MATH 316 Lecture 11

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1 Introduction

- Check Canvas announcements regarding midterm and such (2 new announcements)

2 Recap of Last Lecture

Last lecture, we finished up the heat / diffusion equation.

- Homogeneous equations and boundary conditions (Neumann and Dirichlet)
- Inhomogeneous equations and boundary conditions
- Developed general strategies for splitting inhomogeneous equations into a steady state (Which takes care of inhomogeneous parts, incl. boundary conditions) and a transient part (Satisfying a classic homogeneous diffusion equation)

Today's lecture is about a new equation: Wave equation.
Two methods to solve:

- Applying separation of variables
- We'll talk about the second method tomorrow

3 Wave equation

Takes the following form:

$$y_{tt} = a^2 y_{xx}$$

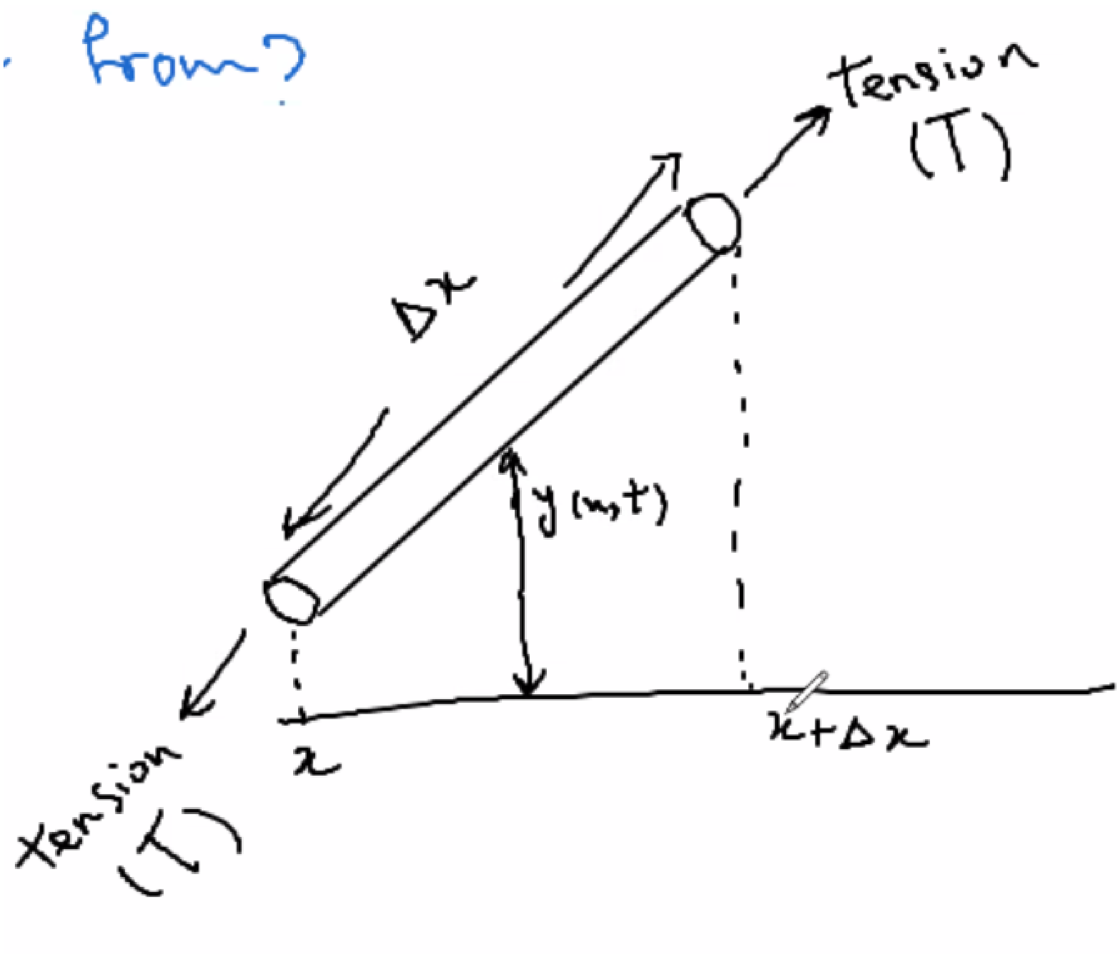
Two derivatives with respect to time, and two derivatives with respect to x.

