MATH 316 Lecture 3

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May 12 2021

1 Singular Points

Singular points are divided into two classes:

- Regular singular points, where we use the Frobenius series solution
- Irregular singular points, which are beyond the scope of the course

If we are to look at the ODE $P(x)y'' + Q(x)y' + R(x)y = 0$, and define the point x_0 as a singular point, we have the following:
The Cauchy-Euler equation is

$$(x-x_0)^2 + \alpha(x-x_0)y' + \beta y = 0 \tag{1}$$

We know that $y = (x-x_0)^r$ (As an example).
How do we make $P(x)y'' + Q(x)y' + R(x)y = 0$ look like(1)?
If we multiply with $(x-x_0)^2$ and divide by $P(x)$, we may get something similar to the Cauchy-Euler.
Then, we get something like this:

$$(x-x_0)^2 y'' + \left\{\frac{Q(x)}{P(x)}(x-x_0)\right\}(x-x_0)y' + \left\{\frac{R(x)}{P(x)}(x-x_0)^2\right\} y = 0 \tag{2}$$

Now, if $\frac{Q(x)}{P(x)}(x-x_0)$ and $\frac{R(x)}{P(x)}(x-x_0)^2$ are analytic at $x = x_0$, then the singularity is not worse than the singularity in the Cauchy-Euler equation (1), **and** x_0 is a "regular singular point". Otherwise, x_0 is an "irregular singular point".
If we start to write the Taylor series for (?),
$\frac{Q(x)}{P(x)}(x-x_0) = p_0 + p_1(x-x_0) + ...$
$\frac{R(x)}{P(x)}(x-x_0)^2 = q_0 + q_1(x-x_0) + ...$
Being analytic means that we need to be able to write the series.
An example of a singular point:

$$x y'' + 2\alpha y' = 0$$

$$\left(\frac{2\alpha}{\cancel{x}}\cancel{x}\right)$$

Here, $x = 0$ is a singular point. (We re-wrote the first equation as the second equation, I believe)

As $x \longrightarrow x_0$, our ODE becomes: $(x-x_0)^2 y'' + p_0(x-x_0)y' + q_0 y = 0$

The corresponding Cauchy-Euler equation solution: $y = (x-x_0)^r$. We need to have finite p_0 and q_0. If p_0 and q_0 are both finite, then x_0 is a regular singular point. Otherwise, it is an irregular singular point.

For regular singular points, the solution we are going to write:

$$\underbrace{y(x) = (x-x_0)^r}_{*} \underbrace{\sum_{n=0}^{\infty} a_n (x-x_0)^n}_{\text{Correction}}$$

*: The singular part of the solution to the corresponding Cauchy-Euler.

1.1 Example

$$x(1+x^2)y'' + 2xy' + (1+x^2)y = 0$$

.

Classify singular points. Here, $p(x) = x(1+x)^2$, $Q(x) = 2x$, and $R(x) = 1+x^2$. Singular points: $\begin{cases} x = 0 \\ x = \pm i \end{cases}$

We need $p(x_0) = 0$ (Take a look at the left hand side if you don't understand!).

Classify them:

$\begin{cases} \lim_{x \to x_0} \frac{Q(x)}{P(x)}(x-x_0) = p_0 \\ \lim_{x \to x_0} \frac{R(x)}{P(x)}(x-x_0)^2 = q_0 \end{cases}$

For $x_0 = 0$, we have $\lim_{x \to 0} \frac{2x}{x(1+x^2)} x = 0 = p_0$

$\lim_{x \to 0} \frac{1+x^2}{x(1+x^2)} x^2 = 0 = q_0$

Now for $x_0 = i$:

$\lim_{x \to i} \frac{2x}{x(1+x^2)}(x-i) = \lim_{x \to i} \frac{2(x-i}{(x-i)(x+i)} = \frac{1}{i} = p_0$

$\lim_{x \to i} \frac{1+x^2}{x(1+x^2)}(x-i)^2 = 0 = q_0$

We see that because both p_0 and q_0 are finite, $x = i$ is also a regular singular point.

