Boundary value problems

Is there a similar setup for BVPs? Let's consider 3 different BVPs:

P1: $y'' + \lambda y = 0,$ for $x \in [0, L]$, with $y(0) = 0 = y(L)$

P2: $y'' + \lambda y = 0,$ for $x \in [0, L]$, with $y'(0) = 0 = y'(L)$

P3: $y'' + \lambda y = 0,$ for $x \in [0, L]$, with $y(0) = y(L)$ and $y'(0) = y'(L)$

Any value of λ for which P1 (P2 or P3) has a non-zero solution is called an **eigenvalue** of P1 (P2 or P3) and the corresponding solution is called and **eigenfunction** of P1 (P2 or P3).

Exercise: find the eigenvalues and eigenfunctions of problems P1, P2 and P3

Fourier Series

Fourier series arise in 3 different situations of relevance to this course:

1. Simple **boundary value problems**, e.g. P1-P3
2. **Partial differential equations** that describe heat flow, waves and diffusion (more later).
3. Some **initial value problems** with less simple periodic forcing, e.g. we are very unlikely to have exactly: $f(t) = F_0 \cos \omega t$, in any real system, but might have a periodic forcing function

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For what follows, let the interval in P1-P3 be the interval $[a.b] = [-L, L]$. The key idea is that an arbitrary function, $f(t)$, defined on $[-L, L]$ can be represented in the following form:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{L} \tag{1}$$

Note that these are the eigenfunctions of problem P3. Outside of the interval, because each function above has period $2L$, the above series must converge to a periodic extension of $f(t)$ of period $2L$

Two immediate questions:

1. Can all functions $f(t)$ be represented in this way, i.e. which functions?
2. How do we find the coefficients a_n and b_n?

Definition: If the series on the right-hand side of (1) converges to a function $f(t)$, then this is called the **Fourier series** of $f(t)$.

Comments:

Firstly, in order for $f(t)$ to have **Fourier series representation** (1), that is valid <u>for all t</u>, it is **necessary** that $f(t)$ be periodic, with period $2L$, i.e.

$$f(t+2L) = f(t) \qquad \forall t$$

Secondly, suppose that $f(t)$ has a Fourier series representation (1). The a_n & b_n are then determined straightforwardly, (see below for a_n).

1. Multiply (1) by: $cos\,\frac{m\pi t}{L}$
2. Integrate both sides of the equation between $[-L, L]$:

$$\int_{-L}^{L} f(t)\, cos\,\frac{m\pi t}{L}\, dt = \int_{-L}^{L} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\, cos\,\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n\, sin\,\frac{n\pi t}{L} \right) cos\,\frac{m\pi t}{L}\, dt$$

Note that:

$$\int_{-L}^{L} cos\,\frac{n\pi t}{L}\, cos\,\frac{m\pi t}{L}\, dt = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

$$\int_{-L}^{L} cos\,\frac{n\pi t}{L}\, sin\,\frac{m\pi t}{L}\, dt = 0$$

$$\int_{-L}^{L} sin\,\frac{n\pi t}{L}\, sin\,\frac{m\pi t}{L}\, dt = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

$$\boxed{\text{Trig identity: } \cos(A)\cos(B) = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)}$$

So for $m \neq n$:

$$\int_{-L}^{L} \cos\left(\frac{n\pi t}{L}\right)\cos\left(\frac{m\pi t}{L}\right) dt$$

$$= \int_{-L}^{L} \frac{1}{2}\left[\cos\left(\frac{(n+m)\pi t}{L}\right) + \cos\left(\frac{(n-m)\pi t}{L}\right)\right] dt$$

$$= \frac{L}{2\pi(n+m)}\left(\sin\left(\frac{(n+m)\pi t}{L}\right)\Big|_{-L}^{L} + \sin\left(\frac{(n-m)\pi t}{L}\right)\Big|_{-L}^{L}\right) = 0$$

$$\boxed{\cos^2(A) = \frac{1}{2}\left(1 + \cos(2A)\right)}$$

So for $m = n$:

$$\int_{-L}^{L} \cos^2\left(\frac{n\pi t}{L}\right) dt = \frac{1}{2}\int_{-L}^{L}\left(1 + \cos\left(\frac{n\pi t}{L}\right)\right) dt$$

$$= \frac{1}{2}\left(t \Big|_{-L}^{L} + \underbrace{\cancel{\frac{L}{2n\pi} \sin\left(\frac{2n\pi t}{L}\right) \Big|_{-L}^{L}}}_{0} \right) = L$$