Physically, $a = \left[\frac{T}{\rho}\right]^{\frac{1}{2}}$ (A string under tension), where T is stress / tension, and ρ is density. Can also be written as $a = \left[\frac{E}{\rho}\right]^{\frac{1}{2}}$ (Elastic bar), where $E =$ elastic stress.

Boundary conditions: We have two x derivatives $\rightarrow$ 2 conditions needed.

Initial conditions: We also need 2 initial conditions (Because we have 2 time derivatives). This is the main difference between the wave equation and the heat equation. $y(x, 0) = C$ (Initial displacement), and $y_t(x, 0) = k$ (initial velocity)

Where did the wave equation come from?



Derivation for small $\left|\frac{\partial y}{\partial x}\right|$
String of density ρ (with units of $[\frac{kh}{m}]$)
Force balance in the y direction:

$$\rho \Delta x \frac{\partial^2 y}{\partial t^2} = m \cdot \vec{a} = \sum F_y$$

$$\rho \Delta x \frac{\partial^2 y}{\partial t^2} = T \frac{\partial y}{\partial x}(x + \Delta x, t) - T \frac{\partial y}{\partial x}(x, t)$$

Divide by Δx and let $\Delta x \to 0$:

$$\Rightarrow \rho \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2}$$

N.B length of element is $\Delta x (1 + (\frac{\partial y}{\partial x})^2)^{\frac{1}{2}} \approx \Delta x$

Wave equation:

$$y_{tt} = a^2 y_{xx}$$

$$\text{BC: } y(0,t) = y(L,t) = 0$$
$$\text{IT: } y(x,0) = f(x) \text{ and } y_t(x,0) = g(x)$$

The idea is to split the solution into two parts.

- Problem 1: Initial velocity, but no displacement of string
  - $w_{tt} = a^2 w_{xx}$
  - $w(0,t) = w(L,t)$
  - $w(x,0) = 0$ for $0 < x < L$
  - $w_t(x,0) = g(x)$ for $0 < x < L$
- Problem 2: Initial displacement, but no velocity of spring
  - $z_{tt} = a^2 z_{xx}$
  - $z_t(x,0) = f(x)$ for $0 < x < L$
  - $z_t(x,0) = 0$ for $0 < x < L$

Solve problems 1 and 2: $y(x,t) = w(x,t) + z(x,t)$

3.1 Step 1: Solving Problem 1

$$w(x,t) = X(x)T(t) \qquad \underbrace{\Rightarrow}_{\text{Substitute into PDE}} \qquad X\ddot{T} = a^2 X'' T$$

Divide by $a^2 XT$:

$$\Rightarrow \frac{1}{a^2}\frac{\ddot{T}}{T} = \frac{X''}{X} = -\lambda$$

where λ is a constant.
First, boundary value problem:

$$X'' + \lambda X = 0$$

Boundary conditions are $w(0,t) = X(0)T(t) \Rightarrow X(0) = 0$, and $w(L,t) = X(L)T(t) = 0 \Rightarrow X(L) = 0$
It is a P1 eigenvalue problem.
Therefore, the eigenvalue problem for $X(x)$ is exactly as for heat / diffusion equation.
Therefore:

$$\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

and

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

where $n \in N$ (Natural numbers; 1,2,3,...)
IVP is:

$$\frac{1}{a^2}\frac{\ddot{T}_n}{T_n} = -\lambda_n \Rightarrow \ddot{T}_n + \left(\frac{an\pi}{L}\right)^2 T_n = 0$$

Therefore:

$$T_n(t) = A_n \cos\left(\frac{an\pi}{L}t\right) + B_n \sin\left(\frac{an\pi}{L}t\right)$$

How about the initial condition?
$w(x,0) = 0 \longrightarrow X(x)T(0) = 0 \Rightarrow T(0) = 0$ and $w_t(x,0) = g(x)$
$T_n(0) = 0 \Rightarrow A_n = 0$

As a result:

$$T_n(t) = B_n \sin\left(\frac{an\pi}{L}t\right)$$

Now, note that PDE, boundary conditions, and $w(x,0)$ are homogeneous. Therefore, we can superimpose solutions.

