

MATH 316 Lecture 2

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1 Solving the problem from last class

We started with a simple example that was solvable using integrating factor, and we are now going to do the Taylor expansion of the **answer**, around the point $x = x_0$. The confusion from last class was that we tried to solve the ODE using a Taylor expansion **of the differential equation**, as opposed to the actual answer.

$$y(x) = y(0) + \frac{y'(0)}{1}x + \frac{y''(0)}{2!}x^2 + \dots$$

$$y(x) = C \left[1 - x + \frac{3}{2}x^2 - \frac{7}{6}x^3 + \dots \right]$$

From the class notes:

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$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n \quad \star$$

(Ex) $y' + (1-2x)y = 0$

Integrating factor : $\mu(x) = e^{\int (1-2x)dx} = e^{x-x^2}$

$$[y e^{x-x^2}]' = 0 \rightarrow y e^{x-x^2} = C \rightarrow y = C e^{-x+x^2}$$

Taylor expand $y(x)$ about the point $x_0=0$

$$y(x) = y(0) + \frac{y'(0)}{1}x + \frac{y''(0)}{2!}x^2 + \dots$$

$$\star y(x) = C \left[1 - x + \frac{3}{2}x^2 - \frac{7}{6}x^3 + \dots \right] \quad \checkmark$$

$y'(0) = -C$

Note that we factored C out of the Taylor expansion. Moving forward, assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$y' + (1-2x)y = 0$$

$$y' + y - 2xy = 0$$

$$\text{letting } y = \sum_{n=0}^{\infty} a_n x^n, \text{ and } y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$\sum_{n=1}^{\infty} a_n \underbrace{x^{n-1}}_{m=n-1; n=1, m=0} + \sum_{n=0}^{\infty} \underbrace{a_n}_{m=n} x^n - 2 \sum_{n=0}^{\infty} \underbrace{a_n}_{m=n+1; n=0, m=1} x^{n+1} = 0$$

Not exactly sure what these variables are doing with the n and m ... I guess they're dummy variables that we're using for each sum.

$$\sum_{m=0}^{\infty} a_{m+1}(m+1)x^m + \sum_{m=0}^{\infty} a_m x^m - 2 \sum_{m=1}^{\infty} a_{m-1} x^m = 0$$

Peel-off:

$$a_1 x^0 + a_0 x^0 + \sum_{m=1}^{\infty} [a_{m+1}(m+1) + a_m - 2a_{m-1}] x^m = 0$$

Note that x^0, x, x^2, \dots, x^n are linearly independent.

$$\begin{array}{l} x^0] \quad a_1 + a_0 = 0 \rightarrow a_1 = -a_0 \\ x^m] \quad a_{m+1}(m+1) + a_m - 2a_{m-1} = 0 \\ m \geq 1 \end{array}$$

Q: Why do we need to shift the indices?

A: We want to get to a single sigma – A single sum. If we don't do it, we aren't able to get to a single relation. Indices must start at the same point to be able to combine sums.

From the relation $a_{m+1}(m+1) + a_m - 2a_{m-1} = 0$, we can find the relation:

$$a_{m+1} = \frac{-a_m + 2a_{m-1}}{m+1}$$

$$m = 1 : a_2 = \frac{-a_1 + 2a_0}{2} = \frac{3}{2}a_0$$

$$m = 2 : a_3 = \frac{-a_2 + 2a_1}{3} = \frac{-7}{6}a_0$$

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$y(x) = a_0 - a_0 x + \frac{3}{2}a_0 x^2 - \frac{7}{6}a_0 x^3 + \dots$$

$$y(x) = a_0 \left[1 - x + \frac{3}{2}x^2 - \frac{7}{6}x^3 + \dots \right]$$

= Taylor expansion of the direct solution

2 Example 2

$$xy' + (2-x)y = 0$$

$$y' + \frac{2-x}{x}y = 0$$

Solving this using integrating factor method, we find the following:

$$\mu(x) = e^{\int \frac{1-x}{x} dx} = e^{2 \ln x - x} = x^2 e^{-x}$$

$$[x^2 e^{-x} y]' = 0 \rightarrow y = \frac{C}{x^2 e^{-x}} = C x^{-2} e^x$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$x y' + 2y - x y = 0$$