Subs. into $*$:

$$\int_{-L}^{L} f(t) \cos\left(\frac{m\pi t}{L}\right) dt = a_0 \frac{L}{m\pi} \sin\left(\frac{m\pi t}{L}\right)\Big|_{-L}^{L} + \underbrace{a_m L}_{\text{for } n=m} + 0$$

$$\Rightarrow \quad a_m = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{m\pi t}{L}\right) dt \qquad \therefore$$

$$\boxed{\text{Trig identity: } \quad \sin A \cdot \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)}$$

if $m \neq n$:

$$\int_{-L}^{L} \sin\left(\frac{n\pi t}{L}\right) \sin\left(\frac{m\pi t}{L}\right) dt$$

$$= \int_{-L}^{L} \frac{1}{2}\left(\cos\left(\frac{(n-m)\pi t}{L}\right) - \cos\left(\frac{(n+m)\pi t}{L}\right)\right) dt$$

$$= \frac{1}{2}\left[\frac{L}{(n-m)\pi} \sin\left(\frac{\pi t(n-m)}{L}\right)\Bigg|_{-L}^{L} - \frac{L}{(n+m)\pi} \sin\left(\frac{(n+m)\pi t}{L}\right)\Bigg|_{-L}^{L}\right]$$

$$= 0$$

$$\boxed{\sin^2 A = \frac{1}{2}\left(1 - \cos 2A\right)}$$

if $m = n$:

$$\int_{-L}^{L} \sin^2\left(\frac{n\pi t}{L}\right) dt = \frac{1}{2}\int_{-L}^{L} \left(1 - \cos\left(\frac{2n\pi}{L} t\right)\right) dt =$$

$$= \frac{1}{2}\left(t\Big|_{-L}^{L} - \frac{L}{2n\pi} \underbrace{\sin\left(\frac{2n\pi}{L} t\right)\Bigg|_{-L}^{L}}_{0}\right) = L$$

if you multiply eq (1) by $\sin\left(\frac{m\pi}{L}t\right)$ and integrate $\int_{-L}^{L}$:

$$\Rightarrow \int_{-L}^{L} f(t)\sin\left(\frac{m\pi t}{L}\right)dt = \int_{-L}^{L}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}\right)\sin\frac{m\pi t}{L}\,dt$$

$$\Rightarrow \int_{-L}^{L} f(t)\sin\left(\frac{m\pi t}{L}\right)dt = 0 + 0 + b_m\int_{-L}^{L}\sin^2\left(\frac{m\pi t}{L}\right)dt + 0 + 0 + \cdots$$

only the mth terms are nonzero

$$\Rightarrow \int_{-L}^{L} f(t)\sin\left(\frac{m\pi t}{L}\right)dt = b_m L \Rightarrow b_m = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{m\pi t}{L}\right)dt$$

Therefore, interchanging summation and integration:

$$\int_{-L}^{L} f(t)\cos\frac{m\pi t}{L}\,dt = a_m \int_{-L}^{L} \cos\frac{m\pi t}{L}\cos\frac{m\pi t}{L}\,dt = a_m L$$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{m\pi t}{L}\,dt$$

For the coefficients b_n a similar procedure is possible (exercise).

Thus, we finally have:

$$\boxed{\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(t)dt \\ a_m &= \frac{1}{L}\int_{-L}^{L} f(t)\,cos\frac{m\pi t}{L}dt \qquad m = 1,2,3,\ldots \\ b_m &= \frac{1}{L}\int_{-L}^{L} f(t)\,sin\frac{m\pi t}{L}dt \qquad m = 1,2,3,\ldots \end{aligned}}$$

which are known as the **Euler-Fourier** formulas.