$$w(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{an\pi}{L}t\right)$$

We need to find B_n. To find this, we use the second initial condition.

$$w_t(x,0) = \sum_{n=1}^{\infty} B_n \left(\frac{an\pi}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{an\pi}{L}t\right)$$

$$w_t(x,0) = \sum_{n=1}^{\infty} B_n \left(\frac{an\pi}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = g(x)$$

If we write a Fourier sine series for $g(x)$, we can match up the coefficients:
To make this work, represent $g(x)$ as a Fourier sine series on $[0,L]$:

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \Rightarrow b_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

We know this series converges, so we match up the coefficients.

$$B_n = b_n \tfrac{L}{n\pi a} \text{ for } n \in N.$$

$$\Rightarrow w(x,t) = \sum_{n=1}^{\infty} b_n \frac{L}{n\pi a}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi a t}{L}\right)$$

3.2 Step 2: Solving Problem 2

$$z_{tt} = a^2 z_{xx}$$

Initial boundary conditions: $z(0,t) = Z(L,t) = 0$ and $Z(x,0) = f(x)$; $z_t(x,0) = 0$ for $0 < x < L$
Solution: Similarly, we use separation of variables and we assume that z is a product of X and T:

$$z(x,t) = X(x)T(t)$$

(Note that these are different X and T than in step 1!!)
The solution for the eigenvalue problem is a P1 problem again.

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

$$\lambda_n = \left(\tfrac{n\pi}{L}\right)^2 \text{ for } n \in N$$

For the IVP part,

$$\ddot{T}_n + \left(\frac{n\pi a}{L}\right)^2 T_n = 0$$

$$\Rightarrow T_n(t) = A_n \cos\left(\frac{n\pi a}{L}t\right) + B_n \sin\left(\frac{n\pi a}{L}t\right)$$

Now, $z_t(x,0) = X(x)\dot{T}(0) = 0 \Rightarrow \dot{T}_n(0) = 0 \Rightarrow B_n = 0$
So, let's superimpose the solution:

$$z_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi a}{L}t\right)$$

Use the other initial condition and find A_n:

The other initial condition tells us:

$$f(x) = z(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

Suppose we compute the Fourier sine series for $f(x)$, Then,

$$b_n' = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Note that prime is NOT a derivative, just used to denote that it's a different b_n.

$$\Rightarrow z(x,t) = \sum_{n=1}^{\infty} b_n' \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi a}{L}t\right)$$

3.3 Step 3

$$y(x,t) = x(x,t) + z(x,t)$$

$$y(x,t) = \sum_{n=1}^{\infty}\left[b_n \frac{L}{n\pi a}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi at}{L}\right) + b_n' \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi a}{L}t\right)\right]$$

We can factor[1]:

$$y(x,t) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[\frac{b_n L}{n\pi a}\sin\left(\frac{n\pi a}{L}t\right) + b_n'\cos\left(\frac{n\pi a}{L}t\right)\right]$$

For Neumann boundary conditions, the procedure is exactly the same as Dirichlet boundary conditions. (Using the PDF file posted on Canvas – Wave Equations, under week 4. Posted at the bottom of this document.)

