MATH 316 Lecture 9

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1 Recap

We learned how to write a function as a fourier series, in the following format:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L})$$

We have the following formulas:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})dx$$

2 Solving the Heat / Diffusion Equation

Examples are posted in the pdf slides posted.

2.1 Example 1

Solve the initial boundary value problem (IBVP)

$$\tfrac{\partial u}{\partial t} = \alpha \tfrac{\partial^2 u}{\partial x^2} \quad 0 < x < L; \quad t > 0 \tag{1}$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = f(x)$$

We need to use the method of separation of variables.

$$u(x,t) = X(x)T(t)$$

Taking the partial derivative with respect to t:

$$\rightarrow u_t = X(x)\dot{T}(t)$$

where dots are derivatives with respect to time.

$$u_x = X'(x)T(t)$$

$$u_{xx} = X''(x)T(t)$$

Now, we substitute this into the PDE equation 1

$$X\dot{T} = \alpha X''T$$

Now, we divide by αXT:

$$\frac{\dot{T}}{\alpha T} = \frac{X''}{X}$$

The left hand side of the equation is a function of t, and the right hand side is a function of x. In what condition are they equal?

The only way that theycan both be equal is if:

$$\frac{1}{\alpha}\frac{\dot{T}}{T} = \frac{X''}{X} = -\lambda \tag{2}$$

Where λ is a constant.

Now, let's work on boundary conditions:

$$u(0,t) = X(0)T(t) = 0$$

$$u(L,t) = X(L)T(t) = 0$$

$$\text{Hence, } X(0) = X(L) = 0$$

From 2, we get two equations:

- 1 - BVP
- 2 - IVP

2.1.1 BVP

$$\frac{X''}{X} = -\lambda \Rightarrow X'' + \lambda X = 0$$

with $X(0) = X(L) = 0$. This is a dirichlet boundary condition (BVP type P1)

The solution to P1:

$$X(x) = X_n(x) = C_n \sin(\frac{n\pi x}{L})$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, n \in N$$

Therefore, there is a countably infinite set of $\lambda_n, X_n(x)$ as a solutiojn for the BVP. For each λ_n we find an IVP separately:

2.1.2 IVP

$$\frac{\dot{T}}{\alpha T} = -\lambda_n \longrightarrow T_n(t) = e^{-\lambda_n \alpha t}$$

is the solution to the IVP.

2.1.3 Summary

We found, for $n = 1, 2, 3, ...$ ($n \in N$), we found:

$$u_n(x,t) = X_n(x)T_n(t) = C_n \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$$

This satisfies $u_t = \alpha u_{xx}$ with the conditions $u(0,t) = u(L,t) = 0$

Since the PDE and boundary conditions are homogeneous, we can superimpose solution, i.e.

$$C_k u_k + C_m u_m$$

also satisfies this problem for any constants of C_k and C_m.

Let's extend this idea to ∞:

$$u(x,t) = \sum_{n=1}^{\infty} C_n u_n(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$$

for constants $C_1, C_2, C_3, ...$

How abut initial conditions $u(x,0) = f(x)$?

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \tag{3}$$

How do we find C_n to meet this condition?

We need to find C_n such that $u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$ holds.

If we write $f(x)$ as a fourier sine series, we can match the coefficients.

Let's write $f(x)$ as a fourier sine series on $[0, L]$: i.e.

$$f(x) \approx \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}) \tag{4}$$

where $b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$

$\Rightarrow$ With comparing (3) and (4) $\rightarrow C_n = b_n$

Finally, the solution for IBVP is:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}) e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$$

- Homogeneous boundary conditions (Dirichlet: $u(0,t) = u(L,t) = 0$)
- Neumann: $u_x(0,t) = U_x(L,t) = 0$
- if it's not equal to 0 it's inhomogeneous
- If $u_t = \alpha u_{xx} + G$ it's inhomogeneous

2.2 Example 2

Same as example 1: $f(x) = x(L-x)$, $0 < x \leq L$

To solve, we use the method of separation of variables: $u(x,t) = X(x)T(t)$

Step 1: $u_t = X\dot{T}$; $U_x = X'T$; $U_{xx} = X''T$

Substitute into PDE and separate variables:

$$X\dot{T} = \alpha X''T \longrightarrow \frac{\dot{T}}{\alpha T} = \frac{X''}{X} = -\lambda$$

where λ is a constant.

Step 2: Boundary conditions.

$$u(0,t) = 0 \longrightarrow X(0) = 0$$
$$u(L,t) = 0(L) = 0$$

Step 3: Solve the eigenvalue problem for $X(x)$:
$X'' + \lambda X = 0$, $X(0) = 0 = X(L)$
Hence, the solution:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

$$X_n = \sin(\frac{n\pi}{L}x)$$

where $n \in N$
Step 4: For each λ_n find $T_n(t)$:

$$\frac{1}{\alpha}\frac{\dot{T_n}}{T_n} \rightarrow T_n(t) = e^{-\alpha\lambda_n t} = e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$$

Step 5: use superposition and linearity to construct a general series:

$$u(x,t) = \sum_{n=1}^{\infty} C_n u_n(x,t) = \underbrace{\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}}_{X_n(x)T_n(t)}$$

Step 6: Apply initial conditions:

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi x}{L}) = x(L-x)$$

Write $x(L-x)$ as a Fourier sine series:

$$x(L-x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L})$$

$$b_n = \frac{2}{L}\int_0^L x(L-x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = -\frac{2}{L}x(L-x)\frac{L}{n\pi}\cos(\frac{n\pi x}{L})\Big|_0^L + \frac{2}{n\pi}\int_0^L (L-2x)\cos(\frac{n\pi x}{L})dx$$

$$= 0 + \frac{2}{n\pi}\int_0^L (L-2x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2L}{(n\pi)^2}(L-2x)\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{4L}{(n\pi)^2}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{-4L^2}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L = \frac{4L^2}{(n\pi)^3}\left((-1)^{n+1}+1\right)$$

Step 7: Match the initial condition of the series solution ($C_n = b_n$)

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4L^2}{(n\pi)^3}\left((-1)^{n+1}+1\right)\sin\left(\frac{n\pi x}{L}\right)e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$$

Note that all $\sin(\frac{n\pi x}{L})$ terms are linearly independent (orthogonal)

2.3 Example 3

Similar to example 1 but with Neumann boundary conditions.

Please find the examples in the pdf "Heat / diffusion examples" on Canvas

Solution:

Step 1:

$$u(x,t) = X(x)T(t) \to \frac{\dot{T}}{\alpha T} = \frac{X''}{X} = -\lambda$$

Step 2:

$$u_x(0,t) = X'(0)T(t) = 0 \to X'(0) = 0$$

$$u_x(L,t) = X'(L)T(t) = 0 \to X'(L) = 0$$

Step 3: Solve the BVP with the conditions

$$X'' + \lambda X = 0$$

$$X'(0) = 0 = X'(L)$$

$\Rightarrow$ P2 problem. Cosine series.

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

and

$$X_n(x) = \cos(\frac{n\pi x}{L})$$

for $n \in N$

and $\lambda_0 = 0; X_0(x) = 1$

Step 4: Solving the IVP

For each λ_n and X_n, there is a T_n such that

$$\frac{\dot{T}_n}{T_n} = -\alpha\lambda_n \to T_n = e^{-\alpha\lambda_n t}$$

Step 5:

For $\lambda_0 = 0 \to u_0(x,t) = 1$

For $\lambda_n = \left(\frac{n\pi}{L}\right)^2 \to u_n(x,t) = \cos\left(\frac{n\pi x}{L}\right) e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$

PDE is linear and homogeneous $\Rightarrow$ we may superimpose the solutions in a linear combination:

$$u(x,t) = \sum_{n=0}^{\infty} d_n u_n(x,t) = d_0 + \sum_{n=1}^{\infty} d_n \cos\left(\frac{n\pi x}{L}\right) e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$$

Step 6: Initial conditions

$$u(x,0) = f(x) = d_0 + \sum_{n=1}^{\infty} d_n \cos\left(\frac{n\pi x}{L}\right)$$

If we rewrite as a fourier series (cosine), we find that $d_0 = a_0/2$ and that $d_n = a_n$

if we take the even extension of $f(x)$, to $[-L, 0]$ interval, then we know $f(x)$ has a fourier cosine series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow d_0 = \tfrac{a_0}{2}; \quad d_n = a_n = \tfrac{2}{L} \int_0^L f(x) \cos\left(\tfrac{n\pi x}{L}\right) dx$$

Step 7:
Thus, the solution is

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$$