Chapter 13

Lecture 13

Last time, we solved wave equations using the method of separation of variables, and introduced D'Alembert's solution.

We found that D'Alembert's solution is in the following format:

$$y(x,t) = \frac{1}{2}\left[f(x-at) + f(x+at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)ds$$

This solution solves the following IBVP:

$$y_{tt} = a^2 y_{xx}$$

With initial conditions

$$y(x,0) = f(x), y_t(x,0) = g(x)$$

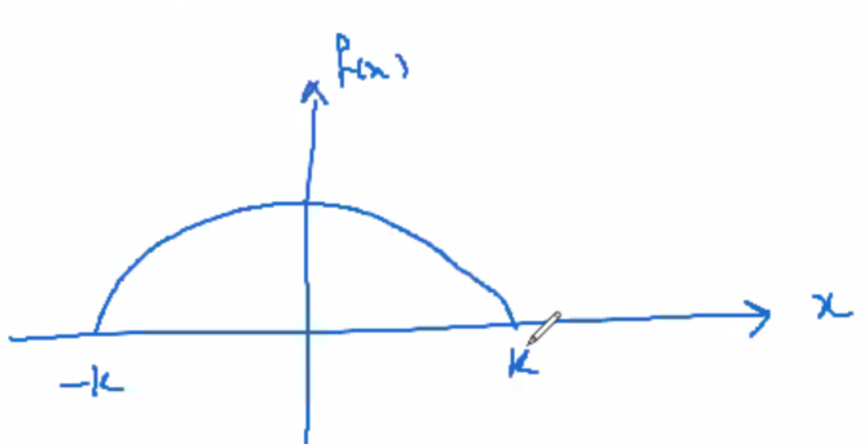
13.1 Why do we use d'Alembert's method?

- It's very graphical (And you might like it more than series solutions)

Recall: Characteristic lines for wave equation are $x - at = \xi$ and $x + at = \eta$ are important for constructing the solution. The initial data, $f(x)$ and $g(x)$ is propagated along these lines at speed of a.

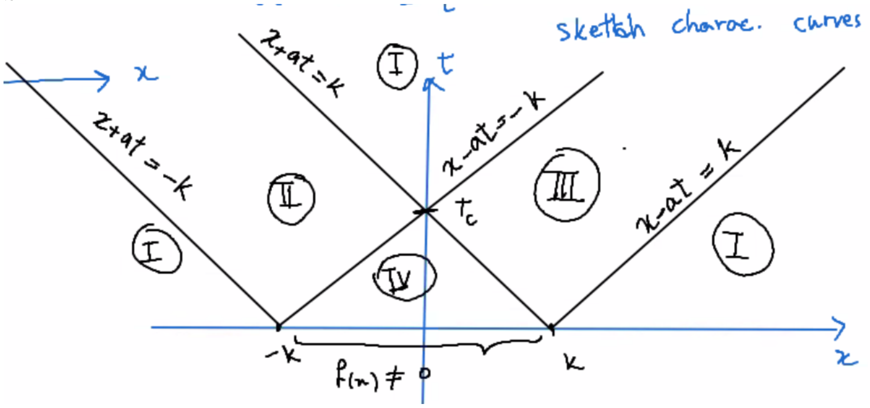
13.2 Region of influence

Let's consider $g(x) = 0$, i.e. no initial velocity. Suppose $y(x,0) = f(x)$ where $f(x)$ is defined on some finite interval $[-k, k]$. For example:



$$y(t) = \frac{1}{2}\left[f(x-at) + f(x+at)\right]$$

Sketch the characteristic curves. The domain is defined from $-k$ to k:



Between $-k$ to k, $f(x) \neq 0$.
T_c is the critical time at which two waves become separate.