Try for $x = -i$:

$\lim_{x \to -i} \frac{2x}{x(1+x^2)}(x+i)$

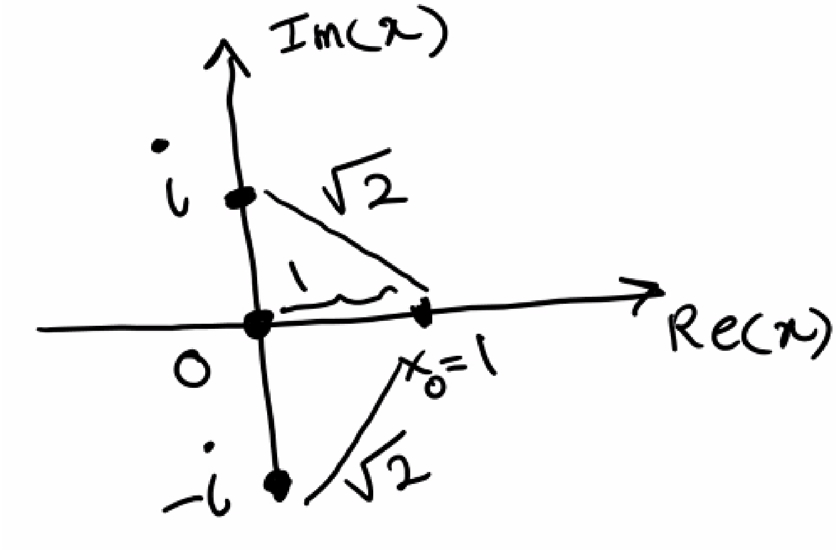
$\lim_{x \to -i} \frac{1+x^2}{x(1+x^2)}(x+i)^2$

yeah uhhhh.... review how to calculate limits.

When we calculate $x_0 = -i$, we get: $p_0 = -\frac{1}{i}$, and $q_0 = 0$. Hence, $x_0 = -i$ is also a regular singular point.

2 Radius of Convergence

The radius of convergence of the series solution is at least equal to the distance from the x_0 to the nearest singular point. In the example that we solved:



$y(x) = \sum_{n=0}^{\infty} a_n(x-1)^n$

$\rho = 1$ is the lower bound estimate. ρ is the radius of convergence. Imagine a circle of radius 1 (as it's the distance to the closest singular point)

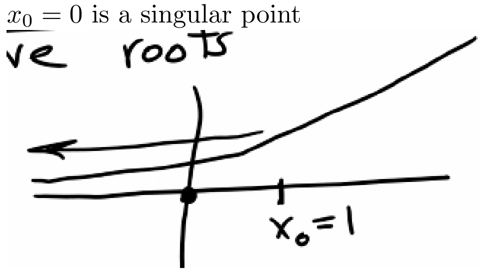
2.0.1 An Example

$$x^2y'' + \alpha xy' + \beta y = 0$$

r_1, r_2 are two positive roots

$$y = C_1x^{r_1} + C_2x^{r_2}$$

$x_0 = 0$ is a singular point



Radius of convergence is infinite. This is a rare case.

3 Frobenius Series

EX:

$$6x^2(1+x)y'' + 5xy' - y = 0$$

$P(x) = 6x^2(1+x)$, $Q(x) = 5x$, $R(x) = -1$

Singular points are $x = 0$ and $x = -1$.

For $x = 0$, let's find if it's irregular or regular:

$\lim_{x\to 0} \frac{Q(x)}{P(x)}(x-x_0) = \lim_{x\to 0} \frac{5x}{6x^2(1+x)}x = \frac{5}{6} = p_0$

$\lim_{x\to 0} \frac{-1}{6x^2(1+x)}x^2 = \frac{-1}{6} = q_0$

Therefore $x_0 = 0$ is a regular singular point.

$$(x-x_0)^2y'' + \frac{5}{6}(x-x_0)y' - \frac{1}{6}y = 0$$

This is for $x_0 = 0$. Hence, $x^2y'' + \frac{5}{6}xy' - \frac{1}{6}y = 0$.