Example 1: Assume that the function $f(t)$, defined by

$$f(t) = \begin{cases} t & -L \le t < 0 \\ 0 & 0 \le t < L \end{cases}$$

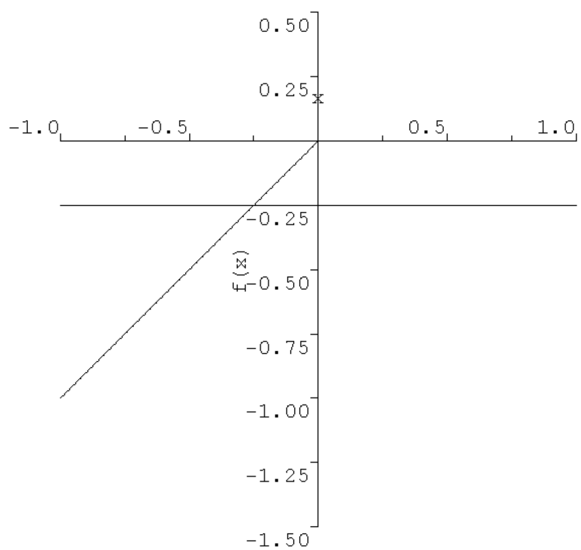
with $f(t + 2L) = f(t)$, has a Fourier series. Sketch the function and find the Fourier series.

Why are we doing this?

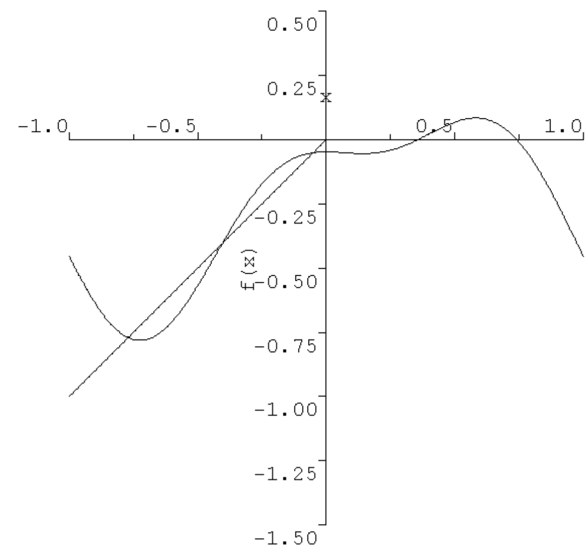
Lets fix $L = 1$ in the above example and plot the partial sums:

$$f(t) \sim -\frac{1}{4} + \sum_{n=1}^{k} \frac{1-(-1)^n}{(n\pi)^2} cos\, n\,\pi t + \sum_{n=1}^{k} \frac{(-1)^{n+1}}{n\pi} sin\, n\,\pi t$$

