MATH 316 Lecture 7

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1 Boundary value Problems

Is there a similar setup for BVPs/ Let's consider 3 different BVPs:

1. P1: $y'' + \lambda y = 0$, for $x \in [0, L]$, with $y(0) = 0 = y(L)$
2. P2: $y'' + \lambda y = 0$, for $x \in [0, L]$, with $y'(0) = 0 = y'(L)$
3. P3: $y'' + \lambda y = 0$, for $x \in [0, L]$, with $y(0) = y(L)$ and $y'(0) = y'(L)$

Any value of λ for which P1 (P2 or P3) has a non-zero solution is called an **eigenvalue** of P1 (P2 or P3) and the corresponding solution is called and **eigenfunction** of P1 (P2 or P3).

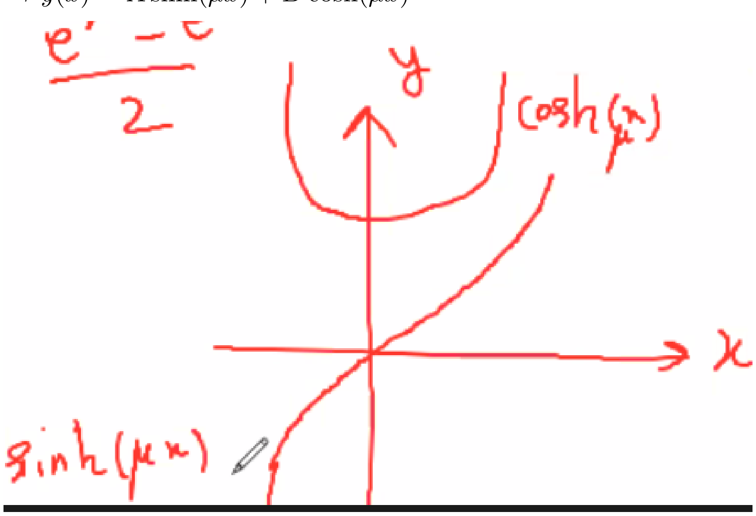
Exercise: find the eigenvalues and eigenfunctions of problems P1, P2 and P3.

1.1 Solving BVP (P1,P2,P3)

P1:

$$y'' + \lambda y = 0, \text{ and } y(0) = 0 = y(L)$$

λ: Eigenvalue. There are three categories that we have to investigate each time we solve such a problem:

1. If λ is negative ($\lambda < 0$):
   - $\lambda = -\mu^2 \to y'' - \mu^2 y = 0$
   - $r^2 - \mu^2 = 0 \to r_1 = \mu, r_2 = -\mu \longrightarrow y(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$
   - Note that $\cosh(\mu x) = \frac{e^{\mu x} + e^{-\mu x}}{2}$ and $\sinh(\mu x) = \frac{e^{\mu x} - e^{-\mu x}}{2} \longrightarrow$
   - $\hookrightarrow y(x) = A \sinh(\mu x) + B \cosh(\mu x)$
   - 

- Note that we have the boundary conditions $y(0) = 0 \rightarrow B = 0$ and $y(L) = 0 \rightarrow A\sinh(\mu L) = 0$
- $\Rightarrow A = 0$ and $\Rightarrow y(x) = 0$ which is trivial

2. If λ is zero: $y'' = 0 \longrightarrow y(x) = Ax + B$
   - $y(0) = 0 \longrightarrow B = 0$
   - $y(L) = 0 \longrightarrow AL = 0 \rightarrow A = 0 \rightarrow y(x) = 0$
   - Therefore it's a trivial solution.