Corresponding Cauchy-Euler equation:
$y = x^r$ and therefore $[6r(r-1) + 5r - 1]\, x^r = 0$.
Hence $6r^2 - r - 1 = 0 \longrightarrow r_{1,2} = \frac{1 \pm \sqrt{1+24}}{12} \longrightarrow r_{1,2} = \frac{1}{2}, \frac{-1}{3}$

Frobenius Series Solution about $x = 0$:

$$y(x) = x^2 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

Now, we only need to replace y, y', y'' in the ODE:

$$6x^2(1+x)y'' + 5xy' - y = 0$$

$$6x^2y'' + 6x^3y'' + 5xy' - y = 0$$

$$6\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} \qquad n=m$$

$$+6\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r+1} \qquad n+1=m \quad n=0, m=1$$

$$+5\sum_{n=0}^{\infty} a_n (n+r) x^{n+r} \qquad n=m$$

$$-\sum_{n=0}^{\infty} a_n x^{n+r} \qquad n=m$$

$$6\sum_{m=0}^{\infty} a_m(m+r)(m+r-1)x^{m+r}$$

$$+6\sum_{m=1}^{\infty} a_{m-1}(m+r-1)(m+r-2)x^{m+r}$$

$$+5\sum_{m=0}^{\infty} a_m(m+r)x^{m+r} - \sum_{m=0}^{\infty} a_m x^{m+r}$$

peel off

$6a_0 r(r-1)x^r + 5a_0 r x^r - a_0 x^r +$

$$\sum_{m=1}^{\infty}\left[6a_m(m+r)(m+r-1) + 6a_{m-1}(m+r-1)(m+r-2) + 5a_m(m+r) - a_m\right]x^{m+r} = 0$$

$x^r\Big]\ a_0\underbrace{\left[6r(r-1)+5r-1\right]}_{\text{C-E eqn}} = 0 \rightarrow r_{1,2} = \frac{1}{2}, -\frac{1}{3}$

$\left.\begin{matrix}x^{m+r}\\ m\geq 1\end{matrix}\right]\ a_m\left[6(m+r)(m+r-1)+5(m+r)-1\right] = -6a_{m-1}(m+r-1)(m+r-2)$

$$a_m = \frac{-6a_{m-1}(m+r-1)(m+r-2)}{(m+r)\left[6m+6r-6+5\right]-1}$$

recursion for r_1 & r_2

$r = -\frac{1}{3}$, $m \geq 1$

$$a_m = \frac{-6a_{m-1}(m-\frac{4}{3})(m-\frac{7}{3})}{(m-\frac{1}{3})(6m-3)-1}$$

$m=1$: $a_1 = \dfrac{-6a_0(-\frac{1}{3})(-\frac{4}{3})}{(+\frac{2}{3})(3)-1} = \dfrac{-8a_0}{3}$

$m=2$: $a_2 = \dfrac{-32a_0}{42(3)}$

$$y_1(x) = a_0 x^{\frac{-1}{3}}\left[1 - \frac{8}{3}x - \frac{32}{3(42)}x^2 + ...\right]$$

$r = \frac{1}{2}$: If we follow the same steps, we get
$y_2(x) = \underbrace{a_0}_{\text{A different } a_0} x^{\frac{1}{2}}\left[1 + \frac{3}{22}x - \frac{22}{22.68}x^2 + ...\right]$
Hence the general solution is $y = C_1 y_1(x) + C_2 y_2(x)$

3.0.1 Convergence: ratio test

$$\sum_{m=0}^{\infty} C_m : \lim_{m\to\infty}|\frac{C_{m+1}}{C_m}| < 1$$

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}$$

Ratio test:

$$\lim_{m\to\infty}|\frac{a_{m+1}x^{m+r+1}}{a_m x^{m+r}}| = |x|\lim_{m\to\infty}|\frac{a_{m=1}}{a_m}$$

For this example:

$$|x|\lim_{m\to\infty}|\frac{a_{m=1}}{a_m}, a_m = \frac{-6a_{m-1}(m+r-1)(m+r-2)}{(m+r)(6m+6r-1)-1}$$

$$|x| \lim_{m\to\infty} \frac{-6a_{m-1}(m+r-1)(m+r-2)}{(m+r)(6m+6r-1)-1} = |x| \cdot 1$$

$|x| < 1$

$-1 < x < 1$

$\rho =$ is at least equal to the distance between $x_0 = 0$ & the nearest singular point, $\rho = 1$

$x_{singular} = 0, -1$