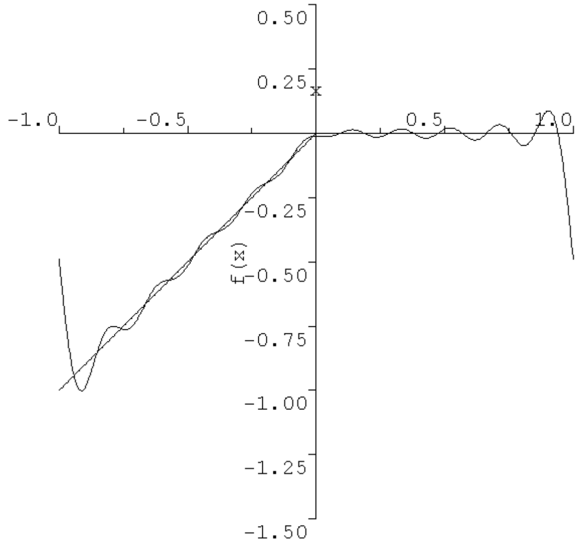
k=0 Constant term only



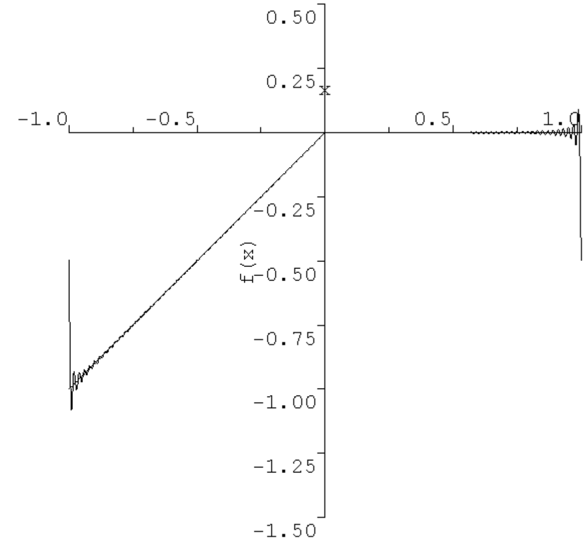
k=2 First 2 trignometric terms



k=10 First 10 terms

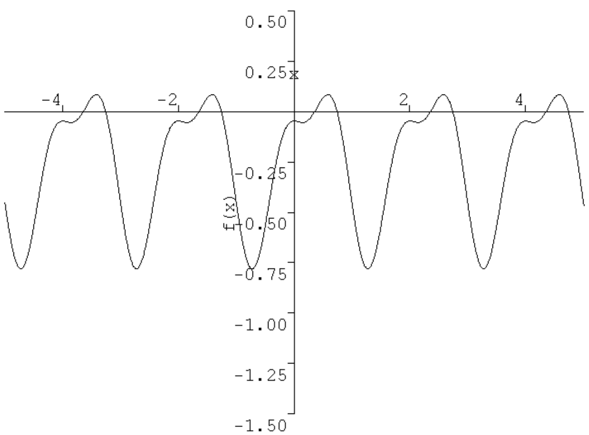


k=100 First 100 terms

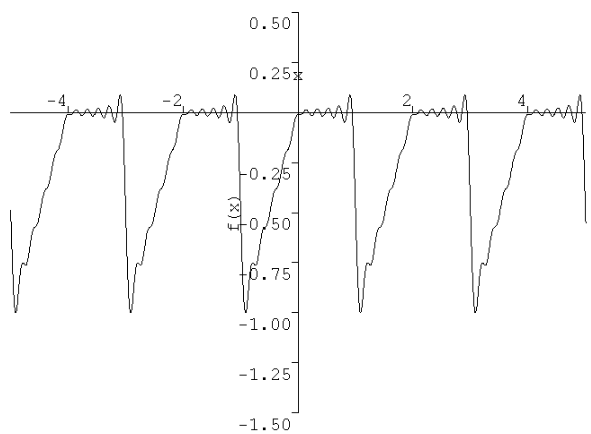


What’s happening over longer interval of t?

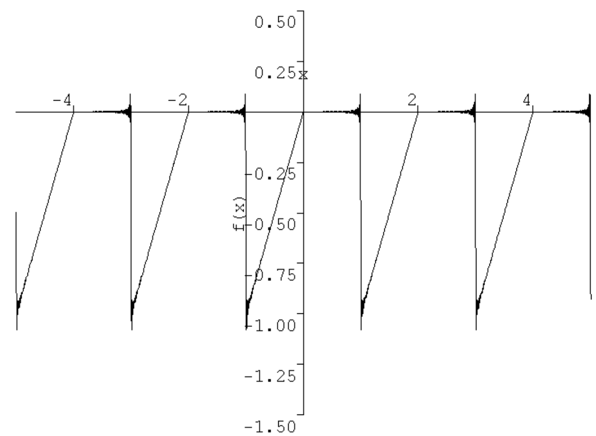
k=2



k=10



k=100



Observations:

1. Take more terms in the series it appears to converge to $f(t)$, (even if $f(t)$ has discontinuities!)
2. The coefficients a_n & b_n that we calculated decrease as $n \to \infty$.
3. Initial coefficient $a_0/2$ is the mean value of $f(t)$
4. Appears to be a slight overshoot at the points of discontinuity of the function $f(t)$

The above are common observations for Fourier series expansions with arbitrary functions $f(t)$.

Fourier Sine and Cosine Series

Our main usage for Fourier series will be in representing a function $f(x)$, over a finite interval $[0, L]$, e.g. the initial temperature in a heat conduction problem. It turns out that there are many possible ways to do this, depending on the particular function.

Odd and even functions:

Suppose that $f(x)$ is defined at $-x$ whenever it is defined at x

- The function $f(x)$ is an **even** function if $f(x) = f(-x)$. Examples: 1, x^2, x^{2n}, $|x|$, $cos\, x$
- The function $f(x)$ is an **odd** function if $f(x) = -f(-x)$. Examples: x, x^3, x^{2n+1}, $sin\, x$

Note: Most functions are neither odd nor even

Simple properties:

1. The sum (difference) and product (quotient) of 2 even functions is an even function
2. The sum (difference) of 2 odd functions is an odd function
3. The product (quotient) of 2 odd functions is an even function
4. The product (quotient) of an odd and an even function is an odd function
5. The sum (difference) of an odd and an even function is neither odd nor even

Integral properties:

1. If $f(x)$ is an even function then: $\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$
2. If $f(x)$ is an odd function then: $\int_{-L}^{L} f(x)\,dx = 0$

The form of the Fourier series for $f(x)$ is different, if $f(x)$ is an odd or an even function.