3. If $\lambda > 0$: $\lambda = \mu^2 \rightarrow y'' + \mu^2 y = 0$
   - $r^2 + \mu^2 = 0 \longrightarrow r = \pm i\mu$
   - $y(x) = A\sin(\mu x) + B\cos(\mu x)$
   - $y(0) = 0 \rightarrow B = 0$
   - $y(L) = 0 \rightarrow A\sin(\mu L) = 0 \rightarrow \begin{array}{c} A = 0(\text{trivial}) \\ \sin(\mu L) = 0 \rightarrow \mu L = n\pi \end{array}$ therefore $\mu = \frac{n\pi}{L}$
   - Eigenvalue: $\lambda = \left(\frac{n\pi}{L}\right)^2$
   - Eigenfunction: $y_n(x) = A_n \sin(\frac{n\pi}{L}x)$

1.2 P2: $y'' + \lambda y = 0$

$y'(0) = 0 = y'(L)$
$r^2 + \lambda = 0$

1. If $\lambda > 0$, $\lambda = \mu^2$
   - $r^2 + \mu^2 = 0 \longrightarrow r = \pm i\mu$
   - $y(x) = A\sin(\mu x) + B\cos(\mu x)$
   - Sub boundary conditions:
   - $y'(0) = A = 0$ and $y'(L) = 0 \longrightarrow -B\mu\sin(\mu L) = 0$
   - This gives us two solutions: $\begin{array}{c} B = 0 \text{ which is trivial} \\ \underbrace{\sin(\mu L) = 0}_{\mu = \frac{n\pi}{L}} \end{array}$
   - $\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$ is the eigenvalue
   - $y_n(x) = B\cos\left(\frac{n\pi}{L}x\right)$ is the eigenfunction

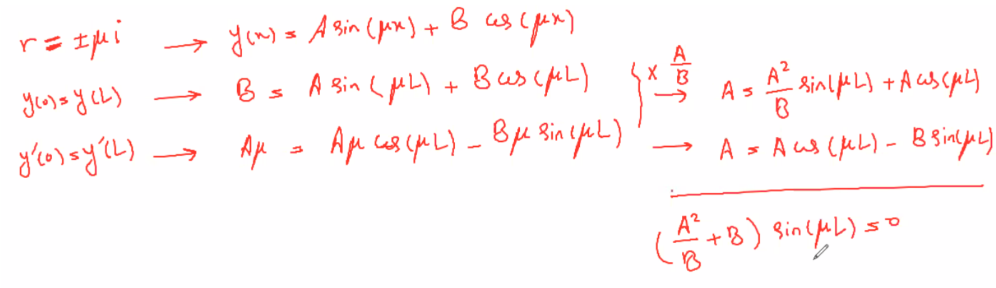
2. If $\lambda < 0$: $\longrightarrow \lambda = -\mu^2$
   - $r^2 - \mu^2 = 0 \longrightarrow r = \pm\mu \longrightarrow y(x) = C_1 e^{\mu x} + C_2 e^{-\mu x} \Rightarrow y(x) = A\sinh(\mu x) + B\cosh(\mu x)$
   - Substituting the boundary conditions into $y(x) = A\sinh(\mu x) + B\cosh(\mu x)$, we get:
   - $y'(0) = 0 \longrightarrow A = 0$
   - $y'(L) = 0 \longrightarrow -\mu B\sinh(\mu L) = 0 \longrightarrow B = 0$ (which means that $y(x) = 0$ which is trivial)

3. If $\lambda = 0 \longrightarrow y'' = 0 \rightarrow y = Ax + B$
   - $y'(0) = 0 \rightarrow A = 0$ and $y'(L) = 0 \rightarrow A = 0$: $\longrightarrow y(x) = B$
   - $\lambda = 0$ is an eigenvalue $\longrightarrow y(x) = 1$ is the eigenfunction
   - For P2 problems: eigenvalues are: $0, \frac{n^2\pi^2}{L^2}$ and eigenfunctions are: $1, \cos\left(\frac{n\pi}{L}x\right)$

