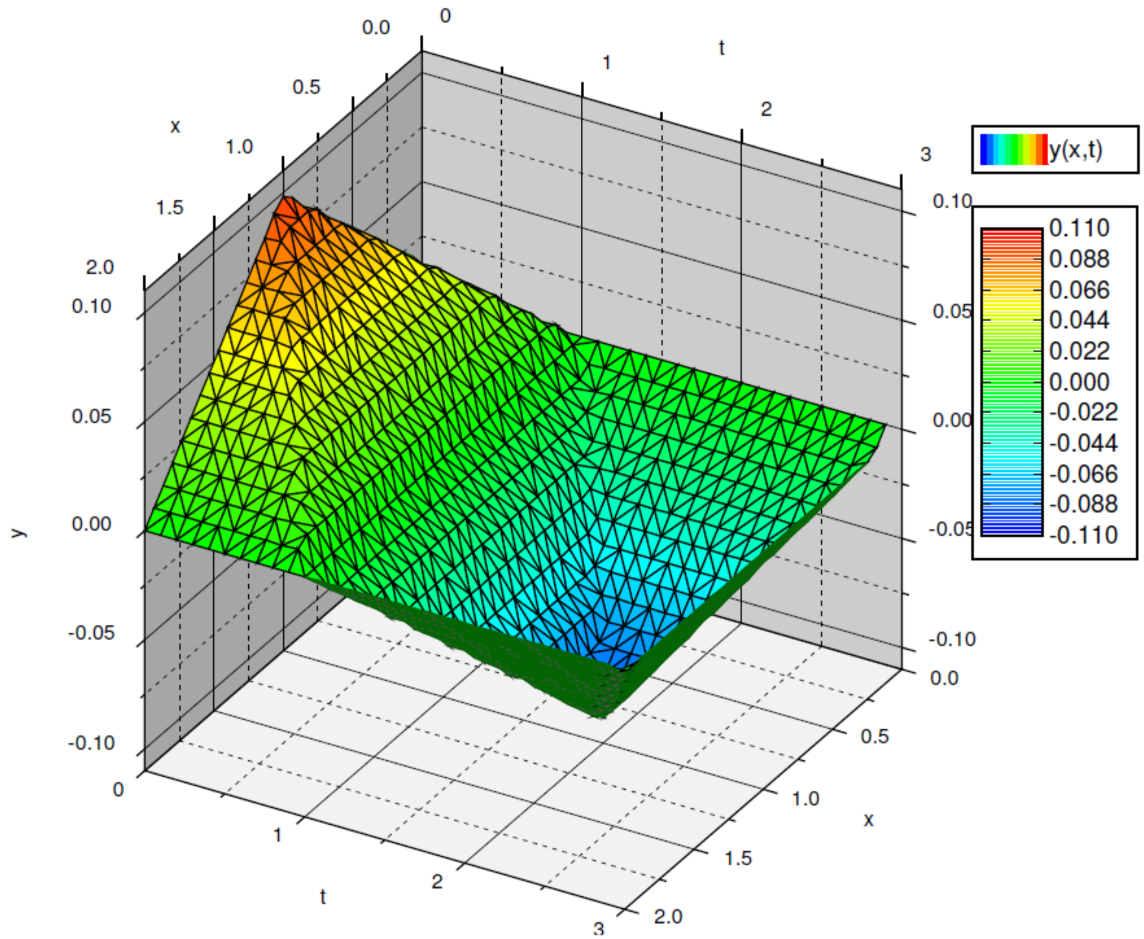


Figure 4.20: Shape of the plucked string for $0 < t < 3$.

Example 9: Solve the IBVP – what makes this one simple?

$$
\begin{aligned}
&y_{tt} = y_{xx} \\
&y(0,t) = 0, \quad y(1,t) = 0, \\
&y(x,0) = 0 \\
&y_t(x,0) = sin\,5\,\pi x
\end{aligned}
$$

Wave Equation with Neumann boundary condition:

$$y_{tt} = a^2 y_{xx}$$
$$y_x(0,t) = 0, \quad y_x(L,t) = 0,$$
$$y(x,0) = f(x)$$
$$y_t(x,0) = g(x)$$

D'Alembert's solution:

$$y_{tt} = a^2 y_{xx}$$

Return to wave equation and see if we can guess a solution of exponential form:

$$y(x,t) = e^{ikx+\sigma t}$$

Why this form?

$$y_1(x,t) = e^{ik(x+at)}$$
$$y_2(x,t) = e^{ik(x-at)}$$

Is this form of solution more general – how about:

$$y_1(x,t) = F(x-at), \qquad y_2(x,t) = G(x+at)$$

Consider a change of variables: $\xi = x - at$, $\eta = x + at$

Suppose initial conditions:

$$y(x,0) = f(x)$$
$$y_t(x,0) = g(x)$$

Finally, D'Alembert's solution:

$$y(x,t) = \frac{1}{2}[f(x-at) + f(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds$$

Above analysis has no boundary conditions!