Fourier Cosine series: Assume that $\boldsymbol{f(x)}$ is piecewise differentiable on $[\boldsymbol{-L, L}]$ and $\boldsymbol{f(x)}$ is an even function. Then $\boldsymbol{f(x)}$ has Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

called the **Fourier cosine series**, with coefficients a_n given by:

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx \qquad n = 0,1,2,3, \ldots$$

Fourier Sine series: Assume that $\boldsymbol{f(x)}$ is piecewise differentiable on $[\boldsymbol{-L,L}]$ and $\boldsymbol{f(x)}$ is an odd function. Then $\boldsymbol{f(x)}$ has Fourier series:

$$f(x) = \sum_{n=1}^{\infty} b_n \, cos \frac{n\pi x}{L}$$

called the **Fourier sine series**, with coefficients b_n given by:

$$b_n = \frac{2}{L}\int_0^L f(x) \, sin \frac{n\pi x}{L} dx \qquad n = 1,2,3,\ldots$$

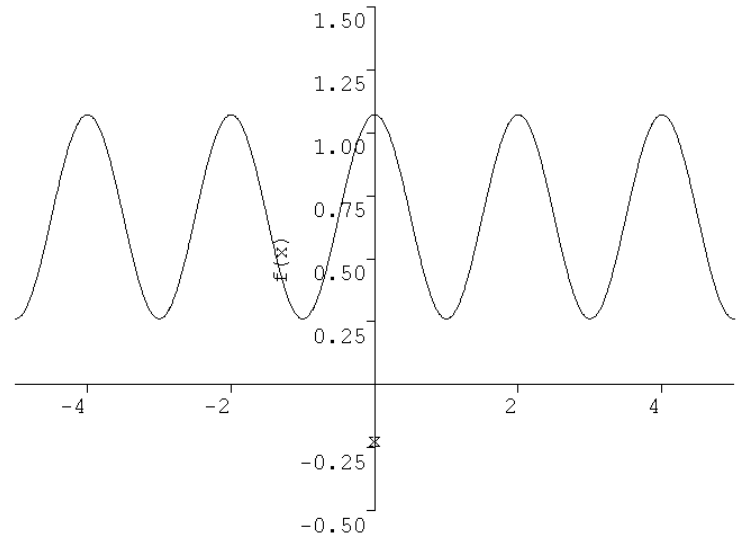
Example 2: Sketch the following functions $f(t)$ & find the Fourier series:

(a) $$f(t) = \begin{Bmatrix} -1 & -1 < t < 0 \\ 1 & 0 \leq t \leq 1 \end{Bmatrix} \quad f(t+2) = f(t)$$

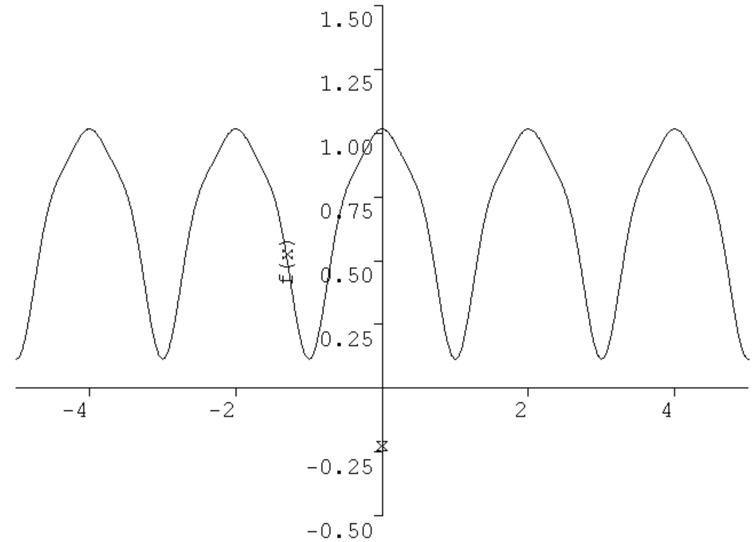
(b) $$f(t) = \begin{Bmatrix} -t & -2 < t < 0 \\ t & 0 \leq t \leq 2 \end{Bmatrix} \quad f(t+4) = f(t)$$

Example 3: Consider the function $f(t) = 1 - t^2$ for $-1 \leq t \leq 1$ with $f(t+2) = f(t)$. Find the Fourier series expansion and plot the k -th partial sums of the Fourier series for $k = 1,3,10,100$