1.3 P3: $y'' + \lambda y = 0$

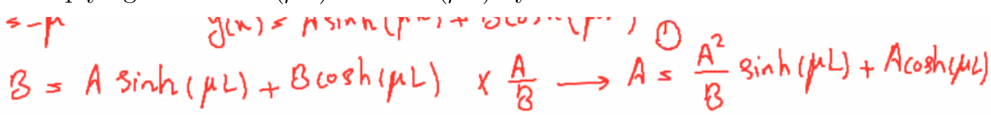
Periodic boundary conditions: $y(0) = y(L)$

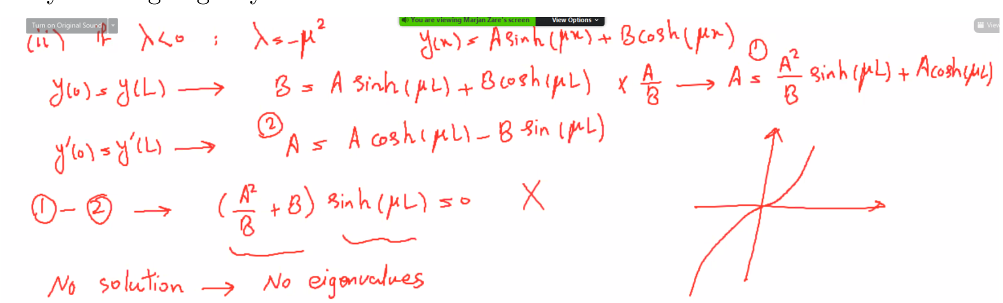
1. If $\lambda > 0$: $\lambda = \mu^2$

   - $r = \pm\mu i \longrightarrow y(x) = A\sin(\mu x) + B\cos(\mu x)$
   - $y(0)\ y(L) \longrightarrow B = A\sin(\mu L) + B\cos(\mu L)$
   - $y'(0) = y'(L) \longrightarrow A\mu = A\mu\cos(\mu L) - B\mu\sin(\mu L)$
   - 

   - We get the last term by using the previous two terms (on the right) and cancelling out.
   - $\sin(\mu L) = 0 \to \mu L = n\pi$. $a \neq 0$ and $B \neq 0$ and therefore $\mu = \frac{n\pi}{L}$
   - Eigenvalues: $\lambda = \left(\frac{n\pi}{L}\right)^2$
   - Eigenfunction is $y_n(x) = A_n\sin(\frac{n\pi}{L}x) + B_n\cos(\frac{n\pi}{L}x)$

2. If $\lambda < 0$: $\lambda = -\mu^2$

   - $y(x) = A\sinh(\mu x) + B\cosh(\mu x)$
   - $y(0) = y(L) \longrightarrow B = A\sinh(\mu L) + B\cosh(\mu L)$
   - $y'(0) = y'(L) \longrightarrow A = A\cosh(\mu L) - B\sinh(\mu L)$
   - Multiplying $B = A\sinh(\mu L) + B\cosh(\mu L)$ by:
   - 

   - Okay this is going way too fast... screenshots it is.
   - 

3. $\lambda = 0$:

   - View screenshot below.

(iii) $\lambda = 0$: $y'' = 0 \longrightarrow y = Ax + B$

$y(0) = B$ & $y(L) = AL + B \longrightarrow A = 0$ } $\longrightarrow y(x) = B$ ✓✓

$y'(0) = y'(L) = A$

$\lambda = 0$ is an eigenvalue

- $y(x) = 1$ is an eigenfunction

2 Fourier Series

Fourier series arise in 3 different situations of relevance to this course:1.Simple boundary value problems, e.g. P1-P32.Partial differential equations that describe heat flow, waves and diffusion (more later).3.Some initial value problems with less simple periodic forcing, e.g. we are very unlikely to have exactly: $f(t) = F_0 \cos(\omega t)$, in any real system, but might have a periodic forcing function.

