316 Notes: Lecture 1

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Some basic, quick notes regarding the course:

- Watch the pre-recorded lectures prior to the actual lectures
- Homeworks: 20%; Midterm: 30%; Final: 50%
- Homeworks are best 4 out of 5
- Exams will be open book, open notes
- Homeworks can be written + scanned, or written using OneNote or LaTeX

1 Brief Overview of ODEs

First Order ODEs

- Separable ODEs (ex: $y' = \frac{dy}{dx} = p(x) \cdot q(y)$
- Linear: $Ly = y' + p(x)y = Q(x)$, where $\underbrace{L}_{\text{linear}} = \frac{d}{dx} + p(x)$

Second Order ODEs:

- Constant coefficient ODE: $Ly = y'' + ay' + by = 0$
- Cauchy-Euler, or Equi-dimensional ODEs: $Ly = x^2y'' + axy' + by = 0$
- L is the linear operator. In the constant coefficient, $L = \frac{d^2}{dx^2} + a\frac{d}{dx} + b$

1.1 Examples

1.1.1 First Order Separable Equation Examples

$\frac{dy}{dx} = p(x)Q(y) \longrightarrow \int \frac{dy}{Q(y)} = \int P(x)dx + C$

Then, we get:

$\frac{dy}{dx} = y\cos(x) \rightarrow \frac{dy}{y} = \cos(x)dx \longrightarrow \ln|y| = \sin(x) + C \rightarrow y = C_1e^{\sin(x)}$

1.1.2 Linear First Order Equation:

$y' + p(x)y = Q(x)$

Multiply both sides by an integrating factor $\mu(x)$

Ex: $y' + \frac{2x}{1+x^2}y = \frac{\cot(x)}{1+x^2}$ $\mu(x)y' + \underbrace{\mu(x)\frac{2x}{1+x^2}y}_{\mu'(x)y} = \frac{\cot(x)}{1+x^2}\mu(x)$ Compare to: $\mu(x)y' + \mu'(x)y = [\mu(x)y(x)]'$

$\frac{d\mu}{dx} = \mu\frac{2x}{1+x^2}$: integrate: $\int \frac{d\mu}{\mu} = \int \frac{2x}{1+x^2}dx + C$

Hence, using integrating factor:

$$\ln|\mu(x)| = \ln(1 + x^2) + C$$

with $\mu(x) = C_1(1+x^2)$

Hence,

$$C_1(1+x^2)y' + C_1 2xy = C_1 \cot(x)$$

$$\frac{d}{dx}\left[(1+x^2)y\right] = \cot(x)$$

$$(1+x^2)y = \int \cot(x) + C = \ln|\sin(x)| + C$$

Hence,

$$y(x) = \frac{\ln|\sin(x)|}{1+x^2} + \frac{C}{1+x^2}$$

1.2 Second Order Constant Coefficient ODE

$$ay'' + by' + cy = 0$$

We start with a guess: e^{rx}, and substitute: $(ar^2 + br + c) \cdot e^{rx} = 0$. Note that $ar^2 + br + c$ is the characteristic equation. We then have two solutions:'

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \Delta = b^2 - 4ac$$

Based on the sign of $Delta$, we have 3 different cases:

- $\Delta > 0$: 2 real, distinct roots
- $\Delta < 0$: 2 complex roots
- $\Delta = 0$: Repeated roots.

Example: $2y'' + 2y' + y = 0$: Guess e^{rx}.

$2r^2 + 2r + 1 = 0 \longrightarrow r_{1,2} = \frac{-2\pm\sqrt{4-8}}{2(2)}, \Delta = -4$. As $-4 < 0$, this is 2 complex roots. We end up with the following solution:

$$e^{\frac{-x}{2}}\left[C_1 e^{\frac{i}{2}x} + C_2 e^{\frac{-i}{2}x}\right]$$

Using Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, we get the following:

$$e^{\frac{-x}{2}}\left[C_1(\cos(\frac{x}{2}) + i\sin(\frac{x}{2})) + C_2(\cos(\frac{x}{2}) - i\sin(\frac{x}{2}))\right]$$

$$e^{\frac{-x}{2}}\left[(C_1 + C_2)\cos(\frac{x}{2}) + i(C_1 + C_2)\sin(\frac{x}{2})\right]$$

$$y(x) = e^{\frac{-x}{2}}\left[A_1 \cos\frac{x}{2} + A_2 \sin\frac{x}{2}\right]$$

Real form of the solution, A1 & A2: $c_1 = a_1 + ib_1$; $c_2 = a_2 + ib_2$

$A_1 = (a_1 + a_2) + i(b_1 + b_2)$, and $A_2 = i(a_1 - a_2) - (b_1 - b_2)$. $b_1 + b_2 = 0$, $a_1 - a_2 = 0$

Example: $y'' - 2y' + y = 0$. Characteristic equation: $r^2 - 2r + 1 = 0 \longrightarrow (r-1)^2 = 0$, and therefore the roots are $r = 1$ repeated. Hence,

$$y = C_1 e^x + C_2 x e^x$$

is the solution of the equation.