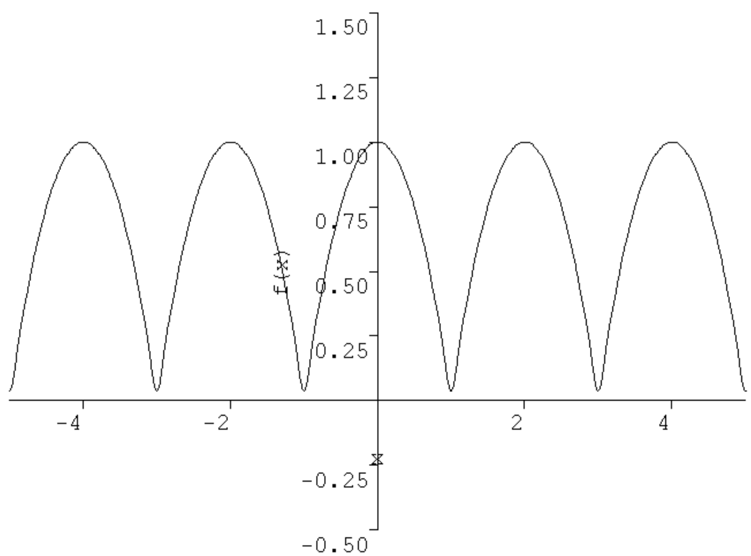
k=1



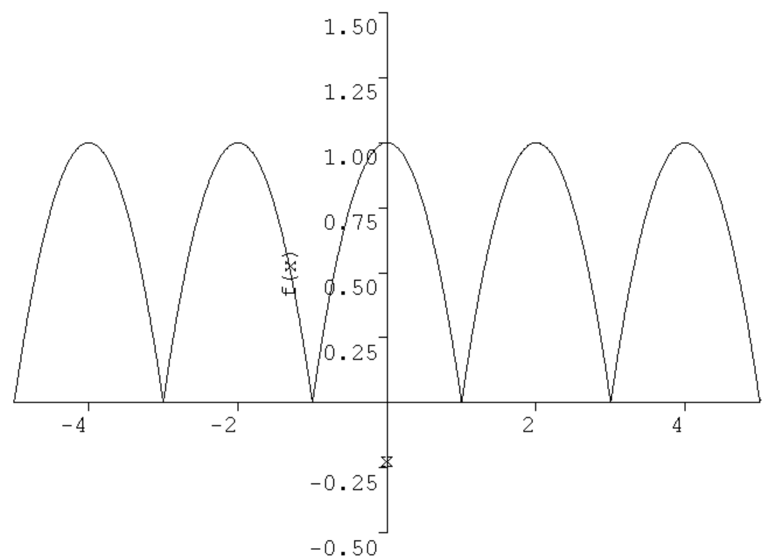
k=3



k=10



k=100



Example 4:

Find the Fourier series for $f(x) = x: \quad -L \leq x \leq L; \;\; f(x + 2L) = f(x)$

Example 5:

Find the Fourier series for $f(x) = |x|: \quad -L \leq x \leq L; \;\; f(x + 2L) = f(x)$

Suppose we wish to represent $f(x)$ on $[0, L]$, but don't care what form it has outside $[0, L]$. Many alternatives exist:

1. Use the Fourier cosine series. This series will converge to the function $g(x)$:
$$g(x) = \begin{cases} f(x) & 0 \leq x \leq L \\ f(-x) & -L < x < 0 \end{cases}$$
$$g(x + 2L) = g(x)$$
which is the even periodic extension of $f(x)$.
2. Use the Fourier sine series. This function will converge to the function $h(x)$:
$$h(x) = \begin{cases} f(x) & 0 < x < L \\ 0 & x = 0, L \\ -f(-x) & -L < x < 0 \end{cases}$$
$$h(x + 2L) = h(x)$$
which is the odd periodic extension of $f(x)$.
3. Define any function $k(x)$ that is piecewise differentiable on $[-L, L]$ and for which: $k(x) = f(x): \quad 0 \leq x \leq L$. Find the Fourier series for $k(x)$. Note that there are infinitely many choices for $k(x)$!

Factors affecting your choice of Fourier series representation:

- Speed of convergence. Generally, slow convergence results from discontinuities; the smoother the function, the faster the convergence.
- Sometimes the problem at hand dictates directly the choice