Let $F_o(x)$ and $G_o(x)$ be the odd 2L-periodic extensions of $f(x)$ and $g(x)$, respectively.

$$y(x,t) = \frac{1}{2}[F_o(x-at) + F_o(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} G_o(\zeta)\, d\zeta$$

What is the relationship between d'Alembert's formula and our separation of variables solution?

$$\boxed{\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} b_n \frac{L}{n\pi a} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi a}{L}t\right) + c_n \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{n\pi a}{L}t\right) \\ &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) \left[b_n \frac{L}{n\pi a} \sin\left(\frac{n\pi a}{L}t\right) + c_n \cos\left(\frac{n\pi a}{L}t\right)\right]. \end{aligned}}$$

Region of influence & domain of dependence:

Example 10: Solve the following IVP using D'Alembert's method

$$y_{tt} = y_{xx}, \qquad -\infty < x < \infty$$

$$y(x,0) = \begin{cases} 1, & |x| < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$y_t(x,0) = 0$$

Example 11: Solve the following IVP using D'Alembert's method

$$y_{tt} = y_{xx},$$
$$y(0,t) = 0, \quad y(1,t) = 0,$$
$$y(x,0) = \begin{cases} 0, & 0 \le x < 0.45 \\ 20(x - 0.45), & 0.45 \le x < 0.5 \\ 20(0.55 - x), & 0.5 \le x < 0.55 \\ 0, & 0.55 \le x \le 1 \end{cases}$$
$$y_t(x,0) = 0$$

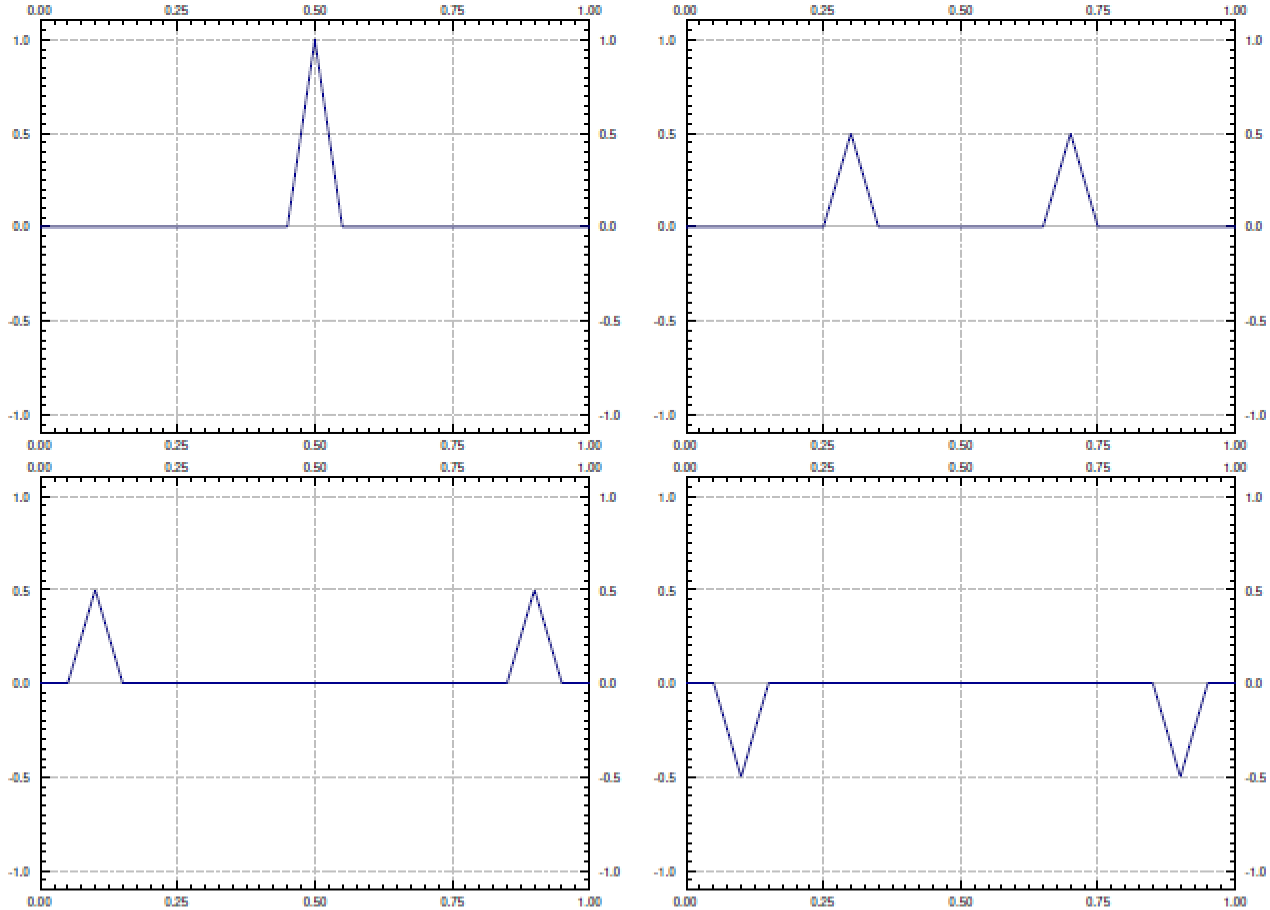


Figure 4.21: Plot of the d'Alembert solution for $t = 0$, $t = 0.2$, $t = 0.4$, and $t = 0.6$.