$$\sum_{n=1}^{\infty} a_n n x^n + 2 \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} + 1 = 0$$

$$\sum_{n=1}^{\infty} a_n n x^n + 2 \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\downarrow$$

$$m = n$$

$$\downarrow$$

$$m = n$$

$$\downarrow$$

$$m = n+1$$

$$m=1, n=0$$

$$\sum_{m=1}^{\infty} a_m m x^m + 2 \sum_{m=0}^{\infty} a_m x^m - \sum_{m=1}^{\infty} a_{m-1} x^m = 0$$

peel off first term

$$2a_0 x^0 + \sum_{m=1}^{\infty} (a_m m + 2a_m - a_{m-1}) x^m = 0$$

$$x^0] 2a_0 = 0 \rightarrow a_0 = 0$$

$$\left. \begin{matrix} x^m \\ m \geq 1 \end{matrix} \right] a_m(m+2) = a_{m-1}$$

$$\text{note that } a_m = \frac{a_{m-1}}{m+2} \text{ for } m \geq 1$$

$$m=1 : a_1 = \frac{a_0}{3} = 0$$

$$m = 2 : a_2 = \frac{a_4}{4} = 0$$

This is a trivial solution; they are all zero (and will continue to be).
 $y = cx^{-2}e^x$ is the general solution.

$$= C \underbrace{x^{-2}}_{\text{Capture the singularity}} \underbrace{\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right]}_{\sum_{n=0}^{\infty} a_n x^n}$$

$$y(x) = \underbrace{x^r}_{\text{Capture singularity}} \sum_{n=0}^{\infty} a_n x^n$$

This brings us to the **Forbenius Series**.

3 Forbenius Series

Let's define ordinary & singular points:

A linear 2nd order ODE:

$$P(x)y'' + Q(x)y' + r(x)y = 0$$

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0$$

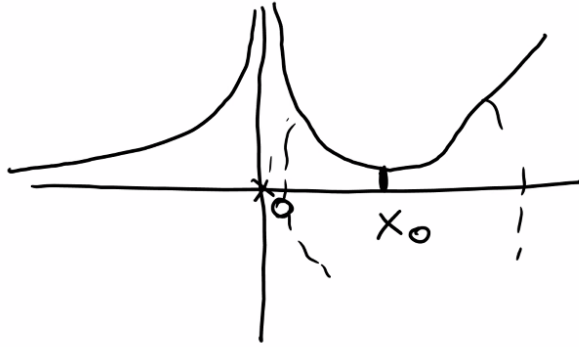
Letting $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$. If they are both **analytic** at the point x_0 . i.e. they have Taylor expansions about x_0 , x_0 is an ordinary point. Otherwise, x_0 is a singular point.

Example: $\left. \begin{array}{l} p(x) = \frac{1}{x} \\ p(x) = \ln(x) \end{array} \right\}$ At $x = 0$, not analytic.

Analytic means that it is expressible as a power series around x_0 . It means that it is infinitely differentiable around x_0 .

3.0.1 Quick notes

- A power series solution is possible for all ordinary points (similar to the first example we saw), but not all singular points.
- For singular points, we introduce the Forbenius Series. However, this only works for some singular points.
- Singular points results in the change of the nature of the ODE. Ordinary points exists in the domain of $p(x)$ and $q(x)$.
- The radius of convergence of the power series is at least as large as the distance from the x_0 to the nearest singular point.
 - For example, when we had $y = Cx^{-2}e^x$, we realize that $x = 0$ is a singular point.
 - We can see this in both the answer as well as the ODE.
 - When we plot the function, it looks like this:



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- The dotted circles around x_0 is the radius of convergence.

Example:

$$y' + (1 - 2x)y = 0$$

$$y(x) = Ce^{-x+x^2}$$

There are no singular points. Hence the radius of convergence is infinity.

3.1 Singular Points

Singular points are divided into two classes:

- Regular singular points, that we use the Forbenius series solution for
- Irregular singular points (Beyond this course). For these, we **cannot** use Forbenius series.