

# MATH 316 Lecture 12

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## 1 Wave Equations: Neumann Boundary Conditions

Solve IBVP:  $y_{tt} = a^2 y_{xx}$

Boundary conditions:  $y_x(0, t) = y_x(L, t) = 0 \Rightarrow$  Homogeneous Neumann boundary conditions.

Initial conditions:  $y(x, 0) = f(x)$ ,  $y_t(x, 0) = 0 = g(x)$  (Therefore zero initial velocity, with specified initial displacement).

Split into two problems (w and z). Because  $g(x) = 0$ , we only have the z equation. For the solution, we use separation of variables.<sup>1</sup>

$$\left. \begin{aligned} X'' + \lambda X &= 0 \\ \Rightarrow X_n(x) &= \cos\left(\frac{n\pi x}{L}\right) \\ \lambda_n &= \left(\frac{n\pi}{L}\right)^2 \\ X_0(x) &= 1 \Rightarrow \lambda_0 = 0 \end{aligned} \right| \begin{aligned} \ddot{T} + a^2 \lambda T &= 0 \\ \dot{T}_n(0) &= 0 \\ T_n(t) &= A_n \cos\left(\frac{n\pi a}{L} t\right) \\ T_0(t) &= 1 \end{aligned}$$

The solution would be:

$$y(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi a}{L} t\right)$$

Note that  $A_0$  is the multiplication of  $X_0 T_0$  terms.

$$\text{At } t = 0 \longrightarrow y(x, 0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) = f(x)$$

We construct Fourier cosine series for  $f(x)$ .

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \\ \Rightarrow A_0 &= \frac{a_0}{2}, A_n = a_n \end{aligned}$$

**Note: We can do a similar solution if initial velocity is given.**

### 1.1 Recap

- Introduced wave equation
- Developed separation of variables method to find its solution
  - Dirichlet and Neumann boundary conditions
  - Examples and normal modes

Now: New method.

- New look at the wave equation and we solve the wave equation using **D'Alembert's solution**.

<sup>1</sup>Note that left and right side of table are entirely separate.

## 2 D'Alembert's Solution

$$y_{tt} = a^2 y_{xx}$$

Let's see if we can guess a solution of exponential format.

$$y(x, t) = e^{ikx + \sigma t}$$

where  $k$  and  $\sigma$  are constants.<sup>2</sup>

Substitute the guessed solution into the PDE.

$$y_{tt} = \sigma^2 e^{ikx + \sigma t}$$

$$y_{xx} = -k^2 e^{ikx + \sigma t}$$

Now, substitute this into the PDE:

$$(\sigma^2 + a^2 k^2) e^{ikx + \sigma t} = 0$$

$$\Rightarrow \sigma = \pm ika$$

$$y_1(x, t) = e^{ik(x - at)}$$

$$y_2(x, t) = e^{ik(x + at)}$$

$x \pm at$  are known as characteristics, these are lines in  $x$  and  $t$  along which the initial conditions (and general information) travels.

The question here is this: Can this form of solution be more general such that we can apply it to any wave equation?

$$y_1(x, t) = F(x - at), y_2(x, t) = G(x + at)$$

Can we find a general equation that satisfies the wave equation?

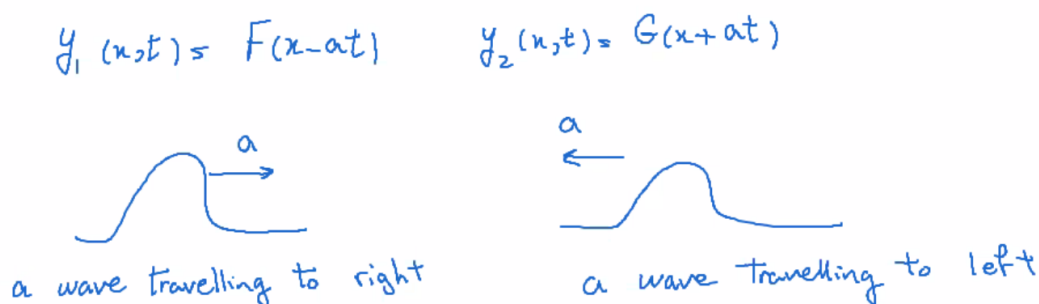


Figure 1:  $F(x - at)$  is a wave travelling to the right with a speed of  $a$ .  $G(x + at)$  is a wave travelling to the left with a speed of  $a$ .

Hence, a general solution:

$$y(x, t) = F(x - at) + G(x + at)$$

Does it satisfy the PDE?