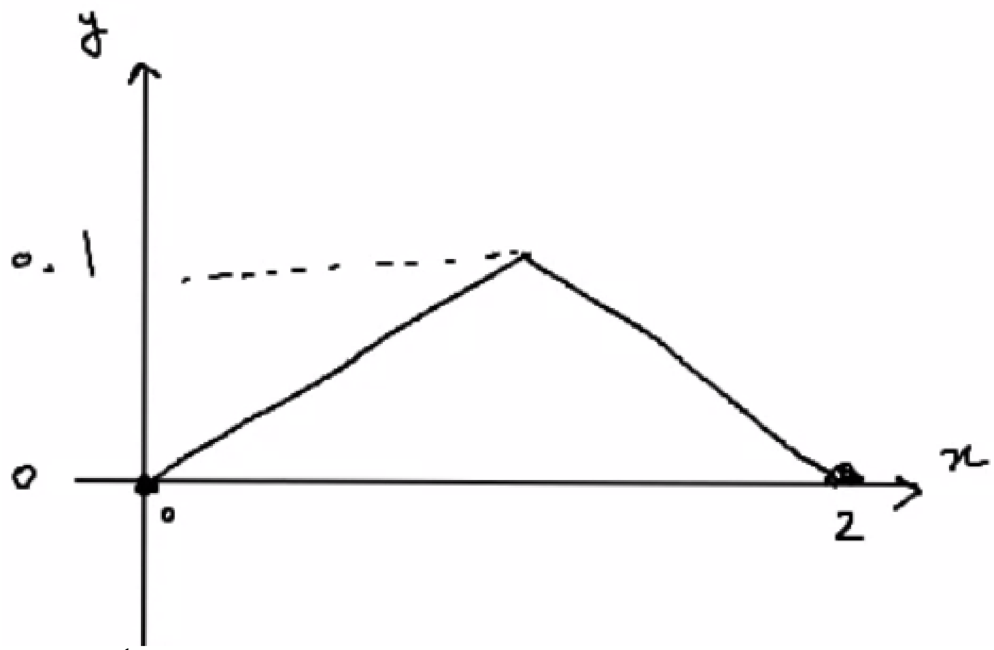
4 Example 8

(Example 8 of the pdf)

Solve the IBVP $y_{tt} = y_{xx}$

Initial conditions: $y(0,t) = y(2,t)$

[1] Again, b_n' is not a derivative!



$y(x,0) = \begin{cases} 0.1x & 0 \leq x \leq 1 \\ 0.1(2-x) & 1 \leq x \leq 1 \end{cases}$
$y_t(x,0) = 0 = g(x) \Rightarrow b_n = 0 \ \forall n$
Solution: a = 1 and L = 2. (There is only initialdisplacement $z(x,t)$ problem)

$$b'_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx = 0.1\int_0^1 x\sin\left(\frac{n\pi x}{2}\right) + 0.1\int_1^2 (2-x)\sin\left(\frac{n\pi x}{2}\right)$$

$$b'_n = -\frac{0.2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + 0 + \frac{0.4}{(n\pi)^2}\cdot\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1 - 0 + \frac{0.2}{n\pi}\cos(\frac{n\pi}{2}) - \frac{0.4}{(n\pi)^2}\sin\left(\frac{n\pi x}{2}\right)\Big|_1^2$$

$$b'_n = \frac{0.8}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

$$y(x,t) = \sum_{n=1}^{\infty}\frac{0.8}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{2}\right)\cos\left(\frac{n\pi}{2}t\right)$$

N.B[2]:
$\sin(\frac{n\pi}{2}) = 1, 0, -1, 0, 1, 0, ...$ for $n \in N$
Thus, we could write $n = 2k-1$ and $b'_k = (-1)^{k+1}$ for $k \in N$

$$y(x,t) = \sum_{k=1}^{\infty}\frac{0.8(-1)^{k+1}}{(2k-1)^2\pi^2}\sin\left(\frac{(2k-1)\pi}{2}x\right)\cos\left(\frac{(2k-1)\pi}{2}t\right)$$

5 Example 9

$$y_{tt} = y_{xx}$$

$y(0,t) = 0$ and $y(1,t) = 0$
$y(x,0) = 0$ and $y_t(x,0) = \sin(5\pi x)$
Solution: a = 1 and L = 1 and $f(x) = 0 \Rightarrow$ we only have the velocity problem to solve.
Need to find the Fourier sine series for $g(x)$:

[2]Nota bene. Used to denote an important point.

$$\sum_{n=1}^{\infty} b_n \sin(n\pi x) = g(x) = \sin(5\pi x)$$

But this is already a fourier series. For any n value $\neq 5$, $b_n = 0$. $b_5 = 1$.
Therefore, the general solution

$$y(x,t) = \frac{1}{5\pi} \sin(5\pi x) \sin(5\pi t)$$

6 Wave Equations PDF

Wave Equation

The wave equation takes the form:

$$\boldsymbol{y_{tt} = a^2 y_{xx}}$$

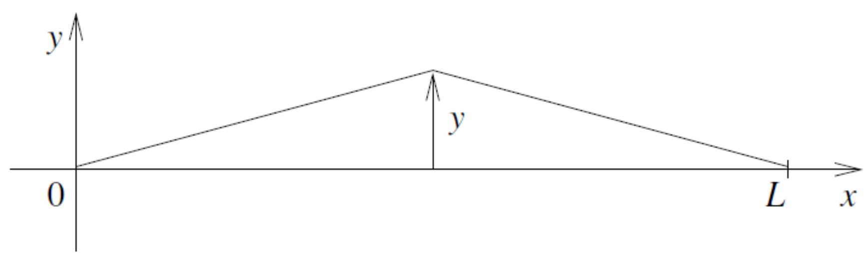
Physically, $a = [T/\rho]^{0.5}$ (a string under tension) or $a = [E/\rho]^{0.5}$ (elastic bar)

Boundary conditions?

Initial conditons?

Typical IBVP for the wave equation looks like this:

$$y_{tt} = a^2 y_{xx}$$
$$y(0,t) = 0, \quad y(L,t) = 0,$$
$$y(x,0) = f(x)$$
$$y_t(x,0) = g(x)$$



Superposition and separation of variables

1. Split $y(x,t)$ and the initial "data" into 2 problems:

$$y(x,t) = w(x,t) + z(x,t)$$

Problem 1: initial velocity, but no displacement of string

$$\begin{aligned}
&w_{tt} = a^2 w_{xx}, \\
&w(0,t) = w(L,t) = 0, \\
&w(x,0) = 0 && \text{for } 0 < x < L, \\
&w_t(x,0) = g(x) && \text{for } 0 < x < L.
\end{aligned}$$

Problem 2: initial displacement, but no velocity of string

$$\begin{aligned}
&z_{tt} = a^2 z_{xx}, \\
&z(0,t) = z(L,t) = 0, \\
&z(x,0) = f(x) && \text{for } 0 < x < L, \\
&z_t(x,0) = 0 && \text{for } 0 < x < L.
\end{aligned}$$

2. Solve each problem by separation of variables

Exactly analogous procedure for Neumann boundary conditions

Period and frequency of the nth mode:

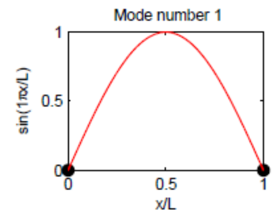
Modes of vibration:

- Note these are standing waves of wavelength $\lambda_n = 2L/n$

- Each mode: $n+1$ positions at which displacement is zero

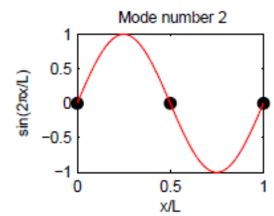
I: The fundamental mode of vibration with 2 nodes

$$X_1(x) = \sin\left(\frac{\pi x}{L}\right)$$



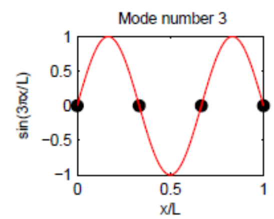
II: The second mode of vibration or first overtone with 3 nodes

$$X_2(x) = \sin\left(\frac{2\pi x}{L}\right)$$



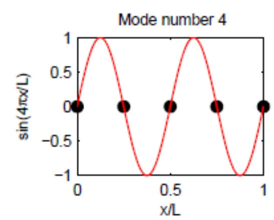
III: The third mode of vibration with 4 nodes

$$X_3(x) = \sin\left(\frac{3\pi x}{L}\right)$$



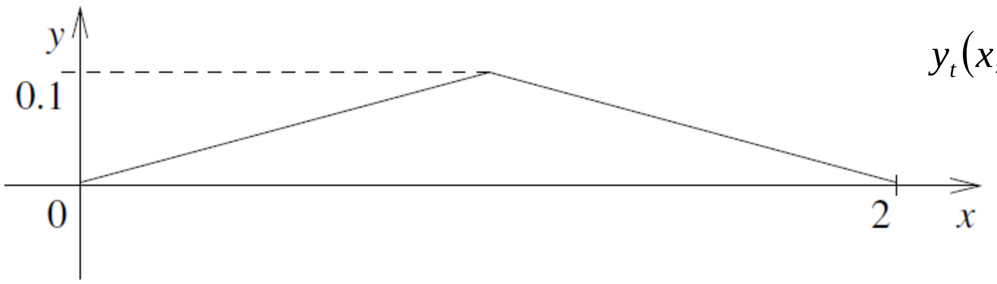
IV: The fourth mode of vibration with 5 nodes