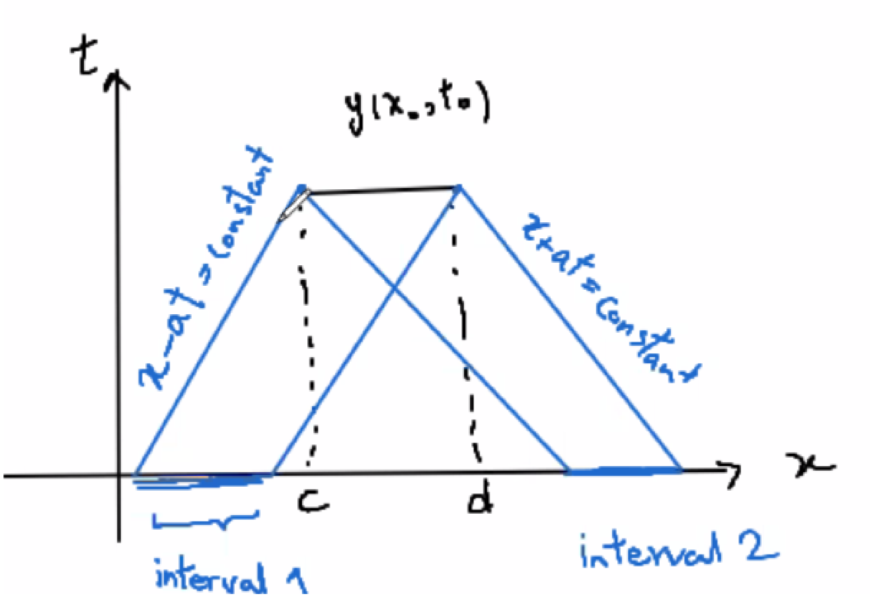
Regions

- Region 1:
  - $y = 0$. (No initial displacement)
- Region 2:
  - $y(x,t) = \frac{1}{2}f(x+at)$.
- Region 3:
  - $y(x,t) = \frac{1}{2}f(x-at)$
- Region 4:
  - Combination of both. We get characteristic lines from both 2 and 3:
  - $y(x,t) = \frac{1}{2}\left[f(x+at) + f(x-at)\right]$

Regions 2, 3, and 4 are called regions of influence of $f(x)$ for $x \in [-k, k]$.

13.3 Domain of Dependence

Suppose you know the solution $y(x_0, t_0)$ is the solution for $x \in [c, d]$.



Domain of dependence of the solution depends only on the initial data in interval 1, propagating right, and those data in interval 2 that are propagating left.

13.4 Examples

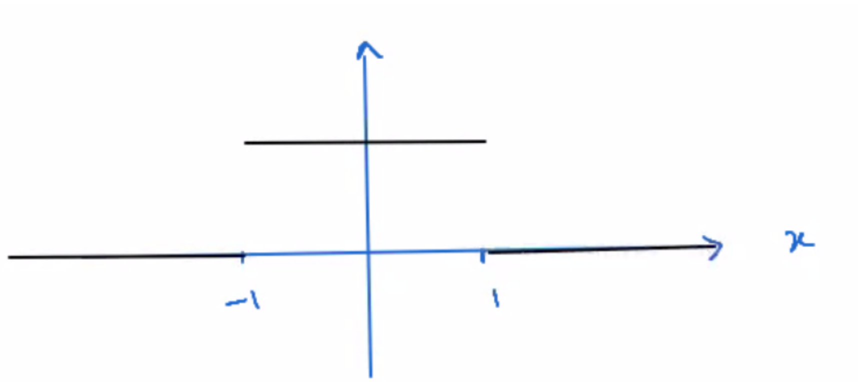
13.4.1 Example 10

Assume we are given the wave equation:

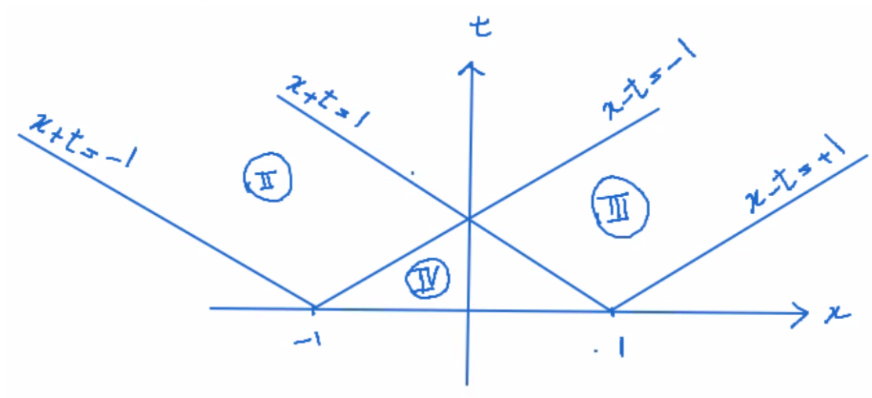
$$y_{tt} = y_{xx}, -\infty < x < \infty$$

$$y(x,0) = \begin{cases} 1 & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$y_t(x,0) = 0 = g(x)$$



From the D'Alembert formula, $y(x,t) = \frac{1}{2}\left[f(x-t) + f(x+t)\right]$ given $a = 1$.



$$\text{At region 2, } y(x,t) = \tfrac{1}{2}f(x+t) = \tfrac{1}{2}$$

$$\text{At region 3, } y(x,t) = \tfrac{1}{2}f(x-t) = \tfrac{1}{2}$$

$$\text{At region 4, } y(x,t) = \tfrac{1}{2}\left[f(x+t) + f(x-t)\right] = 1$$

13.4.2 Example 11

$$y_{tt} = y_{xx}$$

$$y(0,t) = 0, y(1,t) = 0$$

Note: it has a boundary condition above.