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<sup>2</sup>Try this guess solution with heat solution! You will find that it does work for heat equations.

$$y(x, 0) = f(x) \Rightarrow F(x) + G(x) = f(x) \quad (1)$$

$$y_t(x, 0) = g(x) \Rightarrow -aF'(x) + aG'(x) = g(x) \quad (2)$$

We get (2) from:

$$-aF(x) + aG(x) = \int_0^x g(s)ds + A$$

$$(1)xa + 2 \Rightarrow 2aG(x) + f(x) = \int_0^x g(s)ds + A$$

$$\Rightarrow G(x) = \frac{1}{2}f(x) + \frac{1}{2a} \int_0^x g(s)ds + \frac{A}{2a}$$

To find  $F(x)$ :

$$(1)xa - (2) \Rightarrow 2aF(x) = af(x) - \int_0^x g(s)ds - A$$

$$\Rightarrow F(x) = \frac{1}{2}f(x) - \frac{1}{2a} \int_0^x g(s)ds - \frac{A}{2a}$$

Now, substitute these into the general solution: (plug into  $y(x, t) = F(x - at) + G(x + at)$ )

This gives us:

$$\frac{1}{2}f(x - at) - \frac{1}{2a} \int_0^{x-at} g(s)ds + \frac{1}{2}f(x + at) + \frac{1}{2a} \int_0^{x+at} g(s)ds$$

Note that  $-\frac{A}{2a}$  and  $\frac{A}{2a}$  cancel.

$$y(x, t) = \frac{1}{2} \left[ \underbrace{f(x - at)}_{\text{half of init cond travels right}} + \underbrace{f(x + at)}_{\text{half of init cond travels left}} \right] + \frac{1}{2a} \int_{x-at}^{x+at} g(s)ds \quad (1)$$

N.B. Above analysis has no boundary conditions:  $-\infty < x < \infty$

What if the problem has boundary condition?

Let  $F^o(x)$  and  $G^o(x)$  be the odd<sup>3</sup> 2L-periodic extension of  $f(x)$  and  $g(x)$  respectively:

$$y(x, t) = \frac{1}{2} [F^o(x - at) + F^o(x + at)] + \frac{1}{2a} \int_{x-at}^{x+at} G^o(s)ds$$

Boundary conditions: <sup>4</sup>

$$y(0, t) = \frac{1}{2} \left[ \underbrace{F^o(-at) + F^o(at)}_{=0} + \underbrace{\frac{1}{2a} \int_{-at}^{at} G^o(s)ds}_{=0} \right] = 0$$

What's the relationship between D'Alembert's formula and the separation of variables?

$$y(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[ b_n \frac{L}{n\pi a} \sin\left(\frac{n\pi a}{L}t\right) + b'_n \cos\left(\frac{n\pi a}{L}t\right) \right]$$

Recall trig formulae:

$$\sin(A) \sin(B) = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$$

<sup>3</sup>(Assumes Dirichlet boundary conditions)

<sup>4</sup>Note that we are using the properties of odd functions to cancel out both F and G.

$$\sin(A) \cos(B) = \frac{1}{2} [\sin(A - B) + \sin(A + B)]$$

Let's apply these:

$$\begin{aligned} y(x, t) &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{b_n L}{n\pi a} \underbrace{\left[ \cos\left(\frac{n\pi}{L}(x - at)\right) - \cos\left(\frac{n\pi}{L}(x + at)\right) \right]}_{\text{Let's write this in integral format}} \rightarrow \\ &\hookrightarrow +b'_n \left[ \sin\left(\frac{n\pi}{L}(x - at)\right) - \sin\left(\frac{n\pi}{L}(x + at)\right) \right] \\ y(x, t) &= \frac{1}{2} \sum_{n=1}^{\infty} b'_n \left[ \sin\left(\frac{n\pi}{L}(x - at)\right) - \sin\left(\frac{n\pi}{L}(x + at)\right) \right] \rightarrow \\ &\hookrightarrow + \frac{1}{2a} \sum_{n=1}^{\infty} b_n \int_{x-at}^{x+at} \sin\left(\frac{n\pi s}{L}\right) ds \end{aligned}$$

Recall:

$$\sum_{n=1}^{\infty} b'_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

and

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = g(x)$$

$\Rightarrow$  d'Alembert's solution:

$$y(x, t) = \frac{1}{2} [f(x - at) + f(x + at)] + \frac{1}{2a} \int_{x-at}^{x+at} g(s) ds$$

Both methods give similar solution.