For what follows, let the interval in P1-P3 be the interval $[a, b] = [-L, L]$. The key idea is that an arbitrary function, $f(t)$, defined on $[-L, L]$ can be represented in the following form:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi t}{L}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi t}{L}) \tag{1}$$

Note that these are the eigenfunctions of problem P3. Outside of the interval,because each function above has period $2L$, the above series must converge to a periodic extension of $f(t)$ of period $2L$.

Two immediate questions:

1. Can all functions $f(t)$ be represented in this way, i.e. which functions?
2. How do we find the coefficients a_n and b_n?

Definition:If the series on the right-hand side of (1) converges to a function $f(t)$ then this is called the Fourier series of $f(t)$.

Comments:

Firstly, in order for $f(t)$ to have Fourier series representation(1), that is valid for all t it is necessary that $f(t)$ is periodic, with period $2L$, i.e.

$$f(t + 2L) = f(t) \; \forall t$$

Secondly, suppose that $f(t)$ has a Fourier series representation (1). Then a_n and b_n are determined straightforwardly. See below for a_n:

1. Multiply (1) by $\cos(\frac{m\pi t}{L})$
2. Integrate both sides of the equation between $[-L, L]$:

$$\int_{-L}^{L} f(t) \cos \frac{m\pi t}{L} dt = \int_{-L}^{L} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{L} \right) \cos \frac{m\pi t}{L} dt$$

Note that:

$$\begin{array}{l} \int_{-L}^{L} \cos \frac{n\pi t}{L} \cos \frac{m\pi t}{L} dt = \left\{ \begin{array}{ll} 0 & m \neq n \\ L & m = n \end{array} \right. \\ \int_{-L}^{L} \cos \frac{n\pi t}{L} \sin \frac{m\pi t}{L} dt = 0 \\ \int_{-L}^{L} \sin \frac{n\pi t}{L} \sin \frac{m\pi t}{L} dt = \left\{ \begin{array}{ll} 0 & m \neq n \\ L & m = n \end{array} \right. \end{array}$$

Therefore, interchanging summation and integration:

$$\int_{-L}^{L} f(t) \cos \frac{m\pi t}{L} dt = a_m \int_{-L}^{L} \cos \frac{m\pi t}{L} \cos \frac{m\pi t}{L} dt = a_m L$$
$$a_m = \frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{m\pi t}{L} dt$$

For the coefficients b_n a similar procedure is possible (exercise).
Thus, we finally have:

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(t) dt$$
$$a_m = \frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{m\pi t}{L} dt \quad m = 1, 2, 3, \ldots$$
$$b_m = \frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{m\pi t}{L} dt \quad m = 1, 2, 3, \ldots$$

which are known as the **Euler-Fourier series**.

2.1 Example 1

Assumer that the function $f(t)$, defined by $(t) = \begin{cases} t & -L \leq t < 0 \\ 0 & 0 \leq t < L \end{cases}$ with $f(t + 2L) = f(t)$, has a fourier series. Sketch the function and find the fourier series.

Solution:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi t}{L}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi t}{L})$$

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(t) dt = \frac{1}{L} \int_{-L}^{0} t dt = \frac{-L}{2}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos(\frac{n\pi t}{L}) dt = \frac{1}{L} \int_{-L}^{0} t \cos(\frac{n\pi t}{L}) dt$$

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Using integration by parts, with $u = t$ ($du = dt$) and $dv = \cos(\frac{n\pi t}{L})dt$, with $v = \frac{L}{n\pi} \sin(\frac{n\pi t}{L})$, we get the following:

$$a_n = \frac{1}{L} \left[\frac{tL}{n\pi} \frac{\sin(n\pi t)}{L} \bigg|_{-L}^{0} - \int_{-L}^{0} \frac{L}{n\pi} \frac{\sin(n\pi t}{L} dt \right.$$
$$= \frac{1}{n\pi} \left[\frac{L}{n\pi} \cos(\frac{n\pi t}{L} |_{-L}^{0} \right] = \frac{L}{n^2 \pi^2} (1 - \cos(n\pi)$$