$$X_4(x) = \sin\left(\frac{4\pi x}{L}\right)$$



Example 8: Solve the IBVP

$$
\begin{aligned}
&y_{tt} = y_{xx} \\
&y(0,t) = 0, \quad y(2,t) = 0, \\
&y(x,0) = \begin{cases} 0.1x & 0 \le x \le 1 \\ 0.1(2-x) & 1 < x \le 2 \end{cases} \\
&y_t(x,0) = 0
\end{aligned}
$$



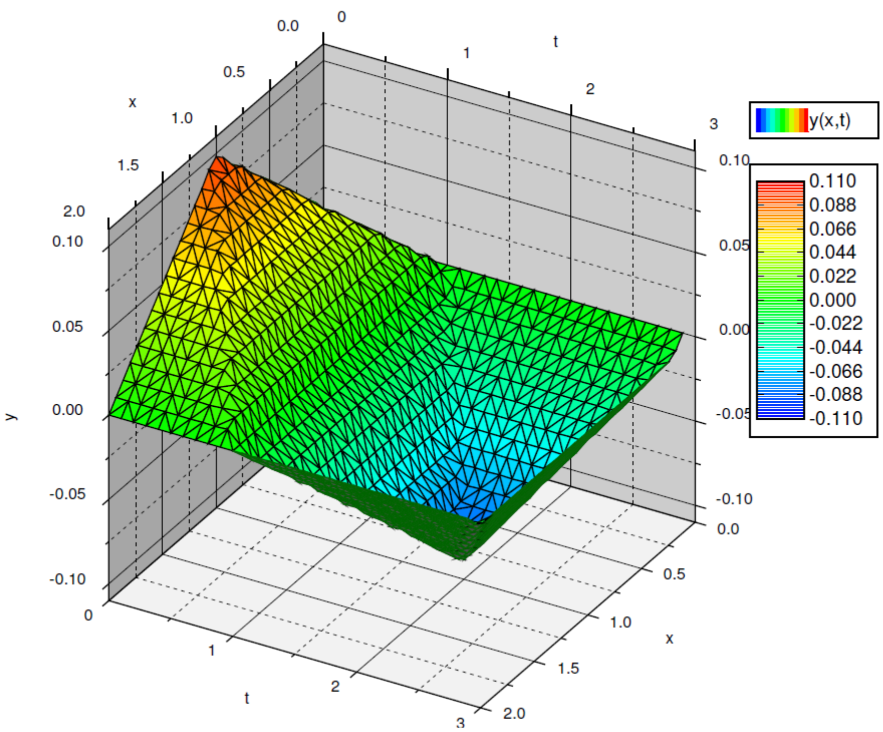


Figure 4.20: Shape of the plucked string for $0 < t < 3$.

Example 9: Solve the IBVP – what makes this one simple?

$$y_{tt} = y_{xx}$$
$$y(0,t) = 0, \quad y(1,t) = 0,$$
$$y(x,0) = 0$$
$$y_t(x,0) = sin\, 5\,\pi x$$

Wave Equation with Neumann boundary condition:

$$y_{tt} = a^2 y_{xx}$$
$$y_x(0,t) = 0, \quad y_x(L,t) = 0,$$
$$y(x,0) = f(x)$$
$$y_t(x,0) = g(x)$$

D'Alembert's solution:

$$y_{tt} = a^2 y_{xx}$$

Return to wave equation and see if we can guess a solution of exponential form:

$$y(x,t) = e^{ikx+\sigma t}$$

Why this form?

$$y_1(x,t) = e^{ik(x+at)}$$
$$y_2(x,t) = e^{ik(x-at)}$$

Is this form of solution more general – how about:

$$y_1(x,t) = F(x-at), \qquad y_2(x,t) = G(x+at)$$

Consider a change of variables: $\xi = x - at$, $\eta = x + at$

Suppose initial conditions:

$$y(x,0) = f(x)$$
$$y_t(x,0) = g(x)$$

Finally, D'Alembert's solution:

$$y(x,t) = \frac{1}{2}[f(x-at) + f(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\, ds$$

Above analysis has no boundary conditions!