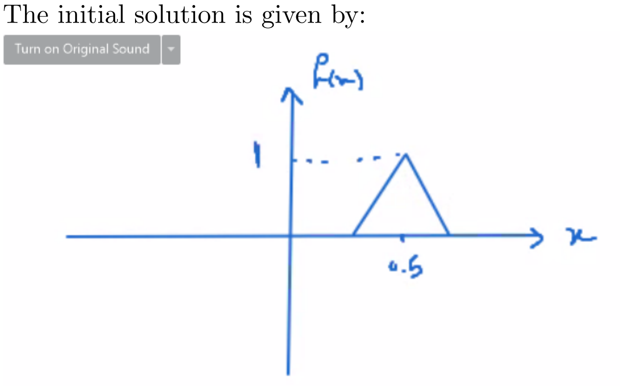
$$y(x,0) = \begin{cases} 0 & 0.00 \le x < 0.45 \\ 10(x-0.45) & 0.45 \le x < 0.50 \\ 20(0.55-x) & 0.50 \le x < 0.55 \\ 0 & 0.55 \le x \le 1.00 \end{cases}$$

$$y_t(x,0) = 0 = g(x)$$

For Dirichlet we took the odd periodic extension and apply the D'Alembert solution. [1]

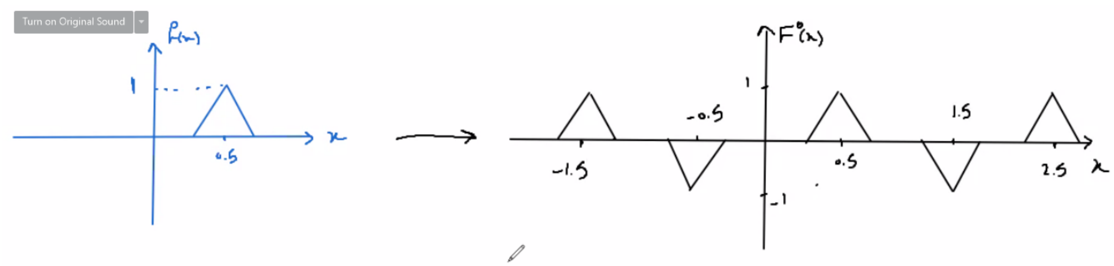
$$y(x,t) = \frac{1}{2}\left(\underbrace{F^o(x-t)}_{\text{moving right}} + \underbrace{F^o(x+t)}_{\text{moving left}}\right)$$

The initial solution is given by:

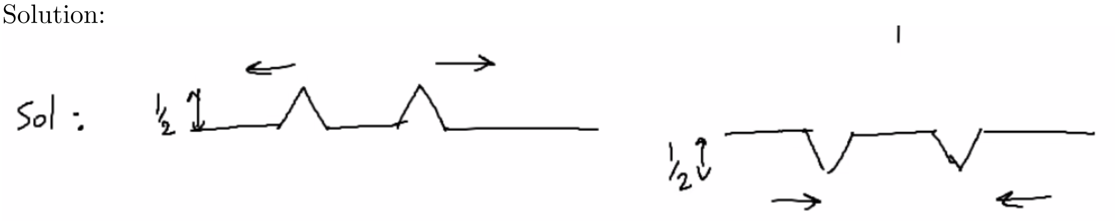


Odd extension:

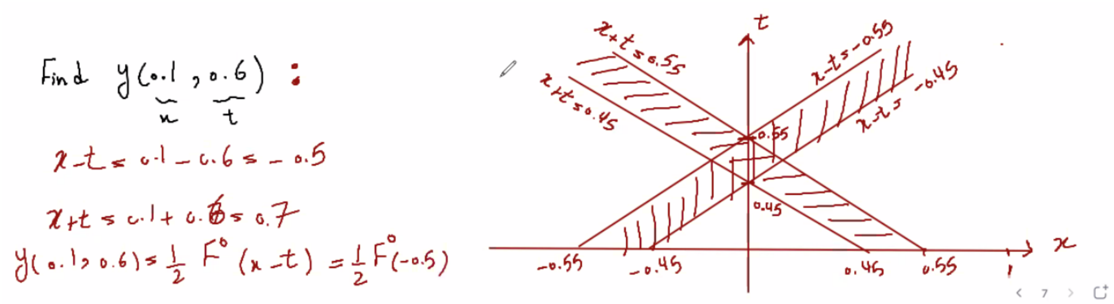
[1] Note that F^o is the odd extension.