1.2.1 Cauchy - Euler Eqn

$$Ly = x^2y'' + \alpha xy' + \beta y = 0$$

Guess: $y = x^r$
EX: $2x^2y'' - xy' + y = 0$. $y(x) = x^r, y'(x) = rx^{r-1}, y'' = r(r-1)x^{r-2}$.
Therefore,

$$2r(r-1)x^r - rx^r + x^r = 0$$

which is equivalent to

$$[2r(r-1) - r + 1]\, x^2 = 0$$

$$2r(r-1) - r + 1 = 0 \longrightarrow 2r^2 - 3r + 1 = 0$$

$$r_{1,2} = \frac{3 \pm \sqrt{9-8}}{2(2)} = 1, \frac{1}{2}$$

Hence, the solution to the equation is $y(x) = C_1x + C_2x^{\frac{1}{2}}$

EX2: $x^2y'' - xy' + y = 0$
$y(x) = x^r$ hence, we get: $r(r-1)x^r - rx^r + x^r = 0$.
Therefore $[r(r-1) - r + 1]\, x^r = 0$.

$$r^2 - 2r + 1 = (r-1)^2 = 0 \rightarrow r = 1$$

$$y_1 = x, y_2 = \ln(x) \cdot x$$

$$y(x) = C_1x + C_2x\ln(x)$$

(?) $Ly = 0, L\frac{d}{dr}y(x,r) = 0, \frac{d}{dr}(x^r) = x^r\ln(x)$

EX3: $x^2y'' - xy' + 5y = 0$. $y(x) = x^r$, and therefore $r(r-1)x^2 - rx^r + 5x^r = 0$
$[r(r-1) - r + 5]\, x^r = 0$, hence, $r^2 - 2r + 5 = 0$. We then get the general solution of the following:

$$y(x) = C_1x^{1+2i} + C_2x^{1-2i}$$

This can be re-written as the following:

$$y(x) = x\left[C_1e^{2i\ln(x)} + C_2e^{-2i\ln(x)}\right]$$

And hence as the following:

$$y(x) = x\,[C_1(\cos(2\ln(x) + i\sin(2\ln(x)) + C_2(\cos(2\ln(x)) - i\sin(2\ln(x)))]$$

$$= x\,[(c_1 + c_2)\cos(2\ln(x)) + i(c_1 - c_2)\sin(2\ln(x))]$$

$$= x\,[A_1\cos(2\ln(x)) + A_2\sin(2\ln(x))]$$

A_1 and A_2 are real.

EX4: Solve the IVP

$$x^2y'' - 3xy' + 4y = 0, y(1) = 1, y'(1) = 1$$

If we let $y(x) = x^r$, then we get the following: $y(x) = x^r, y'(x) = rx^{r-1}, y'' = r(r-1)x^{r-2}$

Plugging this into our equations:
$x^2r(r-1)x^{r-2} - 3xrx^{r-1} + 4x^r = 0$

Solving, we discover we have a repeated root of 2. Hence, the general solution is $y(x) = C_1x^2 + C_2x^2\ln(x)$. Plugging in the initial conditions, we find that $C_1 = 1$, and $C_2 = -1$ and hence the solution is $y(x) = x^2 - x^2\ln(x)$

2 Series Solutions of ODEs

We use power series expansion to solve variable coefficient linear ODEs. Remember that a function $f(x)$ can be approximated by a polynomial of degree n, such that $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$. As the degree of n increases, the approximation improves. Hence, $f(x) = \sum_{m=0}^{\infty} a_m x^m$. Or, in general, we can approximate $f(x)$ by a power series expanded about a point x_0, writing it as $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$. We can remember that when we had the Taylor series, $a_n = \frac{f^{(n)}(x_0)}{n!}$. So, we have: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}x(0)}{n!}(x-x_0)^n$.

2.0.1 Example: $y' + (1-2x)y = 0$

Using the integrating factor, $\mu(x) = e^{\int(1-2x)dx} = e^{x-x^2}$.

$\left[ye^{x-x^2}\right]' = 0 \longrightarrow ye^{x-x^2} = C \rightarrow y = Ce^{-x+x^2}$.

Use Taylor expansion $y(x)$ about the point $x_0 = 0$.

In order to do this, we write it as the sum described above:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}x(0)}{n!}(x-x_0)^n$$

We can then let $y = \sum_{n=0}^{\infty} \frac{f^{(n)}x(0)}{n!}x^n$ and $y' = \sum_{n=1}^{\infty} \frac{f^{(n)}x(0)}{n!}nx^{n-1}$, and therefore we can write our ODE as

$$\sum_{n=1}^{\infty} \frac{f^{(n)}x(0)}{n!}x^{n-1} + (1-2x)\sum_{n=0}^{\infty} \frac{f^{(n)}x(0)}{n!}x^n = 0$$