Let $F_o(x)$ and $G_o(x)$ be the odd 2L-periodic extensions of $f(x)$ and $g(x)$, respectively.

$$y(x,t) = \frac{1}{2}[F_o(x-at) + F_o(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} G_o(\zeta)\, d\zeta$$

What is the relationship between d'Alembert's formula and our separation of variables solution?

$$\boxed{\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} b_n \frac{L}{n\pi a} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi a}{L}t\right) + c_n \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{n\pi a}{L}t\right) \\ &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) \left[b_n \frac{L}{n\pi a} \sin\left(\frac{n\pi a}{L}t\right) + c_n \cos\left(\frac{n\pi a}{L}t\right)\right]. \end{aligned}}$$

Region of influence & domain of dependence:

Example 10: Solve the following IVP using D'Alembert's method

$$y_{tt} = y_{xx}, \qquad -\infty < x < \infty$$
$$y(x,0) = \begin{cases} 1, & |x| < 1 \\ 0, & \text{otherwise} \end{cases}$$
$$y_t(x,0) = 0$$

Example 11: Solve the following IVP using D'Alembert's method

$$
\begin{aligned}
&y_{tt} = y_{xx}, \\
&y(0,t) = 0, \quad y(1,t) = 0, \\
&y(x,0) = \begin{cases} 0, & 0 \le x < 0.45 \\ 20(x-0.45), & 0.45 \le x < 0.5 \\ 20(0.55-x), & 0.5 \le x < 0.55 \\ 0, & 0.55 \le x \le 1 \end{cases} \\
&y_t(x,0) = 0
\end{aligned}
$$

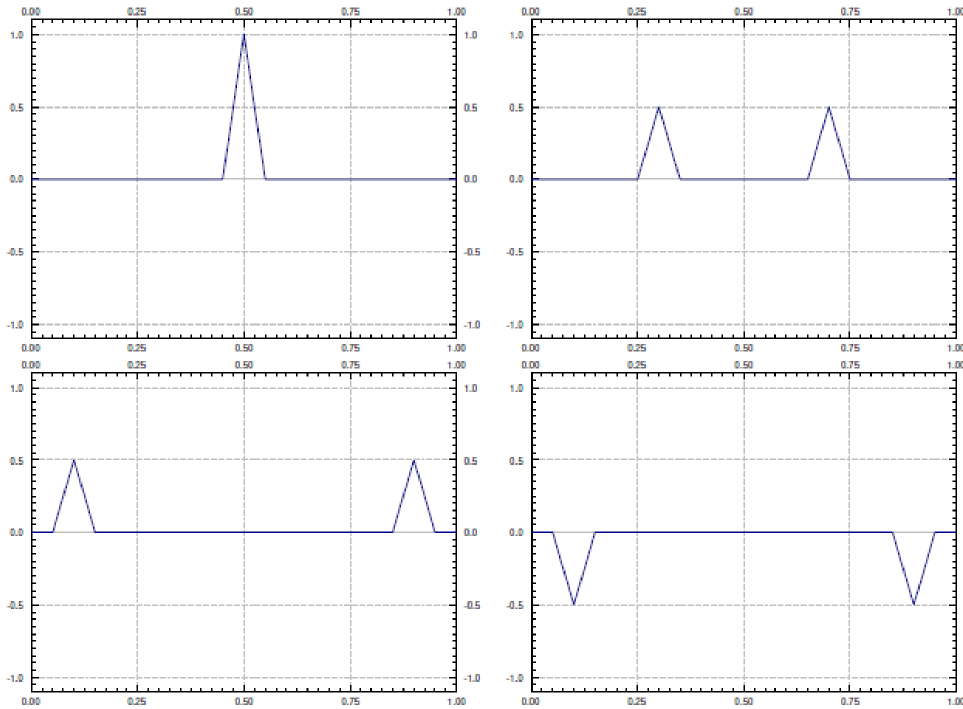


Figure 4.21: Plot of the d'Alembert solution for $t = 0$, $t = 0.2$, $t = 0.4$, and $t = 0.6$.