Solution:



Find $y(x, t) = y(0, 1, 0.6)$:



Note that $\frac{1}{2}F^o(-0.5) = -\frac{1}{2}f(0.5) = -\frac{1}{2}(20(0.55 - 0.5)) = -\frac{1}{2}$

13.5 Laplace Equations PDF

Firstly, for Laplace equations, we go through the 6 page pdf attached below.

Laplace's Equation in 2-Dimensional Regions

Laplace's Equation arises in many situations, e.g.

- Steady Heat Flow in a 2-D region

$$\rho c_p \frac{\partial T}{\partial t} = K\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \qquad (x, y) \in [0, L] \times [0, L]$$
$$T(0, y, t) = 10, \quad T(L, y, t) = 20$$
$$T(x, 0, t) = 10 + 10x/L, \quad T(L, y, t) = 10 + 10y/L$$

  At sufficiently long times we have seen the solutions tend to decay exponentially fast, to a steady temperature solution:

$$0 = K\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \qquad (x, y) \in [0, L] \times [0, L]$$
$$T(0, y) = 10, \quad T(L, y) = 20$$
$$T(x, 0) = 10 + 10x/L, \quad T(L, y) = 10 + 10y/L$$

- Steady diffusion problems (as above, with T replaced by a concentration C)
- Steady wave problems
- Potential Flow (e.g. irrotational inviscid flow), modelling for example, flows around aerofoils, cylinders, etc.. $\boldsymbol{u} = \nabla\varphi$, where the velocity potential φ satisfies:

$$0 = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \qquad + \text{ BC's}$$

Two-dimensional region is typically a rectangle or a circle, (or even outside of a circle = aerofoil), but could (in principle) be a more arbitrary shape, denoted Ω.

Therefore, we consider:

$$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad (x, y) \in \Omega$$

The boundary of Ω is denoted $\partial\Omega$. Two types of boundary conditions are prescribed on $\partial\Omega$:

1. **Dirichlet conditions**: this means that u is given on $\partial\Omega$
2. **Neumann conditions**: this means that $\frac{\partial u}{\partial n}$ is given on $\partial\Omega$, where $\boldsymbol{n}$ denotes the unit normal vector to $\partial\Omega$.

The method we use to solve Laplace's equation in symmetric regions is separation of variables

Example 12: Find the solution to Laplace's equation in the rectangle: $\Omega = [0, a] \times [0, b]$, satisfying the following boundary conditions:

$$u(0, y) = 0, \quad u(a, y) = 0, \quad y \in [0, b]$$
$$u(x, 0) = f(x), \quad u(x, b) = 0, \quad x \in [0, a]$$

Example 13: Find the solution to Laplace's equation in the rectangle: $\Omega = [0, a] \times [0, b]$, satisfying the following boundary conditions:

$$\frac{\partial u}{\partial x}(0, y) = 0, \quad \frac{\partial u}{\partial x}(a, y) = g(y), \quad y \in [0, b]$$
$$u(x, 0) = f(x), \quad u(x, b) = 0, \quad x \in [0, a]$$

Example 14: Laplace's equation in cylindrical polar coordinates is:

$$0 = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \qquad (r, \theta) \in \Omega$$

Solve Laplace's equation inside the circular region $\Omega = \{r : r \in (0, a)\}$ subject to the following boundary conditions:

$$u(a, \theta) = f(\theta) \quad \theta \in [0, 2\pi]$$
$$u(r, \theta) \text{ bounded as } r \to 0$$

Example 15: Show that $\varphi(r,\theta) = -U\left(r + \frac{a^2}{r}\right)\cos\theta + \frac{\kappa\theta}{2\pi}$, is a solution to Laplace's equation in cylindrical polar coordinates:

$$0 = \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} \qquad (r,\theta) \in \Omega$$

where $\Omega = \{r : r > a\}$. What is the condition satisfied by $\varphi(r,\theta)$ at $r = a$. This solution represents the potential flow around a moving cylinder (a circular aerofoil). Find the velocity field corresponding to the potential $\varphi(r,\theta)$.

13.6 Laplace Equations Notes

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \to \Delta u = 0$$

where Δ is the laplacian.
The main idea is to split into 4 problems, each with 3 homogeneous boundary conditions.
[Using N,S,